TRAVELLING AND ROTATING SOLUTIONS TO THE GENERALIZED INVISCID SURFACE QUASI-GEOSTROPHIC EQUATION

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ABSTRACT. For the generalized surface quasi-geostrophic equation

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta = 0, \quad \text{in } \mathbb{R}^2\times(0,T), \\ u = \nabla^\perp\psi, \quad \psi = (-\Delta)^{-s}\theta \quad \text{in } \mathbb{R}^2\times(0,T), \end{cases}$$

$0 < s < 1$, we consider for $k \geq 1$ the problem of finding a family of k-vortex solutions $\theta_\varepsilon(x,t)$ such that as $\varepsilon \to 0$

$$\theta_\varepsilon(x,t) \rightharpoonup \sum_{j=1}^k m_j\delta(x-\xi_j(t))$$

for suitable trajectories for the vortices $x = \xi_j(t)$. We find such solutions in the special cases of vortices travelling with constant speed along one axis or rotating with same speed around the origin. In those cases the problem is reduced to a fractional elliptic equation which is treated with singular perturbation methods. A key element in our construction is a proof of the non-degeneracy of the radial ground state for the so-called fractional plasma problem

$$(-\Delta)^s W = (W-1)_+^\gamma, \quad \text{in } \mathbb{R}^2, \quad 1 < \gamma < \frac{1+s}{1-s}$$

whose existence and uniqueness have recently been proven in [11].

1. INTRODUCTION

In this paper we consider the problem

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta = 0, \quad \text{in } \mathbb{R}^2\times(0,T) \\ u = \nabla^\perp\psi, \quad \psi = (-\Delta)^{-s}\theta \quad \text{in } \mathbb{R}^2\times(0,T), \end{cases} \tag{1.1}$$

where $0 < s < 1$ and $(a_1,a_2)^\perp = (a_2,-a_1)$, which is known as the modified or generalized surface quasi-geostrophic equation. Here θ is the active scalar being transported by the velocity field u generated by θ and ψ is the stream function. The operator $(-\Delta)^{-s}$ in $\mathbb{R}^n$ is the standard inverse of the fractional laplacian and is given by the expression

$$(-\Delta)^{-s}\theta(x) = \int_{\mathbb{R}^n} G_s(x-y)\theta(y)\,ds, \quad G_s(z) = \frac{c_{n,s}}{|z|^{n-2s}} \tag{1.2}$$

where

$$c_{n,s} = \pi^{-\frac{n}{2}}2^{-2s}\frac{\Gamma(\frac{n-2s}{2})}{\Gamma(s)}. \tag{1.3}$$

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The case $s = \frac{1}{2}$ in (1.1) corresponds to the surface quasi-geostrophic (SQG) equation while the limit $s \uparrow 1$ corresponds to the 2D Euler equation. In the formulation (1.1) we assume that θ is sufficiently regular so that ψ is C^1.

Equations (1.1) for $s = \frac{1}{2}$ first appeared as models of geophysical flows. After the classical work by Constantin, Majda and Tabak [12], who pointed out its formal mathematical analogies with the three dimensional Euler equation, these equations have been widely investigated.

The Cauchy problem for (1.1) is a delicate matter. For $0 < s < 1$, local well-posedness is known for sufficiently regular initial data, see [9, 10, 12, 22] and the references therein. Large class of initial data (patches) may produce finite time singularities, see [21, 22].

Of special interest are solutions of (1.1) with highly concentrated values of the active scalar $\theta(x,t)$ around a finite number of points $\xi_1(t), \dots, \xi_k(t)$ which are idealized as regular solutions that approximate a singular object of the form

$$\sum_{i=1}^{k} m_i \delta(x - \xi_i(t)), \tag{1.4}$$

where $\delta(x)$ is the standard Dirac mass at the origin. The constants m_i are called the intensities of the vortices $\xi_i(t)$. In the case $s = 1$, corresponding to the 2D Euler equation, these solutions represent fluids with sharply concentrated vorticities around the points $\xi_i(t)$. In this setting the problem is classical and traces back Kirchhoff. The location of the limiting point vortices is found by formal substitution, leading to the Hamiltonian system

$$\dot{\xi}_j(t) = \frac{1}{2\pi} \sum_{i \neq j}^{k} m_i \frac{(\xi_i(t) - \xi_j(t))^{\perp}}{|\xi_i(t) - \xi_j(t)|^2} \quad j = 1, \dots, k. \tag{1.5}$$

Finding regular solutions that approximate the superposition of point vortices (1.4) for a given solution of system (1.5), is the classical *vortex desingularization problem*. See the works [24, 15, 16, 7, 32] and references therein.

For the generalized SQG equation (1.1), the point vortex model corresponding to a solution of the form (1.4) becomes

$$\dot{\xi}_j(t) = \frac{1}{2^{2s-1}\pi} \frac{\Gamma(2-s)}{\Gamma(s)} \sum_{i \neq j} m_i \frac{(\xi_i(t) - \xi_j(t))^{\perp}}{|\xi_i(t) - \xi_j(t)|^{4-2s}}, \quad j = 1, \dots, k, \tag{1.6}$$

see the recent work by Rosenzweig [31] and references therein.

The purpose of this paper is to construct regular solutions $\theta(x,t)$ which resemble a superposition of point vortices of the form (1.4), where the k-tuple $(\xi_1(t), \dots, \xi_k(t))$ represents a solution of system (1.6) which *does not change form* as time evolves. More precisely, we focus on traveling and rotating solutions of system (1.6).

A traveling solution of (1.6) is one of the form

$$\xi_j(t) = b_j + cte_2 \tag{1.7}$$

where $b_1, \dots, b_k$ are points in $\mathbb{R}^2$, the constant $c \in \mathbb{R}$ is the speed and without loss of generality we take the travel direction to be $e_2 = (0,1)$. Then (1.6) reduces to the system

$$ce_2 = \frac{\Gamma(2-s)}{2^{2s-1}\pi\Gamma(s)} \sum_{i\neq j} m_i \frac{(b_i - b_j)^\perp}{|b_i - b_j|^{4-2s}}, \quad j = 1, \dots, k. \tag{1.8}$$

A rotating solution of (1.6) is one of the form

$$\xi_j(t) = Q_{\alpha t} b_j, \quad Q_{\alpha t} = \begin{bmatrix} \cos(\alpha t) & -\sin(\alpha t) \\ \sin(\alpha t) & \cos(\alpha t) \end{bmatrix} \tag{1.9}$$

and $b_1, \dots, b_k \in \mathbb{R}^2$. These are solutions of (1.6) if

$$\alpha b_j = -\frac{\Gamma(2-s)}{2^{2s-1}\pi\Gamma(s)} \sum_{i\neq j} m_i \frac{b_i - b_j}{|b_i - b_j|^{4-2s}}, \quad j = 1, \dots, k. \tag{1.10}$$

For simplicity, we will concentrate on the most elementary solutions to (1.8) and (1.10). For (1.8) we consider the traveling vortex pair, namely the solution with $k = 2$, and

$$\begin{aligned} &b_1 = de_1, \quad b_2 = -de_1, \quad e_1 = (1, 0), \quad m_1 = -m_2 = m, \\ &c = -\frac{\Gamma(2-s)}{4\pi\Gamma(s)} \frac{m}{d^{3-2s}}, \end{aligned} \tag{1.11}$$

where $d > 0$.

In the case of rotating solutions, we consider the rotating polygon with equal masses, that is, for $k \geq 2$,

$$\begin{aligned} &b_j = \rho e^{2\pi i \frac{j}{k}}, \quad m_j = m, \quad j = 0, \dots, k-1, \\ &\alpha = \frac{m}{\rho^{2-2s}} \frac{\Gamma(2-s)}{2^{s+1}\pi\Gamma(s)} \sum_{l=1}^{k-1} \frac{1}{(1 - \cos(\frac{2\pi l}{k}))^{1-s}}, \end{aligned} \tag{1.12}$$

where $\rho > 0$.

1.1. **Main results.** In analogy with the solution (1.7) of (1.6), we look for traveling solutions to (1.1) by requiring that

$$\theta(x_1, x_2, t) = \Theta(x_1, x_2 - ct), \tag{1.13}$$

for some profile function $\Theta(x_1, x_2)$ defined on $\mathbb{R}^2$. In this case, the generalized SQG equation (1.1) can be rewritten as the stationary problem

$$(\nabla^\perp \Psi - ce_2) \cdot \nabla \Theta = 0, \quad \Psi = (-\Delta)^{-s} \Theta. \tag{1.14}$$

The condition that θ approximates (1.4) now becomes

$$\Theta(x) \approx \sum_{j=1}^{k} m_j \delta(x - b_j). \tag{1.15}$$

Similarly, associated to solutions (1.9) of system (1.6), we look for rotating solutions $\theta(x)$ of (1.1) close to (1.4), by requiring that

$$\theta(x, t) = \Theta(Q_{-\alpha t} x), \quad x \in \mathbb{R}^2, \tag{1.16}$$

with $\Theta(x)$ also having the concentration behavior (1.15). Then (1.1) becomes

$$(\nabla^\perp \Psi + \alpha x^\perp) \cdot \nabla \Theta = 0, \quad \Psi = (-\Delta)^{-s} \Theta. \tag{1.17}$$

Our first result states the existence of a traveling solution concentrated near the vortex pair associated to the solution (1.7), (1.11) of system (1.8).

Theorem 1.1. *Consider the traveling vortex pair given by* (1.11)*. Then for $\varepsilon > 0$ small there is a solution θ_ε of* (1.1) *of the form* (1.13) *such that Θ_ε is $C^1(\mathbb{R}^2)$, and*

$$\Theta_\varepsilon(x) \rightharpoonup m\delta(x-b_1) - m\delta(x-b_2) \quad \text{as } \varepsilon \to 0, \quad \operatorname{supp}\Theta_\varepsilon \subset \bigcup_{j=1}^{2} B_{C\varepsilon}(b_j),$$

where the convergence is in the sense of measures and $C > 0$ is a constant.

Similarly, we obtain rotating concentrated solutions near the vertices of the rotating polygon solution (1.9), (1.12) of (1.10).

Theorem 1.2. *Consider the rotating polygon given by* (1.12)*. Then for $\varepsilon > 0$ small there is a solution θ_ε of* (1.1) *of the form* (1.16) *such that Θ_ε is $C^1(\mathbb{R}^2)$,*

$$\Theta_\varepsilon(x) \rightharpoonup m\sum_{j=1}^{k}\delta(x-b_j) \quad \text{as } \varepsilon \to 0, \quad \operatorname{supp}\Theta_\varepsilon \subset \bigcup_{j=1}^{k} B_{C\varepsilon}(b_j),$$

where the convergence is in the sense of measures and $C > 0$ is a constant.

A natural way of obtaining solutions to the stationary problem (1.14) is to locally impose that $\Theta(x) = f(\Psi(x) + cx_1)$ for a sufficiently regular function $f(u)$ so that (1.14) locally becomes the elliptic equation

$$(-\Delta)^s\Psi = f(\Psi + cx_1). \tag{1.18}$$

Similarly, locally imposing $\Theta(x) = f(\Psi(x) + \alpha\frac{|x|^2}{2})$, problem (1.17) becomes

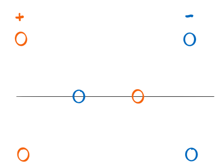
$$(-\Delta)^s\Psi = f\Big(\Psi + \alpha\frac{|x|^2}{2}\Big). \tag{1.19}$$

Using this observation, Gravejat and Smets [20] have recently found solutions $\Theta(x)$ to problem (1.14) for $s = \frac{1}{2}$, with compact support and odd symmetry in x_1. They use variational techniques applied to a suitable class of subcritical nonlinearities.

1.2. **Extensions.** Traveling solutions with multiple vortices can be found under suitable non-degeneracy conditions for solutions of system (1.8). For instance, following [23], we can consider configurations with k vortices with intensities 1 located at points $p_1, \dots, p_k$ and k vortices with intensities -1 located at $q_1, \dots, q_k$. We say that $\mathbf{b} = (\mathbf{p}, \mathbf{q}) = (p_1, \dots, p_k, q_1, \dots, q_k)$ is an array of traveling vortices if it satisfies the following conditions.

$$\begin{cases} \displaystyle\sum_{j\neq i}\frac{p_i-p_j}{|p_i-p_j|^{4-2s}} - \sum_{l=1}^{k}\frac{p_i-q_l}{|p_i-q_l|^{4-2s}} = c\frac{2^{2s-1}\pi\Gamma(s)}{\Gamma(2-s)}e_1, \quad i=1,\dots,k, \\ \displaystyle\sum_{l\neq m}\frac{q_m-q_l}{|q_m-q_l|^{4-2s}} - \sum_{j=1}^{k}\frac{q_m-p_j}{|q_m-p_j|^{4-2s}} = -c\frac{2^{2s-1}\pi\Gamma(s)}{\Gamma(2-s)}e_1, \quad m=1,\dots,k, \end{cases} \tag{1.20}$$

FIGURE 1. Figure 1. Six vortices



where $c \in \mathbb{R}$. The set of points $(\mathbf{p}, \mathbf{q})$ is called a symmetric array of traveling vortices if in addition to (1.20) it satisfies

$$\begin{cases} q_i = -\bar{p}_i, \quad i = 1, \dots, k; \\ \text{there exists } j_0 \text{ such that } p_{2j-1} = \bar{p}_{2j}, \quad j = 1, \dots, j_0 \\ \text{and } \mathrm{Im}(p_j) = 0, \text{for } j = 2j_0 + 1, \dots, k. \end{cases} \tag{1.21}$$

(Here $\bar{z}$ denotes the conjugate of z and $Im(z)$ is the imaginary part of z.) A symmetric array of traveling vortices $(\mathbf{p}, \mathbf{q})$ is called nondegenerate if the linearization map at $(\mathbf{p}, \mathbf{q})$ among points satisfying (1.21) has only trivial kernel.

When $s = 1$ these definitions are introduced in [23]. There it is shown that the roots of certain Adler-Moser polynomials are nondegenerate symmetric arrays of traveling vortices with $k = \frac{n(n+1)}{2}$ for some integer n. As a consequence they constructed multiple vortices to the traveling wave equation to Gross-Pitaevskii equation.

By a perturbation argument in s, we also obtain that for each fixed $k = \frac{n(n+1)}{2}$ there exist nondegenerate symmetric arrays of traveling vortices when $s < 1, |1 - s| \ll 1$. Since the forces are analytic in s, we infer that except finite number of s, for each fixed $k = \frac{n(n+1)}{2}$, there exist a unique nondegenerate symmetric array of traveling vortices. For general $s \in (0, 1)$, it is an interesting and challenging question to find nondegenerate symmetric arrays of traveling vortices. In the special case $s = \frac{1}{2}$ (the SQG case), we can use MatLab[1] to compute numerically the existence of six nondegenerate symmetric arrays of traveling vortices (See Figure 1): $p_1 = -\bar{q}_1 = (-1.026, 0.563), p_2 = -\bar{q}_2 = (-1.026, -0.563), p_3 = -\bar{q}_3 = (0.368, 0)$.

We state the following theorem on the existence of multiple vortex-anti vortex solutions to (1.14).

Theorem 1.3. *Let $(p_1, \dots, p_k, q_1, \dots, q_k)$ be a nondegenerate symmetric array of traveling vortices. Then for $\varepsilon > 0$ small, there exists a solution Θ_ε to (1.14) such that*

$$\Theta_\varepsilon(x) \rightharpoonup \sum_{j=1}^{k} \delta(x - p_j) - \sum_{j=1}^{k} \delta(x - q_j)$$

[1]We thank Prof. Yong Liu for the computations.

as $\varepsilon \to 0$. *Moreover* $\operatorname{supp}\Theta_\varepsilon \subset \bigcup_{j=1}^{k} B_{C\varepsilon}(b_j)$ *and has* Θ_ε *has the symmetries*

$$\Theta_\varepsilon(x_1,x_2) = -\Theta_\varepsilon(-x_1,x_2) = \Theta_\varepsilon(x_1,-x_2) \quad \text{for all } (x_1,x_2)\in\mathbb{R}^2.$$

A similar result can be found for the Euler equation case $s = 1$, applying the results of [7].

More general traveling solutions than those in Theorem 1.3 can also be found. Let us consider the functional of k points $\mathbf{b} = (b_1,\dots,b_k)\in\mathbb{R}^{2k}$,

$$I(\mathbf{b}) = c\sum_{i=1}^{k} m_i(b_i\cdot e_1) + \frac{\Gamma(1-s)}{\pi 2^{2s-1}\Gamma(s)}\sum_{i\neq j}\frac{m_i m_j}{|b_i-b_j|^{2-2s}}.$$

Critical points of $I(\mathbf{b})$ correspond to solutions of system (1.8). The functional I is invariant under translations along the e_2-direction, and therefore all critical points of I are degenerate. We say that a critical point $\mathbf{b}$ of I is non-degenerate up to vertical translations, if $D^2I(\mathbf{b})$ has a one-dimensional kernel. The following result holds.

Theorem 1.4. *If* $\mathbf{b}$ *is a critical point* non-degenerate up to vertical translations *then there exists a solution* $\Theta_\varepsilon(x)$ *of Equation (1.14) such that* $\operatorname{supp}\Theta_\varepsilon \subset \bigcup_{j=1}^{k} B_{C\varepsilon}(b_j)$ *and*

$$\Theta_\varepsilon(x) \rightharpoonup \sum_{j=1}^{k} m_j\delta(x-b_j) \quad \text{as } \varepsilon\to 0.$$

A similar result holds in the case of rotating solutions. Let $k\geq 2$ be an integer, $m_1,\dots,m_k\in\mathbb{R}$ and $\alpha\neq 0$. Let us consider the energy functional

$$J(\mathbf{b}) = \frac{\alpha}{2}\sum_{i=1}^{k} m_i|b_i|^2 + \frac{\Gamma(1-s)}{\pi 2^{2s-1}\Gamma(s)}\sum_{i\neq j}\frac{m_i m_j}{|b_i-b_j|^{2-2s}},$$

where $\mathbf{b} = (b_1,\dots,b_k)\in\mathbb{R}^{2k}$. Critical points of J correspond to solutions of (1.10). Since this functional is invariant under rotations around the origin, its critical points are always degenerate. We say that a critical point $\mathbf{b}$ is non-degenerate up to rotations if $D^2J(\mathbf{b})$ has a one-dimensional kernel.

Theorem 1.5. *Let* $\mathbf{b} = (b_1,\dots,b_k)$ *be a critical point of* J *that is non-degenerate up to rotations. Then for* $\varepsilon>0$ *small there exists a solution* Θ_ε *to* (1.17) *such that*

$$\Theta_\varepsilon(x) \rightharpoonup \sum_{j=1}^{k} m_j\delta(x-b_j)$$

as $\varepsilon\to 0$. *Moreover* $\operatorname{supp}\Theta_\varepsilon \subset \bigcup_{j=1}^{k} B_{C\varepsilon}(b_j)$.

In [6] a related problem in gravitation theory, consisting in the desingularization of rotating point masses in a continuous model of stellar dynamics, has been considered. The result obtained there is similar to Theorem 1.5 for $s=\frac{1}{2}$. Critical points of J for $s=\frac{1}{2}$ are called relative equilibria for the N-body problem. Their study is classical in celestial mechanics. In particular, it is known that for almost every choice of masses, critical points are non-degenerate up rotations and their number is estimated, see [29]. See also [25, 30, 26, 27, 28] and references therein.

1.3. **The fractional plasma problem.** The proof of theorems 1.1–1.5 consists of gluing highly concentrated solutions of special elliptic equations of the form (1.18) and (1.19). For this purpose we will use suitably scaled radial solutions of the so-called fractional plasma problem. In any space dimension $n \geq 2$, this is the semi-linear elliptic problem

$$\begin{cases} (-\Delta)^s W = (W-1)_+^\gamma \quad \text{in } \mathbb{R}^n, \\ \quad W(x) \to 0 \quad \text{as } |x| \to \infty, \end{cases} \tag{1.22}$$

where $s \in (0,1)$, $1 \leq \gamma < \frac{n+2s}{n-2s}$. Problem (1.22) arises in the context of aggregation-diffusion equations, see [8].

When $s = 1$ this free boundary problem has been studied in [33, 34, 2, 5, 18].

For $s = 1$ solutions are radially symmetric up to translations and can be analyzed using ODE techniques. This is not possible in the nonlocal case $s \in (0,1)$ and the analysis becomes substantially harder. Recently, in [11] it has been proven that (1.22) has a unique radial solution. The proof relies on an application of a monotonicity formula developed for the fractional Schrödinger equation in [19].

In this paper, we will use this ground state to (1.22) to construct solutions by a Lyapunov-Schmidt reduction. Our first contribution in this paper is to derive the non-degeneracy of the radial ground state solution of (1.22).

The paper is organized as follows. In section 2 we describe the elliptic equation we use to prove Theorems 1.1, 1.3, and 1.4 and the form of the solution at main order. In Section 3, we introduce the radial ground state solution to (1.22) and study the non-degeneracy of the linearized operator around it. Section 4 is devoted to the proof of Theorem 1.1, with some some arguments deferred for later: a solvability theory for the linearized equation is developed in section 5 and some computations associated to the nonlinear problem are in section 6. We give some ideas of the proofs of Theorems 1.2–1.5 in section 7.

2. An elliptic equation for concentrated solutions of (1.14)

To prove Theorems 1.1, 1.3 and 1.4 we need to find a family of solutions $\Theta_\varepsilon(x)$ to the equation

$$(\nabla^\perp \Psi - ce_2) \cdot \nabla \Theta = 0, \quad \Psi = (-\Delta)^{-s}\Theta, \tag{2.1}$$

such that

$$\Theta_\varepsilon(x) \rightharpoonup \sum_{j=1}^k m_j \delta(x - b_j^0),$$

for given intensities m_j and a solution $\mathbf{b}^0 = (b_1^0, \dots, b_k^0)$ of system (1.8). In order to achieve this we consider the following elliptic problem.

$$\begin{cases} (-\Delta)^s \psi = \varepsilon^{(2-2s)\gamma-2} \displaystyle\sum_{j=1}^k \sigma_j (\sigma_j(\psi + cx_1) - \varepsilon^{2s-2}\lambda_j)_+^\gamma \chi_{B_\delta(b_j)} \quad \text{in } \mathbb{R}^2, \\ \psi(x) \to 0 \quad \text{as } |x| \to \infty, \end{cases} \tag{2.2}$$

where we take $\sigma_j = +1$ if $m_j > 0$ and $= -1$ if $m_j < 0$. The scalars λ_j will be suitably chosen later on. We also assume that $1 < \gamma < \frac{2+2s}{2-2s}$. The number $\delta > 0$

is fixed so that the balls $B_\delta(b_j)$ are disjoint and the points $b_1, \dots, b_k$ are close to $b_1^0, \dots, b_k^0$.

We look for sufficiently smooth solutions ψ of problem (2.2) such that

$$\Theta_\varepsilon(x) := \varepsilon^{(2-2s)\gamma-2} \sum_{j=1}^{k} \sigma_j(\sigma_j(\psi + cx_1) - \varepsilon^{2s-2}\lambda_j)_+^\gamma \chi_{B_\delta(b_j)}(x)$$

has its support contained in $\bigcup_{j=1}^k B_\delta(b_j)$. We readily check that the latter condition guarantees that $\Theta_\varepsilon(x)$ solves (2.1).

We will find a solution of problem (2.2) which at main order looks like a superposition of sharply scaled similar radial profiles centered near each of the points b_j. In [11] it was proved that there exists a unique radial solution $W(y)$ of the problem

$$\begin{cases} (-\Delta)^s W = (W-1)_+^\gamma & \text{in } \mathbb{R}^2, \\ W(y) \to 0 & \text{as } |y| \to \infty, \end{cases} \tag{2.3}$$

where $0 < s < 1$ and $1 < \gamma < \frac{2+2s}{2-2s}$. This solution is understood as a $W \in L^\infty(\mathbb{R}^2)$ with $W(y) \to 0$ as $|y| \to \infty$ that satisfies the integral equation

$$W = (-\Delta)^{-s}[(W-1)_+^\gamma] \quad \text{in } \mathbb{R}^2. \tag{2.4}$$

It turns out that $W \in C^{1,\beta}(\mathbb{R}^2)$ and it has has the precise asymptotic behavior

$$W(y) = M_\gamma c_{2,s} |y|^{-(2-2s)}(1 + o(1)) \quad \text{as } |y| \to \infty, \tag{2.5}$$

where $M_\gamma = \int_{\mathbb{R}^n} (W-1)_+^\gamma dy > 0$ and $c_{2,s}$ is given in (1.3).

We look for a solution of (2.2) that looks approximately like

$$\psi_0(x) = \varepsilon^{2s-2} \sum_{j=1}^{k} \sigma_j \mu_j^{-\frac{2s}{\gamma-1}} W\Big(\frac{x-b_j}{\varepsilon\mu_j}\Big), \tag{2.6}$$

where μ_j are positive constants. Then as $\varepsilon \to 0$ we have

$$(-\Delta)^s \psi_0(x) \rightharpoonup M_\gamma \sum_{j=1}^{k} \sigma_j \mu_j^{2(1-\frac{s\gamma}{\gamma-1})} \delta(x - b_j).$$

Therefore we fix $\mu_j > 0$ such that

$$M_\gamma \sigma_j \mu_j^{2(1-\frac{s\gamma}{\gamma-1})} = m_j, \quad j = 1, \dots, k, \tag{2.7}$$

which is possible if in addition we assume that $\gamma \neq \frac{1}{1-s}$.

We compute, for $x \in B_\delta(b_l)$, assuming for simplicity that $\sigma_l = 1$,

$$(-\Delta)^s \psi_0 - \varepsilon^{(2-2s)\gamma-2} \sum_{j=1}^{k} \sigma_j(\sigma_j(\psi_0 + cx_1) - \varepsilon^{2s-2}\lambda_j)_+^\gamma \chi_{B_\delta(b_j)} \tag{2.8}$$

$$= \varepsilon^{-2} \mu_l^{-\frac{2s\gamma}{\gamma-1}} \Big[\Big(W\Big(\frac{x-b_l}{\varepsilon\mu_l}\Big) - 1 \Big)_+^\gamma$$

$$- \Big(W\Big(\frac{x-b_l}{\varepsilon\mu_l}\Big) + \sum_{j\neq l} \sigma_j \mu_l^{\frac{2s}{\gamma-1}} \mu_j^{-\frac{2s}{\gamma-1}} W\Big(\frac{x-b_j}{\varepsilon\mu_j}\Big) + c\mu_l^{\frac{2s}{\gamma-1}} \varepsilon^{2-2s} x_1 - \mu_l^{\frac{2s}{\gamma-1}} \lambda_l \Big)_+^\gamma \Big].$$

We note that ψ_0 is a good approximation to a solution to (2.2) if the parameters λ_j are chosen such that

$$\sum_{j\neq l}\sigma_j\mu_l^{\frac{2s}{\gamma-1}}\mu_j^{-\frac{2s}{\gamma-1}}W\Big(\frac{b_l-b_j}{\varepsilon\mu_j}\Big)+c\mu_l^{\frac{2s}{\gamma-1}}\varepsilon^{2-2s}b_{l,1}-\mu_l^{\frac{2s}{\gamma-1}}\lambda_l=-1. \tag{2.9}$$

Similarly, if $\sigma_l=-1$ we impose that

$$-\sum_{j\neq l}\sigma_j\mu_l^{\frac{2s}{\gamma-1}}\mu_j^{-\frac{2s}{\gamma-1}}W\Big(\frac{b_l-b_j}{\varepsilon\mu_j}\Big)-c\mu_l^{\frac{2s}{\gamma-1}}\varepsilon^{2-2s}b_{l,1}-\mu_l^{\frac{2s}{\gamma-1}}\lambda_l=-1. \tag{2.10}$$

Using the expansion (2.5) we get that

$$\lambda_l=\mu_l^{-\frac{2s}{\gamma-1}}+O(\varepsilon^{2-2s})$$

as $\varepsilon\to0$. With this choice of λ_l, we find that the error of approximation created by ψ_0, has the estimate

$$\begin{aligned}&\varepsilon^2\Big\{(-\Delta)^s\psi_0-\varepsilon^{(2-2s)\gamma-2}\sum_{j=1}^k\sigma_j(\sigma_j(\psi_0+cx_1)-\varepsilon^{2-2s}\lambda_j)_+^{\gamma}\chi_{B_\delta(b_j)}\Big\}\\&=O(\varepsilon^{3-2s})\sum_{j=1}^k\chi_{B_{C\varepsilon}(b_j)}.\end{aligned} \tag{2.11}$$

The proof of Theorems 1.1, 1.3 and (1.4) consists in finding a solution of problem (2.2) as a suitable small perturbation of the function ψ_0 defined above. Linearizing the problem around ψ_0 and a Lyapunov-Schmidt reduction procedure, transforms the problem into one of adjusting the points b_j as a small perturbation of a solution of system (1.8). Nondegeneracy of the reduced limiting problem and that of the linearized elliptic operator play a crucial role in the complete proof. The proofs of Theorems 1.2 and 1.5 follow from similar considerations.

Of central importance will be the understanding of invertibility properties of the linearized operator of equation (2.3), namely

$$\begin{cases}L_0[\phi]:=(-\Delta)^s\phi-\gamma(W(y)-1)_+^{\gamma-1}\phi,\\ \phi(y)\to0\quad\text{as }|y|\to\infty.\end{cases}$$

We will prove in Section 3 that the only decaying elements in the kernel of L_0 are the ones associated to the invariance of (2.3) under translations.

We will proceed with the detailed proof of Theorem 1.1 in section 4 below.

3. The nondegeneracy of the building blocks

Recently, in [11], the authors studied the fractional plasma equation:

$$\begin{cases}(-\Delta)^sW=(W-1)_+^{\gamma}\quad\text{in }\mathbb{R}^n,\\ \quad W(y)\to0\quad\text{as }|y|\to\infty,\end{cases} \tag{3.1}$$

in the sense (2.4). where $s\in(0,1)$, $1\le\gamma<\frac{n+2s}{n-2s}$. and they proved the following:

Theorem 3.1 (Theorem 1.1 in [11]). *There exists a unique radial solution W to equation (3.1). W is of class $C^{1,\beta}$ for some $\beta>0$. Moreover, this ground state solution satisfies:*

(i) *$W(x)=W(|x|)$ is radial symmetric and decreasing in $r=|x|$;*

(ii) *It satisfies the following asymptotic behavior:*

$$W(x) = M_\gamma c_{n,s}|x|^{-(n-2s)}(1+o(1)) \quad \text{as } |x| \to \infty, \tag{3.2}$$

where $M_\gamma = \int_{\mathbb{R}^n}(W-1)_+^\gamma dx > 0$. *Moreover,*

$$W'(|x|) = -(n-2s)M_\gamma c_{n,s}|x|^{-(n-2s)-1}(1+o(1)) \quad \text{as } |x| \to \infty. \tag{3.3}$$

In this section, we will study the linearized operator of this equation around the ground state, namely,

$$L_0[\phi] = (-\Delta)^s\phi - \gamma(W-1)_+^{\gamma-1}\phi. \tag{3.4}$$

We are interested in characterizing the set of all solutions $\phi \in L^\infty(\mathbb{R}^n)$ of the problem

$$\begin{cases} L_0[\phi] = 0 & \text{in } \mathbb{R}^n, \\ \phi(y) \to 0 & \text{as } |y| \to \infty, \end{cases} \tag{3.5}$$

to which we make sense in $L^\infty(\mathbb{R}^n)$ written in the form

$$\phi = (-\Delta)^{-s}[V\phi]. \tag{3.6}$$

where

$$V(x) = \gamma(W(x)-1)_+^\gamma. \tag{3.7}$$

It actually follows from result of [17] that if $\phi \in L^\infty$ satisfies (3.5) in the distributional sense, then it must satisfy (3.6).

It follows from the regularity analysis in [19] that solutions $\phi \in L^\infty(\mathbb{R}^n)$ of Problem (3.7) are of class $C^{1,\beta}$ with $\beta > 2s$ which makes the equation (3.5) satisfied in the classical sense.

Besides, as in [4, 19] it follows that the s-harmonic extension of $\phi(x)$ defined as

$$\Phi(x,y) = k_{n,s}\int_{\mathbb{R}^n}\frac{y^{2s}}{(y^2+|x-z|^2)^{\frac{n+2s}{2}}}\phi(z)\,dz, \quad k_{n,s}\int_{\mathbb{R}^n}\frac{dz}{(1+|z|^2)^{\frac{n+2s}{2}}} = 1$$

satisfies in the classical sense

$$\begin{cases} \partial_{yy}\Phi + \dfrac{1-2s}{y}\partial_y\Phi + \Delta_{\mathbb{R}^n}\Phi = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ -\tilde{d}_s \lim\limits_{y\to 0} y^{1-2s}\partial_y\Phi = V(x)\phi & \text{on } \mathbb{R}^n, \end{cases} \tag{3.8}$$

where $\tilde{d}_s = -\frac{2^{2s-1}\Gamma(s)}{s\Gamma(-s)}$ and $\Phi|_{y=0} = \phi$. We are using the notation $(x,y) \in \mathbb{R}^{n+1}$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}$ and $\mathbb{R}^{n+1}_+ = \{(x,y) \in \mathbb{R}^{n+1} | y > 0\}$. The latter limit holds uniformly in the C^β sense in x. In fact, for some $C > 0$ and all $y > 0$ we have that

$$\|\nabla_x\Phi(\cdot,y)\|_{C^\beta(\mathbb{R}^n)} + \|y^{1-2s}\partial_y\Phi(\cdot,y)\|_{C^\beta(\mathbb{R}^n)} \le C.$$

By the invariance of (3.1) under translations, every directional derivative of W annihilate L_0. In fact

$$L_0[\frac{\partial W}{\partial x_j}] = 0, \quad j = 1, \dots, n.$$

Equation (3.1) is also invariant under the dilations

$$\lambda^{\frac{2s}{\gamma-1}}(W(\lambda x)-1)+1 \quad \text{where} \quad \lambda > 0.$$

Thus

$$z_0(x) = \frac{d}{d\lambda}\left[\lambda^{\frac{2s}{\gamma-1}}(W(\lambda x)-1)\right]_{\lambda=1}$$

also satisfies $L_0[z_0]=0$. Note that $z_0 \in L^\infty(\mathbb{R}^n)$, but $\lim_{|x|\to\infty} z_0(x) = -\frac{2s}{p-1} \neq 0$.

The main result here is the following.

Proposition 3.2. *If $\phi \in L^\infty(\mathbb{R}^n)$ is a solution to (3.5), then ϕ is a linear combination of $\frac{\partial W}{\partial x_1}, \dots, \frac{\partial W}{\partial x_n}$.*

Proof of Proposition 3.2. Let $\phi \in L^\infty(\mathbb{R}^n)$ be a solution to (3.5). Since $\phi \in L^\infty(\mathbb{R}^n)$ satisfy (3.6), we have that

$$|\phi(x)| \le C(1+|x|)^{-(n-2s)}.$$

Let $\Phi = \Phi(x,y)$ be its s-harmonic extension, which is regular and satisfies (3.8).

Denote by μ_i, $i=0,1,\dots$ the i-th eigenvalue of $-\Delta_{\mathbb{S}^{n-1}}$, repeated according to multiplicity and arranged in increasing order, and by $E_i(\theta)$ the corresponding eigenfunction. In particular E_0 is constant and $E_1,\dots,E_n$ are the coordinate functions x_i normalized. Then we can write Φ using Fourier decomposition as

$$\Phi(x,y) = \sum_{i=0}^{\infty} \Phi_i(x,y), \quad \Phi_i(x,y) = \psi_i(r,y)E_i(\theta), \tag{3.9}$$

where $x = r\theta$, $r=|x|$, $\theta \in \mathbb{S}^{n-1}$, $y>0$, so that Φ_i is the Fourier mode i of Φ. The convergence of the series in (3.9) is in the locally uniform and C^1 senses.

We observe that $L_0[\phi_i]=0$ where $\phi_i(x) = \Phi_i(x,0)$. We have also that ψ_i satisfies:

$$\begin{cases} \partial_{yy}\psi_i + \dfrac{1-2s}{y}\partial_y\psi_i + \partial_{rr}\psi_i + \dfrac{n-1}{r}\partial_r\psi_i - \dfrac{\mu_i}{r^2}\psi_i = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ -\tilde{d}_s \lim\limits_{y\to 0} y^{1-2s}\partial_y\psi_i = V(r)\psi_i & \text{on } \mathbb{R}^n. \end{cases} \tag{3.10}$$

The latter limit holds uniformly in the C^β sense in r.

Step 1. First we consider the mode zero case, that is, ϕ_0 with the notation (3.9), which is radial element of the kernel of L_0.

We claim that $\phi_0 = az_0$ for some constant $a\in\mathbb{R}$. Indeed, consider the function $u := \phi_0 - az_0$ where a is chosen so that $u(0)=0$. We note that $L_0[u]=0$ and that $u \in L^\infty(\mathbb{R}^n)$. We will use the argument in [19] to prove that $u\equiv 0$.

The ground state W of (3.1) constructed in Theorem 3.1 satisfies that there exists a unique $R_0>0$ such that $W(R_0)=0$. By standard estimates for the extension problem (3.8), $W(r)$ is smooth for $r\neq R_0$ (see for example the appendix in [19] and [3]). Hence $V(r)$ is smooth for $r \neq R_0$. Let U denote the extension of u to $\mathbb{R}^{n+1}_+$ solving (3.8). Following [19, 3], for $r>0$ let

$$H(r) = \tilde{d}_s \int_0^\infty \frac{y^{1-2s}}{2}[(\partial_r U(r,y))^2 - (\partial_y U(r,y))^2]\,dy + \frac{1}{2}V(r)u(r)^2.$$

We note that $H(r)$ is well defined and continuous for all $r\ge 0$ and smooth for $r\neq R_0$. We also observe that $H(r)\to 0$ as $r\to\infty$, since $V(r)=0$ for $r>R$. As in

[19] we have

$$H'(r) = -\tilde{d}_s \frac{n-1}{r} \int_0^\infty y^{1-2s} (\partial_r U(r,y))^2 \, dy + \frac{1}{2} V'(r) u(r)^2 \le 0 \quad \text{for } r > 0, \ r \ne R_0. \tag{3.11}$$

Since $H(0) \le 0$ and $H(r) \to 0$ as $r \to \infty$ we conclude that $H(r) = 0$ for all $r > 0$. Hence $H'(r) = 0$, and using again (3.11) we find that $U(r,z)$ must be constant. Since $u(0) = 0$, we conclude that $u \equiv 0$, that is, $\phi_0 = a z_0$. But since $\lim_{r\to\infty} z_0(r) \ne 0$ and $\lim_{r\to\infty} \phi_0(r) = 0$, we deduce that $a = 0$, so $\phi_0 \equiv 0$.

Step 2. The modes $1, \ldots, n$. We consider ϕ_i, $i = 1, \ldots, n$ with the notation (3.9), which are elements in the kernel of L_0 in Fourier mode i. We denote by Φ_i the extension of ϕ_i and write $\Phi_i(x,y) = \psi_i(r,y) E_i(\theta)$ as in (3.9).

Differentiating the equation (3.1) we get

$$L_0[z_i] = 0, \quad z_i = -\frac{\partial W}{\partial x_i}.$$

Let Z_i be the extension of of z_i solving (3.8) and write $Z_i = Z_*(r,y) E_i(\theta)$ so that Z_* solves (3.10). Note that $z_i = W'(r) \frac{x_i}{|x|}$ and that $W'(r) < 0$ by the results in [11]. By the strong maximum principle $Z_*(r,y) > 0$ for all $r > 0$ and all $y \ge 0$. We consider the function $\varphi = \frac{\psi_i}{Z_*}$ and note that it satisfies

$$\frac{1}{r^{n-1}} \partial_r (r^{n-1} Z_*^2 \partial_r \varphi) + \frac{1}{y^{1-2s}} \partial_y (y^{1-2s} Z_*^2 \partial_y \varphi) = 0, \quad r > 0, \quad y > 0.$$

We multiply this equation by $\varphi r^{n-1} y^{1-2s}$ and integrate in the region $D_{\epsilon,R} = \{(r,y) | r > 0, \ y > 0, \ \epsilon^2 < r^2 + y^2 < R^2\}$ where $0 < \epsilon < R$, to find that

$$\int_{D_{\epsilon,R}} Z_*^2 [(\partial_r \varphi)^2 + (\partial_y \varphi)^2] r^{n-1} y^{1-2s} \, dr dy = -I_\epsilon + I_R$$

where

$$I_\rho = \int_{\mathcal{C}_\rho} Z_*^2 [r \varphi \partial_r \varphi + y \varphi \partial_y \varphi] \frac{r^{n-1} y^{1-2s}}{(r^2 + y^2)^{\frac{1}{2}}} \, d\ell,$$

where $\mathcal{C}_\rho = \{(r,y) | r > 0, y > 0, r^2 + y^2 = \rho^2\}$ and ℓ is arclength in (r,y). We claim that

$$\lim_{R\to\infty} I_R = 0, \quad \lim_{\epsilon\to 0} I_\epsilon = 0. \tag{3.12}$$

To prove these statements, we note first that

$$Z_*^2 \varphi \partial_r \varphi = \psi_i \partial_r \psi_i - \frac{\psi_i}{Z^*} \psi_i \partial_r Z_*, \quad Z_*^2 \varphi \partial_y \varphi = \psi_i \partial_y \psi_i - \frac{\psi_i}{Z^*} \psi_i \partial_y Z_*.$$

We recall some estimates for first R we observe that using the Poisson kernel for the operator $\Delta_x + \frac{1-2s}{y} \partial_y$ [4] we have that

$$Z_*(r,y) \ge c \frac{y^{2s}}{(r^2 + y^2)^{\frac{n+2s}{2}}}, \quad |\psi_i(r,y)| \le C \frac{y^{2s}}{(r^2 + y^2)^{\frac{n+2s}{2}}},$$

for $r^2 + y^2$ large, where $c > 0$ is a constant. Using these estimates with similar ones for the derivatives we find that for R large

$$|I_R| \le C R^{-n-1-2s},$$

where C is a constant. This proves the first limit in (3.12).

For the estimate of I_ϵ when $\epsilon > 0$ is small, we first observe that

$$Z_*(r,y) \geq cr$$

for $r^2 + y^2 < \delta^2$ and some $c > 0$, $\delta > 0$. This is proved using the maximum principle applied to the equation (3.10) with the subsolution $r + br^2$ where $b > 0$. We also have the estimates

$$y^{1-2s}|\partial_y \psi_i| + y^{1-2s}|\partial_y Z_*| \leq C$$

for $r^2 + y^2 < \delta^2$, which follow from [3][Lemma 4.5]. Then

$$|I_\epsilon| \leq C\epsilon^{n-2s} \to 0$$

as $\epsilon \to 0$. This proves the second limit in (3.12).

Using (3.12) we deduce that

$$\int_{\{r>0,\, y>0\}} Z_*^2[(\partial_r \varphi)^2 + (\partial_y \varphi)^2] r^{n-1} y^{1-2s}\, drdy = 0,$$

which implies that φ is constant. We deduce then that $\phi_i = c_i z_i$ for some constant c_i.

Step 3. The remaining modes $m \geq n+1$. We use an integral estimate as in [14, 13].

As before, let $z_*(r) = -W'(r) > 0$ and $Z_*(r,y)$ be its extension. Consider ϕ_m with $m \geq n+1$ and Φ_m its extension, which we write as $\Phi_m(x,y) = \psi_m(r,y)E_m(\theta)$.

Let us rewrite (3.10) as

$$\begin{cases} \dfrac{1}{y^{1-2s}}\partial_y(y^{1-2s}\partial_y \psi_m) + \dfrac{1}{r^{n-1}}\partial_r(r^{n-1}\partial_r \psi_m) = \mu_m \dfrac{\psi_m}{r^2} & \text{in } \mathbb{R}^{n+1}_+, \\ -\tilde{d}_s \lim\limits_{y\to 0} y^{1-2s}\partial_y \psi_m = V(r)\psi_m & \text{on } \partial\mathbb{R}^{n+1}_+. \end{cases}$$

Let us write $\psi_m = \varphi Z_*$. Arguing as in the previous step now we find the identity

$$\frac{1}{Z_*^2}\frac{1}{y^{1-2s}}\partial_y(y^{1-2s}\partial_y \phi) + \frac{1}{Z_*^2}\frac{1}{r^{n_1}}\partial_r(r^{n-1}\partial_r \phi) = \frac{\mu_m - \mu_1}{r^2}\varphi.$$

As in the previous step, testing this equation against φ and integrating we get

$$\int_0^\infty \int_0^\infty [Z_*^2(\partial_r \phi) y^{1-2s} r^{n-1} + Z_*^2(\partial_y \phi) y^{1-2s} r^{n-1}]\, dydr \\ + \int_0^\infty \int_0^\infty \frac{\mu_m - \mu_1}{r^2} Z_*^2 \varphi^2 y^{1-2s} r^{n-1} dydr = 0.$$

Since $\mu_m > \mu_1$ we find that $\varphi = 0$, hence $\psi_m = 0$ for $m \geq n+1$. This completes the proof of Proposition 3.2.

□

We point out that a different proof can be done using the ODE techniques for fractional problems as in [1]. The monotonicity of $V(r)$ is not needed.

4. Construction of a vortex-anti vortex pair: the proof of Theorem 1.1

Let us consider the traveling vortex-anti vortex pair described by (1.11), centered at $b_1 = (d, 0)$, $b_2 = (-d, 0)$ with masses m and $-m$. For the construction of the solution stated in Theorem 1.1 we take the formulation described in Section 2 and specialize it to the following problem

$$\begin{cases} (-\Delta)^s \psi = \varepsilon^{(2-2s)\gamma-2}\big((\psi + cx_1 - \varepsilon^{2-2s}\lambda)_+^\gamma \chi_{B_\delta(b_1)} - (-\psi - cx_1 - \varepsilon^{2-2s}\lambda)_+^\gamma \chi_{B_\delta(b_2)}\big) \\ \qquad\qquad \text{in } \mathbb{R}^2, \\ \psi(x) \to 0 \quad \text{as } |x| \to \infty, \end{cases} \tag{4.1}$$

where we have taken the same λ for both points.

Given the symmetries of the problem, it is natural to construct a solution with

$$\psi(x_1, x_2) = -\psi(-x_1, x_2) = \psi(x_1, -x_2) \quad \text{for all } (x_1, x_2) \in \mathbb{R}^2. \tag{4.2}$$

Following (2.6) we take as a first approximation of a solution to (4.1)

$$\psi_0(x) = \varepsilon^{2s-2}\mu^{-\frac{2s}{\gamma-1}} W\Big(\frac{x - b_1}{\varepsilon\mu}\Big) - \varepsilon^{2s-2}\mu^{-\frac{2s}{\gamma-1}} W\Big(\frac{x - b_2}{\varepsilon\mu}\Big), \tag{4.3}$$

where W is the ground state of (2.3) and $\mu > 0$ is defined as in relation (2.7) by

$$M_\gamma \mu^{2(1-\frac{s\gamma}{\gamma-1})} = m. \tag{4.4}$$

Let us write

$$\begin{aligned} S(\psi) = (-\Delta)^s \psi - \varepsilon^{(2-2s)\gamma-2}\big[&(\psi + cx_1 - \varepsilon^{2-2s}\lambda)_+^\gamma \chi_{B_\delta(b_1)} \\ &- (-\psi - cx_1 - \varepsilon^{2-2s}\lambda)_+^\gamma \chi_{B_\delta(b_2)}\big], \end{aligned}$$

so that problem (4.1) is equivalent to $S(\psi) = 0$ and $\psi(x) \to 0$ as $|x| \to \infty$. We recall that the scalar λ is defined by (2.9), which in the case of a vortex-anti vortex pair becomes

$$W\Big(\frac{b_1 - b_2}{\varepsilon\mu}\Big) - c\varepsilon^{2-2s}\mu^{\frac{2s}{\gamma-1}} d + \mu^{\frac{2s}{\gamma-1}}\lambda = 1, \tag{4.5}$$

since $b_1 = (d, 0)$, $b_2 = (-d, 0)$. This gives $\lambda = \lambda(c, d)$ with the form

$$\lambda = \mu^{-\frac{2s}{\gamma-1}} + O(\varepsilon^{2-2s}) > 0.$$

With this choice of λ, we obtain from (2.11) that for $y \in B_{\delta/(\mu\varepsilon)}(b_1')$

$$\begin{aligned} S(\psi_0) = \varepsilon^{-2}\mu^{-\frac{2s\gamma}{\gamma-1}}\Big[&(W(y - b_1') - 1)_+^\gamma \\ &- \big(W(y - b_1') - 1 + W(b_1' - b_2') - W(y - b_2') \\ &+ c\varepsilon^{3-2s}\mu^{\frac{2s}{\gamma-1}+1}(y_1 - d)\big)_+^\gamma \chi_{\{y \in B_{\delta/(\varepsilon\mu)}(0)\}}\Big], \end{aligned} \tag{4.6}$$

where

$$y = \frac{x}{\varepsilon\mu} = (y_1, y_2) \in \mathbb{R}^2, \quad b_j' = \frac{b_j}{\varepsilon\mu}.$$

We will work with the parameters $c, \mu > 0$ fixed and

$$d \in \left(d_0, \frac{1}{d_0}\right) \tag{4.7}$$

to be adjusted, where $d_0 > 0$ is fixed small. Then we see that

$$S(\psi_0) = O\big(\varepsilon^{-2}\mu^{-\frac{2s\gamma}{\gamma-1}}\varepsilon^{3-2s}\chi_{B_{C\varepsilon}(0)}\big),$$

for some constant $C > 0$.

It will be convenient to work with the unknown v defined by

$$\psi(x) = \varepsilon^{2s-2}\mu^{-\frac{2s}{\gamma-1}} v\Big(\frac{y}{\varepsilon\mu}\Big).$$

We note that

$$S(\psi) = \varepsilon^{-2}\mu^{-\frac{2s\gamma}{\gamma-1}}\big[(-\Delta_y)^s v - f(y,v)\big],$$

where

$$\begin{aligned} f(y,v) = {} & \big(v + c\varepsilon^{3-2s}\mu^{\frac{2s}{\gamma-1}+1}y_1 - \mu^{\frac{2s}{\gamma-1}}\lambda\big)_+^{\gamma}\chi_{B_{\delta/(\varepsilon\mu)}(b_1')} \\ & - \big(-v - c\varepsilon^{3-2s}\mu^{\frac{2s}{\gamma-1}+1}y_1 - \mu^{\frac{2s}{\gamma-1}}\lambda\big)_+^{\gamma}\chi_{B_{\delta/(\varepsilon\mu)}(b_2')}. \end{aligned}$$

Thus (4.1) becomes the nonlinear problem

$$\begin{cases} (-\Delta)^s v = f(y,v) & \text{in } \mathbb{R}^2 \\ v(y) \to 0 & \text{as } |y| \to \infty, \end{cases} \tag{4.8}$$

which we interpret as

$$v = (-\Delta)^{-s} f(y,v) \quad \text{in } \mathbb{R}^2.$$

The ansatz (4.3) takes the form

$$v_0(y) = W(y - b_1') - W(y - b_2'). \tag{4.9}$$

We look for a solution of (4.8) of the form $v = v_0 + \phi$. Then equation (4.8) is equivalent to

$$L[\phi] = -E + N[\phi] \tag{4.10}$$

where

$$\begin{aligned} L[\phi] &= (-\Delta)^s\phi - V(y)\phi, \quad V(y) = f_v(y, v_0) \\ E &= (-\Delta_y)^s v_0 - f(y, v_0) \\ N[\phi] &= f(y, v_0 + \phi) - f(y, v_0) - f_v(y, v_0)\phi. \end{aligned} \tag{4.11}$$

We interpret this equation as the fixed point problem in $L^\infty(\mathbb{R}^2)$

$$\phi - (-\Delta)^{-s}[V\phi] = (-\Delta)^{-s}[N(\phi) - E].$$

The symmetries (4.2) allow us to invert the operator L up to one parameter, associated with the function

$$Z = Z_1 + Z_2, \quad Z_1(y) = \frac{\partial W}{\partial y_1}(y - b_1'), \quad Z_2(y) = \frac{\partial W}{\partial y_1}(y - b_2').$$

Note that the function Z is odd in y_1 and even in y_2. Let us consider the projected linear problem

$$\begin{cases} L[\phi] = h(y) + aV(y)Z(y) \quad \text{in } \mathbb{R}^2, \\ \displaystyle\int_{\mathbb{R}^2} VZ\phi \, dy = 0, \\ \phi(y) \to 0 \quad \text{as } |y| \to \infty \end{cases} \tag{4.12}$$

where a is the constant such that $\int_{\mathbb{R}^2}(h + aVZ)Z = 0$, namely

$$a = -\frac{\int_{\mathbb{R}^2} hZ}{\int_{\mathbb{R}^2} VZ^2} \tag{4.13}$$

We introduce the following weighted L^∞ norm

$$\|\phi\|_\beta := \sup_{y \in \mathbb{R}^2} \rho(y)^{-\beta} |\phi(y)| \tag{4.14}$$

where

$$\rho(y) = \frac{1}{1 + |y - b_1'|} + \frac{1}{1 + |y - b_2'|}.$$

In order to deal with the linear problem (4.12), we will use the norms:

$$\|\phi\|_* = \|\phi\|_{2-2s}\,, \qquad \|h\|_{**} = \|h\|_{2+\sigma}\,, \tag{4.15}$$

where $0 < \sigma < 1$.

We have the following:

Proposition 4.1. *Assume that d satisfies (4.7) and h satisfies $\|h\|_{**} < \infty$ and the symmetries (4.2). Then for $\varepsilon > 0$ small there exists a unique solution $\phi = T_d(h)$ of (4.12), which defines a linear operator of h and there exists $C > 0$ independent of ε such that*

$$\|\phi\|_* + |a| \le C\|h\|_{**}.$$

Moreover ϕ satisfies the symmetries (4.2).

We prove this proposition in Section 5.

Instead of solving problem (4.10) directly, we consider the nonlinear projected problem

$$\begin{cases} L[\phi] = -E + N[\phi] + aVZ, \quad \text{in } \mathbb{R}^2 \\ \displaystyle\int_{\mathbb{R}^2} VZ\phi \, dy = 0, \\ \phi(y) \to 0 \quad \text{as } |y| \to \infty. \end{cases} \tag{4.16}$$

Proposition 4.2. *Assume that d satisfies (4.7). There is $r_0 > 0$ such that for $\varepsilon > 0$ small there exists a unique solution $\phi = \phi_d$ to (4.16) in the ball $\|\phi\|_* \le r_0$. Moreover it satisfies*

$$\|\phi_d\|_* \le C\varepsilon^{3-2s}$$

and ϕ_d is continuous with respect to $d \in (d_0, 1/d_0)$.

The proof of this Proposition is in section 6.

We have obtained a solution $v_d = v_0 + \varphi$ of

$$\begin{cases} (-\Delta)^s v_d = f(y, v_d) + a_d VZ, \quad \text{in } \mathbb{R}^2 \\ \quad v_d(y) \to 0 \quad \text{as } |y| \to \infty, \end{cases} \tag{4.17}$$

for some parameter a_d. The fully solvability of (4.8) is reduced to finding d such that $a_d = 0$.

Multiplying (4.17) by Z and integrating over $\mathbb{R}^2$, we have that

$$a_d \int_{\mathbb{R}^2} VZ^2 dx = \int_{\mathbb{R}^2} [(-\Delta)^s v_d - f(y, v_d)] Z dy.$$

Thus $a_d = 0$ is reduced to

$$\int_{\mathbb{R}^2} [(-\Delta)^s v_d - f(y, v_d)]\, Z\, dy = 0.$$

We have the following:

Proposition 4.3. *If d satisfies (4.7), then*

$$\int_{\mathbb{R}^2} [(-\Delta)^s v_d - f(y, v_d)]\, Z\, dy = c_0 \varepsilon^{3-2s} \Big(\frac{1}{d^{3-2s}} + c_1 \Big) + O(\varepsilon^{(3-2s)\min(\gamma,2)}) + O(\varepsilon^{4-2s}),$$

where $c_0 \neq 0$ is a constant and $c_1 = \frac{c}{m} \frac{4\pi\Gamma(s)}{\Gamma(2-s)}$ (here c and m are the parameters in (1.11)).

The proof of this proposition is in section 6.

Proof of Theorem 1.1. The equation $a_d = 0$ is reduced to

$$\frac{1}{d^{3-2s}} - c_1 + g(d) = 0,$$

where g is continuous and $g(d) = O(\varepsilon^{(3-2s)\min(\gamma,2)}) + O(\varepsilon^{4-2s})$ as $\varepsilon \to 0$. Therefore we can find a solution $d = c_1^{-\frac{1}{3-2s}}(1 + o(1))$. □

5. Linear theory

Here we prove Proposition 4.1.

Proof of Proposition 4.1. Let us consider the space X defined as

$$X = \Big\{ \phi \in L^\infty(\mathbb{R}^2) \ / \ \|\phi\|_{2-2s-\sigma'} < +\infty,\ \phi \text{ satisfies (4.2)},\ \int_{\mathbb{R}^2} V\, Z\, \phi\, dy = 0 \Big\},$$

which endowed with the norm $\| \ \|_X = \| \ \|_{2-2s-\sigma'}$ norm becomes a Banach space. Norms $\| \ \|_\beta$ were defined in (4.14). Here $2 - 2s - \sigma' > 0$. Thus we want to solve the equation

$$\phi = K(\phi) + A[h], \quad \phi \in X \tag{5.1}$$

where

$$K[\phi] = (-\Delta)^{-s}(V\phi), \quad A[h] = (-\Delta)^{-s}(h + aVZ)$$

with a given by (4.13). We recall that the operator $(-\Delta)^{-s}$ is defined by (1.2). We directly check that

$$\|(-\Delta)^{-s} h\|_{2-2s} \le C \|h\|_{**}. \tag{5.2}$$

Besides, we have that for any $\phi \in L^\infty(\mathbb{R}^2)$,

$$\|K(\phi)\|_{2-2s} \le C \|\phi\|_{L^\infty(\mathbb{R}^2)}. \tag{5.3}$$

Let us also notice that the operators A and K respect symmetries (4.2). We also see that

$$\int_{\mathbb{R}^2} A[h] VZ = \int_{\mathbb{R}^2} (h + aVZ)(-\Delta)^{-s}[VZ] = \int_{\mathbb{R}^2} [h + aVZ] Z = 0$$

and, similarly
$$\int_{\mathbb{R}^2} K[\phi] VZ = \int_{\mathbb{R}^2} V\phi Z = 0.$$
Hence A and K take values in the space X. Using Arzela-Ascoli's theorem, the latter property yields compactness of the operator K in X since $\|\phi\|_X = \|\phi\|_{2-2s-\sigma'}$. On the other hand the equation
$$\phi - K[\phi] = 0, \quad \phi \in X$$
only has the trivial solution thanks to conditions (4.2), $\int_{\mathbb{R}^2} V\phi Z = 0$ and Proposition 4.1. Fredhom's alternative then yields existence of a unique solution of problem (5.1), which satisfies
$$\|\phi\|_X \le C\|A[h]\|_X.$$
Using the equations and estimates (5.2), (5.3) we get the desired estimate
$$\|\phi\|_* \le C\|h\|_{**}.$$
The proof is concluded. $\square$

6. Proof of Propositions 4.2 and 4.3

Proof of Proposition 4.2. Let T_d be the operator obtained in Proposition 4.1 that to h with h with $\|h\|_{**} < \infty$ and satisfying the symmetries (4.2) associates $\phi = T_d[h]$ solution to (4.12) with the estimate
$$\|T_d[h]\|_* \le C_1\|h\|_{**}.$$

Let $X_* = \{\, \phi \in L^\infty(\mathbb{R}^2) \mid \|\phi\|_* < \infty,\ \phi \text{ satisfies (4.2)} \,\}$ be endowed with $\|\ \|_*$. Let
$$\mathcal{A} : X_* \to X_*, \quad \mathcal{A}[\varphi] = T_d[-E + N[\phi]].$$
Then (4.16) is equivalent to solve the fixed point problem
$$\phi = \mathcal{A}[\phi],$$
which we set-up in the closed ball
$$\mathcal{B} = \{\, \phi \in X_* \mid \|\phi\|_* \le r_0 \,\},$$
where $r_0 > 0$ is to be determined later.

From (4.6) we see that
$$\|E\|_{**} \le C\varepsilon^{3-2s}.$$

We claim that for $\phi \in \mathcal{B}$
$$|N[\phi]| \le C|\phi|^{\min(\gamma,2)} \chi_{B_{R_0}(b_1') \cup B_{R_0}(b_2')}, \tag{6.1}$$
where R_0 is a large fixed constant. Indeed, $N[\phi]$, defined in (4.11), can be written as
$$N[\phi] = N_1[\phi] + N_2[\phi]$$
where
$$\begin{aligned} N_1[\phi] = \chi_{B_{\delta/(\varepsilon\mu)}(b_1')} \Big[&\big(v_0 + \phi + c\varepsilon^{3-2s}\mu^{\frac{2s}{\gamma-1}+1} y_1 - \mu^{\frac{2s}{\gamma-1}}\lambda\big)_+^\gamma \\ &- \big(v_0 + c\varepsilon^{3-2s}\mu^{\frac{2s}{\gamma-1}+1} y_1 - \mu^{\frac{2s}{\gamma-1}}\lambda\big)_+^\gamma \\ &- \gamma\big(v_0 + c\varepsilon^{3-2s}\mu^{\frac{2s}{\gamma-1}+1} y_1 - \mu^{\frac{2s}{\gamma-1}}\lambda\big)_+^{\gamma-1}\phi \Big], \end{aligned}$$

with an analogous formula for $N_2[\phi]$. Using the definition of v_0 (4.9) and the choice of λ (4.5) we see that

$$N_1[\phi] = \chi_{B_{\delta/(\varepsilon\mu)}(b_1')}\Big[(W(y-b_1') - 1 + \mathcal{R} + \phi)_+^\gamma - (W(y-b_1') - 1 + \mathcal{R})_+^\gamma \\ - \gamma(W(y-b_1') - 1 + \mathcal{R})_+^{\gamma-1}\phi\Big],$$

where $\mathcal{R} = O(\varepsilon^{3-2s}|y-b_1'|)$. We deduce from here that

$$|N_1[\phi]| \le C|\phi|^{\min(\gamma,2)}$$

and also that the support of $N_1[\phi]$ is contained in the ball $B_{R_0}(b_1')$ for some R_0 large fixed (assuming $\varepsilon > 0$ is small). We have similar estimates for $N_2[\phi]$ and we deduce (6.1). From (6.1) we get

$$\|N[\phi]\|_{**} \le C_2\|\phi\|_*^{\min(\gamma,2)}.$$

So for $\phi \in \mathcal{B}$ we have

$$\|\mathcal{A}[\phi]\|_* \le C_1(\|E\|_{**} + \|N[\phi]\|_{**}) \le C_1C\varepsilon^{3-2s} + C_1C_2r_0^{\min(\gamma,2)}.$$

We choose $r_0 > 0$ small so that $C_1C_2r_0^{\min(\gamma,2)} \le \frac{1}{2}r_0$. Then we work with $\varepsilon > 0$ small so that $C_1C\varepsilon^{3-2s} \le \frac{1}{2}r_0$. This shows that $\mathcal{A}$ maps $\mathcal{B}$ into itself.

Also from the expression of N, we have

$$|N(\phi_1) - N(\phi_2)| \le C\big(|\phi_1|^{\min(\gamma-1,1)} + |\phi_2|^{\min(\gamma-1,1)}\big)|\phi_1 - \phi_2|\chi_{B_{R_0}(b_1')\cup B_{R_0}(b_2')}$$

where R_0 is a large fixed constant. This implies that for ϕ_1, $\phi_2 \in \mathcal{B}$,

$$\|N(\phi_1) - N(\phi_2)\|_{**} \le Cr_0^{\min(\gamma-1,1)}\|\phi_1 - \phi_2\|_*.$$

Hence

$$\|\mathcal{A}(\phi_1) - \mathcal{A}(\phi_2)\|_* \le C_1(\|N(\phi_1) - N(\phi_2)\|_{**}) \le CC_1r_0^{\min(\gamma-1,1)}\|\phi_1 - \phi_2\|_*.$$

If r_0 is small we obtain that $\mathcal{A}$ is a contraction mapping from $\mathcal{B}$ into itslef and then problem (4.16) admits a unique solution $\phi_d \in \mathcal{B}$.

From the proof above and the estimate for E, one has

$$\|\phi_d\|_* \le C\|E\|_{**} \le C\varepsilon^{3-2s}.$$

Since E and V in (4.16) depend continuously on d, the fixed point characterization of ϕ_d shows that it is continuous with respect to d.

□

Proof of Proposition 4.3. We let ϕ denote the solution of (4.16) obtained in proposition 4.2 and $v = v_0 + \phi$. Since

$$(-\Delta)^s v - f(y,v) = L[\phi] + E - N[\phi]$$

we have

$$\int_{\mathbb{R}^2} [(-\Delta)^s v_d - f(y,v_d)]\, Z\, dy = \int_{\mathbb{R}^2} L[\phi] Z\, dy + \int_{\mathbb{R}^2} EZ\, dy + \int_{\mathbb{R}^2} N[\phi] Z\, dy.$$

Let us consider the term $\int_{\mathbb{R}^2} EZ\, dy$. From (2.8) and the choice of λ (4.5) we have

$$E = E_1 + E_2$$

where

$$E_1(y)=\chi_{B_{R_0}(b_1')}\Big[(W(y-b_1')-1)_+^{\gamma} \\ -\Big(W(y-b_1')-1+W(b_1'-b_2')-W(y-b_2')+c\varepsilon^{3-2s}\mu^{\frac{2s}{\gamma-1}+1}(y_1-\frac{d}{\varepsilon\mu})\Big)_+^{\gamma}\Big]$$

and

$$E_2(y_1,y_2)=-E_1(y_1,y_2).$$

Directly we have

$$|E_j|\le C\varepsilon^{3-2s}\chi_{B_{R_0}(b_j')}. \tag{6.2}$$

Writing

$$\int_{\mathbb{R}^2}EZ\,dy=\int_{\mathbb{R}^2}E_1Z_1\,dy+\int_{\mathbb{R}^2}E_1Z_2\,dy+\int_{\mathbb{R}^2}E_2Z_1\,dy+\int_{\mathbb{R}^2}E_2Z_2\,dy$$

we see that

$$\int_{\mathbb{R}^2}E_1Z_1\,dy=\int_{\mathbb{R}^2}E_2Z_2\,dy,$$

and

$$\int_{\mathbb{R}^2}E_1Z_2\,dy=O(\varepsilon^{2(3-2s)}),\quad \int_{\mathbb{R}^2}E_2Z_1\,dy=O(\varepsilon^{2(3-2s)}),$$

where we have used (6.2) and (3.3). Therefore we need only to compute $\int_{\mathbb{R}^2}E_1Z_1\,dy$. For $y\in B_{R_0}(b_1')$ we have

$$E_1(y)=-\gamma(W(y-b_1')-1)_+^{\gamma-1}\Big[W(b_1'-b_2')-W(y-b_2')+c\varepsilon^{3-2s}\mu^{\frac{2s}{\gamma-1}+1}(y_1-\frac{d}{\varepsilon\mu})\Big] \\ +O(\varepsilon^{(3-2s)(\gamma-1)}).$$

So, integrating by parts

$$\int_{\mathbb{R}^2}E_1Z_1\,dy=\int_{\mathbb{R}^2}(W(y-b_1')-1)_+^{\gamma}\left[-\partial_{y_1}W(y-b_2')+c\varepsilon^{3-2s}\mu^{\frac{2s}{\gamma-1}+1}\right]\,dy$$

and using the expansion (3.2)

$$\begin{aligned}\int_{\mathbb{R}^2}&E_1Z_1\,dy\\ &=\int_{\mathbb{R}^2}(W(y-b_1')-1)_+^{\gamma}\Big[(2-2s)M_\gamma c_{2,s}\Big(\frac{\varepsilon\mu}{2d}\Big)^{3-2s}+c\varepsilon^{3-2s}\mu^{\frac{2s}{\gamma-1}+1}+O(\varepsilon^{4-2s})\Big]\,dy\\ &=c_0\varepsilon^{3-2s}\Big[\frac{1}{d^{3-2s}}+\frac{c}{m}4\pi\frac{\Gamma(s)}{\Gamma(2-s)}+O(\varepsilon)\Big],\end{aligned}$$

for some constant $c_0\neq 0$, where we have used (4.4). Therefore

$$\int_{\mathbb{R}^2}EZ\,dy=2c_0\varepsilon^{3-2s}\Big[\frac{1}{d^{3-2s}}+\frac{c}{m}4\pi\frac{\Gamma(s)}{\Gamma(2-s)}+O(\varepsilon)\Big]. \tag{6.3}$$

Nest we consider $\int_{\mathbb{R}^2} N[\phi] Z\, dy$. Using that $\|\phi\|_* \le C\varepsilon^{3-2s}$ and (6.1) we get

$$\begin{aligned}\left|\int_{\mathbb{R}^2} N[\phi] Z\, dy\right| &= \left|\int_{\mathbb{R}^2} \left(f(y, v_0 + \phi) - f(y, v_0) - f_v(y, v_0)\phi\right) Z\, dy\right| \\ &\le \int_{B_{R_0}(b_1') \cup B_{R_0}(b_2')} |\phi|^{\min(\gamma,2)} Z\, dy \\ &\le C\varepsilon^{(3-2s)\min(\gamma,2)}. \end{aligned} \tag{6.4}$$

Next, for the integral involving $L[\phi]$, we have

$$\begin{aligned}\int_{\mathbb{R}^2} L[\phi] Z\, dy &= \int_{\mathbb{R}^2} \phi L[Z]\, dy \\ &= \int_{\mathbb{R}^2} \phi[\gamma(W(y-b_1') - 1)_+^{\gamma-1} Z_1 + \gamma(W(y-b_2') - 1)_+^{\gamma-1} Z_2 - f_v(y, v_0) Z] dy.\end{aligned}$$

But

$$\gamma(W(y-b_1') - 1)_+^{\gamma-1} Z_1 + \gamma(W(y-b_2') - 1)_+^{\gamma-1} Z_2 - f_v(y, v_0) Z = A_1 + A_2$$

where

$$A_1 = \gamma(W(y-b_1') - 1)_+^{\gamma-1} Z_1 - \gamma(W(y-b_1') - 1 + \mathcal{R}_1)_+^{\gamma} \chi_{B_{R_0}(b_1')}(Z_1 + Z_2)$$

$$A_1 = \gamma(W(y-b_2') - 1)_+^{\gamma-1} Z_2 - \gamma(W(y-b_2') - 1 + \mathcal{R}_1)_+^{\gamma} \chi_{B_{R_0}(b_2')}(Z_1 + Z_2)$$

and $\mathcal{R}_1 = O(\varepsilon^{3-2s}|y - b_1'|)$ and $\mathcal{R}_2 = O(\varepsilon^{3-2s}|y - b_2'|)$. Using the decay of W' (3.3) we find that

$$|A_1| \le C\varepsilon^{(3-2s)\min(\gamma-1,1)} \chi_{B_{R_0}(b_1')}, \quad |A_2| \le C\varepsilon^{(3-2s)\min(\gamma-1,1)} \chi_{B_{R_0}(b_2')},$$

for a possible larger R_0. Using that $\|\phi\|_* \le C\varepsilon^{3-2s}$ we find that

$$\left|\int_{\mathbb{R}^2} L[\phi] Z\, dy\right| \le C\varepsilon^{(3-2s)\min(\gamma,2)}. \tag{6.5}$$

Putting together (6.3), (6.4) and (6.5) we obtain the desired conclusion.

□

7. On Theorems 1.2–1.5

The proofs of the remaining results follow similar lines as those above, so we only present a sketch of the necessary changes.

Concerning Theorem 1.3, let us first formally derive the balancing conditions (1.20). As described in section 2, we consider the elliptic problem

$$\begin{cases} (-\Delta)^s \psi = \varepsilon^{(2-2s)\gamma-2} \displaystyle\sum_{j=1}^{k} (\psi + cx_1 - \varepsilon^{2s-2}\lambda_j^+)_+^{\gamma} \chi_{B_\delta(p_j)} \\ \qquad\qquad - \varepsilon^{(2-2s)\gamma-2} \displaystyle\sum_{l=1}^{k} (-\psi - cx_1 - \varepsilon^{2s-2}\lambda_l^-)_+^{\gamma} \chi_{B_\delta(q_l)} \quad \text{in } \mathbb{R}^2, \\ \psi(x) \to 0 \quad \text{as } |x| \to \infty, \end{cases} \tag{7.1}$$

and look a solution that at main order is approximated by

$$\psi_0(x) = \varepsilon^{2s-2} \mu^{-\frac{2s}{\gamma-1}} \Big[\sum_{j=1}^{k} W\Big(\frac{x - p_j}{\varepsilon\mu}\Big) - \sum_{l=1}^{k} W\Big(\frac{x - q_l}{\varepsilon\mu}\Big)\Big],$$

where, as in (2.7), $\mu > 0$ is such that $M_\gamma \mu^{2(1-\frac{s\gamma}{\gamma-1})} = 1$, and λ_j^+, λ_l^- are as in (2.9), (2.10). With these choices, the error of approximation, defined by

$$
\begin{aligned}
E = (-\Delta)^s \psi - \varepsilon^{(2-2s)\gamma-2} \sum_{j=1}^k (\psi + cx_1 - \varepsilon^{2s-2}\lambda_j^+)_+^\gamma \chi_{B_\delta(p_j)} \\
+ \varepsilon^{(2-2s)\gamma-2} \sum_{l=1}^k (-\psi - cx_1 - \varepsilon^{2s-2}\lambda_l^-)_+^\gamma \chi_{B_\delta(q_l)}
\end{aligned}
$$

has the form, for x near p_i:

$$
\begin{aligned}
E = \varepsilon^{-2}\mu^{-\frac{2s\gamma}{\gamma-1}} \Big[\Big(& W\Big(\frac{x-p_i}{\varepsilon\mu}\Big) - 1 \Big)_+^\gamma \\
& - \Big(W\Big(\frac{x-p_i}{\varepsilon\mu}\Big) + \sum_{j\neq i} \Big(W\Big(\frac{x-p_j}{\varepsilon\mu}\Big) - W\Big(\frac{p_i-p_j}{\varepsilon\mu}\Big)\Big) \\
& - \sum_{l=1}^k \Big(W\Big(\frac{x-q_l}{\varepsilon\mu}\Big) - W\Big(\frac{p_i-q_l}{\varepsilon\mu}\Big)\Big) \\
& + c\mu^{\frac{2s}{\gamma-1}} \varepsilon^{2-2s}(x_1 - p_{i,1})\Big)_+^\gamma \Big].
\end{aligned}
$$

Changing $x = \varepsilon\mu y$ and expanding in ε gives

$$
\begin{aligned}
&\sum_{j\neq i} \Big(W\Big(\frac{x-p_j}{\varepsilon\mu}\Big) - W\Big(\frac{p_i-p_j}{\varepsilon\mu}\Big)\Big) - \sum_{l=1}^k \Big(W\Big(\frac{x-q_l}{\varepsilon\mu}\Big) - W\Big(\frac{p_i-q_l}{\varepsilon\mu}\Big)\Big) \\
&\quad + c\mu^{\frac{2s}{\gamma-1}} \varepsilon^{2-2s}(x_1 - p_{i,1}) \\
&\sim c\varepsilon^{3-2s} \left\{ -\sum_{j\neq i} \frac{(p_i-p_j)\cdot y}{|p_i-p_j|^{4-2s}} + \sum_{l=1}^k \frac{(p_i-q_l)\cdot y}{|p_i-q_l|^{4-2s}} + \frac{c}{m} \frac{2^{2s-1}\pi\Gamma(s)}{\Gamma(2-s)} y\cdot e_1 \right\} + O(\varepsilon^{4-2s}).
\end{aligned}
$$

We want that the first order expansion vanishes, which leads to the equation

$$
\sum_{j\neq i} \frac{p_i-p_j}{|p_i-p_j|^{4-2s}} - \sum_{l=1}^k \frac{p_i-q_l}{|p_i-q_l|^{4-2s}} = c\frac{2^{2s-1}\pi\Gamma(s)}{\Gamma(2-s)} e_1,
$$

for any $i = 1, \ldots, k$.

A similar computation for x near q_m leads to

$$
\sum_{l\neq m} \frac{q_m-q_l}{|q_m-q_l|^{4-2s}} - \sum_{j=1}^k \frac{q_m-p_j}{|q_m-p_j|^{4-2s}} = -c\frac{2^{2s-1}\pi\Gamma(s)}{\Gamma(2-s)} e_1.
$$

To prove that if $(\mathbf{p}, \mathbf{q})$ is a nondegenerate symmetric array of traveling vortices there exists a solution of (7.1) close to ψ_0, we work in the symmetry class

$$
\Psi(x_1, x_2) = -\Psi(-x_1, x_2) = \Psi(x_1, -x_2). \tag{7.2}
$$

Following the same proof as in that of Theorem 1.1 and utilizing the non-degeneracy conditions under the symmetry condition (7.2), which is guaranteed by the symmetry condition (1.21) on $(\mathbf{p}, \mathbf{q})$, we obtain Theorem 1.3.

The proof of Theorem 1.4 is similar. In that case the final adjustment of the points $\mathbf{b} = (b_1, \ldots, b_j)$ is found as a small perturbation of a given $\mathbf{b}_0$, critical point of $I(\mathbf{b})$ which is non-degenerate up to vertical translations. Indeed, the points $\mathbf{b} = (b_1, \dots, b_j)$ obey an equation of the form

$$\nabla_{\mathbf{b}} I(\mathbf{b}) + \mathcal{N}(\mathbf{b}) = \mathbf{0}$$

where $\mathcal{N}(\mathbf{b})$ is an ε- small term, which is invariant under vertical translations. A standard degree argument involving a local orthogonal decomposition of $\mathbf{b}$ yields the desired result.

In the case of the rotating solutions as in Theorems 1.2 and 1.5, we need to find a family of solutions $\Theta_\varepsilon(x)$ to the equation

$$(\nabla^\perp \Psi + \alpha x^\perp) \cdot \nabla\Theta = 0, \quad \Psi = (-\Delta)^{-s}\Theta,$$

such that

$$\Theta_\varepsilon(x) \rightharpoonup \sum_{j=1}^{k} m_j \delta(x - b_j^0),$$

for given intensities m_j and a solution $\mathbf{b}^0 = (b_1^0, \dots, b_k^0)$ of system (1.10). To achieve this we consider the elliptic problem

$$\begin{cases} (-\Delta)^s \psi = \varepsilon^{(2-2s)\gamma - 2} \displaystyle\sum_{j=1}^{k} \sigma_j \Big(\sigma_j \Big(\psi + \alpha \frac{|x|^2}{2}\Big) - \varepsilon^{2s-2}\lambda_j\Big)_+^\gamma \chi_{B_\delta(b_j)} & \text{in } \mathbb{R}^2, \\ \psi(x) \to 0 \quad \text{as } |x| \to \infty, \end{cases}$$

where $1 < \gamma < \frac{2+2s}{2-2s}$, $\gamma \neq \frac{1}{1-s}$, $\sigma_j = +1$ if $m_j > 0$ and $= -1$ if $m_j < 0$. The choice of λ_j is done similarly as in the case of the traveling solutions and we have $\lambda_j = \mu_j^{-\frac{2s}{\gamma-1}} + O(\varepsilon^{2-2s})$. The points $b_1, \dots, b_k$ are close to $b_1^0, \dots, b_k^0$, and $\delta > 0$ is fixed so that the balls $B_\delta(b_j)$ are disjoint.

The ansatz ψ_0 is the same as in (2.6) with μ as in (2.7). The proof of Theorem 1.2 is then a direct adaptation of the proof of Theorem 1.1.

Theorem 1.5 similarly follows after a reduction to a problem of the form

$$\nabla_{\mathbf{b}} J(\mathbf{b}) + \mathcal{N}(\mathbf{b}) = \mathbf{0}$$

where now $\mathcal{N}(\mathbf{b})$ is a small ε-perturbation which is invariant under rotations.

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