A Geometric Variational Problem with Logrithmic-Quadratic Interaction

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Abstract

A geometric variational problem is defined on subsets of a prescribed measure in the entire plane. The functional of the problem consists of two terms: the perimeter of the input subset and an interaction integral with a kernel that is the sum of a logarithmic function and a quadratic function. This kernel is bounded below and tends to infinity at zero and infinity. A single disc is always a critical point of the functional but its stability depends on the parameters of the problem. When the parameters are in a suitable range there exist assemblies of multiple perturbed discs that are stable critical points.

Key words. logarithmic-quadratic interaction, stability of a single disc, disc assemblies as critical points.

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1 Introduction

We study a geometric variational problem

$$\mathcal{J}(\Omega) = \mathcal{P}(\Omega) + \frac{\gamma}{2}\int_\Omega\int_\Omega K(|x-y|)\,dx\,dy. \tag{1.1}$$

defined on the admissible class

$$\mathcal{A} = \{\Omega \subset \mathbb{R}^2 :\ \Omega \text{ is Lebesgue measurable and } |\Omega| = m\},\ m > 0; \tag{1.2}$$

namely $\mathcal{A}$ comprises of measurable subsets of $\mathbb{R}^2$ of the prescribed positive measure m. This m is the first parameter of the problem. Here $|\cdot|$ stands for the Lebesgue measure on $\mathbb{R}^2$.

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In (1.1), $\mathcal{P}(\Omega)$ stands for the perimeter of Ω. If Ω is enclosed by piecewise C^1 curves, then $\mathcal{P}(\Omega)$ is the total length of these curves. In general, if Ω is merely measurable, then

$$\mathcal{P}(\Omega) = \sup \left\{ \int_\Omega \operatorname{div} g(x)\, dx : \ g \in C_0^1(\mathbb{R}^2, \mathbb{R}^2), \ |g(x)| \leq 1 \ \forall x \in \mathbb{R}^2 \right\}. \tag{1.3}$$

Here $\operatorname{div} g$ is the divergence of the C^1 vector field g with compact support and $|g(x)| = \sqrt{\sum_{j=1}^2 g_j^2(x)}$ stands for the Euclidean norm of the vector $g(x) \in \mathbb{R}^2$; see, for instance [4]. This term models a growth force. It prefers connected Ω's with small perimeters like a disc. If it were the only term in (1.1), then we would just have the standard isoperimetric problem.

The function K in the integral is given by

$$K(t) = \frac{1}{2\pi} \log \frac{1}{t} + t^2, \ t > 0 \tag{1.4}$$

We call K a logarithmic-quadratic interaction because it is the sum of the logrithmic function $\frac{1}{2\pi} \log \frac{1}{t}$ and the quadratic function t^2. If we view K as the potential of a force field, then the force is repulsive in short distance and attractive in long distance. In (1.1) the logarithmic part of K works as an inhibition force. It likes to break the set Ω into disconnected small pieces. The quadratic term in K prevents disconnected pices of Ω from moving too far away from each other.

The constant

$$\gamma > 0 \tag{1.5}$$

is another paramter of the problem. By tuning γ we can quantitatively adjust the strength of the perimeter term of $\mathcal{J}$ versus the strength of the integral term.

Our study of problem (1.1) is partially motivated by the nanostructures of diblock copolymers. A diblock copolymer molecule is a linear sub-chain of A-monomers grafted covalently to another sub-chain of B-monomers. Because of the repulsion between the unlike monomers, the different type sub-chains tend to segregate, but as they are chemically bonded in chain molecules, segregation of sub-chains cannot lead to a macroscopic phase separation. Only a local micro-phase separation occurs: micro-domains rich in A monomers and micro-domains rich in B monomers emerge as a result. These micro-domains form patterns known as morphological phases. The widely observed morphological phases in diblock copolymers are the lamellar phase, the cylindrical phase, and the spherical phase [2].

When temperature is low, the A-monomers and the B-monomers in a diblock copolymer separate fully. The A-monomers form a subset in the sample space and the B-monomers form the compliment subset. In the cylindrical phase the subset of the minority monomers is a union of many parallel cylinders whose cross sections are discs of approximately the same radius. The first step to study the cylindrical phase is to isolate one cylinder and consider a cross section which is approximately a disc. A single disc in $\mathbb{R}^2$ is thus a building block and it can be analyzed by problem (1.1).

Ohta and Kawasaki [8] proposed a density functional theory to study diblock copolymers. In the strong segregation region where A-monomers and B-monomers separate completely, their theory is reduced to a geometric variational problem by a Γ-convergence argument [11]. The free energy of a diblock copolymer sample on a bounded domain D takes the form

$$\mathcal{J}_{OK}(\Omega) = \mathcal{P}_D(\Omega) + \frac{\gamma}{2} \int_\Omega (-\Delta)^{-1}(\chi_\Omega - \omega)(x)\, dx \tag{1.6}$$

where Ω is a subset of the domain D of the prescribed measure equal to $\omega|D|$, $\omega \in (0,1)$. In (1.6) $\mathcal{P}_D(\Omega)$ is the perimeter of Ω in D, $(-\Delta)^{-1}$ is the inverse of the negative Laplace operator with the zero Neumann boundary condition (or the periodic boundary condition if D is a flat torus), and χ_Ω is the characteristic function of Ω. This model has been studied extensively in the mathematical community in recent years. Many critical points of $\mathcal{J}_{OK}$ have been found that phenomenologically match experimental data [11, 12, 14, 6, 1, 3, 7, 5, 9, 10].

Since the problem (1.6) is formulated on a bounded domain D, the operator $(-\Delta)^{-1}$ depends on the shape of D. It is an integral operator defined with the help of Green's function G of $(-\Delta)^{-1}$:

$$(-\Delta)^{-1}(\chi_\Omega - \omega)(x) = \int_\Omega G(x,y)\,dy \tag{1.7}$$

Consequently the second term in (1.6) can be written as

$$\int_\Omega (-\Delta)^{-1}(\chi_\Omega - \omega)(x)\,dx = \int_\Omega \int_\Omega G(x,y)\,dxdy \tag{1.8}$$

In two dimensions, Green's function G can be written as a sum of two parts:

$$G(x,y) = \frac{1}{2\pi}\log\frac{1}{|x-y|} + R(x,y) \tag{1.9}$$

where the logarithmic part is the fundamental solution of $-\Delta$ in $\mathbb{R}^2$, and $R(x,y)$ is the regular part of G, a smooth function dependent on the geometry of D.

It is often necessary to derive a model from $\mathcal{J}_{OK}$ that is defined on the entire space $\mathbb{R}^d$, $d = 1$, 2, or 3, instead of a bounded domain D. For example, we may want to zoom in to study a small part of a diblock copolymer system. Or we may have a physical or biological species of finite size living in an infinite sea. A natural approach is to drop the regular part R and consider, in two dimensions,

$$\mathcal{J}_{\mathbb{R}^2}(\Omega) = \mathcal{P}(\Omega) + \frac{\gamma}{2}\int_\Omega\int_\Omega \frac{1}{2\pi}\log\frac{1}{|x-y|}\,dxdy \tag{1.10}$$

This was done in three dimensions by Ren and Wei in [15, 16, 17] to study torus like structures, and in two dimensions by Ren and Zhang to study the stability of single disc and single ball configurations [18].

However $\mathcal{J}_{\mathbb{R}^2}$ in (1.10) has one shortcoming: the energy functional is not bounded below because $\log\frac{1}{|x-y|} \to -\infty$ if $|x-y| \to \infty$. If one takes Ω to be the union of two discs, say $B(\xi_1, r_1)$ and $B(\xi_2, r_2)$, and sends $|\xi_1 - \xi_2| \to \infty$, then $\mathcal{J}_{\mathbb{R}^2}(B(\xi_1, r_1) \cup B(\xi_2, r_2)) \to -\infty$. In other words, the system likes to push disconnected pieces infinitely away from each other.

Here we fix this problem by adding a quadratic term into the logrithmic kernel in (1.10) and consider $\mathcal{J}$ of (1.1) with a new kernel $K(t)$ of (1.4). The quadratic term t^2 is more dominate when t is large, so that $K(t) \to \infty$ when $t \to \infty$. This will prevent disconnected pieces from moving too far away from each other. Apart from this improvement, most interesting features of $\mathcal{J}_{\mathbb{R}^2}$ are preserved in $\mathcal{J}$.

In this paper we study two problems. First we consider a single disc B_ρ in $\mathbb{R}^2$ of radius ρ satisfying the area constraint in (1.2), i.e. $\pi\rho^2 = m$. It is easily seen to be a stationary point of $\mathcal{J}$. We ask whether B_ρ is stable. To have a precise notion of stability, we will identify perturbations of B_ρ as functions ϕ. The disc B_ρ corresponds to $\phi = 0$. This technique turns the geometric variational problem (1.1) to a variational problem with long range interaction on a function space, and transforms the critical point equation (1.18) of $\mathcal{J}$ to an integro-differential equation. This approach will involve three function spaces, $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ defined in (2.4), (2.5), and (2.6) respectively. The functional $\mathcal{J}$ then becomes a functional defined on a neighborhood of 0 in $\mathcal{Y}$; the first variation of $\mathcal{J}$, denoted $\mathcal{J}'$, becomes a nonlinear operator from a neighborhood of 0 in $\mathcal{X}$ to $\mathcal{Z}$; the second variation at B_ρ, denobed $\mathcal{J}''(0)$, is a linear operator from $\mathcal{X}$ to $\mathcal{Z}$. In Theorem 2.1 we find the eigenvalues of $\mathcal{J}''(0)$. These eigenvalues are used to interprete the stability of B_ρ. In Theorem 2.2 we prove that if ρ is greater than or equal to $\frac{1}{2\sqrt{\pi}}$, then B_ρ is stable for any $\gamma > 0$; if ρ is less than $\frac{1}{2\sqrt{\pi}}$, then there exists a treshold value $\beta_\rho > 0$ such that B_ρ is stable if $\gamma\rho^3 < \beta_\rho$ and unstable if $\gamma\rho^3 > \beta_\rho$.

The second problem is about critical points of $\mathcal{J}$ that are assemblies of multiple perturbed discs. As we have explained that $\mathcal{J}_{\mathbb{R}^2}$, which has no quadratic term in the kernel, likes to push disconnected piece away from each other, any union of multiple discs cannot be stable in $\mathcal{J}_{\mathbb{R}^2}$. However, with the quadratic term in the kernel, $\mathcal{J}$ behaves much better. We prove in Theorem 3.1 that for any integer $N \geq 2$, there is a range for parameters m and γ, where m is small, γ is suitably large, and $\mathcal{J}$ admits a stable stationary point which is a union of N perturbed discs of approximately the same radius. The centers of these discs are close to the global minimum of a function F defined in terms of the kernel K in (3.2).

The first problem is studied in section 2 and the second problem in sections 3 through 5. In section 6 we present some numerical minimization results of the function F.

We end the introduction with a review of the equation for critical points of $\mathcal{J}$ in the standard setting. Let Ω be a subset of $\mathbb{R}^2$ with sufficiently smooth boundary $\partial\Omega$. The inward pointing unit normal vector on $\partial\Omega$ is denoted $\mathbf{N}$. A deformation of Ω is a smooth function $S : \mathbb{R}^2 \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^2$, $\varepsilon_0 > 0$, such that $S(x, 0) = x$ for every $x \in \mathbb{R}^2$, and $S(\cdot, \varepsilon) : \mathbb{R}^2 \to \mathbb{R}^2$ is a diffeomorphism for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. The infinitesimal element of the deformation is

$$\mathbf{X}(x) = \frac{\partial S(x,\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0}, \quad \forall x \in \mathbb{R}^2 \tag{1.11}$$

The image of Ω under $S(\cdot, \varepsilon)$ is denoted Ω_ε. Then $\Omega_0 = \Omega$.

There is a first variation formula:

$$\frac{d\mathcal{J}(\Omega_\varepsilon)}{d\varepsilon}\Big|_{\varepsilon=0} = -\int_{\partial\Omega} (\kappa(\partial\Omega) + \gamma K[\Omega])\, \mathbf{N}\cdot\mathbf{X}\, ds, \tag{1.12}$$

In (1.12) $\kappa(\partial\Omega)$ is the curvature of $\partial\Omega$ with respect to the inward pointing normal vector $\mathbf{N}$. In particular, if Ω were convex, $\kappa(\partial\Omega)$ would be non-negative. Also $K[\Omega]$ denotes a function defined on $\mathbb{R}^2$ given by

$$K[\Omega](x) = \int_\Omega K(x-y)\, dy \tag{1.13}$$

The integral on the right side of (1.12) is taken against the arc length element ds.

Another useful formula related to the deformation S is

$$\frac{d|\Omega_\varepsilon|}{d\varepsilon}\Big|_{\varepsilon=0} = -\int_{\partial\Omega} \mathbf{N}\cdot\mathbf{X}\, ds \tag{1.14}$$

For applications in material systems with mass constraint we require that Ω in (1.1) be a measurable set of the fixed measure as in (1.2). Then deformations of Ω must be measure preserving; namely $|\Omega_\varepsilon| = |\Omega| = m$ and (1.14) implies

$$\int_{\partial\Omega} \mathbf{N}\cdot\mathbf{X}\, ds = 0 \tag{1.15}$$

We say that Ω is a critical point of $\mathcal{J}$ if

$$\frac{d\mathcal{J}(\Omega_\varepsilon)}{d\varepsilon}\Big|_{\varepsilon=0} = 0 \tag{1.16}$$

for any deformation S of Ω that preserves the measure of Ω_ε. Then by (1.12), (1.15) and (1.16), we deduce that

$$\int_{\partial\Omega} (\kappa(\partial\Omega) + \gamma K[\Omega])\, \mathbf{N}\cdot\mathbf{X}\, ds = 0 \text{ whenever } \int_{\partial\Omega} \mathbf{N}\cdot\mathbf{X}\, ds = 0. \tag{1.17}$$

This yields the equation for critical points of $\mathcal{J}$:

$$\kappa(\partial\Omega) + \gamma K[\Omega] = C \text{ on } \partial\Omega \tag{1.18}$$

where $C \in \mathbb{R}$ is a Lagrange multiplier corresponding to the constraint $|\Omega| = m$, or condition (1.15).

2 The single disc

The single disc is special. Denote by B_ρ the disc of radius $\rho > 0$. Without the loss of generality, we assume that B_ρ is centered at the origin of $\mathbb{R}^2$. Since B_ρ must satisfy the constraint $|\Omega| = m$ in (1.2),

$$\pi\rho^2 = m. \tag{2.1}$$

In this section we replace m by ρ as the first parameter of the problem.

Since the curvature of the circle ∂B_ρ is just the inverse of the radius ρ:

$$\kappa(\partial B_\rho) = \frac{1}{\rho}, \tag{2.2}$$

and $K[B_\rho]$ is a radially symmetric function and hence a constant on ∂B_ρ, B_ρ satisfies the critical point equation (1.18):

$$\kappa(\partial B_\rho) + \gamma K[B_\rho] = C \text{ on } \partial B_\rho. \tag{2.3}$$

Therefore it is a critical point of $\mathcal{J}$.

We identify perturbations of B_ρ with functions in some Hilbert spaces. Define

$$\mathcal{X} = \left\{ \phi \in W^{2,2}(S^1) : \int_0^{2\pi} \phi(\theta)\, d\theta = 0 \right\} \tag{2.4}$$

$$\mathcal{Y} = \left\{ \phi \in W^{1,2}(S^1) : \int_0^{2\pi} \phi(\theta)\, d\theta = 0 \right\} \tag{2.5}$$

$$\mathcal{Z} = \left\{ \phi \in L^2(S^1) : \int_0^{2\pi} \phi(\theta)\, d\theta = 0 \right\} \tag{2.6}$$

Here S^1 is the unit circle in $\mathbb{R}^2$ centered at the origin, $L^2(S^1)$ is the L^2-Lebesgue space on S^1, $W^{1,2}(S^1)$ is the $W^{1,2}$-Sobolev space on S^1, and $W^{2,2}(S^1)$ is the $W^{2,2}$-Sobolev space on S^1. Note that

$$\mathcal{X} \subset \mathcal{Y} \subset \mathcal{Z} \subset L^2(S^1). \tag{2.7}$$

The inner product of $L^2(S^1)$ is denoted

$$\langle \phi, \psi \rangle = \int_0^{2\pi} \phi(\theta)\psi(\theta)\, d\theta, \tag{2.8}$$

which is inherited by $\mathcal{Z}$. The norms of $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are given respectively by

$$\|\phi\|_{\mathcal{X}}^2 = \langle \phi'', \phi'' \rangle + \langle \phi', \phi' \rangle + \langle \phi, \phi \rangle \tag{2.9}$$

$$\|\phi\|_{\mathcal{Y}}^2 = \langle \phi', \phi' \rangle + \langle \phi, \phi \rangle \tag{2.10}$$

$$\|\phi\|_{\mathcal{Z}}^2 = \langle \phi, \phi \rangle, \tag{2.11}$$

Then

$$\Omega_\phi = \bigcup_{\theta \in S^1} \left\{ te^{i\theta} : \ t \in \left[0, (\rho^2 + 2\phi(\theta))^{1/2}\right] \right\} \tag{2.12}$$

defines a set if $\phi \in \mathcal{X}$, $\mathcal{Y}$, or $\mathcal{Z}$, and $\rho^2 + \phi(\theta) \geq 0$ for every $\theta \in S^1$.

Let $\delta_0 > 0$ and consider $\phi \in \mathcal{Y}$ such that

$$\|\phi\|_{\mathcal{Y}} \leq \delta_0 \rho^2. \tag{2.13}$$

Then for every $\theta \in S^1$,

$$\rho^2 + 2\phi(\theta) \geq \rho^2 - 2\|\phi\|_{L^\infty} \geq \rho^2 - 2\tilde{C}\|\phi\|_{\mathcal{Y}} \geq \rho^2 - 2\tilde{C}\delta_0\rho^2 = (1 - 2\tilde{C}\delta_0)\rho^2 \tag{2.14}$$

where $\tilde{C}$ is a constant in the Sobolev embedding $W^{1,2}(S^1,\mathbb{R}) \to L^\infty(S^1,\mathbb{R})$; namely

$$\|f\|_{L^\infty} \leq \tilde{C}\|f\|_{W^{1,2}},\ \forall f \in W^{1,2}(S^1,\mathbb{R}). \tag{2.15}$$

If we make δ_0 small so that

$$1 - 2\tilde{C}\delta_0 > 0, \tag{2.16}$$

then $\rho^2 + 2\phi(\theta) > 0$ and ϕ defines a perturbed disc Ω_ϕ.

Also note that

$$|\Omega_\phi| = \int_0^{2\pi}\int_0^{(\rho^2+2\phi(\theta))^{1/2}} t\,dtd\theta = \int_0^{2\pi} \frac{\rho^2+2\phi(\theta)}{2}\,d\theta = \pi\rho^2 + \int_0^{2\pi}\phi(\theta)\,d\theta, \tag{2.17}$$

so the constraint (2.1) becomes the condition

$$\int_0^{2\pi}\phi(\theta)\,d\theta = 0 \tag{2.18}$$

in (2.4), (2.5), (2.6).

Now we treat $\mathcal{J}$ as a functional of ϕ and write

$$\mathcal{J}(\phi) = \mathcal{J}(\Omega_\phi). \tag{2.19}$$

More specifically, in terms of ϕ, the two terms in $\mathcal{J}(\phi)$ become

$$\mathcal{P}(\Omega_\phi) = \int_0^{2\pi}\sqrt{\rho^2+2\phi(\theta)+\frac{(\phi'(\theta))^2}{\rho^2+2\phi(\theta)}}\,d\theta \tag{2.20}$$

$$\int_{\Omega_\theta}\int_{\Omega_\theta} K(|x-y|)\,dxdy = \int_0^{2\pi}\int_0^{\sqrt{\rho^2+2\phi(\theta)}}\int_0^{2\pi}\int_0^{\sqrt{\rho^2+2\phi(\omega)}} K(|te^{i\theta}-\tau e^{i\omega}|)t\tau\,d\tau d\omega dt d\theta. \tag{2.21}$$

In this paper we identify $\mathbb{R}^2$ with $\mathbb{C}$ and write $e^{i\theta}$ in stead of $(\cos\theta,\sin\theta)$ for simplicity.

Note that if $\phi = 0$, then

$$\Omega_0 = B_\rho \tag{2.22}$$

and

$$\mathcal{J}(0) = \mathcal{J}(B_\rho). \tag{2.23}$$

This functional $\mathcal{J}$ of ϕ is defined in a neighborhood of 0 in $\mathcal{Y}$:

$$\mathrm{Dom}(\mathcal{J}) = \{\phi \in \mathcal{Y} : \|\phi\|_{\mathcal{Y}} \leq \delta_0\rho^2\} \tag{2.24}$$

where δ_0 is given in (2.13) and satisfies (2.16).

The first variation of $\mathcal{J}$, denoted by $\mathcal{J}'$, may be regarded as a nonlinear operator from a subset of $\mathcal{X}$ to $\mathcal{Z}$ so that

$$\frac{d\mathcal{J}(\phi+\varepsilon\psi)}{d\varepsilon}\Big|_{\varepsilon=0} = \langle \mathcal{J}'(\phi),\psi\rangle \tag{2.25}$$

The domain of $\mathcal{J}'$ is

$$\mathrm{Dom}(\mathcal{J}') = \{\phi \in \mathcal{X} : \|\phi\|_{\mathcal{X}} \leq \delta_0\rho^2\} \tag{2.26}$$

Here δ_0 is the same as in the definition of $\mathrm{Dom}(\mathcal{J})$, but ϕ is taken to be in $\mathcal{X}$ instead of $\mathcal{Y}$. Clearly $\mathrm{Dom}(\mathcal{J}') \subset \mathrm{Dom}(\mathcal{J})$.

Calculations show that

$$\frac{d\mathcal{J}(\phi+\varepsilon\psi)}{d\varepsilon}\Big|_{\varepsilon=0} = \int_0^{2\pi}\left(\kappa(\phi)(\theta) + \gamma K[\phi](\theta)\right)\psi(\theta)\,d\theta \tag{2.27}$$

where

$$\kappa(\phi)(\theta) = \frac{\rho^2 + 2\phi(\theta) + \frac{3(\phi'(\theta))^2}{\rho^2+2\phi(\theta)} - \phi''(\theta)}{\left(\rho^2 + 2\phi(\theta) + \frac{(\phi'(\theta))^2}{\rho^2+2\phi(\theta)}\right)^{3/2}}, \tag{2.28}$$

$$K[\phi](\theta) = \int_{\Omega_\phi} K\left(\left|(\rho^2 + 2\phi(\theta))^{1/2} e^{i\theta} - y\right|\right)\, dy. \tag{2.29}$$

Note that $\kappa(\phi)$ is the curvature of $\partial\Omega_\phi$ with respect to the inward pointing normal vector. Comparing (2.25) and (2.27) we find that

$$\langle \mathcal{J}'(\phi), \psi\rangle = \langle \kappa(\phi) + \gamma K[\phi], \psi\rangle \tag{2.30}$$

for all $\psi \in \mathcal{X}$. Since ψ is subject to the condition $\int_0^{2\pi} \psi(\theta)\, d\theta = 0$, (2.30) implies that there exists $C \in \mathbb{R}$ such that

$$\mathcal{J}'(\phi) = \kappa(\phi) + \gamma K[\phi] - C. \tag{2.31}$$

Note that $\mathcal{J}'(\phi)$ is in $\mathcal{Z}$, but $\kappa(\phi) + \gamma K[\phi]$ is in $L^2(S^1)$, not necessarily in $\mathcal{Z}$.

It is convenient to introduce a congruence relation $\cong$ in $L^2(S^1)$. For $\psi, \eta \in L^2(S^1)$, we say that $\psi \cong \eta$ if there exists $C \in \mathbb{R}$ such that

$$\psi - \eta = C. \tag{2.32}$$

The constant C can be found from ψ and η by averaging:

$$\frac{1}{2\pi}\int_0^{2\pi} \psi(\theta)\, d\theta - \frac{1}{2\pi}\int_0^{2\pi} \eta(\theta)\, d\theta = C. \tag{2.33}$$

The second variation of $\mathcal{J}$, denoted by $\mathcal{J}''$, is a map from $\mathrm{Dom}(\mathcal{J}')$ to the space of bounded linear operators from $\mathcal{X}$ to $\mathcal{Z}$. Note that $\mathcal{J}''$ has the same domain as $\mathcal{J}'$: $\mathrm{Dom}(\mathcal{J}'') = \mathrm{Dom}(\mathcal{J}')$. At each $\phi \in \mathrm{Dom}(\mathcal{J}')$, $\mathcal{J}''(\phi)$ is a linear operator from $\mathcal{X}$ to $\mathcal{Z}$ such that

$$\frac{d^2\mathcal{J}(\phi + \varepsilon\psi)}{d\varepsilon^2}\Big|_{\varepsilon=0} = \langle \mathcal{J}''(\phi)(\psi), \psi\rangle \tag{2.34}$$

for all $\psi \in \mathcal{X}$. In this section we only need the second variation at B_ρ, i.e. $\phi = 0$. Calculations show that

$$\begin{aligned} &\mathcal{J}''(0)(\psi)(\theta) \cong \rho^{-3}(-\psi''(\theta) - \psi(\theta)) \\ &\quad + \gamma\left[\int_0^{2\pi} K\left(|\rho e^{i\theta} - \rho e^{i\omega}|\right)\psi(\omega)\, d\omega + \left(\rho^{-1}\int_{B_\rho} K'\left(|\rho e^{i\theta} - y|\right)\frac{(\rho e^{i\theta} - y)\cdot e^{i\theta}}{|\rho e^{i\theta} - y|}\, dy\right)\psi(\theta)\right]. \end{aligned} \tag{2.35}$$

Note that $\mathcal{J}''(0)$ is a self-adjoint operator defined on $\mathcal{X} \subset \mathcal{Z}$. Its spectrum consists of eigenvalues only. The following theorem gives all the eigenvalues of this operator.

Theorem 2.1. *At B_ρ, the eigenvalues of $\mathcal{J}''(0)$ are $\lambda(n)$, $n = 1, 2, 3, ...$, given by*

$$\begin{aligned} &\lambda(1) = 0, \\ &\lambda(n) = \rho^{-3}(n^2 - 1) + \gamma\left(\frac{1}{2n} - \frac{1}{2} + 2\pi\rho^2\right),\ n \geq 2 \end{aligned}$$

and the corresponding eigen spaces are $E(n) = \{c_1 \cos n\theta + c_2 \sin n\theta :\ c_1, c_2 \in \mathbb{R}\}$.

Proof. We write

$$K(t) = L(t) + Q(t), \quad \text{where } L(t) = \frac{1}{2\pi}\log\frac{1}{t}, \quad Q(t) = t^2. \tag{2.36}$$

Let $\psi(\theta) = e^{in\theta}$, $n = \pm 1, \pm 2, \ldots$. The theorem follows from the following computation.

$$e^{in\theta} \to \rho^{-3}((-e^{in\theta})'' - e^{in\theta}) = \rho^{-3}(n^2-1)e^{in\theta} \tag{2.37}$$

$$e^{in\theta} \to \int_0^{2\pi} L\left(|\rho e^{i\theta} - \rho e^{i\omega}|\right) e^{in\omega}\, d\omega = \frac{1}{2|n|}e^{in\theta} \tag{2.38}$$

$$e^{in\theta} \to \int_0^{2\pi} Q\left(|\rho e^{i\theta} - \rho e^{i\omega}|\right) e^{in\omega}\, d\omega = \begin{cases} -2\pi\rho^2 e^{i\theta} & \text{if } n = 1 \\ -2\pi\rho^2 e^{-i\theta} & \text{if } n = -1 \\ 0 & \text{if } |n| \neq 1 \end{cases} \tag{2.39}$$

$$e^{in\theta} \to \left(\rho^{-1}\int_{B_\rho} L'\left(|\rho e^{i\theta} - y|\right)\frac{(\rho e^{i\theta} - y)\cdot e^{i\theta}}{|\rho e^{i\theta} - y|}\, dy\right) e^{in\theta} = -\frac{1}{2}e^{in\theta} \tag{2.40}$$

$$e^{in\theta} \to \left(\rho^{-1}\int_{B_\rho} Q'\left(|\rho e^{i\theta} - y|\right)\frac{(\rho e^{i\theta} - y)\cdot e^{i\theta}}{|\rho e^{i\theta} - y|}\, dy\right) e^{in\theta} = 2\pi\rho^2 e^{in\theta}. \tag{2.41}$$

Here (2.38) and (2.40) may be less obvious. Because of the Fourier series

$$\log|1 - e^{i\eta}| = -\sum_{n=1}^{\infty}\frac{\cos n\eta}{n} = -\sum_{n=-\infty, n\neq 0}^{\infty}\frac{e^{in\eta}}{2|n|}, \tag{2.42}$$

one finds

$$e^{in\theta} \to \int_0^{2\pi}\log|1 - e^{i(\theta-\omega)}|e^{in\omega}\, d\omega = \begin{cases} -\frac{\pi}{|n|}e^{in\theta}, & \text{if } n \neq 0 \\ 0, & \text{if } n = 0 \end{cases}. \tag{2.43}$$

This proves (2.38). For the integral in (2.40), note

$$\rho^{-1}\int_{B_\rho} L'\left(|\rho e^{i\theta} - y|\right)\frac{(\rho e^{i\theta} - y)\cdot e^{i\theta}}{|\rho e^{i\theta} - y|}\, dy = -\frac{1}{2\pi\rho}\int_{B_\rho}\frac{(\rho e^{i\theta} - y)\cdot e^{i\theta}}{|\rho e^{i\theta} - y|^2}\, dy \tag{2.44}$$

$$= -\frac{1}{2\pi}\int_{B_1(0)}\frac{(e^{i\theta} - Y)\cdot e^{i\theta}}{|e^{i\theta} - Y|^2}\, dY. \tag{2.45}$$

Let $Y = e^{i\theta}(1 - Z)$, and $Z = re^{i\beta}$. The disc $B_1(0)$ now becomes $B_1(1)$, the disc centered at $1 \in \mathbb{C} \equiv \mathbb{R}^2$ of radius 1. Its boundary is parametrized in the polar coordinates by $r = 2\cos\beta$. Then we have

$$\int_{B_1(0)}\frac{(e^{i\theta} - Y)\cdot e^{i\theta}}{|e^{i\theta} - Y|^2}\, dY = \int_{B_1(1)}\frac{e^{i\theta}Z\cdot e^{i\theta}}{|Z|^2}\, dZ = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\beta}\cos\beta\, dr d\beta = \pi, \tag{2.46}$$

and (2.40) follows. □

The zero eigenvalue $\lambda(1) = 0$ associated with the eigenfunctions $\cos\theta$ and $\sin\theta$ is the result of the translation invariance of $\mathcal{J}$: for any $\Omega \in \mathcal{A}$,

$$\mathcal{J}(\Omega) = \mathcal{J}(T_h\Omega) \tag{2.47}$$

for every $h \in \mathbb{R}^2$ where $T_h\Omega = \{x + h : \ x \in \Omega\} \subset \mathbb{R}^2$ is a translate of Ω by h.

The stability of B_ρ is determined by the remaining eigenvalues. If all the remaining eigenvalues are positive, then B_ρ is a stable critical point; if one of the remaining eigenvalue is negative, then B_ρ is an unstable critical point. The next theorem gives the stability of B_ρ in terms of the two parameters: ρ and γ.

Theorem 2.2. *The stability of B_ρ follows from the following statements.*

1. *If $\rho \geq \frac{1}{2\sqrt{\pi}}$, then for all $\gamma > 0$, $\lambda(n) > 0$, $n = 2, 3, ...$*
2. *If $\rho < \frac{1}{2\sqrt{\pi}}$, then there exists $\beta_\rho > 0$ such that all $\lambda(n) > 0$, $n = 2, 3, ...,$ if $\gamma\rho^3 < \beta_\rho$.*
3. *If $\rho < \frac{1}{2\sqrt{\pi}}$ and $\gamma\rho^3 > \beta_\rho$, then at least one $\lambda(n)$ is negative.*
4. *If $\rho < \frac{1}{2\sqrt{\pi}}$ and $\gamma\rho^3 = \beta_\rho$, then all $\lambda(n) \geq 0$ and at least one $\lambda(n)$, $n \geq 2$, equals 0.*

Proof. By Theorem 2.1,

$$\lambda(n) = \rho^{-3}(n^2 - 1) + \gamma\left(\frac{1}{2n} - \frac{1}{2} + 2\pi\rho^2\right), \ n = 2, 3, ... \tag{2.48}$$

If $\rho \geq \frac{1}{2\sqrt{\pi}}$, then $-\frac{1}{2} + 2\pi\rho^2 \geq 0$, and hence all $\lambda(n) > 0$, $n = 2, 3, ...,$ which proves part 1.

Let $\rho < \frac{1}{2\sqrt{\pi}}$. Introduce

$$g_\rho(n) = \frac{-\frac{1}{2n} + \frac{1}{2} - 2\pi\rho^2}{n^2 - 1}, \ n = 2, 3, ... \tag{2.49}$$

so that

$$\lambda(n) = \gamma(n^2 - 1)\left(\frac{1}{\gamma\rho^3} - g_\rho(n)\right), \ n = 2, 3, ... \tag{2.50}$$

by (2.48). Since $\frac{1}{2} - 2\pi\rho^2 > 0$, $g_\rho(n) > 0$ if n is sufficiently large. Also $g_\rho(n) \to 0$ as $n \to \infty$. Hence g_ρ achieves a positive maximum value at some $n_\rho \in \{2, 3, ...\}$. Define $\beta_\rho > 0$ such that

$$\frac{1}{\beta_\rho} = \max\{g_\rho(n) : \ n = 2, 3, ...\} = g_\rho(n_\rho). \tag{2.51}$$

If $\gamma\rho^3 < \beta_\rho$, then

$$\frac{1}{\gamma\rho^3} > g_\rho(n), \ n = 2, 3, ... \tag{2.52}$$

and all $\lambda(n) > 0$, proving part 2.

If $\gamma\rho^3 > \beta_\rho$, then

$$\frac{1}{\gamma\rho^3} < g_\rho(n_\rho) \tag{2.53}$$

and $\lambda(n_\rho) < 0$, proving part 3.

If $\gamma\rho^3 = \beta_\rho$, then

$$\frac{1}{\gamma\rho^3} \geq g_\rho(n), \ n = 2, 3, ..., \text{ and } \frac{1}{\gamma\rho^3} = g_\rho(n_\rho). \tag{2.54}$$

Therefore $\lambda(n) \geq 0$, $n = 2, 3, ...,$ and $\lambda(n_\rho) = 0$, proving part 4. □

3 Multiple disc assemblies

Now we proceed to build critical points of $\mathcal{J}$ that are assemblies of perturbed discs. Let $N \in \{2, 3, ...\}$, $\rho > 0$, and write the constraint $|\Omega| = m$ on the measure of Ω as

$$|\Omega| = N\pi\rho^2 \tag{3.1}$$

Henceforth we replace the parameter m in (1.2) by ρ and N. The main result is the following theorem which is proved in this and the next two sections.

Theorem 3.1. *Let N be an integer ≥ 2 . For each $\eta > 0$, there exists $\delta > 0$, depending on N and η only, such that if*

1. $\rho < \delta$,
2. $\gamma\rho^3 < 12 - \eta$,
3. $\gamma\rho^3 \log \frac{1}{\rho} > 1 + \eta$,

then there exists a critical point of $\mathcal{J}$ satisfying the constraint (3.1)*. Moreover, the following properties hold.*

1. *This critical point is the union of N disconnected components, and each component is close to a disc of radius ρ centered at $\xi_{\rho,j}$, $j = 1, 2, ..., N$.*
2. *As $\rho \to 0$, any accumulation point of $(\xi_{\rho,1}, \xi_{\rho,2}, ..., \xi_{\rho,N})$ is a global minimum of the function*

$$F(\xi_1, \xi_2, ..., \xi_N) = \sum_{j=1}^{N} \sum_{k=1,\neq j}^{N} K(|\xi_j - \xi_k|). \tag{3.2}$$

3. *This critical point is stable in a sense.*

From now on N is a fixed positive integer greater than or equal to 2. Take N discs $B(\xi_j, r_j)$ centered at ξ_j of radius r_j subject to the constraint (3.1), i.e.

$$\sum_{j=1}^{N} r_j^2 = N\rho^2. \tag{3.3}$$

We introduce two positive numbers δ_1 and δ_2 to specify the range of (ξ, r). For now we only require $\delta_1 > 0$ and $0 < \delta_2 < 1$, but more conditions on them will be added later. The ξ_j's must satisfy

$$4\delta_1 \leq |\xi_j - \xi_k| \leq \frac{1}{4\delta_1} \text{ for all } j \neq k \tag{3.4}$$

and the r_j's satisfy

$$|r_j - \rho| \leq \delta_2\rho \tag{3.5}$$

The discs $B(\xi_j, r_j)$ must be mutually disjoint. For $x_j \in B(\xi_j, r_j)$ and $x_k \in B(\xi_k, r_k)$, $j \neq k$

$$\begin{aligned} |x_j - x_k| &\geq |\xi_j - \xi_k| - r_j - r_k \geq 4\delta_1 - (\rho + |r_j - \rho|) - (\rho + |r_k - \rho|) \geq 4\delta_1 - (\rho + \delta_2\rho) - (\rho + \delta_2\rho) \\ &= 4\delta_1 - 2(1 + \delta_2)\rho \end{aligned} \tag{3.6}$$

Hence the $B(\xi_j, r_j)$'s are disjoint if

$$4\delta_1 - 2(1 + \delta_2)\rho > 0 \tag{3.7}$$

which is accomplished if ρ is sufficiently small.

Lemma 3.2.

$$\begin{aligned} \mathcal{J}(\cup_{j=1}^{N} B(\xi_j, r_j)) &= \sum_{j=1}^{N} 2\pi r_j + \frac{\gamma}{2} \sum_{j=1}^{N} \sum_{k=1}^{N} \int_{B(\xi_j, r_j)} \int_{B(\xi_j, r_j)} K(|x - y|)\, dx\, dy \\ &= \sum_{j=1}^{N} 2\pi r_j + \frac{\gamma}{2} \left[\sum_{j=1}^{N} \left(\frac{\pi r_j^4}{2} \log \frac{1}{r_j} + \frac{\pi r_j^4}{8} + \pi^2 r_j^6 \right) \right. \\ &\quad \left. + \sum_{j=1}^{N} \sum_{k=1,\neq j}^{N} \left(\frac{\pi r_j^2 r_k^2}{2} \log \frac{1}{|\xi_j - \xi_k|} + \pi^2 r_j^2 r_k^2 |\xi_j - \xi_k|^2 + \frac{\pi^2 r_j^2 r_k^2 (r_j^2 + r_k^2)}{2} \right) \right] \end{aligned}$$

Proof. Clearly

$$\mathcal{P}(\cup_{j=1}^N B(\xi_j, r_j)) = \sum_{j=1}^N 2\pi r_j. \tag{3.8}$$

Let $X = te^{i\theta}$, $Y = e^{i\theta}Z$, and $Z = \rho e^{i\beta}$. If $0 \le t \le 1$, then, with the help of

$$\log|1-\rho e^{i\beta}| = -\sum_{k=1}^\infty \frac{\rho^k \cos k\beta}{k}, \tag{3.9}$$

we compute

$$\begin{aligned}
&\int_{B_1} \log|X-Y|\,dY \\
&= \int_0^1\int_0^{2\pi} \log|t-\rho e^{i\beta}|\rho\,d\beta d\rho \\
&= \int_0^t\int_0^{2\pi} \log|t-\rho e^{i\beta}|\rho\,d\beta d\rho + \int_t^1\int_0^{2\pi} \log|t-\rho e^{i\beta}|\rho\,d\beta d\rho \\
&= \int_0^t\int_0^{2\pi} \left(\log t + \log|1-\frac{\rho}{t}e^{i\beta}|\right)\rho\,d\beta d\rho + \int_t^1\int_0^{2\pi} \left(\log\rho + \log|1-\frac{t}{\rho}e^{-i\beta}|\right)\rho\,d\beta d\rho \\
&= \int_0^t\int_0^{2\pi} \left(\log t - \sum_{k=1}^\infty \left(\frac{\rho}{t}\right)^k \frac{\cos k\beta}{k}\right)\rho\,d\beta d\rho + \int_t^1\int_0^{2\pi} \left(\log\rho - \sum_{k=1}^\infty \left(\frac{t}{\rho}\right)^k \frac{\cos k\beta}{k}\right)\rho\,d\beta d\rho \\
&= \int_0^t 2\pi(\log t)\rho\,d\rho + \int_t^1 2\pi\rho\log\rho\,d\rho \\
&= \pi t^2\log t + 2\pi\left(-\frac{1}{4} - \frac{t^2}{2}\log t + \frac{t^2}{4}\right) \\
&= \frac{\pi}{2}(t^2-1).
\end{aligned}$$

If $X = te^{i\theta}$ with $t > 1$, then the calculations above change to

$$\begin{aligned}
\int_{B_1} \log|X-Y|\,dY &= \int_0^1\int_0^{2\pi} \log|t-\rho e^{i\beta}|\rho\,d\beta d\rho \\
&= \int_0^1\int_0^{2\pi} \left(\log t + \log|1-\frac{\rho}{t}e^{i\beta}|\right)\rho\,d\beta d\rho \\
&= \int_0^1\int_0^{2\pi} \left(\log t - \sum_{k=1}^\infty \left(\frac{\rho}{t}\right)^k \frac{\cos k\beta}{k}\right)\rho\,d\beta d\rho \\
&= \int_0^1 2\pi(\log t)\rho\,d\rho \\
&= \pi\log t
\end{aligned}$$

Therefore

$$\int_{B_1} \frac{1}{2\pi}\log\frac{1}{|X-Y|}\,dY = \begin{cases} \frac{1}{4}\left(1-|X|^2\right), & \text{if } 0 \le |X| \le 1 \\ \frac{1}{2}\log\frac{1}{|X|}, & \text{if } 1 < |X| \end{cases} \tag{3.10}$$

Also

$$\int_{B_1} |X-Y|^2\,dY = \pi|X|^2 + \frac{\pi}{2}. \tag{3.11}$$

Hence

$$\int_{B_1} K(|X-Y|)\,dY = \begin{Bmatrix} \frac{1}{4}\left(1-|X|^2\right), & \text{if } 0 \le |X| \le 1 \\ \frac{1}{2}\log\frac{1}{|X|}, & \text{if } 1 < |X| \end{Bmatrix} + \pi|X|^2 + \frac{\pi}{2} \tag{3.12}$$

Consequently

$$\int_{B_1}\int_{B_1} K(|X-Y|)\,dXdY = \int_{B_1} \frac{1-|X|^2}{4}\,dX + \int_{B_1}\left(\pi|X|^2+\frac{\pi}{2}\right)\,dX = \frac{\pi}{8}+\pi^2 \tag{3.13}$$

More generally,

$$\int_{B(\xi_j,r_j)} K(|x-y|)\,dy = \begin{Bmatrix} \frac{r_j^2}{2}\log\frac{1}{r_j} + \frac{r_j^2}{4}\left(1-\frac{|x-\xi_j|^2}{r_j^2}\right), & \text{if } 0 \le |x-\xi_j| \le r_j \\ \frac{r_j^2}{2}\log\frac{1}{|x-\xi_j|}, & \text{if } r_j < |x-\xi_j| \end{Bmatrix} + \pi r_j^2|x-\xi_j|^2 + \frac{\pi r_j^4}{2}. \tag{3.14}$$

Then

$$\int_{B(\xi_j,r_j)}\int_{B(\xi_j,r_j)} K(|x-y|)\ dxdy = \frac{\pi r_j^4}{2}\log\frac{1}{r_j} + \frac{\pi r_j^4}{8} + \pi^2 r_j^6 \tag{3.15}$$

$$\int_{B(\xi_j,r_j)}\int_{B(\xi_k,r_k)} K(|x-y|)\ dxdy = \frac{\pi r_j^2 r_k^2}{2}\log\frac{1}{|\xi_j-\xi_k|} + \pi^2 r_j^2 r_k^2|\xi_j-\xi_k|^2 + \frac{\pi^2 r_j^2 r_k^2(r_j^2+r_k^2)}{2} \tag{3.16}$$

where $j \neq k$, and the lemma follows from (3.8), (3.15), and (3.16). □

Now we introduce perturbations of $\cup_{j=1}^N B(\xi_j,r_j)$ and we proceed along the same line as in the single disc case. Define

$$\mathcal{X} = \left\{\phi = (\phi_1,\phi_2,...,\phi_N) \in W^{2,2}(S^1,\mathbb{R}^N):\ \sum_{j=1}^N \int_0^{2\pi} \phi_j(\theta)\,d\theta = 0\right\} \tag{3.17}$$

$$\mathcal{Y} = \left\{\phi = (\phi_1,\phi_2,...,\phi_N) \in W^{1,2}(S^1,\mathbb{R}^N):\ \sum_{j=1}^N \int_0^{2\pi} \phi_j(\theta)\,d\theta = 0\right\} \tag{3.18}$$

$$\mathcal{Z} = \left\{\phi = (\phi_1,\phi_2,...,\phi_N) \in L^2(S^1,\mathbb{R}^N):\ \sum_{j=1}^N \int_0^{2\pi} \phi_j(\theta)\,d\theta = 0\right\}. \tag{3.19}$$

Note that

$$\mathcal{X} \subset \mathcal{Y} \subset \mathcal{Z} \subset L^2(S^1,\mathbb{R}^N). \tag{3.20}$$

The inner product of $L^2(S^1,\mathbb{R}^N)$ is denoted

$$\langle \phi,\psi\rangle = \sum_{j=1}^N \int_0^{2\pi} \phi_j(\theta)\psi_j(\theta)\,d\theta, \tag{3.21}$$

and $\mathcal{Z}$ is the subspace of $L^2(S^1,\mathbb{R}^N)$ perpendicular to $(1,1,...,1)$. The norms of $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are given respectively by

$$\|\phi\|_{\mathcal{X}}^2 = \langle \phi'',\phi''\rangle + \langle \phi',\phi'\rangle + \langle \phi,\phi\rangle \tag{3.22}$$

$$\|\phi\|_{\mathcal{Y}}^2 = \langle \phi',\phi'\rangle + \langle \phi,\phi\rangle \tag{3.23}$$

$$\|\phi\|_{\mathcal{Z}}^2 = \langle \phi,\phi\rangle, \tag{3.24}$$

Given N distinct discs $B(\xi_j, r_j)$ centered at ξ_j of radius r_j considered before, define a perturbation Ω_ϕ by

$$\Omega_\phi = \bigcup_{j=1}^{N} \Omega_{\phi_j}, \quad \Omega_{\phi_j} = \bigcup_{\theta \in S^1} \left\{ \xi_j + t e^{i\theta} : \ t \in \left[0, \left(r_j^2 + 2\phi_j(\theta) \right)^{1/2} \right] \right\} \tag{3.25}$$

for $\phi \in \mathcal{Y}$ sufficiently small.

We must ensure that each perturbed disc Ω_{ϕ_j} is well defined; namely

$$r_j^2 + 2\phi_j(\theta) \geq 0, \ \forall \theta \in S^1, \ j = 1, 2, ..., N \tag{3.26}$$

This is one reason why the $\mathcal{Y}$-norm of ϕ has to be small. To quantify this condition assume

$$\|\phi\|_{\mathcal{Y}} \leq \delta_0 \rho^2 \tag{3.27}$$

where δ_0 is to be determined. We have, for every $\theta \in S^1$,

$$\begin{aligned} r_j^2 + 2\phi_j(\theta) &\geq r_j^2 - 2\|\phi_j\|_{L^\infty} \geq r_j^2 - 2\tilde{C}\|\phi\|_{\mathcal{Y}} \geq r_j^2 - 2\tilde{C}\delta_0\rho^2 \\ &\geq (\rho - |r_j - \rho|)^2 - 2\tilde{C}\delta_0\rho^2 \geq (\rho - \delta_2\rho)^2 - 2\tilde{C}\delta_0\rho^2 \\ &= \left((1-\delta_2)^2 - 2\tilde{C}\delta_0 \right) \rho^2 \end{aligned} \tag{3.28}$$

Here $\tilde{C}$ is the same Sobolev embedding constant as in (2.15). If we make δ_0 small enough so that

$$(1-\delta_2)^2 - 2\tilde{C}\delta_0 > 0, \tag{3.29}$$

then each Ω_{ϕ_j} is a well defined perturbed disc. Condition (3.29) is met provided δ_0 is small in comparison to δ_2. Henceforth δ_0 is taken to satisfy (3.29); this δ_0 is analogous to but different from the δ_0 in the previous section.

We also need to be certain that Ω_{ϕ_j} do not intersect Ω_{ϕ_k} whenever $j \neq k$. Let $x_j \in \Omega_{\phi_j}$. Note that

$$\begin{aligned} |x_j - \xi_j|^2 &\leq r_j^2 + 2\|\phi_j\|_{L^\infty} \leq (\rho + |r_j - \rho|)^2 + 2\tilde{C}\delta_0\rho^2 \leq (1+\delta_2)^2\rho^2 + 2\tilde{C}\delta_0\rho^2 \\ &\leq \left((1+\delta_2)^2 + 2\tilde{C}\delta_0 \right) \rho^2 \end{aligned} \tag{3.30}$$

and consequently

$$\begin{aligned} |x_j - x_k| &\geq |\xi_j - \xi_k| - |x_j - \xi_j| - |x_k - \xi_k| \\ &\geq 4\delta_1 - 2\sqrt{(1+\delta_2)^2 + 2\tilde{C}\delta_0}\ \rho. \end{aligned} \tag{3.31}$$

Hence if we strengthen the condition (3.7) to

$$4\delta_1 - 2\sqrt{(1+\delta_2)^2 + 2\tilde{C}\delta_0}\ \rho > 0, \tag{3.32}$$

then Ω_{ϕ_j} does not intersect Ω_{ϕ_k} whenever $j \neq k$, and (3.32) can be achieved if ρ is small.

Now we treat $\mathcal{J}$ as a functional of ϕ and write

$$\mathcal{J}(\phi) = \mathcal{J}(\Omega_\phi). \tag{3.33}$$

Note that if $\phi = (0, 0, ..., 0)$ which we simply denote by 0, then

$$\Omega_0 = \bigcup_{j=1}^{N} B(\xi_j, r_j) \tag{3.34}$$

and

$$\mathcal{J}(0) = \mathcal{J}(\bigcup_{j=1}^{N} B(\xi_j, r_j)) \tag{3.35}$$

which is given in Lemma 3.2. This functional $\mathcal{J}$ of ϕ is defined for small ρ and the domain of $\mathcal{J}$ is a small neighborhood of 0 in $\mathcal{Y}$:

$$\mathrm{Dom}(\mathcal{J}) = \{\phi \in \mathcal{Y} : \|\phi\|_{\mathcal{Y}} \leq \delta_0 \rho^2\} \tag{3.36}$$

where δ_0 is given in (3.27) and satisfies (3.29). Then as we have just explained when ρ is small, every ϕ in $\mathrm{Dom}(\mathcal{J})$ represents a perturbation of $\cup_{j=1}^{N} B(\xi_j, r_j)$.

The first variation of $\mathcal{J}$, denoted by $\mathcal{J}'$, may be regarded as a nonlinear operator from a subset of $\mathcal{X}$ to $\mathcal{Z}$ so that

$$\frac{d\mathcal{J}(\phi + \varepsilon\psi)}{d\varepsilon}\Big|_{\varepsilon=0} = \langle \mathcal{J}'(\phi), \psi \rangle, \ \forall \psi \in \mathcal{X}. \tag{3.37}$$

The domain of $\mathcal{J}'$ is

$$\mathrm{Dom}(\mathcal{J}') = \{\phi \in \mathcal{X} : \|\phi\|_{\mathcal{X}} \leq \delta_0 \rho^2\}. \tag{3.38}$$

Here δ_0 is the same as in the definition of $\mathrm{Dom}(\mathcal{J})$, but ϕ is taken to be in $\mathcal{X}$ instead of $\mathcal{Y}$. Clearly $\mathrm{Dom}(\mathcal{J}') \subset \mathrm{Dom}(\mathcal{J})$.

Calculations show that

$$\frac{d\mathcal{J}(\phi + \varepsilon\psi)}{d\varepsilon}\Big|_{\varepsilon=0} = \sum_{j=1}^{K} \int_0^{2\pi} \left(\kappa_j(\phi_j)(\theta) + \gamma K_j[\phi](\theta)\right) \psi_j(\theta)\, d\theta \tag{3.39}$$

where

$$\kappa_j(\phi_j)(\theta) = \frac{r_j^2 + 2\phi_j(\theta) + \frac{3(\phi_j'(\theta))^2}{r_j^2 + 2\phi_j(\theta)} - \phi_j''(\theta)}{\left(r_j^2 + 2\phi_j(\theta) + \frac{(\phi_j'(\theta))^2}{r_j^2 + 2\phi_j(\theta)}\right)^{3/2}} \tag{3.40}$$

$$K_j[\phi](\theta) = \int_{\Omega_\phi} K\left(\left|\xi_j + (r_j^2 + 2\phi_j(\theta))^{1/2} e^{i\theta} - y\right|\right) dy. \tag{3.41}$$

Note that $\kappa_j(\phi_j)$ is the curvature of $\partial\Omega_{\phi_j}$. Let us define $\kappa(\phi)$ and $K[\phi]$, both in $L^2(S^2, \mathbb{R}^N)$, by

$$\kappa(\phi) = (\kappa_1(\phi_1), \kappa_2(\phi_2), ..., \kappa_N(\phi_N)) \tag{3.42}$$

$$K[\phi] = (K_1[\phi], K_2[\phi], ..., K_N[\phi])\,. \tag{3.43}$$

Then

$$\frac{d\mathcal{J}(\phi + \varepsilon\psi)}{d\varepsilon}\Big|_{\varepsilon=0} = \langle \kappa(\phi) + \gamma K[\phi], \psi \rangle. \tag{3.44}$$

Comparing (3.37) and (3.44) we find that

$$\langle \mathcal{J}'(\phi), \psi \rangle = \langle \kappa(\phi) + \gamma K[\phi], \psi \rangle \tag{3.45}$$

for all $\psi \in \mathcal{X}$. Since $\psi \perp (1, 1, ..., 1)$, $\mathcal{J}'(\phi)$ and $\kappa(\phi) + \gamma K[\phi]$ differ by a scalar multiple of $(1, 1, ..., 1)$; namely there exists $C \in \mathbb{R}$ such that

$$\mathcal{J}_j'(\phi) = \kappa_j(\phi_j) + \gamma K_j[\phi] - C, \ j = 1, 2, ..., N. \tag{3.46}$$

Analogous to the setting in the previous section, we introduce a congruence relation $\cong$ for members in $L^2(S^1, \mathbb{R}^N)$. This time $\psi \cong \eta$ if there exists $C \in \mathbb{R}$ such that

$$\psi_j - \eta_j = C \text{ for all } j = 1, 2, ..., N. \tag{3.47}$$

We may also abuse this notation and write $\psi_j \cong \eta_j$, $j = 1, 2, ..., N$, in place of $\psi \cong \eta$. Under this notation (3.46) becomes

$$\mathcal{J}'(\phi) \cong \kappa(\phi) + \gamma K[\phi]. \tag{3.48}$$

Our approach to solve the equation

$$\mathcal{J}'(\phi) = 0 \tag{3.49}$$

is based on a type of Lyapunov-Schmidt reduction argument and consists of two steps. First in section 4 we find a "pseudo-solution" which solves (3.49) up to a finite dimensional subspace. Then in the second step we find an exact solution in section 5.

The pseudo-solution is found in a space $\mathcal{X}_\flat$ which is a subspace of $\mathcal{X}$; namely

$$\mathcal{X}_\flat = \mathcal{X} \cap \mathcal{Z}_\flat \tag{3.50}$$

where

$$\mathcal{Z}_\flat = \left\{ \phi \in \mathcal{Z} : \int_0^{2\pi} \phi_j(\theta)\, d\theta = \int_0^{2\pi} \phi_j(\theta) \cos\theta\, d\theta = \int_0^{2\pi} \phi_j(\theta) \sin\theta\, d\theta = 0,\ j = 1, 2, ..., N \right\}. \tag{3.51}$$

If $\phi \in \mathcal{X}_\flat \cap \mathrm{Dom}(\mathcal{J}')$, then in terms of the set Ω_ϕ, the condition

$$\int_0^{2\pi} \phi_j(\theta)\, d\theta = 0 \tag{3.52}$$

means that the measure of each component Ω_{θ_j} equals πr_j^2. The condition

$$\int_0^{2\pi} \phi_j(\theta) \cos\theta\, d\theta = \int_0^{2\pi} \phi_j(\theta) \sin\theta\, d\theta = 0 \tag{3.53}$$

says that Ω_{ϕ_j} is "centered" at ξ_j.

Let Π be the orthogonal projection operator from $\mathcal{Z}$ to $\mathcal{Z}_\flat$.

We find $\mathcal{J}'(0)$, the first variation of $\mathcal{S}$ at $\phi = 0$, and estimate $\Pi\mathcal{J}'(0)$.

Lemma 3.3.

$$\mathcal{J}_j'(0) \cong \frac{1}{r_j} + \gamma \left[\frac{r_j^2}{2} \log\frac{1}{r_j} + \frac{3\pi r_j^4}{2} + \sum_{k=1,\neq j}^{N} \left(\frac{r_k^2}{2} \log \frac{1}{|\xi_j + r_j e^{i\theta} - \xi_k|} + \pi r_k^2 |\xi_j + r_j e^{i\theta} - \xi_k|^2 + \frac{\pi r_k^4}{2} \right) \right]$$

Consequently there exist $C_1' > 0$ and $C_1 > 0$ such that

$$\|\Pi\mathcal{J}'(0)\|_{\mathcal{Z}} \leq C_1' \gamma \rho^4 \leq C_1 \rho.$$

Proof. Since $\mathcal{J}'(0) \cong \kappa(0) + \gamma K[0]$, one finds

$$\kappa_j(0)(\theta) = \frac{1}{r_j}, \tag{3.54}$$

and, from (3.14) and (3.41),

$$K_j[0](\theta) = \sum_{k=1}^{N} \int_{B(\xi_k, r_k)} K(|\xi_j + r_j e^{i\theta} - y|)\, dy$$
$$= \frac{r_j^2}{2} \log \frac{1}{r_j} + \frac{3\pi r_j^4}{2} + \sum_{k=1, k\neq j}^{N} \left[\frac{r_k^2}{2} \log \frac{1}{|\xi_j + r_j e^{i\theta} - \xi_k|} + \pi r_k^2 |\xi_j + r_j e^{i\theta} - \xi_k|^2 + \frac{\pi r_k^4}{2} \right]. \tag{3.55}$$

Expand

$$\log |\xi_j + r_j e^{i\theta} - \xi_k| = \log |\xi_j - \xi_k| + \frac{\xi_j - \xi_k}{|\xi_j - \xi_k|^2} \cdot r_j e^{i\theta} + O(r_j^2) \tag{3.56}$$

$$|\xi_j + r_j e^{i\theta} - \xi_k|^2 = |\xi_j - \xi_k|^2 + 2(\xi_j - \xi_k) \cdot r_j e^{i\theta} + r_j^2. \tag{3.57}$$

When Π is applied, constant terms and terms that just involve $\cos\theta$ and $\sin\theta$ vanish and we arrive at the conclusion of the lemma. Note that $C_1' \gamma \rho^4 \leq C_1 \rho$ because $\gamma \rho^3 < 12 - \eta$ which is the second condition of Theorem 3.1. □

The second derivative of $\mathcal{J}$ at $\phi \in \mathrm{Dom}(\mathcal{J}'') = \mathrm{Dom}(\mathcal{J}')$, denoted by $\mathcal{J}''(\phi)$, is a linear operator from $\mathcal{X}$ to $\mathcal{Z}$ so that

$$\frac{d^2 \mathcal{J}(\phi + \varepsilon \psi)}{d\varepsilon^2}\Big|_{\varepsilon=0} = \langle \mathcal{J}''(\phi)(\psi), \psi \rangle \tag{3.58}$$

for all $\psi \in \mathcal{X}$. Calculations show that the second variation at 0 is

$$\mathcal{J}_j''(0)(\psi)(\theta) \cong r_j^{-3}(-\psi_j''(\theta) - \psi_j(\theta)) + \gamma \left[\sum_{k=1}^{N} \int_0^{2\pi} K\left(|\xi_j + r_j e^{i\theta} - \xi_k - r_k e^{i\omega}|\right) \psi_k(\omega)\, d\omega \right.$$
$$\left. + \sum_{k=1}^{N} \left(r_j^{-1} \int_{B(\xi_k, r_k)} K'\left(|\xi_j + r_j e^{i\theta} - y|\right) \frac{(\xi_j + r_j e^{i\theta} - y) \cdot e^{i\theta}}{|\xi_j + r_j e^{i\theta} - y|}\, dy \right) \psi_j(\theta) \right]. \tag{3.59}$$

To find a pseudo-solution, we need to study $\Pi \mathcal{J}''(0)|_{\mathcal{X}_\flat}$ from $\mathcal{X}_\flat$ to $\mathcal{Z}_\flat$, which is the restriction of $\mathcal{J}''(0)$ to $\mathcal{X}_\flat$ composed with Π. For simplicity we denote this operator just by $\Pi \mathcal{J}''(0)$.

Lemma 3.4. *There exists $c_2 > 0$ such that for every $\psi \in \mathcal{X}_\flat$,*

1. $$\langle \Pi \mathcal{J}''(0)(\psi), \psi \rangle \geq c_2 \rho^{-3} \|\psi\|_{\mathcal{Y}}^2$$

2. $$\|\Pi \mathcal{J}''(0)(\psi)\|_{\mathcal{Z}} \geq c_2 \rho^{-3} \|\psi\|_{\mathcal{X}}.$$

The operator $\Pi \mathcal{J}''(0)$ is bounded, one-to-one, and onto from $\mathcal{X}_\flat$ to $\mathcal{Z}_\flat$ with a bounded inverse. The second assertion means that the norm of the inverse operator $(\Pi \mathcal{J}''(0))^{-1} : \mathcal{Z}_\flat \to \mathcal{X}_\flat$ is bounded by $\frac{1}{c_2}\rho^3$: $\|(\Pi \mathcal{J}''(0))^{-1}\| \leq \frac{1}{c_2}\rho^3$.

Proof. We decompose $\mathcal{J}''(0)$ into the sum of two operators

$$\mathcal{J}''(0) = \mathcal{L} + \mathcal{M} \tag{3.60}$$

The operator $\mathcal{L}$ is the main part of $\mathcal{J}''(0)$, given by

$$\mathcal{L}_j(\psi)(\theta) \cong r_j^{-3}(-\psi_j''(\theta) - \psi_j(\theta)) + \gamma \left[\int_0^{2\pi} L(|r_j e^{i\theta} - r_j e^{i\omega}|) \psi_j(\omega)\, d\omega \right.$$
$$\left. + \left(r_j^{-1} \int_{B(0, r_j)} L'(|r_j e^{i\theta} - y|) \frac{(r_j e^{i\theta} - y) \cdot e^{i\theta}}{|r_j e^{i\theta} - y|}\, dy \right) \psi_j(\theta) \right]. \tag{3.61}$$

The operator $\mathcal{M}$ is the minor part given by

$$
\begin{aligned}
\mathcal{M}_j(\psi)(\theta) \cong \gamma \Bigg[& \int_0^{2\pi} Q\left(|r_j e^{i\theta} - r_j e^{i\omega}|\right) \psi_j(\omega)\, d\omega \\
& + \left(r_j^{-1} \int_{B(0,r_j)} Q'\left(|r_j e^{i\theta} - y|\right) \frac{(r_j e^{i\theta} - y)\cdot e^{i\theta}}{|r_j e^{i\theta} - y|}\, dy \right) \psi_j(\theta) \\
& + \sum_{k=1,\neq j}^{N} \int_0^{2\pi} K\left(|\xi_j + r_j e^{i\theta} - \xi_k - r_k e^{i\omega}|\right) \psi_k(\omega)\, d\omega \\
& + \sum_{k=1,\neq j}^{N} \left(r_j^{-1} \int_{B(\xi_k,r_k)} K'\left(|\xi_j + r_j e^{i\theta} - y|\right) \frac{(\xi_j + r_j e^{i\theta} - y)\cdot e^{i\theta}}{|\xi_j + r_j e^{i\theta} - y|}\, dy \right) \psi_j(\theta) \Bigg]. \qquad (3.62)
\end{aligned}
$$

Decompose $\mathcal{Z}$ into

$$
\mathcal{Z} = \bigoplus_{n=0}^{\infty} \mathcal{Z}_n \qquad (3.63)
$$

$$
\mathcal{Z}_n = \left\{ A\cos n\theta + B\sin\theta :\ A, B \in \mathbb{R}^N \right\},\ \text{if } n \geq 1 \qquad (3.64)
$$

$$
\mathcal{Z}_0 = \left\{ A \in \mathbb{R}^n :\ \sum_{j=1}^{N} A_j = 0 \right\}. \qquad (3.65)
$$

Then

$$
\mathcal{Z}_\flat = \bigoplus_{n=2}^{\infty} \mathcal{Z}_n \qquad (3.66)
$$

We see from (2.37), (2.38), (2.40) that for each $n \geq 1$, $\mathcal{Z}_n$ is an invariant subspece of the operator $\mathcal{L}$, and $\mathcal{L}$ is diagonalized in $\mathcal{Z}_n$. There are N eigenvalues in this subspace given by

$$
\lambda(n,j) = r_j^{-3}(n^2-1) + \gamma\left(\frac{1}{2n} - \frac{1}{2}\right),\ j = 1, 2, ..., N \qquad (3.67)
$$

with two corresponding eigenvectors $e_j \cos n\theta$ and $e_j \sin n\theta$ where e_j is the j-th standard unit vector in $\mathbb{R}^N$.

In the case of $\mathcal{Z}_0$, note that

$$
1 \to r_j^{-3}(1'' - 1) = -r_j^{-3} \qquad (3.68)
$$

$$
1 \to \int_0^{2\pi} L(r_j e^{i\theta} - r_j e^{i\omega}|) 1\, d\omega = \log \frac{1}{r_j} \qquad (3.69)
$$

$$
1 \to \left(r_j^{-1} \int_{B(0,r_j)} L'\left(|r_j e^{i\theta} - y|\right) \frac{(r_j e^{i\theta} - y)\cdot e^{i\theta}}{|r_j e^{i\theta} - y|}\, dy \right) 1 = \left(-\frac{1}{2}\right) 1, \qquad (3.70)
$$

so $\mathcal{Z}_0$ is also an invariant subspace of $\mathcal{L}$, but $\mathcal{L}$ is not yet diagonalized in $\mathcal{Z}_0$. There are $N-1$ eigenvalues $\lambda(0,j)$, $j = 1, 2, ..., N-1$, in this subspace, but we do not need to find them in this work, since we only need to study $\mathcal{L}$ on $\mathcal{X}_\flat \subset \mathcal{Z}_\flat$ and $\mathcal{Z}_\flat \cap \mathcal{Z}_0 = \{0\}$.

Because of (3.66), for every $\psi \in \mathcal{X}_\flat$ there exist $A_{n,j}, B_{n,j} \in \mathbb{R}$ such that

$$\psi(\theta) = \sum_{n=2}^{\infty} \sum_{j=1}^{N} (A_{n,j} e_j \cos n\theta + B_{n,j} e_j \sin n\theta) \tag{3.71}$$

$$\mathcal{L}(\psi)(\theta) = \sum_{n=2}^{\infty} \sum_{j=1}^{N} \lambda(n,j) (A_{n,j} e_j \cos n\theta + B_{n,j} e_j \sin n\theta) \tag{3.72}$$

$$\langle \mathcal{L}(\psi), \psi \rangle = \sum_{n=2}^{\infty} \sum_{j=1}^{N} \lambda(n,j) \pi (A_{n,j}^2 + B_{n,j}^2) \tag{3.73}$$

$$\langle \mathcal{L}(\psi), \mathcal{L}\psi \rangle = \sum_{n=2}^{\infty} \sum_{j=1}^{N} \lambda^2(n,j) \pi (A_{n,j}^2 + B_{n,j}^2) \tag{3.74}$$

Also for $n \geq 2$,

$$\lambda(n,j) = r_j^{-3} n^2 \left(\frac{n^2-1}{n^2} \right) \left(1 - \frac{\gamma r_j^3}{2n(n+1)} \right) \tag{3.75}$$

$$\geq r_j^{-3} n^2 \left(1 - \frac{1}{2^2} \right) \left(1 - \frac{12 - \frac{\eta}{2}}{2 \cdot 2 \cdot (2+1)} \right) \tag{3.76}$$

$$= r_j^{-3} n^2 \left(\frac{\eta}{32} \right) \tag{3.77}$$

To reach (3.76) we need the inequality

$$\gamma r_j^3 \leq 12 - \frac{\eta}{2}. \tag{3.78}$$

Recall $\gamma \rho^3 < 12 - \eta$, condition 2 of Theorem 3.1, and also r_j satisfies (3.5), $|r_j - \rho| \leq \delta_2 \rho$. Hence (3.78) holds if δ_2 is sufficiently small. Therefore

$$\langle \mathcal{L}(\psi), \psi \rangle \geq \left(\frac{\eta}{32} \right) \sum_{n=1}^{\infty} \sum_{j=1}^{N} r_j^{-3} \pi n^2 \left(A_{n,j}^2 + B_{n,j}^2 \right) \tag{3.79}$$

$$\langle \mathcal{L}(\psi), \mathcal{L}(\psi) \rangle \geq \left(\frac{\eta}{32} \right)^2 \sum_{n=1}^{\infty} \sum_{j=1}^{N} r_j^{-6} \pi n^4 \left(A_{n,j}^2 + B_{n,j}^2 \right) \tag{3.80}$$

On the other hand

$$\|\psi\|_{\mathcal{Z}}^2 = \sum_{n=2}^{\infty} \sum_{j=1}^{N} \pi \left(A_{n,j}^2 + B_{n,j}^2 \right) \tag{3.81}$$

$$\|\psi\|_{\mathcal{Y}}^2 = \sum_{n=2}^{\infty} \sum_{j=1}^{N} \pi (n^2+1) \left(A_{n,j}^2 + B_{n,j}^2 \right) \tag{3.82}$$

$$\|\psi\|_{\mathcal{X}}^2 = \sum_{n=2}^{\infty} \sum_{j=1}^{N} \pi (n^4 + n^2 + 1) \left(A_{n,j}^2 + B_{n,j}^2 \right) \tag{3.83}$$

Hence there exists $c_2 > 0$ such that for all $\psi \in \mathcal{X}_\flat$,

$$\langle \mathcal{L}(\psi), \psi \rangle \geq 2c_2 \rho^{-3} \|\psi\|_{\mathcal{Y}}^2, \text{ and } \|\mathcal{L}(\psi)\|_{\mathcal{Z}} \geq 2c_2 \rho^{-3} \|\psi\|_{\mathcal{X}} \tag{3.84}$$

Next we estimate $\mathcal{M}$. We can find $C_2' > 0$ such that for every $\psi \in \mathcal{X}_\flat$, the terms in (3.62) satisfy

$$\left| \int_0^{2\pi} Q\left(|r_j e^{i\theta} - r_j e^{i\omega}|\right) \psi_j(\omega)\, d\omega \right| \le C_2' \rho^2 \|\psi_j\|_{L^2} \tag{3.85}$$

$$r_j^{-1} \int_{B(0,r_j)} Q'\left(|r_j e^{i\theta} - y|\right) \frac{\left(r_j e^{i\theta} - y\right) \cdot e^{i\theta}}{|r_j e^{i\theta} - y|}\, dy = 2\pi r_j^2 \tag{3.86}$$

$$\left| \int_0^{2\pi} K\left(|\xi_j + r_j e^{i\theta} - \xi_k - r_k e^{i\omega}|\right) \psi_k(\omega)\, d\omega \right| \le C_2' \rho \|\psi_j\|_{L^2} \tag{3.87}$$

$$\left| r_j^{-1} \int_{B(\xi_k, r_k)} K'\left(|\xi_j + r_j e^{i\theta} - y|\right) \frac{\left(\xi_j + r_j e^{i\theta} - y\right) \cdot e^{i\theta}}{|\xi_j + r_j e^{i\theta} - y|}\, dy \right| \le C_2' \rho \tag{3.88}$$

uniformly with respect to θ. Here (3.87) may be less obvious. It holds because $\int_0^{2\pi} \psi_k(\omega)\, d\omega = 0$ when $\psi \in \mathcal{X}_\flat$,

$$\begin{aligned} &\int_0^{2\pi} K\left(|\xi_j + r_j e^{i\theta} - \xi_k - r_k e^{i\omega}|\right) \psi_k(\omega)\, d\omega \\ &= \int_0^{2\pi} \left(K\left(|\xi_j + r_j e^{i\theta} - \xi_k - r_k e^{i\omega}|\right) - K(|\xi_j - \xi_k|) \right) \psi_k(\omega)\, d\omega \end{aligned} \tag{3.89}$$

and

$$K\left(|\xi_j + r_j e^{i\theta} - \xi_k - r_k e^{i\omega}|\right) - K(|\xi_j - \xi_k|) = O(\rho) \tag{3.90}$$

uniformly with respect to θ and ω. By (3.85), (3.86), (3.87), and (3.88), we deduce that there exists $C_2 > 0$ such that for all $\psi \in \mathcal{X}_\flat$,

$$\|\Pi \mathcal{M}(\psi)\|_{\mathcal{Z}} \le \|\mathcal{M}(\psi)\|_{\mathcal{Z}} \le C_2 \gamma \rho \|\psi\|_{\mathcal{Z}} \tag{3.91}$$

On $\mathcal{X}_\flat$, since $\Pi\mathcal{L} = \mathcal{L}$, $\Pi \mathcal{J}''(0) = \mathcal{L} + \Pi\mathcal{M}$. Then by (3.84) and (3.91), for all $\psi \in \mathcal{X}_\flat$,

$$\begin{aligned} \langle \Pi \mathcal{J}''(0)(\psi), \psi \rangle &= \langle \mathcal{L}(\psi), \psi \rangle + \langle \Pi \mathcal{M}(\psi), \psi \rangle \\ &\ge 2c_2 \rho^{-3} \|\psi\|_{\mathcal{Y}}^2 - C_2 \gamma \rho \|\psi\|_{\mathcal{Z}}^2 \\ &\ge (2c_2 \rho^{-3} - C_2 \gamma \rho) \|\psi\|_{\mathcal{Y}}^2 \\ &\ge c_2 \rho^{-3} \|\psi\|_{\mathcal{Y}}^2 \end{aligned} \tag{3.92}$$

$$\begin{aligned} \|\Pi \mathcal{J}''(0)(\psi)\|_{\mathcal{Z}} &\ge \|\mathcal{L}(\psi)\|_{\mathcal{Z}} - \|\mathcal{M}(\psi)\|_{\mathcal{Z}} \\ &\ge 2c_2 \rho^{-3} \|\psi\|_{\mathcal{X}} - C_2 \gamma \rho \|\psi\|_{\mathcal{Z}} \\ &\ge (2c_2 \rho^{-3} - C_2 \gamma \rho) \|\psi\|_{\mathcal{X}} \\ &\ge c_2 \rho^{-3} \|\psi\|_{\mathcal{X}} \end{aligned} \tag{3.93}$$

if ρ is sufficiently small. Again we have used $\gamma \rho^3 < 12 - \eta$. This proves part 1 and part 2.

A weaker version of part 2 is

$$\|\Pi \mathcal{J}''(0)(u)\|_{\mathcal{Z}} \ge c_2 \rho^{-3} \|u\|_{\mathcal{Z}}, \text{ for all } u \in \mathcal{X}_\flat, \tag{3.94}$$

It implies that $\Pi \mathcal{J}''(0)$ is one-to-one.

Let $v \in \mathcal{Z}_\flat$ be perpendicular to the range of $\Pi \mathcal{J}''(0)$, i.e. $\langle \Pi \mathcal{J}''(0)(u), v \rangle = 0$ for all $u \in \mathcal{X}_\flat$. Since $\Pi \mathcal{J}''(0)$ is a self-adjoint operator on $\mathcal{Z}_\flat$ with the domain $\mathcal{X}_\flat \subset \mathcal{Z}_\flat$, one deduces that $v \in \mathcal{X}_\flat$ and $\Pi \mathcal{J}''(0)(v) = 0$. By the injectiveness of $\Pi \mathcal{J}''(0)$, $v = 0$. Hence the range of $\Pi \mathcal{J}''(0)$ is dense in $\mathcal{Z}_\flat$.

To show that $\Pi \mathcal{J}''(0)$ is surjective, let $w \in \mathcal{Z}_\flat$. There exist $u_n \in \mathcal{X}_\flat$ such that $\Pi \mathcal{J}''(0)(u_n) \to w$ in $\mathcal{Z}_\flat$. Therefore $\Pi \mathcal{J}''(0)(u_n)$ is a Cauchy sequence in $\mathcal{Z}_\flat$. By (3.94), u_n is also a Cauchy sequence in $\mathcal{Z}_\flat$. There exists $u \in \mathcal{Z}_b$ such that $u_n \to u$ in $\mathcal{Z}_b$. As a self-adjoint operator, $\Pi \mathcal{J}''(0)$ has a closed graph in $\mathcal{Z}_\flat \times \mathcal{Z}_\flat$, so (u, w) is on this graph. Hence $u \in \mathcal{X}_b$ and $\Pi \mathcal{J}''(0)(u) = w$; This proves the last statement. □

Also needed is an estimate on the third variation of $\mathcal{J}$.

Lemma 3.5. *There exist $C_3' > 0$ and $C_3 > 0$ such that for all $\phi \in Dom(\mathcal{J}')$, the following estimates hold for all $u \in \mathcal{X}$ and $v \in \mathcal{X}$,*

1. $$|\langle \mathcal{J}'''(\phi)(u,v),v\rangle| \leq C_3' \left(\rho^{-5}+\gamma\rho^{-2}\right) \|u\|_{\mathcal{X}} \|v\|_{\mathcal{Y}}^2 \leq C_3\rho^{-5} \|u\|_{\mathcal{X}} \|v\|_{\mathcal{Y}}^2 ,$$

2. $$||\mathcal{J}'''(\phi)(u,v)||_{\mathcal{Z}} \leq C_3' \left(\rho^{-5}+\gamma\rho^{-2}\right) \|u\|_{\mathcal{X}} \|v\|_{\mathcal{X}} \leq C_3\rho^{-5} \|u\|_{\mathcal{X}} \|v\|_{\mathcal{X}} .$$

Proof. The proof of this lemma is similar to that of [13, Lemma 3.2] and [12, Lemma 6.1] and is omitted. □

We close this section with a remark on our setting of $\mathcal{J}$ as a functional defined on $\operatorname{Dom}\mathcal{J} \subset \mathcal{X}$. This setup addresses perturbations of $\cup_{j=1}^N B(\xi_j,r_j)$, and is dependent on $\xi_1,\xi_2,...,\xi_N$, and $r_1,r_2,...,r_N$. We do not emphasize this dependence in our notations in this section or the next section, but we will do so in section 5 to exploit this dependence.

4 A pseudo-solution

In this section we solve the equation

$$\Pi\mathcal{J}'(\phi)=0,\ \phi\in\mathcal{X}_\flat\cap\operatorname{Dom}(\mathcal{J}'). \tag{4.1}$$

Any ϕ that solves this equation is termed a pseudo-solution. More explicitly, $\phi=(\phi_1,\phi_2,...,\phi_N)$ is a pseudo-solution if $\phi\in\mathcal{X}_\flat\cap\operatorname{Dom}(\mathcal{J}')$ and

$$\kappa_j(\phi_j)(\theta)+\gamma K_j[\phi](\theta)=A_j\cos\theta+B_j\sin\theta+C_j,\ j=1,2,...,N, \tag{4.2}$$

for some $A_j,B_j,C_j\in\mathbb{R}$.

Lemma 4.1. *When ρ is sufficiently small, there exists $\varphi\in\mathcal{X}_\flat\cap\operatorname{Dom}(\mathcal{J}')$ such that $\Pi\mathcal{J}'(\varphi)=0$. Moreover,*

$$\|\varphi\|_{\mathcal{X}}\leq\frac{2C_1}{c_2}\rho^4.$$

Recall that C_1 comes from Lemma 3.3 and c_2 comes from Lemma 3.4.

Proof. Expand $\mathcal{J}'(\phi)$ as

$$\mathcal{J}'(\phi)=\mathcal{J}'(0)+\mathcal{J}''(0)(\phi)+\mathcal{R}(\phi) \tag{4.3}$$

where $\mathcal{R}(\phi)$, defined by (4.3), will be shown to be a higher order term. Turn the equation (4.1) to a fixed point form:

$$\phi=\mathcal{T}(\phi) \tag{4.4}$$

where

$$\mathcal{T}(\phi)=-(\Pi\mathcal{J}''(0))^{-1}(\Pi\mathcal{J}'(0)+\Pi\mathcal{R}(\phi)) \tag{4.5}$$

is an operator defined on

$$\mathcal{W}=\{\phi\in\mathcal{X}_\flat:\|\phi\|_{\mathcal{X}}\leq\epsilon\rho^2\}, \tag{4.6}$$

and

$$\epsilon=\min\left\{\frac{c_2}{4C_3},\frac{\delta_0}{2}\right\}. \tag{4.7}$$

Recall that c_2 is from Lemma 3.4, C_3 is from Lemma 3.5, and δ_0 is from (3.27) satisfying (3.29). Since $\epsilon < \delta_0$, members in $\mathcal{W}$ all represent assemblies of perturbed discs and $\mathcal{T}$ is well defined on $\mathcal{W}$.

By Lemmas 3.3 and 3.4.2

$$\|(\Pi \mathcal{J}''(0))^{-1}\Pi \mathcal{J}'(0)\|_{\mathcal{X}} \leq \frac{1}{c_2}\rho^3 C_1 \rho = \frac{C_1}{c_2}\rho^4. \tag{4.8}$$

Lemma 3.5.2 implies that

$$\|\mathcal{R}(\phi)\|_{\mathcal{Z}} \leq \frac{C_3}{2}\rho^{-5}\|\phi\|_{\mathcal{X}}^2. \tag{4.9}$$

and

$$\|(\Pi \mathcal{J}''(0))^{-1}\Pi \mathcal{R}(\phi)\|_{\mathcal{X}} \leq \frac{1}{c_2}\rho^3 \frac{C_3}{2}\rho^{-5}\|\phi\|_{\mathcal{X}}^2 = \frac{C_3}{2c_2}\rho^{-2}\|\phi\|_{\mathcal{X}}^2. \tag{4.10}$$

For $\phi \in \mathcal{W}$, by (4.5), (4.8), and (4.10) one deduces

$$\|\mathcal{T}(\phi)\|_{\mathcal{X}} \leq \frac{C_1}{c_2}\rho^4 + \frac{C_3}{2c_2}\rho^2\epsilon^2 = \Big(\frac{C_1}{c_2}\rho^2 + \frac{C_3}{2c_2}\epsilon^2\Big)\rho^2. \tag{4.11}$$

Now we require ρ to be sufficiently small so that

$$\frac{C_1}{c_2}\rho^2 < \frac{\epsilon}{2} \tag{4.12}$$

and consequently, with the help of (4.7),

$$\|\mathcal{T}(\phi)\|_{\mathcal{X}} \leq \left(\frac{\epsilon}{2} + \frac{\epsilon}{2}\frac{C_3}{c_2}\epsilon\right)\rho^2 \leq \epsilon\rho^2. \tag{4.13}$$

Therefore $\mathcal{T}$ maps $\mathcal{W}$ into itself.

Next show that $\mathcal{T}$ is a contraction. Let $\phi, \psi \in \mathcal{W}$. First note that

$$\mathcal{T}(\phi) - \mathcal{T}(\psi) = -(\Pi \mathcal{J}''(0))^{-1}\left(\Pi\left(\mathcal{R}(\phi) - \mathcal{R}(\psi)\right)\right). \tag{4.14}$$

Because

$$\mathcal{R}(\phi) - \mathcal{R}(\psi) = \mathcal{J}'(\phi) - \mathcal{J}'(\psi) - \mathcal{J}''(0)(\phi - \psi), \tag{4.15}$$

one finds, with the help of Lemma 3.5.2, that

$$\begin{aligned}
\|\mathcal{R}(\phi) - \mathcal{R}(\psi)\|_{\mathcal{Z}} &\leq \|\mathcal{J}'(\phi) - \mathcal{J}'(\psi) - \mathcal{J}''(\psi)(\phi-\psi)\|_{\mathcal{Z}} + \|\mathcal{J}''(\psi)(\phi-\psi) - \mathcal{J}''(0)(\phi-\psi)\|_{\mathcal{Z}} \\
&\leq \frac{C_3}{2}\rho^{-5}\|\phi-\psi\|_{\mathcal{X}}^2 + C_3\rho^{-5}\|\psi\|_{\mathcal{X}}\|\phi-\psi\|_{\mathcal{X}} \\
&\leq C_3\rho^{-5}\left(\frac{1}{2}\|\phi-\psi\|_{\mathcal{X}} + \|\psi\|_{\mathcal{X}}\right)\|\phi-\psi\|_{\mathcal{X}} \\
&\leq 2C_3\epsilon\rho^{-3}\|\phi-\psi\|_{\mathcal{X}}.
\end{aligned} \tag{4.16}$$

Then Lemma 3.4.2 and (4.7) imply that

$$\|\mathcal{T}(\phi) - \mathcal{T}(\psi)\|_{\mathcal{X}} \leq \frac{2\epsilon\, C_3}{c_2}\|\phi-\psi\|_{\mathcal{X}} \leq \frac{1}{2}\|\phi-\psi\|_{\mathcal{X}}. \tag{4.17}$$

Hence $\mathcal{T}$ is a contraction mapping, and a unique fixed point, which we denote by φ, exists in $\mathcal{W}$.

By the definition of $\mathcal{W}$, $\|\varphi\|_{\mathcal{X}} = O(\rho^2)$. However, this can be improved to order $O(\rho^4)$, if one revisits the equation $\varphi = \mathcal{T}(\varphi)$ and derives from (4.5), (4.8) and (4.10) that

$$\|\varphi\|_{\mathcal{X}} \le \|(\Pi\mathcal{J}''(0))^{-1}\Pi\mathcal{J}'(0)\|_{\mathcal{X}} + \|(\Pi\mathcal{J}''(0))^{-1}\Pi\mathcal{R}(\varphi)\|_{\mathcal{X}} \le \frac{C_1}{c_2}\rho^4 + \frac{C_3}{2c_2}\rho^{-2}\|\varphi\|_{\mathcal{X}}^2.$$

Rewrite the above as

$$\left(1 - \frac{C_3}{2c_2}\rho^{-2}\|\varphi\|_{\mathcal{X}}\right)\|\varphi\|_{\mathcal{X}} \le \frac{C_1}{c_2}\rho^4. \tag{4.18}$$

In (4.18) estimate

$$\frac{C_3}{2c_2}\rho^{-2}\|\varphi\|_{\mathcal{X}} \le \frac{C_3}{2c_2}\epsilon \le \frac{1}{8} \tag{4.19}$$

by (4.7). The estimate of φ follows from (4.18) and (4.19). □

The next two lemmas show some properties of the pseudo-solution φ. Lemma 4.2.1 says that $\Pi\mathcal{J}''(\varphi)$ is positive definite, so φ locally minimizes $\mathcal{J}$ in $\mathcal{X}_\flat$. Lemma 4.3 gives a good estimate of $\mathcal{J}(\varphi)$ which is very close to $\mathcal{J}(0)$.

Lemma 4.2. *When ρ is sufficiently small, for all $\psi \in \mathcal{X}_\flat$,*

1. $$\langle \Pi\mathcal{J}''(\varphi)(\psi), \psi\rangle \ge \frac{c_2}{2}\rho^{-3}\|\psi\|_{\mathcal{Y}}^2$$

2. $$\|\Pi\mathcal{J}''(\varphi)(\psi)\|_{\mathcal{Z}} \ge \frac{c_2}{2}\rho^{-3}\|\psi\|_{\mathcal{X}}.$$

Proof. By Lemmas 3.4, 3.5 and 4.1,

$$\begin{aligned}\langle \Pi\mathcal{J}''(\varphi)(\psi), \psi\rangle &= \langle \Pi\mathcal{J}''(0)(\psi), \psi\rangle + \langle \Pi(\mathcal{J}''(\varphi) - \mathcal{J}''(0))\psi, \psi\rangle \\ &\ge c_2\rho^{-3}\|\psi\|_{\mathcal{Y}}^2 - C_3\rho^{-5}\|\varphi\|_{\mathcal{X}}\|\psi\|_{\mathcal{Y}}^2 \\ &\ge \left(c_2 - \frac{2C_1C_3}{c_2}\rho^2\right)\rho^{-3}\|u\|_{\mathcal{Y}}^2,\end{aligned}$$

and

$$\begin{aligned}\|\Pi\mathcal{J}''(\varphi)(\psi)\|_{\mathcal{Z}} &\ge \|\Pi\mathcal{J}''(0)(\psi)\|_{\mathcal{Z}} - \|\Pi(\mathcal{J}''(\varphi) - \mathcal{J}''(0))\psi)\|_{\mathcal{Z}} \\ &\ge c_2\rho^{-3}\|\psi\|_{\mathcal{X}} - C_3\rho^{-5}\|\varphi\|_{\mathcal{X}}\|\psi\|_{\mathcal{X}} \\ &\ge \left(c_2 - \frac{2C_1C_3}{c_2}\rho^2\right)\rho^{-3}\|\psi\|_{\mathcal{X}}.\end{aligned}$$

If ρ is sufficiently small, then $\frac{2C_1C_3}{c_2}\rho^2 \le \frac{c_2}{2}$ and both parts of the lemma follow. □

Lemma 4.3. *It holds uniformly with respect to ξ and r that*

$$\mathcal{J}(\varphi) = \mathcal{J}(0) + O(\rho^5).$$

Proof. Expanding $\mathcal{J}(\varphi)$ yields

$$\mathcal{J}(\varphi) = \mathcal{J}(0) + \langle \mathcal{J}'(0), \varphi\rangle + \frac{1}{2}\langle \mathcal{J}''(0)(\varphi), \varphi\rangle + \frac{1}{6}\langle \mathcal{J}'''(t\varphi)(\varphi,\varphi), \varphi\rangle \tag{4.20}$$

for some $t \in (0,1)$. On the other hand expanding $\mathcal{J}'(\varphi)$, and then applying Π give

$$\|\Pi\mathcal{J}'(\varphi) - \Pi\mathcal{J}'(0) - \Pi\mathcal{J}'(0)(\varphi)\|_{\mathcal{Z}} \le \sup_{t\in(0,1)} \frac{1}{2}\|\Pi\mathcal{J}'''(t\varphi)(\varphi,\varphi)\|_{\mathcal{Z}}. \tag{4.21}$$

Since $\Pi\mathcal{J}'(\varphi) = 0$, (4.21) shows that

$$\|\Pi\mathcal{J}'(0) + \Pi\mathcal{J}''(0)(\varphi)\|_{\mathcal{Z}} \le \sup_{t\in(0,1)} \frac{1}{2}\|\Pi\mathcal{J}'''(t\varphi)(\varphi,\varphi)\|_{\mathcal{Z}},$$

which implies that

$$\langle \Pi\mathcal{J}'(0), \varphi\rangle + \langle \Pi\mathcal{J}''(0)(\varphi), \varphi\rangle \le \left(\sup_{t\in(0,1)} \frac{1}{2}\|\Pi\mathcal{J}'''(t\varphi)(\varphi,\varphi)\|_{\mathcal{Z}}\right)\|\varphi\|_{\mathcal{X}}. \tag{4.22}$$

Since $\varphi \in \mathcal{X}_\flat$,

$$\langle \Pi\mathcal{J}'(0), \varphi\rangle = \langle \mathcal{J}'(0), \varphi\rangle, \quad \langle \Pi\mathcal{J}''(0)(\varphi), \varphi\rangle = \langle \mathcal{J}''(0)(\varphi), \varphi\rangle. \tag{4.23}$$

Then (4.22) shows that

$$\langle \mathcal{J}'(0), \varphi\rangle + \langle \mathcal{J}''(0)(\varphi), \varphi\rangle \le \left(\sup_{t\in(0,1)} \frac{1}{2}\|\Pi\mathcal{J}'''(t\varphi)(\varphi,\varphi)\|_{\mathcal{Z}}\right)\|\varphi\|_{\mathcal{X}}. \tag{4.24}$$

By (4.24), (4.20) yields that

$$\left|\mathcal{J}(\varphi) - \mathcal{J}(0) - \frac{1}{2}\langle \mathcal{J}'(0), \varphi\rangle\right| \le \frac{5}{12}\left(\sup_{t\in(0,1)} \|\mathcal{J}'''(t\varphi)(\varphi,\varphi)\|_{\mathcal{Z}}\right)\|\varphi\|_{\mathcal{X}}.$$

Therefore Lemma 3.3, (4.23), Lemma 3.5.2 and Lemma 4.1 imply that

$$\begin{aligned}
|\mathcal{J}(\varphi) - \mathcal{J}(0)| &\le \frac{1}{2}|\langle \mathcal{J}'(0), \varphi\rangle| + \frac{5}{12}\left(\sup_{t\in(0,1)} \|\mathcal{J}'''(t\varphi)(\varphi,\varphi)\|_{\mathcal{Z}}\right)\|\varphi\|_{\mathcal{X}} \\
&= \frac{1}{2}|\langle \Pi\mathcal{J}'(0), \varphi\rangle| + \frac{5}{12}\left(\sup_{t\in(0,1)} \|\mathcal{J}'''(t\varphi)(\varphi,\varphi)\|_{\mathcal{Z}}\right)\|\varphi\|_{\mathcal{X}} \\
&\le \frac{1}{2}C_1\rho\frac{2C_1}{c_2}\rho^4 + \frac{5}{12}C_3\rho^{-5}\left(\frac{2C_1}{c_2}\rho^4\right)^3 \\
&= \frac{C_1^2}{c_2}\rho^5 + \frac{10C_3C_1^3}{3c_2^3}\rho^7.
\end{aligned}$$

This completes the proof. □

5 The reduced problem

In this section we explore the roles played by the centers ξ_j and the radii r_j of $\cup_{j=1}^N B(\xi_j, r_j)$. Write $\xi = (\xi_1, \xi_2, ..., \xi_N)$, $r = (r_1, r_2, ..., r_N)$, and denote the pseudo-solution φ found in the last section by $\varphi(\cdot, \xi, r)$. We will see that if ξ and r are chosen properly, the pseudo-solution turns out to be an exact solution.

The domain for (ξ, r) is defined in (3.4) and (3.5) which we now denote by

$$M = \left\{(\xi, r) \in \mathbb{R}^{3N} :\ 4\delta_1 \le |\xi_j - \xi_k| \le \frac{1}{4\delta_1}\ \forall j \ne k,\ \ |r_j - \rho| \le \delta_2\rho\ \forall j,\ \ \sum_{j=1}^N r_j^2 = N\rho^2\right\}; \tag{5.1}$$

M is an $3N-1$ dimensional submanifold with boundary in $\mathbb{R}^{3N}$. Define a function J by

$$J(\xi, r) = \mathcal{J}(\varphi(\cdot, \xi, r)),\ \ (\xi, r) \in M. \tag{5.2}$$

Lemma 5.1. *If (ξ_c, r_c) in the interior of M is a critical point of the function J, then $\varphi(\cdot, \xi_c, r_c)$ is a critical point of the funcional $\mathcal{J}$.*

Proof. Recall the general first variation formula (1.12) for Ω deformed to Ω_ε:

$$\frac{\partial \mathcal{J}(\Omega_\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0} = -\int_{\partial\Omega} (\kappa + \gamma K[\Omega])\, \mathbf{N}\cdot\mathbf{X}\, ds = -\sum_{j=1}^{N} \int_{\partial\Omega_j} (\kappa(\partial\Omega_j) + \gamma K[\Omega])\, \mathbf{N}_j \cdot \mathbf{X}_j\, ds \tag{5.3}$$

where the Ω_j's are the components of Ω. Let $\Omega_{\varphi(\cdot,\xi,r)}$ be the union of perturbed discs specified by $\varphi(\cdot,\xi,r)$. Since $\Pi\mathcal{J}'(\varphi) = 0$, there exist $A_j(\xi,r)$, $B_j(\xi,r)$, and $C_j(\xi,r)$ such that on $\partial\Omega_{\varphi_j(\cdot,\xi,r)}$

$$\kappa_j(\varphi_j) + \gamma K_j[\varphi] = A_j(\xi,r)\cos\theta + B_j(\xi,r)\sin\theta + C_j(\xi,r) \tag{5.4}$$

Let the boundary of the component $\Omega_{\varphi_j(\cdot,\xi,r)}$ be parametrized by $\mathbf{R}_j$; namely

$$\mathbf{R}_j(\theta) = \xi_j + \sqrt{r_j^2 + 2\varphi_j(\theta,\xi,r)}\, e^{i\theta},\ j = 1, 2, ..., N. \tag{5.5}$$

The unit tangent and normal vectors of $\mathbf{R}_j$ are

$$\mathbf{T}_j(\theta) = \frac{\frac{\partial \mathbf{R}_j(\theta)}{\partial\theta}}{\left|\frac{\partial \mathbf{R}_j(\theta)}{\partial\theta}\right|},\ \text{and } \mathbf{N}_j(\theta) = i\mathbf{T}_j(\theta), \tag{5.6}$$

respectively. Since $ds = \left|\frac{\partial \mathbf{R}_j(\theta)}{\partial\theta}\right| d\theta$,

$$\mathbf{T}_j(\theta)\frac{ds}{d\theta} = \frac{\partial \mathbf{R}_j(\theta)}{\partial\theta} = \frac{\frac{\partial\varphi_j}{\partial\theta}}{\sqrt{r_j^2+2\varphi_j}} e^{i\theta} + \sqrt{r_j^2 + 2\varphi_j}\ i\, e^{i\theta} \tag{5.7}$$

$$\mathbf{N}_j(\theta)\frac{ds}{d\theta} = \frac{\frac{\partial\varphi_j}{\partial\theta}}{\sqrt{r_j^2+2\varphi_j}}\ i\, e^{i\theta} - \sqrt{r_j^2 + 2\varphi_j}\ e^{i\theta}. \tag{5.8}$$

In (5.3), κ_j is the curvature of $\mathbf{R}_j$, and $\mathbf{N}_j$ points inwards.

In this proof we generate deformations by varying (ξ, r) in M. They supply $\mathbf{X}_j$ in (5.3). First take $\xi_{k,1}$, the horizontal coordinate of the k-th center, to be a variable and keep the other centers fixed. This amounts to moving $\Omega_{\varphi_k(\cdot,\xi,r)}$ horizontally while changing the shape of $\Omega_{\varphi(\cdot,\xi,r)}$ slightly. Then

$$\mathbf{X}_k = \frac{\partial \mathbf{R}_k}{\partial \xi_{k,1}} = (1,0) + \frac{\frac{\partial\varphi_k}{\partial\xi_{k,1}}}{\sqrt{r_k^2+2\varphi_k}} e^{i\theta} \tag{5.9}$$

$$\mathbf{N}_k\cdot\mathbf{X}_k\frac{ds}{d\theta} = -\frac{\frac{\partial\varphi_k}{\partial\theta}}{\sqrt{r_k^2+2\varphi_k}}\sin\theta - \sqrt{r_k^2+2\varphi_k}\cos\theta - \frac{\partial\varphi_k}{\partial\xi_{k,1}}, \tag{5.10}$$

$$\mathbf{X}_j = \frac{\partial \mathbf{R}_j}{\partial \xi_{k,1}} = \frac{\frac{\partial\varphi_j}{\partial\xi_{k,1}}}{\sqrt{r_j^2+2\varphi_j}} e^{i\theta},\ j \neq k \tag{5.11}$$

$$\mathbf{N}_j\cdot\mathbf{X}_j\frac{ds}{d\theta} = -\frac{\partial\varphi_j}{\partial\xi_{k,1}},\ j \neq k \tag{5.12}$$

Since $\varphi \in \mathcal{X}_\flat$,

$$\int_0^{2\pi} \varphi_j\, d\theta = \int_0^{2\pi} \varphi_j \cos\theta\, d\theta = \int_0^{2\pi} \varphi_j \sin\theta\, d\theta = 0. \tag{5.13}$$

It follows that

$$\int_0^{2\pi} \frac{\partial\varphi_j}{\partial\xi_{k,1}}\, d\theta = \int_0^{2\pi} \frac{\partial\varphi_j}{\partial\xi_{k,1}} \cos\theta\, d\theta = \int_0^{2\pi} \frac{\partial\varphi_j}{\partial\xi_{k,1}} \sin\theta\, d\theta = 0. \tag{5.14}$$

Because

$$\int_{\partial\Omega_{\varphi,k}} \mathbf{N}_k \cdot \mathbf{X}_k \, ds = \int_0^{2\pi} \left[-\frac{d}{d\theta} \left(\sqrt{r_1^2 + 2\varphi_1} \, \sin\theta \right) - \frac{\partial \varphi_k}{\partial \xi_{k,1}} \right] d\theta = 0, \tag{5.15}$$

$$\int_{\partial\Omega_{\varphi,j}} \mathbf{N}_j \cdot \mathbf{X}_j \, ds = \int_0^{2\pi} -\frac{\partial \varphi_j}{\partial \xi_{k,1}} \, d\theta = 0, \ j \neq k \tag{5.16}$$

by (5.10), (5.12), and (5.14), one deduces from (5.3), (5.4), (5.10), (5.12), (5.14), (5.15), and (5.16),

$$\begin{aligned} \frac{\partial \mathcal{J}(\Omega_{\varphi(\cdot,\xi,r)})}{\partial \xi_{k,1}} &= -\sum_{j=1}^{N} \int_0^{2\pi} \left(A_j(\xi,r)\cos\theta + B_j(\xi,r)\sin\theta + C_j(\xi,r)\right) \mathbf{N}_j \cdot \mathbf{X}_j \, ds. \\ &= \int_0^{2\pi} \left(A_k(\xi,r)\cos\theta + B_k(\xi,r)\sin\theta\right) \left(\frac{\frac{\partial \varphi_k}{\partial \theta}}{\sqrt{r_k^2 + 2\varphi_k}} \sin\theta + \sqrt{r_k^2 + 2\varphi_k} \cos\theta \right) d\theta \\ &= A_k(\xi,r)(O(\rho^3) + \pi r_k) + B_k(\xi,r) O(\rho^3). \end{aligned} \tag{5.17}$$

Here note that, by Lemma 4.1,

$$\varphi_k(\theta,\xi,r) = O(\rho^4), \ \text{and} \ \frac{\partial \varphi_k(\theta,\xi,r)}{\partial \theta} = O(\rho^4) \tag{5.18}$$

uniformly with respect to θ, ξ, and r.

If we vary $\xi_{k,2}$ but hold other parameters, a similar argument shows that

$$\frac{\partial \mathcal{J}(\Omega_{\varphi(\cdot,\xi,r)})}{\partial \xi_{k,2}} = A_k(\xi,r) O(\rho^3) + B_k(\xi,r)(O(\rho^3) + \pi r_k) \tag{5.19}$$

At the critical point (ξ_c, r_c) of the function J,

$$0 = \frac{\partial J(\xi_c, r_c)}{\partial \xi_{k,1}} = \frac{\partial \mathcal{J}(\Omega_{\varphi(\cdot,\xi,r)})}{\partial \xi_{k,1}}\Big|_{(\xi_c,r_c)} = A_k(\xi_c,r_c)(O(\rho^3) + \pi r_{c,k}) + B_k(\xi_c,r_c) O(\rho^3) \tag{5.20}$$

$$0 = \frac{\partial J(\xi_c, r_c)}{\partial \xi_{k,2}} = \frac{\partial \mathcal{J}(\Omega_{\varphi(\cdot,\xi,r)})}{\partial \xi_{k,2}}\Big|_{(\xi_c,r_c)} = A_k(\xi_c,r_c)(O(\rho^3)) + B_k(\xi_c,r_c)(O(\rho^3) + \pi r_{c,k}) \tag{5.21}$$

Hence $A_k(\xi_c, r_c)$ and $B_k(\xi_c, r_c)$ satisfy a homogenous linear 2 by 2 system, and this system is non-singular if ρ is small. Therefore

$$A_k(\xi_c, r_c) = B_k(\xi_c, r_c) = 0. \tag{5.22}$$

Now we are going to vary r_k, but it is more convenient to use $w_k = r_k^2$ instead. Then

$$\mathbf{X}_k = \frac{\partial \mathbf{R}_k}{\partial w_k} = \frac{\frac{\partial \varphi_k}{\partial w_k} + 1}{\sqrt{r_k^2 + 2\varphi_k}} e^{i\theta} \tag{5.23}$$

$$\mathbf{N}_k \cdot \mathbf{X}_k \frac{ds}{d\theta} = -\frac{\partial \varphi_j}{\partial w_k} - 1, \tag{5.24}$$

$$\mathbf{X}_j = \frac{\partial \mathbf{R}_j}{\partial w_k} = \frac{\frac{\partial \varphi_j}{\partial w_k}}{\sqrt{r_j^2 + 2\varphi_j}} e^{i\theta}, \ j \neq k \tag{5.25}$$

$$\mathbf{N}_j \cdot \mathbf{X}_j \frac{ds}{d\theta} = -\frac{\partial \varphi_j}{\partial w_k}, \ j \neq k \tag{5.26}$$

Again by (5.13),

$$\int_0^{2\pi} \frac{\partial \varphi_j}{\partial w_k} \, d\theta = \int_0^{2\pi} \frac{\partial \varphi_j}{\partial w_k} \cos\theta \, d\theta = \int_0^{2\pi} \frac{\partial \varphi_j}{\partial w_k} \sin\theta \, d\theta = 0. \tag{5.27}$$

One deduces from (5.3), (5.4), (5.24), (5.26), and (5.27),

$$\frac{\partial \mathcal{J}(\Omega_{\varphi(\cdot,\xi,r)})}{\partial w_k} = -\sum_{j=1}^{N} \int_0^{2\pi} \left(A_j(\xi,r)\cos\theta + B_j(\xi,r)\sin\theta + C_j(\xi,r)\right) \mathbf{N}_j \cdot \mathbf{X}_j \, ds. \tag{5.28}$$

$$= \int_0^{2\pi} \left(A_k(\xi,r)\cos\theta + B_k(\xi,r)\sin\theta + C_k(\xi,r)\right)(-1)\, d\theta \tag{5.29}$$

$$= -2\pi C_k(\xi,r) \tag{5.30}$$

At the critical point (ξ_c, r_c) of the function J, because of the constraint (3.1), i.e.

$$\sum_{j=1}^{N} w_j = N\rho^2, \tag{5.31}$$

we find

$$\mu = \frac{\partial J(\xi_c, r_c)}{\partial w_k} = \frac{\partial \mathcal{J}(\Omega_{\varphi(\cdot,\xi,r)})}{\partial w_k}\Big|_{(\xi_c,r_c)} = -2\pi C_k(\xi_c, r_c) \tag{5.32}$$

where μ is the Lagrange multiplier corresponding to the constraint. Hence by (5.4), (5.22), and (5.32)

$$\kappa_j(\partial\Omega_{\varphi_j(\cdot,\xi_c,r_c)}) + \gamma K_j[\Omega_{\varphi(\cdot,\xi_c,r_c)}] = -\frac{\mu}{2\pi}, \ j = 1, 2, ..., N. \tag{5.33}$$

This shows that $\Omega_{\varphi(\cdot,\xi_c,r_c)}$ is a critical point of $\mathcal{J}$ according to (1.18). In terms of $\mathcal{J}'$, since

$$\mathcal{J}'(\varphi(\cdot,\xi_c,r_c)) \cong \kappa(\varphi(\cdot,\xi_c,r_c)) + K[\varphi(\cdot,\xi_c,r_c)] \tag{5.34}$$

by (3.48), (5.33) implies

$$\mathcal{J}'(\varphi(\cdot,\xi_c,r_c)) \cong -\frac{\mu}{2\pi}(1, 1, ..., 1) \tag{5.35}$$

This means that $\mathcal{J}'(\varphi(\cdot,\xi_c,r_c))$ is a scalar multiple of $(1, 1, ..., 1)$. But $\mathcal{J}'(\varphi(\cdot,\xi_c,r_c))$ is in $\mathcal{Z}$ which is perpendicular to $(1, 1, ..., 1)$. Hence

$$\mathcal{J}'(\varphi(\cdot,\xi_c,r_c)) = 0. \tag{5.36}$$

This proves the lemma. □

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. Recall that J is defined on M, a $3N-1$ dimensional submanifold with boundary. Although it is a closed subset of $\mathbb{R}^{3N}$, M is unbounded, and hence not compact. However, due to the translation invariance (2.47) of $\mathcal{J}$, we can assume that ξ_1 is the origin and consider the bounded subset

$$M_0 = \{(\xi, r) \in M : \xi_1 = (0, 0)\} \subset M. \tag{5.37}$$

Then M_0 is a compact $3N-3$ dimensional submanifold with boundary in $\mathbb{R}^{3N}$.

Let (ξ_ρ, r_ρ) be a minimum of J on M_0. We proceed to show that when ρ is sufficiently small, (ξ_ρ, r_ρ) is in the interior of M_0 and hence also in the interior of M.

By Lemmas 3.2 and 4.3 we deduce

$$J(\xi, r) = \sum_{j=1}^{N} 2\pi r_j + \frac{\gamma}{2}\left[\sum_{j=1}^{N}\left(\frac{\pi r_j^4}{2}\log\frac{1}{r_j} + \frac{\pi r_j^4}{8}\right) + \sum_{j=1}^{N}\sum_{k=1,\neq j}^{N}\left(\frac{\pi r_j^2 r_k^2}{2}\log\frac{1}{|\xi_j - \xi_k|} + \pi^2 r_j^2 r_k^2 |\xi_j - \xi_k|^2\right)\right] + O(\gamma\rho^6) \tag{5.38}$$

Note that the $O(\rho^5)$ term from Lemma 4.3 is absorbed into the $O(\gamma\rho^6)$ term above since $\gamma\rho^3 < 12 - \eta$.

First we pick out the leading order term so that

$$J(\xi, r) = \sum_{j=1}^{N} 2\pi r_j + \frac{\gamma}{2} \sum_{j=1}^{N} \left(\frac{\pi r_j^4}{2} \log \frac{1}{r_j} \right) + O(\gamma\rho^4) \tag{5.39}$$

Introduce

$$W_j = \left(\frac{r_j}{\rho}\right)^2, \ (1-\delta_2)^2 \le W_j \le (1+\delta_2)^2, \ \sum_{j=1}^{N} W_j = N. \tag{5.40}$$

Then

$$J(\xi, r) = \sum_{j=1}^{N} 2\pi\rho\sqrt{W_j} + \frac{\gamma}{2} \sum_{j=1}^{N} \left(\frac{\pi W_j^2}{2} \rho^4 \log \frac{1}{\rho} \right) + O(\gamma\rho^4) \tag{5.41}$$

$$\frac{1}{\gamma\rho^4 \log\frac{1}{\rho}} J(\xi, r) = \sum_{j=1}^{N} \left(\left(\frac{1}{\gamma\rho^3 \log\frac{1}{\rho}} \right) 2\pi\sqrt{W_j} + \frac{\pi W_j^2}{4} \right) + O\left(\frac{1}{\log\frac{1}{\rho}} \right). \tag{5.42}$$

Hence as $\rho \to 0$,

$$\frac{1}{\gamma\rho^4 \log\frac{1}{\rho}} J(\xi, r) \to 2\pi \sum_{j=1}^{N} \left(\beta\sqrt{W_j} + \frac{W_j^2}{8} \right) \tag{5.43}$$

uniformly with respect to ξ and W, where

$$\frac{1}{\gamma\rho^3 \log\frac{1}{\rho}} \to \beta \in \left[0, \frac{1}{1+\eta}\right] \tag{5.44}$$

as $\rho \to 0$ possibly along a subsequence, since

$$\gamma\rho^3 \log\frac{1}{\rho} > 1 + \eta \tag{5.45}$$

which is condition 3 of Theorem 3.1. Take δ_2 sufficiently small so that the function

$$q \to f(q) = \beta\sqrt{q} + \frac{q^2}{8}, \ q \in \left[(1-\delta_2)^2, (1+\delta_2)^2\right] \tag{5.46}$$

is convex on $[(1-\delta_2)^2, (1+\delta_2)^2]$. This δ_2 exists since

$$f''(1) = \left(-\frac{\beta}{4q^{3/2}} + \frac{1}{4} \right)\Big|_{q=1} = -\frac{\beta - 1}{4} > 0 \tag{5.47}$$

by (5.44). Hence the right side of (5.43) is minimized at $W_1 = W_2 = ... = W_N = 1$ by Jensen's inequality. This implies that

$$\frac{r_\rho}{\rho} \to (1, 1, ..., 1) \text{ as } \rho \to 0. \tag{5.48}$$

Next we return to (5.38) to study ξ_ρ. Recall the function F in Theorem 3.1 whose domain is

$$\mathrm{Dom}(F) = \{\xi = (\xi_1, \xi_2, ..., \xi_N) \in \mathbb{R}^N : \ \xi_j \ne \xi_k \text{ if } j \ne k\} \tag{5.49}$$

Since

$$\lim_{t\to 0+} K(t) = \lim_{t\to\infty} K(t) = \infty, \tag{5.50}$$

F attains a global minimum in the domain of F. Choose δ_1 small enough so that

$$\xi^* \in \left\{ \xi = (\xi_1, \xi_2, ..., \xi_N) \in \mathrm{Dom}(F) : \ 4\delta_1 < |\xi_j - \xi_k| < \frac{1}{4\delta_1} \text{ for all } j \neq k \right\} \tag{5.51}$$

for any global minimum ξ^* of F.

We claim that ξ_ρ converges to a global minimum of F along any convergent subsequence. Suppose this is false. Let $\xi_\rho \to \xi_0$ a $\rho \to 0$, possibly along a subsequence, and ξ^* be a global minimum of F. Then $F(\xi^*) < F(\xi_0)$. Set

$$W_\rho = (W_{\rho,1}, W_{\rho,2}, ..., W_{\rho,N}), \ W_{\rho,j} = \left(\frac{r_\rho}{\rho} \right)^2 \tag{5.52}$$

Then, since $W_\rho \to (1, 1, ..., 1)$ as $\rho \to 0$ by (5.48), (5.38) implies that

$$\begin{aligned}
&\frac{1}{\gamma \rho^4} \left(J(\xi^*, r_\rho) - J(\xi_\rho, r_\rho) \right) \\
&= \frac{1}{2} \sum_{j=1}^{N} \sum_{k=1, \neq j}^{N} \left(\frac{\pi W_{\rho,j} W_{\rho,k}}{2} \log \frac{1}{|\xi_j^* - \xi_k^*|} + \pi^2 W_{\rho,j} W_{\rho,k} |\xi_j^* - \xi_k^*|^2 \right) \\
&\quad - \frac{1}{2} \sum_{j=1}^{N} \sum_{k=1, \neq j}^{N} \left(\frac{\pi W_{\rho,j} W_{\rho,k}}{2} \log \frac{1}{|\xi_{\rho,j} - \xi_{\rho,k}|} + \pi^2 W_{\rho,j} W_{\rho,k} |\xi_j - \xi_k|^2 \right) + O(\rho^2) \\
&\to \frac{\pi^2}{2} F(\xi^*) - \frac{\pi^2}{2} F(\xi_0) < 0
\end{aligned} \tag{5.53}$$

in contradiction to the fact that (ξ_ρ, r_ρ) is a minimum of J.

Because of (5.51), for any global minimum ξ^* of F, $((\xi_1^*, \xi_2^*, ..., \xi_N^*), (\rho, \rho, ..., \rho))$ is in the interior of M. Since ξ_ρ converges to a global minimum of F and (5.48) holds, (ξ_ρ, r_ρ) is in the interior of M when ρ is small. Then Lemma 5.1 asserts that $\varphi(\cdot, \xi_\rho, r_\rho)$ is a critical point of $\mathcal{J}$.

The stability of $\Omega_{\varphi(\cdot, \xi_\rho, r_\rho)}$ comes from its construction. First by Lemma 4.2, $\Omega_{\varphi(\cdot, \xi, r)}$ locally minimizes $\mathcal{J}$ in $\mathcal{X}_\flat$ for each (ξ, r). Then (ξ_ρ, r_ρ) minimizes J among all $(\xi, r) \in M$. As a minimum of minimum, we claim that $\Omega_{\varphi(\cdot, \xi_\rho, r_\rho)}$ is stable. □

6 Discussion

Theorem 3.1 tells us that the critical point $\Omega_{\varphi(\cdot, \xi_\rho, r_\rho)}$ is an assembly of N perturbed discs of approximately the same radius. The centers of these discs are close to a global minimum of F. To get a picture of $\Omega_{\varphi(\cdot, \xi_\rho, r_\rho)}$ we need to find the global minima of F.

When $N = 2$, (ξ_1, ξ_2) is a global minimum of F if and only if $|\xi_1 - \xi_2|$ is the minimum of K; namely

$$|\xi_1 - \xi_2| = \frac{1}{2\sqrt{\pi}} = 0.28209479... \tag{6.1}$$

When $N = 3$, (ξ_1, ξ_2, ξ_3) is a global minimum of F if and only if ξ_1, ξ_2, and ξ_3 are the vertices of an equilateral triangle in $\mathbb{R}^2$ whose side length is $\frac{1}{2\sqrt{\pi}}$.

When $N \geq 4$, we resort to numerical calculations. Figure 1 gives the numerical results for $K = 2, 3, ..., 13$. We also numercially minimize F for large K. Figures 2 and 3 show the results for $K = 100$ and $K = 500$. It looks that when K is large, the small discs fill a large circular region in $\mathbb{R}^2$ with an approximate hexagonal pattern.

If F has local minima, then our numerical computation may find a local minimum instead of a global minimum. But a local minimum of F can still be useful. If F admits a strict local minimum in M_0 in the sense that there exists a neighborhood of the local minimum where F at every other point is strictly greather

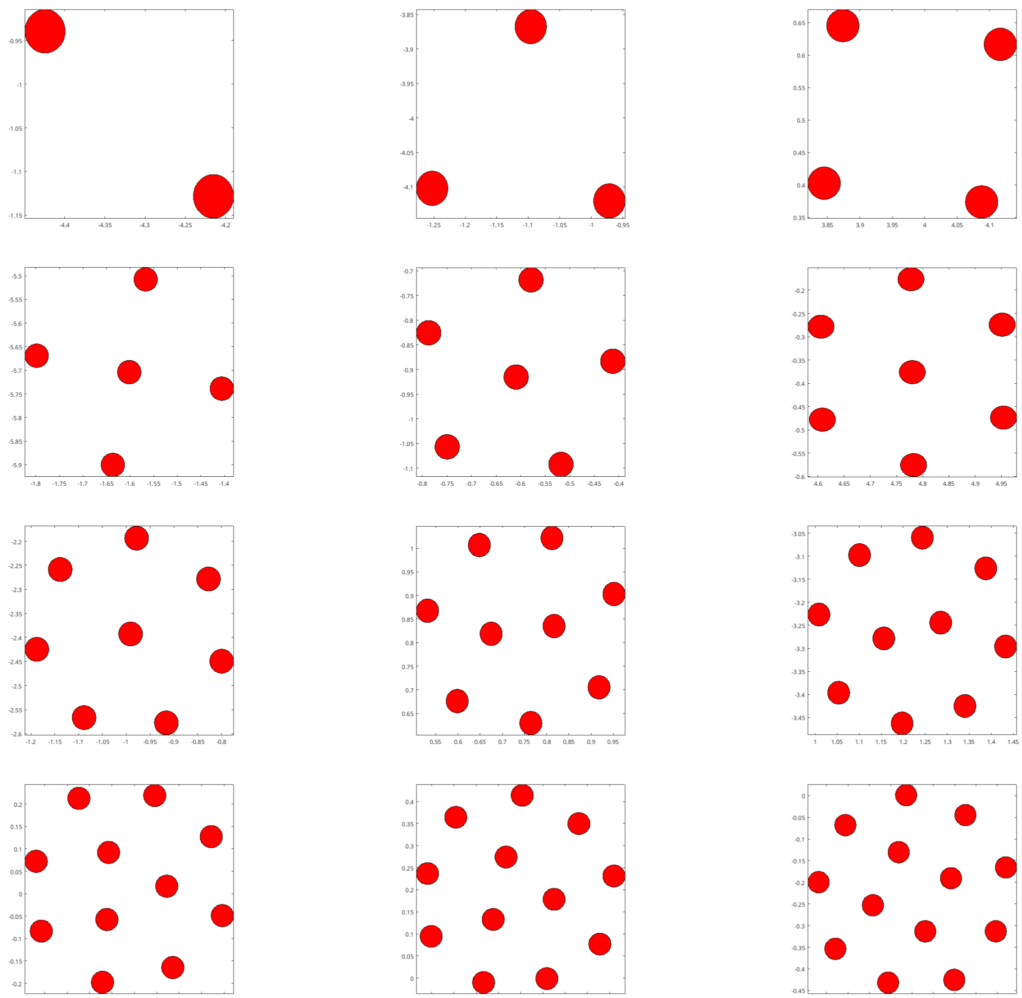

Figure 1: Numerical minima of F for $K = 2, 3, ..., 13$.

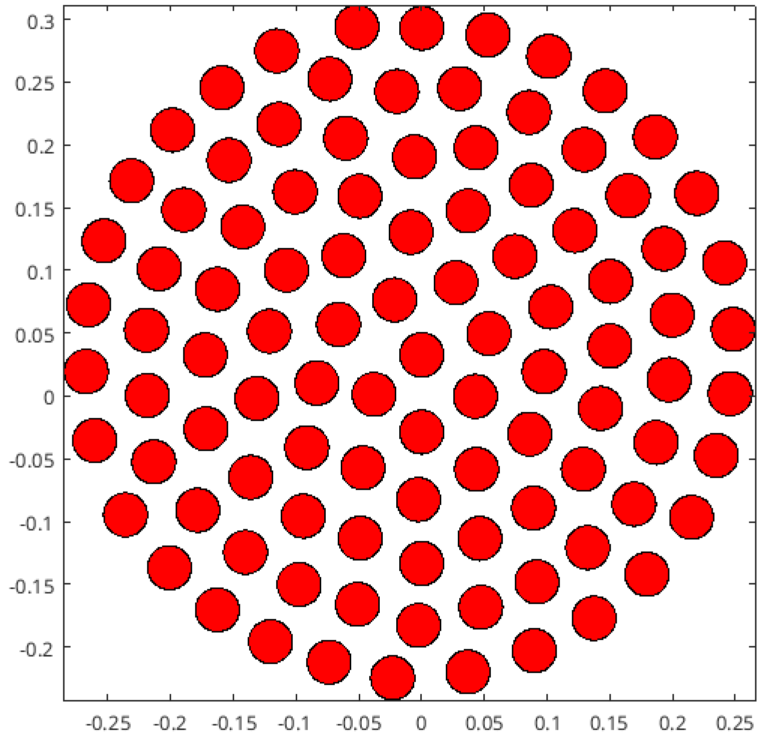

Figure 2: Numerical minimum of F for $K = 100$.

than F at the local minimum, then a slight modification of the argument in section 6 shows that $\mathcal{J}$ has a stable critical point which is an assembly of perturbed discs and the centers of the discs are close to this local minimum of F.

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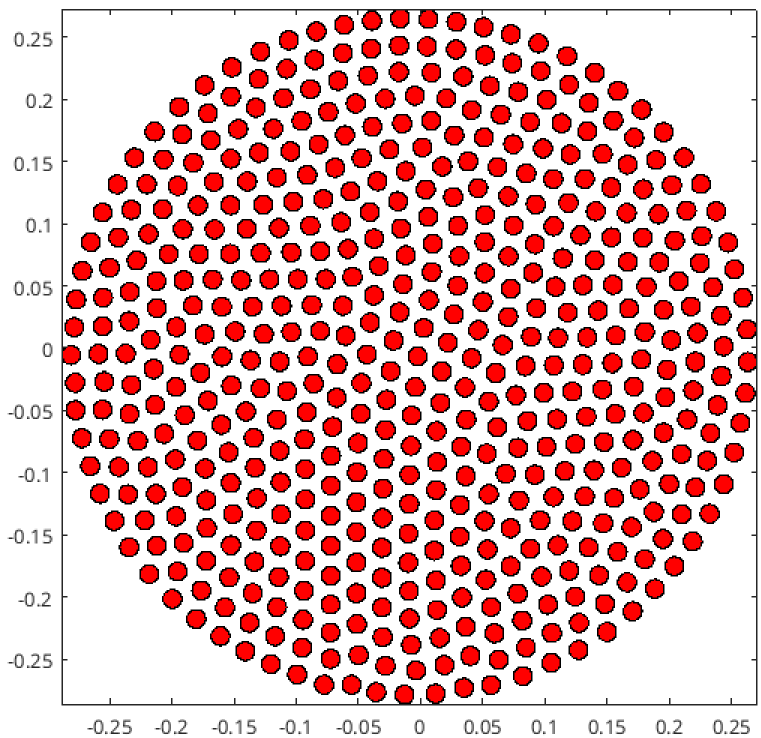

Figure 3: Numerical minimum of F for $K = 500$.

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