SECOND ORDER ESTIMATE ON TRANSITION LAYERS*

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Abstract. In this paper we establish a uniform $C^{2,\theta}$ estimate for level sets of stable solutions to the singularly perturbed Allen-Cahn equation in dimensions $n \leq 10$ (which is optimal). The proof combines two ingredients: one is a reverse application of the infinite dimensional Lyapunov-Schmidt reduction method which enables us to reduce the $C^{2,\theta}$ estimate for these level sets to a corresponding one on solutions of Toda system; the other one uses a small regularity theorem on stable solutions of Toda system to establish various decay estimates, which gives a lower bound on distances between different sheets of solutions to Toda system or level sets of solutions to Allen-Cahn equation.

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1. Introduction

1.1. **Main result.** In this paper, continuing the study in [39], we establish a second order estimate on level sets of stable solutions to the singularly perturbed Allen-Cahn equation

$$\varepsilon\Delta u_\varepsilon=\frac{1}{\varepsilon}W'(u_\varepsilon),\quad |u_\varepsilon|<1\quad\text{in } B_1(0)\subset\mathbb{R}^n. \tag{1.1}$$

Here $W(u)$ is a general double well potential, that is, $W\in C^4([-1,1])$ satisfying

- $W>0$ in $(-1,1)$ and $W(\pm1)=0$;
- $W'(\pm1)=0$ and $W''(-1)=W''(1)=1$; (Note a slight notation difference here with other literatures.)
- there exists only one critical point of W in $(-1,1)$, which we assume to be 0.

A typical model is given by $W(u)=(1-u^2)^2/8$.

Under these assumptions on W, it is known that there exists a unique solution to the following one dimensional problem

$$g''(t)=W'(g(t)),\qquad g(0)=0\qquad\text{and}\quad\lim_{t\to\pm\infty}g(t)=\pm1. \tag{1.2}$$

A solution of (1.1) is stable if for any $\eta\in C_0^\infty(B_1(0))$,

$$\int_{B_1(0)}\left[\varepsilon^2|\nabla\eta|^2+W''(u_\varepsilon)\eta^2\right]\geq 0. \tag{1.3}$$

By Sternberg-Zumbrun [34], the stability condition is equivalent to

$$\int_{B_1(0)}|\nabla\eta|^2\varepsilon|\nabla u_\varepsilon|^2\geq\int_{B_1(0)}\eta^2|B(u_\varepsilon)|^2\varepsilon|\nabla u_\varepsilon|^2,\quad\forall\eta\in C_0^\infty(B_1(0)). \tag{1.4}$$

Here

$$|B(u_\varepsilon)|^2=\begin{cases}\frac{|\nabla^2u_\varepsilon|^2-|\nabla|\nabla u_\varepsilon||^2}{|\nabla u_\varepsilon|^2}, & \text{if } |\nabla u_\varepsilon|\neq 0\\ 0, & \text{otherwise.}\end{cases}$$

It is known that if $|\nabla u_\varepsilon(x)|\neq 0$,

$$|B(u_\varepsilon)(x)|^2=|A_\varepsilon(x)|^2+|\nabla_T\log|\nabla u_\varepsilon(x)||^2, \tag{1.5}$$

where $A_\varepsilon(x)$ is the second fundamental form of the level set $\{u_\varepsilon = u_\varepsilon(x)\}$ and ∇_T denotes the tangential derivative along the level set $\{u_\varepsilon = u_\varepsilon(x)\}$.

The main result of this paper is

Theorem 1.1. *For any $\theta \in (0,1)$, $0 < b_1 \le b_2 < 1$ and $\Lambda > 0$, there exist two constants $C = C(\theta, b_1, b_2, \Lambda)$ and $\varepsilon_* = \varepsilon(\theta, b_1, b_2, \Lambda)$ so that the following holds. Suppose u_ε is a stable solution of (1.1) in $B_1(0) \subset \mathbb{R}^n$ satisfying*

$$|\nabla u_\varepsilon| \neq 0 \quad \text{and} \quad |B(u_\varepsilon)| \le \Lambda, \quad \text{in } \{|u_\varepsilon| \le 1 - b_2\} \cap B_1(0). \tag{1.6}$$

If $n \le 10$ and $\varepsilon \le \varepsilon_*$, then for any $t \in [-1 + b_1, 1 - b_1]$, $\{u_\varepsilon = t\} \cap B_{1/2}(0)$ are smooth hypersurfaces and the C^θ norm of their second fundamental forms are bounded by C. Moreover,

$$|H(u_\varepsilon)| \le C\varepsilon \quad \text{in} \quad B_{1/2}(0), \tag{1.7}$$

where $H(u_\varepsilon)$ denotes the mean curvature of $\{u_\varepsilon = t\}$.

Two corollaries follow from this theorem.

Corollary 1.2. *For any $\theta \in (0,1)$, $b \in (0,1)$, $Q > 0$ and $E > 0$, there exist two constants ε_1 and C_1 so that the following holds. Suppose that u_ε is a sequence of stable solutions of (1.1) in $\mathcal{C}_1 := B_1^{n-1}(0) \times (-1,1) \subset \mathbb{R}^n$ with $\varepsilon \to 0$, satisfying*

- **(H1)** *the energy of u_ε is uniformly bounded, that is,*
$$\int_{\mathcal{C}_1} \left[\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\right] \le E;$$
- **(H2)** *there exists a sequence of $t_\varepsilon \in (-1+b, 1-b)$ such that $\{u_\varepsilon = t_\varepsilon\}$ consists of exactly Q connected components*
$$\Gamma_{\alpha,\varepsilon} = \left\{x_n = f_{\alpha,\varepsilon}(x'), \quad x' := (x_1, \cdots, x_{n-1}) \in B_1^{n-1}\right\}, \quad \alpha = 1, \cdots, Q,$$
where $-1/2 < f_{1,\varepsilon} < f_{2,\varepsilon} < \cdots < f_{Q,\varepsilon} < 1/2$;
- **(H3)** *for each α, $f_{\alpha,\varepsilon}$ converges to a limit f_α in $C^1(B_1^{n-1}(0))$ as $\varepsilon \to 0$.*

If $n \le 10$, then the same conclusion of Theorem 1.1 holds for all u_ε if $\varepsilon \le \varepsilon_1$, with C replaced by C_1.

In the above and hereafter, when saying $\varepsilon \to 0$ we always mean $\lim_{i\to+\infty} \varepsilon_i = 0$.

Corollary 1.3. *For any $\theta \in (0,1)$ and $b \in (0,1)$, there exist three constants ε_2, δ_2 and C_2 so that the following holds. Suppose that u_ε is a sequence of stable solutions of (1.1) in $B_1(0) \subset \mathbb{R}^n$, satisfying for any $x \in \{u_\varepsilon = 0\} \cap B_{1-\varepsilon}(0)$,*

$$\sup_{y \in B_\varepsilon(x)} \left|u_\varepsilon(y) - g\left(\frac{y \cdot e - t}{\varepsilon}\right)\right| \le \delta_2, \tag{1.8}$$

where $e \in \mathbb{R}^n$ is a unit vector and $t \in \mathbb{R}$ is a constant, both depending on x. If Stable Bernstein Conjecture *is true in dimension n, then the same conclusion of Corollary 1.2 holds with ε_1, C_1 replaced by ε_2, C_2.*

Note that it is expected that for $n \le 7$, the *Stable Bernstein Conjecture* should be true, that is, stable minimal hypersurfaces in $\mathbb{R}^n$ are hyperplanes. This however was only known for $n = 3$, see do Carmo and Peng [20], Fischer-Colbrie and Schoen [25] and Pogorelov [33].

Some remarks are in order.

Remark 1.4.
- *The $n=2$ case is essentially contained in our paper [39]. Chodosh and Mantoulidis [9] established the $n=3$ case, which was used in their analysis of Allen-Cahn approximation to minimal surfaces in three dimensional manifold.*
- *Results in this paper were recently used by Gui and authors of this paper [29] to study axially symmetric solutions of Allen-Cahn equation in higher dimensions. Among other results, we show there does not exist axially symmetric solutions in $\mathbb{R}^n$ with finite Morse index, provided $4 \le n \le 10$. As in this paper, the proof relies on the connection between Allen-Cahn equation and Toda system.*
- *Condition* **(H1)** *is a natural energy bound for stable solutions. See Chodosh-Mantoulidis [9], Hutchinson-Tonegawa [30] and Tonegawa-Wickramasekera [36].*
- *The assumption (1.8) says u_ε is close to the one dimensional solution g at $O(\varepsilon)$ scales. This is guaranteed by assumptions* **(H1)**-**(H3)**, *see [39] as well as Section 11.*
- *The dimension bound $n \le 10$ is sharp. If $n \ge 11$, there exists a smooth, radially symmetric, stable solution to the Liouville equation (i.e. two component Toda system)*
$$\Delta f = e^{-f}, \quad \text{in } \mathbb{R}^{n-1}.$$
Agudelo-Del Pino-Wei [1] constructed a family of solutions u_ε of (1.1) in $\mathbb{R}^n$, with its nodal set $\{u_\varepsilon = 0\}$ given by the graph $\{x_n = \pm f_\varepsilon(x')\}$, $x' \in \mathbb{R}^{n-1}$, where
$$f_\varepsilon(x) \approx \varepsilon f\left(\varepsilon^{-\frac{1}{2}} x\right) + \varepsilon |\log \varepsilon|.$$
Clearly we have
$$|\nabla^2 f_\varepsilon(0)| \approx |\nabla^2 f(0)|$$
while for any $x \neq 0$,
$$|\nabla^2 f_\varepsilon(x)| \approx |\nabla f\left(\varepsilon^{-1} x\right)| \to 0, \quad \text{as } \varepsilon \to 0.$$
Hence $\nabla^2 f_\varepsilon$ is not uniformly continuous.
- *The stability condition is also necessary for this second order regularity. Without the stability condition it is not true even in dimension 2. Counterexamples are provided by the multiple end solutions of (1.1) in $\mathbb{R}^2$, constructed by Del Pino-Kowalczyk-Pacard-Wei [15]. By utilizing solutions of Toda system*
$$f_\alpha'' = e^{-(f_\alpha - f_{\alpha-1})} - e^{-(f_{\alpha+1} - f_\alpha)}, \quad \text{on } \mathbb{R}, \quad \alpha = 1, \cdots, Q,$$
they constructed a family of solutions u_ε of (1.1) in $\mathbb{R}^2$, with its nodal set $\{u_\varepsilon = 0\}$ given by the graph of
$$f_{\alpha,\varepsilon}(x) \approx \varepsilon f_\alpha\left(\varepsilon^{-\frac{1}{2}} x\right) + \alpha\varepsilon |\log \varepsilon|.$$
As in the previous case, $\nabla^2 f_\varepsilon$ is not uniformly continuous.
- *Although this second order regularity does not hold any more for $n \ge 11$, a partial regularity result may still hold. For example, under assumptions of Corollary 1.2, there should exist a closed subset in the limit hypersurfaces $\cup_\alpha \{x_n = f_\alpha(x')\}$, which has Hausdorff dimension at most $n-10$, such that in any compact set outside this singular set, uniform second order regularity still holds.*
- *It seems that the second order regularity problem is quite different in nature from the first order regularity problem, i.e. uniform $C^{1,\theta}$ estimates on level sets. See Caffarelli-Cordoba [7] and Tonegawa-Wickramasekera [36]. For example, it can*

be checked that the above counterexamples (constructed in [1] and [15]) to second order regularity still enjoy a uniform $C^{1,\theta}$ estimate.

- *We do not touch any aspect on higher order regularity (e.g. $C^{k,\theta}$ regularity for $k \geq 3$) of level sets. It will be interesting to obtain such a result even for the multiplicity one case.*

1.2. **Outline of proof.** The proof of Theorem 1.1 consists of the following three steps.

Step 1. Reverse infinite dimensional Lyapunov-Schmidt reduction. To prove the second order regularity of level sets, as in [39] we use a reverse version of the infinite dimensional Lyapunov-Schmidt reduction method to show that these level sets satisfy a Toda system and then utilize information on Toda system to finish the proof.

This reduction method has been used by Del Pino, Kowalczyk, Wei and many others to build solutions of Allen-Cahn equations from minimal hypersurfaces or Toda system. In particular, Del Pino, Kowalczyk and Wei [17] constructed counterexamples to De Giorgi conjecture in $\mathbb{R}^9$ by this method, and in [16] they found the connection between Toda system and clustering interfaces in Allen-Cahn equation.

As in [39], our use of this reduction method will be in a reverse order, that is, instead of going from minimal hypersurfaces or Toda system to Allen-Cahn equation, a Toda system will be derived from Allen-Cahn equation by using this reduction method. Such a reverse version has been used by Del Pino, Kowalczyk and Pacard in [14], Gui, Liu and Wei in [28] to analyse refined asymptotic behavior of entire solutions to Allen-Cahn equation. But our main concern here (and in [39]) is on the clustering interface case and the treatment is different in many places.

The reduction method proceeds as follows. First by the assumptions in Theorem 1.1 (or Corollary 1.2 or 1.3), u_ε is close to the one dimensional profile in $O(\varepsilon)$ neighborhood of each connected component of $\{u_\varepsilon = 0\}$, see Section 2. Therefore the solution has the form

$$u_\varepsilon = \sum_\alpha g_{\alpha,\varepsilon} + \phi_\varepsilon, \tag{1.9}$$

where $g_{\alpha,\varepsilon}$ is the one dimensional solution in composition with the distance function to $\Gamma_{\alpha,\varepsilon}$, a connected component of $\{u_\varepsilon = 0\}$, and ϕ_ε is a small error between our solution u_ε and the approximate solution $\sum_\alpha g_{\alpha,\varepsilon}$.

Writing u_ε in this way, the single equation for u_ε, (1.1), is almost decoupled into two equations: one is the equation for the level set $\{u_\varepsilon = 0\}$ and the other one is an equation for ϕ_ε. Such a decoupling is possible by choosing an optimal approximation in (1.9), which then implies that ϕ_ε lies in the subspace orthogonal to the kernel space at $\sum_\alpha g_{\alpha,\varepsilon}$, see Proposition 4.1 for a precise statement. To this end, it is necessary to take a small perturbation in normal directions of each $\Gamma_{\alpha,\varepsilon}$ so that $g_{\alpha,\varepsilon}$ is the optimal approximation to u_ε along each normal line. Here it is convenient to introduce Fermi coordinates with these $\Gamma_{\alpha,\varepsilon}$ and rewrite everything in these coordinates, see Section 3-4.

Since $\Gamma_{\alpha,\varepsilon}$ are far from each other and they are almost parallel, the interaction pattern between different $g_{\alpha,\varepsilon}$, which represents the interaction between different components of $\{u_\varepsilon = 0\}$, can be determined by using asymptotic expansions of the one dimensional profile at infinity. This gives the equation for $\Gamma_{\alpha,\varepsilon}$,

$$H_{\alpha,\varepsilon} = \frac{2A^2_{(-1)^{\alpha-1}}}{\sigma_0 \varepsilon} e^{-\frac{|d_{\alpha-1,\varepsilon}|}{\varepsilon}} - \frac{2A^2_{(-1)^{\alpha}}}{\sigma_0 \varepsilon} e^{-\frac{|d_{\alpha+1,\varepsilon}|}{\varepsilon}} + \text{higher order terms}, \tag{1.10}$$

where $A_{\pm 1}$ and σ_0 are constants (see Appendix A for the definition), $H_{\alpha,\varepsilon}$ is the mean curvature of $\Gamma_{\alpha,\varepsilon}$, $|d_{\alpha-1,\varepsilon}|$ and $|d_{\alpha+1,\varepsilon}|$ are distances to $\Gamma_{\alpha-1,\varepsilon}$ and $\Gamma_{\alpha+1,\varepsilon}$ respectively, see Section 5 for a precise statement.

Higher order terms in (1.10) involve some terms containing ϕ_ε. In order to get a good reduction problem, a precise estimate on ϕ_ε is needed. This is established in Section 6 and Section 7. Since ϕ_ε is known to be a small perturbation, it satisfies an almost linearized equation. (This is the reduction procedure, i.e. we partially linearize (1.1) in the ϕ_ε component.) To estimate ϕ_ε, we need to consider two separate cases: the inner problem near $\{u_\varepsilon = 0\}$, and the outer one which is concerned with the part far away from $\{u_\varepsilon = 0\}$. It is important here that these two parts are still almost decoupled, which is guaranteed by the fast decay of the one dimensional profile at infinity.

Sections 2-7 will be devoted to this reduction procedure. It is almost the same with the one in [39], but various simplifications and improvements will be given in this paper.

The main difference is that in [39], it is either assumed that there are only finitely many connected components of transition layers (as in Corollary 1.2) or the distance between different connected components of transition layers has a lower bound in the form $c\varepsilon|\log\varepsilon|$ (see [39, Section 17]), but now both assumptions are removed and we only need the assumption that the distance between different connected components of transition layers to be $\gg \varepsilon$ (see Lemma 2.1 below) as a starting point. Moreover, now we can show that all estimates in this step hold uniformly with respect to the number of connected components of transition layers. Hence no assumption is needed on the number of connected components of transition layers in Theorem 1.1 and Corollary 1.3.

Step 2. Reduction of the stability condition. Now the $C^{2,\theta}$ estimate is reduced to the one on solutions to (1.10). It turns out that this depends in an essential way on lower bounds on $|d_{\alpha-1,\varepsilon}|$ and $|d_{\alpha+1,\varepsilon}|$, as observed in [39]. To get these lower bounds, as explained before, we need the stability condition (1.4).

In Section 8, we show that if u_ε is a stable solution, then solutions to the reduction problem (1.10) satisfies an almost stability condition. This is achieved by choosing test functions in (1.4) to be

$$\sum_\alpha \eta_\alpha g'_{\alpha,\varepsilon},$$

where $\eta_\alpha \in C_0^\infty(\Gamma_{\alpha,\varepsilon})$. In other words, we consider variations along directions tangential to $\{u_\varepsilon = 0\}$.

This choice of test functions in the stability condition was first used in Del Pino, Kowalczyk and Wei [18], where they showed that solutions to Allen-Cahn equation constructed from finite Morse index minimal surfaces in $\mathbb{R}^3$ has the same Morse index. See also [1] and [9, Appendix D] for similar construction.

By this choice of test functions in the stability condition for Allen-Cahn equation, a careful analysis of contributions from tangential parts, normal parts, cross terms and the interaction between different components, leads to a stability condition on solutions to (1.10), see Proposition 8.1.

Step 3. Decay estimates. Finally, a small regularity theorem on stable solutions of (1.10) will be employed to give decay estimates on $e^{-|d_{\alpha-1,\varepsilon}|/\varepsilon}$ in the interior, which then leads to a $C^{2,\theta}$ estimate on (1.10).

This small regularity result has been established by the first author in [37, 38] for stable solutions of the Liouville equation. The method developed therein can be generalized directly to Toda system (1.10). Here the dimension restriction $n \le 10$ appears, due to the fact that this small regularity theorem requires an L^1 smallness assumption on $e^{-|d_{\alpha-1,\varepsilon}|/\varepsilon}$ as a starting point. This L^1 smallness condition holds unconditionally only in $n \leq 10$, which can be proved by an L^p estimate of Farina [21].

However, the L^p estimate works only if we have an elliptic equation (or at least, an elliptic inequality) for *a single function*. Therefore it is necessary to rewrite (1.10) as a standard Toda system.

In this paper, two approaches will employed. The first one is extrinsic and uses the graph representation (with respect to a fixed hyperplane) of $\Gamma_{\alpha,\varepsilon}$. This works well when two such hypersurfaces are very close. Once they are close, they will be almost parallel to each other if we note that they are disjoint embedded hypersurfaces with curvature bounds. This then allows us to represent distances between them by differences of functions, and replace the minimal surface operator in (1.10) by the standard Laplacian operator.

This approach is sufficient for the proof of $C^{2,\theta}$ regularity. However, we can get further refinements. For example, in Chodosh and Mantoulidis [9] it was shown that interaction between different $\Gamma_{\alpha,\varepsilon}$ is dominated by mean curvatures instead of the exponential nonlinearities in Toda system. For this purpose we need a more precise lower bound on intermediate distance between different $\Gamma_{\alpha,\varepsilon}$, and another approach is needed. This one is intrinsic and uses the Jacobi field construction introduced in [9]. Here we fix an $\Gamma_{\alpha,\varepsilon}$ and view other components as graphs of functions defined on this component. Using this we can also construct positive Jacobi fields for the limiting minimal hypersurfaces as in [9].

For $\lambda > 0$, let $u_\varepsilon^\lambda(x) := u_\varepsilon(\lambda x)$. Then u_ε^λ satisfies the equation (1.1) with parameter $\lambda^{-1}\varepsilon$. In particular, $u(x) := u_\varepsilon(\varepsilon x)$ satisfies the unscaled Allen-Cahn equation

$$\Delta u = W'(u). \tag{1.11}$$

Notations: The following notations will be adopted in this paper.

- Sometimes a point $x \in \mathbb{R}^n$ is denoted by (x', x_n) with $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$.
- A constant is called universal if it depends only on the dimension n, the double well potential W and the constants b_1, b_2, Λ in Theorem 1.1. We use C to denote a large universal constant, while c a small universal constant.
- If $A \leq CB$ for a universal constant C, then we denote it by $A \lesssim B$ or $A = O(B)$. If the constant C depends on a parameter K, it is written as $A = O_K(B)$.
- We will fix the symbol $R := \varepsilon^{-1}$.

2. Preliminary analysis

In the following we will only be concerned with one level set of u_ε, $\{u_\varepsilon = 0\}$. It will be clear that our proof goes through without any change when 0 is replaced by any other $t \in [-1 + b_1, 1 - b_1]$, and all of the following estimates are uniform in $t \in [-1 + b_1, 1 - b_1]$.

By standard elliptic regularity theory and our assumption on the double well potential W, $u_\varepsilon \in C^3_{loc}(B_1(0))$. Concerning the regularity of $\{u_\varepsilon = 0\}$, we first prove that different components of it are at least $O(\varepsilon)$ apart. In the following a connected component of $\{u_\varepsilon = 0\}$ is denoted by $\Gamma_{\alpha,\varepsilon}$, where α is the index. The following lemma also shows that the cardinality of the index set is always finite for fixed ε, although it could go to infinity as $\varepsilon \to 0$.

Lemma 2.1. *For any α and $x_\varepsilon \in \Gamma_{\alpha,\varepsilon} \cap B_{9/10}(0)$, as $\varepsilon \to 0$, $\tilde{u}_\varepsilon(x) := u_\varepsilon(x_\varepsilon + \varepsilon x)$ converges to a one dimensional solution in $C^2_{loc}(\mathbb{R}^n)$. In particular,*

$$\varepsilon^{-1} dist(x_\varepsilon, \{u_\varepsilon = 0\} \setminus \Gamma_{\alpha,\varepsilon}) \to +\infty \quad \textit{uniformly.} \tag{2.1}$$

Proof. In $B_{\varepsilon^{-1}/10}(0)$, $\tilde{u}_\varepsilon(x)$ satisfies the Allen-Cahn equation (1.11). By standard elliptic regularity theory, $\tilde{u}_\varepsilon(x)$ is uniformly bounded in $C^3_{loc}(\mathbb{R}^n)$. By Arzela-Ascoli theorem, as $\varepsilon \to 0$, it converges to a limit function $\tilde{u}_\infty$ in $C^2_{loc}(\mathbb{R}^n)$. Clearly $\tilde{u}_\infty$ is a stable solution of (1.11) in $\mathbb{R}^n$.

Since $\tilde{u}_\varepsilon(0) = 0$, $\tilde{u}_\infty(0) = 0$. By our assumption on the double well potential W, the constant solution 0 is not stable. Hence $\tilde{u}_\infty$ is a non-constant solution. As a consequence, by unique continuation principle, the critical set $\{\nabla \tilde{u}_\infty = 0\}$ has zero Lebesgue measure.

By (1.6),

$$\left|B\left(\tilde{u}_\varepsilon\right)\right| \leq \Lambda\varepsilon, \quad \text{in } \{|\tilde{u}_\varepsilon| \leq 1-b\} \cap B_{\varepsilon^{-1}/9}(0).$$

By the convergence of $\tilde{u}_\varepsilon$, we can pass this inequality to the limit in any compact set outside $\{\nabla \tilde{u}_\infty = 0\}$, which leads to $\left|B\left(\tilde{u}_\infty\right)\right| \equiv 0$ in $\{|\tilde{u}_\infty| \leq 1-b\}$. Hence by (1.5) and Sard theorem, almost all level sets $\{\tilde{u}_\infty = t\}$ (for $t \in [-1+b, 1-b]$) are hyperplanes. Then it is directly verified that these hyperplanes are parallel to each other and $\tilde{u}_\infty$ is one dimensional (along the normal direction to these hyperplanes). Because $\tilde{u}$ is stable, it must be the monotone one, g.

Let $\widetilde{\Gamma}_{\alpha,\varepsilon} := \varepsilon^{-1}\left(\Gamma_{\alpha,\varepsilon} - x_\varepsilon\right)$. By the convergence of $\tilde{u}_{\alpha,\varepsilon}$, in any compact set of $\mathbb{R}^n$, $\widetilde{\Gamma}_{\alpha,\varepsilon}$ converge to $\{\tilde{u}_\infty = 0\}$ in the Hausdorff distance. Since $\{\tilde{u}_\infty = 0\}$ is a single hyperplane, we get

$$\operatorname{dist}\left(0, \{\tilde{u}_\varepsilon = 0\} \setminus \widetilde{\Gamma}_{\alpha,\varepsilon}\right) \to +\infty,$$

where the convergence rate depends only on ε. This gives (2.1). □

The above proof implies that the Implicit Function Theorem can be applied to u_ε at $O(\varepsilon)$ scales, which gives the C^3 regularity of $\{u_\varepsilon = 0\}$. Of course it is not known whether there exists a uniform bound independent of ε.

The following lemma can be proved by combining the curvature bound (1.6) with the fact that different connected components of $\{u_\varepsilon = u_\varepsilon(0)\}$ are disjoint. (This fact has been used a lot in minimal surface theory, see for instance [10].)

Lemma 2.2. *There exist two universal constants $r_g \in (0, 1/20)$ and $C(r_g, \Lambda)$ so that the following holds. For any $x_* \in \{|u_\varepsilon| \leq 1-b\} \cap B_{9/10}(0)$, in a suitable coordinate system, $\{u_\varepsilon = u_\varepsilon(x_*)\} \cap B_{r_g}(x_*)$ is a family of graphs $\cup_\alpha \{x_n = f_{\alpha,\varepsilon}(x')\}$, where $f_{\alpha,\varepsilon} \in C^3(B^{n-1}_{2r_g}(x'_*))$ satisfy $\|f_{\alpha,\varepsilon}\|_{C^{1,1}(B^{n-1}_{2r_g}(x'_*))} \leq C(r_g, \Lambda)$.*

3. Fermi coordinates

3.1. **Definition.** For simplicity of presentation, we now work in the stretched version and do not write the dependence on ε explicitly.

Recall that $R = \varepsilon^{-1}$, and $u(x) = u_\varepsilon(\varepsilon x)$ satisfies the Allen-Cahn equation (1.11) in $B_R(0)$. Its nodal set $\{u = 0\}$ consists of finitely many connected components, Γ_α.

By our assumption, for each α, the second fundamental form A_α of Γ_α satisfies

$$|A_\alpha(y)| \leq \Lambda\varepsilon, \quad \forall y \in \Gamma_\alpha \cap B_R(0). \tag{3.1}$$

We will assume Λ is sufficiently small, perhaps after restricting u_ε to a small ball and then rescaling the radius of this ball to 1.

Let y be a local coordinate of Γ_α. The Fermi coordinate is defined as $(y, z) \mapsto x$, where $x = y + zN_\alpha(y)$. Here $N_\alpha(y)$ is a unit normal vector to Γ_α, z is the signed distance to Γ_α. By (3.1), Fermi coordinates are well defined and smooth in $B_R(0)$.

By Lemma 2.2 (recall that we have assumed $\Lambda \ll 1$), after a rotation,

$$\Gamma_\alpha \cap B_R(0) = \{x_n = f_\alpha(x')\}, \quad \forall \alpha. \tag{3.2}$$

Therefore a canonical way to choose local coordinates of Γ_α is by letting $y = x'$ for each α. Then the induced metric on Γ_α is

$$g_{\alpha,ij}(y) = \delta_{ij} + \frac{\partial f_\alpha}{\partial y_i}(y)\frac{\partial f_\alpha}{\partial y_j}(y).$$

By Lemma 2.2 and (3.1), we get a universal constant C such that

$$|\nabla f_\alpha| \le C \quad \text{and} \quad |\nabla^2 f_\alpha| \le C\varepsilon, \quad \text{in } B_R^{n-1}(0). \tag{3.3}$$

Sometimes the signed distance to Γ_α is also denoted by d_α. Since $\Gamma_\alpha \cap \Gamma_\beta = \emptyset$ for any $\alpha \neq \beta$, we can choose the sign so that $\{d_\alpha > 0\} \cap \{d_\beta > 0\} \neq \emptyset$ for any $\alpha \neq \beta$.

For any $z \in (-R, R)$, let $\Gamma_{\alpha,z} := \{d_\alpha(x) = z\}$. In particular, $\Gamma_{\alpha,0}$ is just Γ_α. Define the vector field

$$X_i := \frac{\partial}{\partial y^i} + z\frac{\partial N_\alpha}{\partial y^i} = \sum_{j=1}^{n-1}\left(\delta_{ij} - zA_{\alpha,ij}\right)\frac{\partial}{\partial y^j}, \quad 1 \le i \le n-1.$$

The tangent space of $\Gamma_{\alpha,z}$ is spanned by X_i. The Euclidean metric restricted to $\Gamma_{\alpha,z}$ is denoted by $g_{\alpha,ij}(y,z)dy^i \otimes dy^j$, where

$$\begin{aligned} g_{\alpha,ij}(y,z) &= X_i(y,z)\cdot X_j(y,z) \\ &= g_{\alpha,ij}(y,0) - 2z\sum_{k=1}^{n-1} A_{\alpha,ik}(y,0)g_{jk}(y,0) \\ &\quad + z^2\sum_{k,l=1}^{n} g_{\alpha,kl}(y,0)A_{\alpha,ik}(y,0)A_{\alpha,jl}(y,0). \end{aligned} \tag{3.4}$$

The second fundamental form of $\Gamma_{\alpha,z}$ has the form

$$A_\alpha(y,z) = \left[I - zA_\alpha(y,0)\right]^{-1} A_\alpha(y,0). \tag{3.5}$$

3.2. **Some notations.** In the remaining part of this paper the following notations will be employed.

- Given a point X, $\Pi_\alpha(X)$ denotes the nearest point on Γ_α to X, which by our assumption is uniquely determined by X.
- Given a point on Γ_α with local coordinates $(y,0)$ in Fermi coordinates, denote
$$D_\alpha(y) := \min_{\beta\neq\alpha} |d_\beta(y,0)|.$$
- For any $x \in B_R(0)$ and $r \in (0, R-|x|)$, denote
$$A(r;x) := \max_\alpha \max_{y\in\overline{\Gamma_\alpha\cap B_r(x)}} e^{-D_\alpha(y)}.$$
- The covariant derivative on $\Gamma_{\alpha,z}$ with respect to the induced metric is denoted by $\nabla_{\alpha,z}$.
- The area form on $\Gamma_{\alpha,z}$ with respect to the induced metric is denoted by $dA_{\alpha,z} = \lambda_\alpha(y,z)dy$, where $\lambda_\alpha(y,z) = \sqrt{\det\left[g_{\alpha,ij}(y,z)\right]}$.
- We use $B_r^\alpha(y)$ to denote the open ball on Γ_α with center y and radius r, which is measured with respect to intrinsic distance.

- For $\lambda \in \mathbb{R}$, let
$$\mathcal{M}_\alpha^\lambda := \{|d_\alpha| < |d_{\alpha-1}| + \lambda \quad \text{and} \quad |d_\alpha| < |d_{\alpha+1}| + \lambda\}.$$
- In the Fermi coordinates with respect to Γ_α, there exist two continuous functions $\rho_\alpha^\pm(y)$ such that
$$\mathcal{M}_\alpha^0 = \left\{(y,z) : \rho_\alpha^-(y) < z < \rho_\alpha^+(y)\right\}.$$

3.3. **Deviation in z.** In this subsection we collect several estimates on the deviation of various terms in z, when $z \neq 0$. Recall that ε is the upper bound on curvatures of level sets of u, see (3.1).

By (3.1), $|A_\alpha(y,0)| \lesssim \varepsilon$. Thus by (3.5), for $|z| < R$, $|A_\alpha(y,z)| \lesssim \varepsilon$. Concerning bounds on derivatives of A_α, we have (see [39, Lemma 8.1])

Lemma 3.1. *For any $y \in \Gamma_\alpha \cap B_{R-1}(0)$,*
$$|\nabla_{\alpha,0} A_\alpha(y,0)| \lesssim \varepsilon. \tag{3.6}$$

Such an ε-bound on third order derivatives will be crucial for the following proof of this paper, because we will need some very precise $C^{2,\theta}$ bounds on various functions constructed from u.

By (3.5), we get
$$|A_\alpha(y,z) - A_\alpha(y,0)| \lesssim |z||A_\alpha(y,0)|^2 \lesssim \varepsilon^2|z|. \tag{3.7}$$
Similarly, by (3.4), the deviation of metric tensors is
$$\begin{cases} |g_{\alpha,ij}(y,z) - g_{\alpha,ij}(y,0)| \lesssim \varepsilon|z|, \\ |g_\alpha^{ij}(y,z) - g_\alpha^{ij}(y,0)| \lesssim \varepsilon|z|. \end{cases} \tag{3.8}$$
As a consequence, the deviation of mean curvature is
$$|H_\alpha(y,z) - H_\alpha(y,0)| \lesssim \varepsilon^2|z|. \tag{3.9}$$

By (3.3) and (3.6), for any $|z| < R$,
$$\sum_{i,j=1}^{n-1}\left(|\nabla_{\alpha,z} g_{\alpha,ij}(y,z)| + |\nabla_{\alpha,z} g_\alpha^{ij}(y,z)| + |\nabla^2_{\alpha,z} g_{\alpha,ij}(y,z)| + |\nabla^2_{\alpha,z} g_\alpha^{ij}(y,z)|\right) \lesssim \varepsilon. \tag{3.10}$$

The Laplacian operator in Fermi coordinates has the form
$$\Delta_{\mathbb{R}^n} = \Delta_{\alpha,z} - H_\alpha(y,z)\partial_z + \partial_{zz},$$
where $\Delta_{\alpha,z}$ is the Beltrami-Laplace operator on $\Gamma_{\alpha,z}$, that is,
$$\begin{aligned} \Delta_{\alpha,z} &= \sum_{i,j=1}^{n-1} \frac{1}{\sqrt{\det(g_{\alpha,ij}(y,z))}} \frac{\partial}{\partial y_j}\left(\sqrt{\det(g_{\alpha,ij}(y,z))}\, g_\alpha^{ij}(y,z)\frac{\partial}{\partial y_i}\right) \\ &= \sum_{i,j=1}^{n-1} g_\alpha^{ij}(y,z)\frac{\partial^2}{\partial y_i \partial y_j} + \sum_{i=1}^{n-1} b_\alpha^i(y,z)\frac{\partial}{\partial y_i} \end{aligned} \tag{3.11}$$
with
$$b_\alpha^i(y,z) = \frac{1}{2}\sum_{j=1}^{n-1} g_\alpha^{ij}(y,z)\frac{\partial}{\partial y_j}\log\det(g_{\alpha,ij}(y,z)).$$

By (3.8) and (3.11), we get

Lemma 3.2. *For any function $\varphi \in C^2(\Gamma_\alpha)$,*

$$|\Delta_{\alpha,z}\varphi - \Delta_{\alpha,0}\varphi| \lesssim \varepsilon|z|\left(|\nabla^2_{\alpha,0}\varphi| + |\nabla_{\alpha,0}\varphi|\right). \tag{3.12}$$

Finally we recall a commutator estimate, which is [39, Lemma 13.2].

Lemma 3.3. *For any $\varphi \in C^3(\Gamma_\alpha)$ and $i = 1, \cdots, n-1$,*

$$\left|\frac{\partial}{\partial y_i}\Delta_{\alpha,z}\varphi - \Delta_{\alpha,z}\frac{\partial \varphi}{\partial y_i}\right| \lesssim \varepsilon\left(|\nabla^2_{\alpha,0}\varphi| + |\nabla_{\alpha,0}\varphi|\right).$$

The additional ε comes from (3.10), which in turn relies on (3.6).

3.4. **Comparison of distance functions.** The following result is [39, Lemma 8.3]. It describes the error between distances to different Γ_α, when their distances is of the order $O(|\log\varepsilon|)$.

Lemma 3.4. *For any $K > 0$, there exists a constant $C(K)$ so that the following holds. For any $X \in B_{8R/9}(0)$ and $\alpha \neq \beta$, if $|d_\alpha(X)| \le K|\log\varepsilon|$ and $|d_\beta(X)| \le K|\log\varepsilon|$ at the same time, then we have*

$$dist_{\Gamma_\beta}\left(\Pi_\beta \circ \Pi_\alpha(X), \Pi_\beta(X)\right) \le C(K)\varepsilon^{1/2}|\log\varepsilon|^{3/2},$$
$$|d_\beta\left(\Pi_\alpha(X)\right) + d_\alpha\left(\Pi_\beta(X)\right)| \le C(K)\varepsilon^{1/2}|\log\varepsilon|^{3/2},$$
$$|d_\alpha(X) - d_\beta(X) + d_\beta\left(\Pi_\alpha(X)\right)| \le C(K)\varepsilon^{1/2}|\log\varepsilon|^{3/2},$$
$$|d_\alpha(X) - d_\beta(X) - d_\alpha\left(\Pi_\beta(X)\right)| \le C(K)\varepsilon^{1/2}|\log\varepsilon|^{3/2},$$
$$1 - \nabla d_\alpha(X)\cdot\nabla d_\beta(X) \le C(K)\varepsilon^{1/2}|\log\varepsilon|^{3/2},$$

The following lemma is an easy consequence of Lemma 2.2.

Lemma 3.5. *For any $\alpha \neq \beta$, both $\Pi_\beta\lfloor_{\Gamma_\alpha}$ and its inverse are Lipschitz continuous with their Lipschitz constants bounded by a universal constant C.*

Finally, the following fact will be used a lot in this paper.

Lemma 3.6. *For any $y \in \Gamma_\alpha$,*

$$\sum_{\beta\neq\alpha} e^{-|d_\beta(y,0)|} \lesssim e^{-D_\alpha(y)}.$$

Proof. By Lemma 2.1 and Lemma 3.4, there exists a constant $C \gg 1$ such that

$$|d_\beta(y,0)| \ge D_\alpha(y) + C\left(|\beta-\alpha| - 1\right).$$

Summing $e^{-|d_\beta(y,0)|}$ in β we conclude the proof. □

4. An approximate solution

4.1. **Orthogonal decomposition.** Fix a function $\zeta \in C_0^\infty(-2,2)$ with $\zeta \equiv 1$ in $(-1,1)$, $|\zeta'| + |\zeta''| \le 16$. Let

$$\bar{g}(t) = \zeta\left(4|\log\varepsilon|t\right)g(t) + \left[1 - \zeta(4|\log\varepsilon|t)\right]\mathrm{sgn}(t), \quad t \in (-\infty, +\infty).$$

In particular, $\bar{g} \equiv 1$ in $(8|\log\varepsilon|, +\infty)$ and $\bar{g} \equiv -1$ in $(-\infty, -8|\log\varepsilon|)$.

This function $\bar{g}$ is an approximate solution to the one dimensional Allen-Cahn equation, that is,

$$\bar{g}'' = W'(\bar{g}) + \bar{\xi}, \tag{4.1}$$

where $\mathrm{spt}(\bar{\xi}) \in \{4|\log\varepsilon| < |t| < 8|\log\varepsilon|\}$, and $|\bar{\xi}| + |\bar{\xi}'| + |\bar{\xi}''| \lesssim \varepsilon^3$.

We also have (see Appendix A for the definition of σ_0)

$$\int_{-\infty}^{+\infty} \bar{g}'(t)^2 dt = \sigma_0 + O\left(\varepsilon^3\right). \tag{4.2}$$

Without loss of generality, we will always assume u has the same sign as $(-1)^\alpha d_\alpha$ near Γ_α. Given a function $h_\alpha \in C^2(\Gamma_\alpha)$, let

$$g_\alpha(y, z; h_\alpha) := \bar{g}\left((-1)^\alpha \left(z - h_\alpha(y)\right)\right),$$

where (y, z) are Fermi coordinates with respect to Γ_α.

Given a sequence of functions $(h_\alpha) =: h$, define the function $g(y, z; h)$ in the following way: for each α, in $\mathcal{M}^0_\alpha$ it is defined as

$$g(y, z; h) := g_\alpha + \sum_{\beta<\alpha} \left[g_\beta - (-1)^\beta\right] + \sum_{\beta>\alpha} \left[g_\beta + (-1)^\beta\right]. \tag{4.3}$$

By the definition of $\bar{g}$ and Lemma 2.1, there are only finitely many terms in the above sum.

For simplicity of notation, denote

$$g'_\alpha(y, z; h_\alpha) = \bar{g}'\left((-1)^\alpha (z - h_\alpha(y))\right), \quad g''_\alpha(y, z; h_\alpha) = \bar{g}''\left((-1)^\alpha (z - h_\alpha(y))\right), \quad \cdots.$$

Proposition 4.1. *There exists $h_\alpha \in C^0(\Gamma_\alpha \cap B_{9R/10}(0)$ with $\|h_\alpha\|_{C^0(\Gamma_\alpha \cap B_{9R/10}(0)} \ll 1$ for each α, such that for any α and $y \in \Gamma_\alpha \cap B_{7R/8}(0)$,*

$$\int_{-\infty}^{+\infty} \left[u(y, z) - g(y, z; h)\right] g'_\alpha(y, z; h_\alpha) dz = 0, \tag{4.4}$$

where (y, z) are Fermi coordinates with respect to Γ_α.

Proof. We prove this proposition by applying a nonlinear open mapping theorem (see [8, Theorem 1.2.4]) to a C^1 map between two Banach spaces.

The Banach spaces are $\mathcal{X} := \bigoplus_\alpha C^0(\Gamma_\alpha \cap B_{9R/10}(0))$, the direct sum of the spaces of continuous functions on $\Gamma_\alpha \cap B_{9R/10}(0)$ (for those α with this intersection being nonempty), and $\mathcal{Y} := \bigoplus_\alpha C^0(\Gamma_\alpha \cap B_{7R/8}(0))$. In the following we also denote $\mathcal{X}_\alpha := C^0(\Gamma_\alpha \cap B_{9R/10}(0))$ and $\mathcal{Y}_\alpha := C^0(\Gamma_\alpha \cap B_{7R/8}(0))$.

Given $(h_\alpha) \in \mathcal{X}$, define $g(\cdot; h)$ as in (4.3). By (3.1) (recall that we have assumed $\Lambda \ll 1$), for each α, $\Pi_\alpha(B_{8R/9}(0)) \subset \Gamma_\alpha \cap B_{9R/10}(0)$. Hence g_α for each α, and then $g(\cdot; h)$ is well defined in $B_{8R/9}(0)$.

Define the map F as follows: the $\mathcal{Y}_\alpha$ component of $F(h)$ is

$$F_\alpha(h)(y) := \int_{-\infty}^{+\infty} \left[u(y, z) - g(y, z; h)\right] g'_\alpha\left(y, z; h_\alpha\right) dz, \quad \forall y \in \Gamma_\alpha \cap B_{7R/8}(0),$$

where (y, z) denote Fermi coordinates with respect to Γ_α. (Note that z represents different object for different α.) By the definition of g_α, the above integral involves only those $z \in (-10 \log R, 10 \log R)$ provided $\|h\|_{\mathcal{X}} \le 1$, where both g'_α and $g(\cdot; h)$ are well defined for $y \in \Gamma_\alpha \cap B_{7R/8}(0)$.

It is readily verified that F is a C^1 map from the unit ball in $\mathcal{X}$ to $\mathcal{Y}$. Furthermore, for each α, $DF_\alpha(h)\xi$ equals

$$(-1)^\alpha \xi_\alpha(y) \int_{-\infty}^{+\infty} \left[g'_\alpha\left(y, z; h_\alpha\right)^2 - \left(u(y, z) - g(y, z; h)\right) g''_\alpha\left(y, z; h_\alpha\right)\right] dz$$

$$+\sum_{\beta\neq\alpha}(-1)^{\beta}\int_{-\infty}^{+\infty}\xi_{\beta}(\Pi_{\beta}(y,z))g'_{\alpha}(y,z;h_{\alpha})\,g'_{\beta}(y,z;h_{\beta})\,dz.$$

From this formula, we see there exist a family of linear operators $D_\beta F_\alpha$ from $\mathcal{X}_\beta$ to $\mathcal{Y}_\alpha$ such that

$$DF_\alpha(h)\xi=\sum_\beta D_\beta F_\alpha(h)\xi_\beta.$$

By Lemma 2.1,

$$\int_{-\infty}^{+\infty}\Big[g'_{\alpha}(y,z;0)^2-(u(y,z)-g(y,z;0))\,g''_{\alpha}(y,z;0)\Big]\,dz\geq\frac{\sigma_0}{2}, \tag{4.5}$$

$$\Big|\int_{-\infty}^{+\infty}g'_{\alpha}(y,z;0)\,g'_{\beta}(y,z;0)\,dz\Big|\lesssim e^{-|d_\beta(y,0)|}. \tag{4.6}$$

A direct consequence of (4.6) is the operator bound

$$\|D_\beta F_\alpha(0)\|_{\mathcal{X}_\beta\mapsto\mathcal{Y}_\alpha}\lesssim\max_{y\in\Gamma_\alpha\cap B_{7R/8}(0)}e^{-|d_\beta(y,0)|},\quad\forall\beta\neq\alpha. \tag{4.7}$$

By applying (4.5) and constructing a suitable extension operator from $\mathcal{Y}_\alpha$ to $\mathcal{X}_\alpha$, we obtain a right inverse of $D_\alpha F_\alpha(0)$, denoted by $(D_\alpha F_\alpha(0))^{-1}$, which is a bounded linear operator from $\mathcal{Y}_\alpha$ to $\mathcal{X}_\alpha$ with bound depending only on σ_0.

Next we use this to construct a right inverse of $DF(0)$. To this end, for any $(\eta_\alpha)\in\mathcal{Y}$, we want to find a solution $(\xi_\alpha)\in\mathcal{X}$ to the following linear equation

$$D_\alpha F_\alpha(0)\xi_\alpha+\sum_{\beta\neq\alpha}D_\beta F_\alpha(0)\xi_\beta=\eta_\alpha,\quad\forall\alpha.$$

This can be transformed to finding a fixed point of the map defined on $\mathcal{X}$:

$$(\xi_\alpha)\mapsto\left((D_\alpha F_\alpha(0))^{-1}\eta_\alpha-\sum_{\beta\neq\alpha}(D_\alpha F_\alpha(0))^{-1}D_\beta F_\alpha(0)\xi_\beta\right).$$

By the bound on $(D_\alpha F_\alpha(0))^{-1}$, (4.7), Lemma 3.6 and Lemma 2.1, this map is a contraction. The existence and uniqueness of the fixed point then follows from the contraction mapping principle. Moreover, (ξ_α) depends linearly and continuously on (η_α). We define this continuous linear map as the right inverse of $DF(0)$, denoted as $(DF(0))^{-1}$. Note that there is a uniform bound on $\|(DF(0))^{-1}\|_{\mathcal{Y}\mapsto\mathcal{X}}$, which depends only on σ_0.

Following the proof of [8, Theorem 1.2.4], define

$$R(h):=F(h)-DF(0)h.$$

To find h so that $F(h)=0$, it is equivalent to find a solution to the equation

$$DF(0)h=-R(h). \tag{4.8}$$

We use the iteration scheme in the proof of [8, Theorem 1.2.4] to find such an h. First let $h_0=0$. For any $k\geq0$, suppose $h_k\in B_\rho^{\mathcal{X}}(0)$ (ρ to be determined below) has been constructed, let

$$h_{k+1}:=-(DF(0))^{-1}R(h_k). \tag{4.9}$$

From the representation formula for DF_α, there exist $\delta>0$ and $\rho>0$ small enough (but independent of ε) such that

$$\|DF(h)-DF(0)\|_{\mathcal{X}\mapsto\mathcal{Y}}\leq\delta,\quad\forall h\in B_\rho^{\mathcal{X}}(0).$$

Combining this with the uniform bound on $\| (DF(0))^{-1} \|_{\mathcal{Y}\mapsto\mathcal{X}}$, we get, if both h_{k-1} and h_k lie in $B^{\mathcal{X}}_\rho(0)$, then

$$\|h_{k+1} - h_k\|_{\mathcal{X}} \le \frac{1}{2}\|h_k - h_{k-1}\|_{\mathcal{X}}. \tag{4.10}$$

By the same calculation in the proof of [8, Theorem 1.2.4], we see h_k stays in $B^{\mathcal{X}}_\rho(0)$ for all k, provided that $\|F(0)\|_{\mathcal{X}} \ll 1$, which is true for all ε small enough by Lemma 2.1. Then by (4.10), h_k converges to a limit h as $k \to +\infty$. Taking limit in (4.9) we obtain

$$h = -(DF(0))^{-1} R(h).$$

Since $(DF(0))^{-1}$ is the right inverse of $DF(0)$, h is also the solution of (4.8). □

Denote $g_\alpha(y,z) := g_\alpha(y,z;h_\alpha)$ and $g_*(y,z) := g(y,z;h)$, where h is given in the previous proposition. As before we denote

$$g'_\alpha(y,z) = g'_\alpha(y,z;h_\alpha), \quad g''_\alpha(y,z) = g''_\alpha(y,z;h_\alpha), \quad \cdots .$$

Let $\phi := u - g_*$ be the error between the solution u and the approximate solution g_*.

Remark 4.2. *The proof of Proposition 4.1 shows that for each α, $\|h_\alpha\|_{C^0(\Gamma_\alpha\cap B_{7R/8}(0))} = o(1)$. By differentiating (4.4), we can show that $\|h_\alpha\|_{C^3(\Gamma_\alpha\cap B_{7R/8}(0))} = o(1)$ for each α. Combining this fact with Lemma 2.1, we obtain $\|\phi\|_{C^3(B_{7R/8}(0))} = o(1)$.*

In the Fermi coordinates with respect to Γ_α, ϕ satisfies the following equation

$$\begin{aligned} & \Delta_{\alpha,z}\phi - H_\alpha(y,z)\partial_z\phi + \partial_{zz}\phi \\ = \;& W'(g_*+\phi) - \sum_\beta W'(g_\beta) + (-1)^\alpha g'_\alpha\left[H_\alpha(y,z) + \Delta_{\alpha,z}h_\alpha(y)\right] - g''_\alpha|\nabla_{\alpha,z}h_\alpha|^2 \\ + \;& \sum_{\beta\neq\alpha}\left[(-1)^\beta g'_\beta\mathcal{R}_{\beta,1} - g''_\beta\mathcal{R}_{\beta,2}\right] - \sum_\beta \xi_\beta, \end{aligned} \tag{4.11}$$

where for each β, in the Fermi coordinates with respect to Γ_β,

$$\xi_\beta(y,z) = \bar{\xi}\left((-1)^\beta(z - h_\beta(y))\right),$$
$$\mathcal{R}_{\beta,1}(y,z) := H_\beta(y,z) + \Delta_{\beta,z}h_\beta(y),$$
$$\mathcal{R}_{\beta,2}(y,z) := |\nabla_{\beta,z}h_\beta(y)|^2.$$

4.2. **Interaction terms.**

In this subsection we collect several estimates on the interaction term $\mathcal{I} := W'(g_*) - \sum_\beta W'(g_\beta)$ between different components. The following three lemmas correspond to [39, Lemma 9.3-Lemma 9.5]. Because in this paper we assume $W''(\pm1) = 1$, the statement is different in some constants, but the proof is exactly the same.

Lemma 4.3. *In $\mathcal{M}^4_\alpha \cap B_{7R/8-1}(0)$,*

$$\begin{aligned} \mathcal{I} \;=\;& \left[W''(g_\alpha) - 1\right]\left[g_{\alpha-1} - (-1)^{\alpha-1}\right] + \left[W''(g_\alpha) - 1\right]\left[g_{\alpha+1} + (-1)^{\alpha+1}\right] \\ +\;& O\left(e^{-2d_{\alpha-1}} + e^{2d_{\alpha+1}}\right) + O\left(e^{-d_{\alpha-2}-|d_\alpha|} + e^{d_{\alpha+2}-|d_\alpha|}\right). \end{aligned} \tag{4.12}$$

The following upper bound on the interaction term will be used a lot in below.

Lemma 4.4. *For any $(y,z) \in \mathcal{M}^4_\alpha \cap B_{7R/8-1}(0)$,*

$$\left|\mathcal{I}(y,z)\right| \lesssim e^{-D_\alpha(y)} + \varepsilon^2.$$

The Lipschitz norm of interaction terms can also be estimated in a similar way.

Lemma 4.5. *For any* $(y,z) \in \mathcal{M}_\alpha^3 \cap B_{7R/8-1}(0)$,

$$\left\|\mathcal{I}\right\|_{Lip(B_1(y,z))} \lesssim \max_{B_1^\alpha(y)} e^{-D_\alpha} + \varepsilon^2.$$

4.3. **Controls on h using ϕ.** The choice of optimal approximation in Subsection 4.1 has the advantage that h is controlled by ϕ. This will allow us to iterate various elliptic estimates in Section 6 below.

Lemma 4.6. *For each α and $y \in \Gamma_\alpha \cap B_{7R/8-2}(0)$, we have*

$$\|h_\alpha\|_{C^{2,\theta}(B_1^\alpha(y))} \lesssim \|\phi\|_{C^{2,\theta}(B_1(y,0))} + \max_{B_1^\alpha(y)} e^{-D_\alpha} \tag{4.13}$$

and

$$\begin{aligned} \|\nabla_{\alpha,0} h_\alpha\|_{C^{1,\theta}(B_1^\alpha(y))} \quad &\lesssim \quad \|\nabla_{\alpha,0}\phi\|_{C^{1,\theta}(B_1(y,0))} + \varepsilon^{1/6} \max_{B_1^\alpha(y)} e^{-D_\alpha} \\ &+ \quad \left(\max_{\beta: |d_\beta(y,0)| \leq 8|\log\varepsilon|} \|\nabla_{\beta,0} h_\beta\|_{C^{1,\theta}(B_2^\beta(\Pi_\beta(y,0)))}\right)\left(\max_{B_1^\alpha(y)} e^{-D_\alpha}\right). \end{aligned} \tag{4.14}$$

Proof. Fix an α. In the Fermi coordinates with respect to Γ_α, because $u(y,0)=0$,

$$\begin{aligned} \phi(y,0) = -\bar{g}\left((-1)^{\alpha+1} h_\alpha(y)\right) \quad &- \quad \sum_{\beta<\alpha} \left[\bar{g}\left((-1)^\beta \left(d_\beta(y,0) - h_\beta(\Pi_\beta(y,0))\right)\right) - (-1)^\beta\right] \\ &- \quad \sum_{\beta>\alpha} \left[\bar{g}\left((-1)^\beta \left(d_\beta(y,0) - h_\beta(\Pi_\beta(y,0))\right)\right) + (-1)^\beta\right]. \end{aligned} \tag{4.15}$$

Note that for $\beta \neq \alpha$, $|h_\beta(\Pi_\beta(y,0))| \ll 1$. Then using Lemma 3.6, we get

$$|h_\alpha(y)| \lesssim |\phi(y,0)| + \sum_{\beta\neq\alpha} e^{-|d_\beta(y,0)|} \lesssim |\phi(y,0)| + e^{-D_\alpha(y)}. \tag{4.16}$$

Differentiating (4.15), we get

$$\begin{aligned} \nabla_{\alpha,0}\phi(y,0) \quad &= \quad (-1)^\alpha \bar{g}'\left((-1)^{\alpha+1} h_\alpha(y)\right) \nabla_{\alpha,0} h_\alpha(y) \\ &- \quad \sum_{\beta\neq\alpha} (-1)^\beta g_\beta'(y,0) \nabla_{\alpha,0}\left(d_\beta - h_\beta\circ\Pi_\beta\right)(y,0), \end{aligned}$$

and

$$\begin{aligned} \nabla_{\alpha,0}^2\phi(y,0) \quad &= \quad (-1)^\alpha \bar{g}'\left((-1)^{\alpha+1} h_\alpha(y)\right) \nabla_{\alpha,0}^2 h_\alpha(y) \\ &- \quad \bar{g}''\left((-1)^\alpha h_\alpha(y)\right) \nabla_{\alpha,0} h_\alpha(y) \otimes \nabla_{\alpha,0} h_\alpha(y) \\ &- \quad \sum_{\beta\neq\alpha} (-1)^\beta g_\beta'(y,0) \nabla_{\alpha,0}^2 \left(d_\beta - h_\beta\circ\Pi_\beta\right)(y,0) \\ &- \quad \sum_{\beta\neq\alpha} g_\beta''(y,0) \left[\nabla_{\alpha,0}\left(d_\beta - h_\beta\circ\Pi_\beta\right) \otimes \nabla_{\alpha,0}\left(d_\beta - h_\beta\circ\Pi_\beta\right)\right](y,0). \end{aligned}$$

By Lemma 3.4, if $g_\beta'(y,0) \neq 0$,

$$|\nabla_{\alpha,0} d_\beta| = \sqrt{1 - \nabla d_\beta \cdot \nabla d_\alpha} \lesssim \varepsilon^{1/6}.$$

Then by noting that $|\nabla_{\alpha,0}\Pi_\beta|$ is uniformly bounded (see Lemma 3.5), we get

$$|\nabla_{\alpha,0} h_\alpha(y)| \lesssim |\nabla_{\alpha,0}\phi(y,0)| + \sum_{\beta\neq\alpha: |d_\beta(y,0)| \leq 8|\log\varepsilon|} \left(\varepsilon^{1/6} + |\nabla_{\beta,0} h_\beta(\Pi_\beta(y,0))|\right) e^{-|d_\beta(y,0)|}.$$

Summing this in β and applying Lemma 3.6 we obtain

$$|\nabla_{\alpha,0}h_\alpha(y)| \lesssim |\nabla_{\alpha,0}\phi(y,0)| + \left(\varepsilon^{1/6} + \max_{\beta\neq\alpha:|d_\beta(y,0)|\leq 8|\log\varepsilon|} |\nabla_{\beta,0}h_\beta(\Pi_\beta(y,0))|\right) e^{-D_\alpha(y)}.$$

A similar calculation leads to an upper bound on $|\nabla^2_{\alpha,0}h_\alpha(y)|$.

Finally, the Hölder estimate in (4.14) follows by combining the above representation formula, Lemma 4.5 and the bound

$$|\nabla^2_{\alpha,0}d_\beta| \lesssim |\nabla^2 d_\beta| + |A_\alpha| \lesssim \varepsilon.$$

Here it is useful to note (i) $\nabla^2 d_\beta$ is the second fundamental form of $\Gamma_{\beta,z}$, and (ii) $\Pi_\beta(B_1^\alpha(y)) \subset B_2^\beta(\Pi_\beta(y,0))$ if $|d_\beta(y,0)| \leq 8|\log\varepsilon|$ (by Lemma 3.4). □

5. A Toda system

In the Fermi coordinates with respect to Γ_α, multiplying (4.11) by g'_α and integrating in z leads to

$$\begin{aligned}
(5.1)\quad & \int_{-\infty}^{+\infty}\left[g'_\alpha\Delta_{\alpha,z}\phi - H_\alpha(y,z)g'_\alpha\partial_z\phi + g'_\alpha\partial_{zz}\phi\right] \\
= & \int_{-\infty}^{+\infty}\left[W'(g_*+\phi) - \sum_\beta W'(g_\beta)\right]g'_\alpha + (-1)^\alpha\int_{-\infty}^{+\infty}\left[H_\alpha(y,z)+\Delta_{\alpha,z}h_\alpha(y)\right]g'_\alpha(z)^2 \\
- & \int_{-\infty}^{+\infty} g''_\alpha g'_\alpha|\nabla_{\alpha,z}h_\alpha|^2 + \sum_{\beta\neq\alpha}\int_{-\infty}^{+\infty}\left[(-1)^\beta g'_\alpha g'_\beta\mathcal{R}_{\beta,1} - g'_\alpha g''_\beta\mathcal{R}_{\beta,2}\right] - \sum_\beta\int_{-\infty}^{+\infty}\xi_\beta g'_\alpha.
\end{aligned}$$

From this equation we deduce that

$$H_\alpha(y,0) + \Delta_{\alpha,0}h_\alpha(y) = \frac{2A^2_{(-1)^{\alpha-1}}}{\sigma_0}e^{-d_{\alpha-1}(y,0)} - \frac{2A^2_{(-1)^\alpha}}{\sigma_0}e^{d_{\alpha+1}(y,0)} + E^0_\alpha(y), \tag{5.2}$$

where E^0_α is a higher order term. (See Appendix A for the definition of A_1 and A_{-1}.) More precisely, we have

Lemma 5.1. *For any $x \in B_{6R/7}(0)$ and $r \in (0, R/60)$,*

$$\begin{aligned}
\max_\alpha \left\|E^0_\alpha\right\|_{C^\theta(\Gamma_\alpha\cap B_r(x))} \lesssim & \ \varepsilon^2 + \varepsilon^{\frac{1}{3}}A(r+10|\log\varepsilon|;x) + A(r+10|\log\varepsilon|;x)^{\frac{3}{2}} \\
(5.3)\qquad + & \max_\alpha\left\|H_\alpha + \Delta_{\alpha,0}h_\alpha\right\|^2_{C^\theta(\Gamma_\alpha\cap B_{r+10|\log\varepsilon|}(x))} + \|\phi\|^2_{C^{2,\theta}(B_{r+10|\log\varepsilon|}(x))}.
\end{aligned}$$

The proof is given in Appendix B.

Since all terms in the right hand side of (5.3) are of higher order, a direct consequence of this lemma is

Corollary 5.2. *There exists a universal constant $C>0$ such that for any $x \in B_{6R/7}(0)$ and $r \in (0, R/60)$,*

$$\begin{aligned}
& \max_\alpha\left\|H_\alpha + \Delta_{\alpha,0}h_\alpha\right\|_{C^\theta(\Gamma_\alpha\cap B_r(x))} \\
(5.4)\qquad \leq & \ \frac{1}{4}\left(\max_\alpha\left\|H_\alpha + \Delta_{\alpha,0}h_\alpha\right\|_{C^\theta(\Gamma_\alpha\cap B_{r+10|\log\varepsilon|}(x))} + \|\phi\|_{C^{2,\theta}(B_{r+10|\log\varepsilon|}(x))}\right) \\
+ & \ C\varepsilon^2 + CA(r+10|\log\varepsilon|;x).
\end{aligned}$$

6. Estimates on ϕ

In this section we prove the following $C^{2,\theta}$ estimate on ϕ.

Proposition 6.1. *For any $x \in B_{6R/7}(0)$ and $r \in (0, R/60)$,*

$$\max_{\alpha} \left\| H_\alpha + \Delta_{\alpha,0} h_\alpha \right\|_{C^\theta(\Gamma_\alpha \cap B_r(x))} + \|\phi\|_{C^{2,\theta}(B_r(x))} \lesssim \varepsilon^2 + A\left(r + 50|\log\varepsilon|^2; x\right). \tag{6.1}$$

The term ε^2 is optimal and cannot be improved any further in this general setting.

The first order Hölder estimates of ϕ will be established in Subsection 6.1 and Subsection 6.2. The second order Hölder estimate will be proved in Subsection 6.3.

To prove the first order Hölder estimate on ϕ, fix a large constant $L > 0$ (to be determined at the end of Subsection 6.3), for each α define

$$\Omega_\alpha^1 := \{-L < d_\alpha < L\} \cap \mathcal{M}_\alpha^0, \qquad \Omega_\alpha^2 := \{|d_\alpha| > L/2\} \cap \mathcal{M}_\alpha^0,$$

and

$$\Omega_\alpha^3 := \{-L/2 \le d_\alpha \le L/2\} \cap \mathcal{M}_\alpha^0.$$

We will estimate the $C^{1,\theta}$ norm of ϕ in $\Omega_\alpha^1 \cap B_r(x)$ and $\Omega_\alpha^2 \cap B_r(x)$ separately, which correspond to the inner and outer problem respectively. Roughly speaking, in Ω_α^1, ϕ satisfies

$$-\Delta\phi + W''(g_\alpha)\phi = \text{interaction terms} + \text{parallel component} + \text{errors}.$$

Together with the orthogonal condition (4.4) we get a control on ϕ, which is possible by the coercivity of the operator $-\Delta + W''(g)$ in the class of functions satisfying the orthogonal condition (4.4), see for example [19, Proposition 4.1]. In Ω_α^2, ϕ satisfies

$$-\Delta\phi + \phi = \text{interaction terms} + \text{errors}.$$

Hence a control on ϕ is possible by using the decay estimate of the coercive operator $-\Delta + 1$.

6.1. $C^{1,\theta}$ estimate in Ω_α^2.

We start with the easy case. In Ω_α^2, the equation for ϕ can be written in the following way.

Lemma 6.2. *For any α, in Ω_α^2,*

$$\Delta\phi = [1 + o(1)]\,\phi + E_\alpha^2,$$

where

$$\begin{aligned} \|E_\alpha^2\|_{L^\infty(\Omega_\alpha^2 \cap B_r(x))} &\le C\varepsilon^2 + CA\left(r + 10|\log\varepsilon|; x\right) + C\|\phi\|^2_{C^{2,\theta}(B_{r+10|\log\varepsilon|}(x))} \\ &+ Ce^{-cL} \max_\alpha \left\| H_\alpha + \Delta_{\alpha,0} h_\alpha \right\|_{L^\infty(\Gamma_\alpha \cap B_{r+10|\log\varepsilon|}(x))}. \end{aligned}$$

Proof. The L^∞ estimate on E_α^2 is a consequence of the following estimates on those terms in (4.11).

- First we have $W'(g_* + \phi) - W'(g_*) = \left[W''(g_*) + O(\phi)\right]\phi = [1 + o(1)]\,\phi$.
- By Lemma 4.4, $W'(g_*) - \sum_\beta W'(g_\beta) = O\left(e^{-D_\alpha}\right) + O\left(\varepsilon^2\right)$.
- By (3.7)-(3.12), we get

$$\begin{aligned} & g_\alpha' \left[H_\alpha(y,z) + \Delta_{\alpha,z} h_\alpha\right] \\ = \; & g_\alpha' \left[H_\alpha(y,0) + \Delta_{\alpha,0} h_\alpha(y)\right] + g_\alpha' \left[H_\alpha(y,z) - H_\alpha(y,0)\right] + g_\alpha' \left[\Delta_{\alpha,z} h_\alpha - \Delta_{\alpha,0} h_\alpha\right] \\ = \; & O\left(e^{-cL}\left|H_\alpha(y,0) + \Delta_{\alpha,0} h_\alpha(y)\right|\right) + O\left(\varepsilon^2\right) + O\left(|\nabla^2_{\alpha,0} h_\alpha|^2 + |\nabla_{\alpha,0} h_\alpha|^2\right). \end{aligned}$$

  Concerning estimates on the last two terms involving h_α, we use Lemma 4.6.

- Similarly, estimates on $g_\alpha''|\nabla_{\alpha,z}h_\alpha|^2$ follow from Lemma 4.6.
- Those two terms involving $g_\beta'\mathcal{R}_{\beta,1}$ and $g_\beta''\mathcal{R}_{\beta,2}$ can be estimated as in the above two cases, but now in Fermi coordinates with respect to Γ_β. Note that we need only to consider those β satisfying $\Gamma_\beta \cap B_{r+8|\log\varepsilon|}(x) \neq \emptyset$, because otherwise $g_\beta' = 0$ in $B_r(x)$. To put all estimates of $\beta \neq \alpha$ together, we use Lemma 3.6.
- Finally, by definition of ξ_β, $\sum_\beta \xi_\beta = O\left(\varepsilon^3|\log\varepsilon|\right) = O\left(\varepsilon^2\right)$. □

By standard elliptic estimates for the coercive operator $-\Delta + 1$, we deduce that, for any α,

$$\begin{aligned} \|\phi\|_{C^{1,\theta}(\Omega_\alpha^2\cap B_r(x))} &\leq Ce^{-cL}\left(\|\phi\|_{C^{1,\theta}(\Omega_\alpha^2\cap B_{r+10|\log\varepsilon|}(x))} + \|\phi\|_{C^{1,\theta}(\Omega_\alpha^3\cap B_{r+10|\log\varepsilon|}(x))}\right) \\ &+ Ce^{-cL}\max_\beta \left\|H_\beta + \Delta_{\beta,0}h_\beta\right\|_{L^\infty(\Gamma_\beta\cap B_{r+10|\log\varepsilon|}(x))} \\ &+ C\varepsilon^2 + CA\left(r + 10|\log\varepsilon|; x\right). \end{aligned} \tag{6.2}$$

6.2. $C^{1,\theta}$ **estimate in** Ω_α^1**.** In Ω_α^1, the equation for ϕ can be written in the following way.

Lemma 6.3. *In* Ω_α^1,

$$\Delta_{\alpha,0}\phi + \partial_{zz}\phi = W''(g_\alpha)\phi + (-1)^\alpha g_\alpha'\left[H_\alpha(y,0) + \Delta_{\alpha,0}h_\alpha\right] + E_\alpha^1,$$

where for some constant $C(L)$,

$$\|E_\alpha^1\|_{L^\infty(\Omega_\alpha^1\cap B_r(x))} \leq C(L)\varepsilon^2 + C(L)A\left(r + 10|\log\varepsilon|; x\right) + C(L)\|\phi\|^2_{C^{2,\theta}(B_{r+10|\log\varepsilon|}(x))}.$$

Proof. The proof is similar to the one for Lemma 6.2, in particular,

- we use Cauchy inequality and (3.12) to bound $\Delta_{\alpha,z}\phi - \Delta_{\alpha,0}\phi$ (here it is useful to note that $|z| < 2L$ in Ω_α^1);
- we use Cauchy inequality and the fact that $|H_\alpha(y,z)| \lesssim \varepsilon$ to bound $H_\alpha(y,z)\partial_z\phi$;
- we use Lemma 4.4 to bound interaction terms;
- we use Lemma 4.6 to bound those terms involving h_α;
- by the exponential decay of $\bar{g}'$ at infinity and Lemma 3.6, $\sum_{\beta\neq\alpha} g_\beta'\mathcal{R}_{\beta,1}$ and $\sum_{\beta\neq\alpha} g_\beta'\mathcal{R}_{\beta,2}$ are bounded by $e^{-D_\alpha(y)}$ in Ω_α^1. (Although there are constants e^{CL} appearing when we bound g_β' by $O\left(e^{-D_\alpha(y)}\right)$, they can be incorporated because $|\mathcal{R}_{\beta,1}| + |\mathcal{R}_{\beta,2}| \ll 1$ while L, although large, is a fixed constant.) □

Take a function $\xi \in C_0^\infty(-2L, 2L)$ satisfying $\xi \equiv 1$ in $(-L, L)$, $|\xi'| \lesssim L^{-1}$ and $|\xi''| \lesssim L^{-2}$. Let $\phi_\alpha(y,z) := \phi(y,z)\xi(z) - c_\alpha(y)g_\alpha'(y,z)$, where

$$c_\alpha(y) = \frac{\int_{-\infty}^{+\infty}\phi(y,z)\left(\xi(z) - 1\right)g_\alpha'(y,z)dz}{\int_{-\infty}^{+\infty}g_\alpha'(y,z)^2dz}. \tag{6.3}$$

Hence by (4.4) we still have the orthogonal condition

$$\int_{-\infty}^{+\infty}\phi_\alpha(y,z)g_\alpha'(y,z)dz = 0, \quad \forall\, y \in \Gamma_\alpha. \tag{6.4}$$

We have the following estimates on c_α.

Lemma 6.4. *For any* $y \in \Gamma_\alpha$,

$$|c_\alpha(y)| \lesssim e^{-L}\max_{L<|z|<8|\log\varepsilon|}|\phi(y,z)|,$$

$$|\nabla_{\alpha,0}c_\alpha(y)| \lesssim e^{-L}\max_{L<|z|<8|\log\varepsilon|}\left(|\phi(y,z)| + |\nabla_{\alpha,z}\phi(y,z)|\right),$$

$$|\nabla^2_{\alpha,0}c_\alpha(y)| \lesssim e^{-L} \max_{L<|z|<8|\log\varepsilon|}\left(|\phi(y,z)| + |\nabla_{\alpha,z}\phi(y,z)| + |\nabla^2_{\alpha,z}\phi(y,z)|\right).$$

Proof. By (6.3) and the definition of $\bar{g}$ and ξ,

$$\begin{aligned} |c_\alpha(y)| &\lesssim \left(\max_{L<|z|<8|\log\varepsilon|}|\phi(y,z)|\right)\int_L^{+\infty} e^{-z}dz \\ &\lesssim e^{-L}\max_{L<|z|<8|\log\varepsilon|}|\phi(y,z)|. \end{aligned}$$

Differentiating (6.3) gives

$$\begin{aligned} &\nabla_{\alpha,0}c_\alpha(y)\left(\int_{-\infty}^{+\infty} g'_\alpha(y,z)^2dz\right) + c_\alpha(y)\left(\nabla_{\alpha,0}\int_{-\infty}^{+\infty} g'_\alpha(y,z)^2dz\right) \\ = &\int_{-\infty}^{+\infty}\nabla_{\alpha,0}\phi(y,z)\left(\xi(z)-1\right)g'_\alpha(y,z)dz \\ - &(-1)^\alpha\nabla_{\alpha,0}h_\alpha(y)\int_{-\infty}^{+\infty}\phi(y,z)\left(\xi(z)-1\right)g''_\alpha(y,z)dz. \end{aligned}$$

The second estimate follows as above. The third one can be proved in the same way. $\square$

The factor e^{-L} reveals the fact that behavior of ϕ in Ω^2_α has little effect on the behavior of ϕ in Ω^1_α, that is, the inner and the outer parts are almost decoupled.

In Fermi coordinates with respect to Γ_α, the equation satisfied by ϕ_α reads as

$$\Delta_{\alpha,0}\phi_\alpha + \partial_{zz}\phi_\alpha = W''(g_\alpha)\phi_\alpha + p_\alpha(y)g'_\alpha + F_\alpha, \tag{6.5}$$

where

$$p_\alpha(y) = (-1)^\alpha\left[H_\alpha(y,0) + \Delta_{\alpha,0}h_\alpha(y)\right] - \Delta_{\alpha,0}c_\alpha(y)$$

and

$$\begin{aligned} F_\alpha(y,z) &= 2(-1)^\alpha\nabla_{\alpha,0}c_\alpha(y)\cdot\nabla_{\alpha,0}h_\alpha(y)g''_\alpha(y,z) \\ &+ c_\alpha(y)\left[(-1)^{\alpha+1}\Delta_{\alpha,0}h_\alpha(y)g''_\alpha(y,z) + g'''_\alpha(y,z)|\nabla_{\alpha,0}h_\alpha(y)|^2\right] \\ &+ 2\partial_z\phi(y,z)\xi'(z) + \phi(y,z)\xi''(z) + E^1_\alpha(y,z)\xi(z) \\ &+ (-1)^\alpha\left[H_\alpha(y,0) + \Delta_{\alpha,0}h_\alpha(y)\right]g'_\alpha(y,z)\left[\xi(z)-1\right] - c_\alpha(y)\xi'_\alpha(y,z). \end{aligned}$$

Combining this expression with Lemma 6.3, Lemma 6.4 and the definition of ξ, we obtain

Lemma 6.5. *For any $x \in B_{5R/6}(0)$ and $r \in (0, R/60)$,*

$$\begin{aligned} &\|F_\alpha\|_{L^\infty(\Omega^1_\alpha\cap B_r(x))} \\ \le\ &C(L)\varepsilon^2 + C(L)A\left(r+10|\log\varepsilon|;x\right) + C(L)\|\phi\|^2_{C^{2,\theta}(B_{r+10|\log\varepsilon|}(x))} \\ +\ &Ce^{-cL}\left[\max_\beta\left\|H_\beta + \Delta_{\beta,0}h_\beta\right\|_{L^\infty(\Gamma_\beta\cap B_{r+10|\log\varepsilon|}(x))} + \|\phi\|_{C^{2,\theta}(B_{r+10|\log\varepsilon|}(x))}\right]. \end{aligned}$$

By (6.5) and the orthogonal condition (6.4), applying standard estimates on the linearized operator $-\Delta + W''(g)$ (see for example [19, Proposition 4.1]) leads to

$$\begin{aligned} \|\phi_\alpha\|_{C^{1,\theta}(B_r(x))} &\le Ce^{-cL}\|\phi_\alpha\|_{C^{1,\theta}(B_{r+10|\log\varepsilon|}(x))} + Ce^{-cL}\|\phi\|_{C^{2,\theta}(B_{r+10|\log\varepsilon|}(x))} \\ &+ Ce^{-cL}\max_\beta\left\|H_\beta + \Delta_{\beta,0}h_\beta\right\|_{L^\infty(\Gamma_\beta\cap B_{r+12|\log\varepsilon|}(x))} \\ &+ C(L)\varepsilon^2 + C(L)A\left(r+10|\log\varepsilon|;x\right) + C(L)\|\phi\|^2_{C^{2,\theta}(B_{r+10|\log\varepsilon|}(x))}. \end{aligned}$$

Coming back to ϕ, by the fact that $\|\phi\|_{C^{2,\theta}(B_{r+10|\log\varepsilon|}(x))} \ll 1$ and the estimates on c_α in Lemma 6.4, we get

$$\begin{aligned}\|\phi\|_{C^{1,\theta}(\Omega^1_\alpha\cap B_r(x))} &\leq C\varepsilon^2 + CA\left(r+10|\log\varepsilon|;x\right)\\ &+ Ce^{-cL}\|\phi\|_{C^{2,\theta}(B_{r+10|\log\varepsilon|}(x))}\\ &+ Ce^{-cL}\max_\alpha\left\|H_\alpha+\Delta_{\alpha,0}h_\alpha\right\|_{L^\infty(\Gamma_\alpha\cap B_{r+10|\log\varepsilon|}(x))}.\end{aligned}\tag{6.6}$$

Combining (6.2) and (6.6), we obtain

$$\begin{aligned}\|\phi\|_{C^{1,\theta}(B_r(x))} &\leq C(L)\varepsilon^2 + C(L)A\left(r+10|\log\varepsilon|;x\right)\\ &+ Ce^{-cL}\left(\max_\alpha\left\|H_\alpha+\Delta_{\alpha,0}h_\alpha\right\|_{L^\infty(\Gamma_\alpha\cap B_{r+10|\log\varepsilon|}(x))} + \|\phi\|_{C^{2,\theta}(B_{r+10|\log\varepsilon|}(x))}\right).\end{aligned}\tag{6.7}$$

6.3. **Second order Hölder estimates on ϕ.** In this subsection we proceed to second order Hölder estimates. To this end, we need the following Hölder bounds on the right hand side of (4.11).

Lemma 6.6. *For any $x \in B_{5R/6}(0)$ and $r \in (0, R/60)$,*

$$\begin{aligned}&\|\Delta\phi - W''(g_*)\phi\|_{C^\theta(B_r(x))}\\ \lesssim\ & \varepsilon^2 + A\left(r+10|\log\varepsilon|;x\right) + \|\phi\|^2_{C^{2,\theta}(B_{r+10|\log\varepsilon|}(x))}\\ +\ & A\left(r+10|\log\varepsilon|;x\right)^{\frac12}\left(\max_\alpha\|H_\alpha+\Delta_{\alpha,0}h_\alpha\|_{C^\theta(\Gamma_\alpha\cap B_{r+10|\log\varepsilon|}(x))}\right).\end{aligned}$$

The proof is given in Appendix C. Here we just emphasize that, because there are second order derivatives of u in the right hand of (4.11), such an Hölder estimate relies on Lemma 3.1, i.e. an ε-bound on some third order derivatives of u.

Since L is fixed, by Lemma 2.1 and Remark 4.2, for all ε small enough,

$$\|\phi\|_{C^{2,\theta}(B_{r+10|\log\varepsilon|}(x))} + A\left(r+10|\log\varepsilon|;x\right)^{\frac12} \leq Ce^{-cL}.$$

Then by (6.7) and Schauder estimates, we get for all ε small enough,

$$\begin{aligned}\|\phi\|_{C^{2,\theta}(B_r(x))} &\leq Ce^{-cL}\left(\max_\alpha\left\|H_\alpha+\Delta_{\alpha,0}h_\alpha\right\|_{C^\theta(\Gamma_\alpha\cap B_{r+10|\log\varepsilon|}(x))} + \|\phi\|_{C^{2,\theta}(B_{r+10|\log\varepsilon|}(x))}\right)\\ &+ C(L)\varepsilon^2 + C(L)A\left(r+10|\log\varepsilon|;x\right).\end{aligned}\tag{6.8}$$

Combining this estimate with Corollary 5.2, we get

$$\begin{aligned}&\max_\alpha\left\|H_\alpha+\Delta_{\alpha,0}h_\alpha\right\|_{C^\theta(\Gamma_\alpha\cap B_r(x))} + \|\phi\|_{C^{2,\theta}(B_r(x))}\\ \leq\ & \frac12\left(\max_\alpha\left\|H_\alpha+\Delta_{\alpha,0}h_\alpha\right\|_{C^\theta(\Gamma_\alpha\cap B_{r+10|\log\varepsilon|}(x))} + \|\phi\|_{C^{2,\theta}(B_{r+10|\log\varepsilon|}(x))}\right)\\ +\ & C(L)\varepsilon^2 + C(L)A\left(r+10|\log\varepsilon|;x\right).\end{aligned}$$

An iteration of this inequality from $r+50|\log\varepsilon|^2$ to r leads to (6.1). The proof of Proposition 6.1 is thus complete.

7. Improved estimates on horizontal derivatives

In this section we establish an improvement on the $C^{1,\theta}$ estimates of horizontal derivatives of ϕ, $\phi_i := \partial\phi/\partial y_i$. $1 \leq i \leq n-1$.

Proposition 7.1. *For any $x \in B_{5R/6}(0)$ and $r \in (0, R/70)$,*

$$\|\phi_i\|_{C^{1,\theta}(B_r(x))} \lesssim \varepsilon^2 + A\left(r+60|\log\varepsilon|^2;x\right)^{3/2} + \varepsilon^{1/6}A\left(r+60|\log\varepsilon|^2;x\right).$$

To prove Proposition 7.1, as in Section 6 we still estimate ϕ_i in Ω_α^1 and Ω_α^2 separately. To this end, first rewrite (4.11) as

$$\Delta_{\alpha,z}\phi + \partial_{zz}\phi = W''(g_*)\phi + (-1)^\alpha g_\alpha' \left[H_\alpha(y,0) + \Delta_{\alpha,0}h_\alpha(y)\right] + \mathcal{I} + E_\alpha, \tag{7.1}$$

where

$$\begin{aligned} E_\alpha &= H_\alpha(y,z)\partial_z\phi + \left[W'(g_*+\phi) - W'(g_*) - W''(g_*)\phi\right] \\ &+ (-1)^\alpha g_\alpha' \left[H_\alpha(y,z) - H_\alpha(y,0) + \Delta_{\alpha,z}h_\alpha(y) - \Delta_{\alpha,0}h_\alpha(y)\right] \\ &- g_\alpha''|\nabla_{\alpha,z}h_\alpha|^2 + \sum_{\beta\neq\alpha}\left[(-1)^\beta g_\beta'\mathcal{R}_{\beta,1} - g_\beta''\mathcal{R}_{\beta,2}\right] - \sum_\beta \xi_\beta. \end{aligned}$$

The following C^θ bound on E_α is similar to Lemma 6.6. However, since we have removed two main order terms $g_\alpha'\left[H_{\alpha,0} + \Delta_{\alpha,0}h_\alpha\right]$ and $\mathcal{I}$ form E_α, there is no $A\left(r + 10|\log\varepsilon|; x\right)$ term in the right hand side. Therefore, with the help of Proposition 6.1 we get

Lemma 7.2. *For any $x \in B_{5R/6}(0)$ and $r \in (0, R/70)$,*

$$\|E_\alpha\|_{C^\theta(\mathcal{M}_\alpha^0\cap B_r(x))} \lesssim \varepsilon^2 + A\left(r + 50|\log\varepsilon|^2; x\right)^{3/2}.$$

Differentiating (7.1) in y_i, we obtain an equation for ϕ_i, which in Fermi coordinates with respect to Γ_α reads as

$$\Delta_{\alpha,z}\phi_i + \partial_{zz}\phi_i = W''(g_\alpha)\phi_i - (-1)^\alpha g_\alpha'\left[H_{\alpha,i}(y,0) + \Delta_{\alpha,0}h_{\alpha,i}(y)\right] + \partial_{y_i}\mathcal{I} + \partial_{y_i}E_\alpha + E^i, \tag{7.2}$$

where $H_{\alpha,i}(y,0) := \partial_{y_i}H_\alpha(y,0)$, $h_{\alpha,i}(y) := \partial_{y_i}h_\alpha$ and the remainder term

$$\begin{aligned} E^i &= \underbrace{\left(\Delta_{\alpha,z}\phi_i - \partial_{y_i}\Delta_{\alpha,z}\phi\right)}_{I} + \underbrace{\left[W''(g_*) - W''(g_\alpha)\right]\phi_i}_{II} \\ &+ \underbrace{W'''(g_*)\phi\left[\sum_{\beta\neq\alpha}(-1)^\beta g_\beta'\left(\frac{\partial d_\beta}{\partial y_i} - \sum_{j=1}^{n-1}\left(h_{\beta,j}\circ\Pi_\beta\right)\frac{\partial\Pi_\beta^j}{\partial y_i}\right)\right]}_{III} \\ &+ \underbrace{(-1)^\alpha g_\alpha'\left[\partial_{y_i}\Delta_{\alpha,0}h_\alpha(y) - \Delta_{\alpha,0}h_{\alpha,i}(y)\right]}_{IV} \\ &- \underbrace{g_\alpha'' h_{\alpha,i}\left[H_\alpha(y,0) + \Delta_{\alpha,0}h_\alpha(y)\right]}_{V}. \end{aligned}$$

We have the following L^∞ bound on E^i.

Lemma 7.3. *For any $x \in B_{5R/6}(0)$ and $r \in (0, R/70)$,*

$$\|E^i\|_{L^\infty(\mathcal{M}_\alpha^0\cap B_r(x))} \lesssim \varepsilon^2 + A\left(r + 60|\log\varepsilon|^2; x\right)^{3/2} + \varepsilon^{1/6}A\left(r + 60|\log\varepsilon|^2; x\right).$$

Proof. We estimate the five terms one by one.

(1) By Lemma 3.3, for $(y,z) \in \mathcal{M}_\alpha^0 \cap B_r(x)$,

$$|I| \lesssim \varepsilon\left(|\nabla^2_{\alpha,0}\phi(y,z)| + |\nabla_{\alpha,0}\varphi(y,z)|\right) \lesssim \varepsilon^2 + \|\phi\|^2_{C^{2,\theta}(B_r(x))}.$$

(2) For $(y,z) \in \mathcal{M}_\alpha^0 \cap B_r(x)$, by Taylor expansion and Lemma 3.6 we get

$$\left|W''(g_*+\phi) - W''(g_\alpha)\right| \lesssim |\phi| + \sum_{\beta\neq\alpha} g_\beta'$$

$$\lesssim \|\phi\|_{L^\infty(B_r(x))} + \max_{\Gamma_\alpha\cap B_r(x)} e^{-\frac{D_\alpha}{2}} + \varepsilon^2.$$

Hence

$$\begin{aligned}\|II\|_{L^\infty(\mathcal{M}_\alpha^0\cap B_r(x))} &\lesssim \|\phi\|_{C^{2,\theta}(B_r(x))}\left(\|\phi\|_{C^{2,\theta}(B_r(x))} + \max_{\Gamma_\alpha\cap B_r(x)} e^{-\frac{D_\alpha}{2}} + \varepsilon^2\right)\\ &\lesssim \|\phi\|^2_{C^{2,\theta}(B_r(x))} + A(r;x)^{\frac12}\|\phi\|_{C^{2,\theta}(B_r(x))} + \varepsilon^2.\end{aligned}$$

(3) For $\beta\neq\alpha$, if $g'_\beta\neq 0$, by Lemma 3.4,

$$\left|\frac{\partial d_\beta}{\partial y_i}\right| \lesssim \varepsilon^{1/6}. \tag{7.3}$$

By the Cauchy inequality, Lemma 3.6 and Lemma 4.6, we obtain

$$\begin{aligned}\|III\|_{L^\infty(\mathcal{M}_\alpha^0\cap B_r(x))} &\lesssim \|\phi\|^2_{C^{1,\theta}(B_{r+8|\log\varepsilon|}(x))} + A\left(r+8|\log\varepsilon|;x\right)^2\\ &+ \varepsilon^{1/6}\|\phi\|_{C^{1,\theta}(B_{r+8|\log\varepsilon|}(x))}.\end{aligned}$$

(4) By Lemma 3.3, Lemma 4.6 and Proposition 6.1,

$$\|IV\|_{L^\infty(\mathcal{M}_\alpha^0\cap B_r(x))} \lesssim \varepsilon^2 + \|\phi\|^2_{C^{2,\theta}(B_r(x))} + A\left(r;x\right)^2.$$

(5) By the Cauchy inequality and Lemma 4.6,

$$\|V\|_{L^\infty(\mathcal{M}_\alpha^0\cap B_r(x))} \lesssim \|\phi\|^2_{C^{2,\theta}(B_r(x))} + \max_{\Gamma_\alpha\cap B_r(x)} e^{-2D_\alpha} + \|H_\alpha + \Delta_{\alpha,0}h_\alpha\|^2_{L^\infty(\Gamma_\alpha\cap B_r(x))}.$$

Putting these estimates together and applying (6.1) we conclude the proof. ☐

Finally, the order of $\partial_{y_i}\mathcal{I}$ is increased by one due to the appearance of one more term involving horizontal derivatives of ϕ or d_α.

Lemma 7.4. *For any $x\in B_{5R/6}(0)$ and $r\in(0,R/70)$,*

$$\|\partial_{y_i}\mathcal{I}\|_{L^\infty(B_r(x))} \lesssim \varepsilon^2 + A\left(r+60|\log\varepsilon|^2;x\right)^2 + \varepsilon^{1/6}A\left(r+60|\log\varepsilon|^2;x\right).$$

Proof. We have

$$\partial_{y_i}\mathcal{I} = \sum_\beta (-1)^\beta\left[W''(g_*) - W''(g_\beta)\right]g'_\beta\left(\frac{\partial d_\beta}{\partial y_i} - \sum_{j=1}^{n-1} h_{\beta,j}\left(\Pi_\beta(y,z)\right)\frac{\partial \Pi_\beta^j}{\partial y_i}(y,z)\right).$$

Let us first give an estimate on $\left[W''(g_*) - W''(g_\beta)\right]g'_\beta$ in $\mathcal{M}_\alpha^0$. There are two cases.

- If $\beta=\alpha$, we have

$$\left|W''(g_*) - W''(g_\alpha)\right|g'_\alpha \lesssim g'_\alpha\sum_{\beta\neq\alpha}\left(1-g_\beta^2\right) \lesssim e^{-D_\alpha},$$

  where the last inequality is similar to Lemma 4.4.
- If $\beta\neq\alpha$, still as in Lemma 4.4,

$$\begin{aligned}\left[W''(g_*) - W''(g_\beta)\right]g'_\beta &= \left[W''(g_\alpha) - W''(1) + O\left(\sum_{\beta\neq\alpha}\left(1-g_\beta^2\right)\right)\right]g'_\beta\\ &= O\left(\varepsilon^2 e^{-c|\beta-\alpha|}\right) + O\left(e^{-|d_\beta(y,0)|}\right).\end{aligned} \tag{7.4}$$

Hence by Lemma 3.6, we have

$$\sum_\beta (-1)^{\beta+1}\left[W''(g_*)-W''(g_\beta)\right]g'_\beta = O\left(\varepsilon^2\right)+O\left(e^{-D_\alpha}\right). \tag{7.5}$$

Using Lemma 4.6 to bound $h_{\beta,j}$, and Lemma 3.5 to bound $\frac{\partial \Pi_\beta^j}{\partial y_i}$, we see if $g'_\beta \neq 0$, (7.3) holds and

$$\Big|\sum_{j=1}^{n-1} h_{\beta,j}\left(\Pi_\beta(y,z)\right)\frac{\partial \Pi_\beta^j}{\partial y_i}(y,z)\Big| \lesssim A\left(r;x\right)+\|\phi\|_{C^{1,\theta}(B_r(x))}. \tag{7.6}$$

Combining (7.5), (7.3) and (7.6), using the Cauchy inequality and applying (6.1) we conclude the proof. ☐

Differentiating (4.4) we obtain for any α and $y\in\Gamma_\alpha\cap B_r(x)$,

$$\int_{-\infty}^{+\infty}\phi_i g'_\alpha dz = h_{\alpha,i}(y)\int_{-\infty}^{+\infty}\phi g''_\alpha dz = O\left(\|\phi\|^2_{C^1(B_{r+8|\log\varepsilon|}(x))}+\max_{\Gamma_\alpha\cap B_r(x)} e^{-2D_\alpha}\right). \tag{7.7}$$

In view of Lemma 7.2, Lemma 7.3 and Lemma 7.4, combining (7.2) and the almost orthogonal condition (7.7), proceeding as in Section 6 we get Proposition 7.1. Here we first take the decomposition $\phi_i=\phi_{i,1}+\phi_{i,2}+\phi_{i,3}$, where $\phi_{i,1}$, $\phi_{i,2}$ and $\phi_{i,3}$ satisfy the same equation as (7.2), but with inhomogeneous terms $\partial_{y_i}\mathcal{I}$, $\partial_{y_i}E_\alpha$ and E^i respectively. First use coercive estimates for the linear elliptic operator $-\Delta+1$ (in Ω^2_α) and $-\Delta+W''(g_\alpha)$ (in Ω^1_α) to get improved L^2 estimates on these three functions. The $C^{1,\theta}$ estimates for $\phi_{i,1}$ and $\phi_{i,3}$ follow by applying standard $W^{2,p}$ estimate (with p so large that $W^{2,p}$ embeds into $C^{1,\theta}$), and we apply Hölder estimates for first derivatives (see [26, Section 4.5]) to obtain the estimate on $\phi_{i,2}$. The proof of Proposition 7.1 is complete by putting these three estimates together.

Two corollaries follow from Proposition 7.1. Combining Proposition 7.1 with Lemma 4.6, we obtain

Corollary 7.5. *For any $x\in B_{5R/6}(0)$ and $r\in(0,R/70)$,*

$$\max_\alpha \|\nabla h_\alpha\|_{C^{1,\theta}(\Gamma_\alpha\cap B_r(x))}\lesssim \varepsilon^2 + A\left(r+60|\log\varepsilon|^2;x\right)^{3/2}+\varepsilon^{1/6}A\left(r+60|\log\varepsilon|^2;x\right).$$

Substituting estimates in this corollary and Proposition 6.1 into (5.2) we get

Corollary 7.6. *For any $x\in B_{5R/6}(0)$ and $r\in(0,R/70)$, in $B_r(x)$ it holds that*

$$\begin{aligned} H_\alpha(y,0) &= \frac{2A^2_{(-1)^{\alpha-1}}}{\sigma_0}e^{-d_{\alpha-1}(y,0)}-\frac{2A^2_{(-1)^\alpha}}{\sigma_0}e^{d_{\alpha+1}(y,0)}+O\left(\varepsilon^2\right)\\ &+ O\left(A\left(r+60|\log\varepsilon|^2;x\right)^{3/2}\right)+O\left(\varepsilon^{1/6}A\left(r+60|\log\varepsilon|^2;x\right)\right). \end{aligned}$$

8. Reduction of the stability condition

In this section we show that if u is a stable solution of the Allen-Cahn equation, then solutions to the Toda system (5.2) constructed in Section 5 satisfy an almost stable condition.

Given a point $x\in B_{5R/6}(0)$ and $r\in(0,R/70)$, and finitely many functions $\eta_\alpha\in C_0^\infty\left(\Gamma_\alpha\cap B_r(x)\right)$ (provided this intersection is nonempty), using Fermi coordinates with respect to Γ_α we define

$$\varphi_\alpha(y,z):=\eta_\alpha(y)g'_\alpha(y,z).$$

In the following we will view η_α as a function defined in $B_{r+8|\log\varepsilon|}(x)$ by identifying it with $\eta_\alpha\circ\Pi_\alpha$. Since $g'_\alpha\neq 0$ only in an $8|\log\varepsilon|$-neighborhood of Γ_α, φ_α is compactly supported in $B_{r+8|\log\varepsilon|}(x)$.

Let $\varphi := \sum_\alpha \varphi_\alpha$. By definition $\varphi\in C_0^\infty(B_{r+8|\log\varepsilon|}(x))$. The stability condition for u says that

$$\int_{B_{r+8|\log\varepsilon|}(x)}\left[|\nabla\varphi|^2+W''(u)\varphi^2\right]\geq 0.$$

The purpose of this section is to rewrite this inequality as a stability condition for the Toda system (5.2).

Proposition 8.1. *If η_α are given as above, then we have*

$$\begin{aligned}&\sum_\alpha\int_{\Gamma_\alpha}|\nabla_{\alpha,0}\eta_\alpha|^2dA_{\alpha,0}+\mathcal{Q}(\eta)\\&\quad\geq\sum_\alpha\frac{2A^2_{(-1)^{\alpha-1}}}{\sigma_0}\int_{\Gamma_\alpha}e^{-d_{\alpha-1}(y)}\left[\eta_\alpha(y)+\eta_{\alpha-1}\left(\Pi_{\alpha-1}(y,0)\right)\right]^2dA_{\alpha,0},\end{aligned}$$

where by denoting N to be the number of non-zero η_α, we have

$$\begin{aligned}|\mathcal{Q}(\eta)|\quad&\lesssim_N\quad\left[\varepsilon^{\frac14}+A\left(r+60|\log\varepsilon|^2;x\right)^{\frac12}\right]\left(\sum_\alpha\int_{\Gamma_\alpha}|\nabla_{\alpha,0}\eta_\alpha|^2dA_{\alpha,0}\right)\\&+\quad\left[\varepsilon^2+A\left(r+60|\log\varepsilon|^2;x\right)^{\frac76}+\varepsilon^{\frac17}A\left(r+60|\log\varepsilon|^2;x\right)\right]\left(\sum_\alpha\int_{\Gamma_\alpha}\eta_\alpha^2dA_{\alpha,0}\right).\end{aligned}$$

Since

$$\begin{aligned}\int_{B_{r+8|\log\varepsilon|}(x)}\left[|\nabla\varphi|^2+W''(u)\varphi^2\right]\quad&=\quad\sum_\alpha\int_{B_{r+8|\log\varepsilon|}(x)}\left[|\nabla\varphi_\alpha|^2+W''(u)\varphi_\alpha^2\right]\\&+\quad\sum_{\alpha\neq\beta}\int_{B_{r+8|\log\varepsilon|}(x)}\left[\nabla\varphi_\alpha\cdot\nabla\varphi_\beta+W''(u)\varphi_\alpha\varphi_\beta\right],\end{aligned}$$

we first consider the first integrals and estimate the tangential part $\nabla_{\alpha,z}\varphi_\alpha(y,z)$ in Subsection 8.1, then the normal part $\partial_z\varphi_\alpha$ in Subsection 8.2, where an interaction term appears and it is studied in Subsection 8.3, and finally in Subsection 8.4 estimates on cross terms are given. Proposition 8.1 follows by putting these estimates together.

8.1. **The tangential part.** In this subsection we prove

Lemma 8.2. *The horizontal part has the expansion*

$$\int_{B_{r+8|\log\varepsilon|}(x)}\left|\nabla_{\alpha,z}\varphi_\alpha(y,z)\right|^2=\left[\sigma_0+O\left(\varepsilon+A(r;x)\right)\right]\int_{\Gamma_\alpha}|\nabla_{\alpha,0}\eta_\alpha|^2dA_{\alpha,0}+\mathcal{Q}_\alpha(\eta_\alpha),$$

where

$$|\mathcal{Q}_\alpha(\eta_\alpha)|\lesssim\left[\varepsilon^2+A\left(r+60|\log\varepsilon|^2;x\right)^{\frac32}+\varepsilon^{\frac16}A\left(r+60|\log\varepsilon|^2;x\right)\right]\int_{\Gamma_\alpha}\eta_\alpha^2dA_{\alpha,0}.$$

Proof. A direct differentiation shows that in Fermi coordinates with respect to Γ_α,

$$\nabla_{\alpha,z}\varphi_\alpha(y,z)=g'_\alpha(y,z)\nabla_{\alpha,z}\eta_\alpha(y)+(-1)^{\alpha+1}\eta_\alpha(y)g''_\alpha(y,z)\nabla_{\alpha,z}h_\alpha(y).$$

Hence

$$
\begin{aligned}
&\int_{B_{r+8|\log\varepsilon|}(x)} \left|\nabla_{\alpha,z}\varphi_\alpha(y,z)\right|^2 \\
= \ & \underbrace{\int_{-\infty}^{+\infty}\int_{\Gamma_\alpha} |\nabla_{\alpha,z}\eta_\alpha|^2 \left|g'_\alpha\right|^2 \lambda_\alpha dydz}_{I} + \underbrace{\int_{-\infty}^{+\infty}\int_{\Gamma_\alpha} \eta_\alpha^2 |\nabla_{\alpha,z}h_\alpha|^2 \left|g''_\alpha\right|^2 \lambda_\alpha dydz}_{II} \\
+ \ & \underbrace{2(-1)^{\alpha+1}\int_{-\infty}^{+\infty}\int_{\Gamma_{\alpha,z}\cap B_r(x)} \eta_\alpha \left(\nabla_{\alpha,z}\eta_\alpha\cdot\nabla_{\alpha,z}h_\alpha\right) g'_\alpha g''_\alpha \lambda_\alpha dydz}_{III}.
\end{aligned}
$$

These three integrals are estimated in the following way.

(1) By (3.8), we have

$$
|\nabla_{\alpha,z}\eta_\alpha|^2 = \left[1+O\left(\varepsilon|z|\right)\right]|\nabla_{\alpha,0}\eta_\alpha|^2
$$

and

$$
\lambda_\alpha(y,z) = \lambda_\alpha(y,0) + O\left(\varepsilon|z|\right). \tag{8.1}
$$

Hence by the exponential decay of g'_α at infinity, we get

$$
\begin{aligned}
I &= \int_{\Gamma_\alpha} |\nabla_{\alpha,0}\eta_\alpha|^2 \left(\int_{-\infty}^{+\infty}\left|g'_\alpha\right|^2 dz\right) dA_{\alpha,0} \\
&+ O\left(\varepsilon\int_{-\infty}^{+\infty}\int_{\Gamma_\alpha} |\nabla_{\alpha,0}\eta_\alpha|^2 |z| \left|g'_\alpha\right|^2 \lambda_\alpha dydz\right) \\
&= \left[\sigma_0 + O(\varepsilon)\right]\int_{\Gamma_\alpha} |\nabla_{\alpha,0}\eta_\alpha(y)|^2 dA_{\alpha,0}.
\end{aligned}
$$

(2) By (6.1) and Lemma 4.6,

$$
II \lesssim \left[\varepsilon^2 + A(r+60|\log\varepsilon|^2;x)^2\right]\int_{\Gamma_\alpha} \eta_\alpha^2 dA_{\alpha,0}.
$$

(3) Integrating by parts on $\Gamma_{\alpha,z}$ leads to

$$
\begin{aligned}
III &= (-1)^\alpha \int_{-\infty}^{+\infty}\int_{\Gamma_{\alpha,z}} \eta_\alpha^2 \Delta_{\alpha,z} h_\alpha g'_\alpha g''_\alpha \lambda_\alpha dydz \\
&+ \int_{-\infty}^{+\infty}\int_{\Gamma_{\alpha,z}} \eta_\alpha^2 |\nabla_{\alpha,z}h_\alpha|^2 \left(\left|g''_\alpha\right|^2 + g'_\alpha g'''_\alpha\right)\lambda_\alpha dydz.
\end{aligned}
$$

By Corollary 7.5 we get

$$
|III| \lesssim \left[\varepsilon^2 + A\left(r+60|\log\varepsilon|^2;x\right)^{\frac{3}{2}} + \varepsilon^{\frac{1}{6}} A\left(r+60|\log\varepsilon|^2;x\right)\right]\int_{\Gamma_\alpha} \eta_\alpha^2 dA_{\alpha,0}.
$$

Putting all of these together we finish the proof. □

8.2. **The normal part.** As before we have

$$
\partial_z\varphi_\alpha(y,z) = (-1)^\alpha \eta_\alpha(y) g''_\alpha(y,z).
$$

Integrating by parts in z we get

$$
\int_{B_{r+8|\log\varepsilon|}(x)} |\partial_z\varphi_\alpha|^2
$$

$$
\begin{aligned}
&= \int_{\Gamma_\alpha}\int_{-\infty}^{+\infty} \eta_\alpha(y)^2 \big|g_\alpha''(y,z)\big|^2 \lambda_\alpha(y,z)dzdy \\
&= \underbrace{-\int_{\Gamma_\alpha}\eta_\alpha(y)^2\left[\int_{-\infty}^{+\infty} W''(g_\alpha(y,z))\big|g_\alpha'(y,z)\big|^2\lambda_\alpha(y,z)dz\right]dy}_{I} \\
&+ \underbrace{\int_{\Gamma_\alpha}\eta_\alpha(y)^2\left[\frac{1}{2}\int_{-\infty}^{+\infty}\big|g_\alpha'(y,z)\big|^2\partial_{zz}\lambda_\alpha(y,z)dz - \int_{-\infty}^{+\infty} g_\alpha'(y,z)\xi_\alpha'(y,z)\lambda_\alpha(y,z)dz\right]dy}_{II}.
\end{aligned}
$$

By (3.4) and the definition of λ_α we have

$$|\partial_{zz}\lambda_\alpha(y,z)| \lesssim |A_\alpha(y)|^2 \lesssim \varepsilon^2.$$

Using this together with estimates on ξ_α we get

$$|II| \lesssim \varepsilon^2 \int_{\Gamma_\alpha}\eta_\alpha^2 dA_{\alpha,0}.$$

It remains to determine the integral

$$\int_{\Gamma_\alpha}\eta_\alpha(y)^2\int_{-\infty}^{+\infty}\left[W''\left(u(y,z)\right) - W''\left(g_\alpha(y,z)\right)\right]\big|g_\alpha'(y,z)\big|^2\lambda_\alpha(y,z)dzdy,$$

which will be the goal of next subsection.

8.3. **The interaction part.** Multiplying (4.11) by $\eta_\alpha^2 g_\alpha''\lambda_\alpha$ and then integrating in y and z gives

$$I - II + III = IV + V - VI + VII - VIII,$$

where

$$I := \int_{\Gamma_\alpha}\eta_\alpha^2\left[\int_{-\infty}^{+\infty}\Delta_{\alpha,z}\phi g_\alpha''\lambda_\alpha dz\right]dy,$$

$$II := \int_{\Gamma_\alpha}\eta_\alpha^2\left[\int_{-\infty}^{+\infty}H_\alpha(y,z)\phi_z g_\alpha''\lambda_\alpha dz\right]dy,$$

$$III := \int_{\Gamma_\alpha}\eta_\alpha^2\left[\int_{-\infty}^{+\infty}\phi_{zz} g_\alpha''\lambda_\alpha dz\right]dy,$$

$$IV := \int_{\Gamma_\alpha}\eta_\alpha^2\left[\int_{-\infty}^{+\infty}\left(W'(u) - \sum_\beta W'(g_\beta)\right)g_\alpha''\lambda_\alpha dz\right]dy,$$

$$V := (-1)^\alpha\int_{\Gamma_\alpha}\eta_\alpha^2\left[\int_{-\infty}^{+\infty}\left(H_\alpha(y,z) + \Delta_{\alpha,z}h_\alpha(y)\right)g_\alpha' g_\alpha''\lambda_\alpha dz\right]dy,$$

$$VI := \int_{\Gamma_\alpha}\eta_\alpha^2\left[\int_{-\infty}^{+\infty}|g_\alpha''||\nabla_{\alpha,z}h_\alpha|^2\lambda_\alpha dz\right]dy,$$

$$VII := \sum_{\beta\neq\alpha}\int_{\Gamma_\alpha}\eta_\alpha^2\left[\int_{-\infty}^{+\infty}\left[(-1)^\beta g_\beta'\mathcal{R}_{\beta,1} - g_\beta''\mathcal{R}_{\beta,2}\right]g_\alpha''\lambda_\alpha dz\right]dy,$$

$$VIII := \sum_{\beta\neq\alpha}\int_{\Gamma_\alpha}\eta_\alpha^2\left[\int_{-\infty}^{+\infty}\sum_\beta \xi_\beta g_\alpha''\lambda_\alpha dz\right]dy,$$

We need to estimate each of them.

(1) By Proposition 7.1,

$$|I| \lesssim \left[\varepsilon^2 + A\left(r+60|\log\varepsilon|^2;x\right)^{\frac{3}{2}} + \varepsilon^{\frac{1}{6}} A\left(r+60|\log\varepsilon|^2;x\right)\right] \int_{\Gamma_\alpha} \eta_\alpha^2 dA_{\alpha,0}.$$

(2) Because $H_\alpha = O(\varepsilon)$, by (6.1),

$$\begin{aligned} |II| &\lesssim \varepsilon \|\phi\|_{C^{2,\theta}(B_{r+8|\log\varepsilon|}(x))} \int_{\Gamma_\alpha} \eta_\alpha^2 dA_{\alpha,0} \\ &\lesssim \left[\varepsilon^2 + A\left(r+60|\log\varepsilon|^2;x\right)^2\right] \int_{\Gamma_\alpha} \eta_\alpha^2 dA_{\alpha,0}. \end{aligned}$$

(3) Integrating by parts in z gives

$$\begin{aligned} III &= -\int_{\Gamma_\alpha} \eta_\alpha^2 \left[\int_{-\infty}^{+\infty} \phi_z g_\alpha''' \lambda_\alpha dz\right] dy - \int_{\Gamma_\alpha} \eta_\alpha^2 \left[\int_{-\infty}^{+\infty} \phi_z g_\alpha'' \partial_z \lambda_\alpha dz\right] dy \\ &= -\int_{\Gamma_\alpha} \eta_\alpha^2 \left[\int_{-\infty}^{+\infty} W''(g_\alpha) g_\alpha' \phi_z \lambda_\alpha dz\right] dy \\ &- \int_{\Gamma_\alpha} \eta_\alpha^2 \left[\int_{-\infty}^{+\infty} \phi_z \xi_\alpha' \lambda_\alpha dz\right] dy - \int_{\Gamma_\alpha} \eta_\alpha^2 \left[\int_{-\infty}^{+\infty} \phi_z g_\alpha'' \partial_z \lambda_\alpha dz\right] dy. \end{aligned}$$

Because $\xi_\alpha = O\left(\varepsilon^3\right)$, the length $|\{z : \xi_\alpha'(\cdot,z)| \neq 0\}| \lesssim |\log\varepsilon|$ and $\partial_z\lambda_\alpha = O(\varepsilon)$ (see (3.4) and the definition of λ_α), using (6.1) and reasoning as in the previous case we obtain

$$\begin{aligned} III &= -\int_{\Gamma_\alpha} \eta_\alpha^2 \left[\int_{-\infty}^{+\infty} W''(g_\alpha) g_\alpha' \phi_z \lambda_\alpha dz\right] dy \\ &+ O\left(\varepsilon^2 + A\left(r+60|\log\varepsilon|^2;x\right)^2\right) \int_{\Gamma_\alpha} \eta_\alpha^2 dA_{\alpha,0}. \end{aligned} \tag{8.2}$$

(4) Integrating by parts in z leads to

$$\begin{aligned} IV &= -\int_{\Gamma_\alpha} \eta_\alpha^2 \int_{-\infty}^{+\infty} W''(u) g_\alpha' \phi_z \lambda_\alpha dz dy \\ &- (-1)^\alpha \int_{\Gamma_\alpha} \eta_\alpha^2 \int_{-\infty}^{+\infty} \left[W''(u) - W''(g_\alpha)\right] \left|g_\alpha'\right|^2 \lambda_\alpha dz dy \\ &- \sum_{\beta\neq\alpha} (-1)^\beta \int_{\Gamma_\alpha} \eta_\alpha^2 \underbrace{\int_{-\infty}^{+\infty} \left[W''(u) - W''(g_\beta)\right] g_\alpha' g_\beta' \left(\frac{\partial d_\beta}{\partial z} - \frac{\partial}{\partial z}\left(h_\beta \circ \Pi_\beta\right)\right) \lambda_\alpha dz dy}_{IX_\beta} \\ &- \int_{\Gamma_\alpha} \eta_\alpha^2 \underbrace{\int_{-\infty}^{+\infty} \left(W'(u) - \sum_\beta W'(g_\beta)\right) g_\alpha' \partial_z \lambda_\alpha dz dy}_{X}. \end{aligned}$$

The first integral cancel with III (see (8.2)) up to a higher order term. The second integral is the one we want to rewrite in Subsection 8.2.

Estimate on X. First let us estimate the term X. By Taylor expansion we have

$$W'(u) - \sum_\beta W'(g_\beta) = \mathcal{I} + O(\phi) \tag{8.3}$$

Then by (6.1), Lemma 4.4 and the fact that $\partial_z\lambda_\alpha = O(\varepsilon)$, we get

$$|X| \lesssim \varepsilon^2 + A\left(r+60|\log\varepsilon|^2;x\right)^2. \tag{8.4}$$

Estimate on IX_β. It remains to rewrite the integral IX_β. First replace $W''(u)$ by $W''(g_*)$. This introduces an error bounded by

$$\begin{aligned} & O\left(\|\phi\|_{C^{2,\theta}(B_{r+8|\log\varepsilon|}(x))}\right)\int_{-\infty}^{+\infty} g_\alpha' g_\beta'\lambda_\alpha dz \\ \lesssim\ & \varepsilon^2 + A\left(r+60|\log\varepsilon|^2;x\right)^{3/2}. \end{aligned} \tag{8.5}$$

Next, if $g_\alpha'\neq 0$ and $g_\beta'\neq 0$ at the same time, by Lemma 3.4,

$$\frac{\partial d_\beta}{\partial z} = 1 + O\left(\varepsilon^{1/6}\right),$$

$$d_\beta(y,z) = d_\beta(y,0) \pm z + O\left(\varepsilon^{1/3}\right). \tag{8.6}$$

Replace $\frac{\partial d_\beta}{\partial z}$ by 1 and throw away the term involving h_β. This introduces another error controlled by

$$\begin{aligned} & \left[\varepsilon^{\frac16} + A(r+60|\log\varepsilon|^2;x)\right]\int_{-\infty}^{+\infty}\left|W''(g_*)-W''(g_\beta)\right| g_\alpha' g_\beta'\lambda_\alpha dz \\ \lesssim\ & \left[\varepsilon^{\frac16} + A(r+60|\log\varepsilon|^2;x)\right]\int_{-\infty}^{+\infty} g_\alpha' g_\beta'\sum_\gamma g_\gamma' dz \\ \lesssim\ & \left[\varepsilon^{\frac16} A(r+60|\log\varepsilon|^2;x) + A(r+60|\log\varepsilon|^2;x)^2\right]. \end{aligned} \tag{8.7}$$

After these replacements, IX_β is changed into

$$XI_\beta := \int_{-\infty}^{+\infty}\left[W''(g_*)-W''(g_\beta)\right] g_\alpha' g_\beta'\lambda_\alpha dz.$$

Lemma 8.3. *We have*

$$\begin{aligned} XI_{\alpha+1} &= -2A_{(-1)^\alpha}^2 e^{-|d_{\alpha+1}(y,0)|} + O\left(\varepsilon^2\right) \\ &+ O\left(\varepsilon^{\frac14} A(r+60|\log\varepsilon|^2;x)\right) + O\left(A(r+60|\log\varepsilon|^2;x)^{\frac76}\right), \end{aligned} \tag{8.8}$$

$$\begin{aligned} XI_{\alpha-1} &= -2A_{(-1)^{\alpha-1}}^2 e^{-|d_{\alpha-1}(y,0)|} + O\left(\varepsilon^2\right) \\ &+ O\left(\varepsilon^{\frac14} A(r+60|\log\varepsilon|^2;x)\right) + O\left(A(r+60|\log\varepsilon|^2;x)^{\frac76}\right). \end{aligned} \tag{8.9}$$

and for any $|\beta-\alpha|\geq 2$,

$$|XI_\beta| \lesssim e^{-C|\beta-\alpha|} A(r+60|\log\varepsilon|^2;x)^{\frac32}. \tag{8.10}$$

Proof. First note that $XI_\beta\neq 0$ only if $|d_\beta(y,0)|\leq 20|\log\varepsilon|$. In the following we will assume this is always the case.

Case 1. The case $|\beta-\alpha|\geq 2$. If $|\beta-\alpha|\geq 2$, by Lemma 3.4 we have

$$|d_\beta(y,0)| \geq 2\min_\gamma \min_{\Gamma_\gamma\cap B_{r+16|\log\varepsilon|}(x)} D_\gamma + C|\beta-\alpha| - C. \tag{8.11}$$

Since we always have

$$\int_{-\infty}^{+\infty} g'_\alpha(y,z)g'_\beta(y,z)dz \lesssim |d_\beta(y,0)|e^{-|d_\beta(y,0)|}, \tag{8.12}$$

we obtain (8.10) by substituting (8.11) into (8.12).

Case 2. The case $\beta = \alpha \pm 1$. We only consider the case $\beta = \alpha + 1$. By (8.6) we can replace $g_{\alpha+1}(y,z)$ by $\bar{g}\left((-1)^{\alpha+1}\left(d_{\alpha+1}(y,0) \pm z\right)\right)$. We can also replace $\lambda_\alpha(y,z)$ by $\lambda_\alpha(y,0)$. These two procedures lead to an error controlled by

$$\begin{aligned} O\left(\varepsilon^{\frac{1}{3}}\right)\int_{-\infty}^{+\infty} g'_\alpha(y,z)g'_{\alpha+1}(y,z)dz &\lesssim \varepsilon^{\frac{1}{3}}|\log\varepsilon|e^{-|d_{\alpha+1}(y,0)|} \\ &\lesssim \varepsilon^{\frac{1}{4}}A(r+60|\log\varepsilon|^2;x). \end{aligned} \tag{8.13}$$

Here to estimate the above integral, different from (8.12), we use the facts that the length of the interval $\{g'_\alpha(y,\cdot) \neq 0\}$ is at most $8|\log\varepsilon|$, and $g'_\alpha(y,z)g'_{\alpha+1}(y,z) \lesssim e^{-|d_{\alpha+1}(y,0)|}$.

We also note that for any $\gamma \neq \alpha, \alpha+1$,

$$\int_{-\infty}^{+\infty} g'_\alpha(y,z)g'_{\alpha+1}(y,z)g'_\gamma(y,z)dz \lesssim A(r+60|\log\varepsilon|^2;x)^2. \tag{8.14}$$

With the help of Taylor expansion

$$W''(g_*) = W''(g_\alpha + g_{\alpha+1} - 1) + \sum_{\gamma\neq\alpha,\alpha+1} O\left(g'_\gamma\right),$$

(8.14) allows us to replace $W''(g_*)$ by $W''(g_\alpha + g_{\alpha+1} - 1)$. (Recall that we have assumed $(-1)^\alpha = 1$.)

Finally, replacing $\bar{g}$ by g produces an $O(\varepsilon^2)$ error. The remaining one is the main order term in $XI_{\alpha+1}$, which is

$$\begin{aligned} &\int_{-\infty}^{+\infty}\left[W''\left(g(z) + g\left(d_{\alpha+1}(y,0) - z\right) - 1\right) - W''\left(g\left(d_{\alpha+1}(y,0) - z\right)\right)\right] \\ &\qquad \times g'(z)g'\left(d_{\alpha+1}(y,0) - z\right)dz \\ = &\; -2A_1^2 e^{-|d_{\alpha+1}(y,0)|} + O\left(e^{-\frac{7}{6}|d_{\alpha+1}(y,0)|}\right). \end{aligned}$$

where the last step follows from Lemma A.2. Combining this identity with the above estimates of various errors gives (8.8). ☐

Summing estimates in this lemma we obtain

$$\begin{aligned} \sum_{\beta\neq\alpha} IX_\beta &= -2\left[A^2_{(-1)^{\alpha-1}}e^{-d_{\alpha-1}(y,0)} + A^2_{(-1)^\alpha}e^{d_{\alpha+1}(y,0)}\right] \\ &+ O\left(\varepsilon^2 + \varepsilon^{\frac{1}{4}}A\left(r+60|\log\varepsilon|^2;x\right) + A\left(r+60|\log\varepsilon|^2;x\right)^{\frac{7}{6}}\right). \end{aligned} \tag{8.15}$$

Combining (8.4) and (8.15) we get

$$\begin{aligned} IV &= -\int_{\Gamma_\alpha}\eta_\alpha^2\int_{-\infty}^{+\infty} W''(u)g'_\alpha\phi_z\lambda_\alpha dzdy \\ &- \int_{\Gamma_\alpha}\eta_\alpha^2\int_{-\infty}^{+\infty}\left[W''(u) - W''(g_\alpha)\right]\left|g'_\alpha\right|^2\lambda_\alpha dzdy \end{aligned}$$

$$
\begin{aligned}
&- \quad 2\int_{\Gamma_\alpha}\eta_\alpha^2\left[A^2_{(-1)^{\alpha-1}}e^{-d_{\alpha-1}}+A^2_{(-1)^\alpha}e^{d_{\alpha+1}}\right]dA_{\alpha,0}\\
&+ \quad O\left(\varepsilon^2+\varepsilon^{\frac{1}{7}}A\left(r+60|\log\varepsilon|^2;x\right)+A\left(r+60|\log\varepsilon|^2;x\right)^{\frac{7}{6}}\right)\int_{\Gamma_\alpha}\eta_\alpha^2 dA_{\alpha,0}.
\end{aligned}
$$

(5) Integrating by parts in z leads to

$$
\begin{aligned}
V &= \frac{(-1)^{\alpha+1}}{2}\int_{\Gamma_\alpha}\eta_\alpha^2\left[\int_{-\infty}^{+\infty}\frac{\partial}{\partial z}\left(H_\alpha(y,z)+\Delta_{\alpha,z}h_\alpha(y)\right)\left|g_\alpha'\right|^2\lambda_\alpha dz\right]dy\\
&+ \frac{(-1)^{\alpha+1}}{2}\int_{\Gamma_\alpha}\eta_\alpha^2\left[\int_{-\infty}^{+\infty}\left(H_\alpha(y,z)+\Delta_{\alpha,z}h_\alpha(y)\right)\left|g_\alpha'\right|^2\partial_z\lambda_\alpha dz\right]dy.
\end{aligned}
$$

By (3.5),

$$
\frac{\partial}{\partial z}H_\alpha(y,z)=O\left(\varepsilon^2\right).
$$

By (3.4), (3.11), (6.1) and Lemma 4.6,

$$
\begin{aligned}
\left|\frac{\partial}{\partial z}\Delta_{\alpha,z}h_\alpha(y)\right| &\lesssim \varepsilon\left(\left|\nabla^2_{\alpha,0}h_\alpha(y)\right|+\left|\nabla_{\alpha,0}h_\alpha(y)\right|\right)\\
&\lesssim \varepsilon^2+A\left(r+60|\log\varepsilon|^2;x\right)^2.
\end{aligned}
$$

Finally, because $\partial_z\lambda_\alpha=O(\varepsilon)$, we have

$$
\begin{aligned}
&\int_{-\infty}^{+\infty}\left(H_\alpha(y,z)+\Delta_{\alpha,z}h_\alpha(y)\right)\left|g_\alpha'\right|^2\partial_z\lambda_\alpha dz\\
=& \int_{-\infty}^{+\infty}\left(H_\alpha(y,0)+\Delta_{\alpha,0}h_\alpha(y)+O\left(\varepsilon^2|z|\right)+O\left(\varepsilon|z|\right)\right)\left|g_\alpha'\right|^2\partial_z\lambda_\alpha dz\\
=& O\left(\varepsilon^2+A\left(r+60|\log\varepsilon|^2;x\right)^2\right). \quad \text{(By (5.1))}
\end{aligned}
$$

Combining these three estimates we see

$$
|V|\lesssim\left[\varepsilon^2+A\left(r+60|\log\varepsilon|^2;x\right)^2\right]\int_{\Gamma_\alpha}\eta_\alpha^2 dA_{\alpha,0}.
$$

(6) By Lemma 4.6 and (6.1), we get

$$
|VI|\lesssim\left[\varepsilon^2+A\left(r+60|\log\varepsilon|^2;x\right)^{\frac{3}{2}}\right]\int_{\Gamma_\alpha}\eta_\alpha^2 dA_{\alpha,0}.
$$

(7) Following the proof of (B.8), we get a similar estimate on

$$
\int_{-\infty}^{+\infty}\left[(-1)^\beta g_\beta'\mathcal{R}_{\beta,1}-g_\beta''\mathcal{R}_{\beta,2}\right]g_\alpha''\lambda_\alpha dz,
$$

which with Proposition 6.1 then gives

$$
|VII|\lesssim\left[\varepsilon^2+A\left(r+60|\log\varepsilon|^2;x\right)^{\frac{3}{2}}\right]\int_{\Gamma_\alpha}\eta_\alpha^2 dA_{\alpha,0}.
$$

(8) Finally, by the definition of ξ_β and Lemma 3.6, and because $\{g_\alpha'(y,\cdot)\neq 0\}$ has length at most $16|\log\varepsilon|$, we obtain

$$
|VIII|\lesssim\varepsilon^3|\log\varepsilon|\int_{\Gamma_\alpha}\eta_\alpha^2 dA_{\alpha,0}\lesssim\varepsilon^2\int_{\Gamma_\alpha}\eta_\alpha^2 dA_{\alpha,0}.
$$

Combining all of these estimates together, we obtain

$$\begin{aligned}
& \int_{\Gamma_\alpha}\eta_\alpha^2\int_{-\infty}^{+\infty}\left[W''(u)-W''(g_\alpha)\right]\left|g_\alpha'\right|^2\lambda_\alpha dzdy \\
= & -\int_{\Gamma_\alpha}\eta_\alpha^2\int_{-\infty}^{+\infty}\left[W''(u)-W''(g_\alpha)\right]g_\alpha'\phi_z\lambda_\alpha dzdy \\
- & \ 2\int_{\Gamma_\alpha}\eta_\alpha^2\left[A^2_{(-1)^{\alpha-1}}e^{-d_{\alpha-1}}+A^2_{(-1)^\alpha}e^{d_{\alpha+1}}\right]dA_{\alpha,0} \\
+ & \ O\left(\varepsilon^2+\varepsilon^{\frac14}A\left(r+60|\log\varepsilon|^2;x\right)+A\left(r+60|\log\varepsilon|^2;x\right)^{\frac76}\right)\int_{\Gamma_\alpha}\eta_\alpha^2 dA_{\alpha,0}.
\end{aligned}$$

The first integral in the right hand side of this equation is estimated in the following way. As in (8.3) and Lemma 4.4,

$$\left|W''(u)-W''(g_\alpha)\right|\lesssim|\phi|+\sum_{\beta\neq\alpha}\left(1-g_\beta^2\right). \tag{8.16}$$

Then using (6.1) to estimate ϕ and arguing as in the proof of Case (12) in Appendix B, we see this integral is also bounded by

$$O\left(\varepsilon^2+A\left(r+60|\log\varepsilon|^2;x\right)^{\frac43}\right)\int_{\Gamma_\alpha}\eta_\alpha^2 dA_{\alpha,0}.$$

Therefore we arrive at the following form

$$\begin{aligned}
& \int_{\Gamma_\alpha}\eta_\alpha^2\int_{-\infty}^{+\infty}\left[W''(u)-W''(g_\alpha)\right]\left|g_\alpha'\right|^2\lambda_\alpha dzdy \\
= & -2\int_{\Gamma_\alpha}\eta_\alpha(y)^2\left[A^2_{(-1)^{\alpha-1}}e^{-d_{\alpha-1}}+A^2_{(-1)^\alpha}e^{d_{\alpha+1}}\right]dA_{\alpha,0} \\
+ & \ O\left(\varepsilon^2+\varepsilon^{\frac17}A\left(r+60|\log\varepsilon|^2;x\right)+A\left(r+60|\log\varepsilon|^2;x\right)^{\frac76}\right)\int_{\Gamma_\alpha}\eta_\alpha^2 dA_{\alpha,0}.
\end{aligned} \tag{8.17}$$

This completes the reduction of the normal part.

8.4. **Cross terms.** In this subsection we fix an α and estimate the integral of cross terms,

$$\sum_{\beta\neq\alpha}\int_{B_{r+8|\log\varepsilon|}(x)}\left[\nabla\varphi_\alpha\cdot\nabla\varphi_\beta+W''(u)\varphi_\alpha\varphi_\beta\right].$$

Throughout this subsection, Fermi coordinates (y,z) with respect to Γ_α will be used, and we assume $(-1)^\alpha=1$.

First is the tangential part.

Lemma 8.4. *For any $\beta\neq\alpha$,*

$$\begin{aligned}
& \Big|\int_{B_{r+8|\log\varepsilon|}(x)}\nabla_{\alpha,z}\varphi_\alpha\cdot\nabla_{\alpha,z}\varphi_\beta\Big| \\
\lesssim & \ A(r;x)^{\frac12}\left[\int_{\Gamma_\alpha}|\nabla_{\alpha,0}\eta_\alpha|^2 dA_{\alpha,0}+\int_{\Gamma_\beta}|\nabla_{\beta,0}\eta_\beta|^2 dA_{\beta,0}\right] \\
+ & \ \left(\varepsilon^2+\varepsilon^{\frac17}A(r;x)+A(r;x)^{\frac32}\right)\left[\int_{\Gamma_\alpha}\eta_\alpha^2 dA_{\alpha,0}+\int_{\Gamma_\beta}\eta_\beta^2 dA_{\beta,0}\right].
\end{aligned}$$

Proof. In Fermi coordinates with respect to Γ_α, write

$$\begin{aligned}\nabla_{\alpha,z}\varphi_\alpha\cdot\nabla_{\alpha,z}\varphi_\beta &= \left[g_\alpha'\nabla_{\alpha,z}\eta_\alpha-(-1)^\alpha\eta_\alpha g_\alpha''\nabla_{\alpha,z}h_\alpha\right]\\ &\quad\times\left[g_\beta'\nabla_{\alpha,z}\eta_\beta+(-1)^\beta\eta_\beta g_\beta''\left(\nabla_{\alpha,z}d_\beta-\nabla_{\beta,0}h_\beta\circ D_{\alpha,z}\Pi_\beta\right)\right]\end{aligned}$$

In the following we assume $\beta<\alpha$. Γ_α and Γ_β divide $B_{r+8|\log\varepsilon|}(x)$ into three domains, $\Omega^0_{\alpha\beta}$ the one between them, $\Omega^+_{\alpha\beta}$ the one above Γ_α and $\Omega^-_{\alpha\beta}$ the one below Γ_β.

Case 1. In $\Omega^0_{\alpha\beta}$,

$$g_\alpha'(y,z)g_\beta'(y,z)+g_\alpha'(y,z)\left|g_\beta''(y,z)\right|+g_\beta'(y,z)\left|g_\alpha''(y,z)\right|+\left|g_\alpha''(y,z)g_\beta''(y,z)\right|\lesssim e^{-d_\beta(y,0)}.$$

Using Lemma 4.6 and (6.1) to estimate terms involving h, using Lemma 3.4 to estimate $\nabla_{\alpha,z}d_\beta$ (note that if $g_\alpha'(y,z)g_\beta'(y,z)\neq 0$, then $d_\beta(y,0)\leq 16|\log\varepsilon|$), we get

$$\begin{aligned}&\left|\nabla_{\alpha,z}\varphi_\alpha(y,z)\cdot\nabla_{\alpha,z}\varphi_\beta(y,z)\right|\\ \lesssim\ & e^{-d_\beta(y,0)}|\nabla_{\alpha,z}\eta_\alpha||\nabla_{\alpha,z}\eta_\beta|\\ +\ &\left[\varepsilon^{\frac16}A\left(r+60|\log\varepsilon|^2;x\right)+A\left(r+60|\log\varepsilon|^2;x\right)^2\right]e^{-d_\beta(y,0)}\eta_\alpha\eta_\beta\\ +\ &\left[\varepsilon^{\frac16}+A\left(r+60|\log\varepsilon|^2;x\right)\right]e^{-d_\beta(y,0)}\eta_\alpha|\nabla_{\alpha,z}\eta_\beta|\\ +\ &\left[\varepsilon^{\frac16}+A\left(r+60|\log\varepsilon|^2;x\right)\right]e^{-d_\beta(y,0)}\eta_\beta|\nabla_{\alpha,z}\eta_\alpha|.\end{aligned}\tag{8.18}$$

Subcase 1.1. Here we show how to estimate the integral of the first term in the right hand side of (8.18). First by Lemma 3.5 we can replace $\nabla_{\alpha,z}\eta_\alpha$ by $\nabla_{\alpha,0}\eta_\alpha$. Then by Lemma 3.4 and Cauchy inequality we obtain

$$\int_{\Omega^0_{\alpha\beta}}e^{-d_\beta(y,0)}|\nabla_{\alpha,0}\eta_\alpha||\nabla_{\alpha,z}\eta_\beta|\lesssim\int_{\Omega^0_{\alpha\beta}}e^{-d_\beta(y,0)}|\nabla_{\alpha,0}\eta_\alpha|^2+\int_{\Omega^0_{\alpha\beta}}e^{-d_\beta(y,0)}|\nabla_{\alpha,z}\eta_\beta|^2.$$

Since $\Omega^0_{\alpha\beta}\subset\{(y,z):|z|<2d_\beta(y,0)\}$, the first integral is controlled by

$$\int_{\Gamma_\alpha}\left(\int_{-2d_\beta(y,0)}^0 e^{-d_\beta(y,0)}dz\right)|\nabla_{\alpha,0}\eta_\alpha|^2dA_{\alpha,0}\lesssim\left(\max_{\Gamma_\alpha\cap B_r(x)}d_\beta e^{-d_\beta}\right)\int_{\Gamma_\alpha}|\nabla_{\alpha,0}\eta_\alpha|^2dA_{\alpha,0}.$$

The second one can be estimated in the same way by writing it in Fermi coordinates with respect to Γ_β.

Subcase 1.2. To estimate the integral of $\varepsilon^{1/6}e^{-d_\beta(y,0)}\eta_\alpha\eta_\beta$, the above method needs a revision. Here we note that the domain of integration can be restricted to $\{|z|<8|\log\varepsilon|\}\cap\{|d_\beta|<8|\log\varepsilon|\}$, because otherwise g_α' or $g_\beta'=0$. Hence we have

$$\begin{aligned}&\varepsilon^{\frac16}\int_{\Omega^0_{\alpha\beta}\cap\{|z|<8|\log\varepsilon|\}\cap\{|d_\beta(y,z)|<8|\log\varepsilon|\}}e^{-d_\beta(y,0)}\eta_\alpha\eta_\beta\\ \lesssim\ &\varepsilon^{\frac16}\int_{\Omega^0_{\alpha\beta}\cap\{|z|<8|\log\varepsilon|\}}e^{-d_\beta(y,0)}\eta_\alpha^2+\varepsilon^{\frac16}\int_{\Omega^0_{\alpha\beta}\cap\{d_\beta(y,z)<8|\log\varepsilon|\}}e^{-d_\beta(y,0)}\eta_\beta^2.\end{aligned}\tag{8.19}$$

The first integral is rewritten as

$$\begin{aligned}\varepsilon^{\frac16}\int_{\Gamma_\alpha}\left(\int_{-8|\log\varepsilon|}^{8|\log\varepsilon|}e^{-d_\beta(y,0)}dz\right)\eta_\alpha^2dA_{\alpha,0}&\lesssim\varepsilon^{\frac16}|\log\varepsilon|\left(\max_{y\in\Gamma_\alpha\cap B_r(x)}e^{-d_\beta}\right)\int_{\Gamma_\alpha}\eta_\alpha^2dA_{\alpha,0}\\ &\lesssim\varepsilon^{\frac17}A(r;x)\int_{\Gamma_\alpha}\eta_\alpha^2dA_{\alpha,0}.\end{aligned}$$

The second integral in (8.19) and integrals involving $\eta_\alpha|\nabla_{\alpha,z}\eta_\beta|$ as well as $\eta_\beta|\nabla_{\alpha,z}\eta_\alpha|$ can be estimated in a similar way.

Case 2. In $\Omega^+_{\alpha\beta}$, we have

$$g'_\alpha(y,z)g'_\beta(y,z)+g'_\alpha(y,z)\left|g''_\beta(y,z)\right|+g'_\beta(y,z)\left|g''_\alpha(y,z)\right|+\left|g''_\alpha(y,z)g''_\beta(y,z)\right|\lesssim e^{-d_\beta(y,0)-z}.$$

Similar to the above case, we have a bound on $\left|\nabla_{\alpha,z}\varphi_\alpha(y,z)\cdot\nabla_{\alpha,z}\varphi_\beta(y,z)\right|$. By noting that they are nonzero only in the $8|\log\varepsilon|$ neighborhood of $\Gamma_\alpha\cup\Gamma_\beta$, we obtain

$$\begin{aligned}
&\int_{\Omega^+_{\alpha\beta}\cap\{z<8|\log\varepsilon|\}} e^{-d_\beta(y,0)-z}|\nabla_{\alpha,z}\eta_\alpha||\nabla_{\beta,z}\eta_\beta| \\
\lesssim\ & \left(\int_{\Omega^+_{\alpha\beta}\cap\{z<8|\log\varepsilon|\}} e^{-2d_\beta(y,0)-z}|\nabla_{\alpha,0}\eta_\alpha|^2\right)^{\frac12}\left(\int_{\Omega^+_{\alpha\beta}\cap\{z<8|\log\varepsilon|\}} e^{-z}|\nabla_{\beta,z}\eta_\beta|^2\right)^{\frac12} \\
\lesssim\ & \left(\int_{\Gamma_\alpha}\int_0^{8|\log\varepsilon|} e^{-2d_\beta(y,0)-z}|\nabla_{\alpha,0}\eta_\alpha(y)|^2dzdA_{\alpha,0}\right)^{\frac12} \\
& \times\left(\int_0^{8|\log\varepsilon|}\int_{\Gamma_\alpha} e^{-z}|\nabla_{\beta,z}\eta_\beta|^2dA_{\alpha,z}dz\right)^{\frac12} \\
\lesssim\ & \left(\max_{y\in\Gamma_\alpha\cap B_r(x)} e^{-d_\beta(y,0)}\right)\left(\int_{\Gamma_\alpha}|\nabla_{\alpha,0}\eta_\alpha(y)|^2dA_{\alpha,0}\right)^{\frac12}\left(\int_{\Gamma_\beta}|\nabla_{\beta,0}\eta_\beta|^2dA_{\beta,0}\right)^{\frac12}.
\end{aligned}$$

Other terms and integrals in $\Omega^-_{\alpha\beta}$ can be estimated in the same way. □

Next we estimate the normal part.

Lemma 8.5. *For any $\beta\neq\alpha$,*

$$\int_{B_{r+8|\log\varepsilon|}(x)}\partial_z\varphi_\alpha\partial_z\varphi_\beta=-\int_{\Gamma_\alpha}\int_{-\infty}^{+\infty}\eta_\alpha\eta_\beta W''(g_\beta)g'_\alpha g'_\beta\lambda_\alpha dzdy+\mathcal{Q}_{\alpha,\beta}(\eta),$$

where

$$\begin{aligned}
|\mathcal{Q}_{\alpha,\beta}(\eta)| \ \lesssim\ & \left[\varepsilon^2+\varepsilon^{\frac17}A(r;x)+A(r;x)^{\frac32}\right]\left[\int_{\Gamma_\alpha}\eta_\alpha^2dA_{\alpha,0}+\int_{\Gamma_{\beta,0}}\eta_\beta^2dA_{\beta,0}\right] \\
+\ & \varepsilon^{\frac17}A(r;x)\int_{\Gamma_{\beta,0}}|\nabla_{\beta,0}\eta_\beta|^2dA_{\beta,0}.
\end{aligned}$$

Proof. Using Fermi coordinates with respect to Γ_α, we have

(8.20)
$$\partial_z\varphi_\alpha\partial_z\varphi_\beta=(-1)^\beta\eta_\alpha\eta_\beta g''_\alpha g''_\beta\left[\frac{\partial d_\beta}{\partial z}-\nabla h_\beta\cdot\frac{\partial\Pi_\beta}{\partial z}\right]+(-1)^\alpha\eta_\alpha g''_\alpha g'_\beta\left(\nabla_{\beta,0}\eta_\beta\cdot\frac{\partial\Pi_\beta}{\partial z}\right).$$

By Lemma 3.4, if $g''_\alpha g'_\beta\neq0$ or $g''_\alpha g''_\beta\neq0$, then

$$\left|\frac{\partial d_\beta}{\partial z}-1\right|+\left|\frac{\partial\Pi_\beta}{\partial z}\right|\lesssim\varepsilon^{1/6}, \tag{8.21}$$

We can proceed as in the proof of Lemma 8.4 to estimate the integral of

$$\eta_\alpha g''_\alpha\eta_\beta g''_\beta\left(\frac{\partial d_\beta}{\partial z}-1-\nabla h_\beta\cdot\frac{\partial\Pi_\beta}{\partial z}\right)+\eta_\alpha g''_\alpha g'_\beta\left(\nabla_{\beta,0}\eta_\beta\cdot\frac{\partial\Pi_\beta}{\partial z}\right).$$

It remains to determine the integral

$$\int_{B_{r+8|\log\varepsilon|}(x)}\eta_\alpha\eta_\beta g''_\alpha g''_\beta.$$

Write this in Fermi coordinates with respect to Γ_α. Integrating by parts in z leads to

$$\begin{aligned}
&\int_{\Gamma_\alpha}\int_{-\infty}^{+\infty}\eta_\alpha\eta_\beta g''_\alpha g''_\beta\lambda_\alpha dzdy\\
= & \ (-1)^{\beta+1}\int_{\Gamma_\alpha}\int_{-\infty}^{+\infty}\eta_\alpha\eta_\beta W''\left(g_\beta\right)g'_\alpha g'_\beta\lambda_\alpha-\underbrace{\int_{\Gamma_\alpha}\int_{-\infty}^{+\infty}\eta_\alpha\left(\nabla_{\beta,0}\eta_\beta\cdot\frac{\partial\Pi_\beta}{\partial z}\right)g'_\alpha g''_\beta\lambda_\alpha}_{I}\\
- & \ (-1)^{\beta}\underbrace{\int_{\Gamma_\alpha}\int_{-\infty}^{+\infty}\eta_\alpha\eta_\beta g'_\alpha\xi'_\beta\lambda_\alpha}_{II}-\underbrace{\int_{\Gamma_\alpha}\int_{-\infty}^{+\infty}\eta_\alpha\eta_\beta g'_\alpha g''_\beta\partial_z\lambda_\alpha dz}_{III}.
\end{aligned}$$

When $g'_\alpha g''_\beta\neq 0$, by Lemma 3.4,

$$\left|\frac{\partial}{\partial z}\eta_\beta\right|\leq\varepsilon^{\frac{1}{6}}|\nabla_{\beta,0}\eta_\beta|.$$

Hence as in the proof of Lemma 8.4 (here it is useful to observe that $g'_\alpha g''_\beta=0$ outside the $8|\log\varepsilon|$ neighborhood of $\Gamma_\alpha\cup\Gamma_\beta$), we get

$$|I|\lesssim\varepsilon^{\frac{1}{7}}A(r;x)\left[\int_{\Gamma_\alpha}\eta_\alpha^2dA_{\alpha,0}+\int_{\Gamma_\beta}\left|\nabla_{\beta,0}\eta_\beta\right|^2dA_{\beta,0}\right]. \tag{8.22}$$

By the definition of ξ_β, we also have

$$|II|\lesssim\varepsilon^2\left[\int_{\Gamma_\alpha}\eta_\alpha^2dA_{\alpha,0}+\int_{\Gamma_\beta}\eta_\beta^2dA_{\beta,0}\right]. \tag{8.23}$$

Because $\partial_z\lambda_\alpha=O(\varepsilon)$, we get

$$|III|\lesssim\varepsilon|\log\varepsilon|A(r;x)\left[\int_{\Gamma_\alpha}\eta_\alpha^2dA_{\alpha,0}+\int_{\Gamma_\beta}\eta_\beta^2dA_{\beta,0}\right]. \tag{8.24}$$

The conclusion follows by combining (8.22)-(8.24). □

Now we determine the integral in the previous lemma.

Lemma 8.6. *We have*

$$\begin{aligned}
&\sum_{\beta\neq\alpha}\int_{B_{r+8|\log\varepsilon|}(x)}\eta_\alpha\eta_\beta\left[W''(u)-W''(g_\beta)\right]g'_\alpha g'_\beta\\
= & \ -2A^2_{(-1)^\alpha}\int_{\Gamma_\alpha}\eta_\alpha(y)\eta_{\alpha+1}\left(\Pi_{\alpha+1}(y,0)\right)e^{-|d_{\alpha+1}(y,0)|}dA_{\alpha,0}\\
- & \ 2A^2_{(-1)^{\alpha-1}}\int_{\Gamma_\alpha}\eta_\alpha(y)\eta_{\alpha-1}\left(\Pi_{\alpha-1}(y,0)\right)e^{-|d_{\alpha-1}(y,0)|}dA_{\alpha,0}\\
+ & \ O\left(\varepsilon^{\frac{1}{7}}A(r;x)\right)\sum_{\beta\neq\alpha}\int_{\Gamma_\beta}|\nabla_{\beta,0}\eta_\beta|^2dA_{\beta,0}
\end{aligned}$$

$$+ \quad O\left(\varepsilon^2+\varepsilon^{\frac{1}{7}}A(r;x)+A\left(r+60|\log\varepsilon|^2;x\right)^{\frac{7}{6}}\right)\sum_\beta\int_{\Gamma_\beta}\eta_\beta^2 dA_{\beta,0}.$$

Proof. The proof is divided into three steps.

Step 1. First by Taylor expansion and (6.1), proceeding as in the proof of Lemma 8.4 we get

$$\begin{aligned}
&\Big|\int_{B_{r+8|\log\varepsilon|}(x)}\left[W''(u)-W''(g_*)\right]\varphi_\alpha\varphi_\beta\Big|\\
\lesssim\ & \|\phi\|_{L^\infty(B_{r+8|\log\varepsilon|}(x))}\int_{B_{r+8|\log\varepsilon|}(x)}\eta_\alpha\eta_\beta g'_\alpha g'_\beta\\
\lesssim\ & \left[\varepsilon^2+A\left(r+60|\log\varepsilon|^2;x\right)^{\frac{3}{2}}\right]\left[\int_{\Gamma_\alpha}\eta_\alpha^2 dA_{\alpha,0}+\int_{\Gamma_\beta}\eta_\beta^2 dA_{\beta,0}\right].
\end{aligned}$$

Step 2. Next by Lemma 3.6 we have

$$\begin{aligned}
&\int_{\Gamma_\alpha}\eta_\alpha(y)\left(\int_{-\infty}^{+\infty}\left|\eta_\beta\left(\Pi_\beta(y,0)\right)-\eta_\beta\left(\Pi_\beta(y,z)\right)\right|\left[W''(g_*)-W''(g_\beta)\right]g'_\beta g'_\alpha\lambda_\alpha dz\right)dy\\
\lesssim\ & \int_0^1\int_{\Gamma_\alpha}\eta_\alpha(y)\left(\int_{-\infty}^{+\infty}\left|\frac{d}{dt}\eta_\beta\left((1-t)\Pi_\beta(y,0)+t\Pi_\beta(y,z)\right)\right|g'_\beta g'_\alpha\lambda_\alpha dz\right)dy\\
\lesssim\ & \varepsilon^{\frac{1}{6}}\int_0^1\int_{\Gamma_\alpha}\eta_\alpha(y)\left(\int_{-\infty}^{+\infty}\left|\nabla_{\beta,0}\eta_\beta\left((1-t)\Pi_\beta(y,0)+t\Pi_\beta(y,z)\right)\right|g'_\beta g'_\alpha\lambda_\alpha dz\right)dy\\
\lesssim\ & \varepsilon^{\frac{1}{6}}|\log\varepsilon|A(r;x)\left[\int_{\Gamma_\alpha}\eta_\alpha^2 dA_{\alpha,0}+\int_{\Gamma_\beta}\left|\nabla_{\beta,0}\eta_\beta\right|^2 dA_{\beta,0}\right].
\end{aligned}$$

Here we have used the fact that the length of $\{g'_\alpha(y,\cdot)\neq 0\}$ and $\{g'_\beta(y,\cdot)\neq 0\}$ are not larger than $16|\log\varepsilon|$.

Step 3. By the previous two steps, we are left with the integral

$$\sum_{\beta\neq\alpha}\int_{\Gamma_\alpha}\eta_\alpha(y)\eta_\beta\left(\Pi_\beta(y,0)\right)\int_{-\infty}^{+\infty}\left[W''(g_*(y,z))-W''(g_\beta(y,z))\right]g'_\beta(y,z)g'_\alpha(y,z)dzdA_{\alpha,0}.$$

The integrals

$$\int_{-\infty}^{+\infty}\left[W''(g_*(y,z))-W''(g_\beta(y,z))\right]g'_\beta(y,z)g'_\alpha(y,z)dz.$$

can be determined by Lemma 8.3.

Combining estimates in these three steps we finish the proof. □

9. A decay estimate

Recall the definition of $A(r;x)$ in Section 3. In this section we establish the following decay estimate for $A(r;x)$.

Proposition 9.1. *There exist two universal constants $M\gg K\gg 1$ such that for any $r\in[2R/3,5R/6]$, if*

$$\kappa := A(r;0)\geq M\varepsilon^2|\log\varepsilon|, \tag{9.1}$$

then

$$A\left(r-KR_*;0\right)\leq\frac{1}{2}A(r;0),$$

where

$$R_* := \max\left\{\kappa^{-\frac{1}{2}}, 200|\log\varepsilon|^2\right\}.$$

The constants M and K will be determined in Lemma 9.7 (according to the requirements in Lemma 9.6).

9.1. **Reduction to a decay estimate for Toda system.** In this subsection we reduce the proof of Proposition 9.1 to a decay estimate for Toda system.

By (5.2) and Corollary 7.5, for any β and $y \in \Gamma_\beta \cap B_{r-R_*}(0)$,

$$H_\beta(y,0) = \frac{2A^2_{(-1)^{\beta-1}}}{\sigma_0} e^{-d_{\beta-1}(y,0)} - \frac{2A^2_{(-1)^{\beta}}}{\sigma_0} e^{d_{\beta+1}(y,0)} + O\left(\kappa^{7/6}\right) + O\left(\varepsilon^2\right). \tag{9.2}$$

Take an arbitrary index α and $x \in \Gamma_\alpha \cap B_{r-KR_*}(0)$. To prove Proposition 9.1, it suffices to show that

$$e^{-D_\alpha(x)} \le \frac{\kappa}{2}. \tag{9.3}$$

After a rotation and a translation, assume $x = 0$. In the finite cylinder $\mathcal{C}_{R/30}(0) := B^{n-1}_{R/30}(0) \times (-R/30, R/30)$, Γ_α is represented by the graph $\{x_n = f_\alpha(y)\}$, where $y \in B^{n-1}_{R/30}(0)$. Without loss of generality assume

$$f_\alpha(0) = 0, \quad \nabla f_\alpha(0) = 0. \tag{9.4}$$

In the following, we also assume

$$|d_{\alpha-1}(0)| \ge d_{\alpha+1}(0), \quad \text{and} \quad d_{\alpha+1}(0) \le 2|\log\varepsilon|. \tag{9.5}$$

Then by Lemma 2.2, we get a function $f_{\alpha+1}$ such that

$$\Gamma_{\alpha+1} \cap \mathcal{C}_{R/30}(0) = \left\{x_n = f_{\alpha+1}(x')\right\}.$$

Moreover, we have the Lipschitz bound

$$\|\nabla f_\alpha\|_{L^\infty(B^{n-1}_{R/30}(0))} + \|\nabla f_{\alpha+1}\|_{L^\infty(B^{n-1}_{R/30}(0))} \le C. \tag{9.6}$$

Curvature bounds on Γ_α and $\Gamma_{\alpha+1}$ are transformed into

$$\|\nabla^2 f_\alpha\|_{L^\infty(B^{n-1}_{R/30}(0))} + \|\nabla^2 f_{\alpha+1}\|_{L^\infty(B^{n-1}_{R/30}(0))} \lesssim \varepsilon. \tag{9.7}$$

By (9.4) and (9.7), for any $y \in B^{n-1}_{K\kappa^{-1/2}}(0)$,

$$|\nabla f_\alpha(y)| \lesssim \varepsilon|y| \lesssim K\varepsilon\kappa^{-1/2} \lesssim KM^{-1/2}|\log\varepsilon|^{-1/2}. \tag{9.8}$$

Concerning $f_{\alpha+1}$ we have the following estimates. Hereafter a positive constant $\delta < 1/48$ will be fixed.

Lemma 9.2. *For $y \in B^{n-1}_{K\kappa^{-1/2}}(0)$, we have*

$$|\nabla f_{\alpha+1}(y)| = O_K\left(M^{-\delta}|\log\varepsilon|^{-\delta}\right). \tag{9.9}$$

Proof. By (9.5) and (9.6) we have

$$\max_{B^{n-1}_{K\kappa^{-1/2}}(0)} (f_{\alpha+1} - f_\alpha) \le 2|\log\varepsilon| + CK\kappa^{-1/2} \lesssim KM^{-1/2}|\log\varepsilon|^{-1/2}\varepsilon^{-1}.$$

As in Lemma 3.4, combining this bound with (9.7) and noting that $f_{\alpha+1} - f_\alpha > 0$, an interpolation argument gives

$$\max_{B^{n-1}_{K\kappa^{-1/2}}(0)} |\nabla f_{\alpha+1} - \nabla f_\alpha| \lesssim C(K) M^{-\delta} |\log\varepsilon|^{-\delta}.$$

Substituting (9.8) into this estimate we get (9.9). □

The following lemma shows that $d_{\alpha+1}$ is well approximated by vertical distances. The proof uses the fact that under assumptions (9.4) and (9.5), Γ_α and $\Gamma_{\alpha+1}$ are almost parallel and horizontal.

Lemma 9.3. *For $y \in B^{n-1}_{K\kappa^{-1/2}}(0)$, if $e^{-|d_{\alpha+1}(y)|} \geq \varepsilon^2$, then*

$$e^{-|d_{\alpha+1}(y)|} = e^{-(f_{\alpha+1}(y) - f_\alpha(y))} + O_K\left(M^{-1}\kappa\right). \tag{9.10}$$

Proof. Assume the nearest point on $\Gamma_{\alpha+1}$ to $(y, f_\alpha(y))$ is $(y_*, f_{\alpha+1}(y_*))$. Because

$$d_\alpha(y_*, f_{\alpha+1}(y_*)) \leq |d_{\alpha+1}(y, f_\alpha(y))| \leq 2|\log\varepsilon|,$$

using Lemma 3.4 we deduce that

$$|\nabla f_{\alpha+1}(y_*) - \nabla f_\alpha(y)| \lesssim \varepsilon^{1/3}.$$

Combining this estimate with (9.8) and noting the fact that $M^{-1/2}|\log\varepsilon|^{-1/2} \gg \varepsilon^{1/3}$, we get

$$|\nabla f_{\alpha+1}(y_*)| \lesssim K M^{-1/2} |\log\varepsilon|^{-1/2}. \tag{9.11}$$

By the Lipschitz bound on Γ_α and $\Gamma_{\alpha+1}$ (see (9.6)), the vertical distance to $\Gamma_{\alpha+1}$ is comparable to the distance to $\Gamma_{\alpha+1}$, that is,

$$|f_{\alpha+1}(y) - f_\alpha(y)| \lesssim |d_{\alpha+1}(y, f_\alpha(y))| \lesssim |\log\varepsilon|.$$

Then by the triangle inequality we have

$$|y - y_*| \leq |d_{\alpha+1}(y, f_\alpha(y))| + |f_{\alpha+1}(y) - f_\alpha(y)| \lesssim |\log\varepsilon|.$$

By (9.7), we get

$$\mathrm{dist}\left((y, f_{\alpha+1}(y)), T_{(y_*, f_{\alpha+1}(y_*))}\Gamma_{\alpha+1}\right) \lesssim \varepsilon|\log\varepsilon|^2, \tag{9.12}$$

where $T_{(y_*, f_{\alpha+1}(y_*))}\Gamma_{\alpha+1}$ denotes the tangent hyperplane of $\Gamma_{\alpha+1}$ at $(y_*, f_{\alpha+1}(y_*))$.

Let ϑ be the angle between $N_{\alpha+1}(y_*)$ (the upward unit normal vector of $\Gamma_{\alpha+1}$ at $(y_*, f_{\alpha+1}(y_*))$ and the direction $e_{n+1} = (0, \cdots, 0, 1)$. By (9.11), we get

$$|\vartheta| \lesssim K M^{-1/2} |\log\varepsilon|^{-1/2}. \tag{9.13}$$

From this we deduce that

$$\begin{aligned} f_{\alpha+1}(y) - f_\alpha(y) &\leq |d_{\alpha+1}(y)| \left[1 + C(\sin\vartheta)^2\right] + C\varepsilon|\log\varepsilon|^2 \\ &\leq |d_{\alpha+1}(y)| + C(K)M^{-1}. \end{aligned}$$

Then by Taylor expansion and the fact that $e^{d_{\alpha+1}(y)} \leq \kappa$, we obtain (9.10). □

Remark 9.4. *Other parts of our proof require only $\kappa \geq M\varepsilon^2$ with M large enough. The stronger assumption in (9.1) is needed only in the derivation of (9.13) (through estimates in (9.8) and (9.11)), which shows how close the tangent hyperplane at y_* to the hyperplane chosen at the beginning with respect to which Γ_α and $\Gamma_{\alpha+1}$ are viewed as graphs over it.*

By this lemma, after throwing away interaction terms except the one between Γ_α and $\Gamma_{\alpha+1}$, the Toda system (9.2) is rewritten as, for any $y \in B^{n-1}_{K\kappa^{-1/2}}(0)$,

$$\begin{cases} \operatorname{div}\left(\frac{\nabla f_\alpha(y)}{\sqrt{1+|\nabla f_\alpha(y)|^2}}\right) \geq -\frac{2A_1^2}{\sigma_0} e^{-[f_{\alpha+1}(y)-f_\alpha(y)]} + O_K\left(M^{-1}\kappa\right), \\ \operatorname{div}\left(\frac{\nabla f_{\alpha+1}(y)}{\sqrt{1+|\nabla f_{\alpha+1}(y)|^2}}\right) \leq \frac{2A_1^2}{\sigma_0} e^{-[f_{\alpha+1}(y)-f_\alpha(y)]} + O_K\left(M^{-1}\kappa\right), \end{cases} \tag{9.14}$$

Here as before we have assumed $(-1)^\alpha = 1$ and we have also used the fact that if Lemma 9.3 is not applicable at a point y, then both $e^{-[f_{\alpha+1}(y)-f_\alpha(y)]}$ and $e^{-|d_{\alpha+1}(y)|}$ are of the order $O\left(\varepsilon^2\right)$, which can be incorporated into the $O_K\left(M^{-1}\kappa\right)$ term.

Taking difference in (9.14), we obtain the equation for $f_{\alpha+1} - f_\alpha$,

$$\operatorname{div}\left[\mathcal{A}_\alpha \nabla\left(f_{\alpha+1} - f_\alpha\right)\right] \leq \frac{4A_1^2}{\sigma_0} e^{-[f_{\alpha+1}(y)-f_\alpha(y)]} + O_K\left(M^{-1}\kappa\right). \tag{9.15}$$

Here $\mathcal{A}_\alpha$ is an $(n-1)\times(n-1)$ symmetric matrix with entries defined by

$$\int_0^1 \left[\frac{\delta_{ij}}{\sqrt{1+|\nabla f_\alpha^t|^2}} - \frac{\partial_i f_\alpha^t \partial_j f_\alpha^t}{\left(1+|\nabla f_\alpha^t|^2\right)^{3/2}}\right] dt, \quad 1 \leq i,j \leq n-1, \tag{9.16}$$

where $f_\alpha^t := (1-t)f_\alpha + t f_{\alpha+1}$ and δ_{ij} is the Kronecker delta symbol.

In view of (9.8) and (9.9), we have

$$|\mathcal{A}_\alpha(y) - Id| \lesssim M^{-\delta}, \quad \forall y \in B^{n-1}_{K\kappa^{-1/2}}(0). \tag{9.17}$$

Note that here we do not need the full strength of (9.8) and (9.9), but only the smallness of $|\nabla f_\alpha|$ and $|\nabla f_{\alpha+1}|$.

By (9.7), we also have

$$|\nabla \mathcal{A}_\alpha(y)| \lesssim \varepsilon, \quad \forall y \in B^{n-1}_{K\kappa^{-1/2}}(0). \tag{9.18}$$

Define the function

$$v_\alpha(y) := f_{\alpha+1}\left(\kappa^{-1/2}y\right) - f_\alpha\left(\kappa^{-1/2}y\right) - |\log\kappa|, \quad y \in B_K^{n-1}(0).$$

It satisfies

$$\operatorname{div}\left(\widetilde{\mathcal{A}}_\alpha \nabla v_\alpha\right) \leq \frac{4A_1^2}{\sigma_0} e^{-v_\alpha} + O_K\left(M^{-1}\right) \quad \text{in } B_K^{n-1}(0). \tag{9.19}$$

Here $\widetilde{\mathcal{A}}_\alpha(y) := \mathcal{A}_\alpha(\kappa^{-1/2}y)$ still satisfies (9.17) in $B_K^{n-1}(0)$. Since $\kappa \geq \varepsilon^2$, (9.18) implies that

$$|\nabla \widetilde{\mathcal{A}}_\alpha(y)| \leq C, \quad \forall y \in B_K^{n-1}(0). \tag{9.20}$$

9.2. **Completion of the proof of Proposition 9.1.** First we show that v_α is almost stable.

Lemma 9.5. *For any $\widetilde{\eta}_\alpha$ and $\widetilde{\eta}_{\alpha+1} \in C_0^\infty(B_K^{n-1}(0))$,*

$$\begin{aligned} &\left[1+O\left(M^{-\delta}\right)\right] \sum_{\beta=\alpha,\alpha+1} \int_{B_K^{n-1}(0)} |\nabla\widetilde{\eta}_\beta|^2 dy \\ \geq\ & \frac{2A_1^2}{\sigma_0} \int_{B_K^{n-1}(0)} e^{-v_\alpha}\left[\widetilde{\eta}_{\alpha+1} + \widetilde{\eta}_\alpha\right]^2 dy - O\left(M^{-\delta}\right) \sum_{\beta=\alpha,\alpha+1} \int_{B_K^{n-1}(0)} \widetilde{\eta}_\beta^2 dy. \end{aligned} \tag{9.21}$$

Proof. For $y \in B^{n-1}_{K\kappa^{-1/2}}(0)$, let $\eta_\alpha(y) := \widetilde{\eta}_\alpha\left(\kappa^{1/2}y\right)$ and $\eta_{\alpha+1}$ be defined similarly. We will view them as functions on Γ_α (respectively $\Gamma_{\alpha+1}$), by identifying y with $(y, f_\alpha(y))$ etc.

Substituting $(\cdots, 0, \eta_\alpha, \eta_{\alpha+1}, 0, \cdots)$ into Proposition 8.1 gives

$$
\begin{aligned}
&\sum_{\beta=\alpha,\alpha+1}\int_{B^{n-1}_{K\kappa^{-1/2}}(0)}|\nabla\eta_\beta|^2\left[1+O\left(|\nabla f_\beta|^2\right)\right]dy+\mathcal{Q}(\eta_\alpha,\eta_{\alpha+1})\\
\geq\;&\sum_{\beta=\alpha,\alpha+1}\frac{2A_1^2}{\sigma_0}\int_{B^{n-1}_{K\kappa^{-1/2}}(0)}e^{d_{\alpha+1}(y,0)}\left[\eta_\alpha(y)+\eta_{\alpha+1}\left(\Pi_{\alpha+1}(y,0)\right)\right]^2\left[1+O\left(|\nabla f_\beta|^2\right)\right]dy,
\end{aligned}
\tag{9.22}
$$

where

$$
\begin{aligned}
|\mathcal{Q}(\eta)| \;\lesssim\;& \left[\varepsilon^{\frac14}+\kappa^{\frac12}\right]\left(\sum_{\beta=\alpha,\alpha+1}\int_{B^{n-1}_{K\kappa^{-1/2}}(0)}|\nabla\eta_\beta(y)|^2\left[1+O\left(|\nabla f_\beta(y)|^2\right)\right]dy\right)\\
&+\left[\varepsilon^2+\kappa^{\frac76}+\varepsilon^{\frac17}\kappa\right]\left(\sum_{\beta=\alpha,\alpha+1}\int_{B^{n-1}_{K\kappa^{-1/2}}(0)}\eta_\beta(y)^2\left[1+O\left(|\nabla f_\beta(y)|^2\right)\right]dy\right)\\
\lesssim\;& \kappa^{1/8}\left(\sum_{\beta=\alpha,\alpha+1}\int_{B^{n-1}_{K\kappa^{-1/2}}(0)}|\nabla\eta_\beta(y)|^2dy\right)+\frac{\kappa}{M}\left(\sum_{\beta=\alpha,\alpha+1}\int_{B^{n-1}_{K\kappa^{-1/2}}(0)}\eta_\beta(y)^2dy\right),
\end{aligned}
$$

if ε is small enough, which also implies that κ is small enough.

If Lemma 9.3 is not applicable, the first term in the right hand side of (9.22) can be incorporated into $\mathcal{Q}(\eta)$. Otherwise we use Lemma 9.3 to replace $e^{d_{\alpha+1}(y,0)}$ by $e^{-(f_{\alpha+1}(y)-f_\alpha(y))}$ in this term. This introduces an error, but it can be estimated as $\mathcal{Q}(\eta)$. Finally, the area weight $1+O\left(|\nabla f_\beta|^2\right)$ can be estimated by using (9.8) and (9.9). Putting these estimates together (9.22) is rewritten as

$$
\begin{aligned}
&\left[1+O\left(M^{-\delta}\right)+O\left(\kappa^{1/8}\right)\right]\sum_{\beta=\alpha,\alpha+1}\int_{B^{n-1}_{K\kappa^{-1/2}}(0)}|\nabla\eta_\beta|^2dy\\
\geq\;&\frac{2A_1^2}{\sigma_0}\sum_{\beta=\alpha,\alpha+1}\int_{B^{n-1}_{K\kappa^{-1/2}}(0)}e^{-(f_{\alpha+1}(y)-f_\alpha(y))}\left[\eta_\alpha(y)+\eta_{\alpha+1}\left(\Pi_{\alpha+1}(y,0)\right)\right]^2dy\\
-\;&O\left(M^{-1}\kappa\right)\sum_{\beta=\alpha,\alpha+1}\int_{B^{n-1}_{K\kappa^{-1/2}}(0)}\eta_\beta(y)^2dy.
\end{aligned}
$$

By a rescaling and noting that if ε is small enough, then $\kappa \ll M^{-\delta}$, we obtain (9.21). □

From now on we assume the space dimension $n \leq 10$. In these low dimensions we show this almost stability condition implies an L^1 smallness estimate.

Lemma 9.6. *For any $\sigma > 0$, if we have chosen $M \gg K \gg 1$ (depending only on n and σ), then*

$$
\int_{B_2^{n-1}(0)}e^{-v_\alpha}\leq\sigma.
\tag{9.23}
$$

Proof. Let $V_\alpha := e^{-v_\alpha}$. Direct calculation using (9.19) gives

$$
-\operatorname{div}\left(\widetilde{\mathcal{A}}_\alpha\nabla V_\alpha\right)\leq\frac{4A_1^2}{\sigma_0}V_\alpha^2-\left[1+O\left(M^{-\delta}\right)\right]V_\alpha^{-1}|\nabla V_\alpha|^2+CM^{-\delta}V_\alpha.
\tag{9.24}
$$

Following Farina [21], for any $\eta \in C_0^\infty(B_K^{n-1}(0))$ and $q>0$, multiplying (9.24) by $V_\alpha^{2q-1}\eta^2$ and integrating by parts, we get

$$\left[2q - C(K)M^{-\delta}\right]\int_{B_K^{n-1}(0)} V_\alpha^{2q-2}|\nabla V_\alpha|^2\eta^2 - \frac{1}{2q}\int_{B_K^{n-1}(0)} V_\alpha^{2q}\Delta\eta^2 \tag{9.25}$$
$$\leq \frac{4A_1^2}{\sigma_0}\int_{B_K^{n-1}(0)} V_\alpha^{2q+1}\eta^2 + C(K)M^{-\delta}\int_{B_K^{n-1}(0)} V_\alpha^{2q}\eta^2.$$

On the other hand, substituting $\widetilde{\eta}_\alpha = \widetilde{\eta}_{\alpha+1} = \eta$ into Lemma 9.5 gives

$$\left[1 + CM^{-\delta}\right]\int_{B_K^{n-1}(0)} |\nabla\eta|^2 dy + C(K)M^{-\delta}\int_{B_K^{n-1}(0)} \eta^2 dy \geq \frac{4A_1^2}{\sigma_0}\int_{B_K^{n-1}(0)} V_\alpha\eta^2 dy. \tag{9.26}$$

Replacing the test function η by $V_\alpha^q\eta$ in (9.26) leads to

$$\frac{4A_1^2}{\sigma_0}\int V_\alpha^{2q+1}\eta^2 \leq \left[1 + CM^{-\delta}\right] q^2\int V_\alpha^{2q-2}|\nabla V_\alpha|^2\eta^2 \tag{9.27}$$
$$+C\int V_\alpha^{2q}\left(\left|\Delta\eta^2\right| + |\nabla\eta|^2\right) + C(K)M^{-\delta}\int V_\alpha^{2q}\eta^2.$$

Combining (9.25) and (9.27) we get, if

$$2q + C(K)M^{-\delta} > q^2\left(1 + C(K)M^{-\delta}\right),$$

which is true provided $q<2$ and M is large enough, then

$$\int V_\alpha^{2q-2}|\nabla V_\alpha|^2\eta^2 + \int V_\alpha^{2q+1}\eta^2 \leq C(q)\int V_\alpha^{2q}\left(\left|\Delta\eta^2\right| + |\nabla\eta|^2 + C(K)M^{-\delta}\eta^2\right). \tag{9.28}$$

Still as in Farina [21], take a standard cut-off function η and replace η by η^m for some $m \gg 1$ (depending only on q) in (9.28). Then applying the Hölder inequality gives, for any $q < 15/8$,

$$\int_{B_2^{n-1}(0)} \left[V_\alpha^{2q-2}|\nabla V_\alpha|^2 + V_\alpha^{2q+1}\right] \leq C(q)K^{n-1-2(2q+1)} + C(q,K)M^{-\delta}K^{n-1}. \tag{9.29}$$

If $n \leq 10$ and $q > 7/4$, we have

$$n - 1 - 2(2q+1) < 0.$$

First choose a K so large that $C(q)K^{n-1-2(2q+1)} < \sigma^{2q+1}/2C$, then take an M so large that $C(q,K)M^{-\delta}K^{n-1} < \sigma^{2q+1}/2C$, where C is a universal large constant. By this choice we get

$$\int_{B_2^{n-1}(0)} \left[V_\alpha^{2q-2}|\nabla V_\alpha|^2 + V_\alpha^{2q+1}\right] \leq \frac{\sigma^{2q+1}}{C}.$$

An application of Hölder inequality gives (9.23). □

Now we improve this L^1 estimate to an L^∞ estimate. To this end, we need the following decay estimate.

Lemma 9.7. *There exist two universal constants σ_* and $\tau_* \in (0,1/8)$ so that the following holds. For any $y \in B_1^{n-1}(0)$ and $r \in (0,1)$, suppose*

$$r^{3-n}\int_{B_r^{n-1}(y)} V_\alpha \leq \sigma_*, \tag{9.30}$$

then

$$(\tau_* r)^{3-n}\int_{B^{n-1}_{\tau_* r}(y)} V_\alpha \le \frac{1}{2} r^{3-n}\int_{B^{n-1}_r(y)} V_\alpha. \tag{9.31}$$

Proof. The proof follows the method introduced by the first author in [37, 38] with minor modifications. Here we give a sketch of the proof. For simplicity of presentation, we will assume $n \ge 4$. If $n = 2, 3$, it can either be proved directly (for example, by the method in Brezis-Merle [6] if $n = 3$) or by trivially extending v_α to higher dimensions.

Denote

$$\sigma := r^{3-n}\int_{B^{n-1}_r(y)} V_\alpha.$$

Fix the constant K and M according to the previous lemma with the constant σ_*.

Taking $q = 1/2$ in (9.28) gives, for any $\eta \in C_0^\infty(B_2^{n-1}(0))$,

$$\int_{B_2^{n-1}(0)} V_\alpha^2\eta^2 \lesssim \int_{B_2^{n-1}(0)} V_\alpha\left(|\eta||\Delta\eta| + |\nabla\eta|^2 + M^{-\delta}\eta^2\right). \tag{9.32}$$

Let η be a standard cut-off function with $\eta \equiv 1$ in $B^{n-1}_{r/2}(y)$, $\eta \equiv 0$ outside $B^{n-1}_r(y)$, $|\nabla^2\eta| + |\nabla\eta|^2 \lesssim r^{-2}$. With this η in (9.32) we get

$$\begin{aligned}\int_{B^{n-1}_{r/2}(y)} V_\alpha^2 &\lesssim \left[r^{-2} + C(K)M^{-\delta}\right]\int_{B_r(y)} V_\alpha \\ &\lesssim r^{n-5}\sigma.\end{aligned} \tag{9.33}$$

Define the rescaling

$$\bar v_\alpha(z) := v_\alpha(y + rz) - 2\log r, \quad z \in B_2^{n-1}(0).$$

It satisfies

$$\operatorname{div}\left(\bar{\mathcal{A}}_\alpha \nabla \bar v_\alpha\right) \le \frac{4A_1^2}{\sigma_0} e^{-\bar v_\alpha} + C(K)M^{-1}r^2, \tag{9.34}$$

where

$$\bar{\mathcal{A}}_\alpha(z) = \widetilde{\mathcal{A}}_\alpha(y + rz), \quad z \in B_2^{n-1}(0).$$

The above estimates on V_α is transformed into

$$\begin{cases}\displaystyle\int_{B_1^{n-1}(0)} e^{-\bar v_\alpha} = \sigma, \\ \displaystyle\int_{B^{n-1}_{1/2}} e^{-2\bar v_\alpha} \lesssim \sigma.\end{cases} \tag{9.35}$$

Choose an $r_* \in (1/4, 1/2)$ so that

$$\int_{\partial B^{n-1}_{r_*}(0)} e^{-\bar v_\alpha} \le 4\int_{B_1^{n-1}(0)} e^{-\bar v_\alpha} \le 4\sigma. \tag{9.36}$$

Take three functions satisfying the following conditions:

(1) h_α is the solution of

$$\begin{cases}\operatorname{div}\left(\bar{\mathcal{A}}_\alpha\nabla h_\alpha\right) = 0, & \text{in } B^{n-1}_{r_*}(0), \\ h_\alpha = \bar v_\alpha, & \text{on } \partial B^{n-1}_{r_*}(0).\end{cases}$$

(2) w^1_α is the solution of

$$\begin{cases}\operatorname{div}\left(\bar{\mathcal{A}}_\alpha \nabla w^1_\alpha\right)=\dfrac{4A_1^2}{\sigma_0}e^{-\bar{v}_\alpha}, & \text{in } B^{n-1}_{r_*}(0),\\ w^1_\alpha=0, & \text{on } \partial B^{n-1}_{r_*}(0).\end{cases} \tag{9.37}$$

(3) w^2_α is the solution of

$$\begin{cases}\operatorname{div}\left(\bar{\mathcal{A}}_\alpha \nabla w^2_\alpha\right)=C(K)M^{-1}r^2, & \text{in } B^{n-1}_{r_*}(0),\\ w^2_\alpha=0, & \text{on } \partial B^{n-1}_{r_*}(0).\end{cases}$$

By comparison principle, $w^1_\alpha<0$, $w^2_\alpha<0$ and $\bar{v}_\alpha\geq h_\alpha+w^1_\alpha+w^2_\alpha$ in $B^{n-1}_{r_*}(0)$.

The following estimates hold for these three functions.

(1) Because $\operatorname{div}\left(\bar{\mathcal{A}}_\alpha\nabla e^{-h_\alpha}\right)\geq 0$, we get

$$\begin{aligned}\sup_{B^{n-1}_{r_*/2}(0)} e^{-h_\alpha} &\lesssim \int_{\partial B^{n-1}_{r_*}(0)} e^{-\bar{v}_\alpha}\\ &\lesssim \sigma. \quad (\text{by } (9.36))\end{aligned} \tag{9.38}$$

Here the first inequality follows by Green's representation

$$\begin{aligned}e^{-h_\alpha(x)} &= -\int_{B^{n-1}_{r_*}(0)} e^{-h_\alpha(y)}\operatorname{div}\left(\bar{\mathcal{A}}_\alpha\nabla_y G(x,y)\right)dy\\ &= -\int_{B^{n-1}_{r_*}(0)} G(x,y)\operatorname{div}\left(\bar{\mathcal{A}}_\alpha\nabla_y e^{-h_\alpha(y)}\right)dy-\int_{\partial B^{n-1}_{r_*}(0)} e^{-h_\alpha(y)}\bar{\mathcal{A}}_\alpha\nabla_y G(x,y)\cdot\frac{y}{r_*}dA(y)\\ &\lesssim \int_{\partial B^{n-1}_{r_*}(0)} e^{-h_\alpha(y)}|\nabla_y G(x,y)|dA(y)\\ &\lesssim \int_{\partial B^{n-1}_{r_*}(0)} e^{-h_\alpha(y)}dA(y), \quad \forall x\in B^{n-1}_{r_*/2}(0).\end{aligned}$$

In the above, $dA(y)$ denotes the area measure on $\partial B^{n-1}_{r_*}(0)$, $G(x,y)$ denotes the Green function for $\operatorname{div}\left(\bar{\mathcal{A}}_\alpha\nabla\cdot\right)$ in $B^{n-1}_{r_*}(0)$ with homogeneous Dirichlet boundary condition, while the last inequality follows by applying the boundary gradient estimate (see [26, Section 14.1]) (which in turn also relies on upper bounds of $G(x,y)$ as in [31]). These estimates do hold if we note that $\bar{\mathcal{A}}_\alpha$ satisfies both the conditions (9.17) and (9.20), that is,

$$\|\bar{\mathcal{A}}_\alpha-Id\|_{L^\infty(B^{n-1}_2(0))}\leq 1/2, \tag{9.39}$$

and

$$\|\nabla\bar{\mathcal{A}}_\alpha\|_{L^\infty(B^{n-1}_2(0))}\leq C. \tag{9.40}$$

(2) Since $\operatorname{div}\left(\bar{\mathcal{A}}_\alpha\nabla\right)$ is a uniformly elliptic operator (see (9.39)), by [31] we get a $|x-y|^{3-n}$ bound on its Green function. Using this, similar to [37, Lemma 3.4] we get

$$\int_{B^{n-1}_{r_*}(0)}|w^1_\alpha|\lesssim\sigma. \tag{9.41}$$

Next, by (9.40), we can apply global $W^{2,2}$ estimates and (9.35) to get

$$\int_{B^{n-1}_{r_*}(0)} |\nabla^2 w^1_\alpha|^2 \lesssim \sigma. \tag{9.42}$$

As in [38], by Sobolev embedding and interpolation between (9.41) and (9.42), we get a universal constant $\nu > 0$ such that

$$\int_{B^{n-1}_{r_*}(0)} |\nabla w^1_\alpha|^2 \lesssim \sigma^{1+\nu}.$$

Multiplying (9.37) by w^1_α and integrating by parts we obtain

$$-\int_{B^{n-1}_{r_*}(0)} w^1_\alpha e^{-v_\alpha} = \int_{B^{n-1}_{r_*}(0)} \bar{\mathcal{A}}_\alpha \nabla w^1_\alpha \cdot \nabla w^1_\alpha \lesssim \sigma^{1+\nu}. \tag{9.43}$$

(3) For w^2_α, by constructing a comparison function we get

$$w^2_\alpha \geq -C(K)M^{-1}r^2, \quad \text{in } B^{n-1}_{r_*}(0). \tag{9.44}$$

With these three estimates in hand, we proceed to estimate $r^{3-n}\int_{B^{n-1}_r(0)} e^{-\bar{v}_\alpha}$, for any $r < r_*/2$. Decompose this estimate into two parts: $\{w^1_\alpha \geq -\sigma^{\nu/2}\}$ and $\{w^1_\alpha < -\sigma^{\nu/2}\}$.

For the first part we have

$$\begin{aligned} r^{3-n}\int_{B^{n-1}_r(0)\cap\{w^1_\alpha\geq-\sigma^{\nu/2}\}} e^{-\bar{v}_\alpha} &\leq r^{3-n}\int_{B^{n-1}_r(0)\cap\{w^1_\alpha\geq-\sigma^{\nu/2}\}} e^{-h_\alpha} e^{\sigma^{\nu/2}+C(K)M^{-1}r^2} \\ &\lesssim_{K,M} r^{3-n}\int_{B^{n-1}_r(0)} e^{-h_\alpha} \\ &\lesssim_{K,M} r^2\sigma. \qquad \text{(by (9.38))} \end{aligned}$$

The second part can be estimated using (9.43):

$$\begin{aligned} r^{3-n}\int_{B^{n-1}_r(0)\cap\{w^1_\alpha\leq-\sigma^{\nu/2}\}} e^{-\bar{v}_\alpha} &\leq \sigma^{-\nu/2} r^{3-n}\int_{B_r(0)} \left(-w^1_\alpha\right) e^{-\bar{v}_\alpha} \\ &\lesssim r^{3-n}\sigma^{1+\nu/2}. \end{aligned}$$

Putting these together we get

$$r^{3-n}\int_{B^{n-1}_r(0)} e^{-\bar{v}_\alpha} \leq C(K,M)r^2\sigma + C(K,M)r^{3-n}\sigma^{1+\nu/2}.$$

First choose $r = \tau_*$ so small that $C(K,M)\tau_*^2 \leq 1/4$, and then choose σ_* so small that

$$C(K,M)\tau_*^{3-n}\sigma_*^{\nu/2} \leq \frac{1}{4}.$$

By this choice we get

$$\tau_*^{3-n}\int_{B^{n-1}_{\tau_*}(0)} e^{-\bar{v}_\alpha} \leq \frac{\sigma}{2}.$$

Rescaling back to v_α this is (9.31). □

Using this lemma we get

Lemma 9.8. *If*

$$\int_{B_2^{n-1}(0)} V_\alpha \leq \sigma_*, \tag{9.45}$$

then

$$\max_{B_{1/4}^{n-1}(0)} V_\alpha \leq \frac{1}{2}. \tag{9.46}$$

Proof. For any $y \in B_1^{n-1}(0)$,

$$\int_{B_1^{n-1}(y)} V_\alpha \leq \int_{B_2^{n-1}(0)} V_\alpha \leq \sigma_*.$$

Then by the previous lemma, we get

$$\tau_*^{(3-n)(k+1)} \int_{B_{\tau_*^{k+1}}(y)} V_\alpha \leq \frac{1}{2}\tau_*^{(3-n)k} \int_{B_{\tau_*^k}(y)} V_\alpha, \quad \forall k \geq 1. \tag{9.47}$$

An iteration of this estimate gives, for any $y \in B_1^{n-1}(0)$ and $r \in (0,1)$,

$$\int_{B_r^{n-1}(y)} V_\alpha \lesssim \sigma_* r^{n-3+\frac{\log 2}{|\log \tau_*|}}. \tag{9.48}$$

This implies that V_α belongs to the Morrey space $M^{p_*}(B_1^{n-1}(0))$ (following the notation in [26, Section 7.9]), where p_* is defined by $(n-1)/p_* = 2 - \log 2/|\log \tau_*|$.

As in the proof of the previous lemma, take an $r_* \in (3/4, 1)$ such that

$$\int_{\partial B_{r_*}^{n-1}(0)} e^{-v_\alpha} \leq 4\sigma_*,$$

and then define the same functions h_α, w_α^1 and w_α^2 in $B_{r_*}^{n-1}(0)$.

The estimate (9.38) for h_α and the estimate (9.44) for w_α^2 (with $r=1$) still hold. For w_α^1, since $1/p_* < 2/(n-1)$, by [26, Lemma 7.18] now we get

$$|w_\alpha^1| \lesssim \sigma_*, \quad \text{in } B_{r_*}^{n-1}(0).$$

Note that although it is not the standard Laplacian in the equation of w_α^1 but only a uniformly elliptic operator, the proof of [26, Lemma 7.18] still goes through because it requires only a $|x-y|^{3-n}$ bound on the Green function for $\operatorname{div}\left(\bar{\mathcal{A}}_\alpha \nabla\right)$, which is true by [31].

Combining these estimates we get

$$e^{-v_\alpha} \leq e^{-h_\alpha - w_\alpha^1 - w_\alpha^2} \leq Ce^{\sigma_*}\sigma_* < 1/2 \quad \text{in } B_{r_*/2}^{n-1}(0)$$

provided σ_* is sufficiently small. This is (9.46). □

Remark 9.9. *Since we have assumed $n \leq 10$, we can also combine L^p estimates in Lemma 9.6 with standard $W^{2,p}$ estimates to deduce this L^∞ estimate. However, the above two lemmas work in any dimension. They could be useful for the establishment of a partial regularity theory in high dimensions, as mentioned in Remark 1.4.*

In view of Lemma 9.6, Lemma 9.8 is applicable to V_α for all κ small. Rescaling (9.46) back we get (9.3). The proof of Proposition 9.1 is thus complete.

10. Distance bound

In this section we give a lower bound on D_α. Recall that we have set $R := \varepsilon^{-1}$, see the beginning of Section 3.

Proposition 10.1. *There exists a universal constant C such that*

$$A(2R/3;0) \le C\varepsilon^2.$$

10.1. **Non-optimal lower bounds.** Let us first provide three non-optimal lower bounds.

Lemma 10.2. *For any $\theta \in (0,1)$ and ε sufficiently small,*

$$A(19R/24;0) \le \varepsilon^{1+\theta}.$$

Proof. Assume by the contrary $A(19R/24;0) > \varepsilon^{1+\theta}$. Then by the monotone dependence of $A(r;0)$ on r, we have

$$A(r;0) \ge A(19R/24;0) > \varepsilon^{1+\theta}, \quad \forall r \in [19R/24, 5R/6]. \tag{10.1}$$

This allows us to apply Proposition 9.1, which says

$$A\left(r - KR^{\frac{1+\theta}{2}};0\right) \le \frac{1}{2}A(r;0).$$

Here we have used the estimate on the constant R_* in Proposition 9.1, that is, by (10.1) we have $R_* \le R^{\frac{1+\theta}{2}}$.

An iteration of this decay estimate from $r = 5R/6$ to $19R/24$ leads to a contradiction, i.e.

$$A\left(19R/24;0\right) \le 2^{-cK^{-1}R^{\frac{1-\theta}{2}}} A\left(5R/6;0\right) \le \varepsilon^2.$$

In the last inequality we have used $A(5R/6;0) \le 1$, which is a consequence of Lemma 2.1. □

By choosing $\theta > 6/7$ and substituting the above estimate into Corollary 7.6 we get

Corollary 10.3. *For any α and $y \in \Gamma_\alpha \cap B_{19R/24}(0)$,*

$$H_\beta(y,0) = \frac{2A^2_{(-1)^{\beta-1}}}{\sigma_0}e^{-d_{\beta-1}(y,0)} - \frac{2A^2_{(-1)^{\beta}}}{\sigma_0}e^{d_{\beta+1}(y,0)} + O\left(\varepsilon^2\right). \tag{10.2}$$

Lemma 10.4. *For all ε sufficiently small,*

$$A(3R/4;0) \le \varepsilon^2|\log\varepsilon|^2.$$

Proof. By Lemma 10.2, we have

$$A(19R/24;0) \le \varepsilon^{1+\theta}, \tag{10.3}$$

where θ is very close to 1 (to be determined below).

Now assume by the contrary $A(3R/4;0) > \varepsilon^2|\log\varepsilon|^2$. Then by the monotone dependence of $A(r;0)$ on r, we have

$$A(r;0) \ge A(3R/4;0) > \varepsilon^2|\log\varepsilon|^2, \quad \forall r \in [3R/4, 19R/24]. \tag{10.4}$$

Now Proposition 9.1 is applicable, which says

$$A\left(r - K\frac{R}{\log R};0\right) \le \frac{1}{2}A(r;0).$$

Here we have used the estimate on the constant R_* in Proposition 9.1, that is, by (10.4) we have $R_* \le R/\log R$.

In view of (10.3), an iteration of this decay estimate from $r = 19R/24$ to $3R/4$ leads to a contradiction, i.e.

$$A\left(3R/4;0\right) \le 2^{-cK^{-1}\log R} A\left(19R/24;0\right) \le C\varepsilon^{1+\theta+\frac{c\log 2}{K}} \le \varepsilon^2,$$

provided $1+\theta+\frac{c\log 2}{K} > 2$, i.e. θ has been chosen to be very close to 1. □

Lemma 10.5. *There exists a universal constant C such that for all ε sufficiently small,*

$$A(17R/24;0) \le C\varepsilon^2|\log\varepsilon|.$$

Proof. By Lemma 10.4, we have

$$A(3R/4;0) \le \varepsilon^2|\log\varepsilon|^2. \tag{10.5}$$

Now assume by the contrary $A(17R/24;0) \ge M\varepsilon^2|\log\varepsilon|$, where M is the constant in Proposition 9.1. Then by the monotone dependence of $A(r;0)$ on r, we have

$$A(r;0) \ge A(17R/24;0) \ge M\varepsilon^2|\log\varepsilon|, \quad \forall r\in[17R/24,3R/4]. \tag{10.6}$$

Now Proposition 9.1 is applicable, which says

$$A\left(r - K\frac{R}{\sqrt{M\log R}};0\right) \le \frac{1}{2}A(r;0).$$

Here we have used the estimate on the constant R_* in Proposition 9.1, that is, by (10.6) we have $R_* \le R/\sqrt{M\log R}$.

In view of (10.5), an iteration of this decay estimate from $r = 3R/4$ to $17R/24$ leads to a contradiction, i.e.

$$A\left(17R/24;0\right) \le 2^{-cK^{-1}\sqrt{M\log R}} A\left(3R/4;0\right) \le C2^{-cK^{-1}\sqrt{M\log R}}\varepsilon^2|\log\varepsilon|^2 \le \varepsilon^2|\log\varepsilon|.$$

The last inequality follows from the estimate

$$2^{-cK^{-1}\sqrt{M|\log\varepsilon|}}|\log\varepsilon| = 2^{-cK^{-1}\sqrt{M|\log\varepsilon|}+\frac{\log|\log\varepsilon|}{\log 2}} \le 1,$$

which is true if ε is small enough. □

10.2. **Proof of Proposition 10.1.** By Lemma 10.5, now we have

$$A(17R/24;0) \lesssim \varepsilon^2|\log\varepsilon|. \tag{10.7}$$

Hence by Corollary 10.3, for any α,

$$\|H_\alpha\|_{L^\infty(B_{17R/24}(0))} \lesssim \varepsilon^2|\log\varepsilon|. \tag{10.8}$$

Denote $\rho := |d_{\alpha+1}|$. Assume $u > 0$ between Γ_α and $\Gamma_{\alpha+1}$. By [9, Eqn. (2.41), Lemma 2.9 and Appendix A] and (9.2), ρ satisfies

$$\mathcal{L}\rho(y)| + |A_\alpha(y)|^2\rho(y) + \mathcal{N}(\rho) \le \frac{4A_1^2}{\sigma_0}e^{-\rho(y)} + O\left(\varepsilon^2\right). \tag{10.9}$$

Here $\mathcal{L}$ is a linear uniformly elliptic operator defined by

$$\mathcal{L}\varphi := a(y)^{-1}\mathrm{div}_{\alpha,0}\left[a(y)N_\alpha(y)\cdot N_{\alpha+1}(y,\rho(y))\nabla_{\alpha+1,0}\varphi\right],$$

where

$$a(y) := \frac{\lambda_\alpha(y,0)}{\lambda_{\alpha+1}(y,\rho(y))}.$$

The nonlinear error term $\mathcal{N}(\rho)$ satisfies

$$|\mathcal{N}(\rho)(y)| \lesssim \varepsilon^3|\rho(y)|^2 + \varepsilon|\nabla_{\alpha,0}\rho(y)|^2 \tag{10.10}$$

In order to estimate $\mathcal{N}(\rho)$, we need

Lemma 10.6. *For any $y \in \Gamma_\alpha \cap B_{11R/16}(0)$, if $\rho(y) \le 4|\log \varepsilon|$, then*

$$\frac{|\nabla_{\alpha,0}\rho(y)|}{\rho(y)} \lesssim \varepsilon.$$

Proof. Fix an α and a point $y_* \in \Gamma_\alpha \cap B_{11R/16}(0)$ with $\rho(y_*) \le 4|\log \varepsilon|$. Note that by (10.7) we always have

$$\rho(y) \ge 2|\log \varepsilon| - \log|\log \varepsilon| - C, \quad \forall y \in B_{R/100}(y_*). \tag{10.11}$$

Choose a coordinate system such that $\Gamma_\alpha \cap B_{R/100}(y_*)$ and $\Gamma_{\alpha+1} \cap B_{R/100}(y_*)$ are represented by graphs of functions f_α and $f_{\alpha+1}$, $\rho(y_*)$ is attained at $(y_*, f_{\alpha+1}(y_*))$, and $\nabla f_{\alpha+1}(y_*) = 0$. Therefore we have

$$\rho(y_*) = f_{\alpha+1}(y_*) - f_\alpha(y_*) \quad \text{and} \quad \rho(y) \le f_{\alpha+1}(y) - f_\alpha(y), \quad \forall y \ne y_*.$$

Consequently,

$$\nabla\rho(y_*) = \nabla f_{\alpha+1}(y_*) - \nabla f_\alpha(y_*). \tag{10.12}$$

Define

$$\widetilde{\rho}(\widetilde{y}) := \rho(y_*)^{-1}\left[f_{\alpha+1}(y_* + \varepsilon^{-1}\widetilde{y}) - f_\alpha(y_* + \varepsilon^{-1}\widetilde{y})\right], \quad \widetilde{y} \in B^{n-1}_{1/100}(0).$$

Representing $\Gamma_{\alpha+1}$ by the graph of the function ρ over Γ_α, combining (10.8) (for $H_{\alpha+1}$) and (10.11) we obtain

$$\left\|\operatorname{div}\left(\bar{\mathcal{A}}_\alpha \nabla\widetilde{\rho}\right)\right\|_{L^\infty(B^{n-1}_{1/100}(0))} \le C. \tag{10.13}$$

Here $\bar{\mathcal{A}}_\alpha(\widetilde{y}) = \mathcal{A}_\alpha(y_* + \varepsilon^{-1}\widetilde{y})$ with $\mathcal{A}_\alpha$ defined as in (9.16).

Note that $\bar{\mathcal{A}}_\alpha$ is a uniformly elliptic operator, thanks to the Lipschitz bound on f_α and $f_{\alpha+1}$. Moreover, there exists a universal constant C such that

$$\|\nabla\bar{\mathcal{A}}_\alpha\|_{L^\infty(B^{n-1}_{1/100}(0))} \le \varepsilon^{-1}\|\nabla\mathcal{A}_\alpha\|_{L^\infty(B^{n-1}_{R/100}(y_*))} \le C,$$

where in the last inequality we have used the form of $\mathcal{A}_\alpha$ (see (9.16)) and bounds on second order derivatives of f_α and $f_{\alpha+1}$ as in (9.7).

By our assumption, $\widetilde{\rho}(0) = 1$. Since $\widetilde{\rho} > 0$, by Moser's Harnack inequality for inhomogeneous equations (see [26, Theorem 8.17 and 8.18]), there exists a $\sigma > 0$ such that

$$1/2 \le \widetilde{\rho} \le 2, \quad \text{in } B^{n-1}_\sigma(0). \tag{10.14}$$

Now we apply $W^{2,p}$ estimates to (10.13) and then Sobolev embedding to get a universal constant C such that $\|\widetilde{\rho}\|_{C^{1,1/2}(B^{n-1}_\sigma(0))} \le C$. In particular, $|\nabla\widetilde{\rho}(0)| \le C$. Rescaling back and using (10.12) we conclude the proof. □

The same proof, in particular, (10.14) implies that

Corollary 10.7. *There exists a constant $\sigma > 0$ such that if $\rho(y) \le 2|\log \varepsilon|$, then*

$$\sup_{B_{\sigma R}(y)\cap\Gamma_\alpha} \rho \le 4|\log \varepsilon|. \tag{10.15}$$

Substituting (10.15) and Lemma 10.6 into (10.10) we obtain

Corollary 10.8. *If $\rho(y) \le 2|\log \varepsilon|$, then*

$$\sup_{B_{\sigma R}(y)\cap\Gamma_\alpha} |\mathcal{N}(\rho)| \lesssim \varepsilon^3|\log \varepsilon|^2, \tag{10.16}$$

where σ is the same constant in the previous corollary.

With these estimates at hand, now we come to

Proof of Proposition 10.1. Assume by the contrary, there exists a constant $L > 0$ such that for some α and $x \in \Gamma_\alpha \cap B_{2R/3}(0)$, we have

$$\lambda^2 := e^{-\rho(x_*)} \geq L^2\varepsilon^2. \tag{10.17}$$

We now show this leads to a contradiction if L is large enough.

Let

$$\widetilde{\Gamma} := \lambda\left(\Gamma_\alpha - x_*\right), \quad \widetilde{\rho}\left(\widetilde{y}\right) := \rho\left(x_* + \lambda^{-1}\widetilde{y}\right) + 2\log\lambda, \quad \forall \widetilde{y} \in \widetilde{\Gamma} \cap B_L(0). \tag{10.18}$$

Because $|A_\alpha| \lesssim \varepsilon$, by (10.17), we get

$$\left|A_{\widetilde{\Gamma}}\right| \leq \frac{C\varepsilon}{\lambda} \leq \frac{C}{L}. \tag{10.19}$$

Hence $\widetilde{\Gamma}$ is very close to a hyperplane in $B_{\sqrt{L}}(0)$.

By Corollary 10.7,

$$\rho \leq 4|\log\varepsilon| \quad \text{in} \quad \Gamma_\alpha \cap B_{\sigma R}(x_*). \tag{10.20}$$

Hence Corollary 10.8 is applicable, which implies that (10.9) has the form

$$\mathcal{L}\rho \leq \frac{4A_1^2}{\sigma_0}e^{-\rho} + O\left(\varepsilon^2\right) \quad \text{in } \Gamma_\alpha \cap B_{\sigma R}(x_*). \tag{10.21}$$

A rescaling of this equation leads to the one for $\widetilde{\rho}$:

$$\widetilde{\mathcal{L}}\widetilde{\rho} \leq \frac{4A_1^2}{\sigma_0}e^{-\widetilde{\rho}} + O\left(L^{-2}\right) \quad \text{in } \widetilde{\Gamma} \cap B_{\sigma L}(0). \tag{10.22}$$

Here $\widetilde{\mathcal{L}}$ is the rescaling of $\mathcal{L}$, but we rewrite it as

$$\mathcal{L} = a(y)^{-1}\mathrm{div}_{\widetilde{\Gamma}}\left(\widehat{\mathcal{A}}_\alpha(y)\nabla_{\widetilde{\Gamma}}\right),$$

where $|a(y) - 1| \ll 1$ and the matrix $\widehat{\mathcal{A}}_\alpha$ satisfies

$$\|\widehat{\mathcal{A}}_\alpha - Id\|_{L^\infty(\Gamma_\alpha \cap B_{\sigma R/2}(x_*))} \ll 1,$$

by a derivation similar to the one of (9.17). (Note that with (10.20) at hand, we can apply Lemma 3.4 to estimate $N_\alpha \cdot N_{\alpha+1}, D\Pi_{\alpha+1}$ etc.)

As in (9.26), for any $\eta \in C_0^\infty(\widetilde{\Gamma} \cap B_{\sqrt{L}}(0))$, we still have

$$\left(1 + C\varepsilon^{1/8}\right)\int_{\widetilde{\Gamma} \cap B_{\sqrt{L}}(0)} |\nabla\eta|^2 + CL^{-2}\int_{\widetilde{\Gamma} \cap B_{\sqrt{L}}(0)} \eta^2 \geq \frac{4A_1^2}{\sigma_0}\int_{\widetilde{\Gamma} \cap B_{\sqrt{L}}(0)} e^{-\widetilde{\rho}}\eta^2 dy. \tag{10.23}$$

Then proceeding as in Subsection 9.2 we conclude that, if L is large enough, we have

$$\sup_{\widetilde{\Gamma} \cap B_1(0)} e^{-\widetilde{\rho}} \leq \frac{1}{2}.$$

On the other hand, by definition $\widetilde{\rho}(0) = 0$. This is a contradiction. Therefore (10.17) cannot hold and the proof is complete. □

Remark 10.9. *In (10.21), we have thrown away $|A_\alpha(y)|^2\rho(y)$ in (10.9), because it has a favorable sign. This is clearly a special point for the Euclidean metric and does not hold if we work with a general Riemannian metric (because of an additional Ricci curvature* *term). We do not know if Proposition 10.1 can be generalized to that setting. However, as in [9], a weaker estimate can still be proven:*

$$A(2R/3;0) = o\left(\varepsilon^2|\log\varepsilon|\right), \tag{10.24}$$

which is sufficient for the construction of positive Jacobi fields.

This estimate can be proved by the same blow up method as above. The main point is to make sure that it is the Jacobi field part, $\mathcal{L}\rho(y)| + |A_\alpha(y)|^2\rho(y)$, not the exponential nonlinearity $e^{-\rho}$, dominating. Indeed, if (10.24) does not hold, because $|A_\alpha|^2 \sim \varepsilon^2$ and $\rho \sim |\log\varepsilon|$, the Jacobi field part is of the order $\varepsilon^2|\log\varepsilon|$. Hence there is a balancing between these two parts. After a rescaling as in (10.18), we get a Jacobi-Toda system. By further showing that the stability condition is preserved in this rescaling procedure, we get a contradiction as above.

Remark 10.10. *In the extrinsic construction in Section 9, we have to approximate $e^{-\rho}$ by $e^{-(f_{\alpha+1}-f_\alpha)}$, see Lemma 9.3. Taking such an approximation is not so precise. The main problem comes from the estimate in (9.13), which forces us to use an assumption involving $\varepsilon^2|\log\varepsilon|$ instead of ε^2 in (9.1), see Remark 9.4. The intrinsic construction in this section avoids this issue. Furthermore, it is more direct to construct positive Jacobi fields by using this approach, which will be needed in the proof of Corollary 1.3.*

On the other hand, this intrinsic construction works only if we already have a good distance lower bound such as (10.7). This is because we need Lemma 10.6 to estimate $\mathcal{N}(\rho)$ in (10.9), the proof of which in turn relies on (10.7).

11. Proof of main results

In this section we prove Theorem 1.1 and its two corollaries, Corollary 1.2 and 1.3.

Proof of Theorem 1.1. Substituting Proposition 10.1 into (6.1), we get

$$\|\phi\|_{C^{2,\theta}(B_{2R/3}(0))} + \max_\alpha \|H_\alpha + \Delta_{\alpha,0}h_\alpha\|_{C^\theta(\Gamma_\alpha\cap B_{2R/3}(0))} \lesssim \varepsilon^2. \tag{11.1}$$

By Lemma 4.6, for any α,

$$\begin{aligned}\|H_\alpha\|_{C^\theta(\Gamma_\alpha\cap B_{2R/3}(0))} &\lesssim \|\phi\|_{C^{2,\theta}(B_{2R/3}(0))} + \|H_\alpha + \Delta_{\alpha,0}h_\alpha\|_{C^\theta(\Gamma_\alpha\cap B_{2R/3}(0))} + A(2R/3;0)\\ &\lesssim \varepsilon^2.\end{aligned}$$

After rescaling back to u_ε, this says for any connected component of $\{u_\varepsilon = 0\}$, say $\Gamma_{\alpha,\varepsilon}$, its mean curvature satisfies

$$\|H_{\alpha,\varepsilon}\|_{L^\infty(\Gamma_{\alpha,\varepsilon}\cap B_{2/3}(0))} \lesssim \varepsilon, \quad \text{and} \quad \|H_{\alpha,\varepsilon}\|_{C^\theta(\Gamma_{\alpha,\varepsilon}\cap B_{2/3}(0))} \lesssim \varepsilon^{1-\theta}. \tag{11.2}$$

Because $\Gamma_{\alpha,\varepsilon}\cap B_R(0)$ is a Lipschitz graph in some direction (see Lemma 2.2), by standard estimates on the minimal surface equations (see for example [26, Chapter 16] or [27, Appendix C]) we obtain a uniform bound on the C^θ norm of its second fundamental form $A_{\alpha,\varepsilon}$ in $B_{1/2}(0)$.

As mentioned at the beginning of Section 2, all of these estimates hold uniformly for $t \in [-1+b_1, 1-b_1]$. This completes the proof of Theorem 1.1. □

Next we show how Corollary 1.2 and 1.3 follow from Theorem 1.1.

Proof of Corollary 1.2. **Step 1.** First we prove that **(H1)-(H3)** imply (1.8). For any $r < 1$, α and $x_\varepsilon = (x'_\varepsilon, f_{\alpha,\varepsilon}(x'_\varepsilon)) \in \{u_\varepsilon = t_\varepsilon\}$ with $|x'_\varepsilon| \le r$, consider $\widetilde{u}_\varepsilon(x) := u_\varepsilon(x_\varepsilon + \varepsilon x)$

and assume they converge to $\widetilde{u}$ in $C^2_{loc}(\mathbb{R}^n)$. By **(H1)**, $\{\widetilde{u}_\varepsilon = t_\varepsilon\} = \cup_{\beta=1}^{Q}\{x_n = \widetilde{f}_{\beta,\varepsilon}(x')\}$, where

$$\widetilde{f}_{\beta,\varepsilon}(x') = \frac{1}{\varepsilon}\left[f_{\beta,\varepsilon}(x'_\varepsilon + \varepsilon x') - f_{\alpha,\varepsilon}(x'_\varepsilon)\right].$$

By **(H2)**, as $\varepsilon \to 0$, $\widetilde{f}_{\beta,\varepsilon}$ either converges to an affine function or goes to infinity uniformly. Assume $t_\varepsilon \to t_0$ as $\varepsilon \to 0$. There exists a $1 \le Q' \le Q$ such that $\{\widetilde{u} = t_0\}$ consists of Q' parallel hyperplanes, say $\{x_n = s_\beta\}$, $s_1 < \cdots < s_{Q'}$.

Without loss of generality assume $\widetilde{u} > t_0$ in the half space $\{x_n > s_{Q'}\}$. By Dancer [12, 13] and the stability of $\widetilde{u}$, $\widetilde{u}$ is strictly increasing along the x_n direction. Therefore the following limit exists:

$$\bar{u}(x') := \lim_{x_n \to +\infty} \widetilde{u}(x', x_n), \quad \forall x' \in \mathbb{R}^{n-1}. \tag{11.3}$$

Moreover, $t_0 \le \bar{u} \le 1$ and it is a stable solution of (1.11) in $\mathbb{R}^{n-1}$.

By **(H1)** and monotonicity formula for Allen-Cahn equation (see Modica [32] and Hutchinson-Tonegawa [30]), there exists a constant C depending only on E and r such that for any $R > 0$ and x with $\mathrm{dist}(x, \mathcal{C}_1) > r$, if $R\varepsilon < (1-r)/2$, then

$$\int_{B_R(x)} \left[\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\right] \le CR^{n-1}.$$

Letting $\varepsilon \to 0$, by the aforementioned convergence of $\widetilde{u}_\varepsilon$ to $\widetilde{u}$, we obtain with the same constant C,

$$\int_{B_R(x)} \left[\frac{1}{2}|\nabla \widetilde{u}|^2 + W(\widetilde{u})\right] \le CR^{n-1}, \quad \forall B_R(x) \subset \mathbb{R}^n.$$

Take the translation $x \mapsto x - se_n$ and let $s \to +\infty$, by (11.3) we deduce that $\bar{u}$ satisfies the same energy growth bound. However, since $\bar{u}$ does not depend on x_n, this is equivalent to

$$\int_{B_R^{n-1}(x')} \left[\frac{1}{2}|\nabla \bar{u}|^2 + W(\bar{u})\right] \le CR^{n-2}, \quad \forall B_R^{n-1}(x') \subset \mathbb{R}^{n-1}.$$

By considering $\bar{u}_\epsilon(x') := \bar{u}(\epsilon^{-1}x')$ and letting $\epsilon \to 0$, applying the convergence theory in Hutchinson-Tonegawa [30], we see if[1]

$$\lim_{R \to +\infty} R^{2-n} \int_{B_R^{n-1}(0)} \left[\frac{1}{2}|\nabla \bar{u}|^2 + W(\bar{u})\right] > 0, \tag{11.4}$$

then for any ϵ small enough, there exists a point $x'_\epsilon \in \mathbb{R}^{n-1}$ [2] and a large constant $L > 0$ such that $\bar{u}_\epsilon$ is close to the scaled one dimensional profile in $B^{n-1}_{L\epsilon}(x'_\epsilon)$, see [30, Proposition 5.6] and its proof therein. As a consequence, $\bar{u}_\epsilon$ can take values as close to -1 as possible. This is a contradiction with the fact $t_0 \le \bar{u} \le 1$. This contradiction implies that the limit in (11.4) must be 0, which by the monotonicity formula implies that $W(\bar{u}) \equiv 0$. Hence $\bar{u} \equiv 1$ in $\mathbb{R}^{n-1}$.

For any $y' \in \mathbb{R}^{n-1}$, repeating the above argument for $\widetilde{u}(x' - y', x_n)$, we deduce that the convergence in (11.3) is uniform. Then by [3, Theorem 5] (see also [5, Theorem 1.4]), we deduce that $\widetilde{u} = g(x_n - s)$ for some constant s. By unique continuation, $\widetilde{u}(x) \equiv g(x_n - s)$ in the entire space. In particular, there is only one connected component of $\{\widetilde{u} = t_0\}$, that

[1] The existence of this limit is guaranteed by the monotonicity formula.

[2] As in the proof of [30, Theorem 1], (i) we first take a point $x'_\infty \in \mathrm{spt}\|V\|$, where V denotes the nontrivial (thanks to (11.4)) stationary varifold constructed from $\bar{u}_\varepsilon$ and $\|V\|$ is its mass measure, such that V has a unique weak tangent space at x_∞, (ii) then we take a sequence of points $x'_\epsilon \in \{|u_\epsilon| \le 1 - b\}$ converging to x_∞ such that the tilt-excess in $B^{n-1}_{L\epsilon}(x'_\epsilon)$ is small in the sense of [30, Eq. (5.2)].

is, $Q' = 1$. Because g' has a positive lower bound in $\{|g| \leq 1-b\}$, we deduce that Lemma 2.1 still holds.

Step 2. Thus for all ε small, $\nabla u_\varepsilon \neq 0$ in $\{|u_\varepsilon| < 1-b\}$ and hence $|B(u_\varepsilon)|$ is well defined. In order to apply Theorem 1.1, it suffices to establish a uniform bound on $|B(u_\varepsilon)|$ as in (1.6).

To this end, assume by the contrary that

$$\lim_{\varepsilon\to 0} \max_{x\in\{|u_\varepsilon|<1-b\}\cap\mathcal{C}_{2/3}} |B(u_\varepsilon)(x)| = +\infty. \tag{11.5}$$

Let $x_\varepsilon \in \mathcal{C}_1 \cap \{|u_\varepsilon| \leq 1-b\}$ attain the maxima

$$\max_{\mathcal{C}_1\cap\{|u_\varepsilon|\leq 1-b\}} \left(1-|x'|\right)|B(u_\varepsilon)(x)|. \tag{11.6}$$

By **(H1)**, $x_\varepsilon \in \{|x_n| \leq 1/2\}$.

Denote

$$L_\varepsilon := |B(u_\varepsilon)(x_\varepsilon)|, \qquad r_\varepsilon := \left(1-|x'_\varepsilon|\right)/2. \tag{11.7}$$

Then by definition and (11.5),

$$L_\varepsilon r_\varepsilon \geq \frac{1}{6} \sup_{\mathcal{C}_{2/3}\cap\{|u_\varepsilon|\leq 1-b\}} |B(u_\varepsilon)(x)| \to +\infty. \tag{11.8}$$

In particular, $L_\varepsilon \to +\infty$. On the other hand, by (1.8), we get

$$L_\varepsilon = o\left(\frac{1}{\varepsilon}\right). \tag{11.9}$$

By the choice of r_ε at (11.7), we have (here $\mathcal{C}_{r_\varepsilon}(x'_\varepsilon) := B^{n-1}_{r_\varepsilon}(x'_\varepsilon) \times (-1,1)$)

$$\max_{x\in\mathcal{C}_{r_\varepsilon}(x'_\varepsilon)\cap\{|u_\varepsilon|\leq 1-b\}} |B(u_\varepsilon)(x)| \leq 2L_\varepsilon. \tag{11.10}$$

Let $\kappa := L_\varepsilon \varepsilon$ and define $u_\kappa(x) := u_\varepsilon(x_\varepsilon + L_\varepsilon^{-1}x)$. Then u_κ satisfies (1.1) with parameter κ in $B_{L_\varepsilon r_\varepsilon}(0)$. By (11.9), $\kappa \to 0$ as $\varepsilon \to 0$. For any $t \in [-1+b, 1-b]$, the level set $\{u_\kappa = t\}$ consists of Q Lipschitz graphs

$$\left\{x_n = f^t_{\beta,\kappa}(x') := L_\varepsilon\left[f^t_{\beta,\varepsilon}(x'_\varepsilon + L_\varepsilon^{-1}x') - f^t_{\alpha,\varepsilon}(x'_\varepsilon)\right]\right\}, \quad \beta = 1, \cdots, Q, \tag{11.11}$$

where α is chosen so that x_ε lies in the connected component of $\{|u_\varepsilon| \leq 1-b\}$ containing $\Gamma_{\alpha,\varepsilon}$.

By (11.10), we also have

$$|B(u_\kappa)| \leq 2, \quad \text{for } x \in \mathcal{C}_{L_\varepsilon r_\varepsilon} \cap \{|u_\kappa| \leq 1-b\}.$$

Now Theorem 1.1 is applicable to u_κ. Hence $f_{\alpha,\kappa}$ are uniformly bounded in $C^{2,\theta}_{loc}(\mathbb{R}^{n-1})$. After passing to a subsequence, it converges to a limit f_∞, which by (1.7) is an entire solution of the minimal surface equation. Since the rescaling (11.11) preserves the Lipschitz constants, f_∞ is global Lipschitz. By Moser's Liouville theorem on minimal surface equations (see [27, Theorem 17.5]), f_∞ is an affine function. In particular,

$$\nabla^2 f_\infty \equiv 0. \tag{11.12}$$

On the other hand, by the construction we have $|B(u_\kappa)(0)| = 1$. If $n = 2$, as in the proof of [39, Theorem 3.6], we get

$$|B(u_\kappa)(0)|^2 \lesssim \kappa^\theta,$$

a contradiction with (11.12). If $n \geq 3$, we have

$$1 = |B(u_\kappa)(0)|^2 = |\nabla^2 f_{\alpha,\kappa}(0)|^2 + O\left(\kappa^\theta\right).$$

(The only difference here with the $n = 2$ case is that now the Hessian of the distance function to $\Gamma_{\alpha,\kappa}$ does not converge to 0, but its leading order term is exactly $\nabla^2 f_{\alpha,\kappa}(0)$, see (3.7).) This gives

$$\lim_{\kappa \to 0} |\nabla^2 f_{\alpha,\kappa}(0)|^2 = 1,$$

a contradiction with (11.12). This contradiction implies that the assumption (11.5) cannot hold and the proof is thus complete. □

Remark 11.1. *In the above proof, what we need in Step 1 is the one dimensional symmetry of solutions to an half space problem. Some cases met here can be covered by existing results in literatures such as the ones in* [2, 3, 4, 5, 11, 12, 13, 22, 23, 24].

Proof of Corollary 1.3. If δ_2 is sufficiently small in (1.8), by unique continuation principle $\nabla u_\varepsilon \neq 0$ in $\{|u_\varepsilon| \leq 1 - b\}$ and hence $|B(u_\varepsilon)|$ is well defined. As in the proof of Corollary 1.2, the proof is reduced to a uniform bound on $|B(u_\varepsilon)|$.

Assume by the contrary, we perform a similar blow up analysis as in the proof of Corollary 1.2. This gives another sequence of solutions u_κ defined in an expanding domain. Moreover, u_κ satisfies all of the assumptions in Theorem 1.1. Hence the connected component of $\{u_\kappa = 0\}$ passing through 0 converges to a minimal hypersurface in $\mathbb{R}^n$ in a C^1 way (because we have the uniform curvature bound). Denote this minimal hypersurface by Σ. Its second fundamental form satisfies $|A_\Sigma| \leq 3$ and $|A_\Sigma(0)| = 1$ (as in the proof of Corollary 1.3).

We claim that Σ is stable. This then leads to a contradiction if *Stable Bernstein conjecture* is true, which states that Σ must be a hyperplane and hence $A_\Sigma \equiv 0$. The stability of Σ follows from the general analysis in [9]: first if there are at least two interfaces of u_κ both converging to Σ, we can construct a positive Jacobi field on Σ as in [9, Theorem 4.1] (recall that we have Proposition 10.1), which implies the stability of Σ; secondly, if there is only one such an interface, then there exist $\sigma > 0$ and $C > 0$ such that

$$\int_{B_\sigma(0)} \left[\frac{\kappa}{2}|\nabla u_\kappa|^2 + \frac{1}{\kappa} W(u_\kappa)\right] \leq C.$$

Because u_κ is stable, the stability of Σ then follows by applying the main result in [35]. □

Appendix A. Some facts about the one dimensional solution

In this appendix we recall some facts about one dimensional solution of (1.11), see [39, Appendix A] for more details.

It is known that the following identity holds for g,

$$g'(t) = \sqrt{2W(g(t))} > 0, \quad \forall t \in \mathbb{R}. \tag{A.1}$$

Moreover, as $t \to \pm\infty$, $g(t)$ converges exponentially to ± 1 and the following quantity is well defined:

$$\sigma_0 := \int_{-\infty}^{+\infty} \left[\frac{1}{2} g'(t)^2 + W(g(t))\right] dt \in (0, +\infty).$$

In fact, as $t \to \pm\infty$, we have the following expansions: There exists a positive constant A_1 such that for all $t > 0$ large,

$$g(t) = 1 - A_1 e^{-t} + O(e^{-2t}),$$
$$g'(t) = A_1 e^{-t} + O(e^{-2t}),$$
$$g''(t) = -A_1 e^{-t} + O(e^{-2t}),$$

and a similar expansion holds as $t \to -\infty$ with A_1 replaced by another positive constant A_{-1}.

The following two lemmas describe the interaction between two one dimensional profiles. The first one is [39, Lemma A.1].

Lemma A.1. *For all $T > 0$ large, we have the following expansion:*

$$\int_{-\infty}^{+\infty} \left[W''(g(t)) - 1\right] \left[g(-t-T) + 1\right] g'(t)dt = -2A_{-1}^2 e^{-T} + O\left(e^{-\frac{4}{3}T}\right).$$

$$\int_{-\infty}^{+\infty} \left[W''(g(t)) - 1\right] \left[g(T-t) - 1\right] g'(t)dt = 2A_1^2 e^{-T} + O\left(e^{-\frac{4T}{3}}\right).$$

The second one is

Lemma A.2. *For all $T > 0$ large, we have the following expansion:*

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \left[W''\left(g(t) + g(-t-T) - 1\right) - W''\left(g(-t-T)\right)\right] g'(-t-T) g'(t)dt \\
= \quad & -2A_{-1}^2 e^{-T} + O\left(e^{-\frac{7}{6}T}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \left[W''\left(g(t) + g(T-t) - 1\right) - W''\left(g(T-t)\right)\right] g'(T-t) g'(t)dt \\
= \quad & -2A_1^2 e^{-T} + O\left(e^{-\frac{7}{6}T}\right).
\end{aligned}$$

Proof. We only prove the second expansion. The integral will be decomposed into two parts: the first one is $(7T/12, +\infty)$ and the second one is $(-\infty, 7T/12)$.

Step 1. In $(7T/12, +\infty)$, $g(t)$ is very close to 1. By Taylor expansion we have

$$\left|W''\left(g(t) + g(T-t) - 1\right) - W''\left(g(T-t)\right)\right| \lesssim 1 - g(t) \lesssim g'(t).$$

Then by the decay of g', the first part is controlled by

$$\int_{7T/12}^{+\infty} g'(t)^2 g'(T-t) \lesssim e^{-\frac{7}{6}T} \int_{7T/12}^{+\infty} g'(T-t) \lesssim e^{-\frac{7}{6}T}. \tag{A.2}$$

Step 2. In $(-\infty, 7T/12)$, $g(T-t)$ is very close to 1. By Taylor expansion we have

$$\begin{aligned}
& W''\left(g(t) + g(T-t) - 1\right) - W''\left(g(T-t)\right) \\
= \quad & W''\left(g(t)\right) - W''(1) + \left[W'''\left(g(t)\right) - W'''(1)\right]\left(g(T-t) - 1\right) + O\left(\left|g(T-t) - 1\right|^2\right) \\
= \quad & W''\left(g(t)\right) - W''(1) + O\left(g'(t)g'(T-t)\right) + O\left(g'(T-t)^2\right).
\end{aligned}$$

Here we have used the facts that by C^4 regularity of W, $|W'''\left(g(t)\right) - W'''(1)| \lesssim 1 - g(t) \lesssim g'(t)$ and $|g(T-t) - 1| \lesssim g'(T-t)$.

By the exponential decay of g', we get

$$\int_{-\infty}^{+\infty} g'(t)^2 g'(T-t)^2 dt \lesssim e^{-\frac{3}{2}T} \tag{A.3}$$

and

$$\int_{-\infty}^{7T/12} g'(t)g'(T-t)^3 dt \lesssim e^{-\frac{5}{4}T} \int_{-\infty}^{7T/12} g'(t) \lesssim e^{-\frac{5}{4}T}. \tag{A.4}$$

We also have

$$\begin{aligned} \Big| \int_{7T/12}^{+\infty} \left[W''\left(g(t)\right) - W''\left(1\right)\right] g'(t)g'(T-t)\Big| &\lesssim \int_{7T/12}^{+\infty} g'(t)^2 g'(T-t) \\ &\lesssim e^{-\frac{7}{6}T}. \end{aligned} \tag{A.5}$$

Combining (A.2)-(A.5) we see the original integral is equal to

$$\begin{aligned} &\int_{-\infty}^{+\infty} \left[W''(g(t)) - 1\right] g'(t)g'(T-t)dt + O\left(e^{-\frac{7}{6}T}\right) \\ = \;& -2A_1^2 e^{-T} + O\left(e^{-\frac{7}{6}T}\right), \end{aligned}$$

where the last step follows from Lemma A.1, if we notice the sign difference in the expansions of $g'(T-t)$ and $g(T-t)-1$. □

Next we discuss the spectrum of the linearized operator at g,

$$\mathcal{L} = -\frac{d^2}{dt^2} + W''(g(t)).$$

By a direct differentiation we see $g'(t)$ is an eigenfunction of $\mathcal{L}$ corresponding to eigenvalue 0. By (A.1), 0 is the lowest eigenvalue. In other words, g is stable.

Concerning the second eigenvalue, we have

Theorem A.3. *There exists a constant $\mu > 0$ such that for any $\varphi \in H^1(\mathbb{R})$ satisfying*

$$\int_{-\infty}^{+\infty} \varphi(t)g'(t)dt = 0, \tag{A.6}$$

we have

$$\int_{-\infty}^{+\infty} \left[\varphi'(t)^2 + W''(g(t))\varphi(t)^2\right] dt \geq \mu \int_{-\infty}^{+\infty} \varphi(t)^2 dt.$$

This can be proved via a contradiction argument.

Appendix B. Proof of Lemma 5.1

The proof of Lemma 5.1 is similar to the one given in [39, Appendix B]. However, since the setting is a little different (as explained in Step 1 of Subsection 1.2), for reader's convenience, we will include a complete proof.

Before proving Lemma 5.1, we first derive the exponential nonlinearity in Toda system (5.2).

Lemma B.1. *For any $y \in \Gamma_\alpha \cap B_{6R/7}(0)$,*

$$\int_{-\infty}^{+\infty} \mathcal{I}(y,z) g_\alpha'(y,z)dz = (-1)^{\alpha-1}\left[2A^2_{(-1)^{\alpha-1}} e^{-d_{\alpha-1}(y,0)} - 2A^2_{(-1)^{\alpha}} e^{d_{\alpha+1}(y,0)}\right] + \mathcal{E}_\alpha(y),$$

where

$$\begin{aligned}\|\mathcal{E}_\alpha\|_{C^\theta(B_1^\alpha(y))} &\lesssim \varepsilon^2 + \varepsilon^{1/3}\max_{B_1^\alpha(y)} e^{-D_\alpha} + \max_{B_1^\alpha(y)} e^{-\frac{3}{2}D_\alpha} \\ &+ \max_{\beta:|d_\beta(y,0)|\le 8|\log\varepsilon|}\ \max_{B_1^\beta(\Pi_\beta(y,0))} e^{-2D_\beta} + \max_{|z|<8|\log\varepsilon|}\|\phi\|^2_{C^{2,\theta}(B_1(y,z))}.\end{aligned}$$

Proof. To determine the integral $\int_{-\infty}^{+\infty}\mathcal{I}g'_\alpha$, consider for each β, the integral on $(-\infty,+\infty)\cap\mathcal{M}^0_\beta$, which we assume to be an interval $(\rho^-_\beta(y),\rho^+_\beta(y))$.

Step 1. If $\beta\neq\alpha$, by Lemma 4.4, in $(\rho^-_\beta(y),\rho^+_\beta(y))$,

$$|\mathcal{I}| \lesssim e^{-(|d_\beta|+|d_{\beta-1}|)} + e^{-(|d_\beta|+|d_{\beta+1}|)} + \varepsilon^2.$$

We only consider the case $\beta>\alpha$ and estimate

$$\int_{\rho^-_\beta(y)}^{\rho^+_\beta(y)} e^{-(|d_\beta|+|d_{\beta-1}|)}g'_\alpha.$$

If $|z|$, $|d_\beta|$ and $|d_{\beta-1}|$ are all smaller than $8|\log\varepsilon|$ at the same time, by Lemma 3.4,

$$d_\beta(y,z) = z + d_\beta(y,0) + O\left(\varepsilon^{1/3}\right), \tag{B.1}$$

$$d_{\beta-1}(y,z) = z + d_{\beta-1}(y,0) + O\left(\varepsilon^{1/3}\right). \tag{B.2}$$

Note that since $\beta>\alpha$, by our convention on the sign of d_β, we have $z>0$ and $d_\beta(y,0)<d_{\beta-1}(y,0)\le 0$.

By (B.1) and (B.2) we get

$$\begin{aligned}\int_{\rho^-_\beta(y)}^{\rho^+_\beta(y)} e^{-(|d_\beta|+|d_{\beta-1}|)}g'_\alpha &\lesssim \int_{\rho^-_\beta(y)}^{\rho^+_\beta(y)} e^{-\left(|z|+|z+d_{\beta-1}(y,0)|+|z+d_\beta(y,0)|\right)} \\ &\lesssim \int_{\rho^-_\beta(y)}^{-d_\beta(y,0)} e^{-\left(z+d_{\beta-1}(y,0)-d_\beta(y,0)\right)} + \int_{-d_\beta(y,0)}^{\rho^+_\beta(y)} e^{-\left(3z+d_{\beta-1}(y,0)+d_\beta(y,0)\right)} \\ &\lesssim e^{-\left(d_{\beta-1}(y,0)-d_\beta(y,0)\right)-\rho^-_\beta(y)} + e^{-\left(d_{\beta-1}(y,0)-2d_\beta(y,0)\right)}.\end{aligned}$$

By definition,

$$-d_\beta(y,\rho^-_\beta(y)) = d_{\beta-1}(y,\rho^-_\beta(y)).$$

Thus by (B.1) and (B.2),

$$\rho^-_\beta(y) = -\frac{d_{\beta-1}(y,0)+d_\beta(y,0)}{2} + O\left(\varepsilon^{1/3}\right).$$

Substituting this into the above estimate gives

$$\int_{\rho^-_\beta(y)}^{\rho^+_\beta(y)} e^{-(|d_\beta|+|d_{\beta-1}|)}g'_\alpha \lesssim e^{-\frac{1}{2}\left(d_{\beta-1}(y,0)-3d_\beta(y,0)\right)} + e^{-\left(d_{\beta-1}(y,0)-2d_\beta(y,0)\right)}.$$

If $\beta=\alpha+1$, because $d_{\beta-1}(y,0)=0$, the right hand side is bounded by $O\left(e^{\frac{3}{2}d_{\alpha+1}(y,0)}\right)$.

If $\beta\ge\alpha+2$, the right hand side is bounded by $O\left(e^{d_{\alpha+2}(y,0)}\right)$.

Step 2. It remains to consider the integration in $(\rho_\alpha^-(y), \rho_\alpha^+(y))$. In this case we use Lemma 4.3, which gives

$$
\begin{aligned}
\int_{\rho_\alpha^-(y)}^{\rho_\alpha^+(y)} \mathcal{I} g_\alpha' &= \int_{\rho_\alpha^-(y)}^{\rho_\alpha^+(y)} \left[W''(g_\alpha)-1\right]\left[g_{\alpha-1}-(-1)^{\alpha-1}\right] g_\alpha' \\
&+ \int_{\rho_\alpha^-(y)}^{\rho_\alpha^+(y)} \left[W''(g_\alpha)-1\right]\left[g_{\alpha+1}+(-1)^{\alpha-1}\right] g_\alpha' \\
&+ \int_{\rho_\alpha^-(y)}^{\rho_\alpha^+(y)} \left[O\left(e^{-2d_{\alpha-1}}+e^{2d_{\alpha+1}}\right)+O\left(e^{-d_{\alpha-2}-|z|}+e^{d_{\alpha+2}-|z|}\right)\right] g_\alpha'.
\end{aligned}
\tag{B.3}
$$

Because $g_\alpha' \lesssim e^{-|z|}$ and

$$
e^{-2d_{\alpha-1}} \lesssim e^{-2d_{\alpha-1}(y,0)-2z} + \varepsilon^2,
$$

we get

$$
\begin{aligned}
\int_{\rho_\alpha^-(y)}^{\rho_\alpha^+(y)} e^{-2d_{\alpha-1}} g_\alpha' &\lesssim \varepsilon^2 + e^{-2d_{\alpha-1}(y,0)}\left[\int_{\rho_\alpha^-(y)}^{0} e^{-z}dz + \int_0^{\rho_\alpha^+(y)} e^{-3z}dz\right] \\
&\lesssim \varepsilon^2 + e^{-2d_{\alpha-1}(y,0)-\rho_\alpha^-(y)} \\
&\lesssim \varepsilon^2 + e^{-\frac{3}{2}d_{\alpha-1}(y,0)}.
\end{aligned}
$$

Similarly, we have

$$
\int_{\rho_\alpha^-(y)}^{\rho_\alpha^+(y)} e^{2d_{\alpha+1}} g_\alpha' \lesssim \varepsilon^2 + e^{\frac{3}{2}d_{\alpha+1}(y,0)},
$$

$$
\int_{\rho_\alpha^-(y)}^{\rho_\alpha^+(y)} O\left(e^{-d_{\alpha-2}-|z|}+e^{d_{\alpha+2}-|z|}\right) g_\alpha' \lesssim e^{-d_{\alpha-2}} + e^{d_{\alpha+2}}.
$$

To determine the first integral in the right hand side of (B.3), arguing as in Step 1, if both g_α' and $g_{\alpha-1}-(-1)^{\alpha-1}$ are nonzero, then

$$
g_{\alpha-1}(y,z) = \bar{g}\left((-1)^{\alpha-1}\left(z + d_{\alpha-1}(y,0) + h_{\alpha-1}(\Pi_{\alpha-1}(y,z)) + O\left(\varepsilon^{1/3}\right)\right)\right).
$$

Therefore

$$
\begin{aligned}
& \int_{\rho_\alpha^-(y)}^{\rho_\alpha^+(y)} \left[W''(g_\alpha)-1\right]\left(g_{\alpha-1}-(-1)^{\alpha-1}\right) g_\alpha' \\
= & \int_{\rho_\alpha^-(y)}^{\rho_\alpha^+(y)} \left[W''\left(\bar{g}\left((-1)^\alpha(z-h_\alpha(y))\right)\right)-1\right]\bar{g}'\left((-1)^\alpha(z-h_\alpha(y))\right) \\
& \times\left[\bar{g}\left((-1)^{\alpha-1}\left(z+d_{\alpha-1}(y,0)+h_{\alpha-1}(\Pi_{\alpha-1}(y,z))+O\left(\varepsilon^{1/3}\right)\right)\right)-(-1)^{\alpha-1}\right]dz \\
= & \int_{-\infty}^{+\infty} \left[W''\left(\bar{g}\left((-1)^\alpha(z-h_\alpha(y))\right)\right)-1\right]\bar{g}'\left((-1)^\alpha(z-h_\alpha(y))\right) \\
& \times\left[\bar{g}\left((-1)^{\alpha-1}\left(z+d_{\alpha-1}(y,0)+h_{\alpha-1}(\Pi_{\alpha-1}(y,z))+O\left(\varepsilon^{1/3}\right)\right)\right)-(-1)^{\alpha-1}\right]dz \\
+ & O\left(e^{-\frac{3}{2}d_{\alpha-1}(y,0)}\right) \\
= & (-1)^{\alpha-1}2A^2_{(-1)^{\alpha-1}}e^{-d_{\alpha-1}(y,0)} + O\left(e^{-\frac{3}{2}d_{\alpha-1}(y,0)}\right) \qquad \text{(by Lemma A.1)}
\end{aligned}
$$

$$+ \quad O\left(|h_\alpha(y)| + \max_{|z|<8|\log\varepsilon|} |h_{\alpha-1}(\Pi_{\alpha-1}(y,z))| + \varepsilon^{1/3}\right) e^{-d_{\alpha-1}(y,0)}.$$

Step 3. What we have proven says

$$\begin{aligned}\left|\mathcal{E}_\alpha(y)\right| &\lesssim \varepsilon^2 + \left(|h_\alpha(y)| + \max_{|z|<8|\log\varepsilon|} |h_{\alpha-1}(\Pi_{\alpha-1}(y,z))| + \varepsilon^{1/3}\right) e^{-d_{\alpha-1}(y,0)} \\ &+ \left(|h_\alpha(y)| + \max_{|z|<8|\log\varepsilon|} |h_{\alpha+1}(\Pi_{\alpha+1}(y,z))| + \varepsilon^{1/3}\right) e^{d_{\alpha+1}(y,0)} \\ &+ e^{-\frac{3}{2}d_{\alpha-1}(y,0)} + e^{\frac{3}{2}d_{\alpha+1}(y,0)} + e^{-d_{\alpha-2}(y,0)} + e^{d_{\alpha+2}(y,0)}.\end{aligned}$$

Note that for $|z| < 8|\log\varepsilon|$, if $e^{-d_{\alpha-1}(y,0)} \geq \varepsilon^2$, then by Lemma 3.4 we have $\Pi_{\alpha-1}(y,z) \in B_1^{\alpha-1}(\Pi_{\alpha-1}(y,0))$. The same remark applies to $\alpha+1$.

By taking derivatives of $\int_{-\infty}^{+\infty} \mathcal{I} g'_\alpha$ in y, and then using Lemma 4.4 and Lemma 4.6, we see the $C^\theta(B_1^\alpha(y))$ norm (in fact, the Lipschitz norm) of $\mathcal{E}_\alpha$ is bounded by

$$\begin{aligned}&\left(\varepsilon^2 + \max_{B_1^\alpha(y)} e^{-D_\alpha}\right)\left(\sum_{\beta:|d_\beta(y,0)|<8|\log\varepsilon|} \|h_\beta\|_{C^{2,\theta}(B_1^\beta(\Pi_\beta(y,0)))}\right) \\ \lesssim\ & \varepsilon^2 + \max_{\beta:|d_\beta(y,0)|\leq 8|\log\varepsilon|} \max_{B_1^\beta(\Pi_\beta(y,0))} e^{-2D_\beta} + \max_{|z|<8|\log\varepsilon|} \|\phi\|^2_{C^{2,\theta}(B_1(y,z))}.\end{aligned}$$

Similarly, by Lemma 3.4, the $C^\theta(B_1^\alpha(y))$ norm of $2A^2_{(-1)^\alpha} e^{-d_{\alpha-1}(y,0)} - 2A^2_{(-1)^{\alpha-1}} e^{d_{\alpha+1}(y,0)}$ is controlled by $\varepsilon^{1/3} \max_{B_1^\alpha(y)} e^{-D_\alpha}$. This completes the estimate on $\mathcal{E}_\alpha$. □

Now let us prove Lemma 5.1. Differentiating (7.7) twice leads to

$$\int_{-\infty}^{+\infty}\left[\frac{\partial\phi}{\partial y^i} g'_\alpha + (-1)^{\alpha-1}\phi g''_\alpha \frac{\partial h_\alpha}{\partial y^i}\right] = 0 \tag{B.4}$$

and

$$\begin{aligned}&\int_{-\infty}^{+\infty}\left[\frac{\partial^2\phi}{\partial y^i\partial y^j} g'_\alpha + (-1)^{\alpha-1}\frac{\partial\phi}{\partial y^i} g''_\alpha \frac{\partial h_\alpha}{\partial y^j} + (-1)^{\alpha-1}\frac{\partial\phi}{\partial y^j} g''_\alpha \frac{\partial h_\alpha}{\partial y^i}\right] \\ +\ & \int_{-\infty}^{+\infty}\left[(-1)^{\alpha-1}\phi g''_\alpha \frac{\partial^2 h_\alpha}{\partial y^i \partial y^j} + \phi g'''_\alpha \frac{\partial h_\alpha}{\partial y^i}\frac{\partial h_\alpha}{\partial y^j}\right] = 0.\end{aligned} \tag{B.5}$$

Therefore

$$\begin{aligned}\int_{-\infty}^{+\infty} \Delta_{\alpha,0}\phi(y,z) g'_\alpha &= (-1)^\alpha \Delta_{\alpha,0} h_\alpha \int_{-\infty}^{+\infty} \phi g''_\alpha - |\nabla_{\alpha,0} h_\alpha|^2 \int_{-\infty}^{+\infty} \phi g'''_\alpha \\ &+ 2(-1)^{\alpha-1}\int_{-\infty}^{+\infty} g^{ij}_\alpha(y,0)\frac{\partial\phi}{\partial y^i}\frac{\partial h_\alpha}{\partial y^j} g''_\alpha.\end{aligned} \tag{B.6}$$

Substituting (B.6) into (5.1), we obtain

$$\begin{aligned}&\int_{-\infty}^{+\infty}\left(\Delta_{\alpha,z}\phi - \Delta_{\alpha,0}\phi\right) g'_\alpha + (-1)^\alpha\left(\int_{-\infty}^{+\infty}\phi g''_\alpha\right)\Delta_{\alpha,0} h_\alpha - \left(\int_{-\infty}^{+\infty}\phi g'''_\alpha\right)|\nabla_{\alpha,0} h_\alpha(y)|^2 \\ +\ & 2(-1)^{\alpha-1}\int_{-\infty}^{+\infty} g''_\alpha g^{ij}_\alpha(y,0)\frac{\partial\phi}{\partial y_i}\frac{\partial h_\alpha}{\partial y_j} - \int_{-\infty}^{+\infty} H_\alpha(y,z) g'_\alpha \phi_z + \int_{-\infty}^{+\infty}\xi'_\alpha\phi \\ =\ & \int_{-\infty}^{+\infty}\left[W''(g_*) - W''(g_\alpha)\right] g'_\alpha\phi + \int_{-\infty}^{+\infty}\mathcal{R}(\phi) g'_\alpha\end{aligned}$$

$$
\begin{aligned}
&+ \quad \int_{-\infty}^{+\infty} \mathcal{I} g'_\alpha + (-1)^\alpha \left(\int_{-\infty}^{+\infty} |g'_\alpha|^2\right) [H_\alpha(y,0) + \Delta_{\alpha,0} h_\alpha(y)] \\
&+ \quad (-1)^\alpha \int_{-\infty}^{+\infty} |g'_\alpha|^2 [H_\alpha(y,z) - H_\alpha(y,0)] + (-1)^\alpha \int_{-\infty}^{+\infty} |g'_\alpha|^2 [\Delta_{\alpha,z} h_\alpha(y) - \Delta_{\alpha,0} h_\alpha(y)] \\
&+ \quad \frac{1}{2}\left(\int_{-\infty}^{+\infty} |g'_\alpha|^2 \frac{\partial}{\partial z} g^{ij}_\alpha(y,z)\right) \frac{\partial h_\alpha}{\partial y_i}\frac{\partial h_\alpha}{\partial y_j} \\
&+ \quad \sum_{\beta\neq\alpha} (-1)^\beta \int_{-\infty}^{+\infty} g'_\alpha g'_\beta \mathcal{R}_{\beta,1} - \sum_{\beta\neq\alpha}\int_{-\infty}^{+\infty} g'_\alpha g''_\beta \mathcal{R}_{\beta,2} - \sum_\beta \int_{-\infty}^{+\infty} g'_\alpha \xi_\beta.
\end{aligned}
$$

We estimate the Hölder norm of these terms one by one.
(1) By (3.12), we have

$$
\begin{aligned}
\Big\| \int_{-\infty}^{+\infty} (\Delta_{\alpha,z}\phi - \Delta_{\alpha,0}\phi)\, g'_\alpha \Big\|_{C^\theta(B^\alpha_1(y))} &\lesssim \varepsilon \max_{|z|<8|\log\varepsilon|} \|\phi\|_{C^{2,\theta}(B_1(y,z))} \\
&\lesssim \varepsilon^2 + \max_{|z|<8|\log\varepsilon|} \|\phi\|^2_{C^{2,\theta}(B_1(y,z))}.
\end{aligned}
$$

(2) By the exponential decay of $\bar{g}'$ and Lemma 4.6, we have

$$
\begin{aligned}
\Big\| \left(\int_{-\infty}^{+\infty} \phi g''_\alpha\right) \Delta_{\alpha,0} h_\alpha \Big\|_{C^\theta(B^\alpha_1(y))} &\lesssim \|h_\alpha\|_{C^{2,\theta}(B^\alpha_1(y))} \max_{|z|<8|\log\varepsilon|} \|\phi\|_{C^\theta(B_1(y,z))} \\
&\lesssim \|h_\alpha\|^2_{C^{2,\theta}(B^\alpha_1(y))} + \max_{|z|<8|\log\varepsilon|} \|\phi\|^2_{C^\theta(B_1(y,z))} \\
&\lesssim \max_{B^\alpha_1(y)} e^{-2D_\alpha} + \max_{|z|<8|\log\varepsilon|} \|\phi\|^2_{C^{2,\theta}(B_1(y,z))}.
\end{aligned}
$$

(3) By the exponential decay of $\bar{g}'$ and Lemma 4.6, we have

$$
\begin{aligned}
\Big\| \int_{-\infty}^{+\infty} g''_\alpha g^{ij}_\alpha(y,0) \frac{\partial\phi}{\partial y_i}\frac{\partial h_\alpha}{\partial y_j} \Big\|_{C^\theta(B^\alpha_1(y))} &\lesssim \|h_\alpha\|_{C^{1,\theta}(B^\alpha_1(y))} \max_{|z|<8|\log\varepsilon|} \|\phi\|_{C^{1,\theta}(B_1(y,z))} \\
&\lesssim \|h_\alpha\|^2_{C^{1,\theta}(B^\alpha_1(y))} + \max_{|z|<8|\log\varepsilon|} \|\phi\|^2_{C^{1,\theta}(B_1(y,z))} \\
&\lesssim \max_{B^\alpha_1(y)} e^{-2D_\alpha} + \max_{|z|<8|\log\varepsilon|} \|\phi\|^2_{C^{2,\theta}(B_1(y,z))}.
\end{aligned}
$$

(4) By the exponential decay of $\bar{g}'$ and Lemma 4.6, we have

$$
\begin{aligned}
\Big\| \left(\int_{-\infty}^{+\infty} \phi g'''_\alpha\right) |\nabla_{\alpha,0} h_\alpha(y)|^2 \Big\|_{C^\theta(B^\alpha_1(y))} &\lesssim \|h_\alpha\|^2_{C^{1,\theta}(B^\alpha_1(y))} \max_{|z|<8|\log\varepsilon|} \|\phi\|_{C^\theta(B_1(y,z))} \\
&\lesssim \|h_\alpha\|^2_{C^{1,\theta}(B^\alpha_1(y))} \\
&\lesssim \max_{B^\alpha_1(y)} e^{-2D_\alpha} + \|\phi\|^2_{C^{2,\theta}(B_1(y,0))}.
\end{aligned}
$$

(5) By (3.1) and the exponential decay of $\bar{g}'$, we have

$$
\begin{aligned}
\Big\| \int_{-\infty}^{+\infty} H_\alpha(y,z) g'_\alpha \phi_z \Big\|_{C^\theta(B^\alpha_1(y))} &\lesssim \varepsilon \max_{|z|<8|\log\varepsilon|} \|\phi\|_{C^{2,\theta}(B_1(y,z))} \\
&\lesssim \varepsilon^2 + \max_{|z|<8|\log\varepsilon|} \|\phi\|^2_{C^{2,\theta}(B_1(y,z))}.
\end{aligned}
$$

(6) The $C^\theta(B_1^\alpha(y))$ norm of $\int_{-\infty}^{+\infty}\left[W''(g_*)-W''(g_\alpha)\right]g'_\alpha\phi$ is bounded by

$$
\begin{aligned}
&\left[\max_{|z|<8|\log\varepsilon|}\|\phi\|_{C^\theta(B_1(y,z))}\right]\left[\max_{B_1^\alpha(y)}\left(\int_{-\infty}^{+\infty}\left(|g_{\alpha-1}^2-1|+|g_{\alpha+1}^2-1|\right)g'_\alpha\right)\right]\\
\lesssim\ &\left(\max_{|z|<8|\log\varepsilon|}\|\phi\|_{C^\theta(B_1(y,z))}\right)\left(\max_{B_1(y)}D_\alpha e^{-D_\alpha}\right)\\
\lesssim\ &\max_{|z|<8|\log\varepsilon|}\|\phi\|^2_{C^{2,\theta}(B_1(y,z))}+\max_{B_1^\alpha(y)}e^{-\frac{3}{2}D_\alpha}.
\end{aligned}
$$

(7) The $C^\theta(B_1^\alpha(y))$ norm of $\int_{-\infty}^{+\infty}\mathcal{R}(\phi)g'_\alpha$ is bounded by $\max_{|z|<8|\log\varepsilon|}\|\phi\|^2_{C^\theta(B_1(y,z))}$.

(8) By the definition of $\bar{g}$ (see Subsection 4.1), the $C^\theta(B_1^\alpha(y))$ norm of $\int_{-\infty}^{+\infty}|g'_\alpha|^2-\sigma_0$ is bounded by $O\left(\varepsilon^2\right)$.

(9) By (3.9), the $C^\theta(B_1^\alpha(y))$ norm of $\int_{-\infty}^{+\infty}|g'_\alpha|^2\left[H_\alpha(y,z)-H_\alpha(y,0)\right]$ is bounded by $O\left(\varepsilon^2\right)$.

(10) By (3.12) and Lemma 4.6, the $C^\theta(B_1^\alpha(y))$ norm of $\int_{-\infty}^{+\infty}|g'_\alpha|^2\left[\Delta_{\alpha,0}h_\alpha(y)-\Delta_{\alpha,z}h_\alpha(y)\right]$ is bounded by

$$
\varepsilon\|h_\alpha\|_{C^{2,\theta}(B_1^\alpha(y))}\lesssim\varepsilon^2+\|h_\alpha\|^2_{C^{2,\theta}(B_1^\alpha(y))}\lesssim\varepsilon^2+\max_{B_1^\alpha(y)}e^{-2D_\alpha}+\|\phi\|^2_{C^{2,\theta}(B_1(y,0))}.
$$

(11) By (3.4), the $C^\theta(B_1^\alpha(y))$ norm of $\left(\int_{-\infty}^{+\infty}|g'_\alpha|^2\frac{\partial}{\partial z}g_\alpha^{ij}(y,z)\right)\frac{\partial h_\alpha}{\partial y_i}\frac{\partial h_\alpha}{\partial y_j}$ is bounded by

$$
\varepsilon\|h_\alpha\|^2_{C^{1,\theta}(B_1^\alpha(y))}\lesssim\max_{B_1^\alpha(y)}e^{-2D_\alpha}+\|\phi\|^2_{C^{2,\theta}(B_1(y,0))}.
$$

(12) For $\beta\neq\alpha$, if $|d_\beta(y,0)|>8|\log\varepsilon|$, the $C^\theta(B_1^\alpha(y))$ norm of $\int_{-\infty}^{+\infty}g'_\alpha g'_\beta\mathcal{R}_{\beta,1}$ is bounded by $O\left(\varepsilon^2\right)$.

If $|d_\beta(y,0)|\leq 8|\log\varepsilon|$, first note that in Fermi coordinates with respect to Γ_β, we have the decomposition

$$
\begin{aligned}
g'_\alpha g'_\beta\mathcal{R}_{\beta,1}\ =\ &\underbrace{g'_\alpha g'_\beta\left[H_\beta(y,0)+\Delta_{\beta,0}h_\beta(y)\right]}_{I}+\underbrace{g'_\alpha g'_\beta\left[H_\beta(y,z)-H_\beta(y,0)\right]}_{II}\\
+\ &\underbrace{g'_\alpha g'_\beta\left[\Delta_{\beta,z}h_\beta(y)-\Delta_{\beta,0}h_\beta(y)\right]}_{III}.
\end{aligned}
$$

These three terms are estimated in the following way. First we have

$$
\|I\|_{C^\theta(B_1(y,z))}\lesssim e^{-|d_\alpha(y,z)|-|z|}\|H_\beta+\Delta_{\beta,0}h_\beta\|_{C^\theta(B_1^\beta(y))}.
$$

By (3.9), we get

$$
\|II\|_{C^\theta(B_1(y,z))}\lesssim\varepsilon^2|z|e^{-|d_\alpha(y,z)|-|z|},
$$

and by (3.12), we get

$$
\|III\|_{C^\theta(B_1^\beta(y))}\lesssim\varepsilon|z|e^{-|d_\alpha(y,z)|-|z|}\|h_\beta\|_{C^{2,\theta}(B_1^\beta(\Pi_\beta(y,0)))}.
$$

Putting these estimates together and coming back to Fermi coordinates with respect to Γ_α, applying Lemma 3.4 to change distances to be measured with respect to Γ_α, we see the $C^\theta(B_1^\alpha(y))$ norm of $\int_{-\infty}^{+\infty}g'_\alpha g'_\beta\mathcal{R}_{\beta,1}$ is controlled by

$$
\begin{aligned}
\text{(B.7)}\quad &e^{-c|\beta-\alpha|}\varepsilon^2+|d_\beta(y,0)|e^{-|d_\beta(y,0)|}\|H_\beta+\Delta_{\beta,0}h_\beta\|_{C^\theta(B_1^\beta(\Pi_\beta(y,0)))}+\varepsilon|d_\beta(y,0)|^2e^{-|d_\beta(y,0)|}\\
\lesssim\ &e^{-c|\beta-\alpha|}\varepsilon^2+e^{-\frac{3}{2}|d_\beta(y,0)|}+e^{-\frac{3}{4}|d_\beta(y,0)|}\|H_\beta+\Delta_{\beta,0}h_\beta\|_{C^\theta(B_1^\beta(\Pi_\beta(y,0)))}.
\end{aligned}
$$

Summing in $\beta\neq\alpha$ and applying Lemma 3.6 we obtain

$$\begin{aligned}\text{(B.8)}\qquad &\Big\|\sum_{\beta\neq\alpha}\int_{-\infty}^{+\infty}g_\alpha' g_\beta'\mathcal{R}_{\beta,1}\Big\|_{C^\theta(B_1^\alpha(y))}\\ \lesssim\ & \varepsilon^2+\max_{B_1^\alpha(y)}e^{-\frac{3}{2}D_\alpha}+\max_{\beta\neq\alpha:|d_\beta(y,0)|\leq 8|\log\varepsilon|}\|H_\beta+\Delta_{\beta,0}h_\beta\|^2_{C^\theta(B_1^\beta(\Pi_\beta(y,0)))}.\end{aligned}$$

(13) By the same reasoning as in the previous case, for $\beta\neq\alpha$, the $C^\theta(B_1^\alpha(y))$ norm of $\sum_{\beta\neq\alpha}\int_{-\infty}^{+\infty}g_\alpha' g_\beta''\mathcal{R}_{\beta,2}$ is controlled by

$$\begin{aligned}&\varepsilon^2+\max_{|z|<8|\log\varepsilon|}\|\phi\|^2_{C^{2,\theta}(B_1(y,z))}+\sum_{\beta\neq\alpha:|d_\beta(y,0)|\leq 8|\log\varepsilon|}|d_\beta(y,0)|e^{-\left(|d_\beta(y,0)|+2D_\beta(\Pi_\beta(y,0))\right)}\\ \lesssim\ & \varepsilon^2+\max_{\beta:|d_\beta(y,0)|\leq 8|\log\varepsilon|}\ \max_{B_1^\beta(\Pi_\beta(y,0))}e^{-\frac{3}{2}D_\beta}+\max_{|z|<8|\log\varepsilon|}\|\phi\|^2_{C^{2,\theta}(B_1(y,z))}.\end{aligned}$$

(14) By the definition of ξ, the $C^\theta(B_1^\alpha(y))$ norm of $\sum_\beta(-1)^{\beta-1}\int_{-\infty}^{+\infty}g_\alpha'\xi_\beta$ is bounded by $O\left(\varepsilon^2\right)$.

Putting estimates (1)-(14) together gives (5.3).

Appendix C. Proof of Lemma 6.6

We estimate the Hölder norm of the right hand side of (4.11) term by term. Since they exhibit similar patterns on each $\mathcal{M}_\alpha^0$, it is sufficient to consider one of such domains.

(1) Because
$$\mathcal{R}(\phi)=W'(g_*+\phi)-W'(g_*)-W''(g_*)\phi,$$
we get
$$\|\mathcal{R}(\phi)\|_{C^\theta(B_r(x))}\lesssim\|\phi\|^2_{C^\theta(B_r(x))}.$$

(2) By Lemma 4.5, we have
$$\Big\|W'(g_*)-\sum_\beta W'(g_\beta)\Big\|_{C^\theta(\mathcal{M}_\alpha^0\cap B_r(x))}\lesssim\varepsilon^2+A\left(r;x\right).$$

(3) Take the decomposition
$$\begin{aligned}g_\alpha'\left[H_\alpha(y,z)+\Delta_z h_\alpha(y)\right] &= g_\alpha'\left[H_\alpha(y,0)+\Delta_0 h_\alpha(y)\right]\\ &+ g_\alpha'\left[H_\alpha(y,z)-H_\alpha(y,0)\right]+g_\alpha'\left[\Delta_{\alpha,z}h_\alpha(y)-\Delta_{\alpha,0}h_\alpha(y)\right].\end{aligned}$$

First we have
$$\begin{aligned}&\|g_\alpha'(y,z)\left[H_\alpha(y,z)-H_\alpha(y,0)\right]\|_{C^\theta(\mathcal{M}_\alpha^0\cap B_r(x))}\\ \lesssim\ &\|g_\alpha'(y,z)\left[H_\alpha(y,z)-H_\alpha(y,0)\right]\|_{Lip(\mathcal{M}_\alpha^0\cap B_r(x))}\\ \lesssim\ &\max_{(y,z)\in\mathcal{M}_\alpha^0\cap B_r(x)}e^{-|z|}\big|H_\alpha(y,z)-H_\alpha(y,0)\big|\\ &+\max_{(y,z)\in\mathcal{M}_\alpha^0\cap B_r(x)}|z||e^{-|z|}|\big|A_\alpha(y,0)||\nabla_{\alpha,0}A_\alpha(y,0)\big|\\ \lesssim\ &\varepsilon^2,\end{aligned}$$
where in the last step we have used Lemma 3.1.

Next because

$$\begin{aligned}\Delta_{\alpha,z}h_\alpha(y)-\Delta_{\alpha,0}h_\alpha(y) &= \sum_{i,j=1}^{n-1}\left[g_\alpha^{ij}(y,z)-g_\alpha^{ij}(y,0)\right]\frac{\partial^2 h_\alpha}{\partial y_i\partial y_j}\\ &+ \sum_{i=1}^{n-1}\left[b_\alpha^i(y,z)-b_\alpha^i(y,0)\right]\frac{\partial h_\alpha}{\partial y_i},\end{aligned}$$

we get

$$\begin{aligned}&\left\|g_\alpha'\left[\Delta_{\alpha,0}h_\alpha(y)-\Delta_{\alpha,z}h_\alpha(y)\right]\right\|_{C^\theta(\mathcal{M}_\alpha^0\cap B_r(x))}\\ \lesssim\ & \max_{(y,z)\in\mathcal{M}_\alpha^0\cap B_r(x)} e^{-|z|}\left[|g_\alpha^{ij}(y,z)-g_\alpha^{ij}(y,0)||\nabla^2_{\alpha,0}h_\alpha(y)|+|b^i(y,z)-b^i(y,0)||\nabla_{\alpha,0}h_\alpha(y)|\right]\\ +\ & \max_{(y,z)\in\mathcal{M}_\alpha^0\cap B_r(x)} e^{-|z|}\left(|\nabla^2_{\alpha,0}h_\alpha(y)|+|\nabla_{\alpha,0}h_\alpha(y)|\right)\\ &\times\left(\left\|g_\alpha^{ij}(\tilde{y},\tilde{z})-g_\alpha^{ij}(\tilde{y},0)\right\|_{C^\theta(B_1(y,z))}+\left\|b_\alpha^i(\tilde{y},\tilde{z})-b_\alpha^i(\tilde{y},0)\right\|_{C^\theta(B_1(y,z))}\right)\\ +\ & \|h_\alpha\|_{C^{2,\theta}(\Gamma_\alpha\cap B_r(x))}\left(\max_{(y,z)\in\mathcal{M}_\alpha^0\cap B_r(x)} e^{-|z|}\left(|g_\alpha^{ij}(y,z)-g_\alpha^{ij}(y,0)|+|b_\alpha^i(y,z)-b_\alpha^i(y,0)|\right)\right)\\ \lesssim\ & \varepsilon\|h_\alpha\|_{C^{2,\theta}(\Gamma_\alpha\cap B_r(x))}\\ \lesssim\ & \varepsilon^2+\|h_\alpha\|^2_{C^{2,\theta}(\Gamma_\alpha\cap B_r(x))} \qquad \text{(by Cauchy inequality)}\\ \lesssim\ & \varepsilon^2+\|\phi\|^2_{C^{2,\theta}(B_{r+8|\log\varepsilon|}(x))}+\max_{\Gamma_\alpha\cap B_{r+8|\log\varepsilon|}(x)} e^{-2D_\alpha}. \qquad \text{(by Lemma 4.6)}\end{aligned}$$

In the above, in order to estimate the Hölder norm of the metric tensors g_α^{ij} and b_α^i terms in the Beltrami-Laplace operator $\Delta_{\alpha,z}$, we have also used the bound on second fundamental forms in (3.1) and its derivatives in Lemma 3.1.

(4) By Lemma 4.6, we have

$$\begin{aligned}&\|g_\alpha''|\nabla_{\alpha,z}h_\alpha|^2\|_{C^\theta(\mathcal{M}_\alpha^0\cap B_r(x))}\\ \lesssim\ & \|\nabla_{\alpha,0}h_\alpha\|^2_{L^\infty(\Gamma_\alpha\cap B_r(x))}+\|\nabla_{\alpha,0}h_\alpha\|_{L^\infty(\Gamma_\alpha\cap B_r(x))}\|\nabla^2_{\alpha,0}h_\alpha\|_{L^\infty(\Gamma_\alpha\cap B_r(x))}\\ \lesssim\ & \|\phi\|^2_{C^{2,\theta}(B_{r+8|\log\varepsilon|}(x))}+\max_{\Gamma_\alpha\cap B_{r+8|\log\varepsilon|}(x)} e^{-2D_\alpha}.\end{aligned}$$

(5) As in the previous case, we first estimate the Hölder norm of $\mathcal{R}_{\beta,1}$ in Fermi coordinates with respect to Γ_β for each $\beta\neq\alpha$. Coming back to Fermi coordinates with respect to Γ_α and noting that if $g_\beta'\neq 0$, then $|d_\beta(y,z)|<8|\log\varepsilon|$, we obtain

$$\begin{aligned}&\|g_\beta'\mathcal{R}_{\beta,1}\|_{C^\theta(\mathcal{M}_\alpha^0\cap B_r(x))}\\ \lesssim\ & \left(\max_{(y,z)\in\mathcal{M}_\alpha^0\cap B_r(x)} e^{-|d_\beta(y,z)|}\right)\\ &\times\left(\varepsilon^2+\|\phi\|^2_{C^{2,\theta}(B_{r+9|\log\varepsilon|}(x))}+\max_{\Gamma_\alpha\cap B_{r+9|\log\varepsilon|}(x)} e^{-2D_\alpha}\right)\\ +\ & \max_{(y,z)\in\mathcal{M}_\alpha^0\cap B_r(x),|d_\beta(y,z)|\leq 8|\log\varepsilon|} e^{-|d_\beta(y,z)|}\|H_\beta+\Delta_{\beta,0}h_\beta\|_{C^\theta(B_2^\beta(\Pi_\beta(y,0)))}.\end{aligned}$$

Then summing in β and using Lemma 3.6, we get

$$\|\sum_{\beta\neq\alpha} g_\beta'\mathcal{R}_{\beta,1}\|_{C^\theta(\mathcal{M}_\alpha^0\cap B_r(x))}$$

$$\lesssim \varepsilon^2 + \|\phi\|^2_{C^{2,\theta}(B_{r+9|\log\varepsilon|}(x))} + \max_{\Gamma_\alpha \cap B_{r+9|\log\varepsilon|}(x)} e^{-2D_\alpha}$$
$$+ \left(\max_{\Gamma_\alpha \cap B_{r+9|\log\varepsilon|}(x)} e^{-\frac{D_\alpha}{2}}\right)\left(\|H_\alpha + \Delta_{\alpha,0}h_\alpha\|_{C^\theta(B_{r+9|\log\varepsilon|}(x))}\right).$$

(6) Similar to the previous case, we have

$$\sum_{\beta\neq\alpha} \|g'_\beta \mathcal{R}_{\beta,2}\|_{C^\theta(\mathcal{M}^0_\alpha \cap B_r(x))}$$
$$\lesssim \sum_{\beta\neq\alpha} \max_{(y,z)\in\mathcal{M}^0_\alpha\cap B_r(x)} e^{-|d_\beta(y,z)|}$$
$$\times \left(\|\nabla_{\beta,0}h_\beta\|^2_{L^\infty(\Gamma_\beta\cap B_{r+8|\log\varepsilon|}(x))} + \|\nabla^2_{\beta,0}h_\alpha\|^2_{L^\infty(\Gamma_\beta\cap B_{r+8|\log\varepsilon|}(x))}\right)$$
$$\lesssim \sum_{\beta\neq\alpha} \max_{(y,z)\in\mathcal{M}^0_\alpha\cap B_r(x)} e^{-|d_\beta(y,z)|}\left(\varepsilon^2 + \|\phi\|^2_{C^{2,\theta}(B_{r+9|\log\varepsilon|}(x))} + \max_{\Gamma_\alpha \cap B_{r+9|\log\varepsilon|}(x)} e^{-2D_\alpha}\right)$$
$$\lesssim \varepsilon^2 + \|\phi\|^2_{C^{2,\theta}(B_{r+9|\log\varepsilon|}(x))} + \max_{\Gamma_\alpha \cap B_{r+9|\log\varepsilon|}(x)} e^{-2D_\alpha}.$$

(7) For any β, $\xi_\beta(y,z)\neq 0$ only if $|d_\beta(y,z)|\leq 8|\log\varepsilon|$. Hence by Lemma 2.1,

$$\Big\|\sum_\beta \xi_\beta\Big\|_{C^\theta(\mathcal{M}^0_\alpha\cap B_r(x))} \lesssim \varepsilon^3|\log\varepsilon| \lesssim \varepsilon^2.$$

Putting these estimates together we finish the proof of Lemma 6.6.

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