A double bubble assembly as a new phase of a ternary inhibitory system

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Abstract

A ternary inhibitory system is a three component system characterized by two properties: growth and inhibition. A deviation from homogeneity has a strong positive feedback on its further increase. In the meantime a longer ranging confinement mechanism prevents unlimited spreading. Together they lead to a locally self-enhancing and self-organizing process. The model considered here is a planar nonlocal geometric problem derived from the triblock copolymer theory. An assembly of perturbed double bubbles is mathematically constructed as a stable critical point of the free energy functional. Triple junction, a phenomenon that the three components meet at a single point, is a key issue addressed in the construction. Coarsening, an undesirable scenario of excessive micro-domain growth, is prevented by a lower bound on the long range interaction term in the free energy. The proof involves several ideas: perturbation of double bubbles in a restricted class; use of internal variables to remove nonlinear constraints, local minimization in a restricted class formulated as a nonlinear problem on a Hilbert space; and reduction to finite dimensional minimization. This existence theorem predicts a new morphological phase of a double bubble assembly.

1 Introduction

The objective of this paper is to establish the existence of a double bubble assembly as a new morphological phase for a ternary inhibitory system.

The term *morphological phase* comes from the block copolymer theory. An archetype of inhibitory systems, a block copolymer is a soft material characterized by fluid-like disorder on the molecular scale and a high degree of order at a longer length scale. A molecule in a block copolymer is a linear sub-chain of one type monomers grafted covalently to other types of monomers. Because of the repulsion between the unlike monomers, different type sub-chains tend to segregate. However the chemical bond between the sub-chains inhibits macroscopic phase separation. Only a local micro-phase separation occurs, resulting in micro-domains rich in different types of monomers. These micro-domains form patterns known as morphological phases [4].

The ternary inhibitory system considered here was originally derived by the authors in [24] from Nakazawa and Ohta's density functional formulation for triblock copolymers [18]. Let D be a bounded and smooth

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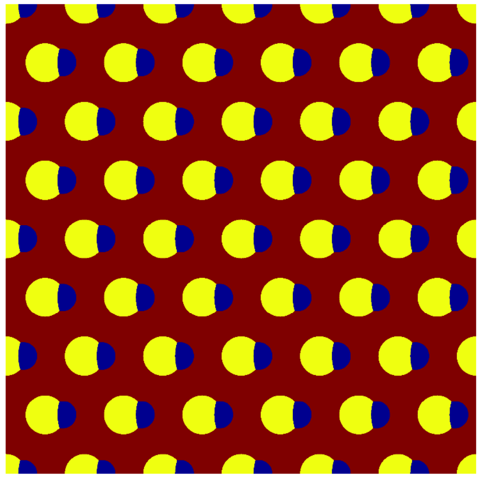

Figure 1: A proposed double bubble phase in a ternary inhibitory system.

open subset of $\mathbb{R}^2$, and ω_1 and ω_2 be two positive numbers such that $\omega_1 + \omega_2 < 1$. For two measurable subsets Ω_1 and Ω_2 of D satisfying $|\Omega_1| = \omega_1|D|$, $|\Omega_2| = \omega_2|D|$, and $|\Omega_1 \cap \Omega_2| = 0$, let $\Omega_3 = D\backslash(\Omega_1 \cup \Omega_2)$. Here $|\Omega_1|$, $|\Omega_2|$ and $|\Omega_1 \cap \Omega_2|$ stands for the area (or the Lebesgue measure) of Ω_1, Ω_2 and $\Omega_1 \cap \Omega_2$ respectively. The free energy of the system is

$$\mathcal{J}(\Omega_1, \Omega_2) = \frac{1}{2}\sum_{i=1}^{3} \mathcal{P}_D(\Omega_i) + \sum_{i,j=1}^{2} \int_D \frac{\gamma_{ij}}{2} \Big((-\Delta)^{-1/2}(\chi_{\Omega_i} - \omega_i)\Big)\Big((-\Delta)^{-1/2}(\chi_{\Omega_j} - \omega_j)\Big)\, dx. \tag{1.1}$$

The first term in (1.1) is responsible for growth. It is the total length of the interfaces separating the three components Ω_1, Ω_2 and Ω_3. Three types of interfaces exist: $\partial\Omega_1\backslash\partial\Omega_2$, the interface separating Ω_1 from Ω_3; $\partial\Omega_2\backslash\partial\Omega_1$, the interface separating Ω_2 from Ω_3; and $\partial\Omega_1 \cap \partial\Omega_2$, the interface separating Ω_1 from Ω_2. One can write the total size of the interfaces of all three types as $\frac{1}{2}(\mathcal{P}_D(\Omega_1) + \mathcal{P}_D(\Omega_2) + \mathcal{P}_D(\Omega_3))$. Here $\mathcal{P}_D(\Omega_i)$ is the perimeter of Ω_i in D. For a set Ω_i with a piecewise C^1 boundary, $\mathcal{P}_D(\Omega_i)$ is simply the length of $\partial\Omega_i \cap D$. For a general Lebesgue measurable subset the perimeter is defined in (2.22). In $\mathcal{P}_D(\Omega_1) + \mathcal{P}_D(\Omega_2) + \mathcal{P}_D(\Omega_3)$, each of $\partial\Omega_1\backslash\partial\Omega_2$, $\partial\Omega_2\backslash\partial\Omega_1$, and $\partial\Omega_1 \cap \partial\Omega_2$ is counted twice. The half is put here to avoid double counting. To make this term small, the Ω_i's like to form large regions separated by curves as short as possible.

The second term in (1.1) provides an inhibition mechanism. The operator $(-\Delta)^{-1/2}$ is the positive square root of the inverse of the $-\Delta$ operator; see (1.6); χ_{Ω_i} is the characteristic function of Ω_i ($\chi_{\Omega_i}(x) = 1$ if $x \in \Omega_i$ and 0 otherwise). The matrix γ_{ij} is symmetric and positive definite for a triblock copolymer. For the second term to be small, the functions χ_{Ω_i} must have frequent fluctuation.

A critical point (Ω_1, Ω_2) of $\mathcal{J}$ is a solution to the following equations:

$$\kappa_1 + \gamma_{11} I_{\Omega_1} + \gamma_{12} I_{\Omega_2} = \lambda_1 \text{ on } \partial\Omega_1\backslash\partial\Omega_2 \tag{1.2}$$

$$\kappa_2 + \gamma_{12} I_{\Omega_1} + \gamma_{22} I_{\Omega_2} = \lambda_2 \text{ on } \partial\Omega_2\backslash\partial\Omega_1 \tag{1.3}$$

$$\kappa_0 + (\gamma_{11} - \gamma_{12}) I_{\Omega_1} + (\gamma_{12} - \gamma_{22}) I_{\Omega_2} = \lambda_1 - \lambda_2 \text{ on } \partial\Omega_1 \cap \partial\Omega_2 \tag{1.4}$$

$$\nu_1 + \nu_2 + \nu_0 = \vec{0} \text{ at } \partial\Omega_1 \cap \partial\Omega_2 \cap \partial\Omega_3. \tag{1.5}$$

Here we assume that Ω_1 and Ω_2 do not touch the boundaries of D. Otherwise we need to add another condition that the boundary of Ω_1 (or Ω_2) meets the boundary of D perpendicularly.

In (1.2)-(1.4) κ_1, κ_2, and κ_0 are the curvatures of the curves $\partial\Omega_1\backslash\partial\Omega_2$, $\partial\Omega_2\backslash\partial\Omega_1$, and $\partial\Omega_1 \cap \partial\Omega_2$, respectively. These are signed curvatures defined with respect to a choice of normal vectors. For instance a circle has positive curvature if the normal vector is inward pointing. On $\partial\Omega_1\backslash\partial\Omega_2$ the normal vector points

Figure 2: Left plot: the $ABC...ABC$ lamellar morphological phase found in triblock copolymers. Right plot: the $ABAB...ABAC$ phase found in homopolymer/diblock copolymer blends.

inward into Ω_1. On $\partial\Omega_2\backslash\partial\Omega_1$, the normal vector points inward into Ω_2. On $\partial\Omega_1 \cap \partial\Omega_2$, the normal vector points from Ω_2 towards Ω_1, i.e. inward with respect to Ω_1 and outward with respect to Ω_2.

Also in (1.2)-(1.4) I_{Ω_1} and I_{Ω_2} are two functions on D determined from Ω_1 and Ω_2 respectively. The function I_{Ω_i}, called an inhibitor, is the solution to Poisson's equation

$$-\Delta I_{\Omega_i} = \chi_{\Omega_i} - \omega_i \ \text{ in } D, \quad \partial_\nu I_{\Omega_i} = 0 \ \text{ on } \partial D, \quad \int_D I_{\Omega_i}(x)\,dx = 0, \tag{1.6}$$

where $\partial_\nu I_{\Omega_i}$ stands for the outward normal derivative of I_{Ω_i} on ∂D. Note that the constraints $|\Omega_i| = \omega_i|D|$ implies that the integral of the right side of the PDE in (1.6) is zero, so the PDE together with the boundary condition is solvable. The solution is unique up to an additive constant. The last condition $\int_D I_{\Omega_i}(x)\,dx = 0$ fixes this constant and selects a particular solution. One also writes $I_{\Omega_i} = (-\Delta)^{-1}(\chi_{\Omega_i} - \omega_i)$, as the outcome of the operator $(-\Delta)^{-1}$ on $\chi_{\Omega_i} - \omega_i$. The operator $(-\Delta)^{-1/2}$ in (1.1) is the positive square root of $(-\Delta)^{-1}$.

The constants λ_1 and λ_2 are Lagrange multipliers corresponding to the constraints $|\Omega_1| = \omega_1|D|$ and $|\Omega_2| = \omega_2|D|$. They are unknown and are to be found with Ω_1 and Ω_2.

In the last equation, (1.5), ν_1, ν_2, and ν_0 are the inward pointing, unit tangent vectors of the curves $\partial\Omega_1\backslash\partial\Omega_2$, $\partial\Omega_2\backslash\partial\Omega_1$, and $\partial\Omega_1 \cap \partial\Omega_2$ at triple points. The requirement that the three unit vectors sum to zero is equivalent to the condition the three curves meet at 120 degree angles.

A morphological phase of the problem (1.1) must be a local minimizer of the functional $\mathcal{J}$, hence a stable solution to (1.2)-(1.5). As a phase in an inhibitory system it should have an approximately periodic pattern. Many patterns have been proposed by physicists as morphological phases based on experiments and numerical simulations; see [4]. Mathematically only two patterns have been known to be local minimizers of $\mathcal{J}$, both of which are one dimensional.

The first was found by the authors in [25] and depicted in the left plot of Figure 2. It is a one dimensional local minimizer of $\mathcal{J}$, consisting of alternating A, B, and C micro-domains. The functional $\mathcal{J}$ is posed on the unit interval with the periodic boundary condition. Cyclic patterns of $3k$, $k \in \mathbb{N}$, micro-domains are all local minimizers of $\mathcal{J}$. Here the matrix γ is positive definite.

Another one dimensional solution, again an energy local minimizer, was found by Choksi and Ren in [6]. It models a diblock copolymer/homopolymer blend. Depicted in the right plot of Figure 2, such a blend is a mixture of a AB diblock copolymer with a homopolymer of monomer species C, where the species C is thermodynamically incompatible with both the A and B monomer species. In the homopolymer a polymer chain consists purely of the monomer species C. Only the AB diblock copolymer has the inhibition property. In this case γ has one positive eigenvalue and one zero eigenvalue.

In this paper we predict a new morphological phase based on an existence theorem. As illustrated in Figure 1, this new phase is an assembly of perturbed double bubbles.

The double bubble is a fascinating geometric structure. It arises as the optimal configuration of the two component isoperimetric problem. Let m_1 and m_2 be two positive numbers. Find two disjoint sets E_1 and E_2 in $\mathbb{R}^n$ such that $|E_1| = m_1$, $|E_2| = m_2$, and the area of $\partial E_1 \cup \partial E_2$ is minimum. The double bubble is the unique solution to this isoperimetric problem by the works of Almgren [3], Taylor [35], Foisy *et al* [9], Hutchings *et al* [11], and Reichardt [22]. In two dimensions the planar double bubble, Figures 4 and 5, is enclosed by three circular arcs that meet at two triple junction points, or triple points. The angles between the arcs at a triple point are all 120 degrees.

A perturbed double bubble assembly is a collection of many disjoint, perturbed double bubbles. A perturbed double bubble deviates from an exact double bubble slightly, due to the impact of the second term in $\mathcal{J}$. All the perturbed double bubbles in the assembly have approximately the same size and shape. In Figure 1 the union of the blue bubbles is taken by the first component Ω_1, the union of the yellow bubbles is occupied by the second component Ω_2, and the rest of the domain is filled by Ω_3.

We introduce a fixed number $m \in (0,1)$ and a small ϵ so that $\omega_1 = \epsilon^2 m$ and $\omega_2 = \epsilon^2(1-m)$. The area constraints $|\Omega_1| = \omega_1|D|$ and $|\Omega_2| = \omega_2|D|$ now take the form

$$|\Omega_1| = m\epsilon^2 \ \text{ and } \ |\Omega_2| = (1-m)\epsilon^2. \tag{1.7}$$

Instead of ω_1 and ω_2, ϵ becomes one parameter of our problem. The fixed number m measures the relative size of $|\Omega_1|$ vs $|\Omega_2|$ since $\frac{|\Omega_1|}{|\Omega_2|} = \frac{m}{1-m}$.

The other parameter is the matrix γ. It must be positive definite and satisfy a uniform positivity condition. Namely, there exists $\iota > 0$ so that $\iota\,\overline{\overline{\lambda}}(\gamma) \leq \overline{\lambda}(\gamma)$ where $\overline{\lambda}(\gamma)$ and $\overline{\overline{\lambda}}(\gamma)$ are the two eigenvalues of γ such that $0 < \overline{\lambda}(\gamma) \leq \overline{\overline{\lambda}}(\gamma)$. The matrix γ must also have a lower bound and an upper bound.

The main result in this paper is the following existence theorem.

Theorem 1.1 *Let $m \in (0,1)$, $n \in \mathbb{N}$, and $\iota \in (0,1]$. There exist positive numbers δ, $\widetilde{\sigma}$, and σ depending on D, m, n, and ι only, such that if the following three conditions hold*

1. $0 < \epsilon < \delta$,
2. $\dfrac{\widetilde{\sigma}}{\epsilon^3 \log\frac{1}{\epsilon}} \leq \overline{\lambda}(\gamma) \leq \overline{\overline{\lambda}}(\gamma) < \dfrac{\sigma}{\epsilon^3}$,
3. $\iota\,\overline{\overline{\lambda}}(\gamma) \leq \overline{\lambda}(\gamma)$,

then there is an assembly of n perturbed double bubbles, satisfying the constraints (1.7), which is a solution to the equations (1.2)-(1.5). Each perturbed double bubble is bounded by three smooth curves that meet at two triple junction points.

This solution is stable in some sense. If $n = 1$, the lower bound $\frac{\widetilde{\sigma}}{\epsilon^3 \log\frac{1}{\epsilon}} \leq \overline{\lambda}(\gamma)$ is not needed.

The proof of Theorem 1.1 reveals several properties of the solution. One of them is that all the perturbed double bubbles in the solution have almost the same size and shape.

Another property is that the locations of the double bubbles in the assembly are determined asymptotically by a Green's function. Let G be the Green's function of the $-\Delta$ operator on D with the Neumann boundary condition; namely $G(x,y)$ as a function of x satisfies

$$-\Delta G(\cdot,y) = \delta(\cdot - y) - \frac{1}{|D|} \text{ in } D;\ \ \partial_\nu G(\cdot,y) = 0 \text{ on } \partial D;\ \ \int_D G(x,y)\,dy = 0 \tag{1.8}$$

for each $y \in D$. One can write G as a sum of two terms:

$$G(x,y) = \frac{1}{2\pi}\log\frac{1}{|x-y|} + R(x,y). \tag{1.9}$$

The first term $\frac{1}{2\pi}\log\frac{1}{|x-y|}$ is the fundamental solution of the Laplace operator; the second term R is the regular part of the Green's function, a smooth function of $(x,y) \in D \times D$.

For n distinct points ξ^k, $k = 1,2,...,n$, in D let

$$F(\xi^1,...,\xi^n) = \sum_{k=1}^n R(\xi^k,\xi^k) + \sum_{k\neq l} G(\xi^k,\xi^l). \tag{1.10}$$

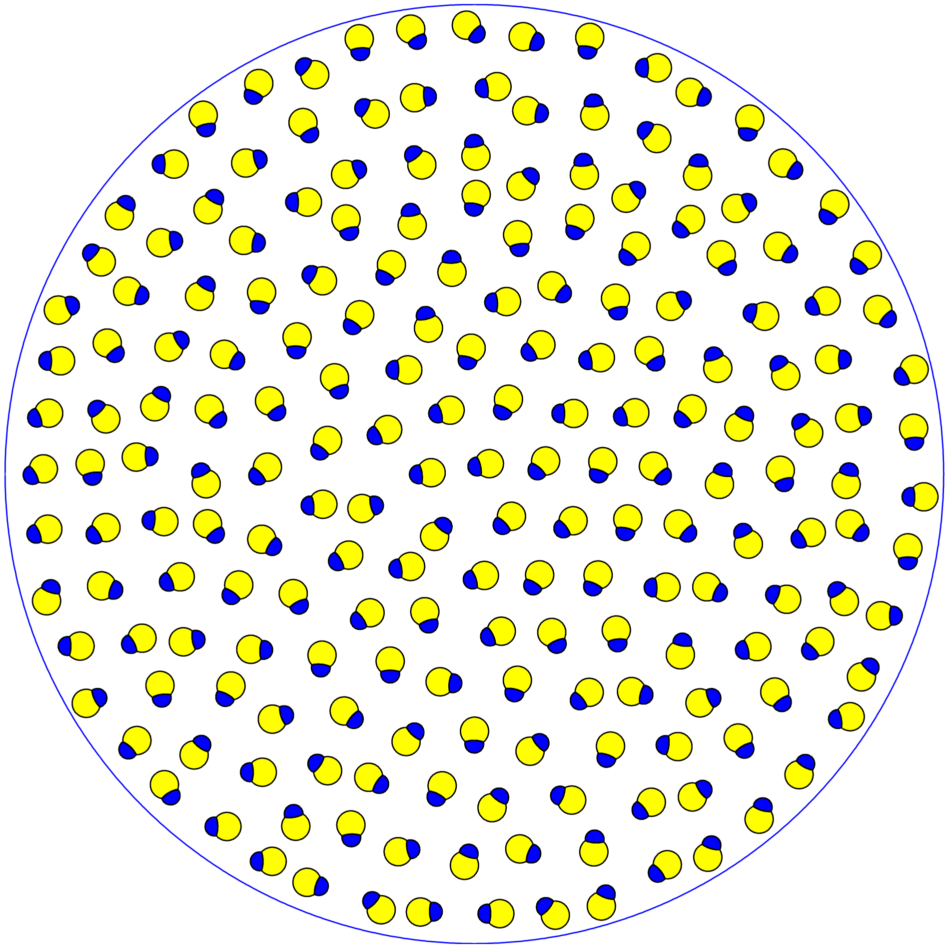

Figure 3: An illustration of the solution found in Theorem 1.1 in the case that D is the unit disc and $n = 200$. The locations of the perturbed double bubbles are determined by minimizing F given in (1.10).

It is known that $R(x,x) \to \infty$ if $x \to \partial D$. Consequently $F(\xi^1, ..., \xi^n) \to \infty$, if one of the ξ^k's approaches ∂D, or if the distance of two points ξ^k and ξ^l approaches 0. This ensures that F is minimized by n distinct points in D. It is proved in Section 6 that if the perturbed double bubbles in the solution are located at points $\xi^{*,1}$, $\xi^{*,2}$, ..., $\xi^{*,n}$, and $(\xi^{*.1}, ..., \xi^{*,n}) \to (\xi^{\circ,1}, ..., \xi^{\circ,n})$ as $\epsilon \to 0$ and $|\gamma|\epsilon^3 \to 0$, possibly along a subsequence, then

$$F(\xi^{\circ,1}, \xi^{\circ,2}, ..., \xi^{\circ,n}) = \min\{F(\xi^1, \xi^2, ..., \xi^n) : \ \xi^1, \xi^2, ..., \xi^n \in D, \ \ \xi^k \neq \xi^l \text{ if } k \neq l\}. \tag{1.11}$$

Therefore the perturbed double bubbles are found near points that minimize the function F. Here $|\gamma|$ can be any norm of the matrix γ. If one takes $|\gamma|$ to be the operator norm, then $|\gamma| = \overline{\overline{\lambda}}(\gamma)$.

If D is the unit disc, the Green's function is known explicitly:

$$G(x,y) = \frac{1}{2\pi} \log \frac{1}{|x-y|} + \frac{1}{2\pi}\Big[\frac{|x|^2}{2} + \frac{|y|^2}{2} + \log \frac{1}{|x\overline{y} - 1|}\Big] - \frac{3}{8\pi} \tag{1.12}$$

where $\overline{y}$ denoted the complex conjugate of $y \in D \subset \mathbb{R}^2 \cong \mathbb{C}$ and $x\overline{y}$ is the complex product of x and $\overline{y}$. Consequently F is also known explicitly.

Figure 3 shows a double bubble assembly with D being the unit disc and $n = 200$. The locations of the perturbed double bubbles in this picture are determined by numerical minimization of F. Away from the boundary of D, the double bubbles organize themselves in a hexagonal pattern. However the proof of Theorem 1.1 does not tell what the directions of the perturbed double bubbles are, so the directions of the double bubbles shown in Figure 3 do not reflect the directions of the double bubbles in a real solution.

The proof of Theorem 1.1 consists of several steps. In the first step, done in Section 2, one constructs an assembly of exact double bubbles and compute its energy. Take n exact double bubbles B^k whose two bubbles are B_1^k and B_2^k for $k = 1, 2, ..., n$. The area of B_i^k is w_i^k. Take n distinct points ξ^k in D and n angles $\theta^k \in \mathbb{S}$, where $\mathbb{S}$ is the unit circle. Scale down each B^k by a factor ϵ, rotate by the angle θ^k and place it in D centered at ξ^k. This small double bubble in D is denoted $T^k(B^k)$, and the collection $(T^1(B^1), T^2(B^2), ..., T^n(B^n))$ is an assembly of exact double bubbles denoted by $T(B)$. This $T(B)$ depends on $\xi = (\xi^1, ..., \xi^n)$, $\theta = (\theta^1, ..., \theta^n)$, and $w = \{w_i^k\}$. In Lemma 2.1 one finds the energy of $T(B)$.

In the second step, perturb each B^k in a special way to define a restricted class of perturbed double bubble assemblies. There are actually two parts in the perturbation, explained in Section 2. First move the two triple points of B^k vertically in opposite directions by the same amount. Connect the new triple points by three circular arcs. The two sets bounded by the new arcs still have the areas w_1^k and w_2^k respectively and the radii ρ_i^k of the new arcs still satisfy the condition $(\rho_1^k)^{-1} - (\rho_2^k)^1 = (\rho_0^k)^{-1}$. However the 120 degree angle condition at triple points no longer holds for the new arcs. In the second part of the restricted perturbation, the arcs are changed to more general curves, while the areas of the two enclosed sets remain to be w_i^k and the triple points are unchanged. This perturbed double bubble is denoted P^k. It is scaled down by ϵ and mapped into D by the same T^k. The collection $T(P) = (T^1(P^1), T^2(P^2), ..., T^n(P^n))$ is an assembly of perturbed double bubbles. All assemblies obtained this way form a class, the restricted class of perturbed double bubbles, which is determined by ξ, θ, and w.

It turns out that each assembly in a restricted class is identified by an element of a Hilbert space $\mathcal{Y}$ defined in Section 2. The element consists of $3n$ functions ϕ_i^k and n numbers η^k for $k = 1, 2, ..., n$ and $i = 1, 2, 0$. Collectively they are denoted by (ϕ, η) where $\phi = (\phi^1, \phi^2, ..., \phi^n)$, $\phi^k = (\phi_1^k, \phi_2^k, \phi_0^k)$, and $\eta = (\eta^1, \eta^2, ..., \eta^n)$. Within the restricted class $\mathcal{J}$ becomes a functional on $\mathcal{Y}$. Sections 3, 4, and 5 culminate in Lemma 5.1, which states that in each restricted class there is an element (ϕ^*, η^*) that locally minimizes $\mathcal{J}$ within the restricted class. This third step is most technical, involving an error estimate of the exact double bubble assembly $T(B)$, proving the positivity of the second variation of $\mathcal{J}$ at $T(B)$, and a fixed point argument. In Lemma 5.3 it is shown that (ϕ^*, η^*) satisfies a weakened version of (1.2)-(1.4) where the constants λ_1 and λ_2 may vary from one perturbed double bubble to another perturbed double bubble in the assembly.

To fix this problem and also to have the 120 degree angle condition (1.5) satisfied, revisit the restricted class of perturbed double bubble assemblies. Since this class is specified by (ξ, θ, w), the energy minimizing element (ϕ^*, η^*) in this class should be denoted by $(\phi^*(\cdot, \xi, \theta, w), \eta^*(\xi, \theta, w))$. In the fourth step one finds the energy $\mathcal{J}(\phi^*(\cdot, \xi, \theta, w), \eta^*(\xi, \theta, w))$ of this element in Lemma 6.1. Treating this quantity as a function of ξ, θ, and w, one minimizes it with respect to (ξ, θ, w) and finds a minimum (ξ^*, θ^*, w^*) in Lemma 6.2. If one uses the restricted class of assemblies specified by this particular (ξ^*, θ^*, w^*), then it is proved in Section 6 that the locally energy minimizing element $(\phi^*(\cdot, \xi^*, \theta^*, w^*), \eta^*(\xi^*, \theta^*, w^*))$ solves (1.2)-(1.4) exactly and also satisfies the 120 degree angle condition (1.5) at triple points.

The idea of using a restricted class of perturbed double bubbles first appeared in the authors' work [31]. There it was shown that when $m = \frac{1}{2}$, $\mathcal{J}$ admits a local minimizer that shapes like a single, symmetric double bubble. This method was later improved by the authors in [32], where the condition $m = \frac{1}{2}$ is relaxed to $m \in (0, 1)$. The single double bubble solution constructed there is asymmetric if $m \neq \frac{1}{2}$. As only one double bubble is considered in those papers, the lower bound $\frac{\widetilde{\sigma}}{\epsilon^3 \log \frac{1}{\epsilon}} \leq \overline{\lambda}(\gamma)$ in Theorem 1.1 is not needed. In Theorem 1.1 this lower bound is used to prevent coarsening. If coarsening occurs, some pieces of a constituent component grows bigger while some other pieces of the same component shrink and disappear. There must be at least two perturbed double bubbles in an assembly for coarsening to be possible. The lower bound also forces the perturbed double bubbles in the solution to have approximately the same shape and size.

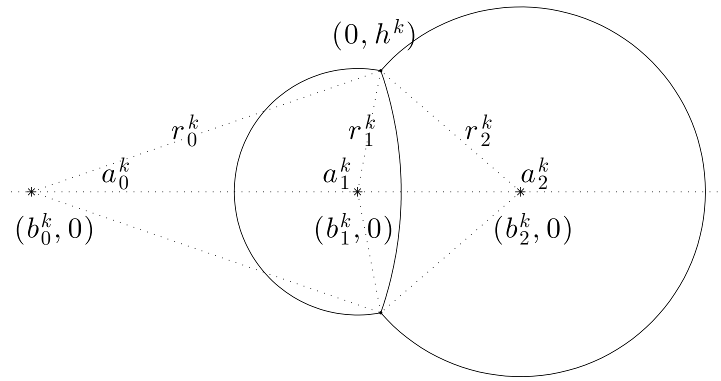

Figure 4: An asymmetric exact double bubble with angles a_i^k, radii r_i^k, and centers $(b_i^k, 0)$. One of the two triple points is $(0, h^k)$.

2 Exact double bubble assemblies and restricted perturbations

We start with n exact double bubbles, denoted by B^1, ..., B^n. Each double bubble B^k is a pair of two adjacent sets B_1^k and B_2^k. The area of B_i^k is denoted by w_i^k:

$$|B_i^k| = w_i^k, \quad i = 1, 2, \;\; k = 1, 2, ..., n. \tag{2.1}$$

These two numbers, w_1^k and w_2^k, completely determine the double bubble B^k. The w_i^k's stay in the set $\overline{W}$ which is the closure of

$$W = \{(w_i^k) \in \mathbb{R}^{2n} : \; \frac{m}{2n} < w_1^k < \frac{2m}{n}, \; \frac{1-m}{2n} < w_2^k < \frac{2(1-m)}{n}, \; \forall k; \; \sum_{k=1}^{n} w_1^k = m, \; \sum_{k=1}^{n} w_2^k = 1-m\}. \tag{2.2}$$

Initially w_i^k are fixed. Later they will vary in $\overline{W}$. Of course w_i^k can vary only if $n \geq 1$. If $n = 1$, the case studied in [31, 32], there is no need to introduce w_i^k.

The set B_1^k is bounded by two circular arcs of radii r_1^k and r_0^k. One arc, whose radius is r_0^k, is also on the boundary of B_2^k. The rest of the boundary of B_2^k is another circular arc whose radius is r_2^k.

There are actually two cases to consider. The first is the asymmetric case, depicted in Figure 4, where the area of B_1^k is different from the area of B_2^k. If the left bubble B_1^k is smaller than the right bubble B_2^k, i.e. $w_1^k < w_2^k$, then

$$r_1^k < r_2^k, \tag{2.3}$$

and the three radii satisfy the condition

$$(r_1^k)^{-1} - (r_2^k)^{-1} = (r_0^k)^{-1}. \tag{2.4}$$

If $w_1^k > w_2^k$, then (2.4) changes to

$$(r_1^k)^{-1} - (r_2^k)^{-1} = -(r_0^k)^{-1}. \tag{2.5}$$

From now on when dealing with an asymmetric double bubble or a perturbation of an asymmetric double bubble, we assume, without the loss of generality, that $w_1^k < w_2^k$. The other case, $w_1^k > w_2^k$ can always be handled in a similar way.

The two points where the three arcs meet are termed triple junction points, or triple points. At these points the three arcs meet at 120 degree angles. Denote by a_1^k, a_2^k, and a_0^k the angles associated with the three arcs, Figure 4. The 120 degree angle condition and (2.3) imply that, if $w_1^k < w_2^k$,

$$a_1^k = \frac{2\pi}{3} - a_0^k, \quad a_2^k = \frac{2\pi}{3} + a_0^k, \quad a_0^k \in \Big(0, \frac{\pi}{3}\Big). \tag{2.6}$$

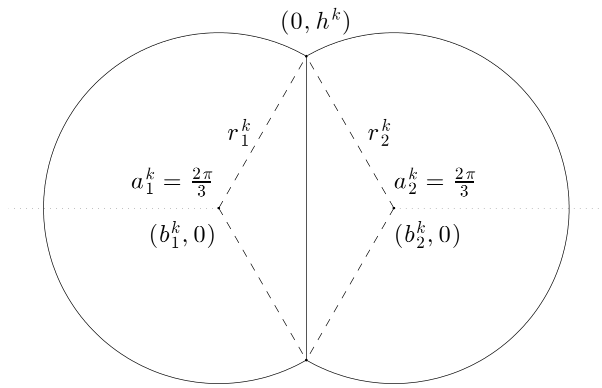

Figure 5: A symmetric exact double bubble where $r_1^k = r_2^k$ and $a_1^k = a_2^k = \frac{2\pi}{3}$.

The conditions (2.1) can be expressed as

$$(r_1^k)^2(a_1^k - \cos a_1^k \sin a_1^k) + (r_0^k)^2(a_0^k - \cos a_0^k \sin a_0^k) = w_1^k \quad (2.7)$$

$$(r_2^k)^2(a_2^k - \cos a_2^k \sin a_2^k) - (r_0^k)^2(a_0^k - \cos a_0^k \sin a_0^k) = w_2^k. \quad (2.8)$$

The second case is the symmetric case where B_1^k and B_2^k have the same area, i.e. $w_1^k = w_2^k$. Then

$$r_1^k = r_2^k \text{ and } r_0^k = \infty. \quad (2.9)$$

The middle arc becomes a straight line segment. The three arcs still meet at 120 degree angles, Figure 5. In this case

$$a_1^k = a_2^k = \frac{2\pi}{3},\ a_0^k = 0 \quad (2.10)$$

and (2.7) and (2.8) become

$$(r_1^k)^2(a_1^k - \cos a_1^k \sin a_1^k) = (r_2^k)^2(a_2^k - \cos a_2^k \sin a_2^k) = w_1^k = w_2^k. \quad (2.11)$$

Place the exact double bubble $B^k = (B_1^k, B_2^k)$ in $\mathbb{R}^2$ so that the triple points are $(0, h^k)$ and $(0, -h^k)$ where

$$h^k = r_i^k \sin a_i^k, \quad i = 1, 2, 0 \quad (2.12)$$

is positive. Moreover the centers of the three arcs are denoted $(b_i^k, 0)$, $i = 1, 2, 0$, respectively. In the symmetric case (2.12) holds for $i = 1, 2$, and the center $(b_0^k, 0)$ of the middle arc is at infinity.

Each $T_{\epsilon,\xi^k,\theta^k}$ is an affine transformation given by

$$T_{\epsilon,\xi^k,\theta^k}\hat{x} = \epsilon e^{\mathrm{i}\theta^k}\hat{x} + \xi^k.$$

In this paper we identify $\mathbb{R}^2$ with $\mathbb{C}$ to use the complex multiplication, like $e^{\mathrm{i}\theta^k}\hat{x}$ above, to simplify notation. Often $T_{\epsilon,\xi^k,\theta^k}$ is simplified to T^k, and $T(B)$, which stands for $(T^1(B^1), T^2(B^2), ..., T^n(B^n))$ where $T^k(B^k) = (T^k(B_1^k), T^k(B_2^k))$, is an assembly of exact double bubbles. One also sets $T(B_i) = \cup_{k=1}^n T^k(B_i^k)$ for $i = 1, 2$.

The double bubbles $T^k(B^k)$ must all be inside D and do not intersect each other. Recall the function F defined in (1.10). The domain of F is

$$\Xi = \{\xi = (\xi^1, \xi^2, ..., \xi^n) :\ \xi^k \in D\ \forall\ k = 1, 2, ..., n,\ \xi^k \neq \xi^l\ \forall\ k \neq l\}. \quad (2.13)$$

Since $F(\xi) \to \infty$ as $\xi \to \partial\Xi$ where Ξ is viewed as a subset of $\mathbb{R}^{2n}$, one can find a small enough $\overline{\delta} > 0$ such that

$$\min_{\xi\in\Xi} F(\xi) < \min_{\xi\in\Xi\setminus\Xi_{\overline{\delta}}} F(\xi). \quad (2.14)$$

Here $\Xi_{\overline{\delta}}$ is a subset of Ξ defined as

$$\Xi_{\overline{\delta}} = \{\xi \in \Xi : \ d(\xi^k, \partial D) > \overline{\delta} \ \forall \ k, \ d(\xi^k, \xi^l) > 2\overline{\delta} \ \forall \ k \neq l\}. \tag{2.15}$$

In (2.15) "d" stands for the Euclidean distance in $\mathbb{R}^2$. The centers ξ^k of the double bubbles $T^k(B^k)$ will always be in the closure of $\Xi_{\overline{\delta}}$:

$$\xi = (\xi^1, \xi^2, ..., \xi^n) \in \overline{\Xi_{\overline{\delta}}}. \tag{2.16}$$

At this point we state our initial requirement on δ which is the bound for ϵ. The number δ must be small enough so that

$$0 < 2\max\{r_1^k, r_2^k\}\delta < \frac{\overline{\delta}}{2} \tag{2.17}$$

holds for the radii r_1^k and r_2^k of any double bubble B^k for which $|B_1^k| = w_1^k \in [\frac{m}{2n}, \frac{2m}{n}]$ and $|B_2^k| = w_2^k \in [\frac{1-m}{2n}, \frac{2(1-m)}{n}]$. In other words (2.17) holds uniformly with respect to all double bubble B^k as long as w_1^k and w_2^k are in the specified ranges.

With this choice of δ and with $\epsilon < \delta$, let $z^k \in T^k(B^k)$. Then for any $x \in \partial D$,

$$d(x, z^k) \geq d(x, \xi^k) - d(\xi^k, z^k) \geq \overline{\delta} - 2\max\{r_1^k, r_2^k\}\epsilon > \overline{\delta} - 2\max\{r_1^k, r_2^k\}\delta > \frac{\overline{\delta}}{2}. \tag{2.18}$$

For $z^k \in T^k(B^k)$ and $z^l \in T^l(B^l)$ where $k \neq l$,

$$d(z^k, z^l) \geq d(\xi^k, \xi^l) - d(\xi^k, z^k) - d(\xi^l, z^l) \geq 2\overline{\delta} - 2\max\{r_1^k, r_2^k\}\epsilon - 2\max\{r_1^l, r_2^l\}\epsilon > \overline{\delta}. \tag{2.19}$$

Hence each $T^k(B^k)$ is inside D and the $T^k(B^k)$'s do not intersect. More precisely with each $z^k \in T^k(B^k)$ for $k = 1, 2, ..., n$, $z = (z^1, z^2, ..., z^n)$ is in $\Xi_{\overline{\delta}/2}$, where the set $\Xi_{\overline{\delta}/2}$ is defined as in (2.15).

The two terms in (1.1) are denoted by $\mathcal{J}_s$ and $\mathcal{J}_l$ standing for the short and long part of the energy respectively:

$$\mathcal{J}_s(\Omega_1, \Omega_2) = \frac{1}{2}\sum_{i=1}^{3} \mathcal{P}_D(\Omega_i), \tag{2.20}$$

$$\mathcal{J}_l(\Omega_1, \Omega_2) = \sum_{i,j=1}^{2} \int_D \frac{\gamma_{ij}}{2} \Big((-\Delta)^{-1/2}(\chi_{\Omega_i} - \omega_i)\Big)\Big((-\Delta)^{-1/2}(\chi_{\Omega_j} - \omega_j)\Big)\, dx. \tag{2.21}$$

For a Lebesgue measurable subset E of D the perimeter is defined by

$$\mathcal{P}_D(E) = \sup\Big\{ \int_E \operatorname{div} g(x)\, dx : \ g \in C_0^1(D, \mathbb{R}^2), \ |g(x)| \leq 1 \ \forall x \in D \Big\} \tag{2.22}$$

where $\operatorname{div} g$ is the divergence of the C^1 vector field g on D with compact support and $|g(x)|$ stands for the Euclidean norm of the vector $g(x) \in \mathbb{R}^2$; see for instance [8] for more on the notion of perimeter. If Ω_1 and Ω_2 are bounded by piecewise C^1 curves and do not share boundary with D, then $\mathcal{J}_s(\Omega_1, \Omega_2)$ is just the length of $\partial\Omega_1 \cup \partial\Omega_2$. With the help of the Green's function one can write $\mathcal{J}_l$ in an alternative form:

$$\mathcal{J}_l(\Omega_1, \Omega_2) = \sum_{i,j=1}^{2} \frac{\gamma_{ij}}{2} \int_{\Omega_i}\int_{\Omega_j} G(x, y)\, dxdy \tag{2.23}$$

which is more amenable to computation.

Lemma 2.1 *The energy $\mathcal{J}(T(B))$ of the exact double bubble assembly $T(B)$ is estimated as follows.*

$$\Big|\mathcal{J}(T(B)) - \Big\{\epsilon \sum_{k=1}^{n}\sum_{i=0}^{3} 2a_i^k r_i^k + \Big(\log\frac{1}{\epsilon}\Big)\epsilon^4 \sum_{k=1}^{n}\sum_{i,j=1}^{2}\frac{\gamma_{ij}w_i^k w_j^k}{4\pi} + \epsilon^4\sum_{k=1}^{n}\sum_{i,j=1}^{2}\frac{\gamma_{ij}}{2}\int_{B_i^k}\int_{B_j^k}\frac{1}{2\pi}\log\frac{1}{|\hat{x}-\hat{y}|}\,d\hat{x}d\hat{y}$$

$$+\epsilon^4\sum_{k=1}^{n}\sum_{i,j=1}^{2}\frac{\gamma_{ij}}{2}w_i^k w_j^k R(\xi^k,\xi^k) + \epsilon^4\sum_{k\neq l}\sum_{i,j=1}^{2}\frac{\gamma_{ij}}{2}w_i^k w_j^l G(\xi^k,\xi^l)\Big\}\Big|$$

$$\leq \epsilon^5\sum_{k=1}^{n}\sum_{i,j=1}^{2}\frac{\gamma_{ij}}{2}w_i^k w_j^k 4A_R\max\{r_1^k,r_2^k\} + \epsilon^5\sum_{k\neq l}\sum_{i,j=1}^{2}\frac{\gamma_{ij}}{2}w_i^k w_j^l 4A_G\max\{r_1^k,r_2^k,r_1^l,r_2^l\} = O(|\gamma|\epsilon^5).$$

The constants A_R and A_G above are given by

$$A_R = \max\Big\{|\nabla R(x,y)|:\ x,y\in D,\ d(x,\partial D)\geq\frac{\overline{\delta}}{2},\quad d(y,\partial D)\geq\frac{\overline{\delta}}{2}\Big\} \tag{2.24}$$

$$A_G = \max\Big\{|\nabla G(x,y)|:\ x,y\in D,\ d(x,\partial D)\geq\frac{\overline{\delta}}{2},\quad d(y,\partial D)\geq\frac{\overline{\delta}}{2},\ d(x,y)\geq\overline{\delta}\Big\} \tag{2.25}$$

where $\overline{\delta}$ is given in (2.14).

Here $O(|\gamma|\epsilon^5)$ stands for a quantity that can be bounded by $C|\gamma|\epsilon^5$ for some constant C is a constant that depend at most on D, m and n. This convention is practiced throughout the paper.

Proof. By the remark following (2.22) and (2.23),

$$\begin{aligned}
\mathcal{J}(T(B)) &= \epsilon\sum_{k=1}^{n}\sum_{i=0}^{3}2a_i^k r_i^k + \sum_{i,j=1}^{2}\frac{\gamma_{ij}}{2}\int_{\cup_{k=1}^n T^k(B_i^k)}\int_{\cup_{l=1}^n T^l(B_j^l)} G(x,y)\,dxdy \\
&= \epsilon\sum_{k=1}^{n}\sum_{i=0}^{3}2a_i^k r_i^k + \sum_{k=1}^{n}\sum_{i,j=1}^{2}\frac{\gamma_{ij}}{2}\int_{T^k(B_i^k)}\int_{T^k(B_j^k)}\frac{1}{2\pi}\log\frac{1}{|x-y|}\,dxdy \\
&\quad+\sum_{k=1}^{n}\sum_{i,j=1}^{2}\frac{\gamma_{ij}}{2}\int_{T^k(B_i^k)}\int_{T^k(B_j^k)}R(x,y)\,dxdy \\
&\quad+\sum_{k\neq l}\sum_{i,j=1}^{2}\frac{\gamma_{ij}}{2}\int_{T^k(B_i^k)}\int_{T^l(B_j^l)}G(x,y)\,dxdy \\
&= \epsilon\sum_{k=1}^{n}\sum_{i=0}^{3}2a_i^k r_i^k + \Big(\log\frac{1}{\epsilon}\Big)\epsilon^4\sum_{k=1}^{n}\sum_{i,j=1}^{2}\frac{\gamma_{ij}w_i^k w_j^k}{4\pi} \\
&\quad+\epsilon^4\sum_{k=1}^{n}\sum_{i,j=1}^{2}\frac{\gamma_{ij}}{2}\int_{B_i^k}\int_{B_j^k}\frac{1}{2\pi}\log\frac{1}{|\hat{x}-\hat{y}|}\,d\hat{x}d\hat{y} \\
&\quad+\epsilon^4\sum_{k=1}^{n}\sum_{i,j=1}^{2}\frac{\gamma_{ij}}{2}\int_{B_i^k}\int_{B_j^k}R(T^k\hat{x},T^k\hat{y})\,d\hat{x}d\hat{y} \\
&\quad+\epsilon^4\sum_{k\neq l}\sum_{i,j=1}^{2}\frac{\gamma_{ij}}{2}\int_{B_i^k}\int_{B_j^l}G(T^k\hat{x},T^l\hat{y})\,d\hat{x}d\hat{y}.
\end{aligned}$$

By the mean value theorem,

$$|R(T^k\hat{x},T^k\hat{y}) - R(\xi^k,\xi^k)|$$

$$
\begin{aligned}
&= |\nabla R(\xi^k+\tau\epsilon e^{\mathrm{i}\theta^k}\hat{x},\xi^k+\tau\epsilon e^{\mathrm{i}\theta^k}\hat{y})\cdot\epsilon e^{\mathrm{i}\theta^k}\hat{x}+\tilde{\nabla}R(\xi^k+\tau\epsilon e^{\mathrm{i}\theta^k}\hat{x},\xi^k+\tau\epsilon e^{\mathrm{i}\theta^k}\hat{y})\cdot\epsilon e^{\mathrm{i}\theta^k}\hat{y}| \\
&\le 4A_R\max\{r_1^k,r_2^k\}\epsilon \\
&|G(T^k\hat{x},T^l\hat{y})-G(\xi^k,\xi^l)| \\
&= |\nabla G(\xi^k+\tau\epsilon e^{\mathrm{i}\theta^k}\hat{x},\xi^l+\tau\epsilon e^{\mathrm{i}\theta^l}\hat{y})\cdot\epsilon e^{\mathrm{i}\theta^k}\hat{y}+\tilde{\nabla}G(\xi^k+\tau\epsilon e^{\mathrm{i}\theta^k}\hat{x},\xi^l+\tau\epsilon e^{\mathrm{i}\theta^l}\hat{y})\cdot\epsilon e^{\mathrm{i}\theta^l}\hat{y}| \\
&\le 4A_G\max\{r_1^k,r_2^k,r_1^l,r_2^l\}\epsilon,
\end{aligned}
$$

from which the lemma follows. □

The following two lemmas can be proved by direct computation.

Lemma 2.2 *Let $\mathbf{q}^\varepsilon(t)$ be a deformation of a curve $\mathbf{q}(t)$ with $\mathbf{q}^0=\mathbf{q}$. Then*

$$
\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\int_{-1}^{1}|(\mathbf{q}^\varepsilon)'|\,dt=\mathbf{T}\cdot\mathbf{X}\Big|_{-1}^{1}-\int_{-1}^{1}\kappa\mathbf{N}\cdot\mathbf{X}\,ds.
$$

Here $\int_{-1}^{1}|(\mathbf{q}^\varepsilon)'|\,dt$ is the length of $\mathbf{q}^\varepsilon$, $\mathbf{T}=\frac{\mathbf{q}'}{|\mathbf{q}'|}$, $\mathbf{N}$ is a unit normal vector, $\kappa\mathbf{N}$ is the curvature vector, and $\mathbf{X}(t)=\frac{\partial\mathbf{q}^\varepsilon(t)}{\partial\varepsilon}|_{\varepsilon=0}$ is the infinitesimal element of $\mathbf{q}^\varepsilon$

Lemma 2.3 *Suppose that a bounded domain U is enclosed by a curve ∂U, and U^ε is a deformation of U. Let $\mathbf{X}$ be the infinitesimal element of the deformation of ∂U. Then*

$$
\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\int_{U^\varepsilon}f(x)\,dx=-\int_{\partial U}f(x)\mathbf{N}\cdot\mathbf{X}\,ds
$$

where $\mathbf{N}$ is the inward unit normal vector on ∂U.

Denote a perturbed double bubble assembly by

$$
\Omega=((\Omega_1^1,\Omega_2^1),(\Omega_1^2,\Omega_2^2),...,(\Omega_1^n,\Omega_2^n)); \tag{2.26}
$$

namely for each k, (Ω_1^k,Ω_2^k) forms a perturbed double bubble, which is enclosed by three curves $\mathbf{r}_1^k$, $\mathbf{r}_2^k$, and $\mathbf{r}_0^k$. More precisely $\mathbf{r}_i^k$ parametrizes $\partial\Omega_1^k\backslash\partial\Omega_2^k$, $\partial\Omega_2^k\backslash\partial\Omega_1^k$, and $\partial\Omega_1^k\cap\partial\Omega_2^k$ for $i=1,2,0$ respectively. Here Ω_1^k and Ω_2^k are disjoint, share part of their boundaries, and have two triple points. Later we will consider perturbed assemblies with more specific properties.

The two triple points of Ω^k are

$$
\mathbf{r}_1^k(1)=\mathbf{r}_2^k(1)=\mathbf{r}_0^k(1)\ \text{ and }\ \mathbf{r}_1^k(-1)=\mathbf{r}_2^k(-1)=\mathbf{r}_0^k(-1). \tag{2.27}
$$

The unit tangent vectors of $\mathbf{r}_1^k$, $\mathbf{r}_2^k$, and $\mathbf{r}_0^k$ are denoted $\mathbf{T}_1^k$, $\mathbf{T}_2^k$, and $\mathbf{T}_0^k$ and given by

$$
\mathbf{T}_i^k(t)=\frac{(\mathbf{r}_i^k)'(t)}{|(\mathbf{r}_i^k)'(t)|}. \tag{2.28}
$$

The unit normal vectors to $\mathbf{r}_1^k$, $\mathbf{r}_2^k$, and $\mathbf{r}_0^k$ are $\mathbf{N}_1^k$, $\mathbf{N}_2^k$, and $\mathbf{N}_0^k$ respectively. We adopt the following direction convention: $\mathbf{N}_1^k$ points inward with respect to Ω_1^k, $\mathbf{N}_2^k$ points inward with respect to Ω_2^k, and $\mathbf{N}_0^k$ points from Ω_2^k towards Ω_1^k, i.e. inward with respect to Ω_1^k and outward with respect to Ω_2^k. The curvature of $\mathbf{r}_i^k$ is denoted κ_i^k. Here $\mathbf{N}_i^k$ and κ_i^k conform to the sign convention so that $\kappa_i^k\mathbf{N}_i^k$ is the (orientation independent) curvature vector. Under this sign convention

$$
\frac{d\mathbf{T}_i^k}{ds}=k_i^k\mathbf{N}_i^k \tag{2.29}
$$

where $ds=|(\mathbf{r}_i^k)'(t)|dt$ is the length element. One sets

$$
\Omega_1=\cup_{k=1}^n\Omega_1^k\ \text{ and }\ \Omega_2=\cup_{k=1}^n\Omega_2^k. \tag{2.30}
$$

A deformation of the assembly (2.32) is a family of assemblies

$$\Omega^\varepsilon = ((\Omega_1^{\varepsilon,1}, \Omega_2^{\varepsilon,1}), (\Omega_1^{\varepsilon,2}, \Omega_2^{\varepsilon,2}), ..., (\Omega_1^{\varepsilon,n}, \Omega_2^{\varepsilon,n})) \tag{2.31}$$

for ε in a neighborhood of 0, where for each k, $(\Omega_1^{\varepsilon,k}, \Omega_2^{\varepsilon,k})$ forms a deformation of the perturbed double bubble (Ω_1^k, Ω_2^k). For each k the three curves $\partial\Omega_1^{\varepsilon,k}\backslash\partial\Omega_2^{\varepsilon,k}$, $\partial\Omega_2^{\varepsilon,k}\backslash\partial\Omega_1^{\varepsilon,k}$, and $\partial\Omega_1^{\varepsilon,k} \cup \partial\Omega_2^{\varepsilon,k}$ that enclose $(\Omega_1^{\varepsilon,k}, \Omega_2^{\varepsilon,k})$ are parametrized respectively by $\mathbf{r}_1^{\varepsilon,k}$, $\mathbf{r}_2^{\varepsilon,k}$, and $\mathbf{r}_0^{\varepsilon,k}$ respectively. At $\varepsilon = 0$, $\mathbf{r}_i^{0,k} = \mathbf{r}_i^k$. Again one writes

$$\Omega_1^\varepsilon = \cup_{k=1}^n \Omega_1^{\varepsilon,k} \ \text{ and } \ \Omega_2^\varepsilon = \cup_{k=1}^n \Omega_2^{\varepsilon,k}. \tag{2.32}$$

Define

$$\mathbf{X}_i^k(t) = \frac{\partial \mathbf{r}_i^{\varepsilon,k}(t)}{\partial\varepsilon}\Big|_{\varepsilon=0} \tag{2.33}$$

which is the infinitesimal element of the deformation.

Lemma 2.4 *Let Ω^ε be a deformation of a perturbed double bubble assembly Ω as described above. Then*

$$\begin{aligned}
\frac{d\mathcal{J}_s(\Omega^\varepsilon)}{d\varepsilon}\Big|_{\varepsilon=0} &= \sum_{k=1}^n \Big(\sum_{j=0}^2 (\mathbf{T}_i^k) \cdot \mathbf{X}^k \Big|_{-1}^1 \Big) - \sum_{k=1}^n \int_{\partial\Omega_1^k\backslash\partial\Omega_2^k} \kappa_1^k \mathbf{N}_1^k \cdot \mathbf{X}_1^k \, ds - \sum_{k=1}^n \int_{\partial\Omega_2^k\backslash\partial\Omega_1^k} \kappa_2^k \mathbf{N}_2^k \cdot \mathbf{X}_2^k \, ds \\
&\quad - \sum_{k=1}^n \int_{\partial\Omega_1^k\cap\partial\Omega_2^k} \kappa_0^k \mathbf{N}_0^k \cdot \mathbf{X}_0^k \, ds \qquad (2.34)\\
\frac{d\mathcal{J}_l(\Omega^\varepsilon)}{d\varepsilon}\Big|_{\varepsilon=0} &= -\sum_{k=1}^n \int_{\partial\Omega_1^k\backslash\partial\Omega_2^k} (\gamma_{11} I_{\Omega_1} + \gamma_{12} I_{\Omega_2}) \mathbf{N}_1^k \cdot \mathbf{X}_1^k \, ds - \sum_{k=1}^n \int_{\partial\Omega_2^k\backslash\partial\Omega_1^k} (\gamma_{12} I_{\Omega_1} + \gamma_{22} I_{\Omega_2}) \mathbf{N}_2^k \cdot \mathbf{X}_2^k \, ds \\
&\quad - \sum_{k=1}^n \int_{\partial\Omega_1^k\cap\partial\Omega_2^k} ((\gamma_{11} - \gamma_{12}) I_{\Omega_1} + (\gamma_{12} - \gamma_{22}) I_{\Omega_2}) \mathbf{N}_0^k \cdot \mathbf{X}_0^k \, ds \qquad (2.35)\\
\frac{d|\Omega_1^\varepsilon|}{d\varepsilon}\Big|_{\varepsilon=0} &= -\sum_{k=1}^n \int_{\partial\Omega_1^k\backslash\partial\Omega_2^k} \mathbf{N}_1^k \cdot \mathbf{X}_1^k \, ds - \sum_{k=1}^n \int_{\partial\Omega_1^k\cap\partial\Omega_2^k} \mathbf{N}_0^k \cdot \mathbf{X}_0^k \, ds \qquad (2.36)\\
\frac{d|\Omega_2^\varepsilon|}{d\varepsilon}\Big|_{\varepsilon=0} &= -\sum_{k=1}^n \int_{\partial\Omega_2^k\backslash\partial\Omega_1^k} \mathbf{N}_2^k \cdot \mathbf{X}_2^k \, ds + \sum_{k=1}^n \int_{\partial\Omega_1^k\cap\partial\Omega_2^k} \mathbf{N}_0^k \cdot \mathbf{X}_0^k \, ds. \qquad (2.37)
\end{aligned}$$

In (2.34) $\mathbf{X}^k$ denotes the $\mathbf{X}_i^k$'s at the triple points. Since (2.27) holds for $\mathbf{r}_i^{\varepsilon,k}$, $\mathbf{X}_1^k(-1) = \mathbf{X}_2^k(-1) = \mathbf{X}_0^k(-1)$ and $\mathbf{X}_1^k(1) = \mathbf{X}_2^k(1) = \mathbf{X}_0^k(1)$. Therefore one can drop the subscript i in $\mathbf{X}_i^k(\pm 1)$.

Proof. . The first formula (2.34) follows directly from Lemma 2.2.

To show (2.35), recall I_{Ω_i} from (1.6) which can be written as

$$I_{\Omega_i}(x) = \int_{\Omega_i} G(x,y)\, dy, \quad i = 1, 2, \tag{2.38}$$

in terms of the Green's function. Then the product rule of differentiation implies that

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_{\Omega_i^\varepsilon} \int_{\Omega_j^\varepsilon} G(x,y)\, dx\, dy = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_{\Omega_i^\varepsilon} I_{\Omega_j}(x)\, dx + \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_{\Omega_j^\varepsilon} I_{\Omega_i}(x)\, dx. \tag{2.39}$$

However, Lemma 2.3 shows

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_{\Omega_i^\varepsilon} I_{\Omega_j}(x)\, dx = \begin{cases} -\sum_{k=1}^n \int_{\partial\Omega_1^k\backslash\partial\Omega_2^k} I_{\Omega_j} \mathbf{N}_1^k \cdot \mathbf{X}_1^k \, ds - \sum_{k=1}^n \int_{\partial\Omega_1^k\cap\partial\Omega_2^k} I_{\Omega_j} \mathbf{N}_0^k \cdot \mathbf{X}_0^k \, ds, & i = 1 \\ -\sum_{k=1}^n \int_{\partial\Omega_2^k\backslash\partial\Omega_1^k} I_{\Omega_j} \mathbf{N}_2^k \cdot \mathbf{X}_2^k \, ds + \sum_{k=1}^n \int_{\partial\Omega_1^k\cap\partial\Omega_2^k} I_{\Omega_j} \mathbf{N}_0^k \cdot \mathbf{X}_0^k \, ds, & i = 2 \end{cases}. \tag{2.40}$$

Therefore

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_{\Omega_i^\varepsilon} \int_{\Omega_j^\varepsilon} G(x,y)\, dx\, dy \tag{2.41}$$

$$= \begin{cases} -2\sum_{k=1}^n \int_{\partial\Omega_1^k \setminus \partial\Omega_2^k} I_{\Omega_1} \mathbf{N}_1^k \cdot \mathbf{X}_1^k\, ds - 2\sum_{k=1}^n \int_{\partial\Omega_1^k \cap \partial\Omega_2^k} I_{\Omega_1} \mathbf{N}_0^k \cdot \mathbf{X}_0^k\, ds, & i=j=1 \\ -2\sum_{k=1}^n \int_{\partial\Omega_2^k \setminus \partial\Omega_1^k} I_{\Omega_2} \mathbf{N}_2^k \cdot \mathbf{X}_2^k\, ds + 2\sum_{k=1}^n \int_{\partial\Omega_1^k \cap \partial\Omega_2^k} I_{\Omega_2} \mathbf{N}_0 \cdot \mathbf{X}_0\, ds, & i=j=2 \\ -\sum_{k=1}^n \int_{\partial\Omega_1^k \setminus \partial\Omega_2^k} I_{\Omega_2} \mathbf{N}_1^k \cdot \mathbf{X}_1^k\, ds - \sum_{k=1}^n \int_{\partial\Omega_2^k \setminus \partial\Omega_1^k} I_{\Omega_1} \mathbf{N}_2^k \cdot \mathbf{X}_2^k\, ds & \\ \qquad -\sum_{k=1}^n \int_{\partial\Omega_1^k \cap \partial\Omega_2^k} (I_{\Omega_2} - I_{\Omega_1}) \mathbf{N}_0^k \cdot \mathbf{X}_0^k\, ds, & i=1, j=2 \end{cases}.$$

Hence,

$$\begin{aligned} \frac{d\mathcal{J}_l(\Omega^\varepsilon)}{d\varepsilon}\Big|_{\varepsilon=0} &= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \sum_{i,j=1}^2 \frac{\gamma_{ij}}{2} \int_{\Omega_i^\varepsilon} \int_{\Omega_j^\varepsilon} G(x,y)\, dx\, dy \\ &= -\sum_{k=1}^n \int_{\partial\Omega_1^k \setminus \partial\Omega_2^k} (\gamma_{11} I_{\Omega_1} + \gamma_{12} I_{\Omega_2}) \mathbf{N}_1^k \cdot \mathbf{X}_1^k\, ds - \sum_{k=1}^n \int_{\partial\Omega_2^k \setminus \partial\Omega_1^k} (\gamma_{12} I_{\Omega_1} + \gamma_{22} I_{\Omega_2}) \mathbf{N}_2^k \cdot \mathbf{X}_2^k\, ds \\ &\quad -\sum_{k=1}^n \int_{\partial\Omega_1^k \cap \partial\Omega_2^k} [(\gamma_{11} - \gamma_{12}) I_{\Omega_1} + (\gamma_{12} - \gamma_{22}) I_{\Omega_2}] \mathbf{N}_0^k \cdot \mathbf{X}_0^k\, ds. \end{aligned} \tag{2.42}$$

This proves (2.35).

The formulas (2.36) and (2.37) follow from Lemma 2.3 with $f(x) = 1$. $\square$

We perform a special type of perturbation to each exact double bubble B^k in the assembly $T(B)$ in two steps.

In the first step, move the two triple points $(0, h^k)$ and $(0, -h^k)$ vertically to $(0, \eta^k)$ and $(0, -\eta^k)$ respectively. In the asymmetric case the three circular arcs are perturbed to three new circular arcs whose radii are ρ_1^k, ρ_2^k, and ρ_0^k; the angles a_i^k are perturbed to α_i^k accordingly; see Figure 6. The ρ_i^k's and the α_i^k's are determined from η^k implicitly by solving the following system of equations

$$(\rho_1^k)^2(\alpha_1^k - \cos\alpha_1^k \sin\alpha_1^k) + (\rho_0^k)^2(\alpha_0^k - \cos\alpha_0^k \sin\alpha_0^k) = w_1^k \tag{2.43}$$
$$(\rho_2^k)^2(\alpha_2^k - \cos\alpha_2^k \sin\alpha_2^k) - (\rho_0^k)^2(\alpha_0^k - \cos\alpha_0^k \sin\alpha_0^k) = w_2^k \tag{2.44}$$
$$\rho_i^k \sin\alpha_i^k = \eta^k, \quad i = 1, 2, 0 \tag{2.45}$$
$$(\rho_1^k)^{-1} - (\rho_2^k)^{-1} = (\rho_0^k)^{-1}. \tag{2.46}$$

The regions bounded by the new arcs still have the areas w_1^k and w_2^k; hence the equations (2.43) and (2.44). The centers of the new arcs are denoted $(\beta_i^k, 0)$, $i = 1, 2, 0$.

In the symmetric case, the first step of perturbation turns the middle line segment connecting $(0, h^k)$ to $(0, -h^k)$ to the line segment connecting $(0, \eta^k)$ and $(0, -\eta^k)$. The left and right arcs become arcs of radius $\rho_1^k = \rho_2^k$, and the angles become $\alpha_1^k = \alpha_2^k$. They satisfy the equations

$$(\rho_i^k)^2(\alpha_i^k - \cos\alpha_i^k \sin\alpha_i^k) = w_i^k, \quad i = 1, 2 \tag{2.47}$$
$$\rho_i^k \sin\alpha_i^k = \eta^k, \quad i = 1, 2 \tag{2.48}$$

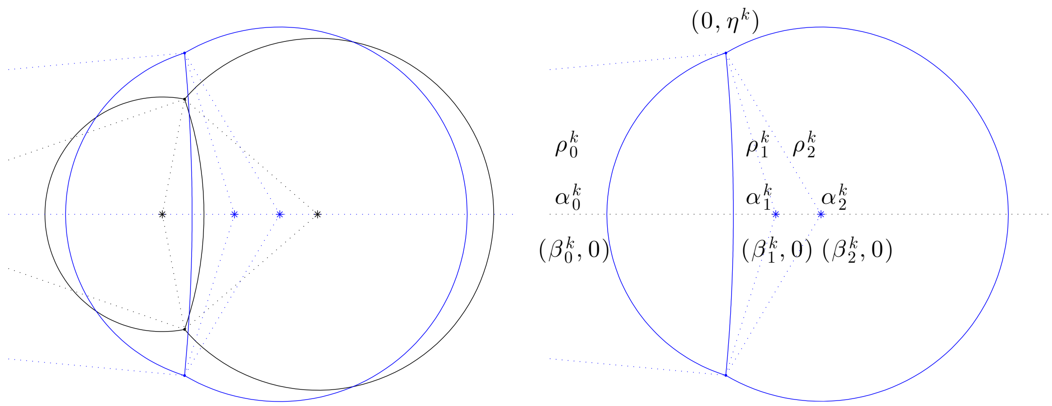

Figure 6: First step of perturbation in the asymmetric case. Left: the exact double bubble is perturbed to a pair of two sets bounded by three circular arcs governed by (2.43) - (2.46). Right: the same perturbed pair without the exact double bubble. Also showing are the angles α_i^k, the radii ρ_i^k, the centers $(\beta_i^k, 0)$, and one triple point $(0, \eta^k)$.

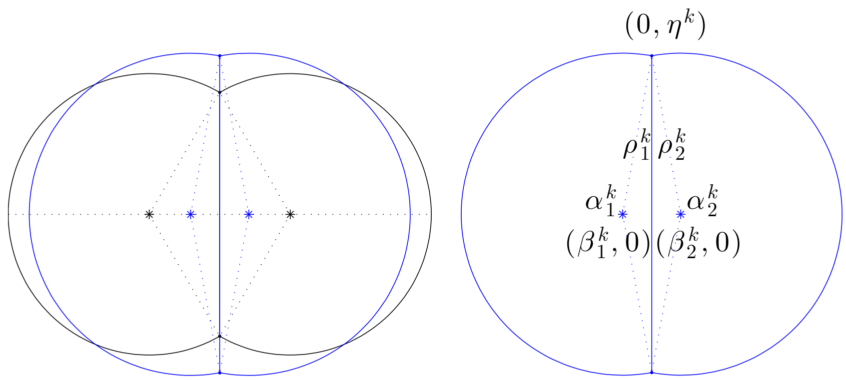

Figure 7: First step of perturbation in the symmetric case.

where $w_1^k = w_2^k$. These equations determine ρ_i^k and α_i^k in terms of η^k.

This step of perturbation is explained in more detail and shown to be well defined when η^k is close to h^k in Appendix B.

In the second step of perturbation we further perturb the shape of the circular arcs. Introduce $3n$ functions $u_i^k(t)$, $k = 1, 2, ..., n$, $i = 1, 2, 0$, for $t \in (-1, 1)$. In the asymmetric case the circular arcs are replaced by curves parametrized by

$$\hat{\mathbf{r}}_1^k(t) = u_1^k(t) e^{\mathrm{i}(\pi - \alpha_1^k t)} + \beta_1^k, \quad \hat{\mathbf{r}}_2^k(t) = u_2^k(t) e^{\mathrm{i}\alpha_2^k t} + \beta_2^k, \quad \hat{\mathbf{r}}_0^k(t) = u_0^k(t) e^{\mathrm{i}\alpha_0^k t} + \beta_0^k, \ k = 1, 2, ..., n; \tag{2.49}$$

see Figure 8. The two triple points correspond to $t = -1$ and $t = 1$, namely

$$\hat{\mathbf{r}}_1^k(-1) = \hat{\mathbf{r}}_2^k(-1) = \hat{\mathbf{r}}_0^k(-1) = -\eta^k \mathrm{i} \ \text{ and } \ \hat{\mathbf{r}}_1^k(1) = \hat{\mathbf{r}}_2^k(1) = \hat{\mathbf{r}}_0^k(1) = \eta^k \mathrm{i}. \tag{2.50}$$

They remain unchanged in this step, so

$$u_i^k(\pm 1) = \rho_i^k. \tag{2.51}$$

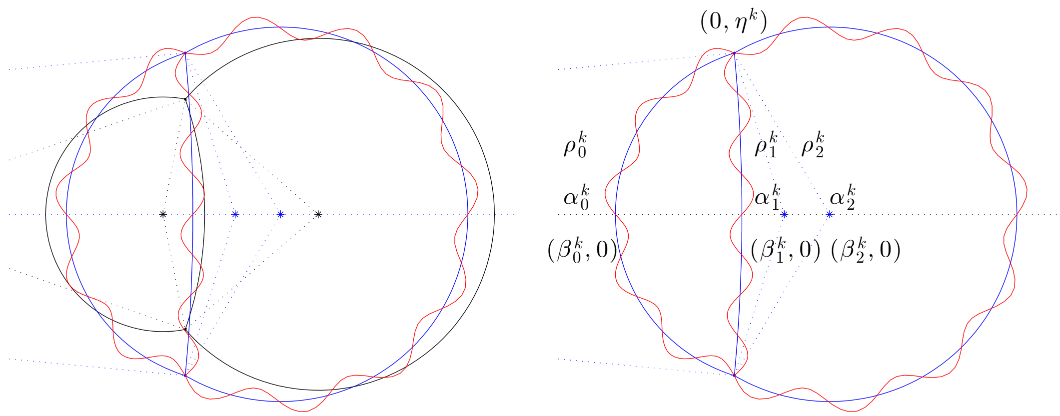

Figure 8: Second step of perturbation in the asymmetric case. Left: The circular arcs obtained in the first step are perturbed to more general curves. Right: the same perturbed double bubble without the exact double bubble showing.

Note that a sector perturbed by u_i^k has the area $\int_{-1}^{1} \frac{\alpha_i^k (u_i^k(t))^2}{2}\, dt$. Since the areas of the newly perturbed regions must still be w_1^k and w_2^k, one requires that

$$\int_{-1}^{1} \frac{\alpha_1^k (u_1^k(t))^2 - (\rho_1^k)^2 \cos\alpha_1^k \sin\alpha_1^k}{2}\, dt + \int_{-1}^{1} \frac{\alpha_0^k (u_0^k(t))^2 - (\rho_0^k)^2 \cos\alpha_0^k \sin\alpha_0^k}{2}\, dt = w_1^k \tag{2.52}$$

$$\int_{-1}^{1} \frac{\alpha_2^k (u_2^k(t))^2 - (\rho_2^k)^2 \cos\alpha_2^k \sin\alpha_2^k}{2}\, dt - \int_{-1}^{1} \frac{\alpha_0^k (u_0^k(t))^2 - (\rho_0^k)^2 \cos\alpha_0^k \sin\alpha_0^k}{2}\, dt = w_2^k. \tag{2.53}$$

In the symmetric case, the middle curve is parametrized differently by

$$\hat{\mathbf{r}}_0^k = u_0^k(t) + \eta^k t \mathrm{i}, \quad \text{where} \quad u_0^k(\pm 1) = 0. \tag{2.54}$$

The constraints (2.52) and (2.53) become

$$\int_{-1}^{1} \frac{\alpha_1^k (u_1^k(t))^2 - (\rho_1^k)^2 \cos\alpha_1^k \sin\alpha_1^k}{2}\, dt + \int_{-1}^{1} \eta^k u_0^k(t)\, dt = w_1^k \tag{2.55}$$

$$\int_{-1}^{1} \frac{\alpha_2^k (u_2^k(t))^2 - (\rho_2^k)^2 \cos\alpha_2^k \sin\alpha_2^k}{2}\, dt - \int_{-1}^{1} \eta^k u_0^k(t)\, dt = w_2^k \tag{2.56}$$

where $\rho_1^k = \rho_2^k$, $\alpha_1^k = \alpha_2^k$, and $w_1^k = w_2^k$.

This perturbed double bubble is denoted $P^k = (P_1^k, P_2^k)$. Its image under $T_{\epsilon,\xi^k,\theta^k}$ is denoted by $T^k(P^k) = (T^k(P_1^k), T^k(P_2^k))$. Collectively one writes $T(P) = (T^1(P^1), ..., T^n(P^n))$ which is an assembly of perturbed double bubbles. Moreover $T(P_i) = \cup_{k=1}^n T^k(P_i^k)$ for $i = 1, 2$. The boundaries of $T^k(P^k)$ are parametrized by

$$\mathbf{r}_i^k(t) = T^k(\hat{\mathbf{r}}_i^k(t)). \tag{2.57}$$

Although the u_i^k's describe the shape of the perturbed double bubble well, the constraints (2.53) are nonlinear and hard to work with. We introduce new variables ϕ_i^k, $k = 1, 2, ..., n$, $i = 1, 2, 0$, in place of u_i^k. In the asymmetric case they are given by

$$\phi_i^k(t) = \frac{\alpha_i^k (u_i^k(t))^2 - \alpha_i^k (\rho_i^k)^2}{2}, \ i = 1, 2, 0. \tag{2.58}$$

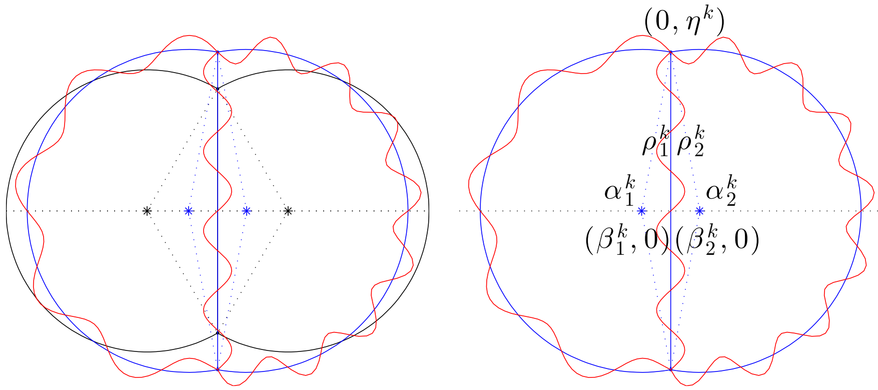

Figure 9: Second step of perturbation in the symmetric case.

In the symmetric case ϕ_0^k is given differently by

$$\phi_0^k(t) = \eta^k u_0^k(t). \tag{2.59}$$

Write ϕ^k for $(\phi_1^k, \phi_2^k, \phi_0^k)$. P^k now depends on (ϕ^k, η^k) in addition to w_1^k and w_2^k, and the scaled down version $T^k(P^k)$ depends on ϵ, ξ^k, θ^k, w_1^k, w_2^k, as well as (ϕ^k, η^k). We call ϕ_i^k and η^k internal variables.

The assembly of perturbed double bubbles $T(P)$ corresponds to the internal variable representation $((\phi^1, \eta^1), ..., (\phi^n, \eta^n))$, which is is simply written as (ϕ, η). By itself ϕ stands for $(\phi^1, \phi^2, ..., \phi^n)$ where $\phi^k = (\phi_1^k, \phi_2^k, \phi_0^k)$, $k = 1, 2, ..., n$; similarly $\eta = (\eta^1, \eta^2, ..., \eta^n)$. The exact double bubble assembly corresponds to $((0, h^1), ..., (0, h^n)) = (0, h)$.

Because ρ_i^k and α_i^k satisfy the conditions (2.43) and (2.44), the area constraints (2.52) and (2.53) become linear constraints

$$\int_{-1}^{1} \phi_1^k(t)\, dt + \int_{-1}^{1} \phi_0^k(t)\, dt = 0 \quad \text{and} \quad \int_{-1}^{1} \phi_2^k(t)\, dt - \int_{-1}^{1} \phi_0^k(t)\, dt = 0 \tag{2.60}$$

on the ϕ_i^k's. The ϕ_i^k's also satisfy the boundary condition

$$\phi_i^k(\pm 1) = 0, \quad k = 1, 2, ..., n, \ i = 1, 2, 0 \tag{2.61}$$

because of (2.51) and (2.54).

In summary each $w \in \overline{W}$ determines n double bubbles B^k for $k = 1, 2, ..., n$. Each $\epsilon < \delta$, $\xi \in \overline{\Xi_{\overline{\delta}}}$ and $\theta \in \mathbb{S}^n$ specify transformations T^k that map the double bubbles B^k to $T^k(B^k)$ inside D to form an exact double bubble assembly $T(B)$. These ϵ, ξ, θ, and w also define a restricted class of perturbed double bubble assemblies. Within this class, each perturbed double bubble assembly $T(P)$ is described by (ϕ, η). The exact double bubble assembly $T(B)$ is in this class and is represented by $(0, h)$, i.e. $\phi_i^k = 0$ and $\eta^k = h^k$ for all $k = 1, 2, ..., n$ and $i = 1, 2, 0$. Other (ϕ, η)'s represent perturbed double bubble assemblies.

For a perturbed double bubble assembly $T(P)$

$$\mathcal{P}_D(T(P_i)) = \sum_{k=1}^{n} \mathcal{P}_D(T^k(P_i^k)),$$

and since each $T^k(P_i^k)$ is bounded by smooth curves $\mathbf{r}_i^k$ and $\mathbf{r}_0^k$, the perimeter $\mathcal{P}_D(T^k(P_i^k))$ is the length of $\mathbf{r}_i^k$ plus the length of $\mathbf{r}_0^k$. The length of $\mathbf{r}_i^k$ is ϵ times the length of $\hat{\mathbf{r}}_i^k$ which is

$$\int_{-1}^{1} \sqrt{((u_i^k)'(t))^2 + (\alpha_i^k)^2 (u_i^k(t))^2}\, dt, \quad i = 1, 2, 0, \tag{2.62}$$

in terms of the variable u_i^k in the asymmetric case. In the symmetric case the length of $\hat{\mathbf{r}}_0^k$ is

$$\int_{-1}^{1} \sqrt{((u_0^k)'(t))^2 + (\eta^k)^2}\, dt. \tag{2.63}$$

In terms of ϕ_i^k (2.62) becomes

$$\int_{-1}^{1} L_i^k((\phi_i^k)', \phi_i^k, \eta^k)\, dt, \quad \text{where} \quad L_i^k((\phi_i^k)', \phi_i^k, \eta^k) = \sqrt{\frac{((\phi_i^k)')^2}{\alpha_i^k(2\phi_i^k + \alpha_i^k(\rho_i^k)^2)} + \alpha_i^k(2\phi_i^k + \alpha_i^k(\rho_i^k)^2)}, \tag{2.64}$$

and (2.63) becomes

$$\int_{-1}^{1} L_0^k((\phi_0^k)', \eta^k)\, dt, \quad \text{where} \quad L_0^k((\phi_0^k)', \eta^k) = \sqrt{\frac{((\phi_0^k)')^2}{(\eta^k)^2} + (\eta^k)^2}. \tag{2.65}$$

By (2.64) and (2.23) the energy of $T(P)$ can be written as

$$\mathcal{J}(T(P)) = \epsilon \sum_{k=1}^{n} \sum_{i=0}^{2} \int_{-1}^{1} L_i^k((\phi_i^k)', \phi_i^k, \eta^k)\, dt + \sum_{k,l=1}^{n} \sum_{i=1}^{2} \frac{\gamma_{ij}}{2} \int_{T^k(P_i^k)} \int_{T^l(P_j^l)} G(x,y)\, dx\, dy. \tag{2.66}$$

To specify the domain of the functional $\mathcal{J}$ in the restricted class of perturbed double bubble assemblies, let

$$\begin{aligned} \mathcal{Y} \;=\; & \{(\phi, \eta) = ((\phi^1, \eta^1), (\phi^2, \eta^2), ..., (\phi^n, \eta^n)) \in (H_0^1((-1,1); \mathbb{R}^3) \times \mathbb{R})^n : \\ & \int_{-1}^{1} (\phi_1^k(t) + \phi_0^k(t))\, dt = \int_{-1}^{1} (\phi_2^k(t) - \phi_0^k(t))\, dt = 0, \;\; k = 1, 2, ..., n\}. \end{aligned} \tag{2.67}$$

This space is equipped with a norm $\|\cdot\|_{\mathcal{Y}}$ derived from the usual H^1 norm; see (3.27).

The functional $\mathcal{J}$ is defined on a neighborhood of $(0, h) \in \mathcal{Y}$; namely there exists $\bar{c} > 0$ such that the domain of $\mathcal{J}$ is the open ball of radius $\bar{c}$ centered at $(0, h)$ in $\mathcal{Y}$:

$$\mathcal{D}(\mathcal{J}) = \{(\phi, \eta) \in \mathcal{Y} : \; \|(\phi, \eta) - (0, h))\|_{\mathcal{Y}} < \bar{c}\}. \tag{2.68}$$

Recall the remark after (2.18) and (2.19) which states for all $\epsilon < \delta$, $\xi \in \overline{\Xi_{\overline{\delta}}}$, $\theta \in \mathbb{S}^n$, and $w \in \overline{W}$, the exactly double bubble assembly $T(B)$ determined by ϵ, ξ, θ and w has the property that $z = (z^1, z^2, ..., z^n) \in \Xi_{\overline{\delta}/2}$ if $z^k \in T^k(B^k)$ for $k = 1, 2, ..., n$. Choose $\bar{c}$ sufficiently small so that for all $\epsilon < \delta$, all $(\xi, \theta, w) \in \overline{\Xi_{\overline{\delta}}} \times \mathbb{S}^n \times \overline{W}$, and all $(\phi, \eta) \in D(\mathcal{J})$, the perturbed double bubble assembly $T(P)$ specified by ϵ, (ξ, θ, w), and (ϕ, η) has the property that $z = (z^1, z^2, ..., z^n) \in \Xi_{\overline{\delta}/4}$ if $z^k \in T^k(P^k)$. Hence the perturbed double bubbles $T^k(P^k)$ in $T(P)$ do not intersect, and all stay in D and away from ∂D.

3 The first variation

Since a perturbed double bubble P^k is described by internal variables ϕ_i^k and η^k, there is an easy way to generate deformations $P^{\varepsilon,k}$. Start with a deformation of $(\phi, \eta) = ((\phi^1, \eta^1), ..., (\phi^n, \eta^n)) \in \mathcal{D}(\mathcal{J})$ in the form:

$$\phi_i^k \to \phi_i^k + \varepsilon \psi_i^k, \quad \eta^k \to \eta^k + \varepsilon \zeta^k, \quad k = 1, 2, ..., n, \; i = 1, 2, 0 \tag{3.1}$$

for $(\psi, \zeta) = ((\psi^1, \zeta^1), ..., (\psi^n, \zeta^n)) \in \mathcal{Y}$. Then in the asymmetric case (2.58) defines a deformation of u_i^k denoted by $u_i^{\varepsilon,k}$ (with $(u_i^k)^0$ being u_i^k), namely by

$$\phi_i^k + \varepsilon \psi_i^k = \frac{\alpha_i^k(\eta^k + \varepsilon\zeta^k)(u_i^{\varepsilon,k})^2 - \alpha_i^k(\eta^k + \varepsilon\zeta^k)(\rho_i^k(\eta^k + \varepsilon\zeta^k))^2}{2}. \tag{3.2}$$

Here α_i^k and ρ_i^k are treated as functions of η^k, and $\alpha_i^k(\eta^k+\varepsilon\zeta^k)$ and $\rho_i^k(\eta^k+\varepsilon\zeta^k)$ are these functions evaluated at $\eta^k+\varepsilon\zeta^k$. Differentiating (3.2) with respect to ε and setting ε to be 0 yield

$$\psi_i^k = \alpha_i^k u_i^k \frac{\partial u_i^{\varepsilon,k}}{\partial \varepsilon}\Big|_{\varepsilon=0} + \frac{(\alpha_i^k)'\zeta^k (u_i^k)^2}{2} - \frac{(\alpha_i^k)'\zeta^k(\rho_i^k)^2}{2} - \alpha_i^k\rho_i^k(\rho_i^k)'\zeta^k. \tag{3.3}$$

Note that since α_i^k, ρ_i^k, and β_i^k depend on η^k,

$$\frac{d\alpha_i^k(\eta^k+\varepsilon\zeta^k)}{d\varepsilon}\Big|_{\varepsilon=0} = (\alpha_i^k)'\zeta^k, \quad \frac{d\rho_i^k(\eta^k+\varepsilon\zeta^k)}{d\varepsilon}\Big|_{\varepsilon=0} = (\rho_i^k)'\zeta^k, \quad \frac{d\beta_i^k(\eta^k+\varepsilon\zeta^k)}{d\varepsilon}\Big|_{\varepsilon=0} = (\beta_i^k)'\zeta^k. \tag{3.4}$$

In (3.3) and (3.4) α_i^k, $(\alpha_i^k)'$, ρ_i^k, $(\rho_i^k)'$ are all functions of η^k and are all evaluated at η^k. In the symmetric case (3.3) becomes

$$\psi_0^k = \eta^k \frac{\partial u_0^{\varepsilon,k}}{\partial\varepsilon}\Big|_{\varepsilon=0} + \zeta^k u_0^k. \tag{3.5}$$

Recall $\mathbf{X}_i^k$ from (2.33), so here in the asymmetric case

$$\mathbf{X}_i^k = \begin{cases} \epsilon e^{\mathrm{i}\theta^k}\Big(\frac{\partial u_1^{\varepsilon,k}}{\partial\varepsilon}\big|_{\varepsilon=0} e^{\mathrm{i}(\pi-\alpha_1^k t)} + u_1^k(\alpha_1^k)'\zeta^k t e^{\mathrm{i}(\pi-\alpha_1^k t)}(-\mathrm{i}) + (\beta_1^k)'\zeta^k\Big) & \text{if} \quad i=1 \\ \epsilon e^{\mathrm{i}\theta^k}\Big(\frac{\partial u_i^{\varepsilon,k}}{\partial\varepsilon}\big|_{\varepsilon=0} e^{\mathrm{i}\alpha_i^k t} + u_i^k(\alpha_i^k)'\zeta^k t e^{\mathrm{i}\alpha_i^k t}\mathrm{i} + (\beta_i^k)'\zeta^k\Big) & \text{if} \quad i=2,0 \end{cases}, \tag{3.6}$$

and in the symmetric case

$$\mathbf{X}_0^k = \epsilon e^{\mathrm{i}\theta^k}\Big(\frac{\partial u_0^{\varepsilon,k}}{\partial\varepsilon}\big|_{\varepsilon=0} + \zeta^k t\mathrm{i}\Big). \tag{3.7}$$

Lemma 3.1 *At the triple points* $\mathbf{X}_i^k(\pm1) = \zeta^k\,\mathbf{X}^{S,k}(\pm1)$ *where* $\mathbf{X}^{S,k}(\pm1) = \pm\epsilon e^{\mathrm{i}\theta^k}\mathrm{i}$.

The superscript S here stands for "stretching".
Proof. In the asymmetric case, by (3.3), since $\psi_i^k(\pm1)=0$, $u_i^k(\pm1)=\rho_i^k$,

$$\frac{\partial u_i^{\varepsilon,k}(\pm1)}{\partial\varepsilon}\Big|_{\varepsilon=0} = (\rho_i^k)'\zeta^k, \quad i=1,2,0,$$

and hence

$$\begin{aligned} \mathbf{X}_i^k(\pm1) &= \begin{cases} \epsilon e^{\mathrm{i}\theta^k}((\rho_1^k)'\zeta^k e^{\mathrm{i}(\pi\mp\alpha_1^k)} \pm \rho_1^k(\alpha_1^k)'\zeta^k e^{\mathrm{i}(\pi\mp\alpha_1^k)}(-\mathrm{i}) + (\beta_1^k)'\zeta^k) & \text{if} \quad i=1 \\ \epsilon e^{\mathrm{i}\theta^k}((\rho_i^k)'\zeta^k e^{\pm\mathrm{i}\alpha_i^k} \pm \rho_i^k(\alpha_i^k)'\zeta^k e^{\pm\mathrm{i}\alpha_i^k}\mathrm{i} + (\beta_i^k)'\zeta^k) & \text{if} \quad i=2,0 \end{cases} \\ &= \begin{cases} \zeta^k\epsilon e^{\mathrm{i}\theta^k}\frac{d(\rho_1^k e^{\mathrm{i}(\pi\mp\alpha_1^k)}+\beta_1^k)}{d\eta^k} & \text{if} \quad i=1 \\ \zeta^k\epsilon e^{\mathrm{i}\theta^k}\frac{d(\rho_i^k e^{\pm\mathrm{i}\alpha_i^k}+\beta_i^k)}{d\eta^k} & \text{if} \quad i=2,0 \end{cases} \\ &= \zeta^k\epsilon e^{\mathrm{i}\theta^k}\frac{d(\pm\eta^k\mathrm{i})}{d\eta^k} = \zeta^k(\pm\epsilon e^{\mathrm{i}\theta^k}\mathrm{i}). \end{aligned}$$

In this proof, $(\rho_i^k)'$, $(\alpha_i^k)'$, and $(\beta_i^k)'$ are derivatives of ρ_i^k, α_i^k, and β_i^k with respect to η^k evaluated at η^k. The same conclusion holds for $\mathbf{X}_0^k(\pm1)$ in the symmetric case. □

Next compute

$$-\mathbf{N}_i^k\cdot\mathbf{X}_i^k\,ds = \begin{cases} ((\mathbf{r}_1^k)'\mathrm{i})\cdot\mathbf{X}_1^k\,dt & \text{if} \quad i=1 \\ -((\mathbf{r}_i^k)'\mathrm{i})\cdot\mathbf{X}_i^k\,dt & \text{if} \quad i=2,0 \end{cases}. \tag{3.8}$$

By (2.57)

$$(\mathbf{r}_i^k)'(t) = \begin{cases} \epsilon e^{\mathrm{i}\theta^k}((u_1^k)'(t)e^{\mathrm{i}(\pi-\alpha_1^k t)} + \alpha_1^k u_1^k(t)e^{\mathrm{i}(\pi-\alpha_1^k t)}(-\mathrm{i})) & \text{if} \quad i=1 \\ \epsilon e^{\mathrm{i}\theta^k}((u_i^k)'(t)e^{\mathrm{i}\alpha_i^k t} + \alpha_i^k u_i^k(t)e^{\mathrm{i}\alpha_i^k t}\mathrm{i}) & \text{if} \quad i=2,0 \end{cases}. \tag{3.9}$$

It follows from (3.6), (3.8) and (3.9) that

$$-\mathbf{N}_i^k \cdot \mathbf{X}_i^k \, ds = \epsilon^2 (\psi_i^k + \mathcal{E}_i^k(\phi_i^k, \eta^k)\zeta^k) \, dt \tag{3.10}$$

where $\mathcal{E}_i^k$ is an operator given, in the asymmetric case, by

$$\mathcal{E}_i^k(\phi_i^k, \eta^k) = \begin{cases} -\frac{(\alpha_1^k)'(u_1^k)^2}{2} - (\alpha_1^k)' u_1^k (u_1^k)' t + \frac{(\alpha_1^k)'(\rho_1^k)^2}{2} + \alpha_1^k \rho_1^k (\rho_1^k)' \\ +(\beta_1^k)' \cdot (\alpha_1^k u_1^k e^{\mathrm{i}(\pi - \alpha_1^k t)} - (u_1^k)' e^{\mathrm{i}(\pi-\alpha_1^k t)}(-\mathrm{i})) & \text{if} \quad i = 1 \\ \\ -\frac{(\alpha_i^k)'(u_i^k)^2}{2} - (\alpha_i^k)' u_i^k (u_i^k)' t + \frac{(\alpha_i^k)'(\rho_i^k)^2}{2} + \alpha_i^k \rho_i^k (\rho_i^k)' \\ +(\beta_i^k)' \cdot (\alpha_i^k u_i^k e^{\mathrm{i}\alpha_i^k t} - (u_i^k)' e^{\mathrm{i}\alpha_i^k t} \mathrm{i}) & \text{if} \quad i = 2, 0 \end{cases} \tag{3.11}$$

where u_i^k is related to ϕ_i^k and η^k via (2.58). In (3.11) $(\alpha_i^k)'$, $(\rho_i^k)'$, and $(\beta_i^k)'$ are derivatives of α_i^k, ρ_i^k, and β_i^k with respect to η^k. All these functions of η^k, namely α_i^k, $(\alpha_i^k)'$, ρ_i^k, $(\rho_i^k)'$, β_i^k, and $(\beta_i^k)'$, are evaluated at η^k. On the other hand $(u_i^k)'$ in (3.11) is just the derivative of $u_i^k(t)$ with respect to t. In the symmetric case

$$\mathcal{E}_0^k(\phi_0^k, \eta^k) = -\frac{\phi_0^k + (\phi_0^k)' t}{\eta^k}. \tag{3.12}$$

Define three more functions of η^k (in the asymmetric case):

$$\mu_i^k = (\rho_i^k)^2 (\alpha_i^k - \cos \alpha_i^k \sin \alpha_i^k), \quad i = 1, 2, 0. \tag{3.13}$$

Geometrically for $i = 1, 2$, μ_i^k is the sum of the area of a sector and the area of a triangle, associated with the left or right arc, after the first step of restricted perturbation, Figure 6. For $i = 0$, μ_0^k is the difference of the area of a sector and the area of a triangle associated with the middle arc. By (2.43) and 2.44) the μ_i^k's satisfy

$$\mu_1^k + \mu_0^k = w_1^k, \quad \mu_2^k - \mu_0^k = w_2^k. \tag{3.14}$$

In the symmetric case μ_1^k and μ_2^k are still given by (3.13), but they are constants, independent of η^k; namely $\mu_1^k = \mu_2^k = w_1^k = w_2^k$ and $\mu_0^k = 0$.

It is straight forward to show the following lemma.

Lemma 3.2 *The operator $\mathcal{E}_i^k$ satisfies the property*

$$\int_{-1}^{1} \mathcal{E}_i^k(\phi_i^k, \eta^k) \, dt = (\mu_i^k)'. \tag{3.15}$$

Moreover

$$\int_{-1}^{1} \mathcal{E}_1^k(\phi_1^k, \eta^k) \, dt + \int_{-1}^{1} \mathcal{E}_0^k(\phi_0^k, \eta^k) \, dt = \int_{-1}^{1} \mathcal{E}_2^k(\phi_2^k, \eta^k) \, dt - \int_{-1}^{1} \mathcal{E}_0^k(\phi_0^k, \eta^k) \, dt = 0. \tag{3.16}$$

Proof. In the asymmetric case, by (3.11),

$$\begin{aligned} \int_{-1}^{1} \mathcal{E}_i^k(\phi^k, \eta^k) \, dt &= \begin{cases} -\frac{(\alpha_1^k)'}{2} t (u_1^k)^2 |_{-1}^1 + (\alpha_1^k)'(\rho_1^k)^2 + 2\alpha_1^k \rho_1^k (\rho_1^k)' - (\beta_1^k)' \cdot u_1^k e^{\mathrm{i}(\pi - \alpha_1^k t)}(-\mathrm{i})|_{-1}^1 \\ -\frac{(\alpha_i^k)'}{2} t (u_i^k)^2 |_{-1}^1 + (\alpha_i^k)'(\rho_i^k)^2 + 2\alpha_i^k \rho_i^k (\rho_i^k)' - (\beta_i^k)' \cdot u_i^k e^{\mathrm{i}\alpha_i^k t} \mathrm{i}|_{-1}^1 \end{cases} \\ &= \begin{cases} 2\alpha_1^k \rho_1^k (\rho_1^k)' - 2\eta^k (\beta_1^k)' & \text{if} \quad i = 1 \\ 2\alpha_i^k \rho_i^k (\rho_i^k)' + 2\eta^k (\beta_i^k)' & \text{if} \quad i = 2, 0 \end{cases}. \end{aligned}$$

On the other hand

$$(\mu_i^k)' = \begin{cases} (\alpha_1^k (\rho_1^k)^2 - \beta_1^k \eta^k)' \\ (\alpha_i^k (\rho_i^k)^2 + \beta_i^k \eta^k)' \end{cases} = \begin{cases} 2\alpha_1^k \rho_1^k (\rho_1^k)' + (\alpha_1^k)'(\rho_1^k)^2 - \beta_1^k - \eta^k (\beta_1^k)' & \text{if} \quad i = 1 \\ 2\alpha_i^k \rho_i^k (\rho_i^k)' + (\alpha_i^k)'(\rho_i^k)^2 + \beta_i^k + \eta^k (\beta_i^k)' & \text{if} \quad i = 2, 0 \end{cases}.$$

Hence

$$(\mu_i^k)' - \int_{-1}^{1} \mathcal{E}_i^k(\phi_i^k,\eta^k)\,dt \;=\; \begin{cases} (\alpha_1^k)'(\rho_1^k)^2 - \beta_1^k + \eta^k(\beta_1^k)' & \text{if} \quad i=1 \\ (\alpha_i^k)'(\rho_i^k)^2 + \beta_i^k - \eta^k(\beta_i^k)' & \text{if} \quad i=2,0 \end{cases} \;=\; \begin{cases} (\alpha_1^k)'(\rho_1^k)^2 - (\beta_1^k)^2(\frac{\eta^k}{\beta_1^k})' \\ (\alpha_i^k)'(\rho_i^k)^2 + (\beta_i^k)^2(\frac{\eta^k}{\beta_i^k})' \end{cases}$$

$$= \; (\alpha_i^k)'(\rho_i^k)^2 - (\beta_i^k)^2(\tan\alpha_i^k)' \;=\; (\alpha_i^k)'(\rho_i^k)^2 - (\beta_i^k)^2(\sec\alpha_i^k)^2(\alpha_i^k)' \;=\; 0.$$

This proves the first part of the lemma. The constraints (3.14) on μ_i^k imply that

$$(\mu_1^k)' + (\mu_0^k)' = (\mu_2^k)' - (\mu_0^k)' = 0$$

from which the second part follows.

The lemma also holds in the symmetric case because

$$\int_{-1}^{1} \mathcal{E}_0^k(\phi_0^k,\eta^k)\,dt = \int_{-1}^{1} -\frac{\phi_0^k + (\phi_0^k)'t}{\eta^k}\,dt = -\frac{\phi_0^k t}{\eta^k}\Big|_{-1}^{1} = 0 = (\mu_0^k)'$$

and the argument in the asymmetric case also applies to $\mathcal{E}_i^k(\phi_i^k,\eta^k)$ for $i=1,2$. $\square$

Let $(\phi,\eta)\in\mathcal{D}(\mathcal{J})$ and $(\psi,\zeta)\in\mathcal{Y}$, and calculate

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\mathcal{J}_s((\phi,\eta)+\varepsilon(\psi,\zeta)), \quad \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\mathcal{J}_l((\phi,\eta)+\varepsilon(\psi,\zeta)).$$

For the former if $\phi_i^k\in H^2(-1,1)$,

$$\begin{aligned}
&\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\mathcal{J}_s((\phi,\eta)+\varepsilon(\psi,\zeta)) \\
&= \epsilon\sum_{k=1}^{n}\sum_{i=0}^{2}\int_{-1}^{1}\Big(\frac{\partial L_i^k((\phi_i^k)',\phi_i^k,\eta^k)}{\partial(\phi_i^k)'}(\psi_i^k)' + \frac{\partial L_i^k((\phi_i^k)',\phi_i^k,\eta^k)}{\partial\phi_i^k}\psi_i^k\Big)\,dt \\
&\quad +\epsilon\sum_{k=1}^{n}\Big(\sum_{i=0}^{2}\int_{-1}^{1}\frac{\partial L_i^k((\phi_i^k)',\phi_i^k,\eta^k)}{\partial\eta^k}\,dt\Big)\zeta^k \\
&= \epsilon\sum_{k=1}^{n}\sum_{i=0}^{2}\int_{-1}^{1}\Big(\frac{d}{dt}\Big(-\frac{\partial L_i^k((\phi_i^k)',\phi_i^k,\eta^k)}{\partial(\phi_i^k)'}\Big) + \frac{\partial L_i^k((\phi_i^k)',\phi_i^k,\eta^k)}{\partial\phi_i^k}\Big)\psi_i^k\,dt \\
&\quad +\epsilon\sum_{k=1}^{n}\Big(\sum_{i=0}^{2}\int_{-1}^{1}\frac{\partial L_i^k((\phi_i^k)',\phi_i^k,\eta^k)}{\partial\eta^k}\,dt\Big)\zeta^k \\
&= \epsilon\sum_{k=1}^{n}\int_{-1}^{1}\sum_{i=0}^{2}\mathcal{K}_i^k(\phi_i^k,\eta^k)\psi_i^k\,dt + \epsilon\sum_{k=1}^{n}\widetilde{\mathcal{K}}^k(\phi^k,\eta^k)\zeta^k \\
&= \sum_{k=1}^{n}\langle\epsilon(\mathcal{K}^k(\phi^k,\eta^k),\widetilde{\mathcal{K}}^k(\phi^k,\eta^k)),(\psi^k,\zeta^k)\rangle. \\
&= \langle\epsilon(\mathcal{K}(\phi,\eta),\widetilde{\mathcal{K}}(\phi,\eta)),(\psi,\zeta)\rangle_n. \qquad (3.17)
\end{aligned}$$

In (3.17) the operator $\mathcal{K}_i^k$ ($k=1,2,...,n$ and $i=0,1,2$) and the functional $\widetilde{\mathcal{K}}^k$ are given by:

$$\mathcal{K}_i^k(\phi_k^k,\eta^k) \;=\; \frac{d}{dt}\Big(-\frac{\partial L_i^k((\phi_i^k)',\phi_i^k,\eta^k)}{\partial(\phi_i^k)'}\Big) + \frac{\partial L_i^k((\phi_i^k)',\phi_i^k,\eta^k)}{\partial\phi_i^k}, \qquad (3.18)$$

$$\widetilde{\mathcal{K}}^k(\phi^k,\eta^k) \;=\; \sum_{i=0}^{2}\int_{-1}^{1}\frac{\partial L_i^k((\phi_i^k)',\phi_i^k,\eta^k)}{\partial\eta^k}\,dt \qquad (3.19)$$

and one writes $\mathcal{K}^k$ for $(\mathcal{K}_1^k, \mathcal{K}_2^k, \mathcal{K}_0^k)$, $\mathcal{K}$ for $((\mathcal{K}_1^1, \mathcal{K}_2^1, \mathcal{K}_0^1), ..., (\mathcal{K}_1^n, \mathcal{K}_2^n, \mathcal{K}_0^n))$ and $\widetilde{\mathcal{K}}$ for $(\widetilde{\mathcal{K}}^1, ..., \widetilde{\mathcal{K}}^n)$.

In (3.17) the inner products $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_n$ come from the Hilbet spaces $L^2((-1,1);\mathbb{R}^3) \times \mathbb{R}$ and $(L^2((-1,1);\mathbb{R}^3) \times \mathbb{R})^n$ respectively:

$$\langle (\phi^k, \eta^k), (\tilde{\phi}^k, \tilde{\eta}^k) \rangle = \sum_{i=0}^{2} \int_{-1}^{1} \phi_i^k(t) \tilde{\phi}_i^k(t)\, dt + \eta^k \tilde{\eta}^k. \tag{3.20}$$

$$\langle (\phi, \eta), (\tilde{\phi}, \tilde{\eta}) \rangle_n = \sum_{k=1}^{n} \sum_{i=0}^{2} \int_{-1}^{1} \phi_i^k(t) \tilde{\phi}_i^k(t)\, dt + \sum_{k=1}^{n} \eta^k \tilde{\eta}^k. \tag{3.21}$$

Comparing (3.17) with (2.34) of Lemma 2.4 and using (3.10) one finds, with the help of Lemma 3.1,

$$\mathcal{K}_i^k = \epsilon \kappa_i^k \quad \text{and } \widetilde{\mathcal{K}}^k = \epsilon^{-1} \Big(\sum_{i=0}^{2} \mathbf{T}_i^k \Big) \cdot \mathbf{X}^{S,k} \Big|_{-1}^{1} + \sum_{i=0}^{2} \int_{-1}^{1} \mathcal{K}_i^k(\phi_i^k, \eta^k) \mathcal{E}_i^k(\phi_i^k, \eta^k)\, dt. \tag{3.22}$$

Moreover, by (3.10), (2.35) of Lemma 2.4 implies

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{J}_l((\phi,\eta) + \varepsilon(\psi,\zeta)) = \sum_{k=1}^{n} \left\langle \epsilon^2 \begin{pmatrix} \gamma_{11} I_{T(P_1)} + \gamma_{12} I_{T(P_2)} \\ \gamma_{12} I_{T(P_1)} + \gamma_{22} I_{T(P_2)} \\ (\gamma_{11} - \gamma_{12}) I_{T(P_1)} + (\gamma_{12} - \gamma_{22}) I_{T(P_2)} \\ \mathcal{Q}^k(\phi,\eta) \end{pmatrix}, \begin{pmatrix} \psi_1^k \\ \psi_2^k \\ \psi_0^k \\ \zeta^k, \end{pmatrix} \right\rangle. \tag{3.23}$$

In (3.23) the functional $\mathcal{Q}^k$ is given by

$$\begin{aligned} \mathcal{Q}^k(\phi,\eta) = \int_{-1}^{1} \big(& (\gamma_{11} I_{T(P_1)} + \gamma_{12} I_{T(P_2)}) \mathcal{E}_1^k(\phi_1^k, \eta^k) + (\gamma_{12} I_{T(P_1)} + \gamma_{22} I_{T(P_2)}) \mathcal{E}_2^k(\phi_2^k, \eta^k) \\ & + ((\gamma_{11} - \gamma_{12}) I_{T(P_1)} + (\gamma_{12} - \gamma_{22}) I_{T(P_2)}) \mathcal{E}_0^k(\phi_0^k, \eta^k) \big)\, dt. \end{aligned} \tag{3.24}$$

In addition to $\mathcal{Y}$ two more spaces are needed in this work:

$$\mathcal{X} = \{(\phi,\eta) \in \mathcal{Y} : \phi_i^k \in H^2(-1,1)\} \tag{3.25}$$

$$\mathcal{Z} = \{(\phi,\eta) : \phi_i^k \in L^2(-1,1),\ \eta^k \in \mathbb{R},\ \int_{-1}^{1} (\phi_1^k + \phi_0^k)\, dt = \int_{-1}^{1} (\phi_2^k - \phi_0^k)\, dt = 0\}. \tag{3.26}$$

Clearly $\mathcal{X} \subset \mathcal{Y} \subset \mathcal{Z} \subset (L^2((-1,1);\mathbb{R}^3) \times \mathbb{R})^n$. The norms of $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are given by

$$\begin{aligned} \|(\phi,\eta)\|_{\mathcal{X}}^2 &= \sum_{k=1}^{n} \sum_{i=0}^{2} \|\phi_i^k\|_{H^2}^2 + \sum_{k=1}^{n} (\eta^k)^2 \\ \|(\phi,\eta)\|_{\mathcal{Y}}^2 &= \sum_{k=1}^{n} \sum_{i=0}^{2} \|\phi_i^k\|_{H^1}^2 + \sum_{k=1}^{n} (\eta^k)^2, \\ \|(\phi,\eta)\|_{\mathcal{Z}}^2 &= \sum_{k=1}^{n} \sum_{i=0}^{2} \|\phi_i^k\|_{L^2}^2 + \sum_{k=1}^{n} (\eta^k)^2 \end{aligned} \tag{3.27}$$

where $\|\cdot\|_{H^1}$ and $\|\cdot\|_{H^2}$ are the usual H^1 and H^2 norms of Sobolev spaces $H^1(-1,1)$ and $H^2(-1,1)$ respectively, and $\|\cdot\|_{L^2}$ is the L^2 norm of $L^2(-1,1)$. Let

$$\Pi : L^2((-1,1);\mathbb{R}^3) \times \mathbb{R} \to \Big\{ (\psi^k, \eta^k) \in L^2((-1,1);\mathbb{R}^3) \times \mathbb{R} : \int_{-1}^{1} (\psi_1^k + \psi_0^k)\, dt = \int_{-1}^{1} (\psi_2^k - \psi_0^k)\, dt = 0 \Big\}$$

be the orthogonal projection operator given by

$$\Pi(\psi^k,\zeta^k)=\begin{pmatrix}\psi_1^k\\ \psi_2^k\\ \psi_0^k\\ \zeta^k\end{pmatrix}-\Big[\int_{-1}^{1}\Big(\frac{\psi_1^k}{3}+\frac{\psi_2^k}{6}+\frac{\psi_0^k}{6}\Big)\,dt\Big]\begin{pmatrix}1\\0\\1\\0\end{pmatrix}-\Big[\int_{-1}^{1}\Big(\frac{\psi_1^k}{6}+\frac{\psi_2^k}{3}-\frac{\psi_0^k}{6}\Big)\,dt\Big]\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}. \quad (3.28)$$

Note that Π has no effect on the fourth component ζ^k.

The gradient of $\mathcal{J}_s$ is an operator $\mathcal{S}_s$ from a neighborhood of $(0,h)$ in $\mathcal{X}$ to $\mathcal{Z}$ such that

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\mathcal{J}_s((\phi,\eta)+\varepsilon(\psi,\zeta))=\langle\mathcal{S}_s(\phi,\eta),(\psi,\zeta)\rangle_n \quad (3.29)$$

for all $(\psi,\zeta)\in\mathcal{X}$. Similarly one defines $\mathcal{S}_l$ and $\mathcal{S}$, the gradients of $\mathcal{J}_l$ and $\mathcal{J}$ respectively. From (3.17) one sees that

$$\mathcal{S}_s(\phi,\eta)=\begin{pmatrix}\mathcal{S}_s^1(\phi^1,\eta^1)\\ \ldots\\ \mathcal{S}_s^n(\phi^n,\eta^n)\end{pmatrix}\quad\text{where}\quad \mathcal{S}_s^k(\phi^k,\eta^k)=\Pi\epsilon\begin{pmatrix}\mathcal{K}_1^k(\phi_1^k,\eta^k)\\ \mathcal{K}_2^k(\phi_2^k,\eta^k)\\ \mathcal{K}_0^k(\phi_0^k,\eta^k)\\ \widetilde{\mathcal{K}}^k(\phi^k,\eta^k)\end{pmatrix}. \quad (3.30)$$

The gradient of $\mathcal{J}_l$ is

$$\mathcal{S}_l(\phi,\eta)=\begin{pmatrix}\mathcal{S}_l^1(\phi,\eta)\\ \ldots\\ \mathcal{S}_l^n(\phi,\eta)\end{pmatrix}\quad\text{where}\quad \mathcal{S}_l^k(\phi,\eta)=\Pi\epsilon^2\begin{pmatrix}\gamma_{11}I_{T(P_1)}+\gamma_{12}I_{T(P_2)}\\ \gamma_{12}I_{T(P_1)}+\gamma_{22}I_{T(P_2)}\\ (\gamma_{11}-\gamma_{12})I_{T(P_1)}+(\gamma_{12}-\gamma_{22})I_{T(P_2)}\\ \mathcal{Q}^k(\phi,\eta)\end{pmatrix}. \quad (3.31)$$

A remark regarding the $I_{T(P_i)}$'s in (3.31) is in order. Recall that each $I_{T(P_i)}$, $i=1,2$, is a function on D given in (1.6), and the set $T(P_i)$ is determined by the internal variables ϕ_i^k, ϕ_0^k and η^k for $k=1,2,...,n$. The $I_{T(P_i)}$'s ($i=1,2$) in the first three components on the right side of (3.31) are now considered as outcomes of the operators

$$\mathcal{I}_{ij}^k:\ (\phi_i,\phi_0,\eta)\to I_{T(P_i)}(\mathbf{r}_j^k(t)),\quad i=1,2,\ j=1,2,0,\ k=1,2,...,n. \quad (3.32)$$

where $j=1,2,0$ corresponds to the first, second, and third component in (3.31) respectively. Note that in (3.32)

$$(\phi_i,\phi_0,\eta)=(\phi_i^1,\phi_0^1,\eta^1)\times(\phi_i^2,\phi_0^2,\eta^2)\times\ldots\times(\phi_i^n,\phi_0^n,\eta^n) \quad (3.33)$$

represents $T(P_i)$.

The gradient of $\mathcal{J}$ is

$$\mathcal{S}=\mathcal{S}_s+\mathcal{S}_l. \quad (3.34)$$

Therefore

$$\mathcal{S}(\phi,\eta)=\begin{pmatrix}\mathcal{S}^1(\phi,\eta)\\ \ldots\\ \mathcal{S}^n(\phi,\eta)\end{pmatrix}$$

where

$$\mathcal{S}^k(\phi,\eta)=\Pi\begin{pmatrix}\epsilon\mathcal{K}_1^k(\phi_1^k,\eta^k)+\epsilon^2(\gamma_{11}I_{T(P_1)}+\gamma_{12}I_{T(P_2)})\\ \epsilon\mathcal{K}_2(\phi_2^k,\eta^k)+\epsilon^2(\gamma_{12}I_{T(P_1)}+\gamma_{22}I_{T(P_2)})\\ \epsilon\mathcal{K}_0^k(\phi_0^k,\eta^k)+\epsilon^2(\gamma_{11}-\gamma_{12})I_{T(P_1)}+\epsilon^2(\gamma_{12}-\gamma_{22})I_{T(P_2)}\\ \epsilon\widetilde{\mathcal{K}}^k(\phi^k,\eta^k)+\epsilon^2\mathcal{Q}^k(\phi,\eta)\end{pmatrix}. \quad (3.35)$$

The domain of $\mathcal{S}$ is taken to be

$$\mathcal{D}(\mathcal{S})=\{(\phi,\eta)\in\mathcal{X}:\ \|(\phi,\eta-h)\|_{\mathcal{X}}<\bar{c}\} \quad (3.36)$$

where $\bar{c}$ in (3.36) is the same as the $\bar{c}$ in (2.68). Consequently, $\mathcal{D}(\mathcal{S})\subset\mathcal{D}(\mathcal{J})$.

Lemma 3.3 *It holds uniformly with respect to t that*

$$\mathcal{S}^k(0,h) = \big(O(|\gamma|\epsilon^4), O(|\gamma|\epsilon^4), O(|\gamma|\epsilon^4), O(|\gamma|\epsilon^4)\big)$$

for all $k = 1, 2, ..., n$. Consequently, there exists $\widetilde{C} > 0$ such that $\|\mathcal{S}(0,h)\|_{\mathcal{Z}} \le \widetilde{C}|\gamma|\epsilon^4$.

Proof. Calculations from (3.18) and (3.19) show that

$$\mathcal{K}_i^k(0,h^k) = \frac{1}{r_i^k} \quad \text{and } \widetilde{\mathcal{K}}^k(0,h^k) = 2\sum_{i=0}^{2} \frac{d(\alpha_i^k \rho_i^k)}{d\eta^k}\Big|_{\eta=h}$$

in the asymmetric case and

$$\widetilde{\mathcal{K}}^k(0,h^k) = \Big(2\sum_{i=1}^{2} \frac{d(\alpha_i^k \rho_i^k)}{d\eta^k} + 2\Big)\Big|_{\eta=h}$$

in the symmetric case. By (B.22) in Appendix B,

$$\widetilde{\mathcal{K}}^k(0,h^k) = 0.$$

Consequently, by the virtue of the projection operator Π and the fact that $\frac{1}{r_1^k} - \frac{1}{r_2^k} = \frac{1}{r_0^k}$,

$$\mathcal{S}_s^k(0,h^k) = \Pi\epsilon \begin{pmatrix} \mathcal{K}_1^k(0,h^k) \\ \mathcal{K}_2^k(0,h^k) \\ \mathcal{K}_0^k(0,h^k) \\ \widetilde{\mathcal{K}}^k(0,h^k) \end{pmatrix} = \Pi\epsilon \begin{pmatrix} 1/r_1^k \\ 1/r_2^k \\ 1/r_0^k \\ 0 \end{pmatrix} = 0. \tag{3.37}$$

Regarding $\mathcal{S}_l^k(0,h)$ let $\hat{\mathbf{r}}_i^k$ be the boundaries of the exact double bubble B^k, i.e.,

$$\hat{\mathbf{r}}_i^k(t) = \begin{cases} r_1^k e^{\mathrm{i}(\pi - a_1^k t)} + b_1^k & \text{if} \quad i=1 \\ r_i^k e^{\mathrm{i} a_i^k t} + b_i^k & \text{if} \quad i = 2, 0 \end{cases}$$

and $\mathbf{r}_i^k$ be the boundary of $T^k(E_i^k)$, i.e.,

$$\mathbf{r}_i^k(t) = \epsilon e^{\mathrm{i}\theta^k} \hat{\mathbf{r}}_i^k(t) + \xi^k.$$

One then deduces

$$\begin{aligned}
\mathcal{I}_{ij}^k(0,0,h) &= \int_{T(B_i)} G(\mathbf{r}_j^k(t), y)\, dy \\
&= \int_{T^k(B_i^k)} \frac{1}{2\pi} \log \frac{1}{|\mathbf{r}_j^k(t) - y|}\, dy + \int_{T^k(B_i^k)} R(\mathbf{r}_j^k(t), y)\, dy + \sum_{l\neq k} \int_{T^l(B_i^l)} G(\mathbf{r}_j^k(t), y)\, dy \\
&= \epsilon^2 \int_{B_i^k} \frac{1}{2\pi} \log \frac{1}{\epsilon|\hat{\mathbf{r}}_j^k(t) - \hat{y}|}\, d\hat{y} + O(\epsilon^2) \\
&= \frac{\epsilon^2}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)|B_i^k| + \epsilon^2 \int_{B_i^k} \frac{1}{2\pi} \log \frac{1}{|\hat{\mathbf{r}}_j(t) - \hat{y}|}\, d\hat{y} + O(\epsilon^2) \\
&= \frac{\epsilon^2}{2\pi}\Big(\log\frac{1}{\epsilon}\Big) w_i^k + O(\epsilon^2).
\end{aligned}$$

Consequently, with the help of (3.16) of Lemma 3.2,

$$\mathcal{Q}^k(0,h) = \int_{-1}^{1} \Big[\gamma_{11}\frac{\epsilon^2}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)w_1^k + \gamma_{12}\frac{\epsilon^2}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)w_2^k\Big]\mathcal{E}_1^k(0,h^k)\, dt$$

$$
\begin{aligned}
&+\int_{-1}^{1}\Big[\gamma_{12}\frac{\epsilon^2}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)w_1^k+\gamma_{22}\frac{\epsilon^2}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)w_2^k\Big]\mathcal{E}_2^k(0,h^k)\,dt\\
&+\int_{-1}^{1}\Big[(\gamma_{11}-\gamma_{12})\frac{\epsilon^2}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)w_1^k+(\gamma_{12}-\gamma_{22})\frac{\epsilon^2}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)w_2^k\Big]\mathcal{E}_0^k(0,h^k)\,dt+O(|\gamma|\epsilon^2)\\
=\;&\frac{\gamma_{11}\epsilon^2}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)w_1^k\int_{-1}^{1}(\mathcal{E}_1^k(0,h^k)+\mathcal{E}_0^k(0,h^k))\,dt\\
&+\frac{\gamma_{12}\epsilon^2}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)w_2^k\int_{-1}^{1}(\mathcal{E}_1^k(0,h^k)+\mathcal{E}_0^k(0,h^k))\,dt\\
&+\frac{\gamma_{12}\epsilon^2}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)w_1^k\int_{-1}^{1}(\mathcal{E}_2^k(0,h^k)-\mathcal{E}_0^k(0,h^k))\,dt\\
&+\frac{\gamma_{22}\epsilon^2}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)w_2^k\int_{-1}^{1}(\mathcal{E}_2^k(0,h^k)-\mathcal{E}_0^k(0,h^k))\,dt+O(|\gamma|\epsilon^2)\\
=\;&O(|\gamma|\epsilon^2).
\end{aligned}
$$

Therefore

$$
\mathcal{S}_l^k(0,h)=\frac{\epsilon^4}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)\Pi\begin{pmatrix}\gamma_{11}w_1^k+\gamma_{12}w_2^k\\ \gamma_{12}w_1^k+\gamma_{22}w_2^k\\ (\gamma_{11}-\gamma_{12})w_1^k+(\gamma_{12}-\gamma_{22})w_2^k\\ 0\end{pmatrix}+O(|\gamma|\epsilon^4)
$$

$$
=\frac{\epsilon^4}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)\left[\gamma_{11}w_1^k\Pi\begin{pmatrix}1\\0\\1\\0\end{pmatrix}+\gamma_{12}w_2^k\Pi\begin{pmatrix}1\\0\\1\\0\end{pmatrix}+\gamma_{12}w_1^k\Pi\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}+\gamma_{22}w_2^k\Pi\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}\right]+O(|\gamma|\epsilon^4)
$$

$$
=\frac{\epsilon^4}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)(\vec{0}+\vec{0}+\vec{0}+\vec{0})+O(|\gamma|\epsilon^4)=O(|\gamma|\epsilon^4). \tag{3.38}
$$

The lemma follows from (3.37) and (3.38). □

4 The second variation

The Fréchet derivative of the operator $\mathcal{S}$ at any $(\phi,\eta)\in\mathcal{D}(\mathcal{S})$ is denoted $\mathcal{S}'(\phi,\eta)$. It is a linear operator from $\mathcal{X}$ to $\mathcal{Z}$. For every $(\psi,\zeta)\in\mathcal{X}$, it yields the second variation of $\mathcal{J}$:

$$
\frac{d^2\mathcal{J}((\phi,\eta)+\varepsilon(\psi,\zeta))}{d\varepsilon^2}\Big|_{\varepsilon=0}=\langle\mathcal{S}'(\phi,\eta)(\psi,\zeta),(\psi,\zeta)\rangle_n. \tag{4.1}
$$

Similar formulas hold if $\mathcal{J}$ is replaced by $\mathcal{J}_s$ and $\mathcal{S}$ replaced by $\mathcal{S}_s$, or $\mathcal{J}$ by $\mathcal{J}_l$ and $\mathcal{S}$ by $\mathcal{S}_l$.

We show that the operator $\mathcal{S}'(0,h)$, the Fréchet derivative of $\mathcal{S}$ at the exact double bubble assembly is positive definite and derives a upper bound for the inverse operator $(\mathcal{S}'(0,h))^{-1}$.

Define the ϵ independent part of $\mathcal{J}_s$ by $\mathcal{P}$ so that $\mathcal{J}_s=\epsilon\mathcal{P}$ where

$$
\mathcal{P}(\phi,\eta)=\sum_{k=1}^{n}\sum_{i=0}^{2}\int_{-1}^{1}L_i^k((\phi_i^k)',\phi_i^k,\eta^k)\,dt,\quad(\phi,\eta)\in\mathcal{Y}. \tag{4.2}
$$

Calculations show that, in the asymmetric case,

$$
\frac{\partial^2L_i^k(0,0,\eta^k)}{\partial((\phi_i^k)')^2}=\frac{1}{(\alpha_i^k\rho_i^k)^3},\quad\frac{\partial^2L_i^k(0,0,\eta^k)}{\partial(\phi_i^k)^2}=-\frac{1}{\alpha_i^k(\rho_i^k)^3},\quad\frac{\partial^2L_i^k(0,0,\eta^k)}{\partial(\eta^k)^2}=\frac{d^2(\alpha_i^k\rho_i^k)}{d(\eta^k)^2}, \tag{4.3}
$$

$$\frac{\partial^2 L_i^k(0,0,\eta^k)}{\partial(\phi_i^k)'\partial\phi_i^k} = 0, \ \frac{\partial^2 L_i^k(0,0,\eta^k)}{\partial(\phi_i^k)'\partial\eta^k} = 0, \ \frac{\partial^2 L_i^k(0,0,\eta^k)}{\partial\phi_i^k\partial\eta^k} = \frac{d}{d\eta^k}\Big(\frac{1}{\rho_i^k}\Big), \tag{4.4}$$

and in the symmetric case

$$\frac{\partial^2 L_0^k(0,\eta^k)}{\partial((\phi_0^k)')^2} = \frac{1}{(\eta^k)^3}, \quad \frac{\partial^2 L_0^k(0,\eta^k)}{\partial(\eta^k)^2} = 0, \quad \frac{\partial^2 L_0^k(0,\eta^k)}{\partial(\phi_0^k)'\partial\eta^k} = 0. \tag{4.5}$$

The second variation of $\mathcal{P}$ at $(\phi,\eta) = (0,h)$ is

$$\frac{d^2\mathcal{P}(0+\varepsilon\psi, h+\varepsilon\zeta)}{d\varepsilon^2}\Big|_{\varepsilon=0} =$$
$$\sum_{k=1}^{n}\sum_{i=0}^{2}\int_{-1}^{1}\Big[\frac{1}{(a_i^k r_i^k)^3}((\psi_i^k)')^2 - \frac{1}{a_i^k(r_i^k)^3}(\psi_i^k)^2 + \frac{d^2(\alpha_i^k\rho_i^k)}{d(\eta^k)^2}\Big|_{\eta^k=h^k}(\zeta^k)^2 + 2\frac{d}{d\eta^k}\Big(\frac{1}{\rho_i^k}\Big)\Big|_{\eta^k=h^k}\psi_i^k\zeta^k\Big]\,dt.$$

However the constraints (2.60) that the ψ_i^k's satisfy and the condition (2.46) on the ρ_i^k's imply

$$\sum_{i=0}^{2}\int_{-1}^{1} 2\frac{d}{d\eta^k}\Big(\frac{1}{\rho_i^k}\Big)\Big|_{\eta^k=h^k}\psi_i^k\zeta^k\,dt = 2\frac{d}{d\eta^k}\Big(\frac{1}{\rho_1^k} - \frac{1}{\rho_2^k} - \frac{1}{\rho_0^k}\Big)\Big|_{\eta^k=h^k}\Big(\int_{-1}^{1}\psi_1^k\,dt\Big)\zeta^k = 0.$$

Hence the integral of the last term vanishes, and

$$\frac{d^2\mathcal{P}(0+\varepsilon\psi, h+\varepsilon\zeta)}{d\varepsilon^2}\Big|_{\varepsilon=0} = \sum_{k=1}^{n}\sum_{i=0}^{2}\int_{-1}^{1}\Big[\frac{1}{(a_i^k r_i^k)^3}((\psi_i^k)')^2 - \frac{1}{a_i^k(r_i^k)^3}(\psi_i^k)^2 + \frac{d^2(\alpha_i^k\rho_i^k)}{d(\eta^k)^2}\Big|_{\eta^k=h^k}(\zeta^k)^2\Big]\,dt. \tag{4.6}$$

This is a quadratic form on $\mathcal{Y}$. A simple lemma is needed at this point.

Lemma 4.1 *Let $q \in (0,\pi)$ and $\Upsilon \in \mathbb{R}$. The inequality*

$$\int_{-1}^{1}((y'(t))^2 - q^2y^2(t))\,dt \geq \frac{\Upsilon^2 q^3}{2(\tan q - q)}$$

holds for all $y \in H_0^1(-1,1)$ that satisfies the constraint $\int_{-1}^{1} y(t)\,dt = \Upsilon$.

The proof of this lemma is given in Appendix A.

Lemma 4.2 *There exists $d > 0$ such that*

$$\frac{d^2\mathcal{P}(0+\varepsilon\psi, h+\varepsilon\zeta)}{d\varepsilon^2}\Big|_{\varepsilon=0} \geq 2d\|(\psi,\zeta)\|_{\mathcal{Y}}^2 \tag{4.7}$$

for all $(\psi,\zeta) \in \mathcal{X}$. In other words for $(\psi,\zeta) \in \mathcal{X}$,

$$\langle \mathcal{S}_s'(0,h)(\psi,\zeta),(\psi,\zeta)\rangle_n \geq 2d\epsilon\|(\psi,\zeta)\|_{\mathcal{Y}}^2. \tag{4.8}$$

Proof. Let

$$\int_{-1}^{1}\psi_0^k\,dt = \Upsilon^k, \quad \int_{-1}^{1}\psi_1^k\,dt = -\Upsilon^k, \quad \int_{-1}^{1}\psi_2^k\,dt = \Upsilon^k \tag{4.9}$$

because of the constraints (2.60). By Lemma 4.1, one deduces

$$
\begin{aligned}
&\frac{d^2\mathcal{P}(0+\varepsilon\psi, h+\varepsilon\zeta)}{d\varepsilon^2}\Big|_{\varepsilon=0} - 2d\sum_{k=1}^{n}\sum_{i=0}^{2}\|\psi_i^k\|_{H^1}^2 \\
= &\sum_{k=1}^{n}\sum_{i=0}^{2}\int_{-1}^{1}\Big[\Big(\frac{1}{(a_i^k r_i^k)^3} - 2d\Big)((\psi_i^k)')^2 - \Big(\frac{1}{a_i^k (r_i^k)^3} + 2d\Big)(\psi_i^k)^2\Big]\,dt + \sum_{k=1}^{n}\sum_{i=0}^{2} 2(\zeta^k)^2\frac{d^2(\alpha_i^k\rho_i^k)}{d(\eta^k)^2}\Big|_{\eta^k=h^k} \\
\geq &\sum_{k=1}^{n}\sum_{i=0}^{2}\frac{\Big(\frac{1}{(a_i^k r_i^k)^3} - 2d\Big)(\Upsilon^k)^2 (q_i^k)^3}{2(\tan q_i^k - q_i^k)} + \sum_{k=1}^{n}\sum_{i=0}^{2} 2(\zeta^k)^2\frac{d^2(\alpha_i^k\rho_i^k)}{d(\eta^k)^2}\Big|_{\eta^k=h^k} \qquad (4.10)
\end{aligned}
$$

where

$$
q_i^k = \sqrt{\frac{\frac{1}{a_i^k (r_i^k)^3} + 2d}{\frac{1}{(a_i^k r_i^k)^3} - 2d}}. \qquad (4.11)
$$

If $d \to 0$, then

$$
\begin{aligned}
\sum_{i=0}^{2}\frac{\Big(\frac{1}{(a_i^k r_i^k)^3} - 2d\Big)(q_i^k)^3}{2(\tan q_i^k - q_i^k)} \quad &\to \quad \frac{1}{2}\sum_{i=0}^{2}\frac{1}{(r_i^k)^3(\tan a_i^k - a_i^k)} \\
&= \quad \frac{1}{2(h^k)^3}\sum_{i=0}^{2}\frac{\sin^3 a_i^k}{\tan a_i^k - a_i^k} \\
&= \quad \frac{1}{2(h^k)^3}\sum_{i=0}^{2}\frac{\cos a_i^k \sin^3 a_i^k}{\sin a_i^k - a_i^k\cos a_i^k}. \qquad (4.12)
\end{aligned}
$$

By Lemma B.1 in Appendix B, (4.12) is positive. Hence for $d > 0$ sufficiently small,

$$
\sum_{i=0}^{2}\frac{\Big(\frac{1}{(a_i^k r_i^k)^3} - 2d\Big)(\Upsilon^k)^2 (q_i^k)^3}{2(\tan q_i^k - q_i^k)} \geq 0 \qquad (4.13)
$$

for all k. By (B.23) in Appendix B,

$$
\sum_{i=0}^{2}\frac{d^2(\alpha_i^k\rho_i^k)}{d(\eta^k)^2}\Big|_{\eta^k=h^k} > 0. \qquad (4.14)
$$

Hence

$$
2(\zeta^k)^2\sum_{i=0}^{2}\frac{d^2(\alpha_i^k\rho_i^k)}{d(\eta^k)^2}\Big|_{\eta^k=h^k} \geq 2d(\zeta^k)^2 \qquad (4.15)
$$

if d is sufficiently small. The lemma now follows from (4.10), (4.13), and (4.15).

The above argument is carried out with the assumption that all exact double bubbles B^k are asymmetric. However if a B^k is symmetric, one simply makes the changes

$$
\alpha_0^k\rho_0^k \to \eta^k, \quad \frac{1}{a_0^k (r_0^k)^3} \to 0, \quad \frac{1}{(a_0^k r_0^k)^3} \to \frac{1}{(h^k)^3} \qquad (4.16)
$$

and the same argument also covers the symmetric case. Note that in the symmetric case (B.31) plays the role of (B.23). □

In the rest of the paper, if only the asymmetric case is presented in a proof, then the same proof will also work for the symmetric case with the suitable modification of (4.16).

From the quadratic form (4.6), one finds the explicit formula for $\mathcal{S}_s'(0,h)$:

$$(\mathcal{S}_s^k)'(0,h^k)(\psi^k,\zeta^k) = \Pi\epsilon \begin{pmatrix} -\frac{1}{(a_1^k r_1^k)^3}(\psi_1^k)'' - \frac{1}{a_1^k (r_1^k)^3}\psi_1^k \\ -\frac{1}{(a_2^k r_2^k)^3}(\psi_2^k)'' - \frac{1}{a_2^k (r_2^k)^3}\psi_2^k \\ -\frac{1}{(a_0^k r_0^k)^3}(\psi_0^k)'' - \frac{1}{a_0^k (r_0^k)^3}\psi_0^k \\ 2(\sum_{i=0}^2 \frac{d^2(\alpha_i^k\rho_i^k)}{d(\eta^k)^2}|_{\eta^k=h^k})\zeta^k \end{pmatrix}. \tag{4.17}$$

Next study

$$(\mathcal{S}_l^k)'(0,h)(\psi,\zeta) = \Pi\epsilon^2 \begin{pmatrix} \gamma_{11}(\mathcal{I}_{11}^k)'(0,0,h)(\psi_1,\psi_0,\zeta) + \gamma_{12}(\mathcal{I}_{21}^k)'(0,0,h)(\psi_2,\psi_0,\zeta) \\ \gamma_{12}(\mathcal{I}_{12}^k)'(0,0,h)(\psi_1,\psi_0,\zeta) + \gamma_{22}(\mathcal{I}_{22}^k)'(0,0,h)(\psi_2,\psi_0,\zeta) \\ (\gamma_{11}-\gamma_{12})(\mathcal{I}_{10}^k)'(0,0,h)(\psi_1,\psi_0,\zeta) + (\gamma_{12}-\gamma_{22})(\mathcal{I}_{20}^k)'(0,0,h)(\psi_2,\psi_0,\zeta) \\ (\mathcal{Q}^k)'(0,h)(\psi,\zeta) \end{pmatrix}. \tag{4.18}$$

Lemma 4.3 *There exists $\check{C} > 0$ depending on D, m and n only such that*

$$\|\mathcal{S}_l'(0,h)(\psi,\zeta)\|_{\mathcal{Z}} \le \check{C}|\gamma|\epsilon^4\|(\psi,\zeta)\|_{\mathcal{Z}}$$

for all $(\psi,\zeta) \in \mathcal{X}$.

Proof. To compute the Fréchet derivatives of $\mathcal{I}_{ij}^k$, deform (ϕ,η) to $(\phi,\eta)+\varepsilon(\psi,\zeta)$ and denote the corresponding deformation of $\mathbf{r}_1^k$, $\mathbf{r}_2^k$ and $\mathbf{r}_0^k$ by $\mathbf{r}_1^{\varepsilon,k}$, $\mathbf{r}_2^{\varepsilon,k}$ and $\mathbf{r}_0^{\varepsilon,k}$ respectively. Then

$$(\mathcal{I}_{ij}^k)'(\phi_i,\phi_0,\eta):\ (\psi_i,\psi_0,\zeta) \to \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0}\int_{T(P_i^\varepsilon)} G(\mathbf{r}_j^k(t),y)\,dy + \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0}\int_{T(P_i)} G(\mathbf{r}_j^{\varepsilon,k},y)\,dy. \tag{4.19}$$

Since (ϕ,η) is $(0,h)$ in this lemma, $T(P)$ becomes $T(B)$ and its deformation is denoted $T(B^\varepsilon)$. Applying Lemma 2.3 to the first term on the left side of (4.19) with the boundaries of $T(B)$ parametrized by

$$\mathbf{r}_1^k(t) = T^k(r_1^k e^{\mathrm{i}(\pi - a_1^k t)} + b_1^k),\quad \mathbf{r}_2^k(t) = T^k(r_2^k e^{\mathrm{i}a_2^k t} + b_2^k),\quad \mathbf{r}_0^k(t) = T^k(r_0^k e^{\mathrm{i}a_0^k t} + b_0^k), \tag{4.20}$$

one obtains

$$\begin{aligned} &\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0}\int_{T(B_i^\varepsilon)} G(\mathbf{r}_j^k(t),y)\,dy \\ &= \begin{cases} -\displaystyle\int_{T(\partial B_1)\backslash T(\partial B_2)} G(\mathbf{r}_j^k(t),\mathbf{r}_1)\mathbf{N}_1\cdot\mathbf{X}_1\,ds - \int_{T(\partial B_1)\cap T(\partial B_2)} G(\mathbf{r}_j^k(t),\mathbf{r}_0)\mathbf{N}_0\cdot\mathbf{X}_0\,ds & \text{if} \quad i=1 \\ -\displaystyle\int_{T(\partial B_2)\backslash T(\partial B_1)} G(\mathbf{r}_j^k(t),\mathbf{r}_2)\mathbf{N}_2\cdot\mathbf{X}_2\,ds + \int_{T(\partial B_1)\cap T(\partial B_2)} G(\mathbf{r}_j^k(t),\mathbf{r}_0)\mathbf{N}_0\cdot\mathbf{X}_0\,ds & \text{if} \quad i=2 \end{cases} \end{aligned} \tag{4.21}$$

In (4.21) some shorthand notations have been used. For instance $T(\partial B_1)\backslash T(\partial B_2)$ stands for the union $\cup_{l=1}^n T^l(\partial B_1^l)\backslash T^l(\partial B_2^l)$; $\mathbf{r}_1$, $\mathbf{N}_1$ and $\mathbf{X}_1$ refer to $\mathbf{r}_1^l$, $\mathbf{N}_1^l$ and $\mathbf{X}_1^l$ respectively on each $T^l(\partial B_1^l)\backslash T^l(\partial B_2^l)$ of the union $T(\partial B_1)\backslash T(\partial B_2)$. With the help of (3.10), one finds that

$$\begin{aligned} &-\int_{T(\partial B_1)\backslash T(\partial B_2)} G(\mathbf{r}_j^k(t),\mathbf{r}_1)\mathbf{N}_1\cdot\mathbf{X}_1\,ds \\ &= \sum_{l=1}^n \epsilon^2\int_{-1}^1 G(\mathbf{r}_j^k(t),\mathbf{r}_1^l(\tau))(\psi_1^l(\tau) + \mathcal{E}_1^l(0,h^l)(\tau)\zeta^l)\,d\tau \end{aligned}$$

$$
\begin{aligned}
&= \int_{-1}^{1} \frac{\epsilon^2}{2\pi}\Big(\log\frac{1}{|\mathbf{r}_j^k(t)-\mathbf{r}_1^k|}\Big)(\psi_1^k+\mathcal{E}_1^k(0,h^k)\zeta^k)\,d\tau + \epsilon^2\int_{-1}^{1} R(\mathbf{r}_j^k(t),\mathbf{r}_1^k)(\psi_1^k+\mathcal{E}_1^k(0,h^k)\zeta^k)\,d\tau \\
&\quad + \sum_{l\neq k}\epsilon^2\int_{-1}^{1} G(\mathbf{r}_j^k(t),\mathbf{r}_1^l)(\psi_1^l+\mathcal{E}_1^l(0,h^l)\zeta^l)\,d\tau \\
&= \frac{\epsilon^2}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)\int_{-1}^{1}(\psi_1^k+\mathcal{E}_1^k(0,h^k)\zeta^k)\,d\tau \\
&\quad + \frac{\epsilon^2}{2\pi}\int_{-1}^{1}\Big(\log\frac{1}{|r_j^k e^{\mathrm{i}a_j^k t}+b_j^k-r_1^k e^{\mathrm{i}(\pi-a_1^k\tau)}-b_1^k|}\Big)(\psi_1^k+\mathcal{E}_1^k(0,h^k)\zeta^k)\,d\tau + O(\epsilon^2)\sum_{l=1}^{n}(\|\psi_1^l\|_{L^2}+|\zeta^l|) \\
&= \frac{\epsilon^2}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)\int_{-1}^{1}(\psi_1^k+\mathcal{E}_1^k(0,h^k)\zeta^k)\,d\tau + O(\epsilon^2)\sum_{l=1}^{n}(\|\psi_1^l\|_{L^2}+|\zeta^l|).
\end{aligned}
$$

The above estimate holds uniformly with respect to t. Also the term $r_j^k e^{\mathrm{i}a_j^k t}$ above is valid if $j=0,2$; if $j=1$, it should be replaced by $r_1^k e^{\mathrm{i}(\pi-a_1^k t)}$. Similar estimates hold for the other three terms in (4.21). By the constraints (2.60) on ψ_i^k and (3.16) of Lemma 3.2 one deduces that

$$
\begin{aligned}
&\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0}\int_{T(B_i^\varepsilon)} G(\mathbf{r}_j^k(t),y)\,dy \\
&= \begin{cases}
\frac{\epsilon^2}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)\int_{-1}^{1}(\psi_1^k+\mathcal{E}_1^k(0,h^k)\zeta^k)\,d\tau + \frac{\epsilon^2}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)\int_{-1}^{1}(\psi_0^k+\mathcal{E}_0^k(0,h^k)\zeta^k)\,d\tau + O(\epsilon^2)\|(\psi,\zeta)\|_{\mathcal{Z}} \\
\frac{\epsilon^2}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)\int_{-1}^{1}(\psi_2^k+\mathcal{E}_2^k(0,h^k)\zeta^k)\,d\tau - \frac{\epsilon^2}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)\int_{-1}^{1}(\psi_0^k+\mathcal{E}_0^k(0,h^k)\zeta^k)\,d\tau + O(\epsilon^2)\|(\psi,\zeta)\|_{\mathcal{Z}}
\end{cases} \\
&= O(\epsilon^2)\|(\psi,\zeta)\|_{\mathcal{Z}}
\end{aligned} \tag{4.22}
$$

holds uniformly with respect to t.

The second part on the right side of (4.19), for $(\phi,\eta)=(0,h)$, is written as

$$
\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0}\int_{T(B_i)} G(\mathbf{r}_j^{\varepsilon,k}(t),y)\,dy = \int_{T(B_i)}\nabla G(\mathbf{r}_j^k(t),y)\cdot\mathbf{X}_j^k(t)\,dy \tag{4.23}
$$

where ∇G stands for the gradient of G with respect to its first argument. Clearly

$$
\int_{T(B_i)}|\nabla G(\mathbf{r}_j^k(t),y)|\,dy = O(\epsilon) \tag{4.24}
$$

holds uniformly with respect to t. Calculations from (3.3) and (3.6) show that

$$
\mathbf{X}_j^k(t) = \begin{cases}
\epsilon e^{\mathrm{i}\theta^k}\Big[\frac{1}{a_1^k r_1^k}(\psi_1^k + a_1^k r_1^k(\rho_1^k)'\zeta^k)e^{\mathrm{i}(\pi-a_1^k t)} + r_1^k(\alpha_1^k)'\zeta^k t e^{\mathrm{i}(\pi-a_1^k t)}(-\mathrm{i}) + (\beta_1^k)'\zeta^k\Big] & \text{if} \quad j=1 \\
\epsilon e^{\mathrm{i}\theta^k}\Big[\frac{1}{a_j^k r_j^k}(\psi_j^k + a_j^k r_j^k(\rho_j^k)'\zeta^k)e^{\mathrm{i}a_j^k t} + r_j^k(\alpha_j^k)'\zeta^k t e^{\mathrm{i}a_j^k t}\mathrm{i} + (\beta_j^k)'\zeta^k\Big] & \text{if} \quad j=2,0
\end{cases} \tag{4.25}
$$

where $(\rho_j^k)'$, $(\alpha_j^k)'$, and $(\beta_j^k)'$ refer to the derivatives of ρ_j^k, α_j^k, and β_j^k with respect to η^k evaluated at h^k, respectively. Then (4.24) and (4.25) imply

$$
\Big\|\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0}\int_{T(B_i)} G(\mathbf{r}_j^\varepsilon(t),y)\,dy\Big\|_{L^2} = O(\epsilon^2)(\|\psi_j^k\|_{L^2}+|\zeta^k|). \tag{4.26}
$$

By (4.22) and (4.26) one finds that

$$\|(\mathcal{I}_{ij}^k)'(0,0,h)(\psi_i,\psi_0,\zeta)\|_{L^2} = O(\epsilon^2)\|(\psi,\zeta)\|_{\mathcal{Z}}. \tag{4.27}$$

This allows us to handle the first three components of $(\mathcal{S}_l^k)'$ in (4.18).

Finally consider $(\mathcal{Q}^k)'$ in the last component of $(\mathcal{S}_l^k)'$. Note that

$$\begin{aligned}
&(\mathcal{Q}^k)'(0,h)(\psi,\zeta)\\
= &\int_{-1}^{1}(\gamma_{11}(\mathcal{I}_{11}^k)'(0,0,h)(\psi_1,\psi_0,\zeta)+\gamma_{12}(\mathcal{I}_{21}^k)'(0,0,h)(\psi_2,\psi_0,\zeta))\mathcal{E}_1^k(0,h^k)\,dt\\
&+\int_{-1}^{1}(\gamma_{12}(\mathcal{I}_{12}^k)'(0,0,h)(\psi_1,\psi_0,\zeta)+\gamma_{22}(\mathcal{I}_{22}^k)'(0,0,h)(\psi_2,\psi_0,\zeta))\mathcal{E}_2^k(0,h^k)\,dt\\
&+\int_{-1}^{1}((\gamma_{11}-\gamma_{12})(\mathcal{I}_{10}^k)'(0,0,h)(\psi_1,\psi_0,\zeta)+(\gamma_{12}-\gamma_{22})(\mathcal{I}_{20}^k)'(0,0,h)(\psi_2,\psi_0,\zeta))\mathcal{E}_0^k(0,h^k)\,dt\\
&+\int_{-1}^{1}(\gamma_{11}\mathcal{I}_{11}^k(0,0,h)+\gamma_{12}\mathcal{I}_{21}^k(0,0,h))(\mathcal{E}_1^k)'(0,h^k)(\psi_1^k,\zeta^k)\,dt\\
&+\int_{-1}^{1}(\gamma_{12}\mathcal{I}_{12}^k(0,0,h)+\gamma_{22}\mathcal{I}_{22}^k(0,0,h))(\mathcal{E}_2^k)'(0,h^k)(\psi_2^k,\zeta^k)\,dt\\
&+\int_{-1}^{1}((\gamma_{11}-\gamma_{12})\mathcal{I}_{10}^k(0,0,h)+(\gamma_{12}-\gamma_{22})\mathcal{I}_{20}^k(0,0,h))(\mathcal{E}_0^k)'(0,h^k)(\psi_0^k,\zeta^k)\,dt.
\end{aligned} \tag{4.28}$$

Denote the six terms on the right side of (4.28) by I, II, III, IV, V, and VI respectively. Then the estimate (4.27) implies that

$$I,\ II,\ III = O(|\gamma|\epsilon^2)\,\|(\psi,\zeta)\|_{\mathcal{Z}}. \tag{4.29}$$

Regarding IV, V, and VI, note that

$$\begin{aligned}
\mathcal{I}_{ij}^k(0,0,h) &= \int_{T(B_i)} G(\mathbf{r}_j^k(t),y)\,dy\\
&= \int_{T^k(B_i^k)}\frac{1}{2\pi}\log\frac{1}{|\mathbf{r}_j^k(t)-y|}\,dy+\int_{T^k(B_i^k)}R(\mathbf{r}_j^k(t),y)\,dy+\sum_{l\neq k}\int_{T^l(B_i^l)}G(\mathbf{r}_j^k(t),y)\,dy\\
&= \frac{|B_i^k|}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)\epsilon^2+\epsilon^2 A_{ij}^k(t)
\end{aligned}$$

where

$$A_{ij}^k(t)=\begin{cases}\displaystyle\int_{B_i^k}\Big(\frac{1}{2\pi}\log\frac{1}{|r_1^k e^{\mathrm{i}(\pi-a_1^k t)}-\hat{y}|}+R(\mathbf{r}_1^k(t),T^k(\hat{y}))\Big)\,d\hat{y}+\sum_{l\neq k}\int_{B_i^l}G(\mathbf{r}_1^k(t),T^l(\hat{y}))\,d\hat{y} & \text{if} \quad j=1\\[2ex] \displaystyle\int_{B_i^k}\Big(\frac{1}{2\pi}\log\frac{1}{|r_j^k e^{\mathrm{i}a_j^k t}-\hat{y}|}+R(\mathbf{r}_j^k(t),T^k(\hat{y}))\Big)\,d\hat{y}+\sum_{l\neq k}\int_{B_i^l}G(\mathbf{r}_j^k(t),T^l(\hat{y}))\,d\hat{y} & \text{if} \quad j=2,0\end{cases}. \tag{4.30}$$

Then

$$\begin{aligned}
&\int_{-1}^{1}\mathcal{I}_{ij}^k(0,0,h)(\mathcal{E}_j^k)'(0,h^k)(\psi_j^k,\zeta^k)\,dt\\
&= \frac{|B_i^k|}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)\epsilon^2\int_{-1}^{1}(\mathcal{E}_j^k)'(0,h^k)(\psi_j^k,\zeta^k)\,dt+\epsilon^2\int_{-1}^{1}A_{ij}^k(t)(\mathcal{E}_j^k)'(0,h^k)(\psi_j^k,\zeta^k)\,dt.
\end{aligned} \tag{4.31}$$

Calculations from (2.58) and (3.11) show that

$$(\mathcal{E}_j^k)'(0,h^k)(\psi_j^k,\zeta^k) = (e_j^k)'(t) \tag{4.32}$$

where $(e_j^k)'(t)$ stands for the derivative of $e_j^k(t)$ with respect to t and

$$e_j^k(t) = \begin{cases} \Big(-\dfrac{(\alpha_1^k)'(h^k)t}{a_1^k} - \dfrac{(\beta_1^k)'(h^k)}{a_1^k r_1^k} \sin a_1^k t\Big)\psi_1^k + \Big(\dfrac{d}{d\eta^k}\Big|_{\eta^k=h^k} (\alpha_1^k \rho_1^k (\rho_1^k)' t - \rho_1^k (\beta_1^k)' \sin \alpha_1^k t)\Big)\zeta^k \\ \Big(-\dfrac{(\alpha_j^k)'(h^k)t}{a_j^k} + \dfrac{(\beta_j^k)'(h^k)}{a_j^k r_j^k} \sin a_j^k t\Big)\psi_j^k + \Big(\dfrac{d}{d\eta^k}\Big|_{\eta^k=h^k} (\alpha_j^k \rho_j^k (\rho_j^k)' t + \rho_j^k (\beta_j^k)' \sin \alpha_j^k t)\Big)\zeta^k \end{cases}. \quad (4.33)$$

One then estimates the second term on the right side of (4.31) via integration by parts:

$$\epsilon^2 \int_{-1}^{1} A_{ij}^k(t)(\mathcal{E}_j^k)'(0,h^k)(\psi_j^k,\zeta^k)\, dt = \epsilon^2 A_{ij}^k(t) e_j^k(t)\Big|_{-1}^{1} - \epsilon^2 \int_{-1}^{1} (A_{ij}^k)'(t) e_j^k(t)\, dt. \quad (4.34)$$

Then

$$\begin{aligned} \epsilon^2 A_{ij}^k(t) e_j^k(t)\Big|_{-1}^{1} &= \epsilon^2 A_{ij}^k(t)\Big[\frac{d}{d\eta^k}\Big|_{\eta^k=h^k} (\alpha_j^k \rho_j^k (\rho_j^k)' t + (-1)^j \rho_j^k (\beta_j^k)' \sin \alpha_j^k t)\Big]\zeta^k\Big|_{-1}^{1} \\ &= O(\epsilon^2)|\zeta^k| \quad (4.35) \\ \Big|\epsilon^2 \int_{-1}^{1} (A_{ij}^k)'(t) e_j^k(t)\, dt\Big| &\le \epsilon^2 \|(A_{ij}^k)'\|_{L^2} \|e_j^k\|_{L^2} \\ &= O(\epsilon^2)(\|\psi_j^k\|_{L^2} + |\zeta^k|) \quad (4.36) \end{aligned}$$

since $(A_{ij}^k)'(t)$ is bounded with respect to t. By (4.31), (4.34), (4.35), and (4.36) one concludes that

$$\int_{-1}^{1} \mathcal{I}_{ij}^k(0,0,h)(\mathcal{E}_j^k)'(0,h^k)(\psi_j^k,\zeta^k)\, dt = \frac{|B_i^k|}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)\epsilon^2 \int_{-1}^{1} (\mathcal{E}_j^k)'(0,h^k)(\psi_j^k,\zeta^k)\, dt + O(\epsilon^2)(\|\psi_j^k\|_{L^2} + |\zeta^k|). \quad (4.37)$$

By (3.16) of Lemma 3.2,

$$\begin{aligned} \int_{-1}^{1} (\mathcal{E}_1^k)'(0,h^k)(\psi_1^k,\zeta^k)\, dt + \int_{-1}^{1} (\mathcal{E}_0^k)'(0,h^k)(\psi_0^k,\zeta^k)\, dt &= 0 \\ \int_{-1}^{1} (\mathcal{E}_2^k)'(0,h^k)(\psi_2^k,\zeta^k)\, dt - \int_{-1}^{1} (\mathcal{E}_0^k)'(0,h^k)(\psi_0^k,\zeta^k)\, dt &= 0. \quad (4.38) \end{aligned}$$

Following (4.37) and (4.38) one arrives at

$$IV + V + VI = O(|\gamma|\epsilon^2)(\sum_{j=0}^{2} \|\psi_j^k\|_{L^2} + |\zeta^k|). \quad (4.39)$$

By (4.29) and (4.39), (4.28) becomes

$$(\mathcal{Q}^k)'(0,h)(\psi,\zeta) = O(|\gamma|\epsilon^2)\|(\psi,\zeta)\|_{\mathcal{Z}}. \quad (4.40)$$

By (4.27) and (4.40) one deduces that there exists $\check{C} > 0$ such that

$$\|\mathcal{S}_l'(0,h)(\psi,\zeta)\|_{\mathcal{Z}} \le \check{C}|\gamma|\epsilon^4 \|(\psi,\zeta)\|_{\mathcal{Z}} \quad (4.41)$$

for all $(\psi,\zeta) \in \mathcal{X}$. □

Lemmas 4.2 and 4.3 give a lower bound on the operator $\mathcal{S}'(0,h)$.

Lemma 4.4 *There exist $d > 0$ and $\sigma > 0$ such that when $|\gamma|\epsilon^3 < \sigma$,*

$$\langle \mathcal{S}'(0,h)(\psi,\zeta),(\psi,\zeta)\rangle_n \geq d\epsilon\|(\psi,\zeta)\|^2_{\mathcal{Y}}$$

for all $(\psi,\zeta) \in \mathcal{X}$.

Proof. Let d be the positive number given in Lemma 4.2 and $\sigma = \frac{d}{\check{C}}$ where $\check{C}$ comes from Lemma 4.3. Then Lemma 4.3 shows that for $|\gamma|\epsilon^3 < \sigma$,

$$\|\mathcal{S}'_l(0,h)(\psi,\zeta)\|_{\mathcal{Z}} \leq \check{C}|\gamma|\epsilon^4\|(\psi,\zeta)\|_{\mathcal{Z}} \leq \check{C}\sigma\epsilon\|(\psi,\zeta)\|_{\mathcal{Z}} = d\epsilon\|(\psi,\zeta)\|_{\mathcal{Z}} \tag{4.42}$$

for all $(\psi,\zeta) \in \mathcal{X}$. By Lemma 4.2 and (4.42)

$$\begin{aligned}\langle \mathcal{S}'(0,h)(\psi,\zeta),(\psi,\zeta)\rangle_n &= \langle \mathcal{S}'_s(0,h)(\psi,\zeta),(\psi,\zeta)\rangle_n + \langle \mathcal{S}'_l(0,h)(\psi,\zeta),(\psi,\zeta)\rangle_n \\ &\geq 2d\epsilon\|(\psi,\zeta)\|^2_{\mathcal{Y}} - d\epsilon\|(\psi,\zeta)\|^2_{\mathcal{Z}} \ \geq \ d\epsilon\|(\psi,\zeta)\|^2_{\mathcal{Y}}\end{aligned}$$

for all $(\psi,\zeta) \in \mathcal{X}$. □

A consequence of the positivity of $\mathcal{S}'(0,h)$ is its invertibility.

Lemma 4.5 *Let σ be the number given in Lemma 4.4.*

1. *There exists $\tilde{d} > 0$ such that if $|\gamma|\epsilon^3 < \sigma$, $\|\mathcal{S}'(0,h)(\psi,\zeta)\|_{\mathcal{Z}} \geq \tilde{d}\epsilon\|(\psi,\zeta)\|_{\mathcal{X}}$ holds for all $(\psi,\zeta) \in \mathcal{X}$.*
2. *The linear map $\mathcal{S}'(0,h)$ is one-to-one and onto from $\mathcal{X}$ to $\mathcal{Z}$; moreover $\|(\mathcal{S}'(0,h))^{-1}\| \leq \frac{1}{\tilde{d}\epsilon}$ where $\|(\mathcal{S}'(0,h))^{-1}\|$ is the operator norm of $(\mathcal{S}'(0,h))^{-1}$.*

Proof. By Lemma 4.4 it is easy to see that if $|\gamma|\epsilon^3 < \sigma$, then for all $(\psi,\zeta) \in \mathcal{X}$

$$\|(\psi,\zeta)\|_{\mathcal{Z}} \leq \frac{1}{d\epsilon}\|\mathcal{S}'(0,h)(\psi,\zeta)\|_{\mathcal{Z}}. \tag{4.43}$$

The first part of Lemma 4.5 asserts that the $\mathcal{Z}$-norm of (ψ,ζ) on the left side of (4.43) can be strengthened to the stronger $\mathcal{X}$-norm, if d is replaced by a possibly smaller $\tilde{d}$.

If part 1 is false, then there exist sequences γ_ν, ϵ_ν, and $(\psi_\nu,\zeta_\nu) \in \mathcal{X}$ such that $|\gamma_\nu|\epsilon_\nu^3 < \sigma$, $\|(\psi_\nu,\zeta_\nu)\|_{\mathcal{X}} = 1$ and with $\epsilon = \epsilon_\nu$ and $\gamma = \gamma_\nu$ in $\mathcal{S}'$,

$$\|\epsilon_\nu^{-1}\mathcal{S}'(0,0)(\psi_\nu,\zeta_\nu)\|_{\mathcal{Z}} \to 0, \quad \text{as } \nu \to \infty. \tag{4.44}$$

By (4.43),

$$\|(\psi_\nu,\zeta_\nu)\|_{\mathcal{Z}} \to 0. \tag{4.45}$$

Moreover, due to the compactness of the embedding $H^2(-1,1) \to C^1[-1,1]$ and $\|(\psi_\nu,\zeta_\nu)\|_{\mathcal{X}} = 1$, $\|\psi^k_{\nu.i}\|_{C^1} \to 0$ and in particular for all k and i,

$$(\psi^k_{\nu,i})'(\pm 1) \to 0 \text{ as } \nu \to \infty. \tag{4.46}$$

Since $\mathcal{S}'(0,h) = \mathcal{S}'_s(0,h) + \mathcal{S}'_l(0,h)$, and (4.42) and (4.45) imply that

$$\|\epsilon_\nu^{-1}\mathcal{S}'_l(0,h)(\psi_\nu,\zeta_\nu)\|_{\mathcal{Z}} \to 0, \tag{4.47}$$

one derives from (4.44) and (4.47) that

$$\|\epsilon_\nu^{-1}\mathcal{S}'_s(0,h)(\psi_\nu,\zeta_\nu)\|_{\mathcal{Z}} \to 0. \tag{4.48}$$

By (4.17) write

$$\epsilon_\nu^{-1}(\mathcal{S}_s^k)'(0,h^k)(\psi_\nu^k,\zeta_\nu^k) = \Pi\begin{pmatrix} -\frac{1}{(a_1^k r_1^k)^3}(\psi_{\nu,1}^k)'' \\ -\frac{1}{(a_2^k r_2^k)^3}(\psi_{\nu,2}^k)'' \\ -\frac{1}{(a_0^k r_0^k)^3}(\psi_{\nu,0}^k)'' \\ 0 \end{pmatrix} + \Pi\begin{pmatrix} -\frac{1}{a_1^k(r_1^k)^3}\psi_{\nu,1}^k \\ -\frac{1}{a_2^k(r_2^k)^3}\psi_{\nu,2}^k \\ -\frac{1}{a_0^k(r_0^k)^3}\psi_{\nu,0}^k \\ 2(\sum_{i=0}^2 \frac{d^2(\alpha_i^k\rho_i^k)}{d(\eta^k)^2}|_{\eta^k=h^k})\zeta^k \end{pmatrix}. \tag{4.49}$$

By (4.45) one finds that

$$\left\|\Pi\begin{pmatrix} -\frac{1}{a_1^k(r_1^k)^3}\psi_{\nu,1}^k \\ -\frac{1}{a_2^k(r_2^k)^3}\psi_{\nu,2}^k \\ -\frac{1}{a_0^k(r_0^k)^3}\psi_{\nu,2}^k \\ 2(\sum_{i=0}^2 \frac{d^2(\alpha_i^k\rho_i^k)}{d(\eta^k)^2}|_{\eta^k=h^k})\zeta^k \end{pmatrix}\right\|_{L^2((-1,1);\mathbb{R}^3)\times\mathbb{R}} \to 0 \tag{4.50}$$

for all k. Then (4.48), (4.49) and (4.50) show that

$$\left\|\Pi\begin{pmatrix} -\frac{1}{(a_1^k r_1^k)^3}(\psi_{\nu,1}^k)'' \\ -\frac{1}{(a_2^k r_2^k)^3}(\psi_{\nu,2}^k)'' \\ -\frac{1}{(a_0^k r_0^k)^3}(\psi_{\nu,0}^k)'' \\ 0 \end{pmatrix}\right\|_{L^2((-1,1);\mathbb{R}^3)\times\mathbb{R}} \to 0 \tag{4.51}$$

for all k. By the definition of Π, (3.28),

$$\begin{aligned} &\Pi\begin{pmatrix} -\frac{1}{(a_1^k r_1^k)^3}(\psi_{\nu,1}^k)'' \\ -\frac{1}{(a_2^k r_2^k)^3}(\psi_{\nu,2}^k)'' \\ -\frac{1}{(a_0^k r_0^k)^3}(\psi_{\nu,0}^k)'' \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{(a_1^k r_1^k)^3}(\psi_{\nu,1}^k)'' \\ -\frac{1}{(a_2^k r_2^k)^3}(\psi_{\nu,2}^k)'' \\ -\frac{1}{(a_0^k r_0^k)^3}(\psi_{\nu,0}^k)'' \\ 0 \end{pmatrix} + \begin{pmatrix} \big(\frac{1}{3(a_1^k r_1^k)^3}(\psi_{\nu,1}^k)' + \frac{1}{6(a_2^k r_2^k)^3}(\psi_{\nu,2}^k)' + \frac{1}{6(a_0^k r_0^k)^3}(\psi_{\nu,0}^k)'\big)\big|_{-1}^1 \\ \big(\frac{1}{6(a_1^k r_1^k)^3}(\psi_{\nu,1}^k)' + \frac{1}{3(a_2^k r_2^k)^3}(\psi_{\nu,2}^k)' - \frac{1}{6(a_0^k r_0^k)^3}(\psi_{\nu,0}^k)'\big)\big|_{-1}^1 \\ \big(\frac{1}{6(a_1^k r_1^k)^3}(\psi_{\nu,1}^k)' - \frac{1}{6(a_2^k r_2^k)^3}(\psi_{\nu,2}^k)' + \frac{1}{3(a_0^k r_0^k)^3}(\psi_{\nu,0}^k)'\big)\big|_{-1}^1 \\ 0 \end{pmatrix}. \end{aligned} \tag{4.52}$$

Moreover, (4.46) implies that

$$\begin{pmatrix} \big(\frac{1}{3(a_1^k r_1^k)^3}(\psi_{\nu,1}^k)' + \frac{1}{6(a_2^k r_2^k)^3}(\psi_{\nu,2}^k)' + \frac{1}{6(a_0^k r_0^k)^3}(\psi_{\nu,0}^k)'\big)\big|_{-1}^1 \\ \big(\frac{1}{6(a_1^k r_1^k)^3}(\psi_{\nu,1}^k)' + \frac{1}{3(a_2^k r_2^k)^3}(\psi_{\nu,2}^k)' - \frac{1}{6(a_0^k r_0^k)^3}(\psi_{\nu,0}^k)'\big)\big|_{-1}^1 \\ \big(\frac{1}{6(a_1^k r_1^k)^3}(\psi_{\nu,1}^k)' - \frac{1}{6(a_2^k r_2^k)^3}(\psi_{\nu,2}^k)' + \frac{1}{3(a_0^k r_0^k)^3}(\psi_{\nu,0}^k)'\big)\big|_{-1}^1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^4. \tag{4.53}$$

Therefore, by (4.51), (4.52) and (4.53), for all k and i

$$\|(\psi_{\nu,i}^k)''\|_{L^2} \to 0 \ \text{ as } \nu \to \infty. \tag{4.54}$$

From (4.45) and (4.54) we deduce that $\|(\psi_\nu,\zeta_\nu)\|_{\mathcal{X}} \to 0$, a contradiction to our assumption at the beginning that $\|(\psi_\nu,\zeta_\nu)\|_{\mathcal{X}} = 1$.

For part 2, it suffices to show that $\mathcal{S}'(0,h)$ is onto. First note that by the standard theory of second order linear differential equations, $\mathcal{S}'(0,h)$ is an unbounded self-adjoint operator on $\mathcal{Z}$ with the domain $\mathcal{X} \subset \mathcal{Z}$. Second if $(\tilde\psi,\tilde\zeta) \in \mathcal{Z}$ is perpendicular to the range of $\mathcal{S}'(0,h)$, i.e. $\langle \mathcal{S}'(0,h)(\psi,\zeta),(\tilde\psi,\tilde\zeta)\rangle = 0$ for all $(\psi,\zeta) \in \mathcal{X}$, then the self-adjointness of $\mathcal{S}'(0,h)$ implies that $(\tilde{\psi},\tilde{\zeta}) \in \mathcal{X}$ and $\mathcal{S}'(0,h)(\tilde{\psi},\tilde{\zeta}) = 0$. By (4.43), $(\tilde{\psi},\tilde{\zeta})$ is zero. Hence, the range of $\mathcal{S}'(0,h)$ is dense in $\mathcal{Z}$. Finally (4.43) implies that the range of $\mathcal{S}'(0,h)$ is a closed subset of $\mathcal{Z}$. Therefore $\mathcal{S}'(0,h)$ is onto. □

Finally in this section we state two properties regarding $\mathcal{S}''$, the second Fréchet derivative of $\mathcal{S}$ or the third variation of $\mathcal{J}$.

Lemma 4.6 *There exists $\widehat{C} > 0$ such that for all $(\phi,\eta) \in \mathcal{D}(\mathcal{S})$,*

$$\|\mathcal{S}''(\phi,\eta)((\tilde{\psi},\tilde{\zeta}),(\psi,\zeta))\|_{\mathcal{Z}} \le \widehat{C}(\epsilon + |\gamma|\epsilon^4)\|(\tilde{\psi},\tilde{\zeta})\|_{\mathcal{X}}\|(\psi,\zeta)\|_{\mathcal{X}}$$

holds for all (ψ,ζ) and $(\tilde{\psi},\tilde{\zeta}) \in \mathcal{X}$.

The proof, which is skipped, is straight forward estimation, similar to the proofs of [27, Lemma 3.2] and [26, Lemma 6.1].

Lemma 4.7 *There exists $\widehat{C} > 0$ such that for all $(\phi,\eta) \in \mathcal{D}(\mathcal{S})$,*

$$|\langle \mathcal{S}''(\phi,\eta)((\tilde{\psi},\tilde{\zeta}),(\psi,\zeta)),(\psi,\zeta)\rangle_n| \le \widehat{C}(\epsilon + |\gamma|\epsilon^4)\|(\tilde{\psi},\tilde{\zeta})\|_{\mathcal{X}}\|(\psi,\zeta)\|^2_{\mathcal{Y}}$$

holds for (ψ,ζ) and $(\tilde{\psi},\tilde{\zeta}) \in \mathcal{X}$.

See [27, Lemma 4.1] or [26, Lemma 7.2] for the proofs of similar formulas.

5 Minimization in a restricted class

For each $(\xi,\theta,w) \in \overline{\Xi_{\tilde{\delta}}} \times \mathbb{S}^n \times \overline{W}$ that specifies the exact double bubbles B^k, $k = 1,2,...,n$, and the transformation $T_{\epsilon,\xi,\theta}$, we find a locally $\mathcal{J}$ minimizing perturbed double bubble in the restricted class of perturbed double bubble assemblies. This restricted class is identified by (ξ,θ,w). One starts by solving

$$\mathcal{S}(\phi,\eta) = 0. \tag{5.1}$$

Lemma 5.1 *There exists $\sigma > 0$ such that (5.1) admits a solution $(\phi^*,\eta^*) \in \mathcal{D}(\mathcal{S}) \subset \mathcal{X}$ satisfying $\|(\phi^*,\eta^*) - (0,h)\|_{\mathcal{X}} \le \frac{2\widetilde{C}|\gamma|\epsilon^3}{\tilde{d}}$, provided $|\gamma|\epsilon^3 < \sigma$.*

Proof. For $(\phi,\eta) \in \mathcal{D}(\mathcal{S})$ write

$$\mathcal{S}(\phi,\eta) = \mathcal{S}(0,h) + \mathcal{S}'(0,h)((\phi,\eta)-(0,h)) + \mathcal{R}(\phi,\eta) \tag{5.2}$$

where $\mathcal{R}(\phi,\eta)$ is a higher order term defined by (5.2). Define an operator $\mathcal{T}$ from $\mathcal{D}(\mathcal{S}) \subset \mathcal{X}$ into $\mathcal{X}$ by

$$\mathcal{T}(\phi,\eta) = (0,h) - (\mathcal{S}'(0,h))^{-1}(\mathcal{S}(0,h) + \mathcal{R}(\phi,\eta)), \tag{5.3}$$

and re-write the equation $\mathcal{S}(\phi,\eta) = 0$ as a fixed point problem $\mathcal{T}(\phi,\eta) = (\phi,\eta)$.

Let $c \in (0,\bar{c})$, where $\bar{c}$ is given in (3.36), and define a closed ball $\mathcal{W} = \{(\phi,\eta) \in \mathcal{X} : \|(\phi,\eta)-(0,h)\|_{\mathcal{X}} \le c\} \subset \mathcal{D}(\mathcal{S})$. For $(\phi,\eta) \in \mathcal{W}$,

$$\|\mathcal{R}(\phi,\eta)\|_{\mathcal{Z}} \le \frac{1}{2}\sup_{\tau\in(0,1)}\|\mathcal{S}''((1-\tau)(0,h)+\tau(\phi,\eta))((\phi,\eta-h),(\phi,\eta-h))\|_{\mathcal{Z}} \le \frac{\widehat{C}(\epsilon+|\gamma|\epsilon^4)}{2}\|(\phi,\eta-h)\|^2_{\mathcal{X}} \tag{5.4}$$

by Lemma 4.6. Then by Lemmas 3.3 and 4.5

$$\begin{aligned}
\|\mathcal{T}(\phi,\eta)-(0,h)\|_{\mathcal{X}} &\le \|(\mathcal{S}'(0,h))^{-1}\|(\|\mathcal{S}(0,h)\|_{\mathcal{Z}} + \|\mathcal{R}(\phi,\eta)\|_{\mathcal{Z}}) \\
&\le \frac{1}{\epsilon\tilde{d}}\Big(\widetilde{C}|\gamma|\epsilon^4 + \frac{\widehat{C}(\epsilon+|\gamma|\epsilon^4)}{2}c^2\Big) \\
&\le \frac{\widetilde{C}\sigma}{\tilde{d}} + \frac{\widehat{C}+\widehat{C}\sigma}{2\tilde{d}}c^2.
\end{aligned} \tag{5.5}$$

Let $(\tilde{\phi}, \tilde{\eta}) \in \mathcal{W}$. Consider

$$\begin{aligned}
\|\mathcal{T}(\phi,\eta) - \mathcal{T}(\tilde{\phi},\tilde{\eta})\|_{\mathcal{X}} &\leq \|(\mathcal{S}'(0,h))^{-1}\| \; \|\mathcal{R}(\phi,\eta) - \mathcal{R}(\tilde{\phi},\tilde{\eta})\|_{\mathcal{Z}} \\
&\leq \frac{1}{\epsilon\tilde{d}}\|\mathcal{S}(\phi,\eta) - \mathcal{S}(\tilde{\phi},\tilde{\eta}) - \mathcal{S}'(0,h)((\phi,\eta)-(\tilde{\phi},\tilde{\eta}))\|_{\mathcal{Z}} \\
&\leq \frac{1}{\epsilon\tilde{d}}\|\mathcal{S}(\phi,\eta) - \mathcal{S}(\tilde{\phi},\tilde{\eta}) - \mathcal{S}'(\tilde{\phi},\tilde{\eta})((\phi,\eta)-(\tilde{\phi},\tilde{\eta}))\|_{\mathcal{Z}} \\
&\quad +\frac{1}{\epsilon\tilde{d}}\|(\mathcal{S}'(\tilde{\phi},\tilde{\eta}) - \mathcal{S}'(0,h))((\phi,\eta)-(\tilde{\phi},\tilde{\eta}))\|_{\mathcal{Z}} \\
&\leq \frac{1}{2\epsilon\tilde{d}}\sup_{\tau\in(0,1)}\|\mathcal{S}''((1-\tau)(\tilde{\phi},\tilde{\eta})+\tau(\phi,\eta))\| \; \|(\phi,\eta)-(\tilde{\phi},\tilde{\eta})\|_{\mathcal{X}}^2 \\
&\quad +\frac{1}{\epsilon\tilde{d}}\sup_{\tau\in(0,1)}\|\mathcal{S}''((1-\tau)(0,h)+\tau(\tilde{\phi},\tilde{\eta}))\| \; \|(\tilde{\phi},\tilde{\eta}-h)\|_{\mathcal{X}}\|(\phi,\eta)-(\tilde{\phi},\tilde{\eta})\|_{\mathcal{X}} \\
&\leq \frac{\widehat{C}(\epsilon+|\gamma|\epsilon^4)}{\epsilon\tilde{d}}\Big(c+c\Big)\|(\phi,\eta)-(\tilde{\phi},\tilde{\eta})\|_{\mathcal{X}} \\
&\leq \frac{2\widehat{C}(1+\sigma)c}{\tilde{d}}\|(\phi,\eta)-(\tilde{\phi},\tilde{\eta})\|_{\mathcal{X}}. \qquad (5.6)
\end{aligned}$$

Take

$$c = \min\Big\{\frac{\tilde{d}}{6\widehat{C}},\ \frac{\bar{c}}{2}\Big\}. \qquad (5.7)$$

Let σ be small enough so that Lemma 4.5 holds, and moreover

$$\sigma \leq \min\Big\{1,\ \frac{\tilde{d}c}{2\widetilde{C}}\Big\}. \qquad (5.8)$$

It follows from (5.5) and (5.6) that

$$\|\mathcal{T}(\phi,\eta)-(0,h)\|_{\mathcal{X}} \leq c \ \text{ and } \ \|\mathcal{T}(\phi,\eta)-\mathcal{T}(\tilde{\phi},\tilde{\eta})\|_{\mathcal{X}} \leq \frac{2}{3}\|(\phi,\eta)-(\tilde{\phi},\tilde{\eta})\|_{\mathcal{X}} \qquad (5.9)$$

for all $(\phi,\eta),\ (\tilde{\phi},\tilde{\eta}) \in \mathcal{W}$. The Contraction Mapping Principle says that $\mathcal{T}$ has a fixed point in $\mathcal{W}$. This fixed point is denoted by (ϕ^*,η^*), and it solves (5.1).

To prove the estimate of (ϕ^*,η^*), revisit the equation $(\phi,\eta)=\mathcal{T}(\phi,\eta)$, satisfied by (ϕ^*,η^*), and derive from (5.3) and (5.4) that

$$\begin{aligned}
\|(\phi^*,\eta^*-h)\|_{\mathcal{X}} &\leq \|(\mathcal{S}'(0,h))^{-1}\|(\|\mathcal{S}(0,h)\|_{\mathcal{Z}} + \|\mathcal{R}(\phi^*,\eta^*)\|_{\mathcal{Z}}) \\
&\leq \frac{1}{\epsilon\tilde{d}}\Big(\widetilde{C}|\gamma|\epsilon^4 + \frac{\widehat{C}(\epsilon+|\gamma|\epsilon^4)}{2}\|(\phi^*,\eta^*-h)\|_{\mathcal{X}}^2\Big).
\end{aligned}$$

Rewrite the above as

$$\Big(1-\frac{\widehat{C}(1+|\gamma|\epsilon^3)}{2\tilde{d}}\|(\phi^*,\eta^*-h)\|_{\mathcal{X}}\Big)\|(\phi^*,\eta^*-h)\|_{\mathcal{X}} \leq \frac{\widetilde{C}|\gamma|\epsilon^3}{\tilde{d}}. \qquad (5.10)$$

In (5.10) estimate

$$\frac{\widehat{C}(1+|\gamma|\epsilon^3)}{2\tilde{d}}\|(\phi^*,\eta^*-h)\|_{\mathcal{X}} \leq \frac{\widehat{C}(1+|\gamma|\epsilon^3)}{2\tilde{d}}c \leq \frac{\widehat{C}c(1+\sigma)}{2\tilde{d}} \leq \frac{1}{6} \qquad (5.11)$$

by (5.7) and (5.8). The estimate of (ϕ^*,η^*) follows from (5.10). □

The first part of the next lemma shows that the assembly (ϕ^*,η^*) is locally energy minimizing, hence stable, within the restricted class of perturbed double bubble assemblies. The second part gives a measurement on the non-degeneracy of (ϕ^*,η^*) within the restricted class.

Lemma 5.2 *1. There exist $\hat{d} > 0$ and $\sigma > 0$ such that if $|\gamma|\epsilon^3 < \sigma$, then the solution (ϕ^*, η^*) found in Lemma 5.1 satisfies $\langle \mathcal{S}'(\phi^*, \eta^*)(\psi, \zeta), (\psi, \zeta)\rangle_n \geq \hat{d}\epsilon \|(\psi, \zeta)\|_{\mathcal{Y}}^2$ for all $(\psi, \zeta) \in \mathcal{X}$.*

2. There exist $\check{d} > 0$ and $\sigma > 0$ such that if $|\gamma|\epsilon^3 < \sigma$, the solution (ϕ^*, η^*) satisfies $\|\mathcal{S}'(\phi^*, \eta^*)(\psi, \zeta)\|_{\mathcal{Z}} \geq \check{d}\epsilon \|(\psi, \zeta)\|_{\mathcal{X}}$ for all $(\psi, \zeta) \in \mathcal{X}$.

Proof. There exists $\tilde{\tau} \in (0, 1)$ such that

$$\begin{aligned}&\langle \mathcal{S}'(\phi^*, \eta^*)(\psi, \zeta), (\psi, \zeta)\rangle_n \\ &= \langle \mathcal{S}'(0, h)(\psi, \zeta), (\psi, \zeta)\rangle_n + \langle \mathcal{S}''((1 - \tilde{\tau})(0, h) + \tilde{\tau}(\phi^*, \eta^*))((\phi^*, \eta^* - h), (\psi, \zeta)), (\psi, \zeta)\rangle_n.\end{aligned}$$

By Lemma 4.7,

$$|\langle \mathcal{S}''((1 - \tilde{\tau})(0, h) + \tilde{\tau}(\phi^*, \eta^*))((\phi^*, \eta^* - h), (\psi, \zeta)), (\psi, \zeta)\rangle_n| \leq \widehat{C}(\epsilon + |\gamma|\epsilon^4)\|(\phi^*, \eta^* - h)\|_{\mathcal{X}} \|(\psi, \zeta)\|_{\mathcal{Y}}^2. \quad (5.12)$$

Consequently by Lemmas 4.4 and 5.1

$$\begin{aligned}\langle \mathcal{S}'(\phi^*, \eta^*)(\psi, \zeta), (\psi, \zeta)\rangle_n &\geq d\epsilon\|(\psi, \zeta)\|_{\mathcal{Y}}^2 - \widehat{C}(\epsilon + |\gamma|\epsilon^4)\frac{2\widetilde{C}|\gamma|\epsilon^3}{\tilde{d}}\|(\psi, \zeta)\|_{\mathcal{Y}}^2 \\ &\geq \epsilon\Big(d - \frac{2\widehat{C}\widetilde{C}(\sigma + \sigma^2)}{\tilde{d}}\Big)\|(\psi, \zeta)\|_{\mathcal{Y}}^2 \geq \frac{d\epsilon}{2}\|(\psi, \zeta)\|_{\mathcal{Y}}^2\end{aligned}$$

if σ is sufficiently small. The first part follows if $\hat{d} = \frac{d}{2}$.

By Lemmas 4.5, 4.6 and 5.1,

$$\begin{aligned}\|\mathcal{S}'(\phi^*, \eta^*)(\psi, \zeta)\|_{\mathcal{Z}} &\geq \|\mathcal{S}'(0, h)(\psi, \zeta)\|_{\mathcal{Z}} - \sup_{\tau \in (0,1)} \|\mathcal{S}''((1 - \tau)(0, h) + \tau(\phi^*, \eta^*))((\phi^*, \eta^* - h), (\psi, \zeta))\|_{\mathcal{Z}} \\ &\geq \tilde{d}\epsilon\|(\psi, \zeta)\|_{\mathcal{X}} - \widehat{C}(\epsilon + |\gamma|\epsilon^4)\|(\phi^*, \eta^* - h)\|_{\mathcal{X}}\|(\psi, \zeta)\|_{\mathcal{X}} \\ &\geq \Big(\tilde{d}\epsilon - \widehat{C}(\epsilon + |\gamma|\epsilon^4)\frac{2\widetilde{C}|\gamma|\epsilon^3}{\tilde{d}}\Big)\|(\psi, \zeta)\|_{\mathcal{Z}} \\ &\geq \epsilon\Big(\tilde{d} - \frac{2\widehat{C}\widetilde{C}(\sigma + \sigma^2)}{\tilde{d}}\Big)\|(\psi, \zeta)\|_{\mathcal{Z}} \geq \frac{\tilde{d}\epsilon}{2}\|(\psi, \zeta)\|_{\mathcal{Z}}\end{aligned}$$

if σ is sufficiently small. Part 2 follows if $\check{d} = \frac{\tilde{d}}{2}$. □

One interprets the equation $\mathcal{S}(\phi^*, \eta^*) = 0$ and proves the following. Let $T(P^*)$ be the assembly represented by (ϕ^*, η^*) and $T(P_i^*) = \cup_{k=1}^n T^k(P_i^{*,k})$ for $i = 1, 2$.

Lemma 5.3 *The perturbed double bubble assembly described by (ϕ^*, η^*) satisfies the equations*

$$\begin{aligned}\epsilon\mathcal{K}_1^k(\phi_1^{*,k}, \eta^{*,k}) + \epsilon^2(\gamma_{11}I_{T(P_1^*)} + \gamma_{12}I_{T(P_2^*)}) &= \lambda_1^k \quad (5.13)\\ \epsilon\mathcal{K}_2^k(\phi_2^{*,k}, \eta^{*,k}) + \epsilon^2(\gamma_{12}I_{T(P_1^*)} + \gamma_{22}I_{T(P_2^*)}) &= \lambda_2^k \quad (5.14)\\ \epsilon\mathcal{K}_0^k(\phi_0^{*,k}, \eta^{*,k}) + \epsilon^2(\gamma_{11} - \gamma_{12})I_{T(P_1^*)} + \epsilon^2(\gamma_{12} - \gamma_{22})I_{T(P_2^*)} &= \lambda_1^k - \lambda_2^k \quad (5.15)\end{aligned}$$

on the boundaries of each perturbed double bubble $T^k(P^{*,k})$, $k = 1, 2, ..., n$. Moreover at the triple points,

$$\sum_{i=0}^{2} \mathbf{T}_i^k \cdot \mathbf{X}^{S,k}\Big|_{-1}^{1} = 0, \quad k = 1, 2, ..., n, \quad (5.16)$$

where the $\mathbf{T}_i^k$'s are unit tangent vectors of the boundaries of the k-th perturbed double bubble and $\mathbf{X}^{S,k}$ is given in Lemma 3.1.

Proof. By the virtue of the projection operator Π, the first three components of each $\mathcal{S}^k$ in (3.35) imply that for each k there exist $\lambda_1^k, \lambda_2^k \in \mathbb{R}$ such that the equations (5.13)-(5.15) hold.

From the fourth component of $\mathcal{S}^k$ in (3.35) one sees that

$$\epsilon\widetilde{\mathcal{K}}^k(\phi^{*,k}, \eta^{*,k}) + \epsilon^2 \mathcal{Q}^k(\phi^*, \eta^*) = 0.$$

By the expression of $\widetilde{\mathcal{K}}^k$ in (3.22) and the definition (3.24) of $\mathcal{Q}^k$, the last equation asserts

$$\sum_{i=0}^{2} \mathbf{T}_i^k \cdot \mathbf{X}^{S,k}\Big|_{-1}^{1} + \int_{-1}^{1} \big(\epsilon\mathcal{K}_1^k(\phi_1^{*,k}, \eta^{*,k}) + \epsilon^2(\gamma_{11} I_{T(P_1^*)} + \gamma_{12} I_{T(P_2^*)})\big)\mathcal{E}_1^k(\phi_1^{*,k}, \eta^{*,k})\, dt$$

$$+ \int_{-1}^{1} \big(\epsilon\mathcal{K}_2^k(\phi_2^{*,k}, \eta^{*,k}) + \epsilon^2(\gamma_{12} I_{T(P_2^*)} + \gamma_{22} I_{T(P_2^*)})\big)\mathcal{E}_2^k(\phi_2^{*,k}, \eta^{*,k})\, dt$$

$$+ \int_{-1}^{1} \big(\epsilon\mathcal{K}_0^k(\phi_0^{*,k}, \eta^{*,k}) + \epsilon^2(\gamma_{11} - \gamma_{12}) I_{T(P_1^*)} + \epsilon^2(\gamma_{12} - \gamma_{22}) I_{T(P_2^*)}\big)\mathcal{E}_0^k(\phi_0^{*,k}, \eta^{*,k})\, dt = 0.$$

The equations (5.13)-(5.15) reduce the last equation to

$$\sum_{i=0}^{2} \mathbf{T}_i^k \cdot \mathbf{X}^{S,k}\Big|_{-1}^{1} + \int_{-1}^{1} \lambda_1^k \mathcal{E}_1^k(\phi_1^{*,k}, \eta^{*,k})\, dt + \int_{-1}^{1} \lambda_2^k \mathcal{E}_2^k(\phi_2^{*,k}, \eta^{*,k})\, dt + \int_{-1}^{1} (\lambda_1^k - \lambda_2^k)\mathcal{E}_0^k(\phi_0^{*,k}, \eta^{*,k})\, dt = 0.$$

Formula (3.16) of Lemma 3.2 further simplifies the above to

$$\sum_{i=0}^{2} \mathbf{T}_i^k \cdot \mathbf{X}^{S,k}\Big|_{-1}^{1} = 0$$

completing the proof. □

Lemma 5.3 does not assert that the perturbed double bubble assembly $T(P^*)$ is a critical point of $\mathcal{J}$. There are two reasons. First the constants λ_i^k in (5.13)-(5.15) depend on k, but the constants λ_i in (1.2)-(1.4) are independent of k. Therefore (5.13)-(5.15) do not imply (1.2)-(1.4). Second the equation (5.16) does not imply (1.5).

The next section will resolve these two issues.

6 Proof of Theorem 1.1

Recall that the locally energy minimizing perturbed double bubble assembly $T(P^*)$ found in the last section was constructed under two conditions:

1. The perturbed double bubbles $P^{*,k}$ are mapped into D by the transformation $T_{\epsilon,\xi,\theta}$ with given $\xi = (\xi^1, ..., \xi^n) \in \overline{\Xi_{\tilde{\delta}}}$ and $\theta = (\theta^1, ..., \theta^n) \in \mathbb{S}^n$.
2. Each $P_i^{*,k}$, for $k = 1, 2, ..., n$ and $i = 1, 2$, has the prescribed area w_i^k.

In this section one minimizes $\mathcal{J}(\phi^*(\cdot, \xi, \theta, w), \eta^*(\xi, \theta, w))$ with respect to $(\xi, \theta, w) \in \overline{\Xi_{\tilde{\delta}}} \times \mathbb{S}^n \times \overline{W}$ to obtain a minimum (ξ^*, θ^*, w^*). With the particular ξ^*, θ^* and w^*, $(\phi^*(\cdot, \xi^*, \theta^*, w^*), \eta^*(\xi^*, \theta^*, w^*))$ will yield the final solution to (1.2)-(1.5).

The first lemma gives an estimate on the difference between the energy of (ϕ^*, η^*) and the energy of the exact double bubble assembly $T(B)$.

Lemma 6.1 *If σ is small, then* $|\mathcal{J}(\phi^*, \eta^*) - \mathcal{J}(0, h)| \leq |\gamma|\epsilon^4\Big(\frac{\widetilde{C}^2}{\tilde{d}}|\gamma|\epsilon^3 + \frac{10\widehat{C}\widetilde{C}^3}{3\tilde{d}^3}(|\gamma|\epsilon^3)^2 + \frac{10\widehat{C}\widetilde{C}^3}{3\tilde{d}^3}(|\gamma|\epsilon^3)^3\Big)$ *holds uniformly for all $(\xi, \theta, w) \in \overline{\Xi_{\tilde{\delta}}} \times \mathbb{S}^n \times \overline{W}$.*

Proof. Expanding $\mathcal{J}(\phi^*,\eta^*)$ yields

$$\begin{aligned}\mathcal{J}(\phi^*,\eta^*) &= \mathcal{J}(0,h)+\langle \mathcal{S}(0,h),(\phi^*,\eta^*-h)\rangle_n+\frac{1}{2}\langle \mathcal{S}'(0,h)(\phi^*,\eta^*-h),(\phi^*,\eta^*-h)\rangle_n \\ &\quad +\frac{1}{6}\langle \mathcal{S}''((1-\tilde{\tau})(0,h)+\tilde{\tau}(\phi^*,\eta^*))((\phi^*,\eta^*-h),(\phi^*,\eta^*-h)),(\phi^*,\eta^*-h)\rangle_n \qquad (6.1)\end{aligned}$$

for some $\tilde{\tau}\in(0,1)$. Also expanding $\mathcal{S}(\phi^*,\eta^*)$ gives

$$\begin{aligned}&\|\mathcal{S}(\phi^*,\eta^*)-\mathcal{S}(0,h)-\mathcal{S}'(0,h)(\phi^*,\eta^*-h)\|_{\mathcal{Z}} \\ \le\ & \sup_{\tau\in(0,1)}\frac{1}{2}\|\mathcal{S}''((1-\tau)(0,h)+\tau(\phi^*,\eta^*))((\phi^*,\eta^*-h),(\phi^*,\eta^*-h))\|_{\mathcal{Z}}. \qquad (6.2)\end{aligned}$$

Since $\mathcal{S}(\phi^*,\eta^*)=0$, (6.2) shows that

$$\|\mathcal{S}(0,h)+\mathcal{S}'(0,h)(\phi^*,\eta^*-h)\|_{\mathcal{Z}}\le \sup_{\tau\in(0,1)}\frac{1}{2}\|\mathcal{S}''((1-\tau)(0,h)+\tau(\phi^*,\eta^*-h))((\phi^*,\eta^*),(\phi^*,\eta^*-h))\|_{\mathcal{Z}},$$

which implies that

$$\begin{aligned}&|\langle \mathcal{S}(0,h),(\phi^*,\eta^*-h)\rangle_n+\langle \mathcal{S}'(0,h)(\phi^*,\eta^*-h),(\phi^*,\eta^*-h)\rangle_n| \\ \le\ & \Big(\frac{1}{2}\sup_{\tau\in(0,1)}\|\mathcal{S}''((1-\tau)(0,h)+\tau(\phi^*,\eta^*))((\phi^*,\eta^*-h),(\phi^*,\eta^*-h))\|_{\mathcal{Z}}\Big)\|(\phi^*,\eta^*-h)\|_{\mathcal{X}}. \qquad (6.3)\end{aligned}$$

By (6.3), (6.1) yields that

$$\begin{aligned}&\Big|\mathcal{J}(\phi^*,\eta^*)-\mathcal{J}(0,h)-\frac{1}{2}\langle \mathcal{S}(0,h),(\phi^*,\eta^*-h)\rangle_n\Big| \\ \le\ & \Big(\frac{5}{12}\sup_{\tau\in(0,1)}\|\mathcal{S}''((1-\tau)(0,h)+\tau(\phi^*,\eta^*))((\phi^*,\eta^*-h),(\phi^*,\eta^*-h))\|_{\mathcal{Z}}\Big)\|(\phi^*,\eta^*-h)\|_{\mathcal{X}}. \qquad (6.4)\end{aligned}$$

Lemmas 3.3, 4.6 and 5.1 show that

$$\begin{aligned}&|\mathcal{J}(\phi^*,\eta^*)-\mathcal{J}(0,h)| \\ \le\ & \frac{1}{2}|\langle \mathcal{S}(0,h),(\phi^*,\eta^*-h)\rangle_n| + \\ & \Big(\frac{5}{12}\sup_{\tau\in(0,1)}\|\mathcal{S}''((1-\tau)(0,h)+\tau(\phi^*,\eta^*))((\phi^*,\eta^*-h),(\phi^*,\eta^*-h))\|_{\mathcal{Z}}\Big)\|(\phi^*,\eta^*-h)\|_{\mathcal{X}} \\ \le\ & \frac{1}{2}(\widetilde{C}|\gamma|\epsilon^4)\frac{2\widetilde{C}|\gamma|\epsilon^3}{\tilde{d}}+\frac{5}{12}\widehat{C}(\epsilon+|\gamma|\epsilon^4)\Big(\frac{2\widetilde{C}|\gamma|\epsilon^3}{\tilde{d}}\Big)^3 \\ =\ & |\gamma|\epsilon^4\Big(\frac{\widetilde{C}^2}{\tilde{d}}|\gamma|\epsilon^3+\frac{10\widehat{C}\widetilde{C}^3}{3\tilde{d}^3}(|\gamma|\epsilon^3)^2+\frac{10\widehat{C}\widetilde{C}^3}{3\tilde{d}^3}(|\gamma|\epsilon^3)^3\Big) \qquad (6.5)\end{aligned}$$

which proves the lemma. □

The solution (ϕ^*,η^*) to (5.1) found in Lemma 5.1 depends on ξ, θ and w. To emphasize this dependence, write $\phi^*=\phi^*(\cdot,\xi,\theta,w)$ and $\eta^*=\eta^*(\xi,\theta,w)$. The exact double bubble $T(B)$ whose internal representation is $(0,h)$ also depends on ξ, θ and w. Now let ξ vary in $\overline{\Xi_{\bar{\delta}}}$, θ vary in $\mathbb{S}^n$, w vary in $\overline{W}$, and set

$$J(\xi,\theta,w)=\mathcal{J}(\phi^*(\cdot,\xi,\theta,w),\eta^*(\xi,\theta,w)). \qquad (6.6)$$

In (6.6) J is treated as functions of $(\xi,\theta,w)\in\overline{\Xi_{\bar{\delta}}}\times\mathbb{S}^n\times\overline{W}$. Since $\overline{\Xi_{\bar{\delta}}}\times\mathbb{S}^n\times\overline{W}$ is compact, J attains at least one minimum. The next lemma shows that such a minimum must be in the interior of the set.

Lemma 6.2 *Let $(\xi^*,\theta^*,w^*) \in \overline{\Xi_{\overline{\delta}}} \times \mathbb{S}^n \times \overline{W}$ be a minimum of J. When δ and σ are sufficiently small and $\widetilde{\sigma}$ is sufficiently large, (ξ^*,θ^*,w^*) must be in $\Xi_{\overline{\delta}} \times \mathbb{S}^n \times W$, the interior of $\overline{\Xi_{\overline{\delta}}} \times \mathbb{S}^n \times \overline{W}$.*

Proof. Suppose

$$(\xi^*,\theta^*,w^*) \to (\xi^\circ,\theta^\circ,w^\circ) \tag{6.7}$$

as $\epsilon \to 0$ and $|\gamma|\epsilon^3 \to 0$, possibly along a subsequence.

First show that $w^\circ = \widetilde{w}$ where

$$\widetilde{w} = \Big(\Big(\frac{m}{n}, \frac{1-m}{n}\Big), ..., \Big(\frac{m}{n}, \frac{1-m}{n}\Big)\Big). \tag{6.8}$$

Let

$$\frac{\epsilon}{\overline{\lambda}(\gamma)\epsilon^4 \log\frac{1}{\epsilon}} \to \Delta \ \text{ and } \ \frac{\gamma}{\overline{\lambda}(\gamma)} \to \Gamma \ \text{ as } \ \epsilon \to 0, \ \text{ possibly along a subsequence.} \tag{6.9}$$

By condition 2 of Theorem 1.1

$$0 \leq \Delta \leq \frac{1}{\widetilde{\sigma}} \tag{6.10}$$

and by condition 3

$$1 \leq \overline{\lambda}(\Gamma) \leq \overline{\overline{\lambda}}(\Gamma) \leq \frac{1}{\iota} \tag{6.11}$$

where $\overline{\lambda}(\Gamma)$ and $\overline{\overline{\lambda}}(\Gamma)$ are the two eigenvalues of Γ. Then by Lemmas 2.1 and 6.1, as $\epsilon \to 0$,

$$\frac{J(\xi,\theta,w)}{\overline{\lambda}(\gamma)\epsilon^4 \log\frac{1}{\epsilon}} \to 2\Delta \sum_{k=1}^n \sum_{i=0}^2 a_i^k r_i^k + \sum_{k=1}^n \sum_{i,j=1}^2 \frac{\Gamma_{ij} w_i^k w_j^k}{4\pi} \tag{6.12}$$

uniformly for $(\xi,\theta,w) \in \overline{\Xi_{\overline{\delta}}} \times \mathbb{S}^n \times \overline{W}$. The right side of (6.12) is a function of w, since a_i^k and r_i^k depend on w_1^k and w_2^k only. If $\widetilde{\sigma}$ is sufficiently large, then Δ is sufficiently small and by Appendix C the right side of (6.12) is minimized at $w = \widetilde{w}$. If w° were not $\widetilde{w}$, then $J(\xi^*,\theta^*,w^*) > J(\xi^*,\theta^*,\widetilde{w})$ when ϵ is sufficiently small, a contradiction to the assumption that (ξ^*,θ^*,w^*) is a minimum of J.

Next show that

$$F(\xi^\circ) = \min_{\xi\in\Xi} F(\xi). \tag{6.13}$$

Let

$$\begin{aligned} H(\xi,\theta) \ &= \ \frac{1}{\overline{\lambda}(\gamma)\epsilon^4}\Big\{J(\xi,\theta,w^*) - \Big[\epsilon \sum_{k=1}^n \sum_{i=0}^3 2a_i^{*,k} r_i^{*,k} + \Big(\log\frac{1}{\epsilon}\Big)\epsilon^4 \sum_{k=1}^n \sum_{i,j=1}^2 \frac{\gamma_{ij} w_i^{*,k} w_j^{*,k}}{4\pi} \\ &\quad + \epsilon^4 \sum_{k=1}^n \sum_{i,j=1}^2 \frac{\gamma_{ij}}{2} \int_{B_i^{*,k}} \int_{B_j^{*,k}} \frac{1}{2\pi} \log\frac{1}{|\hat{x}-\hat{y}|}\, d\hat{x} d\hat{y}\Big]\Big\}. \end{aligned} \tag{6.14}$$

In (6.14) $B^{*,k}$ is the exact double bubble determined by $w_1^{*,k}$ and $w_2^{*,k}$. By Lemmas 2.1 and 6.1, and the fact that $w^* \to \widetilde{w}$, one obtains that, as $\epsilon \to 0$ and $|\gamma|\epsilon^3 \to 0$,

$$H(\xi,\theta) \to \frac{1}{2}\Big(\Gamma_{11}\Big(\frac{m}{n}\Big)^2 + 2\Gamma_{12}\Big(\frac{m}{n}\Big)\Big(\frac{1-m}{n}\Big) + \Gamma_{22}\Big(\frac{1-m}{n}\Big)^2\Big)F(\xi) \tag{6.15}$$

uniformly for $(\xi,\theta) \in \overline{\Xi_{\overline{\delta}}} \times \mathbb{S}^n$. If ξ° were not a minimum of F, then let $\widetilde{\xi}$ be a minimum of F. By (2.14) $\widetilde{\xi}$ must be in $\Xi_{\overline{\delta}}$. One finds that $H(\xi^*,\theta^*) > H(\widetilde{\xi},\theta^*)$ when ϵ and $|\gamma|\epsilon^3$ are sufficiently small, a contradiction to the fact that (ξ^*,θ^*,w^*) is a minimum of J.

Since $\widetilde{w}$ is in W and any minimum of F is attained in $\Xi_{\overline{\delta}}$ by (2.14), one sees that when ϵ and $|\gamma|\epsilon^3$ are sufficiently small (ξ^*,θ^*,w^*) is in $\Xi_{\overline{\delta}} \times \mathbb{S}^n \times W$. □

The proof of Lemma 6.2 is the first instance that requires $\widetilde{\sigma}$ to be large. It is the second time, with (2.17) being the first, when δ is assumed to be small. It is also in this proof that the condition 3 of Theorem 1.1 is first used. From this moment on, δ, σ and $\widetilde{\sigma}$ become dependent on ι.

The dependence of $(\phi^*, \eta^*) = (\phi^*(t, \xi, \theta, w), \eta^*(\xi, \theta, w))$ on ξ_i^k, and θ^k is investigated in the next lemma.

Lemma 6.3 *When σ is sufficiently small, $\|\frac{\partial(\phi^*,\eta^*)}{\partial \xi_i^k}\|_{\mathcal{X}} = O(|\gamma|\epsilon^3)$ where $k = 1, 2, ..., n$ and $i = 1, 2$, and $\|\frac{\partial(\phi^*,\eta^*)}{\partial \theta^k}\|_{\mathcal{X}} = O(|\gamma|\epsilon^4)$ uniformly with respect to all $(\xi, \theta, w) \in \Xi_{\overline{\delta}} \times S^n \times W$.*

Proof. The equation (5.1) is now written as

$$\mathcal{S}(\phi, \eta, \xi, \theta) = 0, \tag{6.16}$$

with the operator $\mathcal{S}$ acting as

$$\mathcal{S} :\ (\phi, \eta) \times (\xi, \theta) \to \mathcal{S}(\phi, \eta, \xi, \theta) \tag{6.17}$$

from $\mathcal{D}(\mathcal{S}) \times D_{\overline{\delta}} \times \mathbb{S}^n$ to $\mathcal{Z}$. Estimate $\frac{D\mathcal{S}(\phi,\alpha,\xi,\theta)}{D\xi_i^k}$ and $\frac{D\mathcal{S}(\phi,\alpha,\xi,\theta)}{D\theta^k}$, the Fréchet derivatives of $\mathcal{S}$ with respect to ξ_i^k and θ^k respectively. Let $T(P)$ be the perturbed double bubble assembly represented by (ϕ, η). Suppose that the boundaries of P^l are $\hat{\mathbf{r}}_j^l(t)$ and the boundaries of $T^l(P^l)$ are $\mathbf{r}_j^l$. Hence $\mathbf{r}_j^l = T^l(\hat{\mathbf{r}}_j^l)$. Note that P^l and $\hat{\mathbf{r}}_j^l$ are independent of ξ and θ. The operator $\mathcal{S}$ acts on ξ and θ via the transformation T, and only the parts involving $I_{T(P_p)}$ in $\mathcal{S}$ depend on ξ and θ as follows:

$$\begin{aligned}
\mathcal{I}_{pj}^l(\phi_p, \phi_0, \eta) &= I_{T(P_p)}(\mathbf{r}_j^l(t)) \ = \int_{T(P_p)} G(\mathbf{r}_j^l(t), y)\, dy \\
&= \int_{T^l(P_p^l)} \frac{1}{2\pi} \log \frac{1}{|\mathbf{r}_j^l(t) - y|} dy + \int_{T^l(P_p^l)} R(\mathbf{r}_j^l(t), y)\, dy + \sum_{q \neq l} \int_{T^q(P_p^q)} G(\mathbf{r}_j^q, y)\, dy \\
&= \int_{P_p^l} \frac{\epsilon^2}{2\pi} \log \frac{1}{\epsilon|\hat{\mathbf{r}}_j^l(t) - \hat{y}|}\, d\hat{y} + \epsilon^2 \int_{P_p^l} R(\epsilon e^{\mathrm{i}\theta^l} \hat{\mathbf{r}}_j^l(t) + \xi^l, \epsilon e^{\mathrm{i}\theta^l} \hat{y} + \xi^l)\, d\hat{y} \\
&\quad + \sum_{q \neq l} \epsilon^2 \int_{P_p^q} G(\epsilon e^{\mathrm{i}\theta^l} \hat{\mathbf{r}}_j^l(t) + \xi^l, \epsilon e^{\mathrm{i}\theta^q} \hat{y} + \xi^q)\, d\hat{y}.
\end{aligned}$$

Then clearly

$$\frac{\partial I_{T(P_p)}}{\partial \xi_i^k} = O(\epsilon^2) \ \text{ and } \ \frac{\partial I_{T(P_p)}}{\partial \theta^k} = O(\epsilon^3) \tag{6.18}$$

hold uniformly with respect to t, ξ, θ, and w. Consequently

$$\Big\|\frac{D\mathcal{S}(\phi, \eta, \xi, \theta)}{D\xi_i^k}\Big\| = O(|\gamma|\epsilon^4) \ \text{ and } \ \Big\|\frac{D\mathcal{S}(\phi, \eta, \xi, \theta)}{D\theta^k}\Big\| = O(|\gamma|\epsilon^5). \tag{6.19}$$

Here the Fréchet derivatives are operators from $\mathbb{R}$ to $\mathcal{Z}$ and the above are estimates on the norms of these operators. On the other hand Lemma 5.2 part 2 shows that at $(\phi^*(\cdot, \xi, \theta, w), \eta^*(\xi, \theta, w))$, the solution found in Lemma 5.1,

$$\Big\|\Big(\frac{D\mathcal{S}(\phi^*, \eta^*, \xi, \theta)}{D(\phi, \eta)}\Big)^{-1}\Big\| \leq \frac{1}{\check{d}\epsilon} \tag{6.20}$$

if σ is small. Note that $\frac{D\mathcal{S}(\phi^*,\eta^*,\xi,\theta)}{D(\phi,\eta)}$ here is the same as $\mathcal{S}'(\phi^*, \eta^*)$ in Lemma 5.2. The implicit function theorem asserts that when σ is small enough,

$$\Big\|\frac{D(\phi^*, \eta^*)}{D\xi_i^k}\Big\| = O(|\gamma|\epsilon^3) \ \text{ and } \ \Big\|\frac{D(\phi^*, \eta^*)}{D\theta^k}\Big\| = O(|\gamma|\epsilon^4). \tag{6.21}$$

Since

$$\Big\|\frac{D(\phi^*,\eta^*)}{D\xi_i^k}\Big\| = \Big\|\frac{\partial(\phi^*,\eta^*)}{\partial\xi_i^k}\Big\|_{\mathcal{X}} \text{ and } \Big\|\frac{D(\phi^*,\eta^*)}{D\theta^k}\Big\| = \Big\|\frac{\partial(\phi^*,\eta^*)}{\partial\theta^k}\Big\|_{\mathcal{X}} \tag{6.22}$$

the lemma follows. □

Finally we complete the proof of the main theorem.

Proof of Theorem 1.1. In Section 5 it is proved that there is an assembly $(\phi^*(\cdot,\xi,\theta,w),\eta^*(\xi,\theta,w))$ for each $(\xi,\theta,w)\in\overline{\Xi_{\overline{\delta}}}\times\mathbb{S}^n\times\overline{W}$ which satisfies the equations (5.13)-(5.15). Lemma 6.2 shows that $J(\xi,\theta,w)=\mathcal{J}(\phi^*(\cdot,\xi,\theta,w),\eta^*(\xi,\theta,w))$ is minimized at $(\xi^*,\theta^*,w^*)\in\Xi_{\overline{\delta}}\times\mathbb{S}^n\times W$.

The proof of Theorem 1.1 is divided into two steps. In the first step one shows that the assembly $(\phi^*(\cdot,\xi^*,\theta^*,w^*),\eta^*(\xi^*,\theta^*,w^*))$ satisfies the triple junction condition (1.5), and in the second step one shows that at $(\phi^*(\cdot,\xi^*,\theta^*,w^*),\eta^*(\xi^*,\theta^*,w^*))$ the constants λ_i^k in equations (5.13)-(5.15) are independent of k.

In the first step of the proof w is taken to be w^*. The dependence on w^* is not explicitly stated in this step. By choosing $(\xi,\theta)\in\Xi_{\overline{\delta}}\times\mathbb{S}^n$ in different ways, one constructs various deformations of $(\phi^*(\cdot,\xi^*,\theta^*),\eta^*(\xi^*,\theta^*))$ to discover properties of $(\phi^*(\cdot,\xi^*,\theta^*),\eta^*(\xi^*,\theta^*))$. These deformations no longer keep assemblies in the restricted class.

First fix k and take $(\xi_1^l,\xi_2^l,\theta^l)=(\xi_1^{*,l},\xi_2^{*,l},\theta^{*,l})+\varepsilon(\delta^{lk},0,0)$ for each l, where $\delta^{lk}=1$ if $l=k$ and $\delta^{lk}=0$ if $l\neq k$. The deformation $(\phi^*(\cdot,\xi,\theta),\eta^*(\xi,\theta))$ with (ξ,θ) chosen this way represent approximately a horizontal translation of the k-th perturbed double bubble in the assembly $(\phi^*(\cdot,\xi^*,\theta^*),\eta^*(\xi^*,\theta^*))$. The infinitesimal element of this deformation is

$$\mathbf{X}_i^{H,k,l}(t)=\frac{\partial\mathbf{r}_i^{*,l}(t,\xi,\theta)}{\partial\xi_1^k}\Big|_{(\xi,\theta)=(\xi^*,\theta^*)} \quad \text{for } l=1,2,...,n,\ i=1,2,0.$$

Here $\mathbf{r}_i^{*,l}(t,\xi,\theta)$, with $i=1,2,0$, form the boundaries of the l-th perturbed double bubble in the assembly $(\phi^*(\cdot,\xi,\theta),\eta^*(\xi,\theta))$. Since

$$\mathbf{r}_i^{*,l}(t,\xi,\theta)=\epsilon e^{\mathrm{i}\theta^l}(u_i^{*,l}e^{\mathrm{i}\alpha_i^l(\eta^{*,l})t}+\beta_i^l(\eta^{*,l}))+\xi^l$$

and $2\phi_i^{*,l}=\alpha_i^l(\eta^{*,l})(u_i^{*,l})^2-\alpha_i^l(\eta^{*,l})(\rho_i^*(\eta^{*,l}))^2$, Lemma 6.3 implies that

$$\mathbf{X}_i^{H,k,l}(t)=\frac{\partial\mathbf{r}_i^{*,l}(t,\xi,\theta)}{\partial\xi_1^k}\Big|_{(\xi,\theta)=(\xi^*,\theta^*)}=\begin{cases}1+O(|\gamma|\epsilon^4) & \text{if } \ l=k\\ O(|\gamma|\epsilon^4) & \text{if } \ l\neq k\end{cases} \tag{6.23}$$

uniformly with respect to t.

Second for every fixed k take $(\xi_1^l,\xi_2^l,\theta^l)=(\xi_1^{*,l},\xi_2^{*,l},\theta^{*,l})+\varepsilon(0,\delta^{lk},0)$ for each l. This is nearly a vertical translation of the k-th perturbed double bubble and the infinitesimal element of this deformation is

$$\mathbf{X}_i^{V,k,l}(t)=\frac{\partial\mathbf{r}_i^{*,l}(t,\xi,\theta)}{\partial\xi_2^k}\Big|_{(\xi,\theta)=(\xi^*,\theta^*)}=\begin{cases}\mathrm{i}+O(|\gamma|\epsilon^4) & \text{if } \ l=k\\ O(|\gamma|\epsilon^4) & \text{if } \ l\neq k\end{cases}. \tag{6.24}$$

Third for every k take $(\xi_1^l,\xi_2^l,\theta^l)=(\xi_1^{*,l},\xi_2^{*,l},\theta^{*,l})+\varepsilon(0,0,\delta^{lk})$ for each l. Then it is almost a rotational deformation of the k-th perturbed double bubble and the infinitesimal element is

$$\mathbf{X}_i^{R,k,l}(t)=\frac{\partial\mathbf{r}_i^{*,l}(t,\xi,\theta)}{\partial\theta^k}\Big|_{(\xi,\theta)=(\xi^*,\theta^*)}=\begin{cases}\mathrm{i}(\mathbf{r}_i^{*,k}(t,\xi^*,\theta^*)-\xi^k)+O(|\gamma|\epsilon^5) & \text{if } \ l=k\\ O(|\gamma|\epsilon^5) & \text{if } \ l\neq k\end{cases}. \tag{6.25}$$

At the triple points they are

$$\mathbf{X}_i^{R,k,l}(\pm1)=\begin{cases}\mp\epsilon e^{\mathrm{i}\theta^k}\eta^{*,k}(\xi^*,\theta^*)+O(|\gamma|\epsilon^5) & \text{if } \ l=k\\ O(|\gamma|\epsilon^5) & \text{if } \ l\neq k\end{cases}. \tag{6.26}$$

Here the estimates $O(|\gamma|\epsilon^5)$ hold uniformly with respect to t.

By Lemma 6.3, since (ξ^*, θ^*) is an interior minimum of J (with $w = w^*$),

$$\frac{\partial J(\xi,\theta)}{\partial \xi_1^k}\Big|_{(\xi,\theta)=(\xi^*,\theta^*)} = \frac{\partial J(\xi,\theta)}{\partial \xi_2^k}\Big|_{(\xi,\theta)=(\xi^*,\theta^*)} = \frac{\partial J(\xi,\theta)}{\partial \theta^k}\Big|_{(\xi,\theta)=\xi^*,\theta^*)} = 0. \tag{6.27}$$

On the other hand Lemma 2.4 shows that $\frac{\partial J(\xi,\theta)}{\partial \xi_1^k}\Big|_{(\xi,\theta)=(\xi^*,\theta^*)}$, $\frac{\partial J(\xi,\theta)}{\partial \xi_2^k}\Big|_{(\xi,\theta)=(\xi^*,\theta^*)}$, and $\frac{\partial J(\xi,\theta)}{\partial \theta^k}\Big|_{(\xi,\theta)=(\xi^*,\theta^*)}$ are equal to

$$\begin{aligned}
&\sum_{l=1}^n \Big((\sum_{i=0}^2 \mathbf{T}_i^l) \cdot \mathbf{X}^l \Big|_{-1}^1 \Big) - \sum_{l=1}^n \int_{\partial T^l(P_1^{*,l}) \backslash \partial T^l(P_2^{*,l})} (\kappa_1 + \gamma_{11} I_{T(P_1^*)} + \gamma_{12} I_{T(P_2^*)}) \mathbf{N}_1^l \cdot \mathbf{X}^l \, ds \\
&- \sum_{l=1}^n \int_{\partial T^l(P_2^{*,l}) \backslash \partial T^l(P_1^{*,l})} (\kappa_2 + \gamma_{12} I_{T(P_1^*)} + \gamma_{22} I_{T(P_2^*)}) \mathbf{N}_2^l \cdot \mathbf{X}^l \, ds \\
&- \sum_{l=1}^n \int_{\partial T^l(P_1^{*,l}) \cap \partial T^l(P_2^{*,l})} (\kappa_0 + (\gamma_{11} - \gamma_{12}) I_{T(P_1^*)} + (\gamma_{12} - \gamma_{22}) I_{T(P_2^*)}) \mathbf{N}_0^l \cdot \mathbf{X}^l \, ds
\end{aligned} \tag{6.28}$$

with $\mathbf{X}$ being $\mathbf{X}^{H,k}$, $\mathbf{X}^{V,k}$, and $\mathbf{X}^{R,k}$ respectively. In (6.28) $\mathbf{T}_i^l$ and $\mathbf{N}_i^l$ are the tangent and normal vectors of the curves $\mathbf{r}_i^{*,l}(t, \xi^*, \theta^*)$. But these curves satisfy the equations (5.13)-(5.15) of Lemma 5.3. Hence, (6.28) is simplified to

$$\begin{aligned}
&\sum_{l=1}^n \Big((\sum_{i=0}^2 \mathbf{T}_i^l) \cdot \mathbf{X}^l \Big|_{-1}^1 \Big) - \sum_{l=1}^n \int_{\partial T^l(P_1^{*,l}) \backslash \partial T^l(P_2^{*,l})} \epsilon^{-2} \lambda_1^l \mathbf{N}_1^l \cdot \mathbf{X}^l \, ds \\
&- \sum_{l=1}^n \int_{\partial T^l(P_2^{*,l}) \backslash \partial T^l(P_1^{*,l})} \epsilon^{-2} \lambda_2^l \mathbf{N}_2^l \cdot \mathbf{X}^l \, ds - \sum_{l=1}^n \int_{\partial T^l(P_1^{*,l}) \cap \partial T^l(P_2^{*,l})} \epsilon^{-2} (\lambda_1^l - \lambda_2^l) \mathbf{N}_0^l \cdot \mathbf{X}^l \, ds.
\end{aligned}$$

By (2.36) and (2.37) of Lemma 2.4, the above is equal to

$$\sum_{l=1}^n \Big((\sum_{i=0}^2 \mathbf{T}_i^l) \cdot \mathbf{X}^l \Big|_{-1}^1 \Big) + \sum_{l=1}^n \epsilon^{-2} \lambda_1^l \frac{d|T^l(P_1^{*,l})|}{d\varepsilon}\Big|_{\varepsilon=0} + \sum_{l=1}^n \epsilon^{-2} \lambda_2^l \frac{d|T^l(P_2^{*,l})|}{d\varepsilon}\Big|_{\varepsilon=0}.$$

When w is fixed at w^*, $|T^l(P_1^{*,l})| = w_1^{*,l}$ and $|T^l(P_2^{*,l})| = w_2^{*,l}$ are constants independent of ε, so the second and the third terms above vanish and one deduces from (6.27) and (6.28) that

$$\sum_{l=1}^n \Big((\sum_{i=0}^2 \mathbf{T}_i^l) \cdot \mathbf{X}^l \Big|_{-1}^1 \Big) = 0 \tag{6.29}$$

for $\mathbf{X}$ equal to $\mathbf{X}^{H,k}$, $\mathbf{X}^{V,k}$, or $\mathbf{X}^{R,k}$.

The equations (6.29) are linear homogeneous equations for variables $\sum_{i=0}^2 \mathbf{T}_i^l(-1)$ and $\sum_{i=0}^2 \mathbf{T}_i^l(1)$. Since $l = 1, 2, ..., n$ and each of $\sum_{i=0}^2 \mathbf{T}_i^l(-1)$ and $\sum_{i=0}^2 \mathbf{T}_i^l(1)$ is a vector in $\mathbb{R}^2$, there are altogether $4n$ variables in (6.29). Since $\mathbf{X}$ can be taken to be $\mathbf{X}^{H,k}$, $\mathbf{X}^{V,k}$, or $\mathbf{X}^{R,k}$ and $k = 1, 2, ..., n$, there are $3n$ equations in (6.29):

$$\sum_{l=1}^n \Big((\sum_{i=0}^2 \mathbf{T}_i^l) \cdot \mathbf{X}^{H,k,l} \Big|_{-1}^1 \Big) = 0, \ \sum_{l=1}^n \Big((\sum_{i=0}^2 \mathbf{T}_i^l) \cdot \mathbf{X}^{V,k,l} \Big|_{-1}^1 \Big) = 0, \ \sum_{l=1}^n \Big((\sum_{i=0}^2 \mathbf{T}_i^l) \cdot \mathbf{X}^{R,k,l} \Big|_{-1}^1 \Big) = 0, \ k = 1, 2, ..., n. \tag{6.30}$$

They are supplemented by n more equations

$$(\sum_{i=0}^2 \mathbf{T}_i^k) \cdot \mathbf{X}^{S,k} \Big|_{-1}^1 = 0, \quad k = 1, 2, ..., n, \quad \text{where} \ \ \mathbf{X}^{S,k}(\pm 1) = \pm \epsilon e^{\mathrm{i}\theta^{*,k}} \mathrm{i} \tag{6.31}$$

obtained in (5.16) of Lemma 5.3. The equations (6.30) and (6.31) form a $4n$ by $4n$ system of linear homogeneous equations. The coefficients of the system are given in (6.23), (6.24), (6.26), and (6.31). One sees that, when ϵ is sufficiently small, they form a non-singular coefficient matrix. Hence

$$(\mathbf{T}_1^k + \mathbf{T}_2^k + \mathbf{T}_0^k)(1) = (\mathbf{T}_1^k + \mathbf{T}_2^k + \mathbf{T}_0^k)(-1) = 0, \quad k = 1, 2, ..., n. \tag{6.32}$$

In (1.5) the ν_i's are the unit inward tangential vectors at the triple points, so $\nu_i = -\mathbf{T}_i$ at each upper triple point corresponding to $t = 1$ and $\nu_i = \mathbf{T}_i$ at each lower triple point corresponding to $t = -1$. Hence (6.32) implies (1.5).

In the second step of the proof take $\xi = \xi^*$ and $\theta = \theta^*$ but vary w in a neighborhood of w^*. In this step the dependence on ξ^* and θ^* is not stated explicitly. For each $k = 1, 2, ..., n$ and $i = 1, 2$, let w be given by $w_j^l = w_j^{*,l} + \varepsilon \delta_{ji}^{lk}$ for $l = 1, 2, ..., n$ and $j = 1, 2$ where δ_{ji}^{lk} is 1 if $l = k$ and $j = i$ and is 0 otherwise. The infinitesimal element of this deformation is denoted $\mathbf{X}_i^{A,k}$. Note that these deformations do not satisfy the constraints $\sum_{k=1}^n w_1^k = m$ and $\sum_{k=1}^n w_2^k = 1 - m$. However, since $J(w)$ is minimized at w^* under these constraints, there exist $\Lambda_1,\ \Lambda_2 \in \mathbb{R}$ such that

$$\frac{\partial J(w)}{\partial w_i^k}\Big|_{w=w^*} = \Lambda_i, \quad k = 1, 2, .., n,\ i = 1, 2. \tag{6.33}$$

On the other hand (2.34)-(2.35) of Lemma 2.4, (6.32), (5.13)-(5.15) of Lemma 5.3, and (2.36)-(2.37) of Lemma 2.4 in turn imply that

$$\begin{aligned}
\frac{\partial J(w)}{\partial w_i^k}\Big|_{w=w^*} &= \sum_{l=1}^n \Big((\sum_{j=0}^2 \mathbf{T}_j^l) \cdot \mathbf{X}_{i,j}^{A,k,l} \Big|_{-1}^1 \Big) + \\
&\quad \sum_{l=1}^n \Big(- \int_{\partial T^l(P_1^{*,l}) \setminus \partial T^l(P_2^{*,l})} (\kappa_1 + \gamma_{11} I_{T(P_1^*)} + \gamma_{12} I_{T(P_2^*)}) \mathbf{N}_1^l \cdot \mathbf{X}_{i,1}^{A,k,l}\, ds \\
&\quad - \int_{\partial T^l(P_2^{*,l}) \setminus \partial T^l(P_1^{*,l})} (\kappa_2 + \gamma_{12} I_{T(P_1^*)} + \gamma_{22} I_{T(P_2^*)}) \mathbf{N}_2^l \cdot \mathbf{X}_{i,2}^{A,k,l}\, ds \\
&\quad - \int_{\partial T^l(P_1^{*,l}) \cap \partial T^l(P_2^{*,l})} (\kappa_0 + (\gamma_{11} - \gamma_{12}) I_{T(P_1^*)} + (\gamma_{12} - \gamma_{22}) I_{T(P_2^*)}) \mathbf{N}_0^l \cdot \mathbf{X}_{i,0}^{A,k,l}\, ds \Big) \\
&= \sum_{l=1}^n \epsilon^{-2} \lambda_1^l \Big(- \int_{\partial T^l(P_1^{*,l}) \setminus \partial T^l(P_2^{*,l})} \mathbf{N}_1^l \cdot \mathbf{X}_{i,1}^{A,k,l}\, ds - \int_{\partial T^l(P_1^{*,l}) \cap \partial T^l(P_2^{*,l})} \mathbf{N}_0^l \cdot \mathbf{X}_{i,0}^{A,k,l}\, ds \Big) \\
&\quad + \sum_{l=1}^n \epsilon^{-2} \lambda_2^l \Big(- \int_{\partial T^l(P_2^{*,l}) \setminus \partial T^l(P_1^{*,l})} \mathbf{N}_2^l \cdot \mathbf{X}_{i,2}^{A,k,l}\, ds + \int_{\partial T^l(P_1^{*,l}) \cap \partial T^l(P_2^{*,l})} \mathbf{N}_0^l \cdot \mathbf{X}_{i,0}^{A,k,l}\, ds \Big) \\
&= \sum_{l=1}^n \epsilon^{-2} \lambda_1^l \frac{\partial w_1^l}{\partial w_i^k} + \sum_{l=1}^n \epsilon^{-2} \lambda_2^l \frac{\partial w_2^l}{\partial w_i^k} \\
&= \epsilon^{-2} \lambda_i^k.
\end{aligned} \tag{6.34}$$

Comparing (6.33) and (6.34) one derives $\epsilon^{-2}\lambda_i^k = \Lambda_i$ for all k. This shows that when $(\xi, \theta, w) = (\xi^*, \theta^*, w^*)$, the λ_i^k's in (5.13)-(5.15) of Lemma 5.3 are independent of k. This establishes (1.2)-(1.4) and completes the second step.

According to Lemma 5.1 the solution $(\phi^*(\cdot, \xi^*, \theta^*, w^*), \eta^*(\xi^*, \theta^*, w^*))$ is found in the space $\mathcal{X}$, so the functions $\phi_i^{*,k}(\cdot, \xi^*, \theta^*, w^*)$ are in $H^2(-1, 1)$. The standard boot-strapping argument applied to the second order integro-differential equations (1.2)-(1.4) shows that the $\phi_i^{*,k}(\cdot, \xi^*, \theta^*, w^*)$'s are all C^∞. Hence the perturbed bubbles in the solution assembly are enclosed by continuous curves that are C^∞ except at the triple points.

A systematic study of stability of solutions to (1.2)-(1.5) is beyond the scope of this paper. Our assertion that the solution $(\phi^*(\cdot, \xi^*, \theta^*, w^*), \eta^*(\xi^*, \theta^*, w^*))$ is stable is interpreted by its local minimization property.

Recall that the solution $(\phi^*(\cdot,\xi^*,\theta^*,w^*),\eta^*(\xi^*,\theta^*,w^*))$ is found in two steps. First for each $(\xi,\theta,w)\in\overline{\Xi_{\bar\delta}}\times\mathbb{S}^n\times\overline{W}$, a fixed point $(\phi^*(\cdot,\xi,\theta,w),\eta^*(\xi,\theta,w))$ is constructed in a restricted class of perturbed double bubble assemblies. This fixed point is shown to be locally minimizing $\mathcal{J}$ in the restricted class in Lemma 5.2 part 1. In the second step $\mathcal{J}$ is minimized among the $(\phi^*(\cdot,\xi,\theta,w),\eta^*(\xi,\theta,w))$'s where (ξ,θ,w) ranges over $\overline{\Xi_{\bar\delta}}\times\mathbb{S}^n\times\overline{W}$, and $(\phi^*(\cdot,\xi^*,\theta^*,w^*),\eta^*(\xi^*,\theta^*,w^*))$ emerges as a minimum. As a minimum of locally minimizing assemblies from restricted classes, $(\phi^*(\cdot,\xi^*,\theta^*,w^*),\eta^*(\xi^*,\theta^*,w^*))$ is locally energy minimizing with respect to both restricted deformations and some non-restricted deformations; hence, we claim that $(\phi^*(\cdot,\xi^*,\theta^*,w^*),\eta^*(\xi^*,\theta^*,w^*))$ is stable.

The deviation of our solution from an exact double bubble assembly is $\|(\phi^*(\xi^*,\theta^*,w^*),\eta^*(\xi^*,\theta^*,w^*))-(0,h)\|_{\mathcal{X}}$ and this quantity is of the order $|\gamma|\epsilon^3$ by Lemma 5.1. Therefore, the smaller $|\gamma|\epsilon^3$ is, the closer the solution is to an exact double bubble assembly. □

7 Discussion

While the locations $\xi^{*,k}$ of the perturbed double bubbles in the solution are near the points that minimize F, the directions $\theta^{*,k}$ of these double bubbles can not be ascertained from our proof. Note that in (6.15) the limit of $H(\xi,\theta)$ is a constant multiple of $F(\xi)$ which does not depend on θ. The directions $\theta^{*,k}$ cannot be determined from this level of convergence. One would have to move to a higher level of convergence to see dependence on θ, but that would require better estimate on the energy of $(\phi^*(\cdot,\xi,\theta,w),\eta^*(\xi,\theta,w))$ than the one in Lemma 6.1. On the other hand not knowing the asymptotic limit of θ^* does not hinder the proof of Theorem 1.1, since θ varies in $\mathbb{S}^n$, a compact manifold without boundary. The other two variables, ξ and w, live in $\overline{\Xi_{\bar\delta}}$ and $\overline{W}$, which are manifolds with boundary, and one must know the dependence of the energy on ξ and w to show that ξ^* and w^* of the minimum are in the interior of these manifolds.

Ideally one likes to find solutions to (1.2)-(1.5) that locally minimizes $\mathcal{J}$ in a natural topology like the one defined by the L^1 norm as follows. For two pairs (Ω_1,Ω_2) and $(\tilde\Omega_1,\tilde\Omega_2)$ of Lebesgue measurable subsets of D satisfying the conditions

$$|\Omega_1|=|\tilde\Omega_1|,\ |\Omega_2|=|\tilde\Omega_2|,\ |\Omega_1\cap\Omega_2|=|\tilde\Omega_1\cap\tilde\Omega_2|=0,$$

define a metric

$$\text{dist}((\Omega_1,\Omega_2),(\tilde\Omega_1,\tilde\Omega_2))=\|\chi_{\Omega_1}-\chi_{\tilde\Omega_1}\|_{L^1(D)}+\|\chi_{\Omega_2}-\chi_{\tilde\Omega_2}\|_{L^1(D)}.$$

The functional $\mathcal{J}$ is lower semi-continuous under this metric. One can prove the existence of a global minimizer by the standard argument. It is also an ideal metric for a Γ-convergence theory to connect the model here to a diffusive interface system; see [25]. However finding local minimizers of $\mathcal{J}$ under this metric is challenging, since any neighborhood defined by the metric contains very irregular elements. We do not know if the solution found in this paper is a local minimizer with respect to this metric.

The functional $\mathcal{J}$ has a simpler counterpart in a binary inhibitory system. Let $\omega\in(0,1)$ and $\gamma>0$. For $\Omega\subset D$ with the fixed area: $|\Omega|=\omega|D|$, the binary free energy of Ω is

$$\mathcal{J}_B(\Omega)=\mathcal{P}_D(\Omega)+\frac{\gamma}{2}\int_D|(-\Delta)^{-1/2}(\chi_\Omega-\omega)|^2\,dx. \tag{7.1}$$

A critical point of this functional satisfies the equation

$$\kappa+\gamma I_\Omega=\lambda \tag{7.2}$$

on $\partial\Omega$. The equation (7.2) or the functional (7.1) may be derived from the Ohta-Kawasaki theory [20] for diblock copolymers; see [19, 23]. The equation can also be derived from the Gierer-Meinhardt system [30]. This binary problem has been studied intensively in recent years. All solutions to (7.2) in one dimension are known to be local minimizers of $\mathcal{J}_B$ [23]. Many solutions in two and three dimensions have been found that match the morphological phases in diblock copolymers [21, 27, 26, 28, 29, 13, 14, 30, 33, 36]. Global minimizers of $\mathcal{J}_B$ are studied in [2, 34, 17, 5, 16, 15, 10] for various parameter ranges. Applications of the second variation of $\mathcal{J}_B$ and its connections to minimality and Gamma-convergence are found in [7, 1, 12].

A relevant result in [26] states that when ω and γ are in a proper range, (7.2) admits a solution that is an assembly of perturbed discs. The discs have approximately the same size, and the centers of the discs nearly minimize the same function F of (1.10).

Appendix A

We prove Lemma 4.1. Let $\mathcal{F}$ be the functional

$$\mathcal{F}(y) = \int_{-1}^{1} ((y'(t))^2 - q^2 y^2(t))\, dt \tag{A.1}$$

for $y \in H_0^1(-1,1)$ and $\int_{-1}^1 y(t)\, dt = \Upsilon$, where $q \in (0,\pi)$.

Step 1: $\mathcal{F}$ is bounded below.

Let $e_1 = (\frac{\pi}{2})^2$, $e_2 = \pi^2$, and $e_3 = (\frac{3\pi}{2})^2$ be the first three eigenvalues of the problem

$$-f'' = ef, \quad f \in H_0^1(-1,1),$$

and $f_1(t) = \cos\frac{\pi t}{2}$ and $f_2(t) = \sin \pi t$ be eigenfunctions corresponding to λ_1 and λ_2. Note that

$$\int_{-1}^1 f_1^2(t)\, dt = \int_{-1}^1 f_2^2(t)\, dt = 1, \quad \text{and} \quad \int_{-1}^1 f_1(t)\, dt = \frac{4}{\pi}, \quad \int_{-1}^1 f_2(t)\, dt = 0.$$

For every $y \in H_0^1(-1,1)$, decompose $y = c_1 f_1 + c_2 f_2 + z$ where $z \in H_0^1(-1,1)$ is perpendicular to f_1 and f_2: $\int_{-1}^1 f_1(t) z(t)\, dt = \int_{-1}^1 f_2(t) z(t)\, dt = 0$. By the variational characterization of the eigenvalues

$$\mathcal{F}(y) = c_1^2(e_1 - q^2) + c_2^2(e_2 - q^2) + \mathcal{F}(z) \geq c_1^2(e_1 - q^2) + c_2^2(e_2 - q^2) + (e_3 - q^2)\int_{-1}^1 z^2(t)\, dt. \tag{A.2}$$

Note

$$\Upsilon = \int_{-1}^1 y(t)\, dt = \frac{4c_1}{\pi} + \int_{-1}^1 z(t)\, dt.$$

Then

$$\Big(\Upsilon - \frac{4c_1}{\pi}\Big)^2 = (\int_{-1}^1 z(t)\, dt)^2 \leq 2\int_{-1}^1 z^2(t)\, dt$$

and

$$\begin{aligned}
\mathcal{F}(y) &\geq c_1^2(e_1 - q^2) + c_2^2(e_2 - q^2) + \frac{1}{2}(e_3 - q^2)\Big(\Upsilon - \frac{4c_1}{\pi}\Big)^2 \\
&= \Big(e_1 - q^2 + (e_3 - q^2)\frac{8}{\pi^2}\Big)c_1^2 - (e_3 - q^2)\Big(\frac{4\Upsilon}{\pi}\Big)c_1 + (e_3 - q^2)\frac{\Upsilon^2}{2} + (e_2 - q^2)c_2^2.
\end{aligned}$$

Since

$$e_1 - \pi^2 + (e_3 - \pi^2)\frac{8}{\pi^2} = -\frac{3\pi^2}{4} + 10 > 0, \quad \text{and } e_2 - q^2 > \pi^2 - \pi^2 = 0,$$

$\mathcal{F}(y)$ is bounded below for all $y \in H_0^1(-1,1)$ with $\int_{-1}^1 y(t)\, dt = \Upsilon$.

Step 2: A minimizing sequence is bounded in $H_0^1(-1,1)$.

Let y_ν be a minimizing sequence. Decompose as above $y_\nu = c_1^\nu f_1 + c_2^\nu f_2 + z_\nu$. Then

$$\mathcal{F}(y_\nu) \geq \Big(e_1 - q^2 + (e_3 - q^2)\frac{8}{\pi^2}\Big)(c_1^\nu)^2 - (e_3 - q^2)\Big(\frac{4\Upsilon}{\pi}\Big)c_1^\nu + (e_3 - q^2)\frac{\Upsilon^2}{2} + (e_2 - q^2)(c_2^\nu)^2.$$

Since $\mathcal{F}(y_\nu)$ is bounded below and above (for y_ν is minimizing), $|c_1^\nu|$ and $|c_2^\nu|$ are bounded with respect to ν. By (A.2), $\int_{-1}^1 z_\nu^2(t)\,dt$ is also bounded. Consequently $\int_{-1}^1 y_\nu^2(t)\,dt$ is bounded. From (A.1) we deduce that $\int_{-1}^1 (y_\nu'(t))^2\,dt$ is bounded. Hence y_ν is bounded in $H_0^1(-1,1)$.

Step 3: A minimizer v exists.

From the minimizing sequence y_ν, there is a subsequence again denoted by y_ν that converges weakly in $H_0^1(-1,1)$ and strongly in $L^2(-1,1)$ to a limit $v \in H_0^1(-1,1)$ with $\int_{-1}^1 v(t) = \Upsilon$. By the weak lower semi-continuity of the H^1 norm,

$$\mathcal{F}(v) \leq \liminf_{\nu\to\infty} \mathcal{F}(y_\nu).$$

Hence v is a minimizer.

Step 4: $\mathcal{F}(v) = \dfrac{\Upsilon^2 q^3}{2(\tan q - q)}$.

As a minimizer, v satisfies the equation $-v'' - q^2 v = \lambda$, $v(\pm 1) = 0$, for some $\lambda \in \mathbb{R}$. Solving the equation, we find $v(t) = C\cos(qt) - \frac{\lambda}{q^2}$, $\lambda = Cq^2\cos q$. Hence $v(t) = C(\cos(qt) - \cos q)$ and $\Upsilon = \int_{-1}^1 v(t)\,dt = C\Big(\frac{2\sin q}{q} - 2\cos q\Big)$. It follows that $C = \frac{\Upsilon}{\frac{2\sin q}{q} - 2\cos q}$ and

$$v(t) = \frac{\Upsilon(\cos(qt) - \cos q)}{\frac{2\sin q}{q} - 2\cos q}.$$

If we multiply the equation for v by v and integrate, then

$$\mathcal{F}(v) = \lambda\int_{-1}^1 v(t)\,dt = \lambda\Upsilon = C\Upsilon q^2\cos q = \frac{\Upsilon^2 q^3}{2(\tan q - q)}.$$

This proves Lemma 4.1.

Appendix B

We explain in more detail the first part of the restricted perturbation. The asymmetric case and the symmetric cases are dealt with differently.

For the asymmetric case, one starts with a somewhat different way to perturb an exact double bubble and later return to the perturbation setting described in Section 2. Since only one double bubble is considered in this appendix, the superscript k will be dropped from notations like r_i^k, a_i^k, etc. One simply writes r_i, a_i.

From an exact double bubble specified by the radii r_i, the angles a_i and the height h of the upper triple point, move the triple points $(0, \pm h)$ vertically by the same distance in the opposite directions to $(0, \eta)$. Connect the new triple points by three arcs with the radii ρ_i, the angles α_i, and the centers $(\beta_i, 0)$ for $i = 1, 2, 0$. However at this point we do not impose the condition $\rho_1^{-1} - \rho_2^{-1} = \rho_0^{-1}$. Hence the choice of ρ_i, α_i, and β_i is not unique.

Define

$$\mu_i = \rho_i^2(\alpha_i - \cos\alpha_i \sin\alpha_i), \quad i = 1, 2, 0 \tag{B.1}$$

as before. Since $\rho_i \sin\alpha_i = \eta$, one can re-write μ_i as

$$\mu_i = \frac{\eta^2(\alpha_i - \cos\alpha_i \sin\alpha_i)}{\sin^2\alpha_i}. \tag{B.2}$$

The μ_i's must still satisfy the area constraints

$$\mu_1 + \mu_0 = w_1, \quad \mu_2 - \mu_0 = w_2. \tag{B.3}$$

If α_i is treated as a function of μ_i and η, implicit differentiation shows that

$$\frac{\partial\alpha_i}{\partial\mu_i} = \frac{\sin^3\alpha_i}{2\eta^2(\sin\alpha_i - \alpha_i\cos\alpha_i)} \tag{B.4}$$

$$\frac{\partial\alpha_i}{\partial\eta} = -\frac{(\alpha_i - \cos\alpha_i\sin\alpha_i)\sin\alpha_i}{\eta(\sin\alpha_i - \alpha_i\cos\alpha_i)}. \tag{B.5}$$

The total length of the three arcs is

$$P = 2\sum_{i=0}^{2}\alpha_i\rho_i = 2\sum_{i=0}^{2}\frac{\eta\alpha_i}{\sin\alpha_i}. \tag{B.6}$$

Since α_i depends on μ_i and η, and the μ_i's are subject to the constraints (B.3), we take μ_0 and η as the independent variables and treat α_i, and P all as functions of μ_0 and η.

Compute $\frac{\partial P}{\partial\mu_0}$. Since

$$\frac{\partial P}{\partial\mu_0} = 2\eta\sum_{i=0}^{2}\frac{\partial}{\partial\alpha_i}\Big(\frac{\alpha_i}{\sin\alpha_i}\Big)\frac{\partial\alpha_i}{\partial\mu_i}\frac{\partial\mu_i}{\partial\mu_0} \tag{B.7}$$

and

$$\frac{\partial}{\partial\alpha_i}\Big(\frac{\alpha_i}{\sin\alpha_i}\Big) = \frac{\sin\alpha_i - \alpha_i\cos\alpha_i}{\sin^2\alpha_i}, \tag{B.8}$$

one deduces by (B.4) and (B.3) that

$$\frac{\partial P}{\partial\mu_0} = \sum_{i=0}^{2}\frac{(-1)^i\sin\alpha_i}{\eta}. \tag{B.9}$$

Note that the right side of (B.9) is $-\rho_1^{-1} + \rho_2^{-1} + \rho_0^{-1}$.

Next compute $\frac{\partial P}{\partial\eta}$. Note that

$$\frac{\partial P}{\partial\eta} = 2\sum_{i=0}^{2}\Big(\frac{\alpha_i}{\sin\alpha_i} + \eta\frac{\partial}{\partial\alpha_i}\Big(\frac{\alpha_i}{\sin\alpha_i}\Big)\frac{\partial\alpha_i}{\partial\eta}\Big).$$

By (B.5) and (B.8) one finds

$$\frac{\partial P}{\partial\eta} = 2\sum_{i=0}^{2}\cos\alpha_i. \tag{B.10}$$

Note that at a critical point where $\frac{\partial P}{\partial\mu_0} = \frac{\partial P}{\partial\eta} = 0$,

$$-\sin\alpha_1 + \sin\alpha_2 + \sin\alpha_0 = 0 \tag{B.11}$$

$$\cos\alpha_1 + \cos\alpha_2 + \cos\alpha_0 = 0 \tag{B.12}$$

which imply that $\alpha_1 = \frac{2\pi}{3} - \alpha_0$ and $\alpha_2 = \frac{2\pi}{3} + \alpha_0$, i.e. an exact double bubble.

Now proceed to calculate the second derivatives of P. First

$$\frac{\partial^2 P}{\partial \mu_0^2} = \frac{\partial}{\partial \mu_0}\Big(\sum_{i=0}^{1} \frac{(-1)^i \sin\alpha_i}{\eta}\Big) = \sum_{i=0}^{2} \frac{(-1)^i \cos\alpha_i}{\eta} \frac{\partial \alpha_i}{\partial \mu_i} \frac{\partial \mu_i}{\partial \mu_0} = \sum_{i=0}^{2} \frac{\cos\alpha_i}{\eta} \frac{\partial \alpha_i}{\partial \mu_i}.$$

By (B.4)

$$\frac{\partial^2 P}{\partial \mu_0^2} = \frac{1}{2\eta^3} \sum_{i=0}^{2} \frac{\cos\alpha_i \sin^3\alpha_i}{\sin\alpha_i - \alpha_i \cos\alpha_i}. \tag{B.13}$$

Next

$$\frac{\partial^2 P}{\partial \mu_0 \partial \eta} = \frac{\partial}{\partial \eta}\Big(\sum_{i=0}^{2} \frac{(-1)^i \sin\alpha_i}{\eta}\Big) = \sum_{i=0}^{2} (-1)^i \Big(-\frac{\sin\alpha_i}{\eta^2} + \frac{\cos\alpha_i}{\eta} \frac{\partial \alpha_i}{\partial \eta}\Big).$$

Using (B.5) one finds

$$\frac{\partial^2 P}{\partial \mu_0 \partial \eta} = -\frac{1}{\eta^2} \sum_{i=0}^{2} \frac{(-1)^i \sin^4\alpha_i}{\sin\alpha_i - \alpha_i \cos\alpha_i}. \tag{B.14}$$

Finally

$$\frac{\partial^2 P}{\partial \eta^2} = \frac{\partial}{\partial \eta}\Big(2\sum_{i=0}^{2} \cos\alpha_i\Big) = -2\sum_{i=0}^{2} \sin\alpha_i \frac{\partial \alpha_i}{\partial \eta}.$$

By (B.5) one derives

$$\frac{\partial^2 P}{\partial \eta^2} = \frac{2}{\eta} \sum_{i=0}^{2} \frac{(\alpha_i - \cos\alpha_i \sin\alpha_i)\sin^2\alpha_i}{\sin\alpha_i - \alpha_i \cos\alpha_i}. \tag{B.15}$$

In summary the Hessian matrix of P is

$$D^2 P = \begin{bmatrix} \frac{1}{2\eta^3} \sum_{i=0}^{2} \frac{\cos\alpha_i \sin^3\alpha_i}{\sin\alpha_i - \alpha_i \cos\alpha_i} & -\frac{1}{\eta^2} \sum_{i=0}^{2} \frac{(-1)^i \sin^4\alpha_i}{\sin\alpha_i - \alpha_i \cos\alpha_i} \\ -\frac{1}{\eta^2} \sum_{i=0}^{2} \frac{(-1)^i \sin^4\alpha_i}{\sin\alpha_i - \alpha_i \cos\alpha_i} & \frac{2}{\eta} \sum_{i=0}^{2} \frac{(\alpha_i - \cos\alpha_i \sin\alpha_i)\sin^2\alpha_i}{\sin\alpha_i - \alpha_i \cos\alpha_i} \end{bmatrix}. \tag{B.16}$$

This matrix is evaluated at the exact double bubble where $\alpha_i = a_i$ and $\eta = h$. The a_i's satisfy $a_1 = \frac{2\pi}{3} - a_0$ and $a_2 = \frac{2\pi}{3} + a_0$.

Lemma B.1 *The matrix $D^2 P$ at the exact double bubble is positive definite.*

This Lemma is proved rigorously in [32, Appendxi B]. Here we offer some numerical evidence by plotting

$$\sum_{i=0}^{2} \frac{\cos a_i \sin^3 a_i}{\sin a_i - a_i \cos a_i} \tag{B.17}$$

and

$$\Big(\sum_{i=0}^{2} \frac{\cos a_i \sin^3 a_i}{\sin a_i - a_i \cos a_i}\Big)\Big(\sum_{i=0}^{2} \frac{(a_i - \cos a_i \sin a_i)\sin^2 a_i}{\sin a_i - a_i \cos a_i}\Big) - \Big(\sum_{i=0}^{2} \frac{(-1)^i \sin^4 a_i}{\sin a_i - a_i \cos a_i}\Big)^2 \tag{B.18}$$

against $a_0 \in (0, \frac{\pi}{3})$ in Figure 10. Both (B.17) and (B.18) are positive, so the matrix $D^2 P$ is positive definite at the exact double bubble.

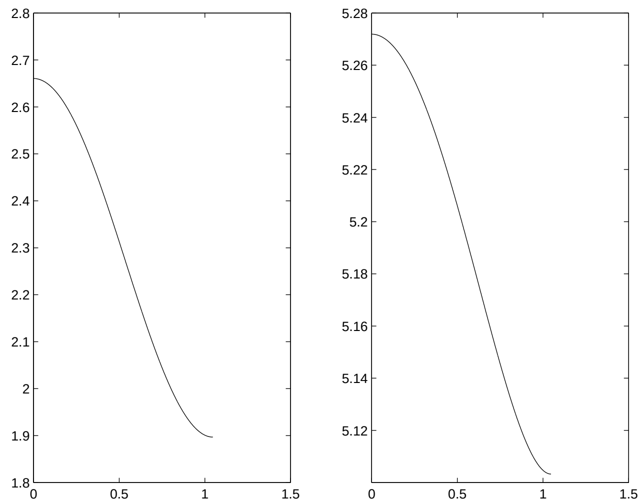

Figure 10: Left plot: the quantity (B.17) against $a_0 \in (0, \frac{\pi}{3})$. Right plot: the quantity (B.18) against a_0.

Lastly we connect the setting here with the setting in the rest of the paper regarding P versus μ_0 and η. After P is treated as a function of μ_0 and η here, one sets up the equation

$$\frac{\partial P(\mu_0, \eta)}{\partial \mu_0} = 0, \tag{B.19}$$

and uses it to define μ_0 as a function of η implicitly. This can be done near the exact double bubble because

$$\frac{\partial^2 P}{\partial \mu_0^2}\Big|_{\mu_0 = r_0^2(a_0 - \cos a_0 \sin a_0),\ \eta = h} \neq 0 \tag{B.20}$$

by Lemma B.1. As seen after (B.9), equation (B.19) is just the condition

$$\rho_1^{-1} - \rho_2^{-1} = \rho_0^{-1}, \tag{B.21}$$

precisely the one requirement, (2.46), in the setting of restricted perturbations in Section 2 that is not implemented in this appendix before (B.19).

Once $\mu_0 = \mu_0(\eta)$ becomes a dependent variable, $P = P(\mu_0(\eta), \eta)$ is a function of η only, and

$$\begin{aligned} \frac{dP}{d\eta} &= \frac{\partial P}{\partial \mu_0}\frac{d\mu_0}{d\eta} + \frac{\partial P}{\partial \eta} = \frac{\partial P}{\partial \eta} \\ \frac{d^2P}{d\eta^2} &= \frac{\partial^2 P}{\partial \mu_0 \partial \eta}\frac{d\mu_0}{d\eta} + \frac{\partial^2 P}{\partial \eta^2} = \frac{\partial^2 P}{\partial \mu_0 \partial \eta}\Big(- \frac{\frac{\partial^2 P}{\partial \mu_0 \partial \eta}}{\frac{\partial^2 P}{\partial \mu_0^2}}\Big) + \frac{\partial^2 P}{\partial \eta^2} \\ &= \frac{\frac{\partial^2 P}{\partial \mu_0^2}\frac{\partial^2 P}{\partial \eta^2} - (\frac{\partial^2 P}{\partial \mu_0 \partial \eta})^2}{\frac{\partial^2 P}{\partial \mu_0^2}}. \end{aligned}$$

Consequently by (B.10),

$$\frac{dP}{d\eta}\Big|_{\eta=h} = \frac{\partial P}{\partial \eta}\Big|_{\mu_0 = r_0^2(a_0 - \cos a_0 \sin a_0),\ \eta = h} = 2\sum_{i=0}^{2} \cos a_i = 0, \tag{B.22}$$

and by Lemma B.1,

$$\frac{d^2P}{d\eta^2}\Big|_{\eta=h} > 0. \tag{B.23}$$

In the symmetric case, we do not use μ_i since they are constants during a restricted perturbation. One starts directly with the equation for the area of the two bubbles

$$\rho_i^2(\alpha_i - \cos\alpha_i \sin\alpha_i) = w_i,\ \ i = 1, 2, \tag{B.24}$$

where $\rho_1 = \rho_2$, $\alpha_1 = \alpha_2$ and $w_1 = w_2$, and the equation

$$\eta = \rho_i \sin\alpha_i. \tag{B.25}$$

Implicit differentiation from (B.24) and (B.25) shows that

$$\frac{d\alpha_i}{d\eta} = -\frac{(\alpha_i - \cos\alpha_i\sin\alpha_i)\sin\alpha_i}{\eta(\sin\alpha_i - \alpha_i\cos\alpha_i)}. \tag{B.26}$$

In this case P is give by

$$P = 2\sum_{i=1}^{2}\alpha_i\rho_i + 2\eta = 2\sum_{i=1}^{2}\frac{\eta\alpha_i}{\sin\alpha_i} + 2\eta \tag{B.27}$$

It follows that

$$\frac{dP}{d\eta} = 2\cos\alpha_1 + 2\cos\alpha_2 + 2. \tag{B.28}$$

Note that at the exact double bubble where α_i is $\frac{2\pi}{3}$,

$$\frac{dP}{d\eta}\Big|_{\eta=h} = 0. \tag{B.29}$$

Moreover

$$\frac{d^2P}{d\eta^2} = \frac{2}{\eta}\sum_{i=1}^{2}\frac{(\alpha_i - \cos\alpha_i\sin\alpha_i)\sin^2\alpha_i}{\sin\alpha_i - \alpha_i\cos\alpha_i}. \tag{B.30}$$

At the exact double bubble

$$\frac{d^2P}{d\eta^2}\Big|_{\eta=h} = \frac{2}{h}\sum_{i=1}^{2}\frac{(\frac{2\pi}{3} - \cos\frac{2\pi}{3}\sin\frac{2\pi}{3})\sin^2\frac{2\pi}{3}}{\sin\frac{2\pi}{3} - \frac{2\pi}{3}\cos\frac{2\pi}{3}} = \frac{3.9631...}{h} > 0. \tag{B.31}$$

Hence (B.23) remains true in the symmetric case. The value in (B.31) may also be obtained from the asymmetric case by taking the $a_0 \to 0$ limit.

Appendix C

Let

$$f(w) = 2\Delta\sum_{k=1}^{n}\sum_{i=0}^{2}a_i^k r_i^k + \frac{1}{4\pi}\sum_{k=1}^{n}\sum_{i,j=1}^{2}\Gamma_{ij}w_i^k w_j^k,\ \ w \in \overline{W}. \tag{C.1}$$

Here a_i^k and r_i^k depend on w_1^k and w_2^k implicitly through the equations

$$\begin{aligned}
(r_1^k)^2(a_1^k - \cos a_1^k\sin a_1^k) + (r_0^k)^2(a_0^k - \cos a_0^k\sin a_0^k) &= w_1^k && \text{(C.2)}\\
(r_2^k)^2(a_2^k - \cos a_2^k\sin a_2^k) - (r_0^k)^2(a_0^k - \cos a_0^k\sin a_0^k) &= w_2^k && \text{(C.3)}\\
r_1^k\sin a_1^k \ = \ r_2^k\sin a_2^k &= r_0^k\sin a_0^k && \text{(C.4)}\\
(r_1^k)^{-1} - (r_2^k)^{-1} &= (r_0^k)^{-1} && \text{(C.5)}\\
\cos a_1^k + \cos a_2^k + \cos a_0^k &= 0 && \text{(C.6)}
\end{aligned}$$

as explained in Appendix B. In this appendix we show that f is minimized at $\widetilde{w}$, given in (6.8), if Δ is sufficiently small.

Because of the constraints

$$\sum_{k=1}^{n} w_1^k = m, \quad \sum_{k=1}^{n} w_2^k = 1-m, \tag{C.7}$$

a critical point of f in W is a solution to the equations

$$\frac{\partial f(w)}{\partial w_1^k} = \Lambda_1, \quad \frac{\partial f(w)}{\partial w_2^k} = \Lambda_2, \quad k = 1, 2, ..., n \tag{C.8}$$

where $\Lambda_1, \ \Lambda_2 \in \mathbb{R}$ are Lagrange multipliers from the constraints (C.7). Since for each k, a_i^k and r_i^k $(i = 1, 2, 0)$ depend on w_1^k and w_2^k only, $\widetilde{w}$ is clearly a critical point of f.

For $w \in \overline{W}$, define

$$f_1(w) = \sum_{k=1}^{n} \sum_{i,j=1}^{2} \Gamma_{ij} w_i^k w_j^k.$$

Let $x_k = w_1^k$ and $y_k = w_2^k$ for $k = 1, 2, ..., n-1$. Then

$$w_1^n = m - (x_1 + x_2 + ... + x_{n-1}) \ \text{ and } w_2^n = 1 - m - (y_1 + y_2 + ... + y_{n-1}).$$

Treating f_1 as a function of $(x_1, ..., x_{n-1})$ and $(y_1, ..., y_{n-1})$, without constraint one differentiates f_1 to find

$$\begin{aligned}
\frac{\partial f_1}{\partial x_k} &= 2\Gamma_{11} x_k + 2\Gamma_{11}\Big(\sum_{l=1}^{n-1} x_l - m\Big) + 2\Gamma_{12} y_k + 2\Gamma_{12}\Big(\sum_{l=1}^{n-1} y_l - (1-m)\Big) \\
\frac{\partial f_1}{\partial y_k} &= 2\Gamma_{12} x_k + 2\Gamma_{12}\Big(\sum_{l=1}^{n-1} x_l - m\Big) + 2\Gamma_{22} y_k + 2\Gamma_{22}\Big(\sum_{l=1}^{n-1} y_l - (1-m)\Big).
\end{aligned}$$

Let $x_k' = x_k - \frac{m}{n}$ and $y_k' = y_k - \frac{1-m}{n}$. Then at a critical point of f_1

$$\begin{bmatrix}
4\Gamma_{11} & 2\Gamma_{11} & ... & 2\Gamma_{11} & 4\Gamma_{12} & 2\Gamma_{12} & ... & 2\Gamma_{12} \\
2\Gamma_{11} & 4\Gamma_{11} & ... & 2\Gamma_{11} & 2\Gamma_{12} & 4\Gamma_{12} & ... & 2\Gamma_{12} \\
... & ... & ... & ... & ... & ... & ... & ... \\
2\Gamma_{11} & 2\Gamma_{11} & ... & 4\Gamma_{11} & 2\Gamma_{12} & 2\Gamma_{12} & ... & 4\Gamma_{12} \\
4\Gamma_{12} & 2\Gamma_{12} & ... & 2\Gamma_{12} & 4\Gamma_{22} & 2\Gamma_{22} & ... & 2\Gamma_{22} \\
2\Gamma_{12} & 4\Gamma_{12} & ... & 2\Gamma_{12} & 2\Gamma_{22} & 4\Gamma_{22} & ... & 2\Gamma_{22} \\
... & ... & ... & ... & ... & ... & ... & ... \\
2\Gamma_{12} & 2\Gamma_{12} & ... & 4\Gamma_{12} & 2\Gamma_{22} & 2\Gamma_{22} & ... & 4\Gamma_{22}
\end{bmatrix}
\begin{bmatrix} x_1' \\ x_2' \\ ... \\ x_{n-1}' \\ y_1' \\ y_2' \\ ... \\ y_{n-1}' \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ ... \\ 0 \\ 0 \\ 0 \\ ... \\ 0 \end{bmatrix}. \tag{C.9}$$

Let

$$x'' = Ax' \ \text{ and } \ y'' = Ay' \tag{C.10}$$

where

$$A = \begin{bmatrix} 4 & 2 & ... & 2 \\ 2 & 4 & ... & 2 \\ ... & ... & ... & ... \\ 2 & 2 & ... & 4 \end{bmatrix} \tag{C.11}$$

is an $n-1$ by $n-1$ matrix. Then the last linear system can be written as

$$\begin{cases} \Gamma_{11} x'' + \Gamma_{12} y'' = 0 \\ \Gamma_{12} x'' + \Gamma_{22} y'' = 0 \end{cases}$$

Since Γ is non-singular,

$$x'' = y'' = \vec{0}. \tag{C.12}$$

Next write the matrix A as

$$A = B + 2I \tag{C.13}$$

where

$$B = \begin{bmatrix} 2 & 2 & ... & 2 \\ 2 & 2 & ... & 2 \\ ... & ... & ... & ... \\ 2 & 2 & ... & 2 \end{bmatrix}$$

and I is the $n-1$ by $n-1$ identity matrix. B is a rank one matrix with two eigenvalues: 0 and $2(n-1)$. The eigenvalue 0 has multiplicity $n-2$ whose eigenvectors span the subspace that is perpendicular to $(1, 1, ..., 1)^T$. The eigenvalue $2(n-1)$ is simple corresponding to the eigenvector $(1, 1, ..., 1)^T$. By (C.13) we deduce that the eigenvalues of A are 2 (of multiplicity $n-2$) and $2n$ (of multiplicity 1).

Hence by (C.10) and (C.12), since A is non-singular,

$$x' = y' = \vec{0}. \tag{C.14}$$

This shows that the only critical point of f_1 is

$$x_k = \frac{m}{n} \text{ and } y_k = \frac{1-m}{n}, \quad k = 1, 2, ..., n-1. \tag{C.15}$$

The second derivative of f_1 with respect to x and y is the same matrix

$$\begin{bmatrix} \Gamma_{11}A & \Gamma_{12}A \\ \Gamma_{12}A & \Gamma_{22}A \end{bmatrix} \tag{C.16}$$

given in (C.9).

For any $(u, v) \in \mathbb{R}^{2(n-1)}$, introduce

$$u' = \sqrt{A}u \text{ and } v' = \sqrt{A}v. \tag{C.17}$$

Here, since A is positive definite, $\sqrt{A}$ is the positive squre root of A. Then consider the quadratic form

$$\begin{aligned} [u^T, v^T] \begin{bmatrix} \Gamma_{11}A & \Gamma_{12}A \\ \Gamma_{12}A & \Gamma_{22}A \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} &= \Gamma_{11}u^T Au + \Gamma_{12}u^T Av + \Gamma_{12}v^T Au + \Gamma_{22}v^T Av \\ &= \Gamma_{11}(u')^T u' + \Gamma_{12}(u')^T v' + \Gamma_{12}(v')^T u' + \Gamma_{22}(v')^T v' \\ &= \sum_{k=1}^{n-1} \left[\Gamma_{11}(u'_k)^2 + 2\Gamma_{12}u'_k v'_k + \Gamma_{22}(v'_k)^2\right] \\ &\geq 0, \end{aligned}$$

since Γ is positive definite. The equality holds only if $u = v = \vec{0}$. This shows that $D^2 f_1$ is everywhere positive definite.

Hence f_1 is minimized at $\widetilde{w}$. Moreover when Δ is small, f is minimized at $\widetilde{w}$.

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