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On the Construction of Single-Peaked Solutions to a Singularly Perturbed Semilinear Dirichlet Problem

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1. INTRODUCTION

The aim of this paper is to construct a family of single-peaked solutions to the singularly elliptic problem

$$\begin{cases} \varepsilon^2 \Delta u - u + u^p = 0 & \text{in } \Omega, \\ u > 0 \text{ in } \Omega \text{ and } u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta = \sum_{i=1}^n (\partial^2/\partial x_i^2)$ is the Laplace operator, Ω is a bounded smooth domain in R^n, $\varepsilon > 0$ is a constant, and the exponent p satisfies $1 < p < (n+2)/(n-2)$ for $n \geqslant 3$ and $1 < p < \infty$ for $n = 2$.

Problem (1.1) arises in various applications, such as chemotaxis, population genetics, and chemical reactor theory, and it has been studied by a number of authors. During the past few years, the question whether the geometry or the topology of Ω was responsible for the solvabity and/or the multiplicity of solutions of problems like (1.1) has been extensively studied; see [6–10]. Especially, in [6] and [7], Benci and Cerami have studied the multiplicity of solutions of (1.1) when ε is sufficiently small, using Category and Morse theory. However, they do not give explicit construction of solutions, nor do they study the properties of the solutions. The first result on spiky solutions of (1.1) is due to Ni and Wei. In [18], we have studied the shape and peak location of "least-energy" solutions. More precisely, we first define the energy as

$$J_\varepsilon(u) = \frac{1}{2} \int_\Omega (\varepsilon^2 |\nabla u|^2 + u^2) - \frac{1}{p+1} \int_\Omega u_+^{p+1}, \tag{1.2}$$

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where $u_+ = \max\{u, 0\}$, for $u \in H_0^1(\Omega)$. The well known *Mountain-Pass Lemma* implies that

$$c_\varepsilon = \inf_{h \in \Gamma} \max_{0 \leqslant t \leqslant 1} J_\varepsilon(h(t)) \tag{1.3}$$

is a positive critical value of J_ε, i.e., $c_\varepsilon = J_\varepsilon(u_\varepsilon)$ and u_ε is a solution of (1.1), where Γ is the set of all continuous paths joining the origin and a fixed nonzero element e in $H_0^1(\Omega)$ with $e \geqslant 0$ and $J_\varepsilon(e) = 0$. It is showed in [18] that J_ε is independent of the choice of e and u_ε is called a "least-energy" solution. We then proved the following:

THEOREM A. *Let u_ε be a least-energy solution to* (1.1). *Then, for ε sufficiently small, we have*

(i) *u_ε has at most one local maximum and it is achieved at exactly one point P_ε in Ω. Moreover, $u_\varepsilon(\cdot + P_\varepsilon) \to 0$ in $C^1_{\text{loc}}(\Omega - P_\varepsilon \backslash \{0\})$ where $\Omega - P_\varepsilon := \{x - P_\varepsilon \mid x \in \Omega\}$ and $u_\varepsilon(P_\varepsilon) \to w(0)$, where w is the unique solution of*

$$\begin{cases} \Delta w - w + w^p = 0 & \text{in } R^n, \\ w > 0,\ w(0) = \max_{z \in R^n} w(z), \\ w(z) \to 0 & \text{as } |z| \to \infty. \end{cases} \tag{1.4}$$

(ii) *$d(P_\varepsilon, \partial\Omega) \to \max_{P \in \Omega} d(P, \partial\Omega)$ as $\varepsilon \to 0$.*

In this paper, we show that a kind of converse of Theorem A is true. We shall construct a family of single-peaked solutions to problem (1.1) for ε sufficiently small at any strictly local maximum point of $d(P, \partial\Omega)$. The precise statement is:

THEOREM 1.1. *Let $P_0 \in \bar{\Omega}$ be a strictly local maximum point of the distance function $d(P, \partial\Omega)$, i.e., there exists a neighborhood $B_\delta(P_0) \subset \Omega$ such that $d(X, \partial\Omega) < d(P_0, \partial\Omega)$ for all $X \in B_\delta(P_0)$, $X \neq P_0$. Then there is an $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$, problem* (1.1) *has a solution u_ε with the property that u_ε has exactly one local maximum point P_ε in Ω, $u_\varepsilon(P_\varepsilon) \to w(0)$ and $u_\varepsilon(\cdot + P_\varepsilon) \to 0$ in $C^1_{\text{loc}}(\bar{\Omega} - P_\varepsilon \backslash \{0\})$, where w is the unique solution of* (1.4). *Moreover, $P_\varepsilon \to P_0$ as $\varepsilon \to 0$.*

A particular example is a domain with k-handles (see Fig. 1). In this case, Theorem 1.1 asserts that there are at least k solutions to problem (1.1) and each handle contributes a single-peaked solution. Note that in this case, the domain has trivial topology. In [11], Dancer studied problem (1.1) in the case of domains with two handles (dumbbell-shaped) and constructed two solutions. However, in [11], it is assumed that the

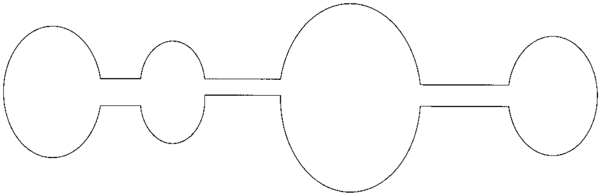

FIG. 1. Domains with handles.

domain is symmetric and the "neck" is sufficiently small. In our theorem, we do not assume any symmetry and the length of the "neck" can be arbitrary. It seems extremely interesting to see how the geometry of the domain plays a role in the existence of "spiky solutions." Partial progress has been done in [27].

Our method in proving Theorem 1.1 is a combination of the "vanishing viscosity method" and the "energy method" developed in [16, 17]. It should be remarked that, in [2, 4], they proved a similar result for the single-peaked boundary spike solutions to a singularly perturbed semilinear Neumann problem. In their case, the mean curvature on the boundary plays an important role. However, in our case, the major difficulty comes from the exponentially smallness in the corrector term of the energy expansion. Traditional techniques such as matched asymptotics do not work here. We believe that this is the first result in constructing "spiky" solutions to problem (1.1).

Remark. (1) By Theorem 1.1, if the function $d(P, \partial\Omega)$ has k strictly local maximum point, then for ε sufficiently small, problem (1.1) has at least k solutions. This, in some cases, is an improvement of the multiplicity results obtained in [6–8] and also answers some questions raised in [6–11].

(2) We note that in [16, 17], Ni and Takagi studied a related problem,

$$\begin{cases} \varepsilon^2 \Delta u - u + u^p = 0,\ 1 < p < \dfrac{n+2}{n-2} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

and obtained results similar to Theorem A. When $p = (n+2)/(n-2)$, similar concentration results have been obtained in [1–3, 15]. More general results have been obtained by [19–23]. Multiplicity of solutions to (1.5) have been studied in [26, 28].

Other kinds of concentrations for other problems are studied in [4, 5, 13, 24–26].

This paper is organized as follows. In Section 2, we state some notation and preliminaries. Section 3 provides a proof of Theorem 1.1. The proofs of some technical lemmas are postponed to Section 4.

Throughout this paper, unless otherwise stated, the letter C will always denote various generic constants which are independent of ε, for ε sufficiently small.

2. NOTATION AND PRELIMINARIES

We shall follow the notation in [12]. Let $P\in\Omega$. We now define $\Omega_{\varepsilon,P}=\{y \mid \varepsilon y+P\in\Omega\}$. Let U be a bounded smooth domain in R^n. We then set $P_U w$ to be the unique solution of

$$\begin{cases} \Delta u - u + w^p = 0, & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases} \tag{2.1}$$

where w is the unique solution of (1.4).

By the Maximum Principle, $0\leqslant P_U w < w$.

Let

$$x=\varepsilon y+P,\ \varphi_\varepsilon, P(y)=w(y)-P_{\Omega_{\varepsilon,P}}w(y)$$

$$\psi_{\varepsilon,P}(x) = -\varepsilon\log\varphi_{\varepsilon,P}(y),\ \beta=\frac{1}{\varepsilon}$$

$$V_{\varepsilon,P}(y)=e^{\beta\varphi_{\varepsilon,P}(P)}\varphi_{\varepsilon,P}(y),\ \psi_\varepsilon(P)=\psi_{\varepsilon,P}(P).$$

It is easy to see that $\psi_{\varepsilon,P}(x)$ is the unique solution of

$$\begin{cases} \varepsilon^2\Delta u - |\nabla u|^2 + 1 = 0, & \text{in } \Omega, \\ u(x) = -\varepsilon\log w\left(\dfrac{x-P}{\varepsilon}\right), & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

The following properties are proved in [18].

PROPOSITION 2.1. (i) *There exists a constant* C_1 *such that*

$$\|\psi_{\varepsilon,P}(x)\|_{L^\infty(\Omega)}\leqslant C_1.$$

(ii) $\psi_{\varepsilon,P}(x)\to\psi_P(x)$ *uniformly on* Ω *as* $\varepsilon\to 0$, *where* $\psi_P(x)$ *in the unique viscosity solution of the Hamilton–Jacobi equation*

$$\begin{cases} |\nabla u|^2=1 & \text{in } \Omega, \\ u(x)=|x-P| & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Indeed, $\psi_P(x)=\inf_{z\in\partial\Omega}(|z-P|+L(x,z))$, *where* $L(x,z)$ *is the infimum of* T *such that there exists* $\xi(s)\in C^{0,1}([0,T],\bar{\Omega})$ *with* $\xi(0)=x$, $\xi(T)=z$ *and* $|d\xi/ds|\leqslant 1$ *a.e., in* $[0,T]$. *Furthermore* $\psi_P(P)=2d(P,\Omega)$.

(iii) *For every sequence* $\varepsilon_k\to 0$, *there is a subsequence* $\varepsilon_{k_l}\to 0$, *such that* $V_{\varepsilon_{k_l},P}\to V_P$ *uniformly on every compact set of* R^n, *where* V_P *is a positive solution of*

$$\begin{cases} \Delta u-u=0 & in\ R^n, \\ u(0)=1,\ u>0 & in\ R^n. \end{cases} \tag{2.4}$$

Furthermore, for any $\sigma_1>0$,

$$\sup_{y\in\bar{\Omega}_{\varepsilon_{k_l},P}} e^{-(1+\sigma_1)|y|}\,|V_{\varepsilon_{k_l},P}(y)-V_P(y)|\to 0 \qquad as \quad \varepsilon_{k_l}\to 0. \tag{2.5}$$

(iv) *Let* V *be an arbitrary solution of* (2.4). *Then we have*

$$2\gamma:=\int_{R^n} w^pV_*=\int_{R^n} w^pV>0, \tag{2.6}$$

where $V_*(r)$ *is the unique positive radial solution of* (2.4).

Remark. It is easy to see that

$$|\psi_{\varepsilon,P}(x)-\psi_{\varepsilon,Q}(x)|\leqslant C\varepsilon\,|\log\varepsilon|+C\,|P-Q|, \tag{2.7}$$

where $P,\ Q\in\Omega$. Hence if $P_\varepsilon\to P\in\Omega$, then

$$|\psi_\varepsilon(P_\varepsilon)-\psi_\varepsilon(P)|\to 0 \qquad \text{as} \quad \varepsilon\to 0.$$

Therefore $\psi_\varepsilon(P_\varepsilon)\to 2d(P,\partial\Omega)$ as $\varepsilon\to 0$.

We also note that in the proof of (2.5) in [18], we actually proved the following fact: for any $\sigma_1>0$, there exists $C>0$, such that

$$V_{\varepsilon,P}(y)\leqslant Ce^{(1+\sigma_1)|y|}, \qquad \text{for all} \quad P\in\bar{B}_\delta(P_0) \text{ and } y \text{ in } \bar{\Omega}_{\varepsilon,P}. \tag{2.8}$$

We now introduce some other notations.

For $u \not\equiv 0$, $u \in W_0^{1,2}(\Omega)$, we define

$$K_\varepsilon(u) = \frac{\varepsilon^2 \int_\Omega |\nabla u|^2 + \int_\Omega u^2}{(\int_\Omega up+1)^{2/(p+1)}}, \quad \beta(u) = \frac{\int_\Omega x u^{p+1}}{\int_\Omega u^{p+1}}, \quad l(u) = \frac{\varepsilon^2 \int_\Omega |\nabla u|^2 + \int_\Omega u^2}{\int_\Omega u^{p+1}},$$

$$\langle u, v \rangle_{W_0^{1,2}(\Omega)} = \int_\Omega \nabla u \cdot \nabla v + \int_\Omega u \cdot v, \tag{2.9}$$

$$Lu = \Delta u - u + p w^{p-1} u.$$

Let P_0 be a fixed strictly local maximum point of the distance function $d(P, \partial\Omega)$. Let $\delta > 0$ be such that $B_{2\delta}(P_0) \subset \bar{\Omega}$. We set

$$B = \{u \in W_0^{1,2}(\Omega) : \beta(u) \in B_\delta(P_0)\} \tag{2.10}$$

(we can choose $\delta > 0$ small such that $d(P, \partial\Omega) < d(P_0, \partial\Omega)$ for all $P \neq P_0$, $P \in B_{2\delta}(P_0)$) and

$$A_\varepsilon = \inf\{K_\varepsilon(u) \mid u \in B\}. \tag{2.11}$$

Let w be the unique solution of (1.4). We set

$$I(w) = \frac{\int_{R^n} |\nabla w|^2 + \int_{R^n} w^2}{(\int_{R^n} w^{p+1})^{2/(p+1)}} = \left(\int_{R^n} w^{p+1} \right)^{(p+1)/(p-1)}. \tag{2.12}$$

Lemma 2.2. *Suppose that the domain of L is $W^{2,r}(R^n)(r>1)$, then* $\ker(L) = \operatorname{span}\{\partial w/\partial y_j;\ j = 1, ..., n\}$.

See [Lemma 4.2, [17]].

Lemma 2.3. *For ε sufficiently small, we have*

$$A_\varepsilon \leqslant \varepsilon^{(p-1)/(p+1)n}\{I(w) + \alpha_1 e^{-\beta\psi_\varepsilon(P_0)} + o(e^{-\beta\psi_\varepsilon(P_0)})\}, \tag{2.13}$$

where $\alpha_1 = 2(\int_{R^n} w^{p+1})^{-2/(p+1)}\gamma$ and γ is defined at (2.6).

Proof. Let $u(x) = P_{\Omega_{\varepsilon,P_0}} w((x - P_0)/\varepsilon) \in W_0^{1,2}(\Omega)$; then

$$\varepsilon^2 \int_\Omega |\nabla u|^2 + \int_\Omega u^2 = \varepsilon^n \left[\int_{\Omega_{\varepsilon,P_0}} |\nabla P_{\Omega_{\varepsilon,P_0}} w|^2 + \int_{\Omega_{\varepsilon,P_0}} |P_{\Omega_{\varepsilon,P_0}} w|^2 \right]$$

$$= \varepsilon^n \int_{\Omega_{\varepsilon,P_0}} w^p P_{\Omega_{\varepsilon,P_0}} w$$

$$= \varepsilon^n \int_{\Omega_{\varepsilon, P_0}} w^p [w - e^{-\beta\psi_\varepsilon(P_0)} V_{\varepsilon, P_0}]$$

$$= \varepsilon^n \left[\int_{\Omega_{\varepsilon, P_0}} w^{p+1} - e^{-\beta\psi_\varepsilon(P_0)} \int_{\Omega_{\varepsilon, P_0}} w^p V_{\varepsilon, P_0} \right].$$

For every sequence $\varepsilon_k \to 0$, there exists a subsequence $\varepsilon_{k_l} \to 0$ such that (2.5) holds. By Lebesgue's Dominated Convergence Theorem

$$\int_{\Omega_{\varepsilon, P_0}} w^p V_{\varepsilon_{k_l}, P_0} \to \int_{R^n} w^p V_{P_0} = 2\gamma.$$

Since γ is independent of the choices of ε_k, we have

$$\int_{\Omega_{\varepsilon, P_0}} w^p V_{\varepsilon, P_0} \to 2\gamma \qquad \text{as} \quad \varepsilon \to 0.$$

It follows that

$$\varepsilon^2 \int_\Omega |\nabla u|^2 + \int_\Omega u^2 = \varepsilon^n \left[\int_{R^n} w^{p+1} - 2\gamma e^{-\beta\psi_\varepsilon(P_0)} + o(e^{-\beta\psi_\varepsilon(P_0)}) \right]. \tag{2.14}$$

On the other hand,

$$\int_\Omega u^{p+1} = \varepsilon^n \int_{\Omega_{\varepsilon, P_0}} (P_{\Omega_{\varepsilon, P_0}} w)^{p+1} \, dy$$

$$= \varepsilon^n \int_{\Omega_{\varepsilon, P_0}} (w^{p+1} - (p+1) \, w_1^p e^{-\beta\psi_\varepsilon(P_0)} V_{\varepsilon, P_0})$$

where $w \geqslant w_1 \geqslant P_{\Omega_{\varepsilon, P_0}} w$. Similarly, Lebesgue's Dominated Convergence Theorem ensures that

$$\int_\Omega u^{p+1} = \varepsilon^n \left[\int_{R^n} w^{p+1} - 2(p+1) \, \gamma e^{-\beta\psi_\varepsilon(P_0)} + o(e^{-\beta\psi_\varepsilon(P_0)}) \right]. \tag{2.15}$$

Combining (2.14) and (2.15), we obtain

$$A_\varepsilon \leqslant K_\varepsilon(u) = \varepsilon^{n(p-1)/(p+1)} [I(w) + \alpha_1 e^{-\beta\psi_\varepsilon(P_0)} + o(e^{-\beta\psi_\varepsilon(P_0)})]$$

since $u \in B$ for ε sufficiently small.

LEMMA 2.4. *Let* $l_0 = \mathrm{dist}(\bar{B}_{2\delta}(P_0), \partial\Omega) > 0$. *Then there exists a positive constant* $C > 0$ *such that*

$$\left| \frac{\partial}{\partial P_i} e^{-(\psi_{\varepsilon, P}(x))/\varepsilon} \right| \leqslant \frac{C}{\varepsilon} e^{-(l_0/\varepsilon)} \qquad \text{for all} \quad x \in \Omega,\ P \in \bar{B}_{2\delta}(P_0). \tag{2.16}$$

Proof. We oberve that $\varphi_{\varepsilon, P(y)} = e^{-(\psi_{\varepsilon, P}(x)/\varepsilon)}$ satisfies the following equation

$$\begin{cases} \Delta u - u = 0 & \text{in } \Omega_{\varepsilon, P}, \\ u = w & \text{on } \partial\Omega_{\varepsilon, P}, \end{cases} \tag{2.17}$$

and $\partial/\partial P_i\ \varphi_{\varepsilon, P}(y)$ satisfies

$$\begin{cases} \Delta u - u = 0 & \text{in } \Omega_{\varepsilon, P}, \\ u = -\dfrac{w'}{\varepsilon} \dfrac{y_i}{|y|} & \text{on } \partial\Omega_{\varepsilon, P}. \end{cases} \tag{2.18}$$

Since $|w'| \leqslant Ce^{-(l_0/\varepsilon)}$ on $\partial\Omega_{\varepsilon, P}$, our assertion follows easily by the Maximum Principle.

Remark. If u is a critical point of K_ε, u satisfies on Ω the equation

$$\varepsilon^2 \Delta u - u + l(u) u^p = 0.$$

By a scaling and elliptic regularity theorem, $(l(u))^{1/(p-1)} u$ is a solution of problem (1.1).

3. PROOF OF THEOREM 1.1

The goal of this section is to obtain a lower bound for A_ε and therefore to prove Theorem 1.1.

We begin with a series of lemmas.

LEMMA 3.2. $A_\varepsilon \geqslant \varepsilon^{(p-1)/(p+1)n} I(w)$.

Proof. It is well known that w is the unique solution of (1.4) and

$$I(w) = \inf \left\{ \frac{\|u\|^2_{W^{1,2}(R^n)}}{\|u\|^2_{L^{p+1}(R^n)}} \,\middle|\, u \in W^{1,2}(R^n),\ u \not\equiv 0 \right\}.$$

Since $p<(n+2)/(n-2)$, A_ε is obtained by a function $u_\varepsilon \in W_0^{1,2}(\Omega)$ and

$$A_\varepsilon = \frac{\varepsilon^2 \int_\Omega |\nabla u|^2 + \int_\Omega u^2}{(\int_\Omega u^{p+1})^{2/(p+1)}}$$

$$= \varepsilon^{(p-1)/(p+1)n} \frac{\int_{\Omega_{\varepsilon,P_0}} |\nabla v_\varepsilon|^2 + \int_{\Omega_{\varepsilon,P_0}} v_\varepsilon^2}{(\int_{\Omega_{\varepsilon,P_0}} v_\varepsilon^{p+1})^{2(p+1)}} \geqslant \varepsilon^{(p-1)/(p+1)n} I(w),$$

where $v_\varepsilon(y) = u_\varepsilon(x) \in W_0^{1,2}(\Omega_{\varepsilon,P_0}) \subset W^{1,2}(R^n)$ and $y = (x - P_0/\varepsilon) \in \Omega_{\varepsilon,P_0}$.

Since A_ε and $\bar{B}_\varepsilon$ are scale invariant and A_ε is obtained by a function u_ε, we may assume that u_ε is a function in $W_0^{1,2}(\Omega)$ such that

$$(1)\quad K_\varepsilon(u_\varepsilon) = A_\varepsilon, \quad u_\varepsilon \in \bar{B}, \tag{3.1}$$

$$(2)\quad \varepsilon^2 \int_\Omega |\nabla u_\varepsilon|^2 + \int_\Omega |u_\varepsilon|^2 = \int_\Omega u_\varepsilon^{p+1}. \tag{3.2}$$

Then we have

LEMMA 3.2. *For any sequence $\varepsilon_k \to 0$, there exists a subsequence $\varepsilon_{k_l} \to 0$ and $P_{\varepsilon_{k_l}} \in \bar{B}_\delta(P_0)$ such that $\|u_{\varepsilon_{k_l}}(\varepsilon_{k_l} \cdot + P_{\varepsilon_{k_l}}) - w\|_{W_0^{1,2}(\Omega_{\varepsilon_{k_l}, P_{\varepsilon_{k_l}}})} \to 0$ as $\varepsilon_{k_l} \to 0$.*

Proof. We define $v_\varepsilon(y) = u_\varepsilon(x) = u_\varepsilon(\varepsilon y + P_0)$ for $y \in \Omega_{\varepsilon,P_0}$ and $v_\varepsilon(y) = 0$ for $y \in \Omega^c_{\varepsilon,P_0}$. Then $\int_{R^n} v_\varepsilon^{p+1} = \int_{\Omega_{\varepsilon,P_0}} v_\varepsilon^{p+1} = \varepsilon^{-n} \int_\Omega u_\varepsilon^{p+1}$.
But by (3.1) and (3.2),

$$A_\varepsilon = \left(\int_\Omega u_\varepsilon^{p+1}\right)^{(p-1)/(p+1)} = \varepsilon^{n(p-1)/(p+1)} \left(\int_{\Omega_{\varepsilon,P_0}} v_\varepsilon^{p+1}\right)^{(p+1)/(p-1)}.$$

By Lemma 3.1 and 2.3, we have

$$\int_{R^n} w^{p+1} \leqslant \int_{\Omega_\varepsilon} v_\varepsilon^{p+1} \leqslant [I(w) + \alpha_1 e^{-\beta\psi_\varepsilon(P_0)} + o(e^{-\beta\psi_\varepsilon(P_0)})]^{(p+1)/(p-1)}.$$

Hence, $\lim_{\varepsilon\to 0} \int_{R^n} v_\varepsilon^{p+1} = \int_{R^n} w^{p+1}$.
Similarly, $\lim_{\varepsilon\to 0} \int_{R^n} |\nabla v_\varepsilon|^2 + v_\varepsilon^2 = \int_{R^n} w^{p+1}$.
By standard concentration compactness argument (see [14] or Appendix in [12]), there exists $\varepsilon_{k_l} \to 0$, $z_{\varepsilon_{k_l}} \in R^n$, such that

$$\|v_{\varepsilon_{k_l}} - w(\cdot - z_{\varepsilon_{k_l}})\|_{H^1(R^n)} \to 0 \qquad \text{as} \quad \varepsilon_{k_l} \to 0. \tag{3.3}$$

Note that

$$\beta(u_\varepsilon)=\frac{\int_\Omega xu_\varepsilon^{p+1}}{\int_\Omega u_\varepsilon^{p+1}}$$

$$=\frac{\int_{R^n}\varepsilon y v_\varepsilon^{p+1}}{\int_{R^n} v_\varepsilon^{p+1}}+P_0$$

We have $\int_{R^n}\varepsilon y v_\varepsilon^{p+1}\in \bar{B}_{\delta\int_{R^n} v_\varepsilon^{p+1}}(P_0)$.

On the other hand,

$$\int_{R^n}\varepsilon y w^{p+1}(y-z_{\varepsilon_{k_l}})\,dy=\int_{R^n}\varepsilon y' w^{p+1}(y')\,dy'+\varepsilon z_{\varepsilon_{k_l}}\int_{R^n} w^{p+1}$$

$$=\varepsilon z_{\varepsilon_{k_l}}\int_{R^n} w^{p+1}.$$

But $\|v_{\varepsilon_{k_l}}-w(\cdot-z_{\varepsilon_{k_l}})\|_{L^{p+1}(\Omega_{\varepsilon_{k_l},P_0})}\to 0$ as $\varepsilon_{k_l}\to 0$. We then have $P_{\varepsilon_{k_l}}:=\varepsilon z_{\varepsilon_{k_l}}+P_0\to P_1\in\bar{B}_\delta(P_0)$ by taking a further subsequence and $\|u_{\varepsilon_{k_l}}(\varepsilon_{k_l}\cdot+P_{\varepsilon_{k_l}})-w\|_{W_0^{1,2}(\Omega_{\varepsilon_{k_l},P_{\varepsilon_{k_l}}})}\to 0$ as $\varepsilon_{k_l}\to 0$.

Corollary 3.3. *For any sequence $\varepsilon_k\to 0$, there exists a subsequence $\varepsilon_{k_l}\to 0$ such that there exists $P'_{\varepsilon_{k_l}}\in\bar{B}_\delta(P_0)$ and*

$$\|u_{\varepsilon_{k_l}}(\varepsilon_{k_l}\cdot+P'_{\varepsilon_{k_l}})-P_{\Omega_{\varepsilon_{k_l}},P'_{\varepsilon_{k_l}}}w\|_{W_0^{1,2}(\Omega_{\varepsilon_{k_l}},P'_{\varepsilon_{k_l}})}\to 0 \text{ as } \varepsilon_{k_l}\to 0.$$

Proof. We use Lemma 3.2 and the properties of $P_{\Omega_{\varepsilon_k},P_{\varepsilon_k}}w$ stated in Section 2.

We now define

$$E_{\varepsilon,P}=\left\{v\in W_0^{1,2}(\Omega_{\varepsilon,P})\;\middle|\;\begin{array}{l}\langle v, P_{\Omega_{\varepsilon},P}w\rangle_{W_0^{1,2}(\Omega_{\varepsilon,P})}\\ =\left\langle v,\dfrac{\partial}{\partial P_i}P_{\Omega_\varepsilon,P}w\right\rangle_{W_0^{1,2}(\Omega_{\varepsilon,P})}\\ =0,\ 1\leqslant i\leqslant n\end{array}\right\}. \tag{3.4}$$

The following lemma will be proved in Section 4.

Lemma 3.4. *For every sequence $\varepsilon_k\to 0$, there exists a subsequence $\varepsilon_{k_l}\to 0$, $C_{k_l}>0$, $P_{k_l}\in\Omega$, $\omega_{k_l}\in E_{\varepsilon_{k_l},P_{k_l}}$ such that as $k_l\to\infty$, $C_{k_l}\to 1$, $P_{k_l}\to\bar{P}\in\bar{B}_\delta(P_0)$ and*

$$u_{\varepsilon_{k_l}}(x)=C_{k_l}P_{\Omega_{\varepsilon_{k_l}},P_{k_l}}w((x-P_{k_l})/\varepsilon_{k_l})/\varepsilon_{k_l})+\omega_{k_l}. \tag{3.5}$$

Moreover, we have

$$A_{\varepsilon_{k_l}} \geqslant \varepsilon_{k_l}^{(p-1/p+1)n}\{I(w)+e^{-\beta_{k_l}\psi_{k_l}(P_{k_l})}\alpha_2+o(e^{-\beta_{k_l}\psi_{k_l}(P_{k_l})})\}, \tag{3.6}$$

where $\alpha_2>0$ *is a positive constant.*

Combining Lemmas 2.3 and 3.4, we can now prove Theorem 1.1 as follows.

Proof of Theorem 1.1. To prove Theorem 1.1, we just need to show that there exists $\varepsilon_0>0$ such that for all $\varepsilon<\varepsilon_0$, $\beta(u_\varepsilon)\in B_\delta(P_0)$. Then we deduce that for $\phi\in W_0^{1,2}(\Omega)$, there exists $\lambda_0=\lambda_0(\varepsilon)>0$ such that

$$\beta(u_\varepsilon+\lambda\phi)\in B_\delta(P_0)$$

for all $|\lambda|<\lambda_0$. This implies that

$$\frac{d}{d\lambda}K_\varepsilon(u_\varepsilon+\lambda\phi)|_{\lambda=0}=0.$$

Hence u_ε is a critical point of K_ε and by (3.2), u_ε is a solution of problem (1.1) in $W_0^{1,2}(\Omega)$ therefore u_ε is a classical solution of problem (1.1).

By the proofs in [18], u_ε has exactly one local maximum point P_ε.By the fact that $\int_\Omega xu_\varepsilon^{p+1}/\int_\Omega u_\varepsilon^{p+1}\in \bar{B}_\delta(P_0)$, we have $P_\varepsilon\to\bar{\bar{P}}\in\bar{B}_\delta(P_0)$. The same proof in [18] shows that $\bar{\bar{P}}=P_0$. Theorem 1.1 follows then. It remains to prove the claim.

Suppose that the claim is not true. That is, there exists $\varepsilon_k\to 0$ such that $\beta(u_{\varepsilon_k})\in\partial B_\delta(P_0)$.

From Corollary 3.3, there exists $\varepsilon_{k_l}\to 0$, $P_{\varepsilon_{k_l}}\to P_1\in\partial B_\delta(P_0)$ and

$$\|u_{\varepsilon_{k_l}}(\varepsilon_{k_l}\cdot+P_{\varepsilon_{k_l}})-P_{\Omega_{\varepsilon_{k_l}},P_{\varepsilon_{k_l}}}w\|_{W_0^{1,2}(\Omega_{\varepsilon_{k_l}},P_{\varepsilon_{k_l}})}\to 0 \tag{3.7}$$

From Corollary 3.3, there exists $\varepsilon_{k_l}\to 0$, ${}_{\varepsilon_{k_l}}\to P_1\in\partial B_\delta(P_0)$ and

$$\|u_{\varepsilon_{k_l}}(\varepsilon_{k_l}\cdot+P_{\varepsilon_{k_l}})-P_{\Omega_{\varepsilon_{k_l}},P_{\varepsilon_{k_l}}}w\|_{W_0^{1,2}(\Omega_{\varepsilon_{k_l}},P_{\varepsilon_{k_l}})}\to 0 \tag{3.7}$$

By Lemma 3.4, there exists a further subsequence $\varepsilon_{k_l'}\to 0$, such that

$$u_{\varepsilon_{k_l'}}(x)=C_{k_l'}P_{\Omega_{\varepsilon_{k_l'}},P_{k_l'}}w((x-P_{k_l'})/\varepsilon_{k_l'})+\omega_{k_l'}. \tag{3.8}$$

and

$$A_{\varepsilon_{k_l'}}\geqslant\varepsilon_{k_l'}^{(p-1/p+1)n}[I(w)+e^{-\beta_{k_l'}\psi_{\varepsilon_{k_l'}}(P_{k_l'})}\alpha_2+o(e^{-\beta_{k_l'}\psi_{\varepsilon_{k_l'}}(P_{k_l'})})]. \tag{3.9}$$

From (2.13) and (3.9), we have

$$\psi_{\varepsilon_{k_l'}}(P_{k_l'}) \geqslant \psi_{\varepsilon_{k_l'}}(P_0) + o(1).$$

By (3.7) and (3.8), we must have $|P_{\varepsilon_{k_l'}} - P_{k_l'}| = o(1)$. Letting $k_l' \to \infty$, we have $d(P_1, \partial\Omega) \geqslant d(P_0, \partial\Omega)$. That is a contradiction.

4. PROOF OF TECHNICAL LEMMAS

Recall that

$$E_{\varepsilon, P} = \left\{ v \in W_0^{1,2}(\Omega_{\varepsilon, P}) \;\middle|\; \begin{aligned} &\langle v, P_{\Omega_{\varepsilon}, P} w \rangle_{W_0^{1,2}(\Omega_{\varepsilon, P})} \\ &= \left\langle v, \frac{\partial}{\partial P_i} P_{\Omega_{\varepsilon}, P} w \right\rangle_{W_0^{1,2}(\Omega_{\varepsilon, P})} \\ &= 0,\ 1 \leqslant i \leqslant n \end{aligned} \right\}. \tag{4.1}$$

We first study the following eigenvalue problem.

LEMMA 4.1. *The eigenvalue problem*

$$\begin{cases} \Delta v - v + \mu w^{p-1} v = 0 \\ v \in W^{1,2}(R^n) \end{cases} \tag{4.2}$$

admits a discrete set of eigenvalues $\nu_1 < \nu_2 \leqslant \nu_3 \leqslant \cdots$ *such that* $\nu_1 = 1$, $\nu_i = p$, $2 \leqslant i \leqslant n+1$, *and* $\nu_{n+2} > p$. *The eigenspaces* V_1 *and* V_p *corresponding to* 1 *and* p *are given by*

$$V_1 = \text{span}\{w\} \tag{4.3}$$

and

$$V_p = \text{span}\left\{ \frac{\partial w}{\partial x_i} \;\middle|\; 1 \leqslant i \leqslant n \right\}. \tag{4.4}$$

Proof. Consider the map i: $W^{1,2}(R^n) \to L^2(w^{p-1}\cdot)$, where $L^2(w^{p-1}\cdot)$ is the Hilbert space with

$$\langle u, v \rangle = \int_{R^n} w^{p-1} u \cdot v$$

Since w is exponentially decaying at ∞, i is compact. Hence there are a discrete number of values $\nu_1 \leqslant \nu_2 \leqslant \cdots$ and functions $v_1, v_2, ...,$ wich are solutions of (4.2).

Let μ be an eigenvalue with $\mu \leqslant p$ and v be a solution of (4.2). As in [11], $v \in C^\infty(R^n)$. Let $\mu_k, e_k(w)$ with $w \in S^{n-1}$ be the eigenvalues and eigenfunctions of the Laplace–Beltrami operator on S^{n-1}. Then

$$\mu_0 = 0 < \mu_1 = \cdots = \mu_n = n-1 < \mu_{n+1} \leqslant \cdots$$

and e_k''s are normalized so that they form a complete orthonormal basis of $L^2(S^{n-1})$.

Put

$$\tilde{v}_k(r) := \int_{S^{n-1}} v(r, w)\, e_k(w)\, dw.$$

Then $\tilde{v} \to 0$ exponentially as $r \to \infty$ and it satisfies

$$\tilde{v}_k''(r) + \frac{n-1}{r}\, \tilde{v}_k' - \tilde{v} + \left(\mu w^{p-1} - \frac{\mu_k}{r^2}\right) \tilde{v}_k = 0, \quad r > 0 \tag{4.5}$$

for $k = 0, 1, 2, \ldots$. We claim that $\tilde{v}_k \equiv 0$ if $k > n$.

Suppose for a contradiction that there is a $\rho_k \in (0, \infty]$ such that $\tilde{v}_k(r) > 0$ for $0 < r < \rho_k$ and $\tilde{v}_k(\rho_k) = 0$. As in [17], multiplying (4.5) with $w'(r)\, r^{n-1}$ and integrate the resulting equation over $0 < r < \rho_k$. We obtain

$$\rho_k^{n-1} \tilde{v}_k w'(\rho_k) + \left(\int_0^{\rho_k} w'(r)\, r^{n-1} \tilde{v}_k\right)(\mu - p)$$

$$+ (n-1-\mu) \int_0^{\rho_k} w' \tilde{v}_k r^{n-3}\, dr = 0.$$

Since $\mu \leqslant p$ and $w'(r) < 0$ for $r \not\equiv 0$, $\tilde{v}_k(\rho_k) \leqslant 0$, we conclude that $\mu_k > n-1$, i.e., $k > n$. Here $\tilde{v}(r, w) = \tilde{v}_0(r) + \sum_{k=1}^n \tilde{v}_k(r)\, e_k(w)$.

It follows then the dimension of the kernel $L_\mu = \Delta - 1 + \mu w^{p-1}$ is at most $n+1$. But note that $\mu = 1$, w is a solution of (4.2), $\mu = p$, $\partial w / \partial x_j$ is a solution of (4.2), and

$$\int_{R^n} w^{p-1} w \frac{\partial w}{\partial x_j} = 0, \qquad \int_{R^n} w \frac{\partial w}{\partial x_j} + \nabla w \cdot \nabla \frac{\partial w}{\partial x_j} = 0.$$

We conclude that $\mu_1 = 1$, $\mu_2 = p = \mu_3 = \cdots = \mu_{n+1}$, and $V_1 = \text{span}\{w\}$, $V_p = \text{span}\{\partial w / \partial x_j\}$.

LEMMA 4.2. *There exist* $\varepsilon_0 > 0$, $\rho > 0$ *such that for any* $\varepsilon < \varepsilon_0$ *and* $P \in \bar{B}_{2\delta}(P_0)$, *we have*

$$\int_{\Omega_{\varepsilon, P}} |\nabla v|^2 + v^2 \geqslant (p+\rho) \int_{\Omega_{\varepsilon, P}} (P_{\Omega_{\varepsilon, P}} w)^{p-1} v^2, \quad \textit{for all } v \in E_{\varepsilon, P}. \tag{4.6}$$

Proof. Suppose on the contrary, there exist $\varepsilon_k \to 0$, $\rho_k \to 0$, $P_k \in \bar{B}_{2\delta}(P_0)$, and $v_k \in E_{\varepsilon_k, P_k}$ so that

$$\int_{\Omega_{\varepsilon_k, P_k}} |\nabla v_k|^2 + v_k^2 \leqslant (p+\rho_k) \int_{\Omega_{\varepsilon_k, P_k}} (P_{\Omega_{\varepsilon, P}} w)^{p-1} v_k^2.$$

Assume that $\int_{\Omega_{\varepsilon_k, P_k}} |\nabla v_k|^2 + v_k^2 = 1$ and extend v_k equal to 0 outside $\Omega_{\varepsilon_k, P_k}$.

Observe that

$$\int_{R^n} |\nabla v_k|^2 + v_k^2 = 1$$

$$\int_{R^n} |\nabla v_k|^2 + v_k^2 \leqslant (p+\rho_k) \int_{R^n} (P_{\Omega_{\varepsilon, P}} w)^{p-1} v_k^2$$

$$\int_{R^n} \nabla v_k \cdot \nabla P_{\Omega_{\varepsilon_k, P_k}} w + v_k \cdot P_{\Omega_{\varepsilon_k, P_k}} w = 0$$

$$\int_{R^n} \nabla v_k \cdot \nabla \left(\frac{\partial}{\partial P_i} P_{\Omega_{\varepsilon_k, P_k}} w \right) + v_k \cdot \frac{\partial}{\partial P_i} P_{\Omega_{\varepsilon_k, P_k}} w = 0.$$

Since $\|v_k\|_{H^1(R^n)} = 1$, there exist $v_0 \in H^1(R^n)$, $v_k \rightharpoonup v_0$ in $H^1(R^n)$, and $v_k \to v_0$ in $H^1_{\mathrm{loc}}(R^n)$.

Hence we have by taking limits (noting that w is exponentially decaying and using Lemma 2.4)

$$1 \leqslant p \int_{R^n} w^{p-1} v_0^2$$

$$\int_{R^n} \nabla v_0 \cdot \nabla w + v_0 \cdot w = 0$$

$$\int_{R^n} \nabla v_0 \cdot \nabla \frac{\partial w}{\partial x_j} + v_0 \frac{\partial w}{\partial x_j} = 0$$

$$\int_{R^n} |\nabla v_0|^2 + v_0^2 \leqslant 1.$$

That is a contradiction to Lemma 4.1.

Let us consider the minimization problem

$$\text{Minimize } \|u_\varepsilon(\varepsilon\cdot+P)-\alpha P_{\Omega_{\varepsilon,P}}w\|_{W_0^{1,2}(\Omega_{\varepsilon,P})}, \tag{4.7}$$

where $\alpha\in(\frac{1}{2},2]$ and $P\in\bar{B}_{2\delta}(P_0)$.

Since $P_{\Omega_{\varepsilon,P}}w$ is continuous about P, (4.7) is achieved and we can write

$$u_\varepsilon(\varepsilon\cdot+P_\varepsilon)=\alpha_\varepsilon P_{\Omega_{\varepsilon,P_\varepsilon}}w+\omega_\varepsilon \tag{4.8}$$

where $\omega_\varepsilon\in E_{\varepsilon,P_\varepsilon}$ and $P_\varepsilon\in\bar{B}_{2\delta}(P_0)$.

By Corollary 3.3, $\|\omega_\varepsilon\|_{W_0^{1,2}(\Omega_{\varepsilon,P_\varepsilon})}\to 0$ as $\varepsilon\to 0$. Moreover,

$$\|u_\varepsilon(\varepsilon\cdot+P_\varepsilon)\|_{W_0^{1,2}(\Omega_{\varepsilon,P_\varepsilon})}=\alpha_\varepsilon\|P_{\Omega_{\varepsilon,P_\varepsilon}}w\|_{W_0^{1,2}(\Omega_{\varepsilon,P_\varepsilon})}+\|\omega_\varepsilon\|_{W_0^{1,2}(\Omega_{\varepsilon,P_\varepsilon})}.$$

Therefore $\alpha_\varepsilon\to 1$ as $\varepsilon\to 0$, since $\|v_\varepsilon\|_{W_0^{1,2}(\Omega_{\varepsilon,P_\varepsilon})}\to\|w\|_{H^1(R^n)}$ and $\|P_{\Omega_{\varepsilon,P_\varepsilon}}w\|_{W_0^{1,2}(\Omega_{\varepsilon,P_\varepsilon})}\to\|w\|_{H^1(R^n)}$.

We are now ready to finish the proof of Lemma 3.4.

Proof of Lemma 3.4. To prove (3.5), we note that by (4.8), we just need to prove that $P_\varepsilon\to\bar{P}\in\bar{B}_\delta(P_0)$ for some $\bar{P}$ and a sequence $\varepsilon=\varepsilon_k\to 0$.

By Corollary 3.3 and (4.8), we have

$$\left\|P_{\Omega_\varepsilon}w\left(\cdot-\frac{P'_\varepsilon}{\varepsilon}\right)-P_{\Omega_\varepsilon}w\left(\cdot-\frac{P_\varepsilon}{\varepsilon}\right)\right\|_{W_0^{1,2}(\Omega_{\varepsilon,P_0})}\to 0$$

as $\varepsilon\to 0$, where $P'_\varepsilon\to P_1\in\bar{B}_\delta(P_0)$.

Assume that $|P'_\varepsilon-P_\varepsilon|\geqslant\nu\geqslant 0$ when ε is sufficiently small, then

$$\left\|P_{\Omega_{\varepsilon,P'_\varepsilon}}w\left(\cdot-\frac{P'_\varepsilon}{\varepsilon}\right)-P_{\Omega_{\varepsilon,P_\varepsilon}}w\left(\cdot-\frac{P_\varepsilon}{\varepsilon}\right)\right\|_{W_0^{1,2}(\Omega_{\varepsilon,P_0})}\to 2\|w\|_{H^1(R^n)}\neq 0,$$

which is a contradiction.

Hence $P_\varepsilon\to P_1\in\bar{B}_\delta(P_0)$ by passing to a subsequence. We now choose $C_{k_l}=\alpha_{\varepsilon_{k_l}}$, $P_{k_l}=P_{\varepsilon_{k_l}}$, and $\bar{P}=P_1$; then the first part of Lemma 3.4 is proved.

From now on, we assume that $\varepsilon=\varepsilon_{k_l}$ and $P_{\varepsilon_{k_l}}\to\bar{P}\in\bar{B}_\delta(P_0)$. To prove (3.6), we need some preparations.

We first calculate

$$\begin{aligned}\int_{\Omega_{\varepsilon,P_\varepsilon}}(P_{\Omega_{\varepsilon,P_\varepsilon}}w)^p\,\omega_\varepsilon&=\int_{\Omega_{\varepsilon,P_\varepsilon}}w^p\omega_\varepsilon+\int_{\Omega\varepsilon,P_\varepsilon}[(P_{\Omega_{\varepsilon,P_\varepsilon}}w)^p-w^p]\,\omega_\varepsilon\\&=I_1+I_2,\end{aligned}$$

where I_1 and I_2 are defined at the least equality.

Let us first estimate (using (2.8))

$$\begin{aligned}
&|(P_{\Omega_{\varepsilon, P_\varepsilon}} w)^p - w^p| \\
&\quad \leqslant C\,|P_{\Omega_{\varepsilon, P_\varepsilon}} w - w||w|^p \leqslant Ce^{-\beta\psi_\varepsilon(P_\varepsilon)} V_{\varepsilon, P_\varepsilon} w^p \\
&\quad \leqslant Ce^{-((1/2)+\delta_1)\,\beta\psi_\varepsilon(P_\varepsilon)} e^{-((1/2)-\delta_1)\,\beta\psi_\varepsilon(P_\varepsilon)} V_{\varepsilon, P_\varepsilon} w^p \\
&\quad \leqslant Ce^{-((1/2)+\delta_1)\,\beta\psi_\varepsilon(P_\varepsilon)} (e^{-\beta\psi_\varepsilon(P_\varepsilon)} V_{\varepsilon, P_\varepsilon})^{(1/2)-\delta_1} \cdot V_{\varepsilon, P_\varepsilon}^{(1/2)+\delta_1} w^p \\
&\quad \leqslant Ce^{-((1/2)+\delta_1)\,\beta\psi_\varepsilon(P_\varepsilon)} w^{p+(1/2)-\delta_1} V_{\varepsilon, P_\varepsilon}^{(1/2)+\delta_1} \\
&\quad \leqslant Ce^{-((1/2)+\delta_1)\,\beta\psi_\varepsilon(P_\varepsilon)} e^{-(p+(1/2)-\delta_1)|y|} e^{(1+\sigma_1)((1/2)+\delta_1)|y|} \\
&\quad \leqslant Ce^{-((1/2)+\delta_1)\,\beta\psi_\varepsilon(P_\varepsilon)} e^{-\delta_2|y|}
\end{aligned}$$

if $\sigma_1 > 0$, $\delta_1 > 0$, $\delta_2 > 0$ are chosen small enough.

Hence

$$I_2 = \int_{\Omega_{\varepsilon, P_\varepsilon}} ((P_{\Omega_{\varepsilon, P_\varepsilon}} w)^p - w^p)\,\omega_\varepsilon \leqslant \|w\|_{L^2(\Omega_{\varepsilon, P_\varepsilon})} \cdot Ce^{-((1/2)+\delta_1)\,\psi_\varepsilon(P_\varepsilon)}$$

$$I_1 = \int_{\Omega_{\varepsilon, P_\varepsilon}} w^p \omega_\varepsilon = \int_{\Omega_{\varepsilon, P_\varepsilon}} (P_{\Omega_{\varepsilon, P_\varepsilon}} w - \Delta P_{\Omega_{\varepsilon, P_\varepsilon}})\,\omega_\varepsilon = 0$$

since $\omega_\varepsilon \in E_{\varepsilon, P_\varepsilon}$.

In conclusion, we have

$$\int_{\Omega_{\varepsilon, P_\varepsilon}} (P_{\Omega_{\varepsilon, P\varepsilon}} w)^p\,\omega_\varepsilon \leqslant Ce^{-((1/2)+\delta_1)\,\psi_\varepsilon(P_\varepsilon)}\,\|\omega_\varepsilon\|_{W_0^{1,2}(\Omega_{\varepsilon, P_\varepsilon})}. \tag{4.9}$$

Second, we calculate by Taylor's expansion,

$$\begin{aligned}
\int_{\Omega_{\varepsilon, P_\varepsilon}} v_\varepsilon^{p+1} &= C_\varepsilon^{p+1}\,|P_{\Omega_{\varepsilon, P_\varepsilon}} w|_{p+1}^{p+1} + C_\varepsilon^p (p+1) \left(\int_{\Omega_{\varepsilon, P_\varepsilon}} P_{\Omega_{\varepsilon, P_\varepsilon}} w\right)^p \omega_\varepsilon \\
&\quad + (p(p+1))/2C_\varepsilon^{p-1} \int_{\Omega_{\varepsilon, P_\varepsilon}} (P_{\Omega_{\varepsilon, P_\varepsilon}} w)^{p-1}\,\omega_\varepsilon^2 + O(\|\omega_\varepsilon\|^r)
\end{aligned}$$

for some $r > 2$, where, for the moment, we denote $|u|_{p+1}^{p+1} = \int_{\Omega_{\varepsilon, p_\varepsilon}} u^{p+1}$ and $\|u\| = \|u\|_{W_0^{1,2}(\Omega_{\varepsilon, P_\varepsilon})}$.

Hence by (4.9),

$$|v_\varepsilon|_{p+1}^{-2} = C_\varepsilon^{-2} \, |P_{\Omega_{\varepsilon, P_\varepsilon}} w|_{p+1}^{-2} \times \left\{ 1 + \frac{p(p+1)}{2C_\varepsilon^2 \, |P_{\Omega_{\varepsilon, P_\varepsilon}} w|_{p+1}^{p+1}} \times \int_{\Omega_{\varepsilon, P_\varepsilon}} (P_{\Omega_{\varepsilon, P_\varepsilon}} w)^{p-1} \, \omega_\varepsilon^2 + O(e^{-((1/2)+\delta_1)\,\beta\psi_\varepsilon(P_\varepsilon)} \, \|\omega_\varepsilon\| + \|\omega_\varepsilon\|^r) \right\}^{-2/p+1}$$

$$= C_\varepsilon^{-2} \, |P_{\Omega_{\varepsilon, P_\varepsilon}} w|_{p+1}^{-2} \times \left\{ 1 - \frac{p \int_{\Omega_{\varepsilon, P_\varepsilon}} (P_{\Omega_{\varepsilon, P_\varepsilon}} w)^{p-1} \, \omega_\varepsilon^2}{2C_\varepsilon^2 \, |P_{\Omega_{\varepsilon, P_\varepsilon}} w|_{p+1}^{p+1}} + O(e^{-((1/2)+\delta_1)\,\beta\psi_\varepsilon(P_\varepsilon)} \, \|\omega_\varepsilon\| + \|\omega_\varepsilon\|^r) \right\}.$$

Thus,

$$K_\varepsilon(u_\varepsilon) = C_\varepsilon^{-2} \, |P_{\Omega_{\varepsilon, P_\varepsilon}} w|_{p+1}^{-2} \, (C_\varepsilon^2 \, \|P_{\Omega_{\varepsilon, P_\varepsilon}} w\|^2 + \|\omega_\varepsilon\|^2) \times \left\{ 1 - \frac{p \int_{\Omega_{\varepsilon, P_\varepsilon}} w)^{p-1} \, \omega_\varepsilon^2}{2C_\varepsilon^2 \, |P_{\Omega_{\varepsilon, P_\varepsilon}} w|_{p+1}^{p+1}} + O(e^{-((1/2)+\delta_1)\,\beta\psi_\varepsilon(P_\varepsilon)} \, \|\omega_\varepsilon\| + \|\omega_\varepsilon\|^r) \right\}$$

$$= \left\{ \frac{\|P_{\Omega_{\varepsilon, P_\varepsilon}} w\|^2}{|P_{\Omega_{\varepsilon, P_\varepsilon}} w|_{p+1}^2} + C_\varepsilon^{-2} \, |P_{\Omega_{\varepsilon, P_\varepsilon}} w|_{p+1}^{-2} \, \|\omega_\varepsilon\|^2 \right\} \times \left\{ 1 - \frac{p \int_{\Omega_{\varepsilon, P_\varepsilon}} (P_{\Omega_{\varepsilon, P_\varepsilon}} w)^{p-1} \, \omega_\varepsilon^2}{2C_\varepsilon^2 \, |P_{\Omega_{\varepsilon, P_\varepsilon}} w|_{p+1}^{p+1}} + O(e^{-((1/2)+\delta_1)\,\beta\psi_\varepsilon(P_\varepsilon)} \, \|\omega_\varepsilon\| + \|\omega_\varepsilon\|^r) \right\}$$

$$= \frac{\|P_{\Omega_{\varepsilon, P_\varepsilon}} w\|^2}{|P_{\Omega_{\varepsilon, P_\varepsilon}} w|_{p+1}^2} + C_\varepsilon^{-2} \, |P_{\Omega_{\varepsilon, P_\varepsilon}} w|_{p+1}^{-2} \, \|\omega_\varepsilon\|^2 - C_\varepsilon^{-2} \, \frac{\|P_{\Omega_{\varepsilon, P_\varepsilon}} w\|^2}{|P_{\Omega_{\varepsilon, P_\varepsilon}}|_{p+1}^2} \cdot \frac{p \int_{\Omega_{\varepsilon, P_\varepsilon}} (P_{\Omega_{\varepsilon, P_\varepsilon}} w)^{p-1} \, \omega_\varepsilon^2}{|P_{\Omega_{\varepsilon, P_\varepsilon}} w|_{p+1}^{p+1}} + O(e^{-((1/2)+\delta_1)\,\beta\psi_\varepsilon(P_\varepsilon)} \, \|\omega_\varepsilon\| + \|\omega_\varepsilon\|^r)$$

$$= \frac{\|P_{\Omega_{\varepsilon, P_\varepsilon}} w\|^2}{|P_{\Omega_{\varepsilon, P_\varepsilon}} w|_{p+1}^2} + C_\varepsilon^{-2} \, |P_{\Omega_{\varepsilon, P_\varepsilon}} w|_{p+1}^{-2} \times \left\{ \|\omega_\varepsilon\|^2 - \frac{\|P_{\Omega_{\varepsilon, P_\varepsilon}} w\|^2}{|P_{\Omega_{\varepsilon, P_\varepsilon}}|_{p+1}^2} \cdot p \int_{\Omega_{\varepsilon, P_\varepsilon}} (P_{\Omega_{\varepsilon, P_\varepsilon}} w)^{p-1} \, \omega_\varepsilon^2 \right\} + O(e^{-((1/2)+\delta_1)\,\beta\psi_\varepsilon(P_\varepsilon)} \, \|\omega_\varepsilon\| + \|\omega_\varepsilon\|^r)$$

$$\geqslant \frac{\|P_{\Omega_{\varepsilon, P_\varepsilon}} w\|^2}{|P_{\Omega_{\varepsilon, P_\varepsilon}} w|_{p+1}^2} + C_\varepsilon^{-2} |P_{\Omega_{\varepsilon, P_\varepsilon}} w|_{p+1}^{-2} \rho_1 \|\omega_\varepsilon\|^2$$
$$+ O(e^{-((1/2)+\delta_1) \beta\psi_\varepsilon(P_\varepsilon)} \|\omega_\varepsilon\| + \|\omega_\varepsilon\|^r) \quad (\text{for some } \rho_1 > 0)$$
$$\geqslant I(w) + \alpha_1 e^{-\beta\psi_\varepsilon(P_\varepsilon)} + o(e^{-\beta\psi_\varepsilon(P_\varepsilon)})$$
$$+ \rho_2 \|\omega_\varepsilon\|^2 + O(e^{-((1/2)+\delta_1) \beta\psi_\varepsilon(P_\varepsilon)} \|\omega_\varepsilon\| + \|\omega_\varepsilon\|^r)$$
$$(\text{for some } \rho_2 > 0). \tag{4.10}$$

By Lemma 2.3 again, we have

$$\rho_2 \|w_\varepsilon\|^2 + O(e^{-((1/2)+\delta_1) \beta\psi_\varepsilon(P_\varepsilon)} \|\omega_\varepsilon\| + \|\omega_\varepsilon\|^r) \leqslant \alpha_1 e^{-\beta\psi_\varepsilon(P_0)} + o(e^{-\beta\psi_\varepsilon(P_0)}).$$

Since $\|\omega_\varepsilon\| \to 0$ as $\varepsilon \to 0$, we obtain

$$\|\omega_\varepsilon\|^2 \leqslant C e^{-\beta\psi_\varepsilon(P_\varepsilon)} + C e^{-\beta\psi_\varepsilon(P_0)}.$$

Substituting into (4.10), we obtain

$$A_\varepsilon \geqslant I(w) + \alpha_1 e^{-\beta\psi_\varepsilon(P_\varepsilon)} + o(e^{-\beta\psi_\varepsilon(P_\varepsilon)}) \tag{4.11}$$

(since for ε sufficiently small, $((1-\delta_1)\, \psi_\varepsilon(P_\varepsilon) < \psi_\varepsilon(P_0))$.

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