STABILITY OF THE CAFFARELLI-KOHN-NIRENBERG INEQUALITY ALONG THE FELLI-SCHNEIDER CURVE: CRITICAL POINTS AT INFINITY

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ABSTRACT. In this paper, we consider the following Caffarelli-Kohn-Nirenberg (CKN for short) inequality

$$\left(\int_{\mathbb{R}^d}|x|^{-b(p+1)}|u|^{p+1}dx\right)^{\frac{2}{p+1}}\leq S_{a,b}\int_{\mathbb{R}^d}|x|^{-2a}|\nabla u|^2dx,$$

where $d\geq 2$, $p=\frac{d+2(1+a-b)}{d-2(1+a-b)}$ with

$$\begin{cases}a<b<a+1, & d=2,\\ a\leq b<a+1, & d\geq 3,\end{cases}$$

$S_{a,b}$ is the optimal constant and $u\in D^{1,2}_a(\mathbb{R}^d)$ with

$$D^{1,2}_a(\mathbb{R}^d)=\left\{u\in D^{1,2}(\mathbb{R}^d)\mid \int_{\mathbb{R}^d}|x|^{-2a}|\nabla u|^2dx<+\infty\right\}.$$

Based on the ideas of [23, 54], we develop a suitable strategy to derive the following **sharp** stability of the critical points at infinity of the above CKN inequality in the degenerate case $b=b_{FS}(a)$ with $a<0$ (the Felli-Schneider curve): let $\nu\in\mathbb{N}$ and $u\in D^{1,2}_a(\mathbb{R}^d)$ be a nonnegative function such that

$$\left(\nu-\frac{1}{2}\right)\left(S^{-1}_{a,b}\right)^{\frac{p+1}{p-1}}<\|u\|^2_{D^{1,2}_a(\mathbb{R}^d)}<\left(\nu+\frac{1}{2}\right)\left(S^{-1}_{a,b}\right)^{\frac{p+1}{p-1}},$$

then

$$\inf_{\overrightarrow{\lambda}_\nu\in\mathbb{R}^\nu}\left\|u-\sum_{j=1}^{\nu}W_{\lambda_j}\right\|_{D^{1,2}_a(\mathbb{R}^d)}\lesssim(\Gamma(u))^{\frac{1}{3}},$$

provided $\Gamma(u)$ sufficiently small, where

$$\Gamma(u)=\|div(|x|^{-a}\nabla u)+|x|^{-b(p+1)}|u|^{p-1}u\|_{D^{-1,2}_a(\mathbb{R}^d)}$$

with $D^{-1,2}_a(\mathbb{R}^d)$ being the dual space of $D^{1,2}_a(\mathbb{R}^d)$, $W_\lambda=\lambda^{-\frac{d-2-2a}{2}}W(\lambda x)$ with $W(x)$ being the unique extremal function of the above CKN inequality which is positive and radial up to dilations and scalar multiplications and $\overrightarrow{\lambda}_\nu=(\lambda_1,\lambda_2,\cdots,\lambda_\nu)$. The above stability is sharp in the sense that the power of the right hand side can not be improved any more. The **significant finding** in our result is that in the degenerate case, the power of the optimal stability is **an absolute constant** $1/3$ (independent of p and ν) which is quite different from the non-degenerate case considered in [23, 78]. We also believe that our strategy of proofs might be useful in studying many other degenerate problems.

Keywords: Caffarelli-Kohn-Nirenberg inequality; Sharp stability; the Felli-Schneider curve; Critical points at infinity.

AMS Subject Classification 2010: 35B09; 35B33; 35B40; 35J20.

1. Introduction

1.1. **Background and Previous Results.** Let $d \geq 2$ be a positive integer and $D^{1,2}_a(\mathbb{R}^d)$ be the Hilbert space given by

$$D^{1,2}_a(\mathbb{R}^d) = \left\{ u \in D^{1,2}(\mathbb{R}^d) \mid \int_{\mathbb{R}^d} |x|^{-2a} |\nabla u|^2 dx < +\infty \right\} \tag{1.1}$$
eqn886

with the inner product

$$\langle u, v \rangle_{D^{1,2}_a(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |x|^{-2a} \nabla u \nabla v dx$$

and the induced norm $\| \cdot \|_{D^{1,2}_a(\mathbb{R}^d)} = \left(\langle \cdot, \cdot \rangle_{D^{1,2}_a(\mathbb{R}^d)} \right)^{\frac{1}{2}}$, where $D^{1,2}(\mathbb{R}^d) = \dot{W}^{1,2}(\mathbb{R}^d)$ is the usual homogeneous Sobolev space (cf. [41, Definition 2.1]) with $D^{-1,2}$ being the dual space. Then the following Caffarelli-Kohn-Nirenberg (CKN for short in what follows) inequality

$$\left(\int_{\mathbb{R}^d} |x|^{-b(p+1)} |u|^{p+1} dx \right)^{\frac{2}{p+1}} \leq S_{a,b} \int_{\mathbb{R}^d} |x|^{-2a} |\nabla u|^2 dx, \tag{1.2}$$
eq0001

which is established by Caffarelli, Kohn and Nirenberg in the celebrated paper [12] in a more general version, holds for all $u \in D^{1,2}_a(\mathbb{R}^d)$, where $d \geq 2$, $p = \frac{d+2(1+a-b)}{d-2(1+a-b)}$ and

$$\begin{cases} a < b < a+1, & d = 2, \\ a \leq b < a+1, & d \geq 3. \end{cases} \tag{1.3}$$
eq0003

As pointed out by Catrina and Wang in [13], a fundamental task in understanding a functional inequality is to study the best constants, existence (and nonexistence) of extremal functions, as well as their qualitative properties and classifications, which have played important roles in many applications by virtue of the complete knowledge on the extremal functions. For the CKN inequality (1.2), it is known that up to dilations $u_\tau(x) = \tau^{a_c - a} u(\tau x)$ and scalar multiplications $Cu(x)$ (also up to translations $u(x+y)$ for the spacial case $a = b = 0$), the radial function $W(x)$ given by

$$W(x) = \left(2(p+1)(a_c - a)^2\right)^{\frac{1}{(p-1)}} \left(1 + |x|^{(a_c - a)(p-1)} \right)^{-\frac{2}{p-1}} \tag{1.4}$$
eq0004

is the unique extremal function of (1.2) in $D^{1,2}_a(\mathbb{R}^d)$ for $d \geq 2$ under the conditions

$$\begin{cases} b_{FS}(a) \leq b < a+1 \text{ and } a < 0, & d \geq 2, \\ a \leq b < a+1 \text{ and } a \geq 0, & d \geq 3, \\ a < b < a+1 \text{ and } a \geq 0, & d = 2, \end{cases} \tag{1.5}$$
eq0003

where

$$b_{FS}(a) = \frac{d(a_c - a)}{2\sqrt{(a_c - a)^2 + (d-1)}} + a - a_c > a$$

is the well known Felli-Schneider curve found in [50] and for the sake of simplicity, we denote $a_c = \frac{d-2}{2}$, as in [30–32]. Precisely, Aubin and Talanti established the existence and classification of the extremal functions of the CKN inequality (1.2) for $a = b = 0$ in [3,76], respectively. As a special case, Lieb established the existence and classification of the extremal functions of the CKN inequality (1.2) for $a = 0$ and $0 < b < 1$ in [64]. Chou and Chu established the existence and classification of the extremal functions of the CKN inequality (1.2) for $a \geq 0$ in [19]. Catrina and Wang established the existence and nonexistence of extremal functions of the CKN inequality (1.2) for $a < 0$ in [13]. Felli and Schneider proved in [50] that extremal functions of the CKN inequality (1.2) must be nonradial if $a < 0$ and $a < b < b_{FS}(a)$. Lin and Wang further proved in [65] that extremal functions of (1.2) must have $\mathcal{O}(N-1)$ symmetry for $a < b < b_{FS}(a)$ with $a < 0$. Dolbeault, Esteban, Loss and Tarantello finally classified the extremal functions of the CKN inequality (1.2) in [32,33] for $a < 0$ and $b_{FS}(a) \leq b < a+1$. Moreover, it is also well known that $W(x)$ is nondegenerate in $D^{1,2}_a(\mathbb{R}^d)$ under the condition (1.5) except $b = b_{FS}(a)$ (cf. [50]). That is, up to scalar multiplications $CV(x)$,

$$V(x) := \nabla W(x) \cdot x - (a_c - a)W(x) = \frac{\partial}{\partial \lambda}\left(\lambda^{-(a_c-a)}W(\lambda x)\right)|_{\lambda=1} \tag{1.6}$$
eq0010

is the only nonzero solution in $D^{1,2}_a(\mathbb{R}^d)$ of the linearization of the Euler-Lagrange equation of the CKN inequality (1.2) around W which is given by

$$-div(|x|^{-a}\nabla u) = p|x|^{-b(p+1)}W^{p-1}u, \quad u \in D^{1,2}_a(\mathbb{R}^d). \tag{1.7}$$
eq0017

However, if the parameters a and b lie on the Felli-Schneider curve, that is, $b = b_{FS}(a)$ with $a < 0$, then the bubble $W(x)$ is degenerate in $D^{1,2}_a(\mathbb{R}^d)$ (cf. [54]). For the sake of simplicity in what follows, we introduce the set

$$\mathcal{Z} = \{cW_\tau(x) \mid c \in \mathbb{R}\backslash\{0\} \text{ and } \tau > 0\}$$

and the usual weighted Lebesgue space $L^{p+1}(|x|^{-b(p+1)}, \mathbb{R}^d)$ with the norm

$$\|u\|_{L^{p+1}(|x|^{-b(p+1)},\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d}|x|^{-b(p+1)}|u|^{p+1}dx\right)^{\frac{1}{p+1}}. \tag{1.8}$$
eqnnewnew0002

As pointed out by Dolbeault and Esteban in [28] (see also Figalli in [40]), once optimal constants are known and the set of extremal functions has been characterised, the next question is to understand stability: which kind of distance is measured by the deficit, that is, the difference of the two terms in the functional inequality, written with the optimal constant. These studies were initialed by Brezis and Lieb in [8] by raising an open question for the classical Sobolev inequality,

$$S\left(\int_{\mathbb{R}^d}|u|^{\frac{2d}{d-2}}dx\right)^{\frac{d-2}{d}} \leq \int_{\mathbb{R}^d}|\nabla u|^2dx, \quad u \in D^{1,2}(\mathbb{R}^d), \tag{1.9}$$
eqin0001

where S is the best Sobolev constant, which was settled partially by Egnell-Pacella-Tricarico in [36] and completely by Bianchi-Egnell in [5] by proving that

$$0 < s_{BE} = \inf_{u \in D^{1,2}(\mathbb{R}^d)\backslash\mathcal{M}} \frac{\|\nabla u\|^2_{L^2(\mathbb{R}^d)} - S\|u\|^2_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}}{(dist_{D^{1,2}}(u,\mathcal{M}))^2}, \tag{1.10}$$
eqin0002

where $\|\cdot\|_{L^p(\mathbb{R}^d)}$ is the usual norm in the Lebesgue space $L^p(\mathbb{R}^d)$ and $dist_{D^{1,2}}(u,\mathcal{M}) = \inf_{v\in\mathcal{M}}\|\nabla u - \nabla v\|_{L^2(\mathbb{R}^d)}$ with

$$\mathcal{M} = \{cU[z,\lambda] \mid c \in \mathbb{R}\backslash\{0\}, z \in \mathbb{R}^d \text{ and } \lambda > 0\},$$

U being the standard Aubin-Talanti bubble and $U[z,\lambda]=\lambda^{\frac{d-2}{2}}U(\lambda(x-z))$. Due to the non-Hilbert property of $W^{1,p}(\mathbb{R}^d)$ for $p\neq 2$, the generalization of the Bianchi-Egnell stability (1.10) to the general L^p-Sobolev inequality takes a long time to introduce new ideas and develop new techniques by Cianchi in [20], Cianchi-Fusco-Maggi-Pratelli in [21], Figalli-Magggi-Pratelli in [47], Figalli-Neumayer in [48], Fusco in [56], Fusco-Maggi-Pratelli in [57], Neumayer in [69] and finally, Figalli and Zhang proved the optimal Bianchi-Egnell stability of the general L^p-Sobolev inequality in [49]. The Bianchi-Egnell type stability like (1.10) was also generalized to many other famous inequalities. Since the literature on this topic is so vast and this direction is not the main topic in our paper, we only refer the readers to [9,16,18,27,34] for the Hardy-Littlewood-Sobolev inequality, [35,67,70,72,74] for the Gagliardo-Nirenberg-Sobolev inequality, [7, 17, 37, 39, 51, 58, 59, 80] for the logarithmic Sobolev inequality, [1, 14, 25, 54, 77, 78] for the Caffarelli-Kohn-Nirenberg inequality, [4, 6, 15, 53, 66] for various different kinds of Sobolev inequalities and [11, 24, 38, 42–46, 55, 60, 68, 71] for many kinds of geometric inequalities. We would like to highlight the survey [28] and the Lecture notes [40,52] to the readers for their detailed introductions and references about the studies on the stability of functional and geometrical inequalities. In particular, the Bianchi-Egnell type stability of the CKN inequality (1.2) reads as follows:

(1) The nondegenerate case ([78,79]). Let $d\geq 2$ and either
   (i) $b_{FS}(a)<b<a+1$ with $a<0$ or
   (ii) $a\leq b<a+1$ with $a\geq 0$ and $a+b>0$ ($a<b$ for $d=2$).
   Then
$$0<c_{BE}=\inf_{u\in D^{1,2}_a(\mathbb{R}^d)\setminus\mathcal{Z}}\frac{\|u\|^2_{D^{1,2}_a(\mathbb{R}^d)}-S^{-1}_{a,b}\|u\|^2_{L^{p+1}(|x|^{-b(p+1)},\mathbb{R}^d)}}{\left(dist_{D^{1,2}_a}(u,\mathcal{Z})\right)^2},$$
   where $dist_{D^{1,2}_a}(u,\mathcal{Z})=\inf_{v\in\mathcal{Z}}\|u-v\|_{D^{1,2}_a(\mathbb{R}^d)}$.

(2) The degenerate case ([54]). Let $d\geq 2$ and $b=b_{FS}(a)$ with $a<0$. Then
$$0<c_{BE}=\inf_{u\in D^{1,2}_a(\mathbb{R}^d)\setminus\mathcal{Z}}\frac{\|u\|^2_{D^{1,2}_a(\mathbb{R}^d)}\left(\|u\|^2_{D^{1,2}_a(\mathbb{R}^d)}-S^{-1}_{a,b}\|u\|^2_{L^{p+1}(|x|^{-b(p+1)},\mathbb{R}^d)}\right)}{\left(dist_{D^{1,2}_a}(u,\mathcal{Z})\right)^4}.$$

We remark that Bianchi and Egnell's arguments for (1.10) depends on the non-degeneracy of the Aubin-Talanti bubble U in $D^{1,2}(\mathbb{R}^d)$. Thus, to establish the Bianchi-Egnell type stability of the CKN inequality (1.2) in the degenerate case, Frank and Peteranderl introduced new ideas and developed new techniques to expand the deficit of the CKN inequality (1.2) up to the fourth order terms in [54], as that in [53]. We would like to mention the paper [10] where Carlen and Figalli proved a quantitative convergence result for the critical mass Keller-Segel system by the Bianchi-Egnell type stability of Gagliardo-Nirenberg-Sobolev inequality and the logarithmic Hardy-Littlewood-Sobolev inequality, which provides the potential applications of the studies on the stability of many other inequalities. We also want to mention the paper [9], where Carlen developed a dual method to establish the stability of functional inequalities. Finally, we also want to mention that in the very recent papers [61–63], Konig proved that s_{BE} is attainable which gives a positive answer to the open question proposed by Dolbeault, Esteban, Figalli, Frank and Loss in [29] and makes the key step in answering the long-standing open question of determining the best constant s_{BE}. Konig's result on s_{BE} has been generalized to c_{BE} in the nondegenerate case in our very recent paper [79]. Moreover, the optimal constant c_{BE} in the degenerate case was determined by Frank and Peteranderl in [54].

On the other hand, it is well known that all critical points at infinity of the corresponding functional of the Sobolev inequality (1.9) are induced by limits of sums of Aubin-Talenti bubbles (at least if we consider only nonnegative functions) which can be precisely stated as follows.

⟨thm0003⟩ **Theorem 1.1.** *(Struwe [75]) Let $d \geq 3$ and $\nu \geq 1$ be positive integers. Let $\{u_n\} \subset D^{1,2}(\mathbb{R}^d)$ be a nonnegative sequence with*

$$\left(\nu - \frac{1}{2}\right) S^{\frac{d}{2}} < \|\nabla u_n\|^2_{L^2(\mathbb{R}^d)} < \left(\nu + \frac{1}{2}\right) S^{\frac{d}{2}},$$

where S is the best Sobolev constant. Assume that $\left\|\Delta u_n + |u_n|^{\frac{4}{d-2}} u_n\right\|_{D^{-1,2}} \to 0$ as $n \to \infty$, then there exist a sequence $(z_1^{(n)}, z_2^{(n)}, \cdots, z_\nu^{(n)})$ of ν-tuples of points in $\mathbb{R}^d$ and a sequence of $(\lambda_1^{(n)}, \lambda_2^{(n)}, \cdots, \lambda_\nu^{(n)})$ of ν-tuples of positive real numbers such that

$$\left\|\nabla u_n - \sum_{i=1}^{\nu} \nabla U[z_i^{(n)}, \lambda_i^{(n)}]\right\|_{L^2(\mathbb{R}^d)} \to 0 \quad \text{as } n \to \infty.$$

Based on the above well-known Struwe decomposition, Ciraolo, Figalli, Glaudo and Maggi proposed the following question on the stability of critical points at infinity of the corresponding functional of the Sobolev inequality (1.9):

(Q) Let $d \geq 3$, $\nu \geq 1$ be positive integers and

$$\mathcal{M}_0^\nu = \left\{ \sum_{i=1}^{\nu} U[z_i, \lambda_i] \mid z_i \in \mathbb{R}^d, \lambda_i > 0 \right\}.$$

If $u \in D^{1,2}(\mathbb{R}^d)$ is nonnegative,

$$\left(\nu - \frac{1}{2}\right) S^{\frac{d}{2}} < \|\nabla u\|^2_{L^2(\mathbb{R}^d)} < \left(\nu + \frac{1}{2}\right) S^{\frac{d}{2}}$$

and $\left\|\Delta u + |u|^{\frac{4}{d-2}} u\right\|_{D^{-1,2}} << 1$, does there exist a constant $C(d, \nu)$ such that

$$dist_{D^{1,2}}(u, \mathcal{M}_0^\nu) \leq C(d, \nu) \left\|\Delta u + |u|^{\frac{4}{d-2}} u\right\|_{D^{-1,2}}?$$

Remark 1.1. *The original question ([41, Problem 1.2]) is more general than (Q) stated here in the sense that, u could be sign-changing if u is close to the sum of $U[z_i, \lambda_i]$ in $D^{1,2}(\mathbb{R}^d)$ where $U[z_i, \lambda_i]$ are weakly interacting (the definition of weakly interaction can be found in [41, Definition 3.1]). We choose to state the question (Q) since it is more close to Theorem 1.1 (Struwe [75]).*

In the recent papers [22,41], Ciraolo, Figalli, Glaudo and Maggi proved the following results by the energy method:

(1) (Ciraolo-Figalli-Maggi [22]) Let $d \geq 3$ and $u \in D^{1,2}(\mathbb{R}^d)$ be positive such that $\|\nabla u\|^2_{L^2(\mathbb{R}^d)} \leq \frac{3}{2}S^{\frac{d}{2}}$ and $\left\|\Delta u + |u|^{\frac{4}{d-2}}u\right\|_{D^{-1,2}} \leq \delta$ for some $\delta > 0$ sufficiently small, then

$$dist_{D^{1,2}}(u, \mathcal{M}^1_0) \lesssim \left\|\Delta u + |u|^{\frac{4}{d-2}}u\right\|_{D^{-1,2}}.$$

(2) (Figalli-Glaudo [41]) Let $u \in D^{1,2}(\mathbb{R}^d)$ be nonnegative and $\nu \geq 2$ be an integer such that

$$\left(\nu - \frac{1}{2}\right) S^{\frac{d}{2}} < \|\nabla u\|^2_{L^2(\mathbb{R}^d)} < \left(\nu + \frac{1}{2}\right) S^{\frac{d}{2}}$$

and $\left\|\Delta u + |u|^{\frac{4}{d-2}}u\right\|_{D^{-1,2}} \leq \delta$ for some $\delta > 0$ sufficiently small, then

$$dist_{D^{1,2}}(u, \mathcal{M}^\nu_0) \lesssim \left\|\Delta u + |u|^{\frac{4}{d-2}}u\right\|_{D^{-1,2}}$$

for $3 \leq d \leq 5$.

It is worth pointing out that a **significant finding** in [41] is that Figalli and Glaudo construct a counterexample for $\nu = 2$ and $d \geq 6$ to show that the answer of the question (Q) for $\nu \geq 2$ and $d \geq 6$ is *negative*! Based on their counterexample for $d \geq 6$, Figalli and Glaudo conjectured in [41] that the stability of critical points at infinity of the corresponding functional of the Sobolev inequality (1.9) should be of the following nonlinear form:

$$dist_{D^{1,2}}(u, \mathcal{M}^\nu_0) \lesssim \begin{cases} \|\Delta u + |u|u\|_{D^{-1,2}}|\ln(\|\Delta u + |u|u\|_{D^{-1,2}})|, & \nu \geq 2 \text{ and } d = 6; \\ \left\|\Delta u + |u|^{\frac{4}{d-2}}u\right\|^{\gamma(d)}_{D^{-1,2}}, & \nu \geq 2 \text{ and } d \geq 7 \end{cases}$$

with $0 < \gamma(d) < 1$. In the recent work [23], the first author, together with Deng and Sun, proved that the stability of critical points at infinity of the corresponding functional of the Sobolev inequality (1.9) is actually of the following nonlinear form by combining the energy method, the reduction argument and the blow-up analysis:

$$dist_{D^{1,2}}(u, \mathcal{M}^\nu_0) \lesssim \begin{cases} \|\Delta u + |u|u\|_{D^{-1,2}}|\ln(\|\Delta u + |u|u\|_{D^{-1,2}})|^{\frac{1}{2}}, & \nu \geq 2 \text{ and } d = 6; \\ \left\|\Delta u + |u|^{\frac{4}{d-2}}u\right\|^{\frac{d+2}{2(d-2)}}_{D^{-1,2}}, & \nu \geq 2 \text{ and } d \geq 7. \end{cases}$$

Moreover, the powers of the right hand sides in the above estimates are shown to be *optimal* in [23] by constructing related examples. We remark that besides its own mathematical interests, the stability of critical points at infinity of the corresponding functional of the Sobolev inequality (1.9) can be used to prove quantitative convergence results for the fast diffusion equation, see, for example [22, 41] and due to the mathematical interests and potential applications, the stability of critical points at infinity of the corresponding functional of other famous functional inequalities have already been established, see, for example by Aryan in [2] and De Nitti and Konig in [26] for the fractional Sobolev inequality, and by us in [78] for the CKN inequality (1.2) in the nondegenerate case. In particular, the stability of critical points at infinity of the corresponding functional of the CKN inequality (1.2) in the nondegenerate case is stated as follows.

⟨thm0002⟩ **Theorem 1.2.** *Let $d \geq 2$ and $\nu \geq 1$ be positive integers and either*

(*i*) $b_{FS}(a) < b < a + 1$ *with* $a < 0$ *or* (*ii*) $a < b < a+1$ *with* $a \geq 0$ *and* $a+b > 0$.

Then for any nonnegative $u \in D^{1,2}_a(\mathbb{R}^d)$ *such that*

$$\left(\nu - \frac{1}{2}\right)(S^{-1}_{a,b})^{\frac{p+1}{p-1}} < \|u\|^2_{D^{1,2}_a(\mathbb{R}^d)} < \left(\nu + \frac{1}{2}\right)(S^{-1}_{a,b})^{\frac{p+1}{p-1}}$$

and $\Gamma(u) \leq \delta$ *with some* $\delta > 0$ *sufficiently small, we have*

$$dist_{D^{1,2}_a}(u, \mathcal{Z}^\nu_0) \lesssim \begin{cases} \Gamma(u), & p > 2 \text{ or } \nu = 1, \\ \Gamma(u)|\log\Gamma(u)|^{\frac{1}{2}}, & p = 2 \text{ and } \nu \geq 2, \\ (\Gamma(u))^{\frac{p}{2}}, & 1 < p < 2 \text{ and } \nu \geq 2, \end{cases} \tag{1.11}$$ eqnnewnew0001

where $\Gamma(u) = \|div(|x|^{-a}\nabla u) + |x|^{-b(p+1)}|u|^{p-1}u\|_{D^{-1,2}_a(\mathbb{R}^d)}$ *and*

$$\mathcal{Z}^\nu_0 = \left\{\sum_{i=1}^{\nu} W_{\tau_i} \mid \tau_i > 0\right\}.$$

Moreover, the powers of the right hand sides in the above estimates are sharp in the sense that there exists $\{u_n\} \subset D^{1,2}_a(\mathbb{R}^d)$ *which are nonnegetive and* $\{\tau_{j,n}\} \subset \mathbb{R}_+ := (0, +\infty)$ *such that*

$$\left\|u_n - \sum_{j=1}^{\nu} W_{\tau_{j,n}}\right\|_{D^{1,2}_a(\mathbb{R}^d)} \sim \begin{cases} \Gamma(u_n), & p > 2 \text{ or } \nu = 1, \\ \Gamma(u_n)|\log\Gamma(u_n)|^{\frac{1}{2}}, & p = 2 \text{ and } \nu \geq 2, \\ (\Gamma(u_n))^{\frac{p}{2}}, & 1 < p < 2 \text{ and } \nu \geq 2. \end{cases}$$

We remark that Theorem 1.2 is a direct generalization of the Ciraolo-Figalli-Maggi, Figalli-Glaudo and Deng-Sun-Wei results in [22, 23, 41] for the Sobolev inequality (1.9) to the CKN inequality (1.2) in the nondegenerate case, which was mainly based on the following Struwe decomposition of critical points at infinity of the corresponding functional of the CKN inequality (1.2).

⟨prop0002⟩ **Proposition 1.1.** *([78, Proposition 3.2] or [13, Lemma 4.2]) Let* $d \geq 2$ *and* $\nu \geq 1$ *be positive integers and either*

(*i*) $b_{FS}(a) \leq b < a+1$ *with* $a < 0$ *or*
(*ii*) $a < b < a+1$ *with* $a \geq 0$ *and* $a+b > 0$.

If $\{w_n\}$ *be a nonnegative sequence with*

$$\left(\nu - \frac{1}{2}\right)(S^{-1}_{a,b})^{\frac{p+1}{p-1}} < \|w_n\|^2_{D^{1,2}_a(\mathbb{R}^d)} < \left(\nu + \frac{1}{2}\right)(S^{-1}_{a,b})^{\frac{p+1}{p-1}}$$

then there exists $\{\tau_{i,n}\} \subset \mathbb{R}_+ := (0, +\infty)$, *satisfying*

$$\min_{i \neq j}\left\{\max\left\{\frac{\tau_{i,n}}{\tau_{j,n}}, \frac{\tau_{j,n}}{\tau_{i,n}}\right\}\right\} \to +\infty$$

as $n \to \infty$ *for* $\nu \geq 2$, *such that*

(1) $w_n = \sum_{i=1}^{\nu} W_{\tau_{i,n}} + o_n(1)$ *in* $D^{1,2}_a(\mathbb{R}^d)$.
(2) $\|w_n\|^2_{D^{1,2}_a(\mathbb{R}^d)} = \nu\|W\|^2_{D^{1,2}_a(\mathbb{R}^d)} + o_n(1)$.

1.2. **Main result.** Since the Struwe decomposition (Proposition 1.1) of critical points at infinity of the corresponding functional of the CKN inequality (1.2) also holds in the degenerate case $b = b_{FS}(a)$ with $a < 0$. It is natural to ask the following question:

($\mathcal{Q}$) Does the stability of critical points at infinity of the corresponding functional of the CKN inequality (1.2) like (1.11) holds true in the degenerate case?

We shall answer the natural question ($\mathcal{Q}$) by proving the following sharp result.

⟨thmn0001⟩ **Theorem 1.3.** *Let $d \geq 2$, $\nu \geq 1$ be positive integers and $b = b_{FS}(a)$ with $a < 0$.*

(1) *Suppose $u \in D_a^{1,2}(\mathbb{R}^d)$ be a nonnegative function such that*

$$\left(\nu - \frac{1}{2}\right)\left(S_{a,b}^{-1}\right)^{\frac{p+1}{p-1}} < \|u\|^2_{D_a^{1,2}(\mathbb{R}^d)} < \left(\nu + \frac{1}{2}\right)\left(S_{a,b}^{-1}\right)^{\frac{p+1}{p-1}}. \tag{1.12}$$ eqqqnew0001

Then we have $dist_{D_a^{1,2}}(u, \mathcal{Z}_0^\nu) \lesssim (\Gamma(u))^{\frac{1}{3}}$, provided $\Gamma(u)$ sufficiently small, where $\Gamma(u)$ and $\mathcal{Z}_0^\nu$ are given in Theorem 1.2.

(2) *There exists $\{u_n\} \subset D_a^{1,2}(\mathbb{R}^d)$, which are nonnegative and satisfies (1.12) with $\nu = 2$ and $\Gamma(u_n) \to 0$, and $\{\tau_{j,n}\} \subset \mathbb{R}_+ := (0, +\infty)$ such that*

$$\left\|u_n - \sum_{j=1}^{2} W_{\tau_{j,n}}\right\|_{D_a^{1,2}(\mathbb{R}^d)} \sim (\Gamma(u_n))^{\frac{1}{3}}.$$

Remark 1.2. (*i*) *Theorem 1.3 is* **rather surprising** *since the optimal power of the stability of critical points at infinity of the corresponding functional of the CKN inequality (1.2) in the degenerate case is an* **absolute constant** *$\frac{1}{3}$ which is independent of p and ν! This is a completely new finding in the studies on the stability of critical points at infinity of the corresponding functional of functional inequalities. This new finding can be explained by the optimal example of the stability stated in (2) of Theorem 1.3 which is constructed in the last section. Roughly speaking, for the two-bubble case as an example, the optimal power of the stability of critical points at infinity of the corresponding functional of the CKN inequality (1.2) depends on two values, the interaction between bubbles which is measured by the distance of these bubbles and the projections on their nontrivial kernels. If the interaction wins the projections then the optimal power of the stability of critical points at infinity of the corresponding functional will be the values in Theorem 1.2 which depends on p and ν. If the projections win the interaction then the optimal power of the stability of critical points at infinity of the corresponding functional will be the absolute constant $\frac{1}{3}$. If the projections and the interaction are comparable then the optimal power of the stability of critical points at infinity of the corresponding functional can be any values between the values of Theorem 1.2 and the absolute constant $\frac{1}{3}$, which depends on the ratio of the projections and the interaction. We refer the readers to Remark 9.1 for more details. Since the function $u \in D_a^{1,2}(\mathbb{R}^d)$ discussed in (1) of Theorem 1.3 is arbitrary, the optimal power of the stability must be the absolute constant $\frac{1}{3}$.*

(*ii*) *In preparing this paper, we knew from personal communications with Professor W. Zou that their group was also working on the question* ($\mathcal{Q}$)

for the one-bubble case. Moreover, we notice that in a very recent paper [81], the optimal stability for the one-bubble case has been established by Zhou and Zou, while for the multi-bubble case only a partial result is obtained by them. Indeed, by assuming that the projections on the nontrivial kernels are much smaller than the interactions, they obtained a stability result with the same exponent in the non-degenerate case (Theorem 1.2), which is just one of the three cases we explained in (i). *However, as explained in* (i), *Theorem 1.3 tells that the most important contribution in the optimal stability comes exactly from the projections on the nontrivial kernels.*

1.3. **Sketch of the proof.** The basic idea in proving Theorem 1.3 is still to apply the Deng-Sun-Wei arguments in [23], as in [78]. Since the bubble W is degenerate now, we need also employ the Frank-Peteranderl strategy in [54]. However, since our problem is in the critical point setting, new ideas and new techniques are also needed to develop. Let us now explain our strategy in proving Theorem 1.3 in what follows.

In the first step, we need to set a good problem. Suppose that $u \in D^{1,2}_a(\mathbb{R}^d)$ be a nonnegative function. We first transform the problem onto the cylinder $\mathcal{C} = \mathbb{R} \times \mathbb{S}^{d-1}$, as usual. Then, based on the Struwe decomposition (Proposition 1.1), the basic idea is to decompose v, the image of the bubble u on the cylinder $\mathcal{C}$, into two parts, as in [22, 23, 41, 78], by considering the following minimizing problem

$$\inf_{\overrightarrow{\alpha}_\nu \in (\mathbb{R}_+)^\nu, \overrightarrow{s}_\nu \in \mathbb{R}^\nu} \left\| v - \sum_{j=1}^{\nu} \alpha_j \Psi_{s_j} \right\|^2,$$

so that we can write $v = \sum_{j=1}^{\nu} \alpha_j^* \Psi_{s_j^*} + \rho$ where the remaining term ρ is orthogonal to $\left\{\Psi_{s_j^*}\right\}$ and $\left\{\partial_t \Psi_{s_j^*}\right\}$ in $H^1(\mathcal{C})$ with Ψ being the image of the bubble W on the cylinder $\mathcal{C}$ and $\Psi_s(t) = \Psi(t-s)$. Since the bubble Ψ is degenerate now, we need further decompose the remaining term ρ and further write

$$v = \sum_{j=1}^{\nu} \alpha_j^* \Psi_{s_j^*} + \left(\sum_{j=1}^{\nu} \sum_{l=1}^{d} \beta_{j,l}^* w_{j,l} \right) + \rho_*,$$

where $\{w_{j,l}\}$ are the nontrivial kernels of $\Psi_{s_j^*}$ and the remaining term ρ_* is orthogonal to $\{\Psi_{s_j^*}\}$, $\left\{\partial_t \Psi_{s_j^*}\right\}$ and $\{w_{j,l}\}$ in $H^1(\mathcal{C})$, as in [54]. Since we are in the critical point setting, the remaining term ρ_* will also satisfy an elliptic equation:

$$\begin{cases} \mathcal{L}(\rho_*) = f + \mathcal{R}_{int} + \mathcal{N}, \quad \text{in } \mathcal{C}, \\ \langle \Psi_{s_j^*}, \rho_* \rangle = \langle \partial_t \Psi_{s_j^*}, \rho_* \rangle = \langle w_{j,l}, \rho_* \rangle = 0 \quad \text{for all } 1 \leq j \leq \nu \text{ and all } 1 \leq l \leq d. \end{cases}$$

Now, our aim is to control $\sum_{j=1}^{\nu} \left|\alpha_j^* - 1\right|$, $\{\beta_{k,l}^*\}$ and $\|\rho_*\|$ by $\|f\|_{H^{-1}}$, which also needs us to control the interaction between bubbles by $\|f\|_{H^{-1}}$ due to the regular interaction $\mathcal{R}_{int}$.

In the second step, we need to expand the nonlinear part $\mathcal{N}$ and the regular interaction $\mathcal{R}_{int}$ in the equation of ρ_* to control $\sum_{j=1}^{\nu} \left|\alpha_j^* - 1\right|$, $\{\beta_{k,l}^*\}$, the interaction between bubbles and $\|\rho_*\|$ by $\|f\|_{H^{-1}}$, as in [23, 78]. Roughly speaking, we shall further decompose ρ_* into two parts, the first part is regular enough so that we can control it very well in any reasonable sense and the second part is (possible) singular due to the (possible) singularity of the data $f \in H^{-1}$ but it can lie in the positive definite part of the linear operator $\mathcal{L}$ and is small enough. We notice that in the functional inequality setting, Frank and Peteranderl have proved in [54] that the optimal Bianchi-Egnell stability of the CKN inequality (1.2) in the degenerate case is quartic and the projection onto nontrivial kernels dominates the remaining term, thus, it is reasonable to expand the nonlinear part $\mathcal{N}$ at least up to the fourth order terms and to ensure that the (possible) singular part of the remaining term ρ_* is smaller than or equal to β_*^4 by decomposing it in a suitable way, where $\beta_* := \max_{k,l}\left|\beta_{k,l}^*\right|$. Keeping this in minds, we expand the nonlinear part $\mathcal{N}$ up to the fourth order terms and pick up all regular parts of $\mathcal{R}_{int}+\mathcal{N}$ which are potentially larger than β_*^4 and solve several linear equations to decompose ρ_* into $\rho_*=\rho_0+\rho_{**}^\perp$, where ρ_0 is the regular part and $\rho_{**}^\perp$ is the (possible) singular part. We remark that we need two sub-steps to pick up all regular parts of $\mathcal{R}_{int}+\mathcal{N}$ which are potentially larger than β_*^4. In the first sub-step, we pick up the leading order terms in this progress which, roughly speaking, behaviors like β_*^2. In the second sub-step, we further pick up the next order terms in this progress which are generated by the leading order terms and roughly speaking, behaviors like β_*^t for $2<t<4$. We remark that in order to pick up all the next order terms which are potentially larger than β_*^4, we need to iterate second sub-step for $\max\left\{\left[\frac{2p-1}{p-1}\right],4\right\}$ times.

In the third step, we need to multiply the equation of ρ_* by $\{\Psi_{s_j^*}\}$, $\{\partial_t\Psi_{s_j^*}\}$ and $\{w_{j,l}\}$, and multiply the equation of $\rho_{**}^\perp$ by $\rho_{**}^\perp$ to establish the relations of $\sum_{j=1}^{\nu}\left|\alpha_j^*-1\right|$, $\{\beta_{k,l}^*\}$, the interaction between bubbles, $\|\rho_*\|$ and $\|f\|_{H^{-1}}$, as in [23,78]. It is worth pointing out that even though we have picked up all regular parts of $\mathcal{R}_{int}+\mathcal{N}$ which are potentially larger than β_*^4, we still need to refine the expansion of the nonlinear part $\mathcal{N}$ for three times, respectively, in estimating $\sum_{j=1}^{\nu}\left|\alpha_j^*-1\right|$, the interaction between bubbles and $\|\rho_{**}^\perp\|$. In particular, to keep the (possible) singular part $\rho_{**}^\perp$ in the desired size, we need to expand the nonlinear part $\mathcal{N}$ up to the $\max\left\{\left[\frac{2p-1}{p-1}\right],6\right\}$th order terms. Moreover, in order to ensure that the (possible) singular part of the remaining term ρ_* is smaller than or equal to β_*^4, we also need to decompose the regular part ρ_0 into several parts, analyze the symmetry of these parts and then full use these symmetry and the orthogonality of $\rho_{**}^\perp$ in the equation of $\rho_{**}^\perp$. However, even though these estimates are good enough, we still can not finally control $\sum_{j=1}^{\nu}\left|\alpha_j^*-1\right|$, $\{\beta_{k,l}^*\}$, the interaction between bubbles and $\|\rho_*\|$ only by $\|f\|_{H^{-1}}$ by only using the above analysis. This is mainly because β_*^4 can only be bounded from above by a very special quartic form, as observed by Frank and Peteranderl in [54]. Thus, we need to find out the right third equation to march this special quartic form and ensure that we will not enlarge the upper bounds in the original estimates of $\sum_{j=1}^{\nu}\left|\alpha_j^*-1\right|$, $\{\beta_{k,l}^*\}$, the interaction between bubbles, $\|\rho_*\|$ and $\|f\|_{H^{-1}}$ in this progress, which is achieved by full using the symmetry and the orthogonality of the remaining term ρ_* once more.

In the final step, we use all above estimates of $\sum_{j=1}^{\nu}\left|\alpha_j^*-1\right|$, $\{\beta_{k,l}^*\}$, the interaction between bubbles, $\|\rho_*\|$ and $\|f\|_{H^{-1}}$ and the estimates of β_*^4 established by Frank and Peteranderl in [54] to derive the desired estimate in (1) of Theorem 1.3. The proof of (2) of Theorem 1.3 is achieved by constructing an example of the case $\nu=2$ and using the good ansatz $\sum_{j=1}^{2}\alpha_j^*\Psi_{s_j^*}+\left(\sum_{j=1}^{2}\sum_{l=1}^{d}\beta_{j,l}^*w_{j,l}\right)+\rho_0$ in the proof of (1) of Theorem 1.3.

We believe that our strategy of proofs may be useful to study many other problems in which degeneracy appears.

1.4. **Structure of this paper.** In section 2, we give some preliminaries. In section 3, we introduce the setting of the problem as stated above by decomposing a given function into three parts, the projection on bubbles, the projection on nontrivial kernels and the remaining term. In section 4, we expand the nonlinear part of the remaining term up to the fourth order terms to pick up all possible leading order terms in it and use these possible leading order terms to decompose the remaining term into two parts, the good regular part and the (possible) singular part, as stated above. In section 5, we refine the expansion of the nonlinear part in the first time by adding the regular part of the remaining term into the ansatz and estimates the differences of the projection on bubbles. In section 6, we refine the expansion of the nonlinear part in the second time by expanding it up to the sixth order terms to estimate the interaction between bubbles. In section 7, we refine the expansion of the nonlinear part in the third time by expanding it up to the max $\left\{\left[\frac{2p-1}{p-1}\right],4\right\}$th order terms to estimate the (possible) singular part in the remaining part. In section 8, we finally estimate the projection on nontrivial kernels and prove (1) of Theorem 1.3. In section 9, we construct an optimal example and prove (2) of Theorem 1.3.

1.5. **Notations.** Throughout this paper, $a \sim b$ means that $C'b \leq a \leq Cb$ and $a \lesssim b$ means that $a \leq Cb$ where C and C' are positive constants. $\sigma \in (0,1)$ is used to denote a positive constant which can be taken arbitrary small if necessary, $(\mathbb{R}_+)^\nu = ((0,+\infty))^\nu$ and we also denote

$$A_{p,l-1} = \frac{\pi_{j=0}^{l-1}(p-j)}{l!} \quad \text{and} \quad n_0 = \min\left\{n \in \mathbb{N} \mid n \geq \left\{\frac{p}{p-1}, 4\right\}\right\}.$$

2. Preliminaries

The CKN inequality (1.2) can be rewritten into the following minimizing problem:

$$S_{a,b}^{-1} = \inf_{u \in D_a^{1,2}(\mathbb{R}^d)\backslash\{0\}} \frac{\|u\|^2_{D_a^{1,2}(\mathbb{R}^d)}}{\|u\|^2_{L^{p+1}(|x|^{-b(p+1)},\mathbb{R}^d)}}, \tag{2.1}$$ eq0002

where $L^{p+1}(|x|^{-b(p+1)},\mathbb{R}^d)$ is the usual weighted Lebesgue space and its usual norm is given by (1.8). The Euler-Lagrange equation of the minimizing problem (2.1) is given by

$$-div(|x|^{-a}\nabla u) = |x|^{-b(p+1)}|u|^{p-1}u, \quad u \in D_a^{1,2}(\mathbb{R}^d). \tag{2.2}$$ eq0018

It is well known (cf. [13, Proposition 2.2]) that $D_a^{1,2}(\mathbb{R}^d)$, the Hilbert space given by (1.1), is isomorphic to the Hilbert space $H^1(\mathcal{C})$ by the transformation

$$u(x) = |x|^{-(a_c-a)}v\left(-\ln|x|, \frac{x}{|x|}\right), \tag{2.3}$$ eq0007

where we recall that we denote $a_c = \frac{d-2}{2}$ as in [30–32], $\mathcal{C} = \mathbb{R}\times\mathbb{S}^{d-1}$ is the standard cylinder, $H^1(\mathcal{C})$ is the Hilbert space with the inner product given by

$$\langle w, v\rangle = \int_{\mathcal{C}} \left(\nabla w\nabla v + (a_c-a)^2 wv\right) d\mu$$

with $d\mu$ the volume element on $\mathcal{C}$ and the induced norm is denoted by $\|\cdot\|$. By (2.3), the minimizing problem (2.1) is equivalent to the following minimizing problem:

$$S_{a,b}^{-1} = \inf_{v\in H^1(\mathcal{C})\backslash\{0\}} \frac{\|v\|^2}{\|v\|^2_{L^{p+1}(\mathcal{C})}}, \tag{2.4}$$ eq0009

where $\|\cdot\|_{L^{p+1}(\mathcal{C})}$ is the usual norm in the Lebesgue space $L^{p+1}(\mathcal{C})$. For the sake of simplicity, we denote

$$L^{p+1} := L^{p+1}(\mathcal{C}) \quad \text{and} \quad H^1 := H^1(\mathcal{C})$$

in what follows. Let $t = -\ln|x|$ and $\theta = \frac{x}{|x|}$ for $x \in \mathbb{R}^N\backslash\{0\}$, then the Euler-Lagrange equation of (2.1) in terms of u given by (2.2) is equivalent to the following Euler-Lagrange equation of (2.4) in terms of v:

$$-\Delta_\theta v - \partial_t^2 v + (a_c - a)^2 v = |v|^{p-1}v, \quad v \in H^1(\mathcal{C}), \tag{2.5}$$ eq0006

where Δ_θ is the Laplace-Beltrami operator on $\mathbb{S}^{d-1}$.

Clearly, minimizers of (2.1) are ground states of (2.2). Moreover, by the transformation (2.3), the linear equation (1.7) can be rewritten as follows:

$$-\Delta_\theta v - \partial_t^2 v + (a_c - a)^2 v = p\Psi^{p-1}v, \quad v \in H^1(\mathcal{C}), \tag{2.6}$$ eq0016

where

$$\Psi(t) = \left(\frac{(p+1)(a_c-a)^2}{2}\right)^{\frac{1}{p-1}} \left(cosh\left(\frac{(a_c-a)(p-1)}{2}t\right)\right)^{-\frac{2}{p-1}} \tag{2.7}$$ eq0026

is the image of $W(x)$ which is given by (1.4) under the transformation (2.3). Since (2.6) is translational invariance, it follows from (1.6) and the transformation (2.3) that

$$\Psi_s'(t) = \Psi'(t-s) = \partial_t\Psi(t-s) = -\partial_s\Psi(t-s)$$

is the only nonzero solution of (2.6) in $H^1(\mathcal{C})$ under the condition (1.5) except $b = b_{FS}(a)$.

For the special case $b = b_{FS}(a)$, the bubble $\Psi(t)$ is degenerate in $H^1(\mathcal{C})$ in the sense that the nonzero solution of (2.6) in $H^1(\mathcal{C})$ is not only generated by the translational invariance of (2.5). Fortunately, we have the following lemma which provides a complete understanding of the solutions of the linear equation (2.6) in $H^1(\mathcal{C})$.

⟨lem0001⟩ **Lemma 2.1.** *([54, Lemma 7]) Let $d \geq 2$, $a < 0$ and $b = b_{FS}(a)$. Then any solution of the linear equation (2.6) in $H^1(\mathcal{C})$ is the linear combination of $\partial_t\Psi$ and $\Psi^{\frac{p+1}{2}}\theta_1, \Psi^{\frac{p+1}{2}}\theta_2, \cdots, \Psi^{\frac{p+1}{2}}\theta_d$, where θ_l are the standard spherical harmonics of degree 1 on $\mathbb{S}^{d-1}$.*

⟨rmk0001⟩ **Remark 2.1.** *As in [54], we call $\partial_t\Psi$ the trivial kernel of the linear equation (2.6) in $H^1(\mathcal{C})$ and call $\Psi^{\frac{p+1}{2}}\theta_1, \Psi^{\frac{p+1}{2}}\theta_2, \cdots, \Psi^{\frac{p+1}{2}}\theta_d$ the nontrivial kernels of the linear equation (2.6) in $H^1(\mathcal{C})$. Moreover, since θ_l are odd on $\mathbb{S}^{d-1}$, $\partial_t\Psi$ is odd in $\mathbb{R}$ and Ψ is even in $\mathbb{R}$, by (2.5) and (2.6), we have the following orthogonal conditions:*

$$\langle\Psi, \partial_t\Psi\rangle = 0, \quad \langle\Psi, w_l\rangle = 0, \quad \langle\partial_t\Psi, w_l\rangle = 0 \quad \textit{and} \quad \langle w_j, w_l\rangle = 0$$

for all $1 \leq j \neq l \leq d$, where $w_l = \Psi^{\frac{p+1}{2}}\theta_l$.

3. Setting of the problem

Let $d \geq 2$, $a < 0$ and $b = b_{FS}(a)$. Then direct calculations show that

$$(a_c - a)^2 = \frac{4(d-1)}{(p+1)^2 - 4}.$$

For the sake of simplicity, we denote that

$$S_{FS} := S_{a,b} \quad \text{and} \quad \Lambda_{FS} := \frac{4(d-1)}{(p+1)^2 - 4}$$

for $d \geq 2$, $a < 0$ and $b = b_{FS}(a)$. Let $v \in H^1(\mathcal{C})$ be a nonnegative function such that

$$\left(\nu - \frac{1}{2}\right)\left(S_{FS}^{-1}\right)^{\frac{p+1}{p-1}} < \|v\|^2 < \left(\nu + \frac{1}{2}\right)\left(S_{FS}^{-1}\right)^{\frac{p+1}{p-1}}$$

for some positive integer $\nu \geq 1$ and denote

$$f := -\Delta_\theta v - \partial_t^2 v + \Lambda_{FS} v - v^p. \tag{3.1}$$
eq0060

Then it is easy to see that $f \in H^{-1}(\mathcal{C})$, where $H^{-1}(\mathcal{C})$ is the dual space of $H^1(\mathcal{C})$. For the sake of simplicity, we denote $H^{-1}(\mathcal{C})$ by H^{-1}.

By Proposition 1.1 and (2.3), there exists $(s_{1,\natural}, s_{2,\natural}, \cdots, s_{\nu,\natural})$ satisfying

$$\min_{i \neq j} |s_{i,\natural} - s_{j,\natural}| \to +\infty \quad \text{as} \quad \|f\|_{H^{-1}} \to 0,$$

such that

$$\left\| v - \sum_{j=1}^{\nu} \Psi_{s_{j,\natural}} \right\|^2 \to 0 \quad \text{as} \quad \|f\|_{H^{-1}} \to 0. \tag{3.2}$$
eqn0005

Thus, we can rewrite

$$v = \sum_{j=1}^{\nu} \Psi_{s_{j,\natural}} + \text{a remaining term}$$

in H^1 as $\|f\|_{H^{-1}} \to 0$. To obtain an optimal decomposition in the above form, let us consider the following minimizing problem:

$$\inf_{\overrightarrow{\alpha}_\nu \in (\mathbb{R}_+)^\nu, \overrightarrow{s}_\nu \in \mathbb{R}^\nu} \left\| v - \sum_{j=1}^{\nu} \alpha_j \Psi_{s_j} \right\|^2, \tag{3.3}$$
eqn0001

where $\overrightarrow{\alpha}_\nu = (\alpha_1, \alpha_2, \cdots, \alpha_\nu)$ and $\overrightarrow{s}_\nu = (s_1, s_2, \cdots, s_\nu)$. By (3.2) and similar arguments used for [78, Proposition 4.1] (see also [54, Proposition 2]), we know that the variational problem (3.3) has minimizers, say $(\overrightarrow{\alpha}^*_\nu, \overrightarrow{s}^*_\nu)$, such that

$$\max_{1 \leq j \leq \nu} |\alpha^*_j - 1| \to 0 \text{ and } \min_{i \neq j} \left|s^*_i - s^*_j\right| \to +\infty \quad \text{as} \quad \|f\|_{H^{-1}} \to 0. \tag{3.4}$$
eqn1005

Thus, we can decompose

$$v = \sum_{j=1}^{\nu} \alpha^*_j \Psi_{s^*_j} + \rho \tag{3.5}$$
eqnewnew0002

where by (3.2),

$$\|\rho\|^2 = \inf_{\vec{\alpha}_\nu\in(\mathbb{R}_+)^\nu, \vec{s}_\nu\in\mathbb{R}^\nu} \left\| v - \sum_{j=1}^{\nu} \alpha_j \Psi_{s_j} \right\|^2 \to 0 \quad \text{as } \|f\|_{H^{-1}} \to 0 \tag{3.6}$$ eqn0008

and by the minimality of $(\vec{\alpha}^*_\nu, \vec{s}^*_\nu)$,

$$\left\langle \rho, \Psi_{s_j^*} \right\rangle = 0 \quad \text{and} \quad \left\langle \rho, \partial_t \Psi_{s_j^*} \right\rangle = 0 \quad \text{for all } 1 \le j \le \nu. \tag{3.7}$$ eqn0004

Since by Lemma 2.1, the linear equation (2.6) has nontrivial kernels in H^1 for $d \ge 2$, $a < 0$ and $b = b_{FS}(a)$, we need further decompose the remaining term as follows:

$$\rho = \left(\sum_{j=1}^{\nu} \sum_{l=1}^{d} \beta^*_{j,l} w_{j,l} \right) + \rho_*, \tag{3.8}$$ eqn0002

where for the sake of simplicity, we denote $w_{j,l} = \Psi_j^{\frac{p+1}{2}} \theta_l = w_l(t - s_j^*)$ and $\{\beta^*_{j,l}\}$ is chosen such that $\langle w_{j,l}, \rho_* \rangle = 0$ for all $1 \le j \le \nu$ and all $1 \le l \le d$. The above facts can be summarized into the following lemma.

⟨lem0002⟩ **Lemma 3.1.** *Let $d \ge 2$, $a < 0$ and $b = b_{FS}(a)$. Then we have the following decomposition of v:*

$$v = \sum_{j=1}^{\nu} \alpha_j^* \Psi_{s_j^*} + \left(\sum_{j=1}^{\nu} \sum_{l=1}^{d} \beta^*_{j,l} w_{j,l} \right) + \rho_*, \tag{3.9}$$ eqnewnew0003

where the remaining term ρ_* satisfies the following orthogonal conditions:

$$\left\langle \rho_*, \Psi_{s_j^*} \right\rangle = 0, \quad \left\langle \rho_*, \partial_t \Psi_{s_j^*} \right\rangle = 0 \quad \textit{and} \quad \langle \rho_*, w_{j,l} \rangle = 0 \tag{3.10}$$ eqn0003

for all $1 \le j \le \nu$ and all $1 \le l \le d$ with

$$\max_{1\le j\le\nu} |\alpha_j^* - 1| \to 0, \quad \min_{i\ne j} \left| s_i^* - s_j^* \right| \to +\infty \quad \textit{and} \quad \|\rho_*\| \to 0 \tag{3.11}$$ eqnewnew1005

as $\|f\|_{H^{-1}} \to 0$. Moreover, we also have

$$\inf_{\vec{\alpha}_\nu\in(\mathbb{R}_+)^\nu, \vec{s}_\nu\in\mathbb{R}^\nu} \left\| v - \sum_{j=1}^{\nu} \alpha_j \Psi_{s_j} \right\|^2 \sim \sum_{k=1}^{\nu} \sum_{l=1}^{d} \left(\beta^*_{k,l} \right)^2 + \|\rho_*\|^2. \tag{3.12}$$ eqnewnew0001

Proof. (3.9) can be obtained by (3.5) and (3.8), directly, while the orthogonal conditions of ρ_* are obtained by the choice of $\{\beta^*_{j,l}\}$, the orthogonal conditions of $\{w_{j,l}\}$ given in Remark 2.1 and the orthogonal condition of ρ given by (3.7). By (3.4) and (3.6), it remains to show that (3.12) holds true as $\|f\|_{H^{-1}} \to 0$. Indeed, by (3.8) and the orthogonal conditions of $w_{j,l}$ given in Remark 2.1 and the orthogonal conditions of ρ_* given by (3.10),

$$\begin{aligned} \|\rho\|^2 &= \sum_{k=1}^{\nu} \sum_{l=1}^{d} \left(\beta^*_{k,l} \right)^2 \|w_d\|^2 + \|\rho_*\|^2 \\ &\quad + 2 \sum_{m,n=1;m<n}^{\nu} \sum_{l=1}^{d} \beta^*_{n,l} \beta^*_{m,l} \langle w_{n,l}, w_{m,l} \rangle, \end{aligned} \tag{3.13}$$ eqn0006

where we have used the invariance of $\mathbb{S}^{d-1}$ and the norm $\|\cdot\|$ under the action of orthogonal matrix $O(d)$. Clearly, by (3.4) and (3.13), it is easy to see that

$$\|\rho\|^2 \sim \sum_{k=1}^{\nu}\sum_{l=1}^{d}\left(\beta_{k,l}^*\right)^2 + \|\rho_*\|^2,$$

which, together with (3.6), implies that (3.12) holds true as $\|f\|_{H^{-1}} \to 0$. □

For the sake of simplicity, we use the notations $\Psi_j := \Psi_{s_j^*}$, $\Psi_j^* = \alpha_j^*\Psi_j$,

$$\mathcal{U} := \sum_{j=1}^{\nu}\Psi_j^*, \quad \mathcal{U}_j = \mathcal{U} - \Psi_j^* = \sum_{i=1;i\neq j}^{\nu}\Psi_i^* \tag{3.14}$$

and

$$\mathcal{V}_j = \sum_{l=1}^{d}\beta_{j,l}^* w_{j,l} = \Psi_j^{\frac{p+1}{2}}\left(\sum_{l=1}^{d}\beta_{j,l}^*\theta_l\right), \quad \mathcal{V} := \sum_{j=1}^{\nu}\mathcal{V}_j. \tag{3.15}$$

Since Ψ_j are solutions of (2.5) and $w_{j,l}$ are solutions of (2.6), by (3.1), (3.5), (3.8) and (3.10), it is easy to see that the remaining term ρ_* satisfies:

$$\begin{cases}\mathcal{L}(\rho_*) = f + \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{N}, \quad \text{in } \mathcal{C},\\ \langle\Psi_j, \rho_*\rangle = \langle\partial_t\Psi_j, \rho_*\rangle = \langle w_{j,l}, \rho_*\rangle = 0 \quad \text{for all } 1\leq j\leq \nu \text{ and all } 1\leq l\leq d,\end{cases} \tag{3.16}$$

where $\mathcal{L}(\rho_*)$ is the linear operator given by

$$\begin{aligned}\mathcal{L}(\rho_*) &= -\partial_t^2\rho_* - \Delta_\theta\rho_* + \Lambda_{FS}\rho_* - p\mathcal{U}^{p-1}\rho_*\\ &= \left(-\partial_t^2\rho_* - \Delta_\theta\rho_* + \Lambda_{FS}\rho_* - p\left(\Psi_j^*\right)^{p-1}\rho_*\right) - p\left(\mathcal{U}^{p-1} - \left(\Psi_j^*\right)^{p-1}\right)\rho_*\\ &:= \mathcal{L}_j(\rho_*) - \mathcal{L}_{j,ex}(\rho_*)\end{aligned} \tag{3.17}$$

for all $j = 1, 2, \cdots, \nu$, $\mathcal{R}_1$ and $\mathcal{R}_2$ are the regular interactions given by

$$\begin{aligned}\mathcal{R}_1 &= \mathcal{U}^p - \sum_{j=1}^{\nu}\left(\Psi_j^*\right)^p + \sum_{j=1}^{\nu}\left(\left(\alpha_j^*\right)^p - \alpha_j^*\right)\Psi_j^p\\ &:= \mathcal{R}_{1,ex} + \sum_{j=1}^{\nu}\mathcal{R}_{1,j}\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}\mathcal{R}_2 &= \sum_{j=1}^{\nu}p\left(\mathcal{U}^{p-1} - \left(\Psi_j^*\right)^{p-1} + \left((\alpha_j^*)^{p-1} - 1\right)\Psi_j^{p-1}\right)\mathcal{V}_j\\ &= \sum_{j=1}^{\nu}p\left(\mathcal{U}^{p-1} - \left(\Psi_j^*\right)^{p-1}\right)\mathcal{V}_j + \sum_{j=1}^{\nu}p\left(\left((\alpha_j^*)^{p-1} - 1\right)\Psi_j^{p-1}\right)\mathcal{V}_j\\ &:= \mathcal{R}_{2,ex} + \sum_{j=1}^{\nu}\mathcal{R}_{2,j},\end{aligned} \tag{3.19}$$

and $\mathcal{N}$ is the only nonlinear part of ρ_* given by

$$\mathcal{N} = \left(\mathcal{U} + \mathcal{V} + \rho_*\right)^p - \mathcal{U}^p - p\mathcal{U}^{p-1}\left(\mathcal{V} + \rho_*\right). \tag{3.20}$$

By (3.12), to establish stability of the CKN inequality in the critical point setting for $d \geq 2$, $a < 0$ and $b = b_{FS}(a)$ as in [23, 41, 78], we shall control $\sum_{j=1}^{\nu} \left|\alpha_j^* - 1\right|$, $\{\beta_{k,l}^*\}$ and $\|\rho_*\|$ by $\|f\|_{H^{-1}}$.

4. Basic expansion of $\mathcal{N}$ and further decomposition of ρ_*

As stated in the introduction, to get optimal control on $\sum_{j=1}^{\nu} \left|\alpha_j^* - 1\right|$, $\{\beta_{k,l}^*\}$ and $\|\rho_*\|$ only by $\|f\|_{H^{-1}}$, we shall apply the ideas in [23] (see also [78]). Roughly speaking, we need to further decompose the remaining term ρ_* into two parts. The first part, say ρ_0, is regular enough in the sense that ρ_0 can be controlled by good weighted L^∞ norms. The second part, say $\rho_{**}^\perp$, is (possible) singular according to the (possible) lack of regularity of $f \in H^{-1}$ which is much smaller than ρ_0 in H^1. For this purpose, we need to firstly expand the nonlinear part $\mathcal{N}$ to pick up all possible leading order terms of the remaining term ρ_*.

4.1. **Basic expansion of $\mathcal{N}$.**

Since by [54, Theorem 1], the optimal Bianchi-Egnell stability of the CKN inequality for $d \geq 2$, $a < 0$ and $b = b_{FS}(a)$ is quartic, it is reasonable to first expand the nonlinear part $\mathcal{N}$ up to the fourth order terms.

⟨lem0003⟩ **Lemma 4.1.** *Let $d \geq 2$, $a < 0$ and $b = b_{FS}(a)$. Then we have the following expansion of the nonlinear part $\mathcal{N}$:*

$$\begin{aligned} \mathcal{N} &= A_{p,1}\mathcal{U}^{p-2}\left(\mathcal{V}^2 + 2\mathcal{V}\rho_*\right) + A_{p,2}\mathcal{U}^{p-3}\left(\mathcal{V}^3 + 3\mathcal{V}^2\rho_*\right) \\ &\quad + \mathcal{O}\left(\mathcal{U}^p\beta_*^4 + \chi_{p\geq 2}|\rho_*|^2 + |\rho_*|^p + \sum_{l=2}^{3} \beta_*^{\frac{2(l-p)_+}{p+1}} |\rho_*|^{l-\frac{2(l-p)_+}{p+1}}\right) \\ &:= \mathcal{N}_* + \mathcal{N}_{rem} \end{aligned} \tag{4.1}$$
eqn0018

in $\mathcal{C}$, where ρ_*, $\mathcal{U}$ and $\mathcal{V}$ are given by (3.9), (3.14) and (3.15), respectively, $\beta_* = \max_{j,l}\left|\beta_{j,l}^*\right|$, $a_\pm = \max\{\pm a, 0\}$ and

$$\chi_{p\geq 2} = \begin{cases} 1, & p \geq 2, \\ 0, & 1 < p < 2. \end{cases}$$

Proof. As in the proof of [54, Lemma 8], we introduce the set

$$\mathcal{A} = \{(\theta, t) \in \mathcal{C} \mid |\rho_*| \leq |\mathcal{V}|\}.$$

Note that by (3.11), (3.15) and $p > 1$, we have

$$|\mathcal{V}| \lesssim \beta_*\mathcal{U}^{\frac{p+1}{2}} \lesssim \beta_*\mathcal{U}. \tag{4.2}$$
eqn0191

Thus, we can apply the ideas in the proof of [54, Lemma 8] to expand the nonlinear part $\mathcal{N}$ in $\mathcal{A}$ and $\mathcal{A}^c$, respectively, as follows:

$$\begin{aligned} \mathcal{N} &= A_{p,1}\mathcal{U}^{p-2}\left(\mathcal{V}^2 + 2\mathcal{V}\rho_*\right) + A_{p,2}\mathcal{U}^{p-3}\left(\mathcal{V}^3 + 3\mathcal{V}^2\rho_*\right) \\ &\quad + \mathcal{O}\left(\mathcal{U}^{p-4}\left(\mathcal{V} + \rho_*\right)^4 + \mathcal{U}^{p-2}|\rho_*|^2\right) \\ &= A_{p,1}\mathcal{U}^{p-2}\left(\mathcal{V}^2 + 2\mathcal{V}\rho_*\right) + A_{p,2}\mathcal{U}^{p-3}\left(\mathcal{V}^3 + 3\mathcal{V}^2\rho_*\right) \\ &\quad + \mathcal{O}\left(\mathcal{U}^{3p-2}\beta_*^4 + \chi_{p\geq 2}|\rho_*|^2 + |\rho_*|^p\right) \end{aligned} \tag{4.3}$$
eqn0009

in $\mathcal{A}$ and

$$\mathcal{N} = \mathcal{O}\left(\chi_{p\geq 2}|\rho_*|^2 + |\rho_*|^p\right) \tag{4.4}$$
eqn0010

in $\mathcal{A}^c$. Since $\frac{2(2-p)}{p+1}\in(0,1)$ for $1<p<2$ and $\frac{2(3-p)}{p+1}\in(0,2)$ for $1<p<3$, by (4.2), we have

$$\sum_{l=2}^{3}\beta_*^{\frac{2(l-p)_+}{p+1}}|\rho_*|^{l-\frac{2(l-p)_+}{p+1}}\gtrsim A_{p,1}\mathcal{U}^{p-2}\left(\mathcal{V}^2+2\left|\mathcal{V}\rho_*\right|\right)+A_{p,2}\mathcal{U}^{p-3}\left(\left|\mathcal{V}\right|^3+3\mathcal{V}^2|\rho_*|\right)\tag{4.5}$$

eqn0017

in $\mathcal{A}^c$. Thus, (4.1) is obtained by combining (4.3), (4.4) and (4.5). □

We need to further expand the nonlinear part $\mathcal{N}_*$ to separate the bubbles, for this purpose, we introduce some necessary notations. For the sake of simplicity and without loss of generality, we assume that

$$-\infty:=s_0^*<s_1^*<s_2^*<\cdots<s_\nu^*<s_{\nu+1}^*:=+\infty.$$

We also denote

$$\tau_j=s_{j+1}^*-s_j^*,\quad \tau=\min_{j=1,2,\cdots,\nu-1}\tau_j\tag{4.6}$$

eqn0240

and

$$\begin{cases}\mathcal{B}_1=\left[s_1^*-\frac{\tau_1}{2},s_1^*+\frac{\tau_1}{2}\right]\times\mathbb{S}^{d-1},\\ \mathcal{B}_j=\left[s_j^*-\frac{\tau_{j-1}}{2},s_j^*+\frac{\tau_j}{2}\right]\times\mathbb{S}^{d-1},\quad 2\le j\le\nu-1,\\ \mathcal{B}_\nu=\left[s_\nu^*-\frac{\tau_{\nu-1}}{2},s_\nu^*+\frac{\tau_{\nu-1}}{2}\right]\times\mathbb{S}^{d-1},\\ \mathcal{B}_*=\cup_{j=1}^{\nu}\mathcal{B}_j.\end{cases}\tag{4.7}$$

eqnnewnew0004

⟨lem0004⟩ **Lemma 4.2.** *Let $d\ge 2$, $a<0$ and $b=b_{FS}(a)$. Then the nonlinear part $\mathcal{N}$ which is given by (3.20) can be further expanded as follows:*

$$\begin{aligned}\mathcal{N}\;=\;&\sum_{j=1}^{\nu}\left(A_{p,1}\left(\Psi_j^*\right)^{p-2}\left(\mathcal{V}_j^2+2\mathcal{V}_j\rho_*\right)+A_{p,2}\left(\Psi_j^*\right)^{p-3}\left(\mathcal{V}_j^3+3\mathcal{V}_j^2\rho_*\right)\right)\chi_{\mathcal{B}_j}\\&+\sum_{j=1}^{\nu}2A_{p,1}\left(\mathcal{U}^{p-2}\mathcal{V}-\left(\Psi_j^*\right)^{p-2}\mathcal{V}_j\right)\rho_*\chi_{\mathcal{B}_j}+\sum_{j=1}^{\nu}\mathcal{O}\left(\beta_*^2\mathcal{U}_j\Psi_j^{2p-3}\left(\Psi_j+\rho_*\right)\right)\chi_{\mathcal{B}_j}\\&+\left(2A_{p,1}\mathcal{U}^{p-2}\mathcal{V}\rho_*+\mathcal{U}^{p-3}\mathcal{V}^2\left(A_{p,1}\mathcal{U}+A_{p,2}\mathcal{V}\right)\right)\chi_{\mathcal{C}\setminus\mathcal{B}_*}\\&+\mathcal{O}\left(\beta_*^2\mathcal{U}^{2(p-1)}\rho_*\chi_{\mathcal{C}\setminus\mathcal{B}_*}\right)+\mathcal{N}_{rem}\end{aligned}\tag{4.8}$$

eqn0045

in $\mathcal{C}$, where $\mathcal{N}_{rem}$ is given in (4.1).

Proof. Since $\mathcal{U}_j>0$ in $\mathcal{C}$ for all $1\le j\le\nu$ by (3.14), by (2.7), (3.15) and the Taylor expansion, we have

$$\begin{aligned}\mathcal{U}^{p-\alpha}\mathcal{V}^{\alpha-1}\;&=\;\left(\Psi_j^*\right)^{p-\alpha}\mathcal{V}_j^{\alpha-1}+\mathcal{O}\left(\Psi_j^{p-\alpha-1}\mathcal{V}_j^{\alpha-1}\mathcal{U}_j+\Psi_j^{p-\alpha}\left|\sum_{i=1;i\ne j}^{\nu}\mathcal{V}_j^{\alpha-2}\mathcal{V}_i\right|\right)\\&=\;\left(\Psi_j^*\right)^{p-\alpha}\mathcal{V}_j^{\alpha-1}+\mathcal{O}\left(\beta_*^{\alpha-1}\left(\Psi_j^{\frac{\alpha(p-1)+p-3}{2}}\mathcal{U}_j+\Psi_j^{\frac{\alpha(p-1)-2}{2}}\mathcal{U}_j^{\frac{p+1}{2}}\right)\right)\end{aligned}$$

in $\mathcal{B}_j$ for all $1\le j\le\nu$ and

$$\mathcal{U}^{p-\alpha}\mathcal{V}^{\alpha-1}=\mathcal{O}\left(\beta_*^{\alpha-1}\mathcal{U}^{\frac{(\alpha+1)(p-1)}{2}}\right)$$

in $\mathcal{C}\backslash\mathcal{B}_*$, where $\alpha=2$ or $\alpha=3$. Similarly,

$$\begin{aligned}\mathcal{U}^{p-\alpha}\mathcal{V}^{\alpha} &= \left(\Psi_j^*\right)^{p-\alpha}\mathcal{V}_j^{\alpha}+\mathcal{O}\left(\Psi_j^{p-\alpha-1}\mathcal{V}_j^{\alpha}\mathcal{U}_j+\Psi_j^{p-\alpha}\left|\sum_{i=1;i\neq j}^{\nu}\mathcal{V}_j^{\alpha-1}\mathcal{V}_i\right|\right)\\ &= \left(\Psi_j^*\right)^{p-\alpha}\mathcal{V}_j^{\alpha}+\mathcal{O}\left(\beta_*^{\alpha}\left(\Psi_j^{\frac{\alpha(p-1)}{2}+p-1}\mathcal{U}_j+\Psi_j^{\frac{(\alpha+1)(p-1)}{2}}\mathcal{U}_j^{\frac{p+1}{2}}\right)\right)\end{aligned}$$

in $\mathcal{B}_j$ for all $1\leq j\leq\nu$ and

$$\mathcal{U}^{p-\alpha}\mathcal{V}^{\alpha}=\mathcal{O}\left(\beta_*^{\alpha}\mathcal{U}^{\frac{\alpha(p-1)}{2}+p}\right)$$

in $\mathcal{C}\backslash\mathcal{B}_*$. Thus, summarizing the above estimates of $\mathcal{U}^{p-\alpha}\mathcal{V}^{\alpha-1}$ and $\mathcal{U}^{p-\alpha}\mathcal{V}^{\alpha}$ in $\mathcal{N}_*$ and by $p>1$ and Lemma 4.1, we have the desired expansion of $\mathcal{N}$ given by (4.8), where $\mathcal{N}_{rem}$ is given in (4.1). □

4.2. **Further decomposition of ρ_*.** Recall that we shall control $\sum_{j=1}^{\nu}\left|\alpha_j^*-1\right|$, $\{\beta_{k,l}^*\}$ and $\|\rho_*\|$ by $\|f\|_{H^{-1}}$. However, due to the regular interactions $\mathcal{R}_1$ and $\mathcal{R}_2$, we have an additional term which is needed to control, that is, the interaction between bubbles. To measure the interaction between bubbles, we denote

$$Q_j=e^{-\sqrt{\Lambda_{FS}}\tau_j},\quad \varphi_{s_j^*}(t)=e^{-\sqrt{\Lambda_{FS}}|t-s_j^*|}\quad\text{and}\quad Q=e^{-\sqrt{\Lambda_{FS}}\tau}\tag{4.9}$$ eqn19993

where $1\leq j\leq\nu$ and τ_j and τ are given by (4.6).

⟨lem0005⟩ **Lemma 4.3.** *Let $d\geq 2$, $a<0$ and $b=b_{FS}(a)$. Then for every $\alpha,\beta\in\mathbb{R}$ such that $\alpha+\beta>0$ and $\beta>0$, we have*

$$\int_{\mathcal{B}_i}\Psi_i^{\alpha}\mathcal{U}_i^{\beta}d\mu\lesssim\begin{cases}Q^{\beta}, & \alpha>\beta,\\ Q^{\beta}\left|\log Q\right|, & \alpha=\beta,\\ Q^{\frac{\alpha+\beta}{2}}, & \alpha<\beta\end{cases}\tag{4.10}$$ eqnew0001

and

$$\int_{\mathcal{C}\backslash\mathcal{B}_*}\Psi_i^{\alpha}\Psi_j^{\beta}d\mu\lesssim Q^{\frac{\alpha+\beta}{2}+\min\{\alpha(i-1),\beta(j-1)\}},\tag{4.11}$$ eqnew0002

where $\mathcal{U}_j$, $\mathcal{B}_j$ and $\mathcal{B}_*$ are given by (3.14) and (4.7), respetcively.

Proof. Recall that

$$s_1^*<s_2^*<\cdots<s_{\nu-1}^*<s_{\nu}^*,$$

thus, by (2.7), (4.6), (4.9) and similar estimates for (4.8), we have

$$\Psi_i^{\alpha}\mathcal{U}_i^{\beta}\sim e^{-\alpha\sqrt{\Lambda_{FS}}(t-s_i^*)}e^{-\beta\sqrt{\Lambda_{FS}}(s_{i+1}^*-t)}\sim Q_i^{\beta}e^{-(\alpha-\beta)\sqrt{\Lambda_{FS}}(t-s_i^*)}\tag{4.12}$$ eqnewnew0006

in the region

$$\mathcal{B}_{i,+}:=\left[s_i^*,s_i^*+\frac{\tau_i}{2}\right]\times\mathbb{S}^{d-1}\tag{4.13}$$ eqnewnew0010

for all $i=1,2,\cdots,\nu-1$, while in the region

$$\mathcal{B}_{i,-}:=\left[s_i^*-\frac{\tau_{i-1}}{2},s_i^*\right]\times\mathbb{S}^{d-1}\tag{4.14}$$ eqnewnew0011

for all $i=2,3,\cdots,\nu$, we have

$$\Psi_i^{\alpha}\mathcal{U}_i^{\beta}\sim e^{-\alpha\sqrt{\Lambda_{FS}}(s_i^*-t)}e^{-\beta\sqrt{\Lambda_{FS}}(t-s_{i-1}^*)}\sim Q_{i-1}^{\beta}e^{-(\alpha-\beta)\sqrt{\Lambda_{FS}}(s_i^*-t)}.\tag{4.15}$$ eqnewnew0007

Thus, by direct calculations, we have

$$\begin{aligned}\int_{\mathcal{B}_i}\Psi_i^\alpha\mathcal{U}_i^\beta d\mu &\sim \int_{\mathcal{B}_{i,+}}\Psi_i^\alpha\Psi_{i+1}^\beta d\mu+\int_{\mathcal{B}_{i,-}}\Psi_i^\alpha\Psi_{i-1}^\beta d\mu\\ &\lesssim \begin{cases}Q^\beta, & \alpha>\beta,\\ Q^\beta\left|\log Q\right|, & \alpha=\beta,\\ Q^{\frac{\alpha+\beta}{2}} & \alpha<\beta,\end{cases}\end{aligned}$$

which implies that (4.10) holds true for $i=2,3,\cdots,\nu-1$. To prove (4.10) for $i=1$ or $i=\nu$ and (4.11), we denote

$$\begin{aligned}\mathcal{B}_{1,-,*} &= (-\infty,s_1^*]\times\mathbb{S}^{d-1}\\ &= \left(-\infty,s_1^*-\frac{\tau_1}{2}\right)\times\mathbb{S}^{d-1}\cup\left[s_1^*-\frac{\tau_1}{2},s_1^*\right]\times\mathbb{S}^{d-1}\\ &:= (\mathcal{C}\backslash\mathcal{B}_*)^-\cup\mathcal{B}_{1,-},\end{aligned}\tag{4.16}$$

eqnewnew0012

and

$$\begin{aligned}\mathcal{B}_{\nu,+,*} &= [s_\nu^*,+\infty)\times\mathbb{S}^{d-1}\\ &= \left[s_\nu^*,s_\nu^*+\frac{\tau_{\nu-1}}{2}\right]\times\mathbb{S}^{d-1}\cup\left(s_\nu^*+\frac{\tau_{\nu-1}}{2},+\infty\right)\times\mathbb{S}^{d-1}\\ &:= \mathcal{B}_{\nu,+}\cup(\mathcal{C}\backslash\mathcal{B}_*)^+.\end{aligned}\tag{4.17}$$

eqnewnew0013

Then by (2.7), (4.6) and similar estimates for (4.8), we have

$$\Psi_i^\alpha\Psi_j^\beta\lesssim\begin{cases}Q^{\min\{\alpha(i-1),\beta(j-1)\}}e^{-(\alpha+\beta)\sqrt{\Lambda_{FS}}(s_1^*-t)}, & \text{in } \mathcal{B}_{1,-,*},\\ Q^{\min\{\alpha(i-1),\beta(j-1)\}}e^{-(\alpha+\beta)\sqrt{\Lambda_{FS}}(t-s_\nu^*)}, & \text{in } \mathcal{B}_{\nu,+,*}.\end{cases}\tag{4.18}$$

eqnewnew0008

Thus, (4.10) for $i=1$ or $i=\nu$ and (4.11) are also obtained by direct calculations as above. □

We remark that in the following of this section, we shall frequently use the linear operator $\mathcal{L}$ which is given by (3.17) and the nontrivial kernels of the bubble Ψ_j in $H^1(\mathcal{C})$, denoted by $w_{j,l}=\Psi_j^{\frac{p+1}{2}}\theta_l$ and given by Lemma 2.1. Now, to further decompose the remaining term ρ_* and pick up a good regular part, let us first consider the following equation:

$$\begin{cases}\mathcal{L}(\overline{\gamma}_{1,ex})=\mathcal{R}_{1,ex}-\sum_{j=1}^{\nu}\Psi_j^{p-1}\left(c_{1,ex,j}\partial_t\Psi_j+\sum_{l=1}^{d}\varsigma_{1,ex,j,l}w_{j,l}\right), & \text{in } \mathcal{C},\\ \langle\partial_t\Psi_j,\overline{\gamma}_{1,ex}\rangle=\langle w_{j,l},\overline{\gamma}_{1,ex}\rangle=0 \quad \text{for all } 1\le j\le\nu \text{ and all } 1\le l\le d,\end{cases}\tag{4.19}$$

eqn0011

where $\mathcal{R}_{1,ex}$ is given by (3.18).

⟨lem0006⟩ **Lemma 4.4.** *Let $d\ge2$, $a<0$ and $b=b_{FS}(a)$. Then (4.19) is uniquely solvable. Moreover, the solution $\overline{\gamma}_{1,ex}$ is even on $\mathbb{S}^{d-1}$ and satisfies*

$$\begin{cases}\|\overline{\gamma}_{1,ex}\|_\sharp\lesssim1, & p\ge3,\\ \|\overline{\gamma}_{1,ex}\|_{\natural,1}\lesssim1, & 1<p<3,\end{cases}\tag{4.20}$$

eqn0047

where the Lagrange multipliers $\{c_{1,ex,j}\}$ and $\{\varsigma_{1,ex,j,l}\}$ are chosen such that the right hand side of the equation (4.19) is orthogonal to $\{\partial_t\Psi_j\}$ and $\{w_{j,l}\}$ in $H^1(\mathcal{C})$,

$$\|\overline{\gamma}_{1,ex}\|_\sharp=\sup_{\mathcal{B}_{i,+}}\frac{|\overline{\gamma}_{1,ex}|}{Q\sum_{i=1}^{\nu}\varphi_{s_i^*}^{1-\sigma}(t)}$$

and

$$\begin{aligned}\|\overline{\gamma}_{1,ex}\|_{\natural,1} &= \sum_{i=1}^{\nu-1}\sup_{(\mathcal{B}_{i,+}\backslash\mathcal{B}_{i,0})\cup(\mathcal{B}_{i+1,-}\backslash\mathcal{B}_{i+1,0})}\frac{|\overline{\gamma}_{1,ex}|}{Q_i(\varphi_{s_i^*}^{p-2}(t)+\varphi_{s_{i+1}^*}^{p-2}(t))}\\ &+\sup_{\cup_{i=1}^{\nu}\mathcal{B}_{i,0}}\frac{|\overline{\gamma}_{1,ex}|}{Q}+\sup_{\mathcal{B}_{\nu,+,*}\backslash\mathcal{B}_{\nu,0}}\frac{|\overline{\gamma}_{1,ex}|}{Q\varphi_{s_\nu^*}^{1-\sigma}(t)}+\sup_{\mathcal{B}_{1,-,*}\backslash\mathcal{B}_{1,0}}\frac{|\overline{\gamma}_{1,ex}|}{Q\varphi_{s_1^*}^{1-\sigma}(t)}\end{aligned}$$

with $\mathcal{B}_{i,\pm}$, $\mathcal{B}_{1,-,*}$ and $\mathcal{B}_{\nu,+,*}$ given by (4.13), (4.14), (4.16) and (4.17), respectively, and

$$\mathcal{B}_{i,0}=[s_i^*-R,s_i^*+R]\times\mathbb{S}^{d-1}$$

for $R>0$ sufficiently large. The Lagrange multipliers also satisfy $|c_{1,ex,j}|\lesssim Q$ for all $1\leq j\leq\nu$ and $\varsigma_{1,ex,j,l}=0$ for all $1\leq j\leq\nu$ and all $1\leq l\leq d$.

Proof. By the Fredholm alternative and the elliptic regularity, it is easy to show the existence and uniqueness of $\overline{\gamma}_{1,ex}$ in $H^2(\mathcal{C})$. Moreover, since $\mathcal{R}_{1,ex}$ is even on $\mathbb{S}^{d-1}$, by uniqueness, we also have that $\gamma_{1,ex}$ is even on $\mathbb{S}^{d-1}$. For the sake of clarity, we divide the remaining proof into three steps.

Step. 1 We estimate $\mathcal{R}_{1,ex}$.

By (3.18) and similar estimates for (4.12), (4.15) and (4.18),

$$\mathcal{R}_{1,ex}\sim\begin{cases}Q_i\varphi_{s_i^*}^{p-2}(t), & \text{in } \mathcal{B}_{i,+} \text{ for } 1\leq i\leq\nu-1,\\ Q_{i-1}\varphi_{s_{i-1}^*}^{p-2}(t), & \text{in } \mathcal{B}_{i,-} \text{ for } 2\leq i\leq\nu,\\ Q_1\varphi_{s_1^*}^{p}(t), & \text{in } \mathcal{B}_{1,-,*},\\ Q_\nu\varphi_{s_\nu^*}^{p}(t), & \text{in } \mathcal{B}_{\nu,+,*}.\end{cases}\tag{4.21}$$ eqnewnew0009

Thus, we have

$$\begin{aligned}1 &\gtrsim \sum_{i=1}^{\nu-1}\sup_{\mathcal{B}_{i,+}}\frac{|\mathcal{R}_{1,ex}|}{Q_i\varphi_{s_i^*}^{p-2}(t)}+\sup_{\mathcal{B}_{\nu,+,*}}\frac{|\mathcal{R}_{1,ex}|}{Q_\nu\varphi_{s_\nu^*}^{1-\sigma}(t)}\\ &+\sum_{i=2}^{\nu}\sup_{\mathcal{B}_{i,-}}\frac{|\mathcal{R}_{1,ex}|}{Q_{i-1}\varphi_{s_i^*}^{p-2}(t)}+\sup_{\mathcal{B}_{1,-,*}}\frac{|\mathcal{R}_{1,ex}|}{Q_1\varphi_{s_1^*}^{1-\sigma}(t)}\end{aligned}\tag{4.22}$$ eqn19997

for $1<p<3$ and

$$\begin{aligned}1 &\gtrsim \sum_{i=1}^{\nu-1}\sup_{\mathcal{B}_{i,+}}\frac{|\mathcal{R}_{1,ex}|}{Q\varphi_{s_i^*}^{1-\sigma}(t)}+\sup_{\mathcal{B}_{\nu,+,*}}\frac{|\mathcal{R}_{1,ex}|}{Q\varphi_{s_\nu^*}^{1-\sigma}(t)}\\ &+\sum_{i=2}^{\nu}\sup_{\mathcal{B}_{i,-}}\frac{|\mathcal{R}_{1,ex}|}{Q\varphi_{s_i^*}^{1-\sigma}(t)}+\sup_{\mathcal{B}_{1,-,*}}\frac{|\mathcal{R}_{1,ex}|}{Q\varphi_{s_1^*}^{1-\sigma}(t)}\end{aligned}\tag{4.23}$$ eqn19996

for $p\geq 3$.

Step. 2 We prove the estimates of the Lagrange multipliers.

By the orthogonal conditions and the oddness of $w_{j,l}$ on $\mathbb{S}^{d-1}$, we have

$$\sum_{i=1}^{\nu}\left\langle\Psi_i^{p-1}\partial_t\Psi_i,\Psi_j^{p-1}\partial_t\Psi_j\right\rangle_{L^2}c_{1,ex,i}=\left\langle\mathcal{R}_{1,ex},\Psi_j^{p-1}\partial_t\Psi_j\right\rangle_{L^2}$$

and

$$\sum_{m=1}^{\nu}\sum_{n=1}^{d}\left\langle\Psi_m^{p-1}w_{m,n},\Psi_j^{p-1}w_{j,l}\right\rangle_{L^2}\varsigma_{1,ex,m,n}=0$$

for all $1\leq j\leq\nu$ and all $1\leq l\leq d$. The matrix $\left[\left\langle\Psi_i^{p-1}\partial_t\Psi_i,\Psi_j^{p-1}\partial_t\Psi_j\right\rangle_{L^2}\right]$ is diagonally dominant by (3.11), thus, by $p>1$, Lemma 4.3 and (4.21),

$$\begin{aligned}|c_{1,ex,j}| &\sim \sum_{l=1}^{\nu}\left|\left\langle\mathcal{R}_{1,ex},\Psi_l^{p-1}\partial_t\Psi_l\right\rangle_{L^2}\right|\\ &\lesssim \sum_{i=1}^{l-1}\int_{\mathcal{B}_{i,+}}Q_i\Psi_i^{p-2}\Psi_l^pd\mu+\sum_{i=l+1}^{\nu}\int_{\mathcal{B}_{i,-}}Q_{i-1}\Psi_i^{p-2}\Psi_l^pd\mu+Q\int_{\mathcal{B}_j}\Psi_j^{2(p-1)}d\mu\\ &\lesssim \int_{\mathcal{B}_{j,+}}\Psi_j^{2p-1}\Psi_{j+1}d\mu+\int_{\mathcal{B}_{j,-}}\Psi_j^{2p-1}\Psi_{j-1}d\mu+Q\int_{\mathcal{B}_j}\Psi_j^{2(p-1)}d\mu\\ &\lesssim Q \end{aligned}\tag{4.24}$$

eqn1047

for all $1\leq j\leq\nu$. Moreover, by (3.11) and the orthogonal conditions of $\{w_l\}$ on $\mathbb{S}^{d-1}$, the matrix $\left[\left\langle\Psi_m^{p-1}w_{m,n},\Psi_j^{p-1}w_{j,l}\right\rangle_{L^2}\right]$ is also diagonally dominant. Thus, it is also easy to see that $\varsigma_{1,ex,j,l}=0$ for all $1\leq j\leq\nu$ and all $1\leq l\leq d$.

Step. 3 We prove the estimate (4.20).

Since it is easy to check that $\varphi_{s_i^*}^{1-\sigma}(t)$ for all $\sigma\in(0,1)$ are supersolutions of the equation $\mathcal{L}(\rho)=0$ in $\mathcal{B}_i\backslash[s_i^*-R,s_i^*+R]\times\mathbb{S}^{d-1}$ for all $1\leq i\leq\nu$ with $R>0$ fixed and large enough, by Lemma 2.1, (4.23) and using $\sum_{i=1}^{\nu}CQ\varphi_{s_i^*}^{1-\sigma}(t)$ for a sufficiently large $C>0$ as the barrier, we can apply the maximum principle in the strong sense and the standard blow-up arguments to (4.19) to derive the desired estimates (4.20) for $p\geq3$. On the other hand, for $1<p<3$, it is also easy to check that $\varphi_{s_i^*}^{p-2}(t)$ for all $p\in(1,3)$ are supersolutions of the equation $\mathcal{L}(\rho)=0$ in $\mathcal{B}_i\backslash[s_i^*-R,s_i^*+R]\times\mathbb{S}^{d-1}$ for all $1\leq i\leq\nu$ with $R>0$ fixed and large enough. Now, by the local regularity, (4.22) and the estimates of the Lagrange multipliers, we have

$$\|\overline{\gamma}_{1,ex}\|_{L^\infty(\widetilde{\mathcal{B}}_{i,0})}\lesssim Q\quad\text{and}\quad\|\overline{\gamma}_{1,ex}\|_{L^\infty(\widetilde{\mathcal{B}}_{i,0,*})}\lesssim Q_i^{\frac{p}{2}},\tag{4.25}$$

eqn19995

where

$$\widetilde{\mathcal{B}}_{i,0}=[s_i^*-2R,s_i^*+2R]\times\mathbb{S}^{d-1}$$

and

$$\widetilde{\mathcal{B}}_{i,0,*}=\left[\frac{s_i^*+s_{i+1}^*}{2}-2R,\frac{s_i^*+s_{i+1}^*}{2}+2R\right]\times\mathbb{S}^{d-1}$$

for $R>0$ sufficiently large. We introduce the sets

$$\mathcal{I}_0=\{1\leq i\leq\nu\mid Q_i\sim Q\}\quad\text{and}\quad\mathcal{I}_{0,*}=\{1\leq i\leq\nu\mid Q_i=o(Q)\}.$$

First of all, for $i \in \mathcal{I}_0$, since

$$Q_i\varphi_{s_i^*}^{p-2} \sim Q_i\varphi_{s_{i+1}^*}^{p-2} \sim \begin{cases} Q, & \text{in } \left((\mathcal{B}_{i,+} \cap \widetilde{\mathcal{B}}_{i,0}) \cup (\mathcal{B}_{i+1,-} \cap \widetilde{\mathcal{B}}_{i,0})\right), \\ Q^{\frac{p}{2}}, & \text{in } \left((\mathcal{B}_{i,+} \cap \widetilde{\mathcal{B}}_{i,0,*}) \cup (\mathcal{B}_{i+1,-} \cap \widetilde{\mathcal{B}}_{i,0,*})\right), \end{cases}$$

by (4.22), (4.25) and the maximum principle, we have

$$\sum_{i \in \mathcal{I}_0} \left(\sup_{\mathcal{B}_{i,+}} \frac{|\overline{\gamma}_{1,ex}|}{Q_i\varphi_{s_i^*}^{p-2}(t)} + \sup_{\mathcal{B}_{i+1,-}} \frac{|\overline{\gamma}_{1,ex}|}{Q_i\varphi_{s_i^*}^{p-2}(t)} \right) \lesssim 1. \tag{4.26}$$ eqnnewnew0009

Secondly, for $i \in \mathcal{I}_{0,*}$, we shall construct a global barrier in $\mathcal{B}_{i,+} \cup \mathcal{B}_{i+1,-}$. Let $\psi_{1,0}$ be the unique solution of the following equation

$$\begin{cases} -\psi'' - 2\sigma\psi' + \sigma^2\psi = 1, \\ \psi(0) = 1, \quad \psi'(0) = 0 \end{cases}$$

and $\psi_{1,*}$ be a solution of the following equation

$$\begin{cases} -\psi'' - 2\sigma\psi' + \sigma^2\psi = 1, \\ \psi'(1) = 0, \end{cases}$$

where $\sigma > 0$ is sufficiently small. Then we have

$$\psi_{1,0}(t) = \sigma^{-2} - \frac{\sigma^{-2} - 1}{2\sqrt{2}} \left((\sqrt{2} - 1)e^{-(\sqrt{2}+1)\sigma t} + (\sqrt{2} + 1)e^{(\sqrt{2}-1)\sigma t} \right)$$

and we can take

$$\psi_{1,*}(t) = \sigma^{-2} + \frac{e^{(\sqrt{2}-1)\sigma}}{\sqrt{2} + 1} \left((\sqrt{2} - 1)e^{-(\sqrt{2}+1)\sigma(t-1)} + (\sqrt{2} + 1)e^{(\sqrt{2}-1)\sigma(t-1)} \right).$$

It is easy to see that $\psi_{1,0}'(t) < 0$ and $\psi_{1,*}'(t) < 0$ for $t \in (0, 1)$ and there exists $t_0 \in (0, 1)$ such that $\psi_{1,0}'(t_0) = \psi_{1,*}'(t_0)$, which implies that

$$\psi_1(t) = \begin{cases} \psi_{1,0}(0) - \psi_{1,0}(t_0) + \psi_{1,*}(t_0) - \psi_{1,*}(1), & t \leq 0, \\ \psi_{1,0}(t) - \psi_{1,0}(t_0) + \psi_{1,*}(t_0) - \psi_{1,*}(1), & 0 < t \leq t_0, \\ \psi_{1,*}(t) - \psi_{1,*}(1), & r_0 < t < 1, \\ 0, & t \geq 1 \end{cases}$$

belongs to $L^\infty(\mathbb{R}) \cap C^1(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$ and is a cut-off function. Let

$$\begin{cases} \psi_{i,+}(t) = \psi_1\left(t - s_i^* - \frac{R}{2} - 1\right), \\ \psi_{i+1,-}(t) = \psi_1\left(s_{i+1}^* + \frac{R}{2} + 1 - t\right). \end{cases}$$

Then by

$$Q_i\varphi_{s_i^*}^{p-2} \sim Q_i\varphi_{s_{i+1}^*}^{p-2} \sim \begin{cases} Q_i = o(Q), & \text{in } \left((\mathcal{B}_{i,+} \cap \widetilde{\mathcal{B}}_{i,0}) \cup (\mathcal{B}_{i+1,-} \cap \widetilde{\mathcal{B}}_{i,0})\right), \\ Q_i^{\frac{p}{2}}, & \text{in } \left((\mathcal{B}_{i,+} \cap \widetilde{\mathcal{B}}_{i,0,*}) \cup (\mathcal{B}_{i+1,-} \cap \widetilde{\mathcal{B}}_{i,0,*})\right), \end{cases}$$

it can be checked that

$$\psi_i = \psi_{i,+}Q + (\psi_{i,+}(0) - \psi_{i,+})Q_i\varphi_{s_i^*}^{p-2}$$

and

$$\psi_{i+1}=\psi_{i+1,-}Q+(\psi_{i+1,-}(0)-\psi_{i+1,-})Q_i\varphi_{s_{i+1}^*}^{p-2}$$

are supersolutions of the equation $\mathcal{L}(\rho)=0$ in $\mathcal{B}_{i,+}\backslash\mathcal{B}_{i,0}$ and $\mathcal{B}_{i+1,-}\backslash\mathcal{B}_{i+1,0}$, respectively. Thus, by (4.22), (4.25) and the maximum principle, we have

$$\sum_{i\in\mathcal{I}_{0,*}}\left(\sup_{\mathcal{B}_{i,+}\backslash\mathcal{B}_{i,0}}\frac{|\overline{\gamma}_{1,ex}|}{Q_i\varphi_{s_i^*}^{p-2}(t)}+\sup_{\mathcal{B}_{i+1,-}\backslash\mathcal{B}_{i+1,0}}\frac{|\overline{\gamma}_{1,ex}|}{Q_i\varphi_{s_{i+1}^*}^{p-2}(t)}+\sup_{\mathcal{B}_{i,0}}\frac{|\overline{\gamma}_{1,ex}|}{Q}\right)\lesssim 1. \quad (4.27)$$ eqnnewnew0010

Finally, by (4.22), (4.25) and the maximum principle, we also have

$$\sup_{\mathcal{B}_{\nu,+,*}}\frac{|\overline{\gamma}_{1,ex}|}{Q\varphi_{s_\nu^*}^{1-\sigma}(t)}+\sup_{\mathcal{B}_{1,-,*}}\frac{|\overline{\gamma}_{1,ex}|}{Q\varphi_{s_1^*}^{1-\sigma}(t)}\lesssim 1. \quad (4.28)$$ eqnnewnew0011

Now, the estimate (4.20) follows from (4.26), (4.27) and (4.28). □

We next consider the following equation:

$$\begin{cases}\mathcal{L}(\gamma_{1,j})=\mathcal{R}_{1,j}-\sum_{i=1}^{\nu}\Psi_i^{p-1}\left(c_{1,j,i}\partial_t\Psi_i+\sum_{l=1}^{d}\varsigma_{1,j,i,l}w_{i,l}\right), \quad \text{in } \mathcal{C},\\ \langle\partial_t\Psi_i,\gamma_{1,j}\rangle=\langle w_{i,l},\gamma_{1,j}\rangle=0 \quad \text{for all } 1\leq i\leq\nu \text{ and all } 1\leq l\leq d,\end{cases} \quad (4.29)$$ eqn0012

where $\mathcal{R}_{1,j}$ is given by (3.18).

⟨lem0007⟩ **Lemma 4.5.** *Let $d\geq 2$, $a<0$ and $b=b_{FS}(a)$. Then (4.29) is uniquely solvable. Moreover, the solution $\gamma_{1,j}$ is even in $\mathbb{R}$ in terms of $t-s_j^*$ and even on $\mathbb{S}^{d-1}$ and satisfies*

$$\sup_{(t,\theta)\in\mathcal{C}}\frac{|\gamma_{1,j}|}{\varphi_{s_j^*}^{1-\sigma}(t)}\lesssim\left|(\alpha_j^*)^{p-1}-1\right|, \quad (4.30)$$ eqn0048

where the Lagrange multipliers $\{c_{1,j,i}\}$ and $\{\varsigma_{1,j,i,l}\}$ are chosen such that the right hand side of the equation (4.29) is orthogonal to $\{\partial_t\Psi_j\}$ and $\{w_{j,l}\}$ in $H^1(\mathcal{C})$. The Lagrange multipliers also satisfy $\varsigma_{1,j,i,l}=0$ and

$$|c_{1,j,i}|\lesssim\begin{cases}\left|(\alpha_j^*)^{p-1}-1\right|Q^p\left|\log Q\right|, & i\neq j,\\ \left|(\alpha_j^*)^{p-1}-1\right|Q^{2p}\left|\log Q\right|^2, & i=j,\end{cases}$$

for all $1\leq j\leq\nu$, all $1\leq l\leq d$ and all $1\leq i\neq j\leq\nu$.

Proof. By the orthogonal conditions and the oddness of $w_{j,l}$ on $\mathbb{S}^{d-1}$, we have

$$\sum_{i=1}^{\nu}\left\langle\Psi_i^{p-1}\partial_t\Psi_i,\Psi_k^{p-1}\partial_t\Psi_k\right\rangle_{L^2}c_{1,j,i}=\left\langle\mathcal{R}_{1,j},\Psi_k^{p-1}\partial_t\Psi_k\right\rangle_{L^2}$$

and

$$\sum_{m=1}^{\nu}\sum_{n=1}^{d}\left\langle\Psi_m^{p-1}w_{m,n},\Psi_j^{p-1}w_{j,l}\right\rangle_{L^2}\varsigma_{1,j,m,n}=0$$

for all $1\leq j,k\leq\nu$ and all $1\leq l\leq d$. Again, the matrix $\left[\left\langle\Psi_i^{p-1}\partial_t\Psi_i,\Psi_j^{p-1}\partial_t\Psi_j\right\rangle_{L^2}\right]$ is diagonally dominant by (3.11). Note that by the oddness of $\partial_t\Psi$ in $\mathbb{R}$, we have

$$\left\langle\mathcal{R}_{1,j},\Psi_j^{p-1}\partial_t\Psi_j\right\rangle_{L^2}=0.$$

Thus, by Lemma 4.3 and (3.18),

$$|c_{1,j,i}| \lesssim \begin{cases} \left|(\alpha_j^*)^{p-1} - 1\right| \int_{\mathcal{C}} \Psi_i^p \Psi_j^p d\mu \lesssim \left|(\alpha_j^*)^{p-1} - 1\right| Q^p \left|\log Q\right|, \quad i \neq j, \\ \sum_{l \neq j} \left|(\alpha_j^*)^{p-1} - 1\right| \left(\int_{\mathcal{C}} \Psi_l^p \Psi_j^p d\mu\right)^2 \lesssim \left|(\alpha_j^*)^{p-1} - 1\right| Q^{2p} \left|\log Q\right|^2, \quad i = j, \end{cases}$$

for all $1 \leq i \leq \nu$. Moreover, by (3.11) and the orthogonal conditions of $\{w_l\}$ on $\mathbb{S}^{d-1}$, the matrix $\left[\left\langle \Psi_m^{p-1} w_{m,n}, \Psi_j^{p-1} w_{j,l} \right\rangle_{L^2}\right]$ is also diagonally dominant. Thus, it is also easy to see that $\varsigma_{1,j,i,l} = 0$ for all $1 \leq i, j \leq \nu$ and all $1 \leq l \leq d$. Now, since $p > 1$, as in the proof of Lemma 4.4, by (2.7), (3.18) and Lemma 2.1, we can use $\widetilde{\varphi} = C\left|(\alpha_j^*)^{p-1} - 1\right| \varphi_{s_j^*}^{1-\sigma}(t)$ for a sufficiently large $C > 0$ as the barrier and apply the maximum principle in the classical sense and the standard blow-up arguments to (4.29) to show the existence and uniqueness of $\gamma_{1,j}$ with the desired estimate (4.30). Moreover, since $\mathcal{R}_{1,j}$ is even in $\mathbb{R}$ in terms of $t - s_j^*$ and even on $\mathbb{S}^{d-1}$, by uniqueness, $\gamma_{1,j}$ is also even in $\mathbb{R}$ in terms of $t - s_j^*$ and even on $\mathbb{S}^{d-1}$. $\square$

We also need to consider the following equation:

$$\begin{cases} \mathcal{L}(\overline{\gamma}_{2,ex}) = \mathcal{R}_{2,ex} - \sum_{j=1}^{\nu} \Psi_j^{p-1}\left(c_{2,ex,j}\partial_t \Psi_j + \sum_{l=1}^{d} \varsigma_{2,ex,j,l} w_{j,l}\right), \quad \text{in } \mathcal{C}, \\ \langle \partial_t \Psi_j, \overline{\gamma}_{2,ex}\rangle = \langle w_{j,l}, \overline{\gamma}_{2,ex}\rangle = 0 \quad \text{for all } 1 \leq j \leq \nu \text{ and all } 1 \leq l \leq d, \end{cases} \tag{4.31}$$ eqn0013

where $\mathcal{R}_{2,ex}$ is given by (3.19).

⟨lem0008⟩ **Lemma 4.6.** *Let $d \geq 2$, $a < 0$ and $b = b_{FS}(a)$. Then (4.31) is uniquely solvable. Moreover, the solution $\overline{\gamma}_{2,ex}$ is odd on $\mathbb{S}^{d-1}$ and satisfies*

$$\begin{cases} \|\overline{\gamma}_{2,ex}\|_\sharp \lesssim \beta_*, \quad p \geq \frac{7}{3}, \\ \|\overline{\gamma}_{2,ex}\|_{\natural,2} \lesssim \beta_*, \quad 1 < p < \frac{7}{3}, \end{cases} \tag{4.32}$$ eqn2047

where the Lagrange multipliers $\{c_{2,ex,j}\}$ and $\{\varsigma_{2,ex,j,l}\}$ are chosen such that the right hand side of the equation (4.31) is orthogonal to $\{\partial_t \Psi_j\}$ and $\{w_{j,l}\}$ in $H^1(\mathcal{C})$, $\|\cdot\|_\sharp$ is given in Lemma 4.4 and

$$\begin{aligned} \|\overline{\gamma}_{2,ex}\|_{\natural,2} &= \sum_{i=1}^{\nu-1} \sup_{(\mathcal{B}_{i,+}\setminus\mathcal{B}_{i,0})\cup(\mathcal{B}_{i+1,-}\setminus\mathcal{B}_{i+1,0})} \frac{|\overline{\gamma}_{2,ex}|}{Q_i(\varphi_{s_i^*}^{\frac{3p-5}{2}}(t) + \varphi_{s_{i+1}^*}^{\frac{3p-5}{2}}(t))} \\ &\quad + \sup_{\cup_{i=1}{}^\nu \mathcal{B}_{i,0}} \frac{|\overline{\gamma}_{2,ex}|}{Q} + \sup_{\mathcal{B}_{\nu,+,*}\setminus\mathcal{B}_{\nu,0}} \frac{|\overline{\gamma}_{2,ex}|}{Q\varphi_{s_\nu^*}^{1-\sigma}(t)} + \sup_{\mathcal{B}_{1,-,*}\setminus\mathcal{B}_{1,0}} \frac{|\overline{\gamma}_{2,ex}|}{Q\varphi_{s_1^*}^{1-\sigma}(t)} \end{aligned}$$

with $\mathcal{B}_{i,\pm}$, $\mathcal{B}_{\nu,+,*}$, $\mathcal{B}_{1,-,*}$ and $\mathcal{B}_{i,0}$ given by (4.13), (4.14), (4.16), (4.17) and Lemma 4.4, respectively. The Lagrange multipliers also satisfy $c_{2,ex,j} = 0$ for all $1 \leq j \leq \nu$ and $|\varsigma_{2,ex,j,l}| \lesssim \beta_* Q$ for all $1 \leq j \leq \nu$ and all $1 \leq l \leq d$.

Proof. Similar to (4.12) and (4.18), by (2.7), (3.15) and (3.19), we have

$$|\mathcal{R}_{2,ex}| \lesssim \beta_*\left(\sum_{i=1}^{\nu} \Psi_i^{\frac{3(p-1)}{2}} \mathcal{U}_i \chi_{\mathcal{B}_i} + \mathcal{U}^{\frac{3p-1}{2}} \chi_{\mathcal{C}\setminus\mathcal{B}_*}\right) \tag{4.33}$$ eqn3147

with $\mathcal{B}_i$ and $\mathcal{B}_*$ given by (4.7). Thus, similar to (4.22) and (4.23),

$$\begin{aligned}\beta_* \gtrsim & \sum_{i=1}^{\nu-1}\sup_{\mathcal{B}_{i,+}}\frac{|\mathcal{R}_{2,ex}|}{Q_i\varphi_{s_i^*}^{\frac{3p-5}{2}}(t)}+\sup_{\mathcal{B}_{\nu,+,*}}\frac{|\mathcal{R}_{2,ex}|}{Q_\nu\varphi_{s_\nu^*}^{1-\sigma}(t)}\\ &+\sum_{i=2}^{\nu}\sup_{\mathcal{B}_{i,-}}\frac{|\mathcal{R}_{2,ex}|}{Q_{i-1}\varphi_{s_i^*}^{\frac{3p-5}{2}}(t)}+\sup_{\mathcal{B}_{1,-}}\frac{|\mathcal{R}_{2,ex}|}{Q_1\varphi_{s_1^*}^{1-\sigma}(t)}\end{aligned} \tag{4.34}$$

eqn29997

for $1<p<\frac{7}{3}$ and $\|\mathcal{R}_{2,ex}\|_\sharp\lesssim\beta_*$ for $p\geq\frac{7}{3}$. By the orthogonal conditions and the oddness of $w_{j,l}$ on $\mathbb{S}^{d-1}$, we have

$$\sum_{i=1}^{\nu}\left\langle\Psi_i^{p-1}\partial_t\Psi_i,\Psi_j^{p-1}\partial_t\Psi_j\right\rangle_{L^2}c_{2,ex,i}=0$$

and

$$\sum_{m=1}^{\nu}\sum_{n=1}^{d}\left\langle\Psi_m^{p-1}w_{m,n},\Psi_j^{p-1}w_{j,l}\right\rangle_{L^2}\varsigma_{2,ex,m,n}=\left\langle\mathcal{R}_{2,ex},\Psi_j^{p-1}w_{j,l}\right\rangle_{L^2}$$

for all $1\leq j\leq\nu$ and all $1\leq l\leq d$. Again, the matrix $\left[\left\langle\Psi_i^{p-1}\partial_t\Psi_i,\Psi_j^{p-1}\partial_t\Psi_j\right\rangle_{L^2}\right]$ is diagonally dominant by (3.11). Thus, $c_{2,ex,j}=0$ for all $1\leq j\leq\nu$. Moreover, the matrix $\left[\left\langle\Psi_m^{p-1}w_{m,n},\Psi_j^{p-1}w_{j,l}\right\rangle_{L^2}\right]$ is also diagonally dominant by (3.11) and the orthogonal conditions of w_l on $\mathbb{S}^{d-1}$. Thus, by Lemma 4.3 and (4.33), we also have

$$\begin{aligned}|\varsigma_{2,ex,j,l}| &\sim \sum_{j=1}^{\nu}\left|\left\langle\mathcal{R}_{2,ex},\Psi_j^{p-1}w_{j,l}\right\rangle_{L^2}\right|\\ &\lesssim \sum_{j=1}^{\nu}\beta_*\left\langle\Psi_{j-1}^{\frac{3(p-1)}{2}}\mathcal{U}_{j-1}\chi_{\mathcal{B}_{j-1,+}}+\Psi_{j+1}^{\frac{3(p-1)}{2}}\mathcal{U}_{j+1}\chi_{\mathcal{B}_{j+1,-}},\Psi_j^{\frac{3p-1}{2}}\right\rangle_{L^2}\\ &\quad+\beta_*\left\langle\Psi_j^{3p-2},\mathcal{U}_j\right\rangle_{L^2(\mathcal{B}_j)}\\ &\sim \beta_*Q.\end{aligned}$$

for all $1\leq j\leq\nu$ and all $1\leq l\leq d$. Moreover, by (3.15) and (3.19), we have

$$\begin{aligned}\mathcal{R}_{2,ex} &= p\sum_{j=1}^{\nu}\left(\mathcal{U}^{p-1}-\left(\Psi_j^*\right)^{p-1}\right)\mathcal{V}_j\\ &= \sum_{l=1}^{d}\left(p\sum_{j=1}^{\nu}\left(\mathcal{U}^{p-1}-\left(\Psi_j^*\right)^{p-1}\right)\Psi_j^{\frac{p+1}{2}}\beta_{j,l}^*\right)\theta_l,\end{aligned}$$

which is odd on $\mathbb{S}^{d-1}$. Now, the rest of the proof, which is devoted to the existence, uniqueness and oddness on $\mathbb{S}^{d-1}$ of $\overline{\gamma}_{2,ex}$ with the desired estimates (4.32), is similar to that of Lemma 4.4, so we omit it here. □

We finally consider the following equation:

$$\begin{cases}\mathcal{L}(\gamma_{\mathcal{N},led,*})=\mathcal{N}_{led}-\sum_{j=1}^{\nu}\Psi_j^{p-1}\left(c_{\mathcal{N},led,j}\partial_t\Psi_j+\sum_{l=1}^{d}\varsigma_{\mathcal{N},led,j,l}w_{j,l}\right), \quad \text{in } \mathcal{C},\\ \langle\partial_t\Psi_j,\gamma_{\mathcal{N},led,*}\rangle=\langle w_{j,l},\gamma_{\mathcal{N},led,*}\rangle=0 \quad \text{for all } 1\leq j\leq\nu \text{ and all } 1\leq l\leq d,\end{cases} \tag{4.35}$$

eqn0015

where

$$\begin{aligned}\mathcal{N}_{led} &= \sum_{j=1}^{\nu}\left(\Psi_j^*\right)^{p-3}\mathcal{V}_j^2\left(A_{p,1}\Psi_j^*+A_{p,2}\mathcal{V}_j\right)\chi_{\mathcal{B}_j}\\ &\quad+\mathcal{U}^{p-3}\mathcal{V}^2\left(A_{p,1}\mathcal{U}+A_{p,2}\mathcal{V}\right)\chi_{\mathcal{C}\backslash\mathcal{B}_*},\end{aligned} \tag{4.36}$$

eqn3045

with $\mathcal{B}_i$ and $\mathcal{B}_*$ given by (4.7).

⟨lem0009⟩ **Lemma 4.7.** *Let $d\geq 2$, $a<0$ and $b=b_{FS}(a)$. Then (4.35) is uniquely solvable. Moreover, the solution $\gamma_{\mathcal{N},led,*}$ satisfies*

$$\sup_{(t,\theta)\in\mathcal{C}}\frac{|\gamma_{\mathcal{N},led,*}|}{\sum_{j=1}^{\nu}\Psi_j^{1-\sigma}(t)}\lesssim\beta_*^2 \tag{4.37}$$

eqn0049

where the Lagrange multipliers $\{c_{\mathcal{N},led,j}\}$ and $\{\varsigma_{\mathcal{N},led,j,l}\}$ are chosen such that the right hand side of the equation (4.35) is orthogonal to $\{\partial_t\Psi_j\}$ and $\{w_{j,l}\}$ in $H^1(\mathcal{C})$. The Lagrange multipliers also satisfy $|c_{\mathcal{N},led,j}|\lesssim\beta_*^2Q^p$ and $|\varsigma_{\mathcal{N},led,j,l}|\lesssim\beta_*^3$ for all $1\leq j\leq\nu$ and all $1\leq l\leq d$.

Proof. Similar to (4.33), by (4.36), we have

$$|\mathcal{N}_{led}|\lesssim\sum_{j=1}^{\nu}\beta_*^2\Psi_j^{2p-1}\chi_{\mathcal{B}_j}+\beta_*^2\mathcal{U}^{2p-1}\chi_{\mathcal{C}\backslash\mathcal{B}_*}.$$

We first estimate the Lagrange multipliers. By the oddness of $w_{j,l}$ on $\mathbb{S}^{d-1}$, we have

$$\begin{aligned}&\sum_{i=1}^{\nu}\left\langle\Psi_i^{p-1}\partial_t\Psi_i,\Psi_j^{p-1}\partial_t\Psi_j\right\rangle_{L^2}c_{\mathcal{N},led,i}\\ =\;& A_{p,1}\left\langle\sum_{i=1}^{\nu}\left(\Psi_i^*\right)^{p-2}\mathcal{V}_i^2\chi_{\mathcal{B}_i}+\mathcal{U}^{p-2}\mathcal{V}^2\chi_{\mathcal{C}\backslash\mathcal{B}_*},\Psi_j^{p-1}\partial_t\Psi_j\right\rangle_{L^2}\end{aligned}$$

and

$$\begin{aligned}&\sum_{m=1}^{\nu}\sum_{n=1}^{d}\left\langle\Psi_m^{p-1}w_{m,n},\Psi_j^{p-1}w_{j,l}\right\rangle_{L^2}\varsigma_{\mathcal{N},led,m,n}\\ =\;& A_{p,2}\left\langle\sum_{i=1}^{\nu}\left(\Psi_i^*\right)^{p-3}\mathcal{V}_i^3\chi_{\mathcal{B}_i}+\mathcal{U}^{p-3}\mathcal{V}^3\chi_{\mathcal{C}\backslash\mathcal{B}_*},\Psi_j^{p-1}w_{j,l}\right\rangle_{L^2}\end{aligned}$$

for all $1\leq j\leq\nu$ and all $1\leq l\leq d$. Thus, similar to (4.24), by Lemma 4.3 and the oddness of $\partial_t\Psi$ in $\mathbb{R}$, we have

$$\begin{aligned}|c_{\mathcal{N},led,j}| &\lesssim \sum_{i=\pm1}\int_{\mathcal{B}_{j+i}}\Psi_{j+i}^{p-2}\mathcal{V}_{j+i}^2\Psi_j^pd\mu+\int_{\mathcal{B}_j}\Psi_j^{p-2}\mathcal{V}_j^2\Psi_j^{p-1}\partial_t\Psi_jd\mu\\ &\lesssim \beta_*^2Q^p+\beta_*^2\int_{s_j^*+\frac{\tau}{2}}^{s_j^*+\frac{\tau_j}{2}}\Psi_j^{3p-1}dt\\ &\lesssim \beta_*^2Q^p\end{aligned} \tag{4.38}$$

eqn1049

and, we have

$$|\varsigma_{\mathcal{N},led,j,l}| \lesssim \left|\left\langle \Psi_j^{p-3}\mathcal{V}_j^3\chi_{\mathcal{B}_j} + \mathcal{U}^{p-3}\mathcal{V}^3\chi_{\mathcal{C}\backslash(\cup_{i=1}^{\nu}\mathcal{B}_i)}, w_{j,l}\right\rangle_{L^2}\right| \lesssim \beta_*^3 \tag{4.39}$$ eqn3049

for all $1 \leq j \leq \nu$ and all $1 \leq l \leq d$. Now, since $p > 1$, as in the proof of Lemma 4.4, by Lemma 2.1 and (4.36), we can use $\widetilde{\varphi} = \sum_{j=1}^{\nu} C\beta_*^2\varphi_{s_j^*}^{1-\sigma}(t)$ for a sufficiently large $C > 0$ as the barrier and apply the maximum principle in the classical sense and the standard blow-up arguments to (4.35) to show the existence and uniqueness of $\gamma_{\mathcal{N},led,*}$ with the desired estimates (4.37). □

By Lemmas 4.4, 4.5, 4.6 and 4.7, we have picked up all possible leading order terms of ρ_* in terms of Q, β_* and $\sum_{j=1}^{\nu}\left|(\alpha_j^*)^{p-1} - 1\right|$. Now, let

$$\rho_{**,0} = \rho_* - \sum_{j=1}^{2}\overline{\gamma}_{j,ex} - \sum_{j=1}^{\nu}\gamma_{1,j} - \gamma_{\mathcal{N},led,*}.$$

Since $\overline{\gamma}_{1,ex}$, $\gamma_{1,j}$ and $\gamma_{\mathcal{N},led,*}$ may have projections on span$\{\Psi_j\}$, we further decompose $\rho_{**,0} = \sum_{j=1}^{\nu}\alpha_{j,0}^{**}\Psi_j + \rho_{**,0}^{\perp}$, where $\{\alpha_{j,0}^{**}\}$ is chosen such that

$$\langle\rho_{**,0}^{\perp}, \Psi_j\rangle = \langle\rho_{**,0}^{\perp}, \partial_t\Psi_j\rangle = \langle\rho_{**,0}^{\perp}, w_{j,l}\rangle = 0$$

for all $1 \leq j \leq \nu$ and all $1 \leq l \leq d$. By the orthogonal conditions of ρ_* given in (3.16) and Lemma 4.6, we have

$$\sum_{l=1}^{\nu}\langle\Psi_l, \Psi_j\rangle\,\alpha_{l,0}^{**} = -\langle\overline{\gamma}_{1,ex}, \Psi_j\rangle - \sum_{i=1}^{\nu}\langle\gamma_{1,i}, \Psi_j\rangle - \langle\gamma_{\mathcal{N},led,*}, \Psi_j\rangle \tag{4.40}$$ eqn0052

for all $1 \leq j \leq \nu$. Since Ψ is a solution of (2.5), by Lemmas 4.3, 4.4, 4.5 and 4.7,

$$\begin{cases} |\langle\overline{\gamma}_{1,ex}, \Psi_j\rangle| = \left|\langle\overline{\gamma}_{1,ex}, \Psi_j^p\rangle_{L^2}\right| \lesssim Q^{1+\sigma}, \\ \left|\left\langle\sum_{i=1}^{\nu}\gamma_{1,i}, \Psi_j\right\rangle\right| = \left|\left\langle\sum_{i=1}^{\nu}\gamma_{1,i}, \Psi_j^p\right\rangle_{L^2}\right| \lesssim \sum_{i=1}^{\nu}\left|(\alpha_i^*)^{p-1} - 1\right|, \\ |\langle\gamma_{\mathcal{N},led,*}, \Psi_j\rangle| = \left|\langle\gamma_{\mathcal{N},led,*}, \Psi_j^p\rangle_{L^2}\right| \lesssim \beta_*^2. \end{cases}$$

Intersecting these estimates into (4.40), we have

$$\sum_{j=1}^{\nu}\left|\alpha_{j,0}^{**}\right| \lesssim Q^{1+\sigma} + \sum_{j=1}^{\nu}\left|\left(\alpha_j^*\right)^{p-1} - 1\right| + \beta_*^2. \tag{4.41}$$ eqn0068

Moreover, by (3.16), (4.19), (4.29), (4.31) and (4.35), $\rho_{**,0}^{\perp}$ satisfies

$$\begin{cases} \mathcal{L}(\rho_{**,0}^{\perp}) = f + \mathcal{R}_{new}, \quad \text{in } \mathcal{C}, \\ \langle\Psi_j, \rho_{**,0}^{\perp}\rangle = \langle\partial_t\Psi_j, \rho_{**,0}^{\perp}\rangle = \langle w_{j,l}, \rho_{**,0}^{\perp}\rangle = 0 \quad \text{for } 1 \leq j \leq \nu \text{ and } 1 \leq l \leq d, \end{cases} \tag{4.42}$$ eqn5114

where by (3.16) and Lemmas 4.2, 4.4 4.5, 4.6 and 4.7,

$$\begin{aligned}\mathcal{R}_{new} &= \sum_{j,i=1}^{\nu}\Psi_i^{p-1}\left((c_{1,ex,i}+c_{1,j,i}+c_{\mathcal{N},led,i})\partial_t\Psi_i+\sum_{l=1}^{d}(\varsigma_{2,ex,i,l}+\varsigma_{\mathcal{N},led,i,l})w_{i,l}\right)\\&\quad+\sum_{j=1}^{\nu}\left(2A_{p,1}\left(\Psi_j^*\right)^{p-2}\mathcal{V}_j+3A_{p,2}\left(\Psi_j^*\right)^{p-3}\mathcal{V}_j^2\right)\rho_*\chi_{\mathcal{B}_j}+\sum_{j=1}^{\nu}\mathcal{R}_{2,j}+\mathcal{N}_{rem}\\&\quad+\sum_{j=1}^{\nu}2A_{p,1}\left(\mathcal{U}^{p-2}\mathcal{V}-\sum_{j=1}^{\nu}\left(\Psi_j^*\right)^{p-2}\mathcal{V}_j\right)\rho_*\chi_{\mathcal{B}_j}\\&\quad+\sum_{j=1}^{\nu}\mathcal{O}\left(\beta_*^2\mathcal{U}_j\Psi_j^{2p-3}\left(\Psi_j+\rho_*\right)\right)\chi_{\mathcal{B}_j}+2A_{p,1}\mathcal{U}^{p-2}\mathcal{V}\rho_*\chi_{\mathcal{C}\setminus\mathcal{B}_*}\\&\quad+\mathcal{O}\left(\beta_*^2\mathcal{U}^{2(p-1)}\rho_*\chi_{\mathcal{C}\setminus\mathcal{B}_*}\right)+\sum_{j=1}^{\nu}p\alpha_{j,0}^{**}\left(\mathcal{U}^{p-1}-\Psi_j^{p-1}\right)\Psi_j,\end{aligned}\tag{4.43}$$ eqn5061

where $\mathcal{B}_i$ and $\mathcal{B}_*$ are given by (4.7).

As we pointed out in the introduction, even though we have picked up all possible leading order terms of ρ_* in terms of Q, β_* and $\sum_{j=1}^{\nu}\left|(\alpha_j^*)^{p-1}-1\right|$ to form a good regular part, the data $\mathcal{R}_{new}$, given by (4.43), is not good enough to control the (possible) singular part $\rho_{**,0}^{\perp}$ in a desired size. This is mainly because the optimal Bianchi-Egnell stability of the CKN inequality for $d\geq 2$, $a<0$ and $b=b_{FS}(a)$ is quartic, as shown in [54, Theorem 1], which implies that we only have the opportunity to control the terms of order β_*^4 from above. Thus, we need to ensure that the (possible) singular part should be smaller than or equal to β_*^4. Keep this in mind, we need to eliminate the lower order terms (compared to the β_*^4 terms) in the data $\mathcal{R}_{new}$. For this purpose, we first need the following decomposition.

⟨lem0010⟩ **Lemma 4.8.** *Let $d\geq 2$, $a<0$ and $b=b_{FS}(a)$. Then we can decompose*

$$\gamma_{\mathcal{N},led,*}=\gamma_{\mathcal{N},led,j}+\gamma_{\mathcal{N},led,rem,j,*},$$

where $\gamma_{\mathcal{N},led,j}$ is even in terms of $t-s_j^*$ and satisfies the equation

$$\begin{cases}\mathcal{L}(\gamma_{\mathcal{N},led,j})=\mathcal{N}_{led,j}-\sum_{i=1}^{\nu}\Psi_i^{p-1}\left(c_{\mathcal{N},led,j,i}\partial_t\Psi_i+\sum_{l=1}^{d}\varsigma_{\mathcal{N},led,j,i,l}w_{i,l}\right), & \text{in } \mathcal{C},\\ \langle\partial_t\Psi_i,\gamma_{\mathcal{N},led,j}\rangle=\langle w_{i,l},\gamma_{\mathcal{N},led,j}\rangle=0 & \text{for all } 1\leq i\leq\nu \text{ and all } 1\leq l\leq d,\end{cases}$$

with $\mathcal{N}_{led,j}=\left(\Psi_j^*\right)^{p-3}\mathcal{V}_j^2\left(A_{p,1}\Psi_j^*+A_{p,2}\mathcal{V}_j\right)$ and

$$\sup_{(t,\theta)\in\mathcal{C}}\frac{|\gamma_{\mathcal{N},led,j}|}{\Psi_j^{1-\sigma}(t)}+\sup_{(t,\theta)\in\mathcal{C}}\frac{|\gamma_{\mathcal{N},led,rem,j,*}|}{\sum_{i=1;i\neq j}^{\nu}\Psi_i^{1-\sigma}(t)}\lesssim\beta_*^2.$$

Moreover, we can decompose $\gamma_{\mathcal{N},led,j}=\gamma_{\mathcal{N},led,j,*}+\gamma_{\mathcal{N},led,j,**}$ with $\gamma_{\mathcal{N},led,j,*}$ being even on $\mathbb{S}^{d-1}$, $\gamma_{\mathcal{N},led,j,**}$ being odd on $\mathbb{S}^{d-1}$ and

$$\sup_{(t,\theta)\in\mathcal{C}}\frac{|\gamma_{\mathcal{N},led,j,*}|}{\beta_*^2\Psi_j^{1-\sigma}(t)}+\sup_{(t,\theta)\in\mathcal{C}}\frac{|\gamma_{\mathcal{N},led,j,**}|}{\beta_*^3\Psi_j^{1-\sigma}(t)}\lesssim 1.$$

Proof. The proof is similar to that of Lemma 4.7 so we omit it. We only point out that since $\mathcal{N}_{led,j}$ is even in terms of $t-s_j^*$, by uniqueness, we also have that

$\gamma_{\mathcal{N},led,j}$ is even in terms of $t-s_j^*$. On the other hand, the decomposition of $\gamma_{\mathcal{N},led,j}$ is generated by the data $\mathcal{N}_{led,j} = A_{p,1}\left(\Psi_j^*\right)^{p-2}\mathcal{V}_j^2 + A_{p,2}\left(\Psi_j^*\right)^{p-3}\mathcal{V}_j^3$. The first part $\gamma_{\mathcal{N},led,j,*}$ is obtained by the data $A_{p,1}\left(\Psi_j^*\right)^{p-2}\mathcal{V}_j^2$ which is even on $\mathbb{S}^{d-1}$ while, the second part $\gamma_{\mathcal{N},led,j,**}$ is obtained by the data $A_{p,2}\left(\Psi_j^*\right)^{p-3}\mathcal{V}_j^3$ which is odd on $\mathbb{S}^{d-1}$. □

By Lemma 4.8, we can eliminate the lower order terms in the data $\mathcal{R}_{new}$ by first considering the following equation:

$$\begin{cases}\mathcal{L}(\rho^{\perp}_{**,1,j}) = \mathcal{R}_{new,*,j} + \sum_{i=1}^{\nu}\Psi_i^{p-1}\left(c_{new,*,j,i}\partial_t\Psi_i + \sum_{l=1}^{d}\varsigma_{new,*,j,i,l}w_{i,l}\right), & \text{in } \mathcal{C},\\ \langle\partial_t\Psi_i, \rho^{\perp}_{**,1,j}\rangle = \langle w_{i,l}, \rho^{\perp}_{**,1,j}\rangle = 0 \quad \text{for all } 1\leq i\leq\nu \text{ and all } 1\leq l\leq d,\end{cases} \tag{4.44}$$ eqn4015

where $\mathcal{R}_{new,*,j} = 2A_{p,1}\left(\Psi_j^*\right)^{p-2}\mathcal{V}_j\gamma_{\mathcal{N},led,j}$.

⟨lem0011⟩ **Lemma 4.9.** *Let $d\geq 2$, $a<0$ and $b=b_{FS}(a)$. Then (4.44) is uniquely solvable. Moreover, the solution $\rho^{\perp}_{**,1,j}$ is even in terms of $t-s_j^*$ and satisfies*

$$\sup_{(t,\theta)\in\mathcal{C}}\frac{|\rho^{\perp}_{**,1,j}|}{\Psi_j^{1-\sigma}(t)} \lesssim \beta_*^3,$$

where the Lagrange multipliers $\{c_{new,*,j,i}\}$ and $\{\varsigma_{new,*,j,i,l}\}$ are chosen such that the right hand side of the equation (4.44) is orthogonal to $\{\partial_t\Psi_j\}$ and $\{w_{j,l}\}$ in $H^1(\mathcal{C})$. The Lagrange multipliers also satisfy $|c_{new,*,j,i}|\lesssim\beta_*^3Q^p$ and $|\varsigma_{new,*,j,i,l}|\lesssim\beta_*^3$ for all $1\leq i,j\leq\nu$ and all $1\leq l\leq d$.

Proof. The proof is also similar to that of Lemma 4.7 so we omit it. Again, we only point out that since $\mathcal{R}_{new,*,j}$ is even in terms of $t-s_j^*$ by Lemma 4.8, by uniqueness, we also have that $\rho^{\perp}_{**,1,j}$ is even in terms of $t-s_j^*$. On the other hand, similar to that of (4.38) and (4.39), by the oddness of $\partial_t\Psi$ in $\mathbb{R}$, we also have

$$\sum_{i=1}^{\nu}|c_{new,*,j,i}|\lesssim\beta_*^3Q^p \quad\text{and}\quad \sum_{i=1}^{\nu}|\varsigma_{new,*,j,i,l}|\lesssim\beta_*^3$$

for all $1\leq i,j\leq\nu$ and all $1\leq l\leq d$. □

Let

$$\rho^{\perp}_{**,1} = \sum_{j=1}^{\nu}\rho^{\perp}_{**,1,j} \quad\text{and}\quad \mathcal{R}_{new,*} = \sum_{j=1}^{\nu}\mathcal{R}_{new,*,j}.$$

Clearly, $\rho^{\perp}_{**,1}$ may also have projections on *span*$\{\Psi_l\}$. Thus, as above, we decompose

$$\rho^{\perp}_{**,1} = \sum_{l=1}^{\nu}\alpha_{l,1}^{**}\Psi_l + \rho^{\perp}_{**,2},$$

where $\{\alpha_{l,1}^{**}\}$ is chosen such that

$$\left\langle\rho^{\perp}_{**,2},\Psi_j\right\rangle = \left\langle\rho^{\perp}_{**,2},\partial_t\Psi_j\right\rangle = \left\langle\rho^{\perp}_{**,2},w_{j,l}\right\rangle = 0. \tag{4.45}$$ eqn2004

Moreover, by (4.44), we know that $\rho^{\perp}_{**,2}$ satisfies the following equation:

$$\begin{cases} \mathcal{L}(\rho^{\perp}_{**,2}) = \mathcal{R}_{new,**} + \sum_{i=1}^{\nu} \Psi_i^{p-1}\left(c_{new,*,i}\partial_t\Psi_i + \sum_{l=1}^{d} \varsigma_{new,*,i,l} w_{i,l}\right), \quad \text{in } \mathcal{C}, \\ \left\langle \Psi_j, \rho^{\perp}_{**,2}\right\rangle = \langle \partial_t\Psi_j, \rho^{\perp}_{**,2}\rangle = 0 \quad \text{for all } 1 \leq j \leq \nu, \\ \langle w_{j,l}, \rho^{\perp}_{**,2}\rangle = 0 \quad \text{for all } 1 \leq j \leq \nu \text{ and all } 1 \leq l \leq d, \end{cases} \tag{4.46}$$ eqn6114

where

$$\mathcal{R}_{new,**} = \sum_{j=1}^{\nu} 2A_{p,1}\left(\Psi_j^*\right)^{p-2}\mathcal{V}_j\gamma_{\mathcal{N},led,j} + \sum_{l=1}^{\nu} p\alpha^{**}_{l,1}\left(\mathcal{U}^{p-1} - \Psi_l^{p-1}\right)\Psi_l. \tag{4.47}$$ eqn6061

⟨lem0012⟩ **Lemma 4.10.** *Let $d \geq 2$, $a < 0$ and $b = b_{FS}(a)$. Then we have*

$$\sum_{j=1}^{\nu}\left|\alpha^{**}_{j,1}\right| \lesssim \beta_*^4 + \beta_*^3 Q^{\frac{1}{2}+\sigma}. \tag{4.48}$$ eqn5068

Moreover, we can decompose $\rho^{\perp}_{**,2} = \rho^{\perp}_{**,2,odd} + \rho^{\perp}_{**,2,rem}$ with $\rho^{\perp}_{**,2,odd}$ being odd on $\mathbb{S}^{d-1}$ and

$$\begin{cases} \sup_{(t,\theta)\in\mathcal{C}} \frac{|\rho^{\perp}_{**,2,odd}|}{\sum_{j=1}^{\nu}\Psi_j^{1-\sigma}(t)} \lesssim \beta_*^3, \\ \|\rho^{\perp}_{**,2,rem}\|_{\sharp} + \sup_{(t,\theta)\in\mathcal{C}} \frac{|\rho^{\perp}_{**,2,rem}|}{\sum_{j=1}^{\nu}\Psi_j^{1-\sigma}(t)} \lesssim \beta_*^4 + \beta_*^3 Q^{\frac{1}{2}+\sigma}, \quad p \geq 3, \\ \|\rho^{\perp}_{**,2,rem}\|_{\natural,1} + \sup_{(t,\theta)\in\mathcal{C}} \frac{|\rho^{\perp}_{**,2,rem}|}{\sum_{j=1}^{\nu}\Psi_j^{1-\sigma}(t)} \lesssim \beta_*^4 + \beta_*^3 Q^{\frac{1}{2}+\sigma}, \quad 1 < p < 3, \end{cases} \tag{4.49}$$ eqn9079

where $\|\cdot\|_{\sharp}$ and $\|\cdot\|_{\natural,1}$ are given in Lemma 4.4.

Proof. By (4.44) and (4.45),

$$\begin{aligned} \sum_{l=1}^{\nu}\langle\Psi_l, \Psi_j\rangle\alpha^{**}_{l,1} &= \langle\mathcal{R}_{new,*}, \Psi_j\rangle_{L^2} + p\left\langle\mathcal{U}^{p-1}\rho^{\perp}_{**,1}, \Psi_j\right\rangle_{L^2} \\ &= \sum_{l=1}^{\nu} p\alpha^{**}_{l,1}\left\langle\mathcal{U}^{p-1}\Psi_l, \Psi_j\right\rangle_{L^2} + p\left\langle\mathcal{U}^{p-1}\rho^{\perp}_{**,2}, \Psi_j\right\rangle_{L^2} \\ &\quad + \langle\mathcal{R}_{new,*}, \Psi_j\rangle_{L^2} \end{aligned} \tag{4.50}$$ eqn5052

for all $1 \leq j \leq \nu$. By the oddness of $w_{j,l}$ on $\mathbb{S}^{d-1}$ and Lemmas 4.3 and 4.8, we have $\left|\langle\mathcal{R}_{new,*}, \Psi_j\rangle_{L^2}\right| \lesssim \beta_*^4$. On the other hand, by Lemma 4.3 and similar estimates of

(4.33),

$$\begin{aligned}\sum_{l=1}^{\nu} p\alpha_{l,1}^{**}\left\langle \mathcal{U}^{p-1}\Psi_l,\Psi_j\right\rangle_{L^2} &= \sum_{l=1}^{\nu} p\alpha_{l,1}^{**}\left\langle \sum_{i=1}^{\nu}\left(\Psi_i^*\right)^{p-1}\chi_{\mathcal{B}_i}\Psi_l,\Psi_j\right\rangle_{L^2}\\ &\quad+\sum_{l=1}^{\nu} p\alpha_{l,1}^{**}\left\langle\left(\mathcal{U}^{p-1}-\sum_{i=1}^{\nu}\left(\Psi_i^*\right)^{p-1}\chi_{\mathcal{B}_i}\right)\Psi_l,\Psi_j\right\rangle_{L^2}\\ &\quad+\sum_{l=1}^{\nu} p\alpha_{l,1}^{**}\left\langle \mathcal{U}^{p-1}\chi_{\mathcal{C}\backslash\mathcal{B}_*}\Psi_l,\Psi_j\right\rangle_{L^2}\\ &= p\|\Psi\|^2\alpha_{j,1}^{**}+\sum_{l=1}^{\nu}\mathcal{O}(Q)\alpha_{l,1}^{**}\end{aligned}$$

and further by (4.45), we have

$$\begin{aligned}\left\langle \mathcal{U}^{p-1}\rho_{**,2}^{\perp},\Psi_j\right\rangle_{L^2} &= \left\langle\left(\mathcal{U}^{p-1}-\left(\Psi_j^*\right)^{p-1}\right)\Psi_j,\rho_{**,2}^{\perp}\right\rangle_{L^2}\\ &= \|\rho_{**,2}^{\perp}\|\times\begin{cases}\mathcal{O}\left(Q\right), & p>2,\\ \mathcal{O}\left(Q\left|\log Q\right|^{\frac{1}{2}}\right), & p=2,\\ \mathcal{O}\left(Q^{\frac{p}{2}}\right), & 1<p<2.\end{cases}\end{aligned}$$

It follows from (4.50) that

$$\sum_{j=1}^{\nu}\left|\alpha_{j,1}^{**}\right|\lesssim\beta_*^4+Q^{\frac{p}{2}\wedge 1}\left|\log Q\right|\|\rho_{**,2}^{\perp}\|.$$

Now, by multiplying (4.46) with $\rho_{**,2}^{\perp}$ on both sides and integrating by parts, we have $\|\rho_{**,2}^{\perp}\|\lesssim\beta_*^3$, which implies that (4.48) holds true. To obtain the estimate (4.49), we shall decompose $\mathcal{R}_{new,**}$ into three parts, where $\mathcal{R}_{new,**}$ is given by (4.47). The first part is given by $\sum_{j=1}^{\nu}2A_{p,1}\left(\Psi_j^*\right)^{p-2}\mathcal{V}_j\gamma_{\mathcal{N},led,j,*}$, which generates the bound

$$\sup_{(t,\theta)\in\mathcal{C}}\frac{|\rho_{**,2,odd}^{\perp}|}{\sum_{j=1}^{\nu}\Psi_j^{1-\sigma}(t)}\lesssim\beta_*^3$$

as that of $\gamma_{\mathcal{N},led,j,*}$, where $\gamma_{\mathcal{N},led,j,*}$ is given by Lemma 4.8. The second part is given by $\sum_{j=1}^{\nu}2A_{p,1}\left(\Psi_j^*\right)^{p-2}\mathcal{V}_j\gamma_{\mathcal{N},led,j,**}$, which generates the bound

$$\sup_{(t,\theta)\in\mathcal{C}}\frac{|\rho_{**,2,rem}^{\perp}|}{\sum_{j=1}^{\nu}\Psi_j^{1-\sigma}(t)}\lesssim\beta_*^4$$

as that of $\gamma_{\mathcal{N},led,j,**}$, where $\gamma_{\mathcal{N},led,j,**}$ is also given by Lemma 4.8. The third part is given by $\sum_{l=1}^{\nu}p\alpha_{l,1}^{**}\left(\mathcal{U}^{p-1}-\Psi_l^{p-1}\right)\Psi_l$, which generates the bound

$$\begin{cases}\|\rho_{**,2,rem}^{\perp}\|_\sharp\lesssim\beta_*^4+\beta_*^3Q^{\frac{1}{2}+\sigma}, & p\geq 3,\\ \|\rho_{**,2,rem}^{\perp}\|_{\natural,1}\lesssim\beta_*^4+\beta_*^3Q^{\frac{1}{2}+\sigma}, & 1<p<3,\end{cases}$$

as that of $\overline{\gamma}_{1,ex}$, where $\overline{\gamma}_{1,ex}$ is given by Lemma 4.4. □

To eliminate the lower order terms in the data $\mathcal{R}_{new}$, we next consider the following equations:

$$\begin{cases}\mathcal{L}(\rho^{\perp}_{**,3,i}) = \mathcal{R}_{3,ex,i} - \sum_{j=1}^{\nu}\Psi_j^{p-1}\left(c_{3,ex,i,j}\partial_t\Psi_j + \sum_{l=1}^{d}\varsigma_{3,ex,i,j,l}w_{j,l}\right), \quad \text{in } \mathcal{C},\\ \langle\partial_t\Psi_j,\rho^{\perp}_{**,3,i}\rangle = \langle w_{j,l},\rho^{\perp}_{**,3,i}\rangle = 0 \quad \text{for all } 1\leq j\leq\nu \text{ and all } 1\leq l\leq d,\end{cases} \tag{4.51}$$

⟨eqn6013⟩

where

$$\mathcal{R}_{3,ex,i} = \begin{cases}\sum_{l=2}^{n_0}A_{p,l-1}\mathcal{U}^{p-l}\overline{\gamma}^l_{1,ex}, \quad i=0,\\ \sum_{l=2}^{n_0}A_{p,l-1}\mathcal{U}^{p-l}\left(\left(\overline{\gamma}_{1,ex}+\rho^{\perp}_{**,3,0,1}\right)^l-\overline{\gamma}^l_{1,ex}\right), \quad i=1,\\ \sum_{l=2}^{n_0}A_{p,l-1}\mathcal{U}^{p-l}\left(\left(\overline{\gamma}_{1,ex}+\sum_{k=0}^{i-1}\rho^{\perp}_{**,3,k,1}\right)^l-\left(\overline{\gamma}_{1,ex}+\sum_{k=0}^{i-2}\rho^{\perp}_{**,3,k,1}\right)^l\right), \quad i\geq 2.\end{cases} \tag{4.52}$$

⟨eqnnewnew0007⟩

with $\overline{\gamma}_{1,ex}$ given by Lemma 4.4.

⟨lem0014⟩ **Lemma 4.11.** *Let $d\geq 2$, $a<0$ and $b=b_{FS}(a)$. Then (4.51) is uniquely solvable for all $i\geq 0$. The solutions $\rho^{\perp}_{**,3,i}$ are even on $\mathbb{S}^{d-1}$ and satisfies*

$$\begin{cases}\|\rho^{\perp}_{**,3,i}\|_{\sharp}\lesssim Q^{((p-1)\wedge 1)(i+1)}, \quad p\geq 3,\\ \|\rho^{\perp}_{**,3,i}\|_{\natural,1}\lesssim Q^{((p-1)\wedge 1)(i+1)}, \quad 1<p<3,\end{cases}$$

where the Lagrange multipliers $\{c_{3,ex,i,j}\}$ and $\{\varsigma_{3,ex,i,j,l}\}$ are chosen such that the right hand side of the equation (4.51) is orthogonal to $\{\partial_t\Psi_j\}$ and $\{w_{j,l}\}$ in $H^1(\mathcal{C})$ and the norms $\|\cdot\|_{\sharp}$ and $\|\cdot\|_{\natural,1}$ are given in Lemma 4.4. The Lagrange multipliers also satisfy $|c_{3,ex,i,j}|\lesssim Q^{1+((p-1)\wedge 1)(i+1)}$ for all $1\leq j\leq\nu$ and $\varsigma_{3,ex,i,j,l}=0$ for all $1\leq j\leq\nu$ and all $1\leq l\leq d$. Moreover, we can write

$$\rho^{\perp}_{**,3,i} = \sum_{j=1}^{\nu}\alpha^{**}_{j,2,i}\Psi_j + \rho^{\perp}_{**,3,i,1}$$

where $\{\alpha^{**}_{j,2,i}\}$ are chosen such that $\langle\Psi_j,\rho^{\perp}_{**,3}\rangle = 0$ for all $1\leq j\leq d$ which also satisfies $\sum_{j=1}^{\nu}\left|\alpha^{**}_{j,2,i}\right|\lesssim Q^{1+((p-1)\wedge 1)(i+1)}|\log Q|$ and

$$\begin{cases}\|\rho^{\perp}_{**,3,i,1}\|_{\sharp}\lesssim Q^{((p-1)\wedge 1)(i+1)}, \quad p\geq 3,\\ \|\rho^{\perp}_{**,3,i,1}\|_{\natural,1}\lesssim Q^{((p-1)\wedge 1)(i+1)}, \quad 1<p<3.\end{cases}$$

Proof. We first prove the conclusion for $i=0$. Since by Lemma 4.4, we always have $|\gamma_{1,ex}| = \mathcal{O}\left(Q^{(p-1)\wedge 1}\mathcal{U}\right)$. Thus, it is easy to see that $|\mathcal{R}_{3,ex,0}|\lesssim\mathcal{U}^{p-2}\overline{\gamma}^2_{1,ex}\lesssim Q^{(p-1)\wedge 1}\left|\overline{\gamma}_{1,ex}\right|$, which together with Lemma 4.4 once more, implies that

$$\begin{cases}\|\mathcal{R}_{3,ex,0}\|_{\sharp}\lesssim Q^{(p-1)\wedge 1}, \quad p\geq 3,\\ \|\mathcal{R}_{3,ex,0}\|_{\natural,1}\lesssim Q^{(p-1)\wedge 1}, \quad 1<p<3.\end{cases}$$

Since $\overline{\gamma}_{1,ex}$ is even on $\mathbb{S}^{d-1}$ which implies that $\mathcal{R}_{3,ex,0}$ is even on $\mathbb{S}^{d-1}$, the rest of the proof of $\rho^{\perp}_{**,3,0}$ and $\{\alpha^{**}_{j,2,0}\}$ are the same as that of Lemmas 4.4 and 4.10, respectively, and we omit it here. The conclusions for $i\geq 1$ then follow from iterating. □

To eliminate the lower order terms in the data $\mathcal{R}_{new}$, we also need to consider the following equation:

$$\begin{cases}\mathcal{L}(\rho^{\perp}_{**,4})=\mathcal{R}_{4,ex}-\sum_{j=1}^{\nu}\Psi_j^{p-1}\left(c_{4,ex,j}\partial_t\Psi_j+\sum_{l=1}^{d}\varsigma_{4,ex,j,l}w_{j,l}\right), & \text{in } \mathcal{C},\\ \langle\partial_t\Psi_j,\rho^{\perp}_{**,4}\rangle=\langle w_{j,l},\rho^{\perp}_{**,4}\rangle=0 \quad \text{for all } 1\leq j\leq\nu \text{ and all } 1\leq l\leq d,\end{cases} \tag{4.53}$$ eqn5013

where $\mathcal{L}$ is given by (3.17), $w_{j,l}=\Psi_j^{\frac{p+1}{2}}\theta_l$ are the nontrivial kernels of the bubble Ψ_j in $H^1(\mathcal{C})$ given by Lemma 2.1 and $\mathcal{R}_{4,ex}=2A_{p,1}\mathcal{U}^{p-2}\mathcal{V}\gamma_{1,ex}$ where

$$\gamma_{1,ex}=\overline{\gamma}_{1,ex}+\sum_{i=0}^{n_0}\rho^{\perp}_{**,3,i,1} \tag{4.54}$$ eqnnewnew1998

with $\overline{\gamma}_{1,ex}$ and $\rho^{\perp}_{**,3,i,1}$ given by Lemmas 4.4 and 4.11, respectively.

⟨lem0013⟩ **Lemma 4.12.** *Let $d\geq 2$, $a<0$ and $b=b_{FS}(a)$. Then (4.53) is uniquely solvable. Moreover, the solution $\rho^{\perp}_{**,4}$ is odd on $\mathbb{S}^{d-1}$ and satisfies*

$$\begin{cases}\|\rho^{\perp}_{**,4}\|_{\sharp}\lesssim\beta_*, & p\geq\frac{7}{3},\\ \|\rho^{\perp}_{**,4}\|_{\natural,2}\lesssim\beta_*, & 1<p<\frac{7}{3},\end{cases} \tag{4.55}$$ eqn4447

where the Lagrange multipliers $\{c_{4,ex,j}\}$ and $\{\varsigma_{4,ex,j,l}\}$ are chosen such that the right hand side of the equation (4.53) is orthogonal to $\{\partial_t\Psi_j\}$ and $\{w_{j,l}\}$ in $H^1(\mathcal{C})$ and the norms $\|\cdot\|_{\sharp}$ and $\|\cdot\|_{\natural,2}$ are given in Lemma 4.6. The Lagrange multipliers also satisfy $c_{4,ex,j}=0$ for all $1\leq j\leq\nu$ and $|\varsigma_{4,ex,j,l}|\lesssim\beta_*Q$ for all $1\leq j\leq\nu$ and $1\leq l\leq d$.

Proof. By Lemmas 4.4 and 4.11, $\gamma_{1,ex}$ is even on $\mathbb{S}^{d-1}$. Moreover, similar to (4.33), by direct calculations and $p>1$, we have

$$\begin{cases}\|\mathcal{R}_{4,ex}\|_{\sharp}\lesssim\beta_*, & p\geq\frac{7}{3},\\ \|\mathcal{R}_{4,ex}\|_{\natural,2}\lesssim\beta_*, & 1<p<\frac{7}{3}.\end{cases} \tag{4.56}$$ eqn4548

The rest of the proof is the same as that of Lemma 4.6, so we omit it here. □

To eliminate the lower order terms in the data $\mathcal{R}_{new}$, we finally need to consider the following equations:

$$\begin{cases}\mathcal{L}(\rho^{\perp}_{**,5,i})=\mathcal{R}_{5,ex,i}-\sum_{j=1}^{\nu}\Psi_j^{p-1}\left(c_{5,ex,i,j}\partial_t\Psi_j+\sum_{l=1}^{d}\varsigma_{5,ex,i,j,l}w_{j,l}\right), & \text{in } \mathcal{C},\\ \langle\partial_t\Psi_j,\rho^{\perp}_{**,5,i}\rangle=\langle w_{j,l},\rho^{\perp}_{**,5,i}\rangle=0 \quad \text{for all } 1\leq j\leq\nu \text{ and all } 1\leq l\leq d,\end{cases} \tag{4.57}$$ eqn7013

where

$$\mathcal{R}_{5,ex,i}=\begin{cases}\sum_{l=2}^{n_0}lA_{p,l-1}\mathcal{U}^{p-l}(\mathcal{V}+\overline{\gamma}_{2,ex}+\rho^{\perp}_{**,4})\gamma_{1,ex}^{l-1}, & i=0,\\ \sum_{l=2}^{n_0}lA_{p,l-1}\mathcal{U}^{p-l}\rho^{\perp}_{**,5,i-1}\gamma_{1,ex}^{l-1}, & i\geq 1,\end{cases} \tag{4.58}$$ eqnnewnew0008

with $\gamma_{1,ex}$ given by (4.54).

⟨lem0015⟩ **Lemma 4.13.** *Let $d \geq 2$, $a < 0$ and $b = b_{FS}(a)$. Then (4.57) is uniquely solvable for all $i \geq 0$. Moreover, the solutions $\rho^{\perp}_{**,5,i}$ are all odd on $\mathbb{S}^{d-1}$ and satisfies*

$$\begin{cases} \|\rho^{\perp}_{**,5,i}\|_{\sharp} \lesssim Q^{((p-1)\wedge 1)i}\beta_*, & p \geq \frac{7}{3}, \\ \|\rho^{\perp}_{**,5,i}\|_{\natural,2} \lesssim Q^{((p-1)\wedge 1)i}\beta_*, & 1 < p < \frac{7}{3}, \end{cases}$$

where the Lagrange multipliers $\{c_{5,ex,i,j}\}$ and $\{\varsigma_{5,ex,i,j,l}\}$ are chosen such that the right hand side of the equation (4.57) is orthogonal to $\{\partial_t\Psi_j\}$ and $\{w_{j,l}\}$ in $H^1(\mathcal{C})$ and the norms $\|\cdot\|_{\sharp}$ and $\|\cdot\|_{\natural,2}$ are given in Lemma 4.6. The Lagrange multipliers also satisfy $c_{5,ex,i,j} = 0$ for all $1 \leq j \leq \nu$ and $|\varsigma_{5,ex,i,j,l}| \lesssim \beta_* Q^{1+(\frac{(p-1)}{2}\wedge 1)i}$ for all $1 \leq j \leq \nu$ and all $1 \leq l \leq d$.

Proof. Again, we first prove the conclusion for $i = 0$. By Lemmas 4.4, 4.11 and 4.12, we have $|\mathcal{R}_{5,ex,0}| \lesssim |\mathcal{U}^{p-2}\mathcal{V}\gamma_{1,ex}| + Q^{(p-1)\wedge 1}|\overline{\gamma}_{2,ex} + \rho^{\perp}_{**,4}|$, which implies that

$$\begin{cases} \|\mathcal{R}_{5,ex,0}\|_{\sharp} \lesssim \beta_*, & p \geq \frac{7}{3}, \\ \|\mathcal{R}_{5,ex,0}\|_{\natural,2} \lesssim \beta_*, & 1 < p < \frac{7}{3}. \end{cases} \tag{4.59}$$ eqnnewnew19997

Since $\gamma_{1,ex}$ is even on $\mathbb{S}^{d-1}$ which implies that $\mathcal{R}_{5,ex,0}$ is odd on $\mathbb{S}^{d-1}$, the rest of the proof for $i = 0$ is the same as that of Lemma 4.6 and we omit it here. Since

$$|\mathcal{R}_{5,ex,i}| \lesssim Q^{(p-1)\wedge 1}\left|\rho^{\perp}_{**,5,i-1}\right|, \tag{4.60}$$ eqnnewnew19996

the conclusions for $i \geq 1$ then follow from iterating. □

We denote

$$\rho^{\perp}_{**,3} = \sum_{l=0}^{n_0}\rho^{\perp}_{**,3,l,1} \quad \text{and} \quad \mathcal{R}_{3,ex} = \sum_{l=0}^{n_0}\mathcal{R}_{3,ex,l} \tag{4.61}$$ eqnnewnew19994

and

$$\rho^{\perp}_{**,5} = \sum_{l=0}^{n_0}\rho^{\perp}_{**,5,l} \quad \text{and} \quad \mathcal{R}_{5,ex} = \sum_{l=0}^{n_0}\mathcal{R}_{5,ex,l} \tag{4.62}$$ eqnnewnew19995

with $\mathcal{R}_{3,ex,l}$ given by (4.52) and $\mathcal{R}_{5,ex,l}$ given by (4.58), respctively. Now, let $\rho^{\perp}_{**} = \rho^{\perp}_{**,0} - \sum_{j=2}^{5}\rho^{\perp}_{**,j}$, then we have the following decomposition of ρ_*.

⟨prop0001⟩ **Proposition 4.1.** *Let $d \geq 2$, $a < 0$ and $b = b_{FS}(a)$. Then we have $\rho_* = \rho_0 + \rho^{\perp}_{**}$, where*

(1) *the regular part $\rho_0 = \gamma_{ex} + \gamma_* + \gamma_{\mathcal{N},led}$ and*
  (*i*) *$\gamma_{ex} = \sum_{l=1}^{2}\gamma_{l,ex}$ with $\gamma_{1,ex} = \overline{\gamma}_{1,ex} + \rho^{\perp}_{**,3}$ even on $\mathbb{S}^{d-1}$ and $\gamma_{2,ex} = \overline{\gamma}_{2,ex} + \rho^{\perp}_{**,4} + \rho^{\perp}_{**,5}$ odd on $\mathbb{S}^{d-1}$ satisfying*

$$\begin{cases} \|\gamma_{ex}\|_{\sharp} + \|\gamma_{1,ex}\|_{\sharp} \lesssim 1, & p \geq 3, \\ \|\gamma_{ex}\|_{\natural,1} + \|\gamma_{1,ex}\|_{\natural,1} \lesssim 1, & 1 < p < 3 \end{cases}$$

and

$$\begin{cases}\|\gamma_{2,ex}\|_\sharp \lesssim \beta_*, \quad p \geq \frac{7}{3},\\ \|\gamma_{2,ex}\|_{\natural,2} \lesssim \beta_*, \quad 1 < p < \frac{7}{3}.\end{cases}$$

(ii) $\gamma_* = \sum_{j=1}^{\nu}\gamma_{j,*}$ *is even on* $\mathbb{S}^{d-1}$ *with* $\gamma_{j,*}$ *even in terms of* $t - s_j^*$ *in* $\mathbb{R}$ *and satisfying*

$$\sup_{(t,\theta)\in\mathcal{C}}\frac{|\gamma_*|}{\sum_{j=1}^{\nu}\Psi_j^{1-\sigma}(t)} \lesssim Q^{1+\sigma} + \sum_{j=1}^{\nu}\left|(\alpha_j^*)^{p-1} - 1\right| + \beta_*^2,$$

where $\gamma_{j,*} = \gamma_{1,j} + \alpha_{j,0}^{**}\Psi_j$.

(iii) $\gamma_{\mathcal{N},led} = \gamma_{\mathcal{N},led,*} + \rho^\perp_{**,2}$, *with the symmetrical part of* $\gamma_{\mathcal{N},led}$ *in terms of* $t - s_j^*$ *given by* $\gamma_{\mathcal{N},led,j} + \rho^\perp_{**,1,j} - \alpha_{j,1}^{**}\Psi_j$ *and the remaining part denoted by* $\gamma_{\mathcal{N},led,rem,j}$, *satisfies the following estimate*

$$\sup_{(t,\theta)\in\mathcal{C}}\frac{|\gamma_{\mathcal{N},led,rem,j}|}{\sum_{i=1;i\neq j}^{\nu}\Psi_j^{1-\sigma}(t)} \lesssim \beta_*^2.$$

Moreover, $\rho^\perp_{**,2} = \rho^\perp_{**,2,odd} + \rho^\perp_{**,2,rem}$ *with* $\rho^\perp_{**,2,odd}$ *being odd on* $\mathbb{S}^{d-1}$ *and* $\gamma_{\mathcal{N},led}$ *satisfies the following estimates*

$$\begin{cases}\frac{\|\rho^\perp_{**,2,rem}\|_\sharp}{\beta_*^4 + \beta_*^3 Q^{\frac12+\sigma}} + \sup_{(t,\theta)\in\mathcal{C}}\frac{\beta_*^{-2}|\gamma_{\mathcal{N},led,*}| + \beta_*^{-3}|\rho^\perp_{**,2,odd}|}{\sum_{j=1}^{\nu}\Psi_j^{1-\sigma}(t)} \lesssim 1, \quad p \geq 3,\\ \frac{\|\rho^\perp_{**,2,rem}\|_{\natural,1}}{\beta_*^4 + \beta_*^3 Q^{\frac12+\sigma}} + \sup_{(t,\theta)\in\mathcal{C}}\frac{\beta_*^{-2}|\gamma_{\mathcal{N},led,*}| + \beta_*^{-3}|\rho^\perp_{**,2,odd}|}{\sum_{j=1}^{\nu}\Psi_j^{1-\sigma}(t)} \lesssim 1, \quad 1 < p < 3.\end{cases}$$

(2) *The singular part* $\rho^\perp_{**}$ *satisfies the following equation:*

$$\begin{cases}\mathcal{L}(\rho^\perp_{**}) = f + \mathcal{R}_{new,0}, \quad \text{in } \mathcal{C},\\ \langle\Psi_j, \rho^\perp_{**}\rangle = \langle\partial_t\Psi_j, \rho^\perp_{**}\rangle = \langle w_{j,l}, \rho^\perp_{**}\rangle = 0 \quad \text{for} 1 \leq j \leq \nu \text{ and } 1 \leq l \leq d,\end{cases} \tag{4.63}$$

eqn1114

where

$$\begin{aligned}\mathcal{R}_{new,0} \quad = \quad & \sum_{i=1}^{\nu}(c_{1,ex,i} + c_{1,j,i} + c_{\mathcal{N},led,i} - c_{3,ex,i} - c_{new,*,i})\Psi_i^{p-1}\partial_t\Psi_i + \sum_{j=1}^{\nu}\mathcal{R}_{2,j}\\ & + \sum_{i=1}^{\nu}\sum_{l=1}^{d}(\varsigma_{2,ex,i,l} + \varsigma_{\mathcal{N},led,i,l} - \varsigma_{4,ex,i,l} - \varsigma_{5,ex,i,l} - \varsigma_{new,*,i,l})\Psi_i^{p-1}w_{i,l}\\ & + \sum_{j=1}^{\nu}\left(2A_{p,1}\left(\Psi_j^*\right)^{p-2}\mathcal{V}_j(\rho_* - \gamma_{1,ex} - \gamma_{\mathcal{N},led,j}) + 3A_{p,2}\left(\Psi_j^*\right)^{p-3}\mathcal{V}_j^2\rho_*\right)\chi_{\mathcal{B}_j}\\ & + \sum_{j=1}^{\nu}\mathcal{O}\left(\beta_*^2\mathcal{U}_j\Psi_j^{2p-3}\left(\Psi_j + \rho_*\right) + \beta_*|\rho_* - \gamma_{1,ex}|\Psi_j^{\frac{3p-5}{2}}\mathcal{U}_j\right)\chi_{\mathcal{B}_j}\\ & + \mathcal{O}\left(\beta_*|\rho_* - \gamma_{1,ex}|\mathcal{U}^{\frac{3(p-1)}{2}}\right)\chi_{\mathcal{C}\setminus\mathcal{B}_*} + \mathcal{O}\left(\beta_*^2\mathcal{U}^{2(p-1)}\gamma_{1,ex}\chi_{\mathcal{C}\setminus\mathcal{B}_*}\right)\\ & + \sum_{j=1}^{\nu}\alpha_j^{**}\left(\mathcal{U}^{p-1} - \Psi_j^{p-1}\right)\Psi_j + \mathcal{N}_{rem} - \mathcal{R}_{3,ex} - \mathcal{R}_{5,ex}\end{aligned} \tag{4.64}$$

eqn0061

with $\mathcal{R}_{3,ex}$ *given by* (4.61), $\mathcal{R}_{5,ex}$ *given by* (4.62) *and* $\alpha_j^{**}=\alpha_{j,0}^{**}-\alpha_{j,1}^{**}-\alpha_{1,2}^{**}$ *with*

$$\sum_{j=1}^{\nu}\left|\alpha_j^{**}\right|\lesssim Q^{1+\sigma}+\sum_{j=1}^{\nu}\left|\left(\alpha_j^*\right)^{p-1}-1\right|+\beta_*^2.$$

Proof. Since $\frac{3p-5}{2}>p-2$ for $p>1$, by (4.32), (4.34), (4.55), (4.56), (4.59) and (4.60) and Lemma 4.13, we also have

$$\|\mathcal{R}_{2,ex}+\mathcal{R}_{4,ex}+\mathcal{R}_{5,ex}\|_{\natural,1}\lesssim\beta_* \quad\text{and}\quad \|\gamma_{2,ex}\|_{\natural,1}\lesssim\beta_* \tag{4.65}$$ eqqnew0009

for $1<p<3$. Thus, the rest proof of (i) of (1) follows from Lemmas 4.4, 4.6, 4.12, 4.11 and 4.13. The conclusion of (ii) of (1) follows from Lemma 4.5 and (4.41). The conclusion of (iii) of (1) follows from Lemmas 4.8, 4.9 and 4.10. The conclusion of (2) follows from (4.41), (4.42), (4.44), (4.46), (4.53), (4.51), (4.57) and Lemma 4.10. □

5. First refined expansion of $\mathcal{N}$ and estimate of $\{\alpha_j^*\}$

As we stated before, inspired by the optimal Bianchi-Egnell stability of the CKN inequality for $d\geq 2$, $a<0$ and $b=b_{FS}(a)$ proved in [54, Theorem 1], we need to eliminate the lower order terms (compared to the β_*^4 terms) to get the desired stability. Thus, we need to refine the expansion of $\mathcal{N}$ since we have picked up a regular part ρ_0 in the remaining term ρ_*.

⟨lemn0001⟩ **Lemma 5.1.** *Let* $d\geq 2$*,* $a<0$ *and* $b=b_{FS}(a)$*. Then the nonlinear part* $\mathcal{N}$*, which is given by* (3.20)*, can be refinedly expanded as follows:*

$$\begin{aligned}
\mathcal{N} &= A_{p,1}\mathcal{U}^{p-2}\left(\overline{\mathcal{V}}^2+2\overline{\mathcal{V}}\rho_{**}^{\perp}\right)+A_{p,2}\mathcal{U}^{p-3}\left(\overline{\mathcal{V}}^3+3\overline{\mathcal{V}}^2\rho_{**}^{\perp}\right)+\overline{\mathcal{N}}_{rem}\\
&= A_{p,1}\mathcal{U}^{p-2}\left(\mathcal{V}^2+2\mathcal{V}\rho_*+\rho_0^2+2\rho_0\rho_{**}^{\perp}\right)+\overline{\mathcal{N}}_{rem}\\
&\quad +A_{p,2}\mathcal{U}^{p-3}\left(\mathcal{V}^3+3\mathcal{V}^2\rho_*+3\mathcal{V}\rho_0^2+\rho_0^3+6\mathcal{V}\rho_0\rho_{**}^{\perp}+3\rho_0^2\rho_{**}^{\perp}\right)\\
&= \mathcal{N}_*+\overline{\mathcal{N}}_{rem}+A_{p,1}\mathcal{U}^{p-2}\left(\rho_0^2+2\rho_0\rho_{**}^{\perp}\right)\\
&\quad +A_{p,2}\mathcal{U}^{p-3}\left(3\mathcal{V}\rho_0^2+\rho_0^3+6\mathcal{V}\rho_0\rho_{**}^{\perp}+3\rho_0^2\rho_{**}^{\perp}\right)\\
&:= \mathcal{N}_*+\overline{\mathcal{N}}_{rem}+\mathcal{N}_0
\end{aligned}$$

in $\mathcal{C}$*, where* $\mathcal{N}_*$ *is given in* (4.1)*,* $\overline{\mathcal{V}}=\mathcal{V}+\rho_0$ *with* ρ_0 *the regular part of* ρ_* *given in* (1) *of Proposition 4.1 and*

$$\begin{aligned}
\overline{\mathcal{N}}_{rem} &= \mathcal{O}\left(\left(\beta_*+Q+\sum_{j=1}^{\nu}\left|\left(\alpha_j^*\right)^{p-1}-1\right|\right)^4+\mathcal{U}^{p-4}\gamma_{ex}^4\right)\\
&\quad +\mathcal{O}\left(\chi_{p\geq 2}\left|\rho_{**}^{\perp}\right|^2+\left|\rho_{**}^{\perp}\right|^p+\left|\rho_{**}^{\perp}\right|^{1+\sigma}+\left|\gamma_*+\gamma_{\mathcal{N},led}\right|^{1+\sigma}\chi_{\mathcal{C}\backslash\widetilde{\mathcal{B}}_{**}}\right)
\end{aligned}$$

where

$$\widetilde{\mathcal{B}}_{**}=\left\{(\theta,t)\in\mathcal{C}\mid\left|\gamma_*+\gamma_{\mathcal{N},led}\right|\leq\frac{1}{2}\mathcal{U}\right\}$$

and γ_* *and* $\gamma_{\mathcal{N},led}$ *are given in* (1) *of Proposition 4.1.*

Proof. We improve the set $\mathcal{A}$, used in the proof of Lemma 4.1, by introducing the set

$$\mathcal{A}_* = \left\{(\theta, t) \in \widetilde{\mathcal{B}}_{**} \mid |\rho_{**}^{\perp}| \leq \left|\overline{\mathcal{V}}\right|\right\},$$

where $\rho_{**}^{\perp}$ is the (possible) singular part of ρ_* given in (2) of Proposition 4.1. Since we always have $|\gamma_{ex}| = o(\mathcal{U})$ by (i) of (1) of Proposition 4.1, by (4.2), $\left|\overline{\mathcal{V}}\right| \leq \frac{3}{4}\mathcal{U}$ in $\widetilde{\mathcal{B}}_{**}$. Thus, by (1) of Proposition 4.1, we can expand $\mathcal{N}$ in $\mathcal{A}_*$ as that of (4.3):

$$\begin{aligned}
\mathcal{N} &= A_{p,1}\mathcal{U}^{p-2}\left(\overline{\mathcal{V}}^2 + 2\overline{\mathcal{V}}\rho_{**}^{\perp}\right) + A_{p,2}\mathcal{U}^{p-3}\left(\overline{\mathcal{V}}^3 + 3\overline{\mathcal{V}}^2\rho_{**}^{\perp}\right) \\
&\quad + \mathcal{O}\left(\mathcal{U}^{p-4}\left(\overline{\mathcal{V}} + \rho_{**}^{\perp}\right)^4 + \mathcal{U}^{p-2}\left|\rho_{**}^{\perp}\right|^2\right) \\
&= A_{p,1}\mathcal{U}^{p-2}\left(\overline{\mathcal{V}}^2 + 2\overline{\mathcal{V}}\rho_{**}^{\perp}\right) + A_{p,2}\mathcal{U}^{p-3}\left(\overline{\mathcal{V}}^3 + 3\overline{\mathcal{V}}^2\rho_{**}^{\perp}\right) \\
&\quad + \mathcal{O}\left(\left(\beta_* + Q + \sum_{j=1}^{\nu}\left|\left(\alpha_j^*\right)^{p-1} - 1\right|\right)^4 + \mathcal{U}^{p-4}\gamma_{ex}^4\right) \\
&\quad + \mathcal{O}\left(\chi_{p\geq 2}|\rho_{**}^{\perp}|^2 + |\rho_{**}^{\perp}|^p\right). \qquad (5.1)
\end{aligned}$$

In $\mathcal{C}\backslash\mathcal{A}_*$, either $\left|\overline{\mathcal{V}}\right| \leq |\rho_{**}^{\perp}|$ which, as that of (4.4), implies that

$$\mathcal{N} = \mathcal{O}\left(\chi_{p\geq 2}|\rho_{**}^{\perp}|^2 + |\rho_{**}^{\perp}|^p\right), \qquad (5.2)$$

or $|\rho_{**}^{\perp}| \leq \left|\overline{\mathcal{V}}\right|$ and $(\theta, t) \in \mathcal{C}\backslash\widetilde{\mathcal{B}}_{**}$ which, together with (ii) and (iii) of (1) of Proposition 4.1, implies that

$$\mathcal{N} = \mathcal{O}\left(|\gamma_* + \gamma_{\mathcal{N},led}|^{2\wedge p}\right). \qquad (5.3)$$

Thus, similar to (4.5), since $\left|\overline{\mathcal{V}}\right| = o\left(\mathcal{U}^{1-\sigma}\right)$ by (1) of Proposition 4.1, we have

$$|\rho_{**}^{\perp}|^{1+\sigma} \gtrsim A_{p,1}\mathcal{U}^{p-2}\left(\overline{\mathcal{V}}^2 + 2\left|\overline{\mathcal{V}}\rho_{**}^{\perp}\right|\right) + A_{p,2}\mathcal{U}^{p-3}\left(\left|\overline{\mathcal{V}}\right|^3 + 3\overline{\mathcal{V}}^2|\rho_{**}^{\perp}|\right) \qquad (5.4)$$

if $\left|\overline{\mathcal{V}}\right| \leq |\rho_{**}^{\perp}|$ in $\mathcal{C}\backslash\mathcal{A}_*$ and

$$|\gamma_* + \gamma_{\mathcal{N},led}|^{1+\sigma} \gtrsim A_{p,1}\mathcal{U}^{p-2}\left(\overline{\mathcal{V}}^2 + 2\left|\overline{\mathcal{V}}\rho_{**}^{\perp}\right|\right) + A_{p,2}\mathcal{U}^{p-3}\left(\left|\overline{\mathcal{V}}\right|^3 + 3\overline{\mathcal{V}}^2|\rho_{**}^{\perp}|\right) \qquad (5.5)$$

if $|\rho_{**}^{\perp}| \leq \left|\overline{\mathcal{V}}\right|$ and $(\theta, t) \in \mathcal{C}\backslash\widetilde{\mathcal{B}}_{**}$ in $\mathcal{C}\backslash\mathcal{A}_*$. The conclusion then follows from (5.1), (5.2), (5.3), (5.4) and (5.5). □

By multiplying (3.16) with Ψ_j on both sides and integrating by parts and by the orthogonal conditions of ρ_* given in (3.16) and the oddness of $\{\mathcal{V}_i\}$ on $\mathbb{S}^{d-1}$, we have

$$\begin{aligned}
-\left\langle f, \Psi_j\right\rangle_{H^1} &= \left\langle \mathcal{R}_{1,j}, \Psi_j\right\rangle_{L^2} + \left\langle \mathcal{N}, \Psi_j\right\rangle_{L^2} + \sum_{i=1;i\neq j}^{\nu}\left\langle \mathcal{R}_{1,i}, \Psi_j\right\rangle_{L^2} \\
&\quad + \left\langle \mathcal{R}_{1,ex}, \Psi_j\right\rangle_{L^2} + \left\langle \mathcal{L}_{j,ex}(\rho_*), \Psi_j\right\rangle_{L^2} \qquad (5.6)
\end{aligned}$$

for all $j = 1, 2, \cdots, \nu$. In what follows, by using the equality (5.6), we shall derive the estimate of $\sum_{j=1}^{\nu}\left(\left(\alpha_j^*\right)^p - \alpha_j^*\right)$.

⟨propn0001⟩ **Proposition 5.1.** *Let $d \geq 2$, $a < 0$ and $b = b_{FS}(a)$. Then we have*

$$\left(\sum_{j=1}^{\nu}\left((\alpha_j^*)^p - \alpha_j^*\right)\right) = -\left(\overline{B}_1 + o(1)\right) Q - \left\langle f, \sum_{j=1}^{\nu} \frac{\Psi_j}{\|\Psi\|^2 + o(1)} \right\rangle - \left(\overline{A}_1 + o(1)\right) \beta_*^2 + \mathcal{O}\left(\|\rho_{**}^{\perp}\|^{1+\sigma}\right),$$

where $\overline{B}_1 > 0$ is a constant and

$$\overline{A}_1 = \lim_{\|f\|_{H^{-1}} \to 0} \frac{\sum_{j=1}^{\nu} A_{p,1}\left\langle \left(\Psi_j^*\right)^{p-2} \mathcal{V}_j^2, \Psi_j \right\rangle_{L^2}}{\beta_*^2 \|\Psi\|^2}.$$

Proof. By the the orthogonal conditions of ρ_* and the oddness of $w_{j,l}$ on $\mathbb{S}^{d-1}$ and $\partial_t \Psi_j$ in $\mathbb{R}$, we also have

$$\langle \mathcal{N}_j, \Psi_j \rangle_{L^2} = A_{p,1}\left\langle \left(\Psi_j^*\right)^{p-2} \mathcal{V}_j^2, \Psi_j \right\rangle_{L^2} + 3A_{p,2}\left\langle \left(\Psi_j^*\right)^{p-3} \mathcal{V}_j^2 \rho_*, \Psi_j \right\rangle_{L^2} \tag{5.7}$$ eqn0025

for all $j = 1, 2, \cdots, \nu$, where

$$\mathcal{N}_j = A_{p,1}\left(\Psi_j^*\right)^{p-2}\left(\mathcal{V}_j^2 + 2\mathcal{V}_j \rho_*\right) + A_{p,2}\left(\Psi_j^*\right)^{p-3}\left(\mathcal{V}_j^3 + 3\mathcal{V}_j^2 \rho_*\right). \tag{5.8}$$ eqnew9999

Intersecting (5.7) into (5.6), we have

$$\begin{aligned} -\sum_{j=1}^{\nu} \langle f, \Psi_j \rangle_{H^1} &= \sum_{j=1}^{\nu}\left((\alpha_j^*)^p - \alpha_j^*\right)\|\Psi\|^2 + \sum_{j=1}^{\nu}\sum_{i=1;i\neq j}^{\nu} \langle \mathcal{R}_{1,i}, \Psi_j \rangle_{L^2} \\ &\quad + \sum_{j=1}^{\nu} \langle \mathcal{N} - \mathcal{N}_j, \Psi_j \rangle_{L^2} + \sum_{j=1}^{\nu} \langle \mathcal{N}_j, \Psi_j \rangle_{L^2} \\ &\quad + \sum_{j=1}^{\nu} \langle \mathcal{R}_{1,ex}, \Psi_j \rangle_{L^2} + \sum_{j=1}^{\nu} \langle \mathcal{L}_{j,ex}(\rho_*), \Psi_j \rangle_{L^2}. \end{aligned} \tag{5.9}$$ eqn0028

The rest of the proof is to estimate every term in (5.9).

Step. 1 The estimate of $\sum_{j=1}^{\nu} \langle \mathcal{N}_j, \Psi_j \rangle_{L^2}$.

By Lemma 3.1,

$$\begin{aligned} \sum_{j=1}^{\nu} \langle \mathcal{N}_j, \Psi_j \rangle_{L^2} &= \sum_{j=1}^{\nu}\left(A_{p,1}\left\langle \left(\Psi_j^*\right)^{p-2} \mathcal{V}_j^2, \Psi_j \right\rangle_{L^2} + 3A_{p,2}\left\langle \left(\Psi_j^*\right)^{p-3} \mathcal{V}_j^2 \rho_*, \Psi_j \right\rangle_{L^2}\right) \\ &= (\overline{A}_{1,*} + o(1))\beta_*^2, \end{aligned}$$

where

$$\overline{A}_{1,*} = \lim_{\|f\|_{H^{-1}} \to 0} \frac{\sum_{j=1}^{\nu} A_{p,1}\left\langle \left(\Psi_j^*\right)^{p-2} \mathcal{V}_j^2, \Psi_j \right\rangle_{L^2}}{\beta_*^2} > 0.$$

Step. 2 The estimate of $\sum_{j=1}^{\nu}\sum_{i=1;i\neq j}^{\nu} \langle \mathcal{R}_{1,i}, \Psi_j \rangle_{L^2}$.

By (3.18) and Lemma 4.3,

$$\sum_{j=1}^{\nu}\sum_{i=1;i\neq j}^{\nu} \langle \mathcal{R}_{1,i}, \Psi_j \rangle_{L^2} = \sum_{j=1}^{\nu}\sum_{i=1;i\neq j}^{\nu}\left((\alpha_i^*)^{p-1} - 1\right)\langle \Psi_i^p, \Psi_j \rangle_{L^2} = o(Q).$$

Step. 3 The estimate of $\sum_{j=1}^{\nu} \langle \mathcal{R}_{1,ex}, \Psi_j \rangle_{L^2}$.

By the Taylor expansion, (3.18) and Lemma 4.3,

$$\begin{aligned}\sum_{j=1}^{\nu}\langle\mathcal{R}_{1,ex},\Psi_j\rangle_{L^2} &= \sum_{j=1}^{\nu}\sum_{i=1}^{\nu}\left\langle p\Psi_i^{p-1}\mathcal{U}_i\chi_{\mathcal{B}_i},\Psi_j\right\rangle_{L^2}\\ &\quad+\mathcal{O}\left(\sum_{j=1}^{\nu}\left\langle\Psi_j^{p-1}\chi_{\mathcal{B}_j},\mathcal{U}_j^2\right\rangle_{L^2}+\|\mathcal{U}\|_{L^{p+1}(\mathcal{C}\backslash\mathcal{B}_*)}^{p+1}\right)\\ &= (\overline{B}_{1,*}+o(1))Q,\end{aligned}$$

where $\overline{B}_{1,*}$ is a positive constant and $\mathcal{B}_i$ and $\mathcal{B}_*$ are given by (4.7).

Step. 4 The estimate of $\sum_{j=1}^{\nu}\langle\mathcal{L}_{j,ex}(\rho_*),\Psi_j\rangle_{L^2}$.

By (3.17) and (1) of Proposition 4.1,

$$\begin{aligned}\langle\mathcal{L}_{j,ex}(\rho_*),\Psi_j\rangle_{L^2} &= \left\langle p\left(\mathcal{U}^{p-1}-(\Psi_j^*)^{p-1}\right)\rho_*,\Psi_j\right\rangle_{L^2}\\ &= \left\langle p\left(\mathcal{U}^{p-1}-(\Psi_j^*)^{p-1}\right)\Psi_j,\rho_0+\rho_{**}^{\perp}\right\rangle_{L^2}.\end{aligned}$$

Similar to (4.33),

$$\left|\left(\mathcal{U}^{p-1}-(\Psi_j^*)^{p-1}\right)\Psi_j\right| \lesssim \left(\sum_{i=1}^{\nu}\Psi_i^{p-1}\mathcal{U}_i\chi_{\mathcal{B}_i}\right)+\mathcal{U}^p\chi_{\mathcal{C}\backslash\mathcal{B}_*}.$$

By Lemma 4.3 and (i) of (1) of Proposition 4.1,

$$\begin{aligned}\left|\left\langle\left(\mathcal{U}^{p-1}-(\Psi_j^*)^{p-1}\right)\Psi_j,\gamma_{ex}\right\rangle_{L^2}\right| &\lesssim \begin{cases}\sum_{i=1}^{\nu}Q\left\langle\Psi_i^{p-\sigma}\chi_{\mathcal{B}_i},\mathcal{U}_i\right\rangle_{L^2}, & p\geq 3,\\ \sum_{i=1}^{\nu}Q\left\langle\Psi_i^{2p-3}\chi_{\mathcal{B}_i},\mathcal{U}_i\right\rangle_{L^2}, & 1<p<3,\end{cases}\\ &= o(Q).\end{aligned}$$

By Lemma 4.3 and (ii) of (1) of Proposition 4.1,

$$\left|\left\langle\left(\mathcal{U}^{p-1}-(\Psi_j^*)^{p-1}\right)\Psi_j,\gamma_*\right\rangle_{L^2}\right| = o\left(\left\langle\Psi_i^{p-\sigma}\chi_{\mathcal{B}_i},\mathcal{U}_i\right\rangle_{L^2}\right)=o(Q).$$

By Lemma 4.3 and (iii) of (1) of Proposition 4.1,

$$\begin{aligned}&\left|\left\langle\left(\mathcal{U}^{p-1}-(\Psi_j^*)^{p-1}\right)\Psi_j,\gamma_{\mathcal{N},led}\right\rangle_{L^2}\right|\\ \lesssim&\ \sum_{i=1}^{\nu}\beta_*^2\left\langle\Psi_i^{p-\sigma}\chi_{\mathcal{B}_i},\mathcal{U}_i\right\rangle_{L^2}+\begin{cases}\sum_{i=1}^{\nu}\beta_*^3Q\left\langle\Psi_i^{p-\sigma}\chi_{\mathcal{B}_i},\mathcal{U}_i\right\rangle_{L^2}, & p\geq 3,\\ \sum_{i=1}^{\nu}\beta_*^3Q\left\langle\Psi_i^{2p-3}\chi_{\mathcal{B}_i},\mathcal{U}_i\right\rangle_{L^2}, & 1<p<3\end{cases}\\ =&\ o(Q).\end{aligned}\tag{5.10}$$

eqnnewnew1999

By Lemma 4.3,

$$\begin{aligned}\left|\left\langle\left(\mathcal{U}^{p-1}-\Psi_j^{p-1}\right)\Psi_j,\rho_{**}^{\perp}\right\rangle_{L^2}\right| &\lesssim \left\langle\sum_{i=1}^{\nu}\Psi_i^{p-1}\mathcal{U}_i\chi_{\mathcal{B}_i},|\rho_{**}^{\perp}|\right\rangle_{L^2}\\ &\lesssim \begin{cases}Q\|\rho_{**}^{\perp}\|, & p>2,\\ Q\left|\log Q\right|^{\frac{1}{2}}\|\rho_{**}^{\perp}\|, & p=2,\\ Q^{\frac{p}{2}}\|\rho_{**}^{\perp}\|, & 1<p<2\end{cases}\\ &= o(Q)+\mathcal{O}\left(\|\rho_{**}^{\perp}\|^{2+\sigma}\right).\end{aligned}$$

Summarizing the above estimates, we have

$$\sum_{j=1}^{\nu}\langle\mathcal{L}_{j,ex}(\rho_*),\Psi_j\rangle_{L^2}=o(Q)+\mathcal{O}\left(\|\rho_{**}^{\perp}\|^{2+\sigma}\right). \tag{5.11}$$ eqn0074

Step. 5 The estimate of $\sum_{j=1}^{\nu}\langle\mathcal{N}-\mathcal{N}_j,\Psi_j\rangle_{L^2}$.
By Lemmas 4.2 and 5.1, (5.8) and the oddness of $\{\mathcal{V}_i\}$ on $\mathbb{S}^{d-1}$, we have

$$\begin{aligned}\langle\mathcal{N}-\mathcal{N}_j,\Psi_j\rangle_{L^2} &= A_{p,1}\sum_{i=1;i\neq j}^{\nu}\left\langle(\Psi_i^*)^{p-2}\left(\mathcal{V}_i^2+2\mathcal{V}_i\rho_*\right)\chi_{\mathcal{B}_i},\Psi_j\right\rangle_{L^2}\\ &+3A_{p,2}\sum_{i=1;i\neq j}^{\nu}\left\langle(\Psi_i^*)^{p-3}\mathcal{V}_i^2\rho_*\chi_{\mathcal{B}_i},\Psi_j\right\rangle_{L^2}\\ &-A_{p,1}\left\langle\left(\Psi_j^*\right)^{p-2}\left(\mathcal{V}_j^2+2\mathcal{V}_j\rho_*\right)\chi_{\mathcal{C}\backslash\mathcal{B}_j},\Psi_j\right\rangle_{L^2}\\ &-3A_{p,2}\left\langle\left(\Psi_j^*\right)^{p-3}\mathcal{V}_j^2\rho_*\chi_{\mathcal{C}\backslash\mathcal{B}_j},\Psi_j\right\rangle_{L^2}\\ &+\sum_{i=1}^{\nu}\left\langle\mathcal{O}\left(\beta_*|\rho_*|\Psi_i^{\frac{3p-5}{2}}\mathcal{U}_i+\beta_*^2\Psi_i^{2p-2}\mathcal{U}_i\right)\chi_{\mathcal{B}_i},\Psi_j\right\rangle_{L^2}\\ &+\left\langle\mathcal{O}\left(\mathcal{U}^{p-2}\mathcal{V}^2+\beta_*\mathcal{U}^{\frac{3(p-1)}{2}}|\rho_*|\right)\chi_{\mathcal{C}\backslash\mathcal{B}_*},\Psi_j\right\rangle_{L^2}\\ &+\langle\overline{\mathcal{N}}_{rem}+\mathcal{N}_0,\Psi_j\rangle_{L^2}.\end{aligned} \tag{5.12}$$ eqnew0020

Step. 5.1 The estimate of $\sum_{i=1;i\neq j}^{\nu}\left\langle(\Psi_i^*)^{p-2}\left(\mathcal{V}_i^2+2\mathcal{V}_i\rho_*\right)\chi_{\mathcal{B}_i},\Psi_j\right\rangle_{L^2}$.
By (1) of Proposition 4.1,

$$\begin{aligned}&\sum_{i=1;i\neq j}^{\nu}\left\langle(\Psi_i^*)^{p-2}\left(\mathcal{V}_i^2+2\mathcal{V}_i\rho_*\right)\chi_{\mathcal{B}_i},\Psi_j\right\rangle_{L^2}\\ =&\sum_{i=1;i\neq j}^{\nu}\left\langle(\Psi_i^*)^{p-2}\left(\mathcal{V}_i^2+2\mathcal{V}_i\gamma_{2,ex}\right)\chi_{\mathcal{B}_i},\Psi_j\right\rangle_{L^2}\\ &+\sum_{i=1;i\neq j}^{\nu}\left\langle(\Psi_i^*)^{p-2}\mathcal{V}_i\left(\gamma_{\mathcal{N},led}+\rho_{**}^{\perp}\right)\chi_{\mathcal{B}_i},\Psi_j\right\rangle_{L^2}.\end{aligned}$$

By Lemma 4.3 and (i) of (1) of Proposition 4.1,

$$\begin{aligned}&\sum_{i=1;i\neq j}^{\nu}\left|\left\langle(\Psi_i^*)^{p-2}\left(\mathcal{V}_i^2+2\mathcal{V}_i\gamma_{2,ex}\right)\chi_{\mathcal{B}_i},\Psi_j\right\rangle_{L^2}\right|\\ \lesssim&\begin{cases}\sum_{i=1;i\neq j}^{\nu}\left\langle\left(\beta_*^2\Psi_i^{2p-1}+\beta_*^2Q\Psi_i^{\frac{3p-1-2\sigma}{2}}\right)\chi_{\mathcal{B}_i},\Psi_j\right\rangle_{L^2}, & p\geq\frac{7}{3},\\ \sum_{i=1;i\neq j}^{\nu}\left\langle\left(\beta_*^2\Psi_i^{2p-1}+\beta_*^2Q\Psi_i^{3p-4}\right)\chi_{\mathcal{B}_i},\Psi_j\right\rangle_{L^2}, & 1<p<\frac{7}{3}\end{cases}\\ =&\ o(Q).\end{aligned}$$

By Lemma 4.3 and (iii) of (1) of Proposition 4.1,

$$\begin{aligned}&\sum_{i=1;i\neq j}^{\nu}\left|\left\langle(\Psi_i^*)^{p-2}\mathcal{V}_i\gamma_{\mathcal{N},led}\chi_{\mathcal{B}_i},\Psi_j\right\rangle_{L^2}\right|\\ \lesssim&\begin{cases}\sum_{i=1;i\neq j}^{\nu}\left\langle\beta_*^3\Psi_i^{\frac{3p-1-2\sigma}{2}}\chi_{\mathcal{B}_i},\Psi_j\right\rangle_{L^2}, & p\geq 3,\\ \sum_{i=1;i\neq j}^{\nu}\left\langle\left(\beta_*^3\Psi_i^{\frac{3p-1-2\sigma}{2}}+\beta_*^3Q\Psi_i^{\frac{5p-7}{2}}\right)\chi_{\mathcal{B}_i},\Psi_j\right\rangle_{L^2}, & 1<p<3\end{cases}\\ =&\ o(Q).\end{aligned}$$

By Lemma 4.3,

$$\begin{aligned}\sum_{i=1;i\neq j}^{\nu}\left|\left\langle(\Psi_i^*)^{p-2}\mathcal{V}_i\rho_{**}^{\perp}\chi_{\mathcal{B}_i},\Psi_j\right\rangle_{L^2}\right| &\lesssim \sum_{i=1;i\neq j}^{\nu}\beta_*\|\rho_{**}^{\perp}\|\left\|\Psi_i^{\frac{3p-3}{2}}\Psi_j\right\|_{L^2(\mathcal{B}_i)}\\ &= o(Q)+\mathcal{O}\left(\|\rho_{**}^{\perp}\|^{2+\sigma}\right).\end{aligned}$$

Summarizing the above estimates, we have

$$\sum_{j=1}^{\nu}\sum_{i=1;i\neq j}^{\nu}\left|\left\langle(\Psi_i^*)^{p-2}\left(\mathcal{V}_i^2+2\mathcal{V}_i\rho_*\right)\chi_{\mathcal{B}_i},\Psi_j\right\rangle_{L^2}\right|=o(Q)+\mathcal{O}\left(\|\rho_{**}^{\perp}\|^{2+\sigma}\right).$$

Step. 5.2 The estimate of $\left\langle\left(\Psi_j^*\right)^{p-2}\left(\mathcal{V}_j^2+2\mathcal{V}_j\rho_*\right)\chi_{\mathcal{C}\setminus\mathcal{B}_j},\Psi_j\right\rangle_{L^2}$.

By Lemma 4.3 and (1) of Proposition 4.1,

$$\begin{aligned}\left|\left\langle\left(\Psi_j^*\right)^{p-2}\left(\mathcal{V}_j^2+2\mathcal{V}_j\rho_*\right)\chi_{\mathcal{C}\setminus\mathcal{B}_j},\Psi_j\right\rangle_{L^2}\right| &\lesssim \beta_*^2Q^p+\beta_*Q^{\frac{3p-1}{4}}\|\rho_{**}^{\perp}\|\\ &\quad+\beta_*\left\langle\Psi_j^{\frac{3p-1}{2}}\chi_{\mathcal{C}\setminus\mathcal{B}_j},|\gamma_{2,ex}+\gamma_{\mathcal{N},led}|\right\rangle_{L^2}.\end{aligned}$$

By Lemma 4.3 and (i) of (1) of Proposition 4.1,

$$\begin{aligned}\left\langle\Psi_j^{\frac{3p-1}{2}}\chi_{\mathcal{C}\setminus\mathcal{B}_j},|\gamma_{2,ex}|\right\rangle_{L^2} &\lesssim \begin{cases}\sum_{i=1;i\neq j}^{\nu}\beta_*Q\left\langle\Psi_i^{1-\sigma}\chi_{\mathcal{B}_i},\Psi_j^{\frac{3p-1}{2}}\right\rangle_{L^2}, & p\geq\frac{7}{3},\\ \sum_{i=1;i\neq j}^{\nu}\beta_*Q\left\langle\Psi_i^{\frac{3p-5}{2}}\chi_{\mathcal{B}_i},\Psi_j^{\frac{3p-1}{2}}\right\rangle_{L^2}, & 1<p<\frac{7}{3}\end{cases}\\ &= o(Q).\end{aligned}$$

By Lemma 4.3 and (iii) of (1) of Proposition 4.1,

$$\begin{aligned}&\left\langle \Psi_j^{\frac{3p-1}{2}}\chi_{\mathcal{C}\backslash\mathcal{B}_j}, |\gamma_{\mathcal{N},led}|\right\rangle_{L^2}\\ \lesssim&\begin{cases}\sum_{i=1;i\neq j}^{\nu}\beta_*^2\left\langle \Psi_i^{1-\sigma}\chi_{\mathcal{B}_i}, \Psi_j^{\frac{3p-1}{2}}\right\rangle_{L^2}, & p\geq 3,\\ \sum_{i=1;i\neq j}^{\nu}\left(\beta_*^2\left\langle \Psi_i^{1-\sigma}\chi_{\mathcal{B}_i}, \Psi_j^{\frac{3p-1}{2}}\right\rangle_{L^2}+\beta_*^3Q\left\langle \Psi_i^{p-2}\chi_{\mathcal{B}_i}, \Psi_j^{\frac{3p-1}{2}}\right\rangle_{L^2}\right), & 1<p<3\end{cases}\\ =&\ o(Q).\end{aligned}$$

Summarizing the above estimates, we have

$$\left\langle \left(\Psi_j^*\right)^{p-2}\left(\mathcal{V}_j^2+2\mathcal{V}_j\rho_*\right)\chi_{\mathcal{C}\backslash\mathcal{B}_j}, \Psi_j\right\rangle_{L^2}=o(Q)+\mathcal{O}\left(\|\rho_{**}^{\perp}\|^{2+\sigma}\right).$$

Step. 5.3 The estimate of $\sum_{i=1;i\neq j}^{\nu}\left\langle (\Psi_i^*)^{p-3}\mathcal{V}_i^2\rho_*\chi_{\mathcal{B}_i}, \Psi_j\right\rangle_{L^2}$.

By (i) of (1) of Proposition 4.1,

$$\begin{aligned}\sum_{i=1;i\neq j}^{\nu}\left\langle (\Psi_i^*)^{p-3}\mathcal{V}_i^2\rho_*\chi_{\mathcal{B}_i}, \Psi_j\right\rangle_{L^2} =& \sum_{i=1;i\neq j}^{\nu}\left\langle (\Psi_i^*)^{p-3}\mathcal{V}_i^2\left(\gamma_{\mathcal{N},led}+\gamma_*\right)\chi_{\mathcal{B}_i}, \Psi_j\right\rangle_{L^2}\\ &+\sum_{i=1;i\neq j}^{\nu}\left\langle (\Psi_i^*)^{p-3}\mathcal{V}_i^2\left(\gamma_{1,ex}+\rho_{**}^{\perp}\right)\chi_{\mathcal{B}_i}, \Psi_j\right\rangle_{L^2}.\end{aligned}$$

By Lemma 4.3 and (i) of (1) of Proposition 4.1,

$$\begin{aligned}\sum_{i=1;i\neq j}^{\nu}\left|\left\langle (\Psi_i^*)^{p-3}\mathcal{V}_i^2\gamma_{1,ex}\chi_{\mathcal{B}_i}, \Psi_j\right\rangle_{L^2}\right| \lesssim& \begin{cases}\beta_*^2Q\left\langle \Psi_i^{2p-1-\sigma}\chi_{\mathcal{B}_i}, \Psi_j\right\rangle_{L^2}, & p\geq 3,\\ \beta_*^2Q\left\langle \Psi_i^{3p-4}\chi_{\mathcal{B}_i}, \Psi_j\right\rangle_{L^2}, & 1<p<3\end{cases}\\ =&\ o(Q).\end{aligned}$$

By Lemma 4.3 and (ii) of (1) of Proposition 4.1,

$$\sum_{i=1;i\neq j}^{\nu}\left|\left\langle (\Psi_i^*)^{p-3}\mathcal{V}_i^2\gamma_*\chi_{\mathcal{B}_i}, \Psi_j\right\rangle_{L^2}\right|=\sum_{i=1;i\neq j}^{\nu}o(\beta_*^2)\left\langle \Psi_i^{2p-1-\sigma}\chi_{\mathcal{B}_i}, \Psi_j\right\rangle_{L^2}=o(Q).$$

Similar to (5.10), by Lemma 4.3 and (iii) of (1) of Proposition 4.1,

$$\sum_{i=1;i\neq j}^{\nu}\left|\left\langle (\Psi_i^*)^{p-3}\mathcal{V}_i^2\gamma_{\mathcal{N},led}\chi_{\mathcal{B}_i}, \Psi_j\right\rangle_{L^2}\right|=o(Q).$$

By Lemma 4.3,

$$\begin{aligned}\sum_{i=1;i\neq j}^{\nu}\left|\left\langle (\Psi_i^*)^{p-3}\mathcal{V}_i^2\rho_{**}^{\perp}\chi_{\mathcal{B}_i}, \Psi_j\right\rangle_{L^2}\right| \lesssim& \beta_*^2\|\rho_{**}^{\perp}\|\left\|\Psi_i^{2p-2}\Psi_j\right\|_{L^2(\mathcal{B}_i)}\\ =&\ o(Q)+\mathcal{O}\left(\|\rho_{**}^{\perp}\|^{2+\sigma}\right).\end{aligned}$$

Summarizing the above estimates, we have

$$\sum_{i=1;i\neq j}^{\nu}\left\langle (\Psi_i^*)^{p-3}\mathcal{V}_i^2\rho_*\chi_{\mathcal{B}_i}, \Psi_j\right\rangle_{L^2}=o(Q)+\mathcal{O}\left(\|\rho_{**}^{\perp}\|^{2+\sigma}\right).$$

Step. 5.4 The estimate of $\left\langle \left(\Psi_j^*\right)^{p-3}\mathcal{V}_j^2\rho_*\chi_{\mathcal{C}\backslash\mathcal{B}_j}, \Psi_j\right\rangle_{L^2}$.

By (i) of (1) of Proposition 4.1,

$$\begin{aligned}\left\langle \left(\Psi_j^*\right)^{p-3}\mathcal{V}_j^2\rho_*\chi_{\mathcal{C}\setminus\mathcal{B}_j}, \Psi_j\right\rangle_{L^2} &= \left\langle \left(\Psi_j^*\right)^{p-3}\mathcal{V}_j^2\left(\gamma_{\mathcal{N},led}+\gamma_*\right)\chi_{\mathcal{C}\setminus\mathcal{B}_j}, \Psi_j\right\rangle_{L^2}\\&+\left\langle \left(\Psi_j^*\right)^{p-3}\mathcal{V}_j^2\left(\gamma_{1,ex}+\rho_{**}^{\perp}\right)\chi_{\mathcal{C}\setminus\mathcal{B}_j}, \Psi_j\right\rangle_{L^2}.\end{aligned}$$

By Lemma 4.3 and (i) of (1) of Proposition 4.1,

$$\begin{aligned}&\left|\left\langle \left(\Psi_j^*\right)^{p-3}\mathcal{V}_j^2\gamma_{1,ex}\chi_{\mathcal{C}\setminus\mathcal{B}_j}, \Psi_j\right\rangle_{L^2}\right|\\ \lesssim&\begin{cases}\sum_{i=1;i\neq j}^{\nu}\beta_*^2Q\left\langle\Psi_i^{1-\sigma}\chi_{\mathcal{B}_i},\Psi_j^{2p-1}\right\rangle_{L^2}, & p\geq 3,\\ \sum_{i=1;i\neq j}^{\nu}\beta_*^2Q\left\langle\Psi_i^{p-2}\chi_{\mathcal{B}_i},\Psi_j^{2p-1}\right\rangle_{L^2}, & 1<p<3\end{cases}\\ =&\ o(Q).\end{aligned}$$

By Lemma 4.3 and (ii) of (1) of Proposition 4.1,

$$\left|\left\langle \left(\Psi_j^*\right)^{p-3}\mathcal{V}_j^2\gamma_*\chi_{\mathcal{C}\setminus\mathcal{B}_j}, \Psi_j\right\rangle_{L^2}\right| = \sum_{i=1;i\neq j}^{\nu}o(\beta_*^2)\left\langle\Psi_i^{1-\sigma}\chi_{\mathcal{B}_i},\Psi_j^{2p-1}\right\rangle_{L^2}=o(Q).$$

Similar to (5.10), by Lemma 4.3 and (iii) of (1) of Proposition 4.1,

$$\left|\left\langle \left(\Psi_j^*\right)^{p-3}\mathcal{V}_j^2\gamma_{\mathcal{N},led}\chi_{\mathcal{C}\setminus\mathcal{B}_j}, \Psi_j\right\rangle_{L^2}\right| = o(Q).$$

By Lemma 4.3,

$$\left|\left\langle \left(\Psi_j^*\right)^{p-3}\mathcal{V}_j^2\rho_{**}^{\perp}\chi_{\mathcal{C}\setminus\mathcal{B}_j}, \Psi_j\right\rangle_{L^2}\right| = o(Q)+\mathcal{O}\left(\|\rho_{**}^{\perp}\|^{2+\sigma}\right).$$

Summarizing the above estimates, we have

$$\left|\left\langle \left(\Psi_j^*\right)^{p-3}\mathcal{V}_j^2\rho_*\chi_{\mathcal{C}\setminus\mathcal{B}_j}, \Psi_j\right\rangle_{L^2}\right| = o(Q)+\mathcal{O}\left(\|\rho_{**}^{\perp}\|^{2+\sigma}\right).$$

Step. 5.5 The estimate of $\sum_{i=1}^{\nu}\left\langle\beta_*\rho_*\Psi_i^{\frac{3p-5}{2}}\mathcal{U}_i\chi_{\mathcal{B}_i},\Psi_j\right\rangle_{L^2}$.

By (1) of Proposition 4.1,

$$\begin{aligned}\sum_{i=1}^{\nu}\left\langle\beta_*\rho_*\Psi_i^{\frac{3p-5}{2}}\mathcal{U}_i\chi_{\mathcal{B}_i},\Psi_j\right\rangle_{L^2} &= \sum_{i=1}^{\nu}\beta_*\left\langle\Psi_i^{\frac{3p-3}{2}}\mathcal{U}_i\chi_{\mathcal{B}_i},\gamma_{\mathcal{N},led}+\gamma_*\right\rangle_{L^2}\\&+\sum_{i=1}^{\nu}\beta_*\left\langle\Psi_i^{\frac{3p-3}{2}}\mathcal{U}_i\chi_{\mathcal{B}_i},\gamma_{ex}+\rho_{**}^{\perp}\right\rangle_{L^2}.\end{aligned}$$

By Lemma 4.3 and (i) of (1) of Proposition 4.1,

$$\begin{aligned}\sum_{i=1}^{\nu}\beta_*\left|\left\langle\Psi_i^{\frac{3p-3}{2}}\mathcal{U}_i\chi_{\mathcal{B}_i},\gamma_{ex}\right\rangle_{L^2}\right| &\lesssim \begin{cases}\sum_{i=1}^{\nu}\beta_*Q\left\langle\Psi_i^{\frac{3p-1-2\sigma}{2}}\chi_{\mathcal{B}_i},\mathcal{U}_i\right\rangle_{L^2}, & p\geq 3,\\ \sum_{i=1}^{\nu}\beta_*Q\left\langle\Psi_i^{\frac{5p-7}{2}}\chi_{\mathcal{B}_i},\mathcal{U}_i\right\rangle_{L^2}, & 1<p<3\end{cases}\\ &= o(Q).\end{aligned}$$

By Lemma 4.3 and (ii) of (1) of Proposition 4.1,

$$\sum_{i=1}^{\nu}\beta_*\left|\left\langle \Psi_i^{\frac{3p-3}{2}}\mathcal{U}_i\chi_{\mathcal{B}_i},\gamma_*\right\rangle_{L^2}\right|=\sum_{i=1}^{\nu}o(\beta_*)\left\langle \Psi_i^{\frac{3p-1-2\sigma}{2}},\mathcal{U}_i\chi_{\mathcal{B}_i}\right\rangle_{L^2}=o(Q).$$

Similar to (**??**), by Lemma 4.3 and (iii) of (1) of Proposition 4.1,

$$\sum_{i=1}^{\nu}\beta_*\left|\left\langle \Psi_i^{\frac{3p-3}{2}}\mathcal{U}_i\chi_{\mathcal{B}_i},\gamma_{\mathcal{N},led}\right\rangle_{L^2}\right|=o(Q).$$

By Lemma 4.3,

$$\begin{aligned}\sum_{i=1}^{\nu}\beta_*\left|\left\langle \Psi_i^{\frac{3p-3}{2}}\mathcal{U}_i\chi_{\mathcal{B}_i},\rho_{**}^{\perp}\right\rangle_{L^2}\right| &\lesssim \sum_{i=1}^{\nu}\|\rho_{**}^{\perp}\|\left\|\Psi_i^{\frac{3p-3}{2}}\mathcal{U}_i\right\|_{L^2(\mathcal{B}_i)}\\ &= o(Q)+\mathcal{O}\left(\|\rho_{**}^{\perp}\|^{2+\sigma}\right).\end{aligned}$$

Summarizing the above estimates, we have

$$\sum_{i=1}^{\nu}\left|\left\langle \beta_*\rho_*\Psi_i^{\frac{3p-5}{2}}\mathcal{U}_i\chi_{\mathcal{B}_i},\Psi_j\right\rangle_{L^2}\right|=o(Q)+\mathcal{O}\left(\|\rho_{**}^{\perp}\|^{2+\sigma}\right).$$

Step. 5.6 The estimate of $\sum_{i=1}^{\nu}\beta_*^2\left\langle \Psi_i^{2p-2}\mathcal{U}_i\chi_{\mathcal{B}_i},\Psi_j\right\rangle_{L^2}$.

By Lemma 4.3,

$$\sum_{i=1}^{\nu}\beta_*^2\left|\left\langle \Psi_i^{2p-2}\mathcal{U}_i\chi_{\mathcal{B}_i},\Psi_j\right\rangle_{L^2}\right|\lesssim\sum_{i=1}^{\nu}\beta_*^2\left\langle \Psi_i^{2p-1}\chi_{\mathcal{B}_i},\mathcal{U}_i\right\rangle_{L^2}=o(Q).$$

Step. 5.7 The estimate of $\left\langle\left(\mathcal{U}^{p-2}\mathcal{V}^2+\beta_*\rho_*\mathcal{U}^{\frac{3(p-1)}{2}}\right)\chi_{\mathcal{C}\backslash\mathcal{B}_*},\Psi_j\right\rangle_{L^2}$.

By (1) of Proposition 4.1,

$$\begin{aligned}&\left|\left\langle\left(\mathcal{U}^{p-2}\mathcal{V}^2+\beta_*\rho_*\mathcal{U}^{\frac{3(p-1)}{2}}\right)\chi_{\mathcal{C}\backslash\mathcal{B}_*},\Psi_j\right\rangle_{L^2}\right|\\ \lesssim&\left\langle\left(\mathcal{U}^{p-2}\mathcal{V}^2+\beta_*|\gamma_{ex}+\gamma_*+\gamma_{\mathcal{N},led}|\mathcal{U}^{\frac{3(p-1)}{2}}\right)\chi_{\mathcal{C}\backslash\mathcal{B}_*},\Psi_j\right\rangle_{L^2}\\ &+\left\langle\beta_*|\rho_{**}^{\perp}|\mathcal{U}^{\frac{3(p-1)}{2}}\chi_{\mathcal{C}\backslash\mathcal{B}_*},\Psi_j\right\rangle_{L^2}.\end{aligned}$$

By Lemma 4.3 and (1) of Proposition 4.1,

$$\begin{aligned}&\left\langle\left(\mathcal{U}^{p-2}\mathcal{V}^2+\beta_*|\gamma_{ex}+\gamma_*+\gamma_{\mathcal{N},led}|\mathcal{U}^{\frac{3(p-1)}{2}}\right)\chi_{\mathcal{C}\backslash\mathcal{B}_*},\Psi_j\right\rangle_{L^2}\\ \lesssim&\ o(\beta_*)\left\|\mathcal{U}^{\frac{3p+1-2\sigma}{2}}\right\|_{L^1(\mathcal{C}\backslash\mathcal{B}_*)}+\beta_*^2\left\|\mathcal{U}^{2p}\right\|_{L^1(\mathcal{C}\backslash\mathcal{B}_*)}\\ =&\ o(Q).\end{aligned}$$

By Lemma 4.3,

$$\begin{aligned}\left\langle\beta_*|\rho_{**}^{\perp}|\mathcal{U}^{\frac{3(p-1)}{2}}\chi_{\mathcal{C}\backslash\mathcal{B}_*},\Psi_j\right\rangle_{L^2} &\lesssim \beta_*\|\rho_{**}^{\perp}\|\left\|\mathcal{U}^{\frac{3p-1}{2}}\right\|_{L^2(\mathcal{C}\backslash\mathcal{B}_*)}\\ &= o(Q)+\mathcal{O}\left(\|\rho_{**}^{\perp}\|^{2+\sigma}\right).\end{aligned}$$

Summarizing the above estimates, we have

$$\left\langle\left(\mathcal{U}^{p-2}\mathcal{V}^2+\beta_*|\rho_*|\mathcal{U}^{\frac{3(p-1)}{2}}\right)\chi_{\mathcal{C}\backslash\mathcal{B}_*},\Psi_j\right\rangle_{L^2}=o(Q)+\mathcal{O}\left(\|\rho_{**}^{\perp}\|^{2+\sigma}\right).$$

Step. 5.8 The estimate of $\langle\mathcal{N}_0,\Psi_j\rangle_{L^2}$.

By (4.2), (1) of Proposition 4.1 and Lemma 5.1,

$$\left|\langle\mathcal{N}_0,\Psi_j\rangle_{L^2}\right|\lesssim\left\langle\mathcal{U}^{p-2-\sigma}\Psi_j,\gamma_{ex}^2+|\gamma_{\mathcal{N},led}+\gamma_*|^2\right\rangle_{L^2}+\|\rho_{**}^{\perp}\|^2.$$

By Lemma 4.3 and (i) of (1) of Proposition 4.1,

$$\begin{aligned}&\langle\mathcal{U}^{p-2-\sigma}\Psi_j,\gamma_{ex}^2\rangle_{L^2}\\ \lesssim&\begin{cases}\sum_{i=1}^{\nu}Q^2\left\langle\Psi_i^{p-3\sigma},\Psi_j\right\rangle_{L^2(\mathcal{B}_i)}, & p\geq 3,\\ \sum_{i=1}^{\nu}Q_i^2\left\langle\Psi_i^{3(p-2)-\sigma},\Psi_j\right\rangle_{L^2(\mathcal{B}_i)}+\mathcal{O}\left(Q^2\right), & 1<p<3\end{cases}\\ =&\ o(Q).\end{aligned}\tag{5.13}$$ eqnew9998

By Lemma 4.3 and (ii) of (1) of Proposition 4.1,

$$\left\langle\mathcal{U}^{p-2}\Psi_j,|\gamma_*|^2\right\rangle_{L^2}\lesssim\left(Q^{1+\sigma}+\beta_*^2+\sum_{l=1}^{\nu}\left|(\alpha_l^*)^{p-1}-1\right|\right)^2.$$

Similar to (5.10), by Lemma 4.3 and (iii) of (1) of Proposition 4.1,

$$\langle\mathcal{U}^{p-2}\Psi_j,\gamma_{\mathcal{N},led}^2\rangle_{L^2}=o(Q)+\mathcal{O}\left(\beta_*^2+\sum_{l=1}^{\nu}\left|(\alpha_l^*)^{p-1}-1\right|\right)^2.$$

Thus, summarizing the above estimates, we have

$$\left|\langle\mathcal{N}_0,\Psi_j\rangle_{L^2}\right|\lesssim\|\rho_{**}^{\perp}\|^2+o(Q)+\left(\beta_*^2+\sum_{l=1}^{\nu}\left|(\alpha_l^*)^{p-1}-1\right|\right)^2.$$

Step. 5.9 The estimate of $\langle\overline{\mathcal{N}}_{rem},\Psi_j\rangle_{L^2}$.
By Lemma 5.1,

$$\begin{aligned}\left|\langle\overline{\mathcal{N}}_{rem},\Psi_j\rangle_{L^2}\right|\ \lesssim\ &\|\rho_{**}^{\perp}\|^{1+\sigma}+\left(\beta_*+\sum_{l=1}^{\nu}\left|(\alpha_l^*)^{p-1}-1\right|\right)^4+\langle\mathcal{U}^{p-4}\gamma_{ex}^4,\Psi_j\rangle_{L^2}\\&+\left\langle|\gamma_*+\gamma_{\mathcal{N},led}|^{1+\sigma},\Psi_j\right\rangle_{L^2(\mathcal{C}\backslash\widetilde{\mathcal{B}}_{**})}+o(Q).\end{aligned}$$

By Lemma 4.3 and (i) of (1) of Proposition 4.1,

$$\begin{aligned}\langle\mathcal{U}^{p-4}\gamma_{ex}^4,\Psi_j\rangle_{L^2}\ \lesssim&\ \begin{cases}\sum_{i=1}^{\nu}Q^4\left\langle\Psi_i^{p-4\sigma},\Psi_j\right\rangle_{L^2(\mathcal{B}_i)}, & p\geq 3,\\ \sum_{i=1}^{\nu}Q^4\left\langle\Psi_i^{5p-12},\Psi_j\right\rangle_{L^2(\mathcal{B}_i)}, & 1<p<3\end{cases}\\ =&\ o(Q).\end{aligned}\tag{5.14}$$ eqnew9998

By the definition of $\widetilde{\mathcal{B}}_{**}$ given in Lemma 5.1, we have

$$\mathcal{C}\backslash\widetilde{\mathcal{B}}_{**}\subset\left(\cup_{j=1}^{\nu}\left(\mathcal{B}_j\cap\mathcal{B}_{j,*}\right)\right)\cup\left(\mathcal{B}_{1,-,*}\cap\mathcal{B}_{1,*}\right)\cup\left(\mathcal{B}_{\nu,+,*}\cap\mathcal{B}_{\nu,*}\right),$$

where $\mathcal{B}_{1,-,*}$ and $\mathcal{B}_{\nu,+,*}$ are given by (4.16) and (4.17), respectively, and

$$\mathcal{B}_{j,*} = \left\{ (\theta, t) \in \mathcal{C} \mid \left|t - s_j^*\right| \gtrsim \frac{\left|\ln\left(Q + \beta_*^2 + \sum_{l=1}^{\nu}\left|(\alpha_l^*)^{p-1} - 1\right|\right)\right|}{\sigma} \right\}.$$

Thus, by (ii) and (iii) of Proposition 4.1,

$$\begin{aligned}\left\langle \left|\gamma_* + \gamma_{\mathcal{N},led}\right|^{1+\sigma}, \Psi_j \right\rangle_{L^2(\mathcal{C}\backslash\widetilde{\mathcal{B}}_{**})} &\lesssim \left(Q + \beta_*^2 + \sum_{l=1}^{\nu}\left|(\alpha_l^*)^{p-1} - 1\right|\right)^{1+\frac{(2-\sigma)\Lambda_{FS}}{\sigma}} \\ &\lesssim \left(Q + \beta_*^2 + \sum_{l=1}^{\nu}\left|(\alpha_l^*)^{p-1} - 1\right|\right)^{6}. \qquad (5.15)\end{aligned}$$

eqnnewnew0005

By summarizing the above estimates, we have

$$\left|\langle \overline{\mathcal{N}}_{rem}, \Psi_j\rangle_{L^2}\right| \lesssim \|\rho_{**}^{\perp}\|^{1+\sigma} + o(Q) + \left(\beta_* + \sum_{l=1}^{\nu}\left|(\alpha_l^*)^{p-1} - 1\right|\right)^4.$$

By summarizing the estimates from Step. 5.1 to Step. 5.9, we have

$$\left|\langle \mathcal{N} - \mathcal{N}_j, \Psi_j\rangle_{L^2}\right| \lesssim \|\rho_{**}^{\perp}\|^{1+\sigma} + o(Q) + \left(\beta_*^2 + \sum_{l=1}^{\nu}\left|(\alpha_l^*)^{p-1} - 1\right|\right)^2.$$

The conclusion then follows from the estimates in Step. 1 to Step.5. □

6. Second refined expansion of $\mathcal{N}$ and estimates of Q

Again, we emphasize that we need to eliminate the lower order terms to get the desired stability. Thus, we need to further refine the expansion of $\overline{\mathcal{N}}_{rem}$, which is the remaining term in the expansion of $\mathcal{N}$ given by Lemma 5.1.

⟨lemn0002⟩ **Lemma 6.1.** *Let $d \geq 2$, $a < 0$ and $b = b_{FS}(a)$. Then $\overline{\mathcal{N}}_{rem}$, the remaining term in the expansion of $\mathcal{N}$ given by Lemma 5.1, can be further expanded as follows:*

$$\begin{aligned}\overline{\mathcal{N}}_{rem} &= A_{p,3}\mathcal{U}^{p-4}\left(\overline{\mathcal{V}}^4 + 4\overline{\mathcal{V}}^3\rho_{**}^{\perp}\right) + A_{p,4}\mathcal{U}^{p-5}\left(\overline{\mathcal{V}}^5 + 5\overline{\mathcal{V}}^4\rho_{**}^{\perp}\right) \\ &\quad +\mathcal{O}\left(\left(\beta_* + Q + \sum_{j=1}^{\nu}\left|(\alpha_j^*)^{p-1} - 1\right|\right)^6 + \mathcal{U}^{p-6}\gamma_{ex}^6\right) \\ &\quad +\mathcal{O}\left(\chi_{p\geq 2}|\rho_{**}^{\perp}|^2 + |\rho_{**}^{\perp}|^p + |\rho_{**}^{\perp}|^{1+\sigma} + |\gamma_* + \gamma_{\mathcal{N},led}|^{1+\sigma}\chi_{\mathcal{C}\backslash\widetilde{\mathcal{B}}_{**}}\right)\end{aligned}$$

in $\mathcal{C}$ where $\overline{\mathcal{V}} = \mathcal{V} + \rho_0$ with $\mathcal{V}$ given by (3.15) and ρ_0 given in Proposition 4.1, $\widetilde{\mathcal{B}}_{**}$ is given in Lemma 5.1 and γ_* and $\gamma_{\mathcal{N},led}$ are given in (1) of Proposition 4.1.

Proof. The proof is a direct application of the Taylor expansion to $\mathcal{N}$ in the sets $\mathcal{A}_*$ and $\mathcal{C}\backslash\mathcal{A}_*$, which is introduced in the proof of Lemma 5.1, up to the sixth order terms as in the proof of Lemma 5.1. □

By multiplying (3.16) with $\partial_t\Psi_j$ on both sides and integrating by parts, the orthogonal conditions of ρ_* given in (3.16) and the oddness of $\{\mathcal{V}_i\}$ on $\mathbb{S}^{d-1}$ and

$\partial_t\Psi_j$ in $\mathbb{R}$, we have

$$-\left\langle f, \sum_{j=1}^{\nu}\partial_t\Psi_j\right\rangle_{H^1} = \sum_{j=1}^{\nu}\sum_{i=1;i\neq j}^{\nu}\langle\mathcal{R}_{1,i},\partial_t\Psi_j\rangle_{L^2}+\sum_{j=1}^{\nu}\langle\mathcal{L}_{j,ex}(\rho_*),\partial_t\Psi_j\rangle_{L^2} + \sum_{j=1}^{\nu}\langle\mathcal{N},\partial_t\Psi_j\rangle_{L^2}+\sum_{j=1}^{\nu}\langle\mathcal{R}_{1,ex},\partial_t\Psi_j\rangle_{L^2}. \qquad (6.1)$$ eqn0022

In what follows, we shall derive the estimate of Q from (6.1).

⟨propn0002⟩ **Proposition 6.1.** *Let $d\geq 2$, $a<0$ and $b=b_{FS}(a)$. Then we have*

$$Q=\mathcal{O}\left(\beta_*^6+\|\rho_{**}^{\perp}\|^{1+\sigma}+\|f\|_{H^{-1}}\right).$$

Proof. By the oddness of $w_{j,l}$ on $\mathbb{S}^{d-1}$ and $\partial_t\Psi_j$ in $\mathbb{R}$, we also have

$$\langle\mathcal{N}_j,\partial_t\Psi_j\rangle_{L^2}=2A_{p,1}\left\langle\left(\Psi_j^*\right)^{p-2}\mathcal{V}_j\rho_*,\partial_t\Psi_j\right\rangle_{L^2}+3A_{p,2}\left\langle\left(\Psi_j^*\right)^{p-3}\mathcal{V}_j^2\rho_*,\partial_t\Psi_j\right\rangle_{L^2} \qquad (6.2)$$ eqn0026

for all $j=1,2,\cdots,\nu$, where $\mathcal{N}_j$ is given by (5.8). Intersecting (6.2) into (6.1), we have

$$-\sum_{j=1}^{\nu}\langle f,\partial_t\Psi_j\rangle_{H^1} = \sum_{j=1}^{\nu}\langle\mathcal{R}_{1,ex},\partial_t\Psi_j\rangle_{L^2}+\sum_{j=1}^{\nu}\langle\mathcal{N}_j,\partial_t\Psi_j\rangle_{L^2} + \sum_{j=1}^{\nu}\langle\mathcal{N}-\mathcal{N}_j,\partial_t\Psi_j\rangle_{L^2}+\sum_{j=1}^{\nu}\langle\mathcal{L}_{j,ex}(\rho_*),\partial_t\Psi_j\rangle_{L^2} + \sum_{j=1}^{\nu}\sum_{i=1;i\neq j}^{\nu}\langle\mathcal{R}_{1,i},\partial_t\Psi_j\rangle_{L^2}. \qquad (6.3)$$ eqn2022

As in the proof of Proposition 4.1, the rest of the proof is to estimate every term in (6.3).

Step. 1 The estimate of $\sum_{j=1}^{\nu}\langle\mathcal{R}_{1,ex},\partial_t\Psi_j\rangle_{L^2}$.

By (2.7), (3.18), Lemma 4.3 and the Taylor expansion,

$$\begin{aligned}\sum_{j=1}^{\nu}\langle\mathcal{R}_{1,ex},\partial_t\Psi_j\rangle_{L^2} &= \sum_{j=1}^{\nu}\sum_{i=1}^{\nu}\int_{\mathcal{B}_i}\left(\mathcal{U}^p-\sum_{l=1}^{\nu}(\Psi_l^*)^p\right)\partial_t\Psi_j d\mu+o(Q)\\ &= \sum_{j=1}^{\nu}p\int_{\mathcal{B}_j}\left(\Psi_j^*\right)^{p-1}\left(\Psi_{j+1}^*+\Psi_{j-1}^*\right)\partial_t\Psi_j d\mu\\ &\quad+\mathcal{O}\left(\sum_{i=1}^{\nu}\int_{\mathcal{B}_i}\Psi_i^{p-1}\mathcal{U}_i^2d\mu\right)+o(Q)\\ &= p\int_{\mathcal{B}_j}\left(\Psi_j^*\right)^{p-1}\left(\Psi_{j+1}^*+\Psi_{j-1}^*\right)\partial_t\Psi_j d\mu+o(Q),\end{aligned}$$

where $\mathcal{B}_i$ is given by (4.7) and by (2.7) and Lemma 4.3 again,

$$\begin{aligned}\sum_{j=1}^{\nu} p\int_{\mathcal{B}_j}\left(\Psi_j^*\right)^{p-1}\left(\Psi_{j+1}^*+\Psi_{j-1}^*\right)\partial_t\Psi_j d\mu &= \sum_{j=1}^{\nu}\frac{\int_{\mathcal{C}}\partial_t\left(\Psi_j^*\right)^p\left(\Psi_{j+1}^*+\Psi_{j-1}^*\right)d\mu}{\alpha_j^*}\\ &\quad+\mathcal{O}\left(Q^{\frac{p+1}{2}}\right)\\ &= -\sum_{j=1}^{\nu}\frac{\int_{\mathcal{C}}\left(\Psi_j^*\right)^p\partial_t\left(\Psi_{j+1}^*+\Psi_{j-1}^*\right)d\mu}{\alpha_j^*}\\ &\quad+\mathcal{O}\left(Q^{\frac{p+1}{2}}\right)\\ &= (B_2+o(1))Q\end{aligned}$$

with $B_2>0$ being a constant. Thus, summarizing the above estimates, we have

$$\sum_{j=1}^{\nu}\left\langle\mathcal{R}_{1,ex},\partial_t\Psi_j\right\rangle_{L^2}=(B_2+o(1))Q.$$

Step. 2 The estimate of $\sum_{j=1}^{\nu}\sum_{i=1;i\neq j}^{\nu}\left\langle\mathcal{R}_{1,i},\partial_t\Psi_j\right\rangle_{L^2}$.
By (3.18) and Lemma 4.3,

$$\left|\sum_{i=1;i\neq j}^{\nu}\left\langle\mathcal{R}_{1,i},\partial_t\Psi_j\right\rangle_{L^2}\right|\lesssim\sum_{i=1;i\neq j}^{\nu}\left|(\alpha_i^*)^{p-1}-1\right|\left\langle\Psi_i^p,\Psi_j\right\rangle_{L^2}=o(Q).$$

Step. 3 The estimate of $\left\langle\mathcal{L}_{j,ex}(\rho_*),\partial_t\Psi_j\right\rangle_{L^2}$.
By (2.7) and (5.11),

$$\left|\left\langle\mathcal{L}_{j,ex}(\rho_*),\partial_t\Psi_j\right\rangle_{L^2}\right|\lesssim\left\langle\left|\mathcal{L}_{j,ex}(\rho_*)\right|,\Psi_j\right\rangle_{L^2}=o\left(Q\right)+\mathcal{O}\left(\|\rho_{**}^{\perp}\|^{2+\sigma}\right).$$

Step. 4 The estimate of $\left\langle\mathcal{N}_j,\partial_t\Psi_j\right\rangle_{L^2}$.
By (6.2) and (1) of Proposition 4.1,

$$\begin{aligned}\left\langle\mathcal{N}_j,\partial_t\Psi_j\right\rangle_{L^2} &= 2A_{p,1}\left\langle\left(\Psi_j^*\right)^{p-2}\mathcal{V}_j\partial_t\Psi_j,\gamma_{2,ex}+\gamma_{\mathcal{N},led,rem,j}+\rho_{**}^{\perp}\right\rangle_{L^2}\\ &\quad+3A_{p,2}\left\langle\left(\Psi_j^*\right)^{p-3}\mathcal{V}_j^2\partial_t\Psi_j,\gamma_{1,ex}+\rho_{**}^{\perp}\right\rangle_{L^2}\\ &\quad+3A_{p,2}\left\langle\left(\Psi_j^*\right)^{p-3}\mathcal{V}_j^2\partial_t\Psi_j,\sum_{l=1;l\neq j}^{\nu}\gamma_{1,l}+\gamma_{\mathcal{N},led,rem,j}\right\rangle_{L^2}.\end{aligned}$$

By (2.7), Lemma 4.3 and (i) of (1) of Proposition 4.1

$$\begin{aligned}\left|\left\langle\left(\Psi_j^*\right)^{p-3}\mathcal{V}_j^2\partial_t\Psi_j,\gamma_{1,ex}\right\rangle_{L^2}\right| &\lesssim \begin{cases}\sum_{i=1}^{\nu}\beta_*^2Q\left\langle\Psi_i^{1-\sigma}\chi_{\mathcal{B}_i},\Psi_j^{2p-1}\right\rangle_{L^2}, & p\geq 3,\\ \sum_{i=1}^{\nu}\beta_*^2Q\left\langle\Psi_i^{p-2}\chi_{\mathcal{B}_i},\Psi_j^{2p-1}\right\rangle_{L^2}, & 1<p<3\end{cases}\\ &= o(Q).\end{aligned}$$

By (2.7), Lemma 4.5 and (iii) of (1) of Proposition 4.1,

$$\begin{aligned}&\left|\left\langle\left(\Psi_j^*\right)^{p-3}\mathcal{V}_j^2\partial_t\Psi_j,\sum_{l=1;l\neq j}^{\nu}\gamma_{1,l}+\gamma_{\mathcal{N},led,rem,j}\right\rangle_{L^2}\right|\\ \lesssim&\sum_{l=1;l\neq j}^{\nu}o\left(\beta_*^2\right)\left\langle\Psi_l^{1-\sigma}\chi_{\mathcal{B}_l},\Psi_j^{2p-1}\right\rangle_{L^2}\\ =&\ o(Q).\end{aligned}$$

By (2.7), Lemma 4.3 and (i) of (1) of Proposition 4.1,

$$\begin{aligned}\left|\left\langle\left(\Psi_j^*\right)^{p-2}\mathcal{V}_j\partial_t\Psi_j,\gamma_{2,ex}\right\rangle_{L^2}\right| &\lesssim \begin{cases}\sum_{i=1}^{\nu}\beta_*^2Q\left\langle\Psi_i^{1-\sigma}\chi_{\mathcal{B}_i},\Psi_j^{\frac{3p-1}{2}}\right\rangle_{L^2}, & p\geq\frac{7}{3},\\ \sum_{i=1}^{\nu}\beta_*^2Q\left\langle\Psi_i^{\frac{3p-5}{2}}\chi_{\mathcal{B}_i},\Psi_j^{\frac{3p-1}{2}}\right\rangle_{L^2}, & 1<p<\frac{7}{3}\end{cases}\\ &= o(Q).\end{aligned}$$

By (2.7), Lemma 4.3 and (iii) of (1) of Proposition 4.1,

$$\left|\left\langle\left(\Psi_j^*\right)^{p-2}\mathcal{V}_j\partial_t\Psi_j,\gamma_{\mathcal{N},led,rem,j}\right\rangle_{L^2}\right|\lesssim\sum_{i=1;i\neq j}^{\nu}\beta_*^3\left\langle\Psi_i^{1-\sigma}\chi_{\mathcal{B}_i},\Psi_j^{\frac{3p-1}{2}}\right\rangle_{L^2}=o(Q).$$

By (2.7) and Lemma 4.3,

$$\begin{aligned}&\left|2A_{p,1}\left\langle\left(\Psi_j^*\right)^{p-2}\mathcal{V}_j\partial_t\Psi_j,\rho_{**}^{\perp}\right\rangle_{L^2}+3A_{p,2}\left\langle\left(\Psi_j^*\right)^{p-3}\mathcal{V}_j^2\partial_t\Psi_j,\rho_{**}^{\perp}\right\rangle_{L^2}\right|\\ \lesssim&\ \beta_*\left(\left\|\Psi_j^{\frac{3p-1}{2}}\right\|_{L^2}+\beta_*\left\|\Psi_j^{2p-1}\right\|_{L^2}\right)\|\rho_{**}^{\perp}\|\\ =&\ \mathcal{O}\left(\beta_*^6+\|\rho_{**}^{\perp}\|^{1+\sigma}\right).\end{aligned}$$

Thus, summarizing the above estimates, we have

$$\langle\mathcal{N}_j,\partial_t\Psi_j\rangle_{L^2}=o(Q)+\mathcal{O}\left(\beta_*^6+\|\rho_{**}^{\perp}\|^{1+\sigma}\right).$$

Step. 5 The estimate of $\langle\mathcal{N}-\mathcal{N}_j,\partial_t\Psi_j\rangle_{L^2}$.
Since $|\partial_t\Psi|\lesssim\Psi$ by (2.7), we can use similar estimates of (5.12) to obtain

$$\left|\left\langle\mathcal{N}-\mathcal{N}_j-\left(\overline{\mathcal{N}}_{rem}+\mathcal{N}_0\right),\partial_t\Psi_j\right\rangle_{L^2}\right|=o(Q)+\mathcal{O}\left(\|\rho_{**}^{\perp}\|^{2+\sigma}\right).$$

Step. 5.1 The estimate of $\langle\mathcal{N}_0,\partial_t\Psi_j\rangle_{L^2}$.
Step. 5.1.1 The estimate of $\left\langle\mathcal{N}_0-\overline{\mathcal{N}}_{0,1},\partial_t\Psi_j\right\rangle_{L^2}$, where $\overline{\mathcal{N}}_{0,1}=A_{p,1}\mathcal{U}^{p-2}\rho_0^2+A_{p,2}\mathcal{U}^{p-3}\left(3\mathcal{V}\rho_0^2+\rho_0^3\right)$.
By (4.2), (1) of Proposition 4.1 and Lemma 5.1,

$$\begin{aligned}\left\langle\mathcal{N}_0-\overline{\mathcal{N}}_{0,1},\partial_t\Psi_j\right\rangle_{L^2} &\lesssim \left\langle\mathcal{U}^{p-2-\sigma}\Psi_j\left(|\gamma_{ex}|+|\gamma_{\mathcal{N},led}+\gamma_*|\right),\left|\rho_{**}^{\perp}\right|\right\rangle_{L^2}\\ &\quad+\left\langle\mathcal{U}^{p-3}\Psi_j\gamma_{ex}^2,\left|\rho_{**}^{\perp}\right|\right\rangle_{L^2}.\end{aligned}$$

By Lemma 4.3 and (i) of (1) of Proposition 4.1,

$$\begin{aligned} &\left\langle \mathcal{U}^{p-2-\sigma}\Psi_j\left|\gamma_{ex}\right|,\left|\rho_{**}^{\perp}\right|\right\rangle_{L^2} \\ \lesssim &\begin{cases} \sum_{i=1}^{\nu} Q\left\|\Psi_i^{p-1-2\sigma}\Psi_j\right\|_{L^2(\mathcal{B}_i)}\|\rho_{**}^{\perp}\|, & p\geq 3,\\ \sum_{i=1}^{\nu} Q_i\left\|\Psi_i^{2(p-2)-\sigma}\Psi_j\right\|_{L^2(\mathcal{B}_i)}\|\rho_{**}^{\perp}\|+o(Q), & 1<p<3 \end{cases}\\ = &\ o(Q)+\mathcal{O}\left(\|\rho_{**}^{\perp}\|^{2+\sigma}\right) \end{aligned}$$

and

$$\begin{aligned} &\left\langle \mathcal{U}^{p-3}\Psi_j\gamma_{ex}^2,\left|\rho_{**}^{\perp}\right|\right\rangle_{L^2} \\ \lesssim &\begin{cases} \sum_{i=1}^{\nu} Q^2\left\|\Psi_i^{p-1-2\sigma}\Psi_j\right\|_{L^2(\mathcal{B}_i)}\|\rho_{**}^{\perp}\|, & p\geq 3,\\ \sum_{i=1}^{\nu} Q_i^2\left\|\Psi_i^{3p-7}\Psi_j\right\|_{L^2(\mathcal{B}_i)}\|\rho_{**}^{\perp}\|+o(Q^2), & 1<p<3 \end{cases}\\ = &\ o(Q)+\mathcal{O}\left(\|\rho_{**}^{\perp}\|^{2+\sigma}\right). \end{aligned}$$

By Lemma 4.3, (ii) of (1) of Proposition 4.1 and Proposition 5.1,

$$\begin{aligned} \left\langle \mathcal{U}^{p-2}\Psi_j\left|\gamma_*\right|,\left|\rho_{**}^{\perp}\right|\right\rangle_{L^2} &\lesssim \left(Q+\beta_*^2+\|\rho_{**}^{\perp}\|^{1+\sigma}+\|f\|_{H^{-1}}\right)\|\rho_{**}^{\perp}\| \\ &= o(Q+\beta_*^6)+\mathcal{O}\left(\|\rho_{**}^{\perp}\|^{1+\sigma}+\|f\|_{H^{-1}}\right). \end{aligned}$$

Similar to (5.10), by Lemma 4.3 and (iii) of (1) of Proposition 4.1,

$$\left\langle \mathcal{U}^{p-2}\Psi_j\left|\gamma_{\mathcal{N},led}\right|,\left|\rho_{**}^{\perp}\right|\right\rangle_{L^2}=o(\beta_*^6+Q)+\mathcal{O}\left(\|\rho_{**}^{\perp}\|^{1+\sigma}+\|f\|_{H^{-1}}\right).$$

Summarizing the above estimates, we have

$$\left\langle \mathcal{N}_0-\overline{\mathcal{N}}_{0,1},\partial_t\Psi_j\right\rangle_{L^2}=o(Q+\beta_*^6)+\mathcal{O}\left(\|\rho_{**}^{\perp}\|^{1+\sigma}+\|f\|_{H^{-1}}\right).$$

Step. 5.1.2 The estimate of $\left\langle \mathcal{U}^{p-2}\rho_0^2,\partial_t\Psi_j\right\rangle_{L^2}$.

By (1) of Proposition 4.1,

$$\begin{aligned} \left\langle \mathcal{U}^{p-2}\rho_0^2,\partial_t\Psi_j\right\rangle_{L^2} &= \left\langle \mathcal{U}^{p-2}\left(\gamma_*+\gamma_{\mathcal{N},led}\right)^2,\partial_t\Psi_j\right\rangle_{L^2}+\left\langle \mathcal{U}^{p-2}\gamma_{ex}^2,\partial_t\Psi_j\right\rangle_{L^2} \\ &\quad +2\left\langle \mathcal{U}^{p-2}\left(\gamma_*+\gamma_{\mathcal{N},led}\right)\gamma_{ex},\partial_t\Psi_j\right\rangle_{L^2}. \end{aligned}$$

By (2.7) and similar estimates of (5.14),

$$\left|\left\langle \mathcal{U}^{p-2}\gamma_{ex}^2,\partial_t\Psi_j\right\rangle_{L^2}\right|=o(Q).$$

By (2.7) and (i) and (ii) of (1) of Proposition 4.1,

$$\begin{aligned} \left|\left\langle \mathcal{U}^{p-2}\gamma_*\gamma_{ex},\partial_t\Psi_j\right\rangle_{L^2}\right| &\lesssim \begin{cases} \sum_{i=1}^{\nu} o\left(Q\left\|\Psi_i^{p-2\sigma}\Psi_j\right\|_{L^1(\mathcal{B}_i)}\right), & p\geq 3,\\ \sum_{i=1}^{\nu} o\left(Q\left\|\Psi_i^{2p-3-\sigma}\Psi_j\right\|_{L^1(\mathcal{B}_i)}\right), & 1<p<3 \end{cases}\\ &= o(Q). \end{aligned}$$

Similar to (5.10), by (2.7) and (i) and (iii) of (1) of Proposition 4.1,

$$\left\langle \mathcal{U}^{p-2}\gamma_{\mathcal{N},led}\gamma_{ex},\partial_t\Psi_j\right\rangle_{L^2}=o(Q).$$

By the oddness of $\partial_t\Psi$ in $\mathbb{R}$ and (ii) and (iii) of (1) of Proposition 4.1,

$$\left\langle \mathcal{U}^{p-2}\left(\gamma_*+\gamma_{\mathcal{N},led}\right)^2, \partial_t\Psi_j\right\rangle_{L^2} = \left\langle \left(\mathcal{U}^{p-2}-\left(\Psi_j^*\right)^{p-2}\right)\partial_t\Psi_j, \mathcal{W}_{sym,j}^2\right\rangle_{L^2} + \left\langle \mathcal{U}^{p-2}\partial_t\Psi_j\mathcal{W}_{*,j}, 2\mathcal{W}_{sym,j}+\mathcal{W}_{*,j}\right\rangle_{L^2},$$

where $\mathcal{W}_{sym,j}=\gamma_{1,j}+\rho_{**,1,j}^{\perp}-\alpha_{j,3}^{**}\Psi_j+\gamma_{\mathcal{N},led,j}$ and $\mathcal{W}_{*,j}=\gamma_*+\gamma_{\mathcal{N},led}-\mathcal{W}_{sym,j}$ with

$$\alpha_{j,3}^{**}=\alpha_{j,0}^{**}-\alpha_{j,1}^{**} \tag{6.4}$$ eqnnewnew199985

and $\alpha_{j,0}^{**}$ and $\alpha_{j,1}^{**}$ being given by (4.40) and Lemma 4.10, respectively. Similar to (4.33), by (2.7),

$$\left|\left(\mathcal{U}^{p-2}-\left(\Psi_j^*\right)^{p-2}\right)\partial_t\Psi_j\right| \lesssim \left(\sum_{i=1}^{\nu}\Psi_i^{p-2}\mathcal{U}_i\chi_{\mathcal{B}_i}\right)+\mathcal{U}^{p-1}\chi_{\mathcal{C}\backslash\mathcal{B}_*}$$

where $\mathcal{B}_*$ is given by (4.7), thus, by Lemmas 4.3, 4.5, 4.8, 4.9, 4.10 and Proposition 5.1,

$$\left\langle \left(\mathcal{U}^{p-2}-\left(\Psi_j^*\right)^{p-2}\right)\partial_t\Psi_j, \mathcal{W}_{sym,j}^2\right\rangle_{L^2} = o\left(\sum_{j=1}^{\nu}\left\langle \Psi_j^{p-2\sigma}\chi_{\mathcal{B}_j}, \mathcal{U}_j\right\rangle_{L^2}\right)=o(Q).$$

Similar to (4.12), by (2.7), Lemma 4.5, (1) of Proposition 4.1 and Proposition 5.1,

$$\begin{aligned}&\left|\partial_t\Psi_j\mathcal{W}_{*,j}\left(2\mathcal{W}_{sym,j}+\mathcal{W}_{*,j}\right)\right|\\ \lesssim\ & \left(\beta_*^2+Q+\|\rho_{**}^{\perp}\|^{1+\sigma}+\|f\|_{H^{-1}}\right)^2 Q^{1-\sigma}\left(\sum_{j=1}^{\nu}\Psi_j^{1-\sigma}\chi_{\mathcal{B}_j}+\mathcal{U}^{1-\sigma}\chi_{\mathcal{C}\backslash\mathcal{B}_*}\right)\end{aligned}$$

in $\mathcal{C}$, thus, we have

$$\left\langle \mathcal{U}^{p-2}\partial_t\Psi_j\mathcal{W}_{*,j}, 2\mathcal{W}_{sym,j}+\mathcal{W}_{*,j}\right\rangle_{L^2}=o(Q+\beta_*^6)+\mathcal{O}\left(\|\rho_{**}^{\perp}\|^{1+\sigma}+\|f\|_{H^{-1}}\right).$$

Summarizing the above estimates, we have

$$\left\langle \mathcal{U}^{p-2}\rho_0^2, \partial_t\Psi_j\right\rangle_{L^2}=o(Q+\beta_*^6)+\mathcal{O}\left(\|\rho_{**}^{\perp}\|^{1+\sigma}+\|f\|_{H^{-1}}\right).$$

Step. 5.1.3 The estimate of $\left\langle \mathcal{U}^{p-3}\mathcal{V}\rho_0^2, \partial_t\Psi_j\right\rangle_{L^2}$.
Clearly, we have

$$\left\langle \mathcal{U}^{p-3}\mathcal{V}\rho_0^2, \partial_t\Psi_j\right\rangle_{L^2}=\left\langle \mathcal{U}^{p-3}\left(\sum_{i=1;i\neq j}^{\nu}\mathcal{V}_i\right)\rho_0^2, \partial_t\Psi_j\right\rangle_{L^2}+\left\langle \mathcal{U}^{p-3}\mathcal{V}_j\rho_0^2, \partial_t\Psi_j\right\rangle_{L^2}.$$

By (4.2) and applying the same symmetry as in the estimate of $\left\langle \mathcal{U}^{p-2}\rho_0^2, \partial_t\Psi_j\right\rangle_{L^2}$, we have

$$\left\langle \mathcal{U}^{p-3}\mathcal{V}_j\rho_0^2, \partial_t\Psi_j\right\rangle_{L^2}=o(Q+\beta_*^6)+\mathcal{O}\left(\|\rho_{**}^{\perp}\|^{1+\sigma}+\|f\|_{H^{-1}}\right).$$

By (2.7) and (1) of Proposition 4.1,

$$\begin{aligned}&\left|\left\langle \mathcal{U}^{p-3}\left(\sum_{i=1;i\neq j}^{\nu}\mathcal{V}_i\right)\rho_0^2, \partial_t\Psi_j\right\rangle_{L^2}\right|\\ \lesssim\ & \left\langle \mathcal{U}^{p-3}\left|\sum_{i=1;i\neq j}^{\nu}\mathcal{V}_i\right|\gamma_{ex}^2, \Psi_j\right\rangle_{L^2}+\left\langle \mathcal{U}^{p-3}\left|\sum_{i=1;i\neq j}^{\nu}\mathcal{V}_i\right|\left(\gamma_*+\gamma_{\mathcal{N},led}\right)^2, \Psi_j\right\rangle_{L^2}.\end{aligned}$$

By Lemma 4.3 and (i) of (1) of Proposition 4.1,

$$\begin{aligned}
&\left\langle \mathcal{U}^{p-3}\left|\sum_{i=1;i\neq j}^{\nu}\mathcal{V}_i\right|\gamma_{ex}^2,\Psi_j\right\rangle_{L^2}\\
\lesssim&\begin{cases}\sum_{i=1}^{\nu}\beta_*Q^2\left\langle\Psi_i^{\frac{3p-1-4\sigma}{2}}\chi_{\mathcal{B}_i},\mathcal{U}_i\right\rangle_{L^2}, & p\geq 3,\\ \sum_{i=1}^{\nu}\beta_*Q_i^2\left\langle\Psi_i^{\frac{7p-13}{2}}\chi_{\mathcal{B}_i},\mathcal{U}_i\right\rangle_{L^2}+o(\beta_*Q^2), & 1<p<3\end{cases}\\
=&\ o(Q).
\end{aligned}$$

By Lemma 4.3, (ii) of (1) of Proposition 4.1 and Proposition 5.1,

$$\left\langle \mathcal{U}^{p-3}\left|\sum_{i=1;i\neq j}^{\nu}\mathcal{V}_i\right|\gamma_{*}^2,\Psi_j\right\rangle_{L^2}=\sum_{i=1}^{\nu}o(\beta_*)\left\langle\Psi_i^{\frac{3p-1-4\sigma}{2}}\chi_{\mathcal{B}_i},\mathcal{U}_i\right\rangle_{L^2}=o(Q).$$

Similar to (5.10), by Lemma 4.3 and (iii) of (1) of Proposition 4.1,

$$\left\langle \mathcal{U}^{p-3}\left|\sum_{i=1;i\neq j}^{\nu}\mathcal{V}_i\right|\gamma_{\mathcal{N},led}^2,\Psi_j\right\rangle_{L^2}=o(Q).$$

Summarizing the above estimates, we have

$$\left\langle\mathcal{U}^{p-3}\mathcal{V}\rho_0^2,\partial_t\Psi_j\right\rangle_{L^2}=o(Q+\beta_*^6)+\mathcal{O}\left(\|\rho_{**}^{\perp}\|^{1+\sigma}+\|f\|_{H^{-1}}\right).$$

Step. 5.1.4 The estimate of $\left\langle\mathcal{U}^{p-3}\rho_0^3,\partial_t\Psi_j\right\rangle_{L^2}$.
By (1) of Proposition 4.1 and Proposition 5.1,

$$\begin{aligned}
\left\langle\mathcal{U}^{p-3}\rho_0^3,\partial_t\Psi_j\right\rangle_{L^2}&=\mathcal{O}\left(Q^{(2p-1)\wedge 3}|\log Q|+\left(\beta_*^2+\|\rho_{**}^{\perp}\|^{1+\sigma}+\|f\|_{H^{-1}}\right)^3\right)\\
&=o(Q)+\mathcal{O}\left(\beta_*^6+\|\rho_{**}^{\perp}\|^{1+\sigma}+\|f\|_{H^{-1}}\right).
\end{aligned}$$

Summarizing the estimates from Step. 5.1.1 to Step. 5.1.4, we have

$$\left\langle\mathcal{N}_0,\partial_t\Psi_j\right\rangle_{L^2}=o(Q)+\mathcal{O}\left(\beta_*^6+\|\rho_{**}^{\perp}\|^{1+\sigma}+\|f\|_{H^{-1}}\right).$$

Step. 5.2 The estimate of $\left\langle\overline{\mathcal{N}}_{rem},\partial_t\Psi_j\right\rangle_{L^2}$.
Similar to (5.14) and (5.15), by Lemma 6.1 and Proposition 5.1,

$$\begin{aligned}
\left\langle\overline{\mathcal{N}}_{rem},\partial_t\Psi_j\right\rangle_{L^2}=&\ A_{p,3}\left\langle\mathcal{U}^{p-4}\left(\overline{\mathcal{V}}^4+4\overline{\mathcal{V}}^3\rho_{**}^{\perp}\right),\partial_t\Psi_j\right\rangle_{L^2}\\
&+A_{p,4}\left\langle\mathcal{U}^{p-5}\left(\overline{\mathcal{V}}^5+5\overline{\mathcal{V}}^4\rho_{**}^{\perp}\right),\partial_t\Psi_j\right\rangle_{L^2}\\
&+o(Q)+\mathcal{O}\left(\beta_*^6+\|\rho_{**}^{\perp}\|^{1+\sigma}+\|f\|_{H^{-1}}\right).
\end{aligned}$$

Step. 5.2.1 The estimates of $\left\langle\mathcal{U}^{p-4}\overline{\mathcal{V}}^3\rho_{**}^{\perp},\partial_t\Psi_j\right\rangle_{L^2}$ and $\left\langle\mathcal{U}^{p-5}\overline{\mathcal{V}}^4\rho_{**}^{\perp},\partial_t\Psi_j\right\rangle_{L^2}$.
Recall that $\overline{\mathcal{V}}=\mathcal{V}+\rho_0$ with $\mathcal{V}$ given by (3.15) and ρ_0 given in Proposition 4.1. By (2.7), (1) of Proposition 4.1 and Proposition 5.1,

$$\begin{aligned}
\left|\left\langle\mathcal{U}^{p-4}\overline{\mathcal{V}}^3\rho_{**}^{\perp},\partial_t\Psi_j\right\rangle_{L^2}\right|&\lesssim\left(Q^{\frac{4p-3}{2}\wedge 3}|\log Q|^{\frac{1}{2}}+\left(\beta_*+\|\rho_{**}^{\perp}\|^{1+\sigma}+\|f\|_{H^{-1}}\right)^3\right)\|\rho_{**}^{\perp}\|\\
&=o(Q+\beta_*^6)+\mathcal{O}\left(\|\rho_{**}^{\perp}\|^{1+\sigma}+\|f\|_{H^{-1}}\right).
\end{aligned}$$

and

$$\begin{aligned}\left|\left\langle \mathcal{U}^{p-5}\overline{\mathcal{V}}^4\rho_{**}^{\perp},\partial_t\Psi_j\right\rangle_{L^2}\right| &\lesssim \left(Q^{\frac{5p-4}{2}\wedge 4}|\log Q|^{\frac{1}{2}}+\left(\beta_*+\|\rho_{**}^{\perp}\|^{1+\sigma}+\|f\|_{H^{-1}}\right)^4\right)\|\rho_{**}^{\perp}\|\\ &= o(Q+\beta_*^6)+\mathcal{O}\left(\|\rho_{**}^{\perp}\|^{1+\sigma}+\|f\|_{H^{-1}}\right).\end{aligned}$$

Step. 5.2.2 The estimates of $\left\langle \mathcal{U}^{p-4}\overline{\mathcal{V}}^4,\partial_t\Psi_j\right\rangle_{L^2}$ and $\left\langle \mathcal{U}^{p-5}\overline{\mathcal{V}}^5,\partial_t\Psi_j\right\rangle_{L^2}$.
By the oddness of $\partial_t\Psi$ in $\mathbb{R}$ and (1) of Proposition 4.1,

$$\begin{aligned}&\left\langle \mathcal{U}^{p-4}\overline{\mathcal{V}}^4,\partial_t\Psi_j\right\rangle_{L^2}\\ =&\left\langle\left(\mathcal{U}^{p-4}-\left(\Psi_j^*\right)^{p-4}\right)\partial_t\Psi_j,\overline{\mathcal{V}}_{sym,j}^4\right\rangle_{L^2}\\ &+\left\langle \mathcal{U}^{p-4}\partial_t\Psi_j\overline{\mathcal{V}}_{*,j},4\overline{\mathcal{V}}_{sym,j}^3+6\overline{\mathcal{V}}_{sym,j}^2\overline{\mathcal{V}}_{*,j}+4\overline{\mathcal{V}}_{sym,j}\overline{\mathcal{V}}_{*,j}^2+\overline{\mathcal{V}}_{*,j}^3\right\rangle_{L^2},\end{aligned}$$

where $\overline{\mathcal{V}}_{sym,j}=\mathcal{V}_j+\gamma_{1,j}+\rho_{**,1,j}^{\perp}-\alpha_{j,3}^{**}\Psi_j+\gamma_{\mathcal{N},led,j}$ and $\overline{\mathcal{V}}_{*,j}=\overline{\mathcal{V}}-\overline{\mathcal{V}}_{sym,j}$ with $\alpha_{j,3}^{**}$ given by (6.4). Similar to (4.33), by (2.7), we have

$$\left|\left(\mathcal{U}^{p-4}-\left(\Psi_j^*\right)^{p-4}\right)\partial_t\Psi_j\right|\lesssim\left(\sum_{i=1}^{\nu}\Psi_i^{p-4}\mathcal{U}_i\chi_{\mathcal{B}_i}\right)+\mathcal{U}^{p-1}\chi_{\mathcal{C}\backslash\mathcal{B}_*},$$

thus, by (3.15), Lemmas 4.3, 4.5, 4.8, 4.9, 4.10 and Proposition 5.1,

$$\left\langle\left(\mathcal{U}^{p-4}-\left(\Psi_j^*\right)^{p-4}\right)\partial_t\Psi_j,\overline{\mathcal{V}}_{sym,j}^4\right\rangle_{L^2}=o\left(\left\langle\Psi_j^{p-4\sigma}\chi_{\mathcal{B}_j},\mathcal{U}_j\right\rangle_{L^2}\right)=o(Q).$$

Again, similar to (4.12), by (2.7), (3.15), Lemma 4.5, (iii) of (1) of Proposition 4.1 and Proposition 5.1,

$$\begin{aligned}&\left|\partial_t\Psi_j\overline{\mathcal{V}}_{*,j}\left(4\overline{\mathcal{V}}_{sym,j}^3+6\overline{\mathcal{V}}_{sym,j}^2\overline{\mathcal{V}}_{*,j}+4\overline{\mathcal{V}}_{sym,j}\overline{\mathcal{V}}_{*,j}^2+\overline{\mathcal{V}}_{*,j}^3\right)\right|\\ \lesssim&\ Q^{1-\sigma}\left(Q+\beta_*+\|\rho_{**}^{\perp}\|^{1+\sigma}+\|f\|_{H^{-1}}\right)^4\left(\sum_{j=1}^{\nu}\Psi_j^{3-3\sigma}\chi_{\mathcal{B}_j}+\mathcal{U}^{3-3\sigma}\chi_{\mathcal{C}\backslash\mathcal{B}_*}\right)\end{aligned}$$

in $\mathcal{C}$, thus, we have

$$\begin{aligned}&\left\langle \mathcal{U}^{p-4}\partial_t\Psi_j\overline{\mathcal{V}}_{*,j},4\overline{\mathcal{V}}_{sym,j}^3+6\overline{\mathcal{V}}_{sym,j}^2\overline{\mathcal{V}}_{*,j}+4\overline{\mathcal{V}}_{sym,j}\overline{\mathcal{V}}_{*,j}^2+\overline{\mathcal{V}}_{*,j}^3\right\rangle_{L^2}\\ \lesssim&\ Q^{1-\sigma}\left(Q+\beta_*+\|\rho_{**}^{\perp}\|^{1+\sigma}+\|f\|_{H^{-1}}\right)^4\\ =&\ o(Q+\beta_*^6)+\mathcal{O}\left(\|\rho_{**}^{\perp}\|^{1+\sigma}+\|f\|_{H^{-1}}\right).\end{aligned}$$

Summarizing the above estimates, we have

$$\left\langle \mathcal{U}^{p-4}\overline{\mathcal{V}}^4,\partial_t\Psi_j\right\rangle_{L^2}=o(Q)+\mathcal{O}\left(\beta_*^6+\|\rho_{**}^{\perp}\|^{1+\sigma}+\|f\|_{H^{-1}}\right).$$

By (4.2) and applying the same symmetry as in the estimate of $\left\langle \mathcal{U}^{p-4}\overline{\mathcal{V}}^4,\partial_t\Psi_j\right\rangle_{L^2}$, we also have

$$\left\langle \mathcal{U}^{p-5}\overline{\mathcal{V}}^5,\partial_t\Psi_j\right\rangle_{L^2}=o(Q+\beta_*^6)+\mathcal{O}\left(\|\rho_{**}^{\perp}\|^{1+\sigma}+\|f\|_{H^{-1}}\right).$$

Summarizing the estimates from Step. 5.2.1 to Step. 5.2.2, we have

$$\left\langle\overline{\mathcal{N}}_{rem},\partial_t\Psi_j\right\rangle_{L^2}=o(Q)+\mathcal{O}\left(\beta_*^6+\|\rho_{**}^{\perp}\|^{1+\sigma}+\|f\|_{H^{-1}}\right).$$

The conclusion then follows from the estimates in Step. 1 to Step. 5. □

7. Finally refined expansion of $\mathcal{N}$ and estimate of $\|\rho_{**}^{\perp}\|$

By the orthogonal conditions of $\rho_{**}^{\perp}$, given by (4.63) and multiplying (4.63) with $\rho_{**}^{\perp}$ on both sides and integrating by parts, we have

$$\|\rho_{**}^{\perp}\|^2 \lesssim \|f\|_{H^{-1}}\|\rho_{**}^{\perp}\| + \left|\left\langle \mathcal{R}_{new,0}, \rho_{**}^{\perp}\right\rangle_{L^2}\right|, \tag{7.1}$$ eqn0060

where $\mathcal{R}_{new,0}$ is given by (4.64). Moreover, we remark that by Lemmas 4.1 and 5.1, we have

$$\mathcal{N}_{rem} = \mathcal{N}_0 + \overline{\mathcal{N}}_{rem}, \tag{7.2}$$ eqnew0093

where $\mathcal{N}_{rem}$ is the remaining term in $\mathcal{R}_{new,0}$. We emphasize once more that we need to eliminate the lower order terms in the data $\mathcal{R}_{new,0}$ to get the desired stability. Thus, we need further decompose the term $\sum_{j=1}^{\nu}\gamma_{1,j}$ which is given in Lemma 4.5.

⟨lemn0003⟩ **Lemma 7.1.** *Let $d \geq 2$, $a < 0$ and $b = b_{FS}(a)$. Then we have the following decomposition*

$$\sum_{j=1}^{\nu}\gamma_{1,j} = \widetilde{\gamma}_{1,*} + \sum_{l=1}^{\nu}\alpha_l^{***}\Psi_l,$$

where $\{\alpha_l^{***}\}$ is chosen such that $\langle\widetilde{\gamma}_{1,*}, \Psi_l\rangle = 0$ for all $1 \leq l \leq \nu$. Moreover, we have the following estimates

$$\|\widetilde{\gamma}_{1,*}\|_{L^\infty} \lesssim \sum_{l=1}^{\nu} Q^{1-\sigma}\left|(\alpha_l^*)^p - \alpha_l^*\right| \quad \text{and} \quad \sum_{j=1}^{\nu}\left|\alpha_j^{***}\right| \lesssim \sum_{l=1}^{\nu}\left|(\alpha_l^*)^p - \alpha_l^*\right|.$$

Proof. It is easy to see that $\widetilde{\gamma}_{1,*}$ satisfies the following equation:

$$\begin{cases}\mathcal{L}(\widetilde{\gamma}_{1,*}) = \mathcal{R}_{1,*} - \sum_{i=1}^{\nu}\Psi_i^{p-1}\left(c_{1,j,i}\partial_t\Psi_i + \sum_{l=1}^{d}\varsigma_{1,j,i,l}w_{i,l}\right), \quad \text{in } \mathcal{C},\\ \langle\partial_t\Psi_j, \widetilde{\gamma}_{1,*}\rangle = \langle w_{j,l}, \widetilde{\gamma}_{1,*}\rangle = 0 \quad \text{for all } 1 \leq j \leq \nu \text{ and } 1 \leq l \leq d,\end{cases} \tag{7.3}$$ eqn9012

where by (3.18),

$$\begin{aligned}\mathcal{R}_{1,*} &= \sum_{l=1}^{\nu}\left(\mathcal{R}_{1,l} - \alpha_l^{***}\left(\Psi_l^p - p\mathcal{U}^{p-1}\Psi_l\right)\right)\\ &= \sum_{l=1}^{\nu}\left((\alpha_l^*)^p - \alpha_l^* - \alpha_l^{***}\left(1 - p\left(\alpha_l^*\right)^{p-1}\right)\right)\Psi_l^p\\ &\quad + \sum_{l=1}^{\nu}p\alpha_l^{***}\left(\mathcal{U}^{p-1} - (\Psi_l^*)^{p-1}\right)\Psi_l.\end{aligned}$$

By Lemma 4.3, (3.18) and similar estimates in the proof of Lemma 4.10, we have

$$\left(1 - p\left(\alpha_j^*\right)^{p-1}\right)\alpha_j^{***} = \left(\left(\alpha_j^*\right)^p - \alpha_j^*\right) + \sum_{l=1}^{\nu}\mathcal{O}(Q)\left((\alpha_l^*)^p - \alpha_l^*\right) + \mathcal{O}\left(Q^{1-\sigma}\right)\|\widetilde{\gamma}_{1,*}\|$$

for all $1 \leq j \leq \nu$. Thus, by Lemma 4.5, (7.3), the orthogonal conditions of $\gamma_{1,j}$ given in (4.29) and the elliptic estimates, we have the desired estimates of $\|\widetilde{\gamma}_{1,*}\|_{L^\infty}$ and $\sum_{j=1}^{\nu}\left|\alpha_j^{***}\right|$. □

To eliminate the lower order terms (compared to the β_*^4 terms) in the data $\mathcal{R}_{new,0}$ to get the desired stability, we need to refine the expansion of $\overline{\mathcal{N}}_{rem}$ for the third time, where $\overline{\mathcal{N}}_{rem}$ is the remaining term in the expansion of $\mathcal{N}$ given by Lemma 5.1.

⟨lemn0004⟩ **Lemma 7.2.** *Let $d \geq 2$, $a < 0$ and $b = b_{FS}(a)$. Then $\overline{\mathcal{N}}_{rem}$, the remaining term in the expansion of $\mathcal{N}$ given by Lemma 5.1, can be further expanded as follows:*

$$\begin{aligned}\overline{\mathcal{N}}_{rem} &= \sum_{l=4}^{n_0} A_{p,l-1}\mathcal{U}^{p-l}\left(\overline{\mathcal{V}}^l + l\overline{\mathcal{V}}^{l-1}\rho_{**}^{\perp}\right) + \mathcal{O}\left(|\gamma_* + \gamma_{\mathcal{N},led}|^{1+\sigma}\chi_{\mathcal{C}\backslash\widetilde{\mathcal{B}}_{**}}\right)\\ &\quad+\mathcal{O}\left(\left(\beta_* + \sum_{j=1}^{\nu}\left|(\alpha_j^*)^{p-1} - 1\right|\right)^{n_0} + Q^{p\wedge 3}\right)\\ &\quad+\mathcal{O}\left(\chi_{p\geq 2}|\rho_{**}^{\perp}|^2 + |\rho_{**}^{\perp}|^p + |\rho_{**}^{\perp}|^{1+\sigma}\right)\end{aligned}$$

in $\mathcal{C}$ where $\overline{\mathcal{V}} = \mathcal{V} + \rho_0$ with $\mathcal{V}$ given by (3.15) and ρ_0 given in Proposition 4.1, $\widetilde{\mathcal{B}}_{**}$ is given in Lemma 5.1 and γ_* and $\gamma_{\mathcal{N},led}$ are given in (1) of Proposition 4.1.

Proof. Since by the choice of n_0 and (i) of (1) of Proposition 4.1, we have

$$\begin{aligned}\left|\mathcal{U}^{p-n_0}\gamma_{ex}^{n_0}\right| &\lesssim \begin{cases} Q^{n_0}, & p \geq 3,\\ Q^{\frac{p+n_0(p-1)}{2}}, & 1 < p < 3\end{cases}\\ &\lesssim Q^{p\wedge 3},\end{aligned}$$

the conclusion can be obtained by applying the Taylor expansion in the sets $\mathcal{A}_*$ and $\mathcal{C}\backslash\mathcal{A}_*$ as in the proof of Lemma 5.1 up to the n_0th order terms. □

In what follows, we shall estimate $\|\rho_{**}^{\perp}\|$ by (7.1).

⟨propn0003⟩ **Proposition 7.1.** *Let $d \geq 2$, $a < 0$ and $b = b_{FS}(a)$. Then we have*

$$\|\rho_{**}^{\perp}\| \lesssim \beta_*^4 + \|f\|_{H^{-1}}.$$

Proof. By (3.19), (4.64), (7.2) and the orthogonal conditions of $\rho_{**}^{\perp}$ given by (4.63),

$$\begin{aligned}&\left\langle \mathcal{R}_{new,0}, \rho_{**}^{\perp}\right\rangle_{L^2}\\ =& \sum_{j=1}^{\nu} 2A_{p,1}\left\langle \left(\Psi_j^*\right)^{p-2}\mathcal{V}_j(\rho_* - \gamma_{1,ex} - \gamma_{\mathcal{N},led,j})\chi_{\mathcal{B}_j}, \rho_{**}^{\perp}\right\rangle_{L^2} + \sum_{j=1}^{\nu}\left\langle \mathcal{R}_{2,j}, \rho_{**}^{\perp}\right\rangle_{L^2}\\ &+\sum_{j=1}^{\nu} 3A_{p,2}\left\langle \left(\Psi_j^*\right)^{p-3}\mathcal{V}_j^2\rho_*\chi_{\mathcal{B}_j}, \rho_{**}^{\perp}\right\rangle_{L^2} + \sum_{j=1}^{\nu}\alpha_j^{**}\left\langle \left(\mathcal{U}^{p-1} - \Psi_j^{p-1}\right)\Psi_j, \rho_{**}^{\perp}\right\rangle_{L^2}\\ &+\sum_{j=1}^{\nu}\left\langle \mathcal{O}\left(\beta_*\mathcal{U}_j\left(\beta_*\Psi_j^{2p-3}\left(\Psi_j + \rho_*\right) + |\rho_* - \gamma_{1,ex}|\Psi_j^{\frac{3p-5}{2}}\right)\right)\chi_{\mathcal{B}_j}, \rho_{**}^{\perp}\right\rangle_{L^2}\\ &+\left\langle \mathcal{O}\left(|\rho_* - \gamma_{1,ex}|\mathcal{U}^{\frac{3p-3}{2}}\right)\chi_{\mathcal{C}\backslash\mathcal{B}_*}, \rho_{**}^{\perp}\right\rangle_{L^2} + \left\langle \mathcal{O}\left(\beta_*^2\mathcal{U}^{2(p-1)}\gamma_{1,ex}\chi_{\mathcal{C}\backslash\mathcal{B}_*}\right), \rho_{**}^{\perp}\right\rangle_{L^2}\\ &+\left\langle \overline{\mathcal{N}}_{rem} + \mathcal{N}_0 - \mathcal{R}_{3,ex} - \mathcal{R}_{5,ex}, \rho_{**}^{\perp}\right\rangle_{L^2},\end{aligned}$$

where $\mathcal{R}_{3,ex}$ and $\mathcal{R}_{5,ex}$ are given by (4.61) and (4.62), respectively, and $\mathcal{B}_i$ and $\mathcal{B}_*$ are given by (4.7).

Step. 1 The estimate of $\left\langle \sum_{j=1}^{\nu}\alpha_j^{**}\left(\mathcal{U}^{p-1} - \Psi_j^{p-1}\right)\Psi_j, \rho_{**}^{\perp}\right\rangle_{L^2}$.

By the orthogonal conditions of $\rho_{**}^{\perp}$ given by (4.63), we have

$$\left\langle \sum_{j=1}^{\nu} \alpha_j^{**} \left(\mathcal{U}^{p-1} - \Psi_j^{p-1} \right) \Psi_j, \rho_{**}^{\perp} \right\rangle_{L^2} = \left\langle \sum_{j=1}^{\nu} \alpha_j^{**} \left(\mathcal{U}^{p-1} - (\Psi_j^*)^{p-1} \right) \Psi_j, \rho_{**}^{\perp} \right\rangle_{L^2}.$$

Similar to (4.33), we have

$$\left| \sum_{j=1}^{\nu} \alpha_j^{**} \left(\mathcal{U}^{p-1} - (\Psi_j^*)^{p-1} \right) \Psi_j \right| \lesssim \sum_{j=1}^{\nu} |\alpha_j^{**}| \Psi_j^{p-1} \mathcal{U}_j \chi_{\mathcal{B}_j} + \left(\sum_{j=1}^{\nu} |\alpha_j^{**}| \right) \mathcal{U}^p \chi_{\mathcal{C} \setminus \mathcal{B}_*}.$$

Thus, by Lemma 4.3, (2) of Proposition 4.1 and Propositions 5.1 and 6.1,

$$\begin{aligned} \left\langle \sum_{j=1}^{\nu} \alpha_j^{**} \left(\mathcal{U}^{p-1} - \Psi_j^{p-1} \right) \Psi_j, \rho_{**}^{\perp} \right\rangle_{L^2} &= \mathcal{O}\left(\sum_{j=1}^{\nu} |\alpha_j^{**}| Q^{\frac{1}{2}+\sigma} \|\rho_{**}^{\perp}\| \right) \\ &\lesssim \beta_*^2 \left(\|\rho_{**}^{\perp}\|^{1+\sigma} + \|f\|_{H^{-1}} \right)^{\frac{1}{2}+\sigma} \|\rho_{**}^{\perp}\| \\ &\quad + \mathcal{O}\left(\left(\beta_*^4 + \|f\|_{H^{-1}} \right) \|\rho_{**}^{\perp}\| \right) + o(\|\rho_{**}^{\perp}\|^2) \\ &= \mathcal{O}\left(\left(\beta_*^4 + \|f\|_{H^{-1}} \right) \|\rho_{**}^{\perp}\| \right) + o(\|\rho_{**}^{\perp}\|^2). \end{aligned}$$

Step. 2 The estimates of

$$\sum_{j=1}^{\nu} \left\langle \beta_*^2 \mathcal{U}_j \Psi_j^{2p-3} \left(\Psi_j + \rho_* \right) \chi_{\mathcal{B}_j}, \rho_{**}^{\perp} \right\rangle_{L^2}$$

and $\left\langle \beta_*(\rho_* - \gamma_{1,ex}) \mathcal{U}_*, \rho_{**}^{\perp} \right\rangle_{L^2}$, where $\mathcal{U}_* = \sum_{j=1}^{\nu} \Psi_j^{\frac{3p-5}{2}} \mathcal{U}_j \chi_{\mathcal{B}_j} + \mathcal{U}^{\frac{3(p-1)}{2}} \chi_{\mathcal{C} \setminus \mathcal{B}_*}$.

By Lemma 4.3 and Proposition 6.1,

$$\begin{aligned} \left\langle \beta_*^2 \mathcal{U}_j \Psi_j^{2p-2} \chi_{\mathcal{B}_j}, \rho_{**}^{\perp} \right\rangle_{L^2} &= \mathcal{O}\left(\beta_*^2 Q^{\frac{1}{2}+\sigma} \|\rho_{**}^{\perp}\| \right) \\ &= \mathcal{O}\left(\beta_*^4 \|\rho_{**}^{\perp}\| + \|f\|_{H^{-1}} \|\rho_{**}^{\perp}\| \right) + o\left(\|\rho_{**}^{\perp}\|^2 \right). \end{aligned}$$

By (1) of Proposition 4.1,

$$\begin{aligned} \sum_{j=1}^{\nu} \left\langle \beta_*^2 \mathcal{U}_j \Psi_j^{2p-3} \rho_* \chi_{\mathcal{B}_j}, \rho_{**}^{\perp} \right\rangle_{L^2} &= \sum_{j=1}^{\nu} \left\langle \beta_*^2 \mathcal{U}_j \Psi_j^{2p-3} (\gamma_{ex} + \gamma_* + \gamma_{\mathcal{N},led}) \chi_{\mathcal{B}_j}, \rho_{**}^{\perp} \right\rangle_{L^2} \\ &\quad + o(\|\rho_{**}^{\perp}\|^2). \end{aligned}$$

By Lemma 4.3, (i) of (1) of Proposition 4.1 and Proposition 6.1,

$$\begin{aligned} &\left| \left\langle \beta_*^2 \mathcal{U}_j \Psi_j^{2p-3} \gamma_{ex} \chi_{\mathcal{B}_j}, \rho_{**}^{\perp} \right\rangle_{L^2} \right| \\ \lesssim &\begin{cases} \left\langle \beta_*^2 \mathcal{U}_j \Psi_j^{2p-2-\sigma} \left(Q_j \chi_{\mathcal{B}_{j,+}} + Q_{j-1} \chi_{\mathcal{B}_{j,-}} \right), |\rho_{**}^{\perp}| \right\rangle_{L^2}, & p \geq 3, \\ \left\langle \beta_*^2 \mathcal{U}_j \Psi_j^{3p-5} \left(Q_j \chi_{\mathcal{B}_{j,+}} + Q_{j-1} \chi_{\mathcal{B}_{j,-}} \right), |\rho_{**}^{\perp}| \right\rangle_{L^2} + \beta_*^2 Q \|\rho_{**}^{\perp}\|, & 1 < p < 3 \end{cases} \\ = &\ \mathcal{O}\left(\beta_*^2 Q^{\frac{1}{2}+\sigma} \|\rho_{**}^{\perp}\| \right) \\ = &\ \mathcal{O}\left(\beta_*^4 \|\rho_{**}^{\perp}\| + \|f\|_{H^{-1}} \|\rho_{**}^{\perp}\| \right) + o\left(\|\rho_{**}^{\perp}\|^2 \right). \end{aligned}$$

Similar to (5.10), by Lemma 4.3, (ii) and (iii) of (1) of Proposition 4.1, Propositions 5.1 and 6.1,

$$\begin{aligned} & \left|\left\langle \beta_*^2\mathcal{U}_j\Psi_j^{2p-3}(\gamma_*+\gamma_{\mathcal{N},led})\chi_{\mathcal{B}_j},\rho_{**}^{\perp}\right\rangle_{L^2}\right| \\ \lesssim\ & o\left(\beta_*^2\right)\left\langle \mathcal{U}_j\Psi_j^{2p-2-\sigma},\left|\rho_{**}^{\perp}\right|\right\rangle_{L^2}+\mathcal{O}\left(\beta_*^2Q^{\frac{1}{2}+\sigma}\|\rho_{**}^{\perp}\|\right) \\ =\ & \mathcal{O}\left(\beta_*^4\|\rho_{**}^{\perp}\|+\|f\|_{H^{-1}}\|\rho_{**}^{\perp}\|\right)+o\left(\|\rho_{**}^{\perp}\|^2\right). \end{aligned}$$

Summarizing the above estimates, we have

$$\sum_{j=1}^{\nu}\left\langle \beta_*^2\mathcal{U}_j\Psi_j^{2p-3}\rho_*\chi_{\mathcal{B}_j},\rho_{**}^{\perp}\right\rangle_{L^2}=\mathcal{O}\left(\beta_*^4\|\rho_{**}^{\perp}\|+\|f\|_{H^{-1}}\|\rho_{**}^{\perp}\|\right)+o\left(\|\rho_{**}^{\perp}\|^2\right).$$

By (1) of Proposition 4.1,

$$\left\langle \beta_*(\rho_*-\gamma_{1,ex})\mathcal{U}_*,\rho_{**}^{\perp}\right\rangle_{L^2}=\beta_*\left\langle \gamma_{2,ex}+\gamma_*+\gamma_{\mathcal{N},led},\mathcal{U}_*\rho_{**}^{\perp}\right\rangle_{L^2}+\beta_*\left\langle \mathcal{U}_*,\left(\rho_{**}^{\perp}\right)^2\right\rangle_{L^2}.$$

Since $\|\mathcal{U}_*\|_{L^\infty}=o(1)$, we have $\beta_*\left\langle \mathcal{U}_*,\left|\rho_{**}^{\perp}\right|^2\right\rangle_{L^2}=o\left(\|\rho_{**}^{\perp}\|^2\right)$. By Lemma 4.3, (i) of (1) of Proposition 4.1 and Proposition 6.1, we have

$$\begin{aligned} & \beta_*\left|\left\langle \gamma_{2,ex},\mathcal{U}_*\rho_{**}^{\perp}\right\rangle_{L^2}\right| \\ \lesssim\ & \begin{cases} \beta_*^2Q\|\rho_{**}^{\perp}\|\left(\sum_{i=1}^{\nu}\left\langle \Psi_i^{3p-3-2\sigma},\mathcal{U}_i^2\right\rangle_{L^2(\mathcal{B}_i)}\right)^{\frac{1}{2}}, & p\geq\frac{7}{3}, \\ \beta_*^2Q\|\rho_{**}^{\perp}\|\left(\sum_{i=1}^{\nu}\left\langle \Psi_i^{6p-10},\mathcal{U}_i^2\right\rangle_{L^2(\mathcal{B}_i)}\right)^{\frac{1}{2}}+o(\beta_*^2Q), & 1<p<\frac{7}{3} \end{cases} \\ \lesssim\ & \beta_*^2Q^{\frac{1}{2}+\sigma}\|\rho_{**}^{\perp}\| \\ =\ & \mathcal{O}\left(\beta_*^4\|\rho_{**}^{\perp}\|+\|f\|_{H^{-1}}\|\rho_{**}^{\perp}\|\right)+o\left(\|\rho_{**}^{\perp}\|^2\right). \end{aligned}$$

By Lemma 4.3, (ii) and (iii) of (1) of Proposition 4.1 and Propositions 5.1 and 6.1, we have

$$\begin{aligned} & \beta_*\left\langle \gamma_*+\gamma_{\mathcal{N},led},\mathcal{U}_*\rho_{**}^{\perp}\right\rangle_{L^2} \\ \lesssim\ & \beta_*^3\|\rho_{**}^{\perp}\|\sum_{j=1}^{\nu}\left(\left\|\Psi_j^{\frac{3(p-1)-2\sigma}{2}}\mathcal{U}_j\right\|_{L^2(\mathcal{B}_j)}\right)+\|\rho_{**}^{\perp}\|^{2+\sigma}+\|f\|_{H^{-1}}\|\rho_{**}^{\perp}\| \\ & +\beta_*^4Q^{\frac{1}{2}+\sigma}\|\rho_{**}^{\perp}\| \\ =\ & \mathcal{O}\left(\beta_*^4\|\rho_{**}^{\perp}\|+\|f\|_{H^{-1}}\|\rho_{**}^{\perp}\|\right)+o\left(\|\rho_{**}^{\perp}\|^2\right). \end{aligned}$$

Summarizing the above estimates, we have

$$\left\langle \beta_*\rho_*\mathcal{U}_*,\rho_{**}^{\perp}\right\rangle_{L^2}=\mathcal{O}\left(\beta_*^4\|\rho_{**}^{\perp}\|+\|f\|_{H^{-1}}\|\rho_{**}^{\perp}\|\right)+o\left(\|\rho_{**}^{\perp}\|^2\right).$$

Step. 3 The estimates of

$$\sum_{j=1}^{\nu}\left\langle \left(\Psi_j^*\right)^{p-2}\mathcal{V}_j(\rho_*-\gamma_{1,ex}-\gamma_{\mathcal{N},led,j})\chi_{\mathcal{B}_j},\rho_{**}^{\perp}\right\rangle_{L^2}$$

and $\sum_{j=1}^{\nu}\left\langle \left(\Psi_j^*\right)^{p-3}\mathcal{V}_j^2\rho_*\chi_{\mathcal{B}_j},\rho_{**}^{\perp}\right\rangle_{L^2}$.

By (1) of Proposition 4.1,

$$\begin{aligned}
&\sum_{j=1}^{\nu}\left\langle\left(2A_{p,1}\left(\Psi_j^*\right)^{p-2}\mathcal{V}_j(\rho_*-\gamma_{1,ex}-\gamma_{\mathcal{N},led,j})+3A_{p,2}\left(\Psi_j^*\right)^{p-3}\mathcal{V}_j^2\rho_*\right)\chi_{\mathcal{B}_j},\rho_{**}^{\perp}\right\rangle_{L^2}\\
=\ &\left\langle\mathcal{V}_*\rho_0,\rho_{**}^{\perp}\right\rangle_{L^2}-\sum_{j=1}^{\nu}\left\langle 2A_{p,1}\left(\Psi_j^*\right)^{p-2}\mathcal{V}_j(\gamma_{1,ex}+\gamma_{\mathcal{N},led,j})\chi_{\mathcal{B}_j},\rho_{**}^{\perp}\right\rangle_{L^2}+o\left(\|\rho_{**}^{\perp}\|^2\right).
\end{aligned}$$

where

$$\mathcal{V}_*=\sum_{j=1}^{\nu}\left(2A_{p,1}\left(\Psi_j^*\right)^{p-2}\mathcal{V}_j+3A_{p,2}\left(\Psi_j^*\right)^{p-3}\mathcal{V}_j^2\right)\chi_{\mathcal{B}_j}.$$

By Lemma 4.3, (i) of (1) of Proposition 4.1 and Propositions 5.1 and 6.1, we have

$$\begin{aligned}
\left|\left\langle\gamma_{2,ex},\mathcal{V}_*\rho_{**}^{\perp}\right\rangle_{L^2}\right| &\lesssim \begin{cases}\beta_*^2\|\rho_{**}^{\perp}\|\left(\sum_{j=1}^{\nu}Q_j\left\|\Psi_j^{\frac{3p-1-2\sigma}{2}}\right\|_{L^2(\mathcal{B}_j)}\right), & p\geq\frac{7}{3},\\ \beta_*^2\|\rho_{**}^{\perp}\|\left(\sum_{j=1}^{\nu-1}Q_j\left\|\Psi_j^{3p-4}\right\|_{L^2(\mathcal{B}_j)}+Q\right), & 1<p<\frac{7}{3}\end{cases}\\
&\lesssim \beta_*^2Q^{\frac{1}{2}+\sigma}\|\rho_{**}^{\perp}\|\\
&= \mathcal{O}\left(\beta_*^4\|\rho_{**}^{\perp}\|+\|f\|_{H^{-1}}\|\rho_{**}^{\perp}\|\right)+o\left(\|\rho_{**}^{\perp}\|^2\right).
\end{aligned}$$

By the orthogonal conditions of $\rho_{**}^{\perp}$ given in (2) of Proposition 4.1, Lemmas 4.5, 4.8 and 4.10, (iii) of (1) of Proposition 4.1 and Propositions 5.1 and 6.1 and Lemma 7.1 that

$$\begin{aligned}
&\sum_{j=1}^{\nu}2A_{p,1}\left\langle\left(\Psi_j^*\right)^{p-2}\mathcal{V}_j(\sum_{l=1}^{\nu}\gamma_{1,l}+\gamma_{\mathcal{N},led}-\gamma_{\mathcal{N},led,j})\chi_{\mathcal{B}_j},\rho_{**}^{\perp}\right\rangle_{L^2}\\
&+\sum_{j=1}^{\nu}3A_{p,2}\left\langle\left(\Psi_j^*\right)^{p-3}\mathcal{V}_j^2\left(\sum_{l=1}^{\nu}\gamma_{1,l}+\gamma_{\mathcal{N},led}\right)\chi_{\mathcal{B}_j},\rho_{**}^{\perp}\right\rangle_{L^2}\\
\lesssim\ &\sum_{j=1}^{\nu}\left|\left\langle\widetilde{\gamma}_{1,*}+\gamma_{\mathcal{N},led,rem,j,*}+\rho_{**,2}^{\perp}+\sum_{l=1;l\neq j}^{\nu}\left(\alpha_l^{***}-\alpha_{l,1}^{**}\right)\Psi_l,\Psi_j^{p-2}\mathcal{V}_j\rho_{**}^{\perp}\right\rangle_{L^2(\mathcal{B}_j)}\right|\\
&+\left|\left\langle\left(\alpha_j^{***}-\alpha_{j,1}^{**}\right)\Psi_j,\Psi_j^{p-2}\mathcal{V}_j\rho_{**}^{\perp}\right\rangle_{L^2(\mathcal{C}\backslash\mathcal{B}_j)}\right|+\beta_*^2\|\rho_{**}^{\perp}\|\left(\beta_*^2+\|\rho_{**}^{\perp}\|^{1+\sigma}+\|f\|_{H^{-1}}\right)\\
\lesssim\ &Q^{\frac{1}{2}+\sigma}\left(\beta_*^2+\|\|f\|_{H^{-1}}+\|\rho_{**}^{\perp}\|^{1+\sigma}\right)\|\rho_{**}^{\perp}\|\\
=\ &\mathcal{O}\left(\beta_*^4\|\rho_{**}^{\perp}\|+\|f\|_{H^{-1}}\|\rho_{**}^{\perp}\|\right)+o\left(\|\rho_{**}^{\perp}\|^2\right).
\end{aligned}$$

Summarizing the above estimates, we have

$$\begin{aligned}
&\left|\sum_{j=1}^{\nu}\left\langle\left(2A_{p,1}\left(\Psi_j^*\right)^{p-2}\mathcal{V}_j(\rho_*-\gamma_{1,ex}-\gamma_{\mathcal{N},led,j})+3A_{p,2}\left(\Psi_j^*\right)^{p-3}\mathcal{V}_j^2\rho_*\right)\chi_{\mathcal{B}_j},\rho_{**}^{\perp}\right\rangle_{L^2}\right|\\
=\ &\mathcal{O}\left(\beta_*^4\|\rho_{**}^{\perp}\|+\|f\|_{H^{-1}}\|\rho_{**}^{\perp}\|\right)+o\left(\|\rho_{**}^{\perp}\|^2\right).
\end{aligned}$$

Step. 4 The estimate of $\left\langle\beta_*^2\mathcal{U}^{2(p-1)}\gamma_{1,ex}\chi_{\mathcal{C}\backslash\mathcal{B}_*},\rho_{**}^{\perp}\right\rangle_{L^2}$.

By (i) of (1) of Proposition 4.1 and Proposition 6.1,

$$\begin{aligned}\left|\left\langle\beta_*^2\mathcal{U}^{2(p-1)}\gamma_{1,ex}\chi_{\mathcal{C}\backslash\mathcal{B}_*},\rho_{**}^\perp\right\rangle_{L^2}\right| &\lesssim \beta_*^2Q^{\frac{1}{2}+\sigma}\|\rho_{**}^\perp\|\\ &= \mathcal{O}\left(\beta_*^4\|\rho_{**}^\perp\|+\|f\|_{H^{-1}}\|\rho_{**}^\perp\|\right)+o\left(\|\rho_{**}^\perp\|^2\right).\end{aligned}$$

Step. 5 The estimate of $\langle\overline{\mathcal{N}}_{rem}+\mathcal{N}_0-\mathcal{R}_{3,ex}-\mathcal{R}_{5,ex},\rho_{**}^\perp\rangle_{L^2}$.
By (1) of Proposition 4.1 and Lemmas 4.11, 4.13 and 7.2, we have

$$\begin{aligned}&\overline{\mathcal{N}}_{rem}+\mathcal{N}_0-\mathcal{R}_{3,ex}-\mathcal{R}_{5,ex}\\ =&\sum_{l=2}^{n_0}A_{p,l-1}\mathcal{U}^{p-l}\left(\gamma_{1,ex}^l-\left(\overline{\gamma}_{1,ex}+\sum_{i=0}^{n_0-1}\rho_{**,3,i,1}^\perp\right)^l\right)\\ &+\sum_{l=2}^{n_0}lA_{p,l-1}\mathcal{U}^{p-l}\gamma_{1,ex}^{l-1}\left(\rho_{5,**,n_0}^\perp+\gamma_*+\gamma_{\mathcal{N},led}\right)\\ &+\sum_{l=2}^{n_0}\sum_{k=2}^{l}C_l^kA_{p,l-1}\mathcal{U}^{p-l}\gamma_{1,ex}^{l-k}\left(\mathcal{V}+\gamma_{2,ex}+\gamma_*+\gamma_{\mathcal{N},led}\right)^k\\ &+\mathcal{O}\left(\beta_*^4+\|\rho_{**}^\perp\|^{1+\sigma}+\|f\|_{H^{-1}}+Q\right)+o(\rho_{**}^\perp)\\ &+\mathcal{O}\left(\chi_{p\geq2}|\rho_{**}^\perp|^2+|\rho_{**}^\perp|^p+|\rho_{**}^\perp|^{1+\sigma}+|\gamma_*+\gamma_{\mathcal{N},led}|^{1+\sigma}\chi_{\mathcal{C}\backslash\widetilde{\mathcal{B}}_{**}}\right),\end{aligned}\tag{7.4}$$ eqnnewnew19991

where $C_l^k=\frac{l!}{k!(l-k)!}$. Thus, similar to (5.15), by (ii) and (iii) of (1) of Proposition 4.1 and Propositions 5.1 and 6.1,

$$\begin{aligned}&\langle\overline{\mathcal{N}}_{rem}+\mathcal{N}_0-\mathcal{R}_{3,ex}-\mathcal{R}_{5,ex},\rho_{**}^\perp\rangle_{L^2}\\ =&\left\langle\sum_{l=2}^{n_0}A_{p,l-1}\mathcal{U}^{p-l}\left(\gamma_{1,ex}^l-\left(\overline{\gamma}_{1,ex}+\sum_{i=0}^{n_0-1}\rho_{**,3,i,1}^\perp\right)^l\right),\rho_{**}^\perp\right\rangle_{L^2}\\ &+\left\langle\sum_{l=2}^{n_0}lA_{p,l-1}\mathcal{U}^{p-l}\gamma_{1,ex}^{l-1}\left(\rho_{5,**,n_0}^\perp+\gamma_*+\gamma_{\mathcal{N},led}\right),\rho_{**}^\perp\right\rangle_{L^2}\\ &+\left\langle\sum_{l=2}^{n_0}\sum_{k=2}^{l}C_l^kA_{p,l-1}\mathcal{U}^{p-l}\gamma_{1,ex}^{l-k}\left(\mathcal{V}+\gamma_{2,ex}+\gamma_*+\gamma_{\mathcal{N},led}\right)^k,\rho_{**}^\perp\right\rangle_{L^2}\\ &+\mathcal{O}\left(\left(\|f\|_{H^{-1}}+\beta_*^4\right)\|\rho_{**}^\perp\|\right)+o(\|\rho_{**}^\perp\|^2).\end{aligned}$$

By Lemma 4.13, the choice of n_0 , (1) of Proposition 4.1 and Proposition 6.1,

$$\begin{aligned}\left|\left\langle\sum_{l=2}^{n_0}lA_{p,l-1}\mathcal{U}^{p-l}\gamma_{1,ex}^{l-1}\rho_{5,**,n_0}^\perp,\rho_{**}^\perp\right\rangle_{L^2}\right| &\lesssim \left\langle\mathcal{U}^{p-2}\left|\gamma_{1,ex}\rho_{5,**,n_0}^\perp\right|,\left|\rho_{**}^\perp\right|\right\rangle_{L^2}\\ &\lesssim Q^{n_0(p-1)}\|\rho_{**}^\perp\|\\ &= \mathcal{O}\left(\beta_*^4+\|f\|_{H^{-1}}\right)\|\rho_{**}^\perp\|+o\left(\|\rho_{**}^\perp\|^2\right).\end{aligned}$$

By (1) of Proposition 4.1 and Propositions 5.1 and 6.1,

$$\begin{aligned}&\left|\left\langle \sum_{l=2}^{n_0} lA_{p,l-1}\mathcal{U}^{p-l}\gamma_{1,ex}^{l-1}\left(\gamma_* + \gamma_{\mathcal{N},led}\right), \rho_{**}^{\perp}\right\rangle_{L^2}\right| \\ \lesssim\ & Q^{\frac{1}{2}+\sigma}\left(\beta_*^2 + \|\rho_{**}^{\perp}\|^{1+\sigma} + \|f\|_{H^{-1}}\right)\|\rho_{**}^{\perp}\| \\ =\ & \mathcal{O}\left(\left(\|f\|_{H^{-1}} + \beta_*^4\right)\|\rho_{**}^{\perp}\|\right) + o(\|\rho_{**}^{\perp}\|^2).\end{aligned}$$

By (1) of Proposition 4.1 and Propositions 5.1 and 6.1,

$$\begin{aligned}&\left|\left\langle \sum_{l=2}^{n_0}\sum_{k=2}^{l} C_l^k A_{p,l-1}\mathcal{U}^{p-l}\gamma_{1,ex}^{l-k}\left(\mathcal{V} + \gamma_* + \gamma_{\mathcal{N},led}\right)^k, \rho_{**}^{\perp}\right\rangle_{L^2}\right| \\ \lesssim\ & \sum_{k=2}^{n_0}\sum_{l=k}^{n_0} C_l^k Q^{\frac{1}{2}+(1+l-k)\sigma}\left(\beta_* + \|\rho_{**}^{\perp}\|^{1+\sigma} + \|f\|_{H^{-1}}\right)^k\|\rho_{**}^{\perp}\| \\ \lesssim\ & \sum_{k=2}^{n_0} Q^{\frac{1}{2}+\sigma}\left(\beta_* + \|\rho_{**}^{\perp}\|^{1+\sigma} + \|f\|_{H^{-1}}\right)^k\|\rho_{**}^{\perp}\| \\ =\ & \mathcal{O}\left(\left(\|f\|_{H^{-1}} + \beta_*^4\right)\|\rho_{**}^{\perp}\|\right) + o(\|\rho_{**}^{\perp}\|^2)\end{aligned}$$

and

$$\begin{aligned}\left|\left\langle \sum_{l=2}^{n_0}\sum_{k=2}^{l} C_l^k A_{p,l-1}\mathcal{U}^{p-l}\gamma_{1,ex}^{l-k}\gamma_{2,ex}^k, \rho_{**}^{\perp}\right\rangle_{L^2}\right| &\lesssim \sum_{k=2}^{n_0}\sum_{l=k}^{n_0} C_l^k Q^{\frac{1}{2}+(1+l+k)\sigma}\beta_*^k\|\rho_{**}^{\perp}\| \\ &\lesssim \sum_{k=2}^{n_0} Q^{\frac{1}{2}+\sigma}\beta_*^k\|\rho_{**}^{\perp}\| \\ &= \mathcal{O}\left(\|f\|_{H^{-1}} + \beta_*^4\right)\|\rho_{**}^{\perp}\| + o(\|\rho_{**}^{\perp}\|^2).\end{aligned}$$

By (i) of (1) of Proposition 4.1 and Proposition 6.1,

$$\begin{aligned}&\left|\left\langle \sum_{l=2}^{n_0} A_{p,l-1}\mathcal{U}^{p-l}\left(\gamma_{1,ex}^l - \left(\overline{\gamma}_{1,ex} + \sum_{i=0}^{n_0-1}\rho_{**,3,i,1}^{\perp}\right)^l\right), \rho_{**}^{\perp}\right\rangle_{L^2}\right| \\ \lesssim\ & \left\langle \mathcal{U}^{p-2}\overline{\Psi}_*^2 Q^{((p-1)\wedge 1)(n_0+1)}, |\rho_{**}^{\perp}|\right\rangle_{L^2} \\ =\ & \mathcal{O}\left(\|f\|_{H^{-1}} + \beta_*^4\right)\|\rho_{**}^{\perp}\| + o(\|\rho_{**}^{\perp}\|^2),\end{aligned}$$

where we use the notation $\overline{\Psi}_*$ to denote the barrier used in the norms $\|\rho_{**,3}^{\perp}\|_{\sharp}$ for $p \geq 3$ and $\|\rho_{**,3}^{\perp}\|_{\natural,1}$ for $1 < p < 3$.

Summarizing the above estimates, we have

$$\left\langle \overline{\mathcal{N}}_{rem} + \mathcal{N}_0 - \mathcal{R}_{3,ex} - \mathcal{R}_{5,ex}, \rho_{**}^{\perp}\right\rangle_{L^2} = \mathcal{O}\left(\left(\|f\|_{H^{-1}} + \beta_*^4\right)\|\rho_{**}^{\perp}\|\right) + o(\|\rho_{**}^{\perp}\|^2).$$

The conclusion then follows from the estimates in Step. 1 to Step. 5. □

8. Estimate of β_* and proof of (1) of Theorem 1.3

By multiplying (3.16) with $\mathcal{V}_j$ on both sides and integrating by parts, the orthogonal conditions of ρ_* and the oddness of $\{\mathcal{V}_i\}$ on $\mathbb{S}^{d-1}$, we have

$$\begin{aligned}-\sum_{j=1}^{\nu}\langle f,\mathcal{V}_j\rangle_{H^1} &= \sum_{j=1}^{\nu}\langle\mathcal{R}_{2,j},\mathcal{V}_j\rangle_{L^2}+\sum_{j=1}^{\nu}\langle\mathcal{N},\mathcal{V}_j\rangle_{L^2}+\sum_{j=1}^{\nu}\langle\mathcal{L}_{j,ex}(\rho_*),\mathcal{V}_j\rangle_{L^2}\\ &\quad+\sum_{j=1}^{\nu}\sum_{i=1;i\neq j}^{\nu}\langle\mathcal{R}_{2,i},\mathcal{V}_j\rangle_{L^2}+\sum_{j=1}^{\nu}\langle\mathcal{R}_{2,ex},\mathcal{V}_j\rangle_{L^2} \qquad (8.1)\end{aligned}$$ eqn0023

for all $j=1,2,\cdots,\nu$.

⟨propn0004⟩ **Proposition 8.1.** *Let $d\geq 2$, $a<0$ and $b=b_{FS}(a)$. Then we have*

$$\sum_{j=1}^{\nu}\Big(p\left((\alpha_j^*)^{p-1}-1\right)\left\|\Psi_j^{p-1}\mathcal{V}_j^2\right\|_{L^1}+\langle\mathcal{N}_j,\mathcal{V}_j\rangle_{L^2}+\langle f,\mathcal{V}_j\rangle\Big)=o(\beta_*^4)+\mathcal{O}\left(\beta_*\|f\|_{H^{-1}}\right),$$

where $\mathcal{N}_j$ is given by (5.8).

Proof. By the oddness of $\{\mathcal{V}_i\}$ on $\mathbb{S}^{d-1}$, we have

$$\begin{aligned}\langle\mathcal{N}_j,\mathcal{V}_j\rangle_{L^2} &= A_{p,2}\left\langle\left(\Psi_j^*\right)^{p-3},\mathcal{V}_j^4\right\rangle_{L^2}+3A_{p,2}\left\langle\left(\Psi_j^*\right)^{p-3}\mathcal{V}_j^3,\rho_*\right\rangle_{L^2}\\ &\quad+2A_{p,1}\left\langle\left(\Psi_j^*\right)^{p-2}\mathcal{V}_j^2,\rho_*\right\rangle_{L^2} \qquad (8.2)\end{aligned}$$ eqn0027

for all $1\leq j\leq\nu$, where $\mathcal{N}_j$ is given by (5.8). By (8.1) and (8.2), we have

$$\begin{aligned}-\sum_{j=1}^{\nu}\langle f,\mathcal{V}_j\rangle_{H^1} &= \sum_{j=1}^{\nu}\langle\mathcal{R}_{2,j},\mathcal{V}_j\rangle_{L^2}+\sum_{j=1}^{\nu}\langle\mathcal{N}_j,\mathcal{V}_j\rangle_{L^2}+\sum_{j=1}^{\nu}\langle\mathcal{L}_{j,ex}(\rho_*),\mathcal{V}_j\rangle_{L^2}\\ &\quad+\sum_{j=1}^{\nu}\langle\mathcal{R}_{2,ex},\mathcal{V}_j\rangle_{L^2}+\sum_{j=1}^{\nu}\langle\mathcal{N}-\mathcal{N}_j,\mathcal{V}_j\rangle_{L^2}\\ &\quad+\sum_{j=1}^{\nu}\sum_{i=1;i\neq j}^{\nu}\langle\mathcal{R}_{2,i},\mathcal{V}_j\rangle_{L^2}. \qquad (8.3)\end{aligned}$$ eqn0029

As in the proof of Proposition 4.1, the rest of the proof is to estimate every term in (8.3).

Step. 1 The estimate of $\sum_{j=1}^{\nu}\langle\mathcal{R}_{2,j},\mathcal{V}_j\rangle_{L^2}$.
By (3.19),

$$\sum_{j=1}^{\nu}\langle\mathcal{R}_{2,j},\mathcal{V}_j\rangle_{L^2}=\sum_{j=1}^{\nu}p\left((\alpha_j^*)^{p-1}-1\right)\left\|\Psi_j^{p-1}\mathcal{V}_j^2\right\|_{L^1}.$$

Step. 2 The estimate of $\sum_{i=1;i\neq j}^{\nu}\langle\mathcal{R}_{2,i},\mathcal{V}_j\rangle_{L^2}$.
By (3.19), Lemma 4.3 and Propositions 5.1 and 6.1,

$$\begin{aligned}\left|\sum_{i=1;i\neq j}^{\nu}\langle\mathcal{R}_{2,i},\mathcal{V}_j\rangle_{L^2}\right| &\lesssim \sum_{i=1;i\neq j}^{\nu}\beta_*^2\left|(\alpha_i^*)^{p-1}-1\right|\left\langle\Psi_i^{\frac{3p-1}{2}},\Psi_j^{\frac{p+1}{2}}\right\rangle_{L^2}\\ &= \mathcal{O}\left(\beta_*^2\left(\beta_*^2+\|\rho_{**}^{\perp}\|^{1+\sigma}+\|f\|_{H^{-1}}\right)^2\right)\\ &= o(\beta_*^4+\|\rho_{**}^{\perp}\|^2)+\mathcal{O}\left(\beta_*\|f\|_{H^{-1}}\right).\end{aligned}$$

Step. 3 The estimate of $\langle \mathcal{R}_{2,ex}, \mathcal{V}_j\rangle_{L^2}$.

By (4.33), Lemma 4.3 and Propositions 5.1 and 6.1,

$$\left|\langle \mathcal{R}_{2,ex}, \mathcal{V}_j\rangle_{L^2}\right| \lesssim \sum_{i=1}^{\nu} \beta_*^2 \left\langle \Psi_i^{2p-1}, \mathcal{U}_i\right\rangle_{L^2(\mathcal{B}_i)} = o(\beta_*^4 + \|\rho_{**}^{\perp}\|^2) + \mathcal{O}\left(\beta_*\|f\|_{H^{-1}}\right),$$

where $\mathcal{B}_i$ is given by (4.7).

Step. 4 The estimate of $\langle \mathcal{L}_{j,ex}(\rho_*), \mathcal{V}_j\rangle_{L^2}$.

By (3.17) and (i) and (ii) of (1) of Proposition 4.1,

$$\begin{aligned}\langle \mathcal{L}_{j,ex}(\rho_*), \mathcal{V}_j\rangle_{L^2} &= p\left\langle \left(\mathcal{U}^{p-1} - \left(\Psi_j^*\right)^{p-1}\right)\rho_*, \mathcal{V}_j\right\rangle_{L^2} \\ &= \mathcal{O}\left(\langle \beta_*\mathcal{U}_{**}, \gamma_{2,ex} + \gamma_{\mathcal{N},led} + \rho_{**}^{\perp}\rangle_{L^2}\right),\end{aligned}$$

where $\mathcal{U}_{**} = \sum_{j=1}^{\nu} \Psi_j^{\frac{3(p-1)}{2}}\mathcal{U}_j\chi_{\mathcal{B}_j} + \mathcal{U}^{\frac{3p-1}{2}}\chi_{\mathcal{C}\setminus\mathcal{B}_*}$ with $\mathcal{B}_*$ given by (4.7). By Lemma 4.3, (i) of (1) of Proposition 4.1 and Proposition 6.1,

$$\begin{aligned}\beta_*\left\langle \mathcal{U}_{**}, \gamma_{2,ex}\right\rangle_{L^2} &\lesssim \begin{cases} \sum_{j=1}^{\nu} \beta_*^2 Q\left\langle \Psi_j^{\frac{3p-1-2\sigma}{2}}, \mathcal{U}_j\right\rangle_{L^2(\mathcal{B}_j)}, & p \geq \frac{7}{3}, \\ \sum_{j=1}^{\nu} \beta_*^2 Q\left\langle \Psi_j^{3p-4}, \mathcal{U}_j\right\rangle_{L^2(\mathcal{B}_j)}, & 1 < p < \frac{7}{3} \end{cases} \\ &= o(\beta_*^4 + \|\rho_{**}^{\perp}\|^2) + \mathcal{O}\left(\beta_*\|f\|_{H^{-1}}\right).\end{aligned}$$

By Lemma 4.3, (iii) of (1) of Proposition 4.1 and Proposition 6.1,

$$\begin{aligned}\beta_*\left\langle \mathcal{U}_{**}, \gamma_{\mathcal{N},led}\right\rangle_{L^2} &\lesssim \beta_*^3 Q + \begin{cases} \sum_{j=1}^{\nu} \beta_*^4 Q\left\langle \Psi_j^{\frac{3p-1-2\sigma}{2}}\chi_{\mathcal{B}_j}, \mathcal{U}_j\right\rangle_{L^2}, & p \geq 3, \\ \sum_{j=1}^{\nu} \beta_*^4 Q\left\langle \Psi_j^{\frac{5p-7}{2}}\chi_{\mathcal{B}_j}, \mathcal{U}_j\right\rangle_{L^2}, & 1 < p < 3 \end{cases} \\ &= o(\beta_*^4 + \|\rho_{**}^{\perp}\|^2) + \mathcal{O}\left(\beta_*\|f\|_{H^{-1}}\right).\end{aligned}$$

By Lemma 4.3 and Proposition 6.1,

$$\begin{aligned}\beta_*\left\langle \mathcal{U}_{**}, |\rho_{**}^{\perp}|\right\rangle_{L^2} &\lesssim \sum_{i=1}^{\nu} \beta_*\left\|\Psi_i^{\frac{3(p-1)}{2}}\mathcal{U}_i\right\|_{L^2(\mathcal{B}_j)}\|\rho_{**}^{\perp}\| \\ &= \mathcal{O}\left(\beta_* Q^{\frac{1}{2}+\sigma}\|\rho_{**}^{\perp}\|\right) \\ &= \mathcal{O}\left(\beta_*\left(\beta_*^{3+\sigma} + \|\rho_{**}^{\perp}\|^{\frac{1}{2}+\sigma} + \|f\|_{H^{-1}}^{\frac{1}{2}+\sigma}\right)\|\rho_{**}^{\perp}\|\right) \\ &= o(\beta_*^4 + \|\rho_{**}^{\perp}\|^2) + \mathcal{O}\left(\beta_*\|f\|_{H^{-1}}\right).\end{aligned}$$

Summarizing the above estimates, we have

$$\langle \mathcal{L}_{j,ex}(\rho_*), \mathcal{V}_j\rangle_{L^2} = o(\beta_*^4 + \|\rho_{**}^{\perp}\|^2) + \mathcal{O}\left(\beta_*\|f\|_{H^{-1}}\right).$$

Step. 5 The estimate of $\langle \mathcal{N} - \mathcal{N}_j, \mathcal{V}_j\rangle_{L^2}$.

Similar to (5.12), by the oddness of $w_{j,l}$ on $\mathbb{S}^{d-1}$,

$$\begin{aligned}\langle \mathcal{N}-\mathcal{N}_j,\mathcal{V}_j\rangle_{L^2} &= \sum_{i=1;i\neq j}^{\nu} A_{p,1}\left\langle (\Psi_i^*)^{p-2}\left(\mathcal{V}_i^2+2\mathcal{V}_i\rho_*\right)\chi_{\mathcal{B}_i},\mathcal{V}_j\right\rangle_{L^2}\\
&\quad+\sum_{i=1;i\neq j}^{\nu} A_{p,2}\left\langle (\Psi_i^*)^{p-3}\left(\mathcal{V}_i^3+3\mathcal{V}_i^2\rho_*\right)\chi_{\mathcal{B}_i},\mathcal{V}_j\right\rangle_{L^2}\\
&\quad+2A_{p,1}\left\langle \left(\Psi_j^*\right)^{p-2}\mathcal{V}_j\rho_*\chi_{\mathcal{C}\setminus\mathcal{B}_j},\mathcal{V}_j\right\rangle_{L^2}\\
&\quad+A_{p,2}\left\langle \left(\Psi_j^*\right)^{p-3}\left(\mathcal{V}_j^3+3\mathcal{V}_j^2\rho_*\right)\chi_{\mathcal{C}\setminus\mathcal{B}_j},\mathcal{V}_j\right\rangle_{L^2}\\
&\quad+\sum_{i=1}^{\nu}\left\langle \mathcal{O}\left(\beta_*|\rho_*|\Psi_i^{\frac{3p-5}{2}}\mathcal{U}_i+\beta_*^2\Psi_i^{2p-2}\mathcal{U}_i\right)\chi_{\mathcal{B}_i},\mathcal{V}_j\right\rangle_{L^2}\\
&\quad+\left\langle \mathcal{O}\left(\mathcal{U}^{p-2}\mathcal{V}^2+\beta_*|\rho_*|\mathcal{U}^{\frac{3(p-1)}{2}}\right)\chi_{\mathcal{C}\setminus\mathcal{B}_*},\mathcal{V}_j\right\rangle_{L^2}\\
&\quad+\langle\overline{\mathcal{N}}_{rem},\mathcal{V}_j\rangle_{L^2}+\langle\mathcal{N}_0,\mathcal{V}_j\rangle_{L^2}.\end{aligned}$$

Step. 5.1 The estimate of $\sum_{i=1;i\neq j}^{\nu}\left\langle (\Psi_i^*)^{p-2}\mathcal{V}_i^2\chi_{\mathcal{B}_i},\mathcal{V}_j\right\rangle_{L^2}$.
By Lemma 4.3 and Proposition 6.1,

$$\begin{aligned}\left|\sum_{i=1;i\neq j}^{\nu}\left\langle (\Psi_i^*)^{p-2}\mathcal{V}_i^2\chi_{\mathcal{B}_i},\mathcal{V}_j\right\rangle_{L^2}\right| &\lesssim \beta_*^3\left\langle \Psi_i^{2p-1},\Psi_j^{\frac{p+1}{2}}\right\rangle_{L^2}\\
&= o(\beta_*^4+\|\rho_{**}^{\perp}\|^2)+\mathcal{O}\left(\beta_*\|f\|_{H^{-1}}\right).\end{aligned}$$

Step. 5.2 The estimate of $\sum_{i=1;i\neq j}^{\nu}\left\langle (\Psi_i^*)^{p-3}\mathcal{V}_i^3\chi_{\mathcal{B}_i},\mathcal{V}_j\right\rangle_{L^2}$.
By (4.2), we also have

$$\begin{aligned}\left|\sum_{i=1;i\neq j}^{\nu}\left\langle (\Psi_i^*)^{p-3}\mathcal{V}_i^3\chi_{\mathcal{B}_i},\mathcal{V}_j\right\rangle_{L^2}\right| &\lesssim \left|\sum_{i=1;i\neq j}^{\nu}\left\langle (\Psi_i^*)^{p-2}\mathcal{V}_i^2\chi_{\mathcal{B}_i},\mathcal{V}_j\right\rangle_{L^2}\right|\\
&= o(\beta_*^4+\|\rho_{**}^{\perp}\|^2)+\mathcal{O}\left(\beta_*\|f\|_{H^{-1}}\right).\end{aligned}$$

Step. 5.3 The estimate of $\sum_{i=1;i\neq j}^{\nu}\left\langle (\Psi_i^*)^{p-2}\mathcal{V}_i\rho_*\chi_{\mathcal{B}_i},\mathcal{V}_j\right\rangle_{L^2}$.
By (1) of Proposition 4.1,

$$\left\langle (\Psi_i^*)^{p-2}\mathcal{V}_i\rho_*\chi_{\mathcal{B}_i},\mathcal{V}_j\right\rangle_{L^2}=\left\langle (\Psi_i^*)^{p-2}\mathcal{V}_i\mathcal{V}_j\chi_{\mathcal{B}_i},\rho_{**}^{\perp}+\gamma_{ex}+\gamma_*+\gamma_{\mathcal{N},led}\right\rangle_{L^2}.$$

By Lemma 4.3, (i) of (1) of Proposition 4.1 and Proposition 6.1,

$$\begin{aligned}\left|\left\langle (\Psi_i^*)^{p-2}\mathcal{V}_i\mathcal{V}_j\chi_{\mathcal{B}_i},\gamma_{ex}\right\rangle_{L^2}\right| &\lesssim \begin{cases}\beta_*^2Q\left\langle \Psi_i^{\frac{3p-1-2\sigma}{2}},\Psi_j^{\frac{p+1}{2}}\right\rangle_{L^2(\mathcal{B}_i)}, & p\geq 3,\\ \beta_*^2Q\left\langle \Psi_i^{\frac{5p-7}{2}},\Psi_j^{\frac{p+1}{2}}\right\rangle_{L^2(\mathcal{B}_i)}, & 1<p<3\end{cases}\\
&= o(\beta_*^4+\|\rho_{**}^{\perp}\|^2)+\mathcal{O}\left(\beta_*\|f\|_{H^{-1}}\right).\end{aligned}$$

Similar to (5.10), by Lemma 4.3, (ii) and (iii) of (1) of Proposition 4.1 and Propositions 5.1 and 6.1,

$$\begin{aligned}\left|\left\langle (\Psi_i^*)^{p-2}\mathcal{V}_i\mathcal{V}_j\chi_{\mathcal{B}_i}, \gamma_*+\gamma_{\mathcal{N},led}\right\rangle_{L^2}\right| &\lesssim \beta_*^4\left\langle \Psi_i^{\frac{3p-1-2\sigma}{2}}, \Psi_j^{\frac{p+1}{2}}\right\rangle_{L^2(\mathcal{B}_i)} \\ &\quad +o(\beta_*^4+\|\rho_{**}^\perp\|^2)+\beta_*\|f\|_{H^{-1}} \\ &= o(\beta_*^4+\|\rho_{**}^\perp\|^2)+\mathcal{O}\left(\beta_*\|f\|_{H^{-1}}\right).\end{aligned}$$

By Lemma 4.3 and Proposition 6.1,

$$\begin{aligned}\left\langle (\Psi_i^*)^{p-2}\mathcal{V}_i\mathcal{V}_j\chi_{\mathcal{B}_i}, \rho_{**}^\perp\right\rangle_{L^2} &\lesssim \beta_*^2\|\rho_{**}^\perp\|\left(\left\langle \Psi_i^{3p-3}, \Psi_j^{p+1}\right\rangle_{L^2(\mathcal{B}_i)}\right)^{\frac{1}{2}} \\ &= o(\beta_*^4+\|\rho_{**}^\perp\|^2)+\mathcal{O}\left(\beta_*\|f\|_{H^{-1}}\right).\end{aligned}$$

Summarizing the above estimates, we have

$$\sum_{i=1;i\neq j}^{\nu}\left\langle (\Psi_i^*)^{p-2}\mathcal{V}_i\rho_*\chi_{\mathcal{B}_i}, \mathcal{V}_j\right\rangle_{L^2} = o(\beta_*^4+\|\rho_{**}^\perp\|^2)+\mathcal{O}\left(\beta_*\|f\|_{H^{-1}}\right).$$

Step. 5.4 The estimate of $\sum_{i=1;i\neq j}^{\nu}\left\langle (\Psi_i^*)^{p-3}\mathcal{V}_i^2\rho_*\chi_{\mathcal{B}_i}, \mathcal{V}_j\right\rangle_{L^2}$.

By (4.2), we have

$$\begin{aligned}\left|\sum_{i=1;i\neq j}^{\nu}\left\langle (\Psi_i^*)^{p-3}\mathcal{V}_i^2\rho_*\chi_{\mathcal{B}_i}, \mathcal{V}_j\right\rangle_{L^2}\right| &\lesssim \left|\sum_{i=1;i\neq j}^{\nu}\left\langle (\Psi_i^*)^{p-2}\mathcal{V}_i\rho_*\chi_{\mathcal{B}_i}, \mathcal{V}_j\right\rangle_{L^2}\right| \\ &= o(\beta_*^4+\|\rho_{**}^\perp\|^2)+\mathcal{O}\left(\beta_*\|f\|_{H^{-1}}\right).\end{aligned}$$

Step. 5.5 The estimate of $\left\langle \left(\Psi_j^*\right)^{p-2}\mathcal{V}_j\rho_*\chi_{\mathcal{C}\backslash\mathcal{B}_j}, \mathcal{V}_j\right\rangle_{L^2}$.

By (1) of Proposition 4.1,

$$\left\langle \left(\Psi_j^*\right)^{p-2}\mathcal{V}_j\rho_*\chi_{\mathcal{C}\backslash\mathcal{B}_j}, \mathcal{V}_j\right\rangle_{L^2} = \left\langle \left(\Psi_j^*\right)^{p-2}\mathcal{V}_j^2\chi_{\mathcal{C}\backslash\mathcal{B}_j}, \rho_{**}^\perp+\gamma_{1,ex}+\gamma_*+\gamma_{\mathcal{N},led}\right\rangle_{L^2}.$$

By Lemma 4.3, (i) of (1) of Proposition 4.1 and Proposition 6.1,

$$\begin{aligned}\left|\left\langle \left(\Psi_j^*\right)^{p-2}\mathcal{V}_j^2\chi_{\mathcal{C}\backslash\mathcal{B}_j}, \gamma_{1,ex}\right\rangle_{L^2}\right| &\lesssim \begin{cases}\sum_{i=1;i\neq j}^{\nu}\beta_*^2 Q\left\langle \Psi_i^{1-\sigma}, \Psi_j^{2p-1}\right\rangle_{L^2(\mathcal{B}_i)}, & p\geq 3,\\ \sum_{i=1;i\neq j}^{\nu}\beta_*^2 Q\left\langle \Psi_i^{p-2}, \Psi_j^{2p-1}\right\rangle_{L^2(\mathcal{B}_i)}, & 1<p<3\end{cases} \\ &= o(\beta_*^4+\|\rho_{**}^\perp\|^2)+\mathcal{O}\left(\beta_*\|f\|_{H^{-1}}\right).\end{aligned}$$

Similar to (5.10), by Lemma 4.3, (ii) and (iii) of (1) of Proposition 4.1 and Propositions 5.1 and 6.1,

$$\begin{aligned}\left\langle \left(\Psi_j^*\right)^{p-2}\mathcal{V}_j^2\chi_{\mathcal{C}\backslash\mathcal{B}_j}, \gamma_*+\gamma_{\mathcal{N},led}\right\rangle_{L^2} &\lesssim \sum_{i=1;i\neq j}^{\nu}\beta_*^4\left\langle \Psi_i^{1-\sigma}, \Psi_j^{2p-1}\right\rangle_{L^2(\mathcal{B}_i)} \\ &\quad +o(\beta_*^4+\|\rho_{**}^\perp\|^2)+\beta_*\|f\|_{H^{-1}} \\ &= o(\beta_*^4+\|\rho_{**}^\perp\|^2)+\mathcal{O}\left(\beta_*\|f\|_{H^{-1}}\right).\end{aligned}$$

By Lemma 4.3 and Proposition 6.1,

$$\begin{aligned}\left|\left\langle \left(\Psi_j^*\right)^{p-2}\mathcal{V}_j^2\chi_{\mathcal{C}\backslash\mathcal{B}_j}, \rho_{**}^\perp\right\rangle_{L^2}\right| &\lesssim \beta_*^2\|\rho_{**}^\perp\|\left(\int_{\mathcal{C}\backslash\mathcal{B}_j}\Psi_j^{4p-2}d\mu\right)^{\frac{1}{2}}\\ &= o(\beta_*^4+\|\rho_{**}^\perp\|^2)+\mathcal{O}\left(\beta_*\|f\|_{H^{-1}}\right).\end{aligned}$$

Summarizing the above estimates, we have

$$\left\langle \left(\Psi_j^*\right)^{p-2}\mathcal{V}_j\rho_*\chi_{\mathcal{C}\backslash\mathcal{B}_j}, \mathcal{V}_j\right\rangle_{L^2} = o(\beta_*^4+\|\rho_{**}^\perp\|^2)+\mathcal{O}\left(\beta_*\|f\|_{H^{-1}}\right).$$

Step. 5.6 The estimate of $\left\langle \left(\Psi_j^*\right)^{p-3}\left(\mathcal{V}_j^3+3\mathcal{V}_j^2\rho_*\right)\chi_{\mathcal{C}\backslash\mathcal{B}_j}, \mathcal{V}_j\right\rangle_{L^2}$.

By Lemma 4.3 and (1) of Proposition 4.1,

$$\begin{aligned}\left\langle \left(\Psi_j^*\right)^{p-3}\left(\mathcal{V}_j^3+3\mathcal{V}_j^2\rho_*\right)\chi_{\mathcal{C}\backslash\mathcal{B}_j}, \mathcal{V}_j\right\rangle_{L^2} &= 3\left\langle \left(\Psi_j^*\right)^{p-3}\mathcal{V}_j^3\chi_{\mathcal{C}\backslash\mathcal{B}_j}, \gamma_{2,ex}+\gamma_{\mathcal{N},led}\right\rangle_{L^2}\\ &\quad 3\left\langle \left(\Psi_j^*\right)^{p-3}\mathcal{V}_j^3\chi_{\mathcal{C}\backslash\mathcal{B}_j}, \rho_{**}^\perp\right\rangle_{L^2}+o(\beta_*^4).\end{aligned}$$

By Lemma 4.3 and (i) of (1) of Proposition 4.1, $\left|\left\langle \left(\Psi_j^*\right)^{p-3}\mathcal{V}_j^3\chi_{\mathcal{C}\backslash\mathcal{B}_j}, \gamma_{2,ex}\right\rangle_{L^2}\right| = o(\beta_*^4)$. By Lemma 4.3 and (iii) of (1) of Proposition 4.1,

$$\left|\left\langle \left(\Psi_j^*\right)^{p-3}\mathcal{V}_j^3\chi_{\mathcal{C}\backslash\mathcal{B}_j}, \gamma_{\mathcal{N},led}\right\rangle_{L^2}\right| = o(\beta_*^4).$$

By Lemma 4.3 and Proposition 6.1,

$$\begin{aligned}\left|\left\langle \left(\Psi_j^*\right)^{p-3}\mathcal{V}_j^3\chi_{\mathcal{C}\backslash\mathcal{B}_j}, \rho_{**}^\perp\right\rangle_{L^2}\right| &\lesssim \beta_*^3\|\rho_{**}^\perp\|\left(\int_{\mathcal{C}\backslash\mathcal{B}_j}\Psi_j^{5p-3}d\mu\right)^{\frac{1}{2}}\\ &= o(\beta_*^4+\|\rho_{**}^\perp\|^2)+\mathcal{O}\left(\beta_*\|f\|_{H^{-1}}\right).\end{aligned}$$

Summarizing the above estimates, we have

$$\left\langle \left(\Psi_j^*\right)^{p-3}\left(\mathcal{V}_j^3+3\mathcal{V}_j^2\rho_*\right)\chi_{\mathcal{C}\backslash\mathcal{B}_j}, \mathcal{V}_j\right\rangle_{L^2} = o(\beta_*^4+\|\rho_{**}^\perp\|^2)+\mathcal{O}\left(\beta_*\|f\|_{H^{-1}}\right).$$

Step. 5.7 The estimate of $\sum_{i=1}^{\nu}\left\langle \beta_*\rho_*\Psi_i^{\frac{3p-5}{2}}\mathcal{U}_i\chi_{\mathcal{B}_i}, \mathcal{V}_j\right\rangle_{L^2}$.

By (1) of Proposition 4.1 and the oddness of $\mathcal{V}_j$ on $\mathbb{S}^{d-1}$,

$$\left|\left\langle \beta_*\rho_*\Psi_i^{\frac{3p-5}{2}}\mathcal{U}_i\chi_{\mathcal{B}_i}, \mathcal{V}_j\right\rangle_{L^2}\right| \lesssim \beta_*^2\left\langle \Psi_i^{2p-2}\mathcal{U}_i\chi_{\mathcal{B}_i}, \left|\rho_{**}^\perp+\gamma_{2,ex}+\gamma_{\mathcal{N},led}\right|\right\rangle_{L^2}.$$

By Lemma 4.3, (i) of (1) of Proposition 4.1, and Proposition 6.1,

$$\begin{aligned}\left|\beta_*^2\left\langle \Psi_i^{2p-2}\mathcal{U}_i\chi_{\mathcal{B}_i}, \gamma_{2,ex}\right\rangle_{L^2}\right| &\lesssim \begin{cases}\beta_*^3 Q\left\langle \Psi_i^{2p-1-\sigma}, \mathcal{U}_i\right\rangle_{L^2(\mathcal{B}_{i,+})}, & p\geq\frac{7}{3},\\ \beta_*^3 Q\left\langle \Psi_i^{\frac{7p-9}{2}}, \mathcal{U}_i\right\rangle_{L^2(\mathcal{B}_{i,+})}, & 1<p<\frac{7}{3}\end{cases}\\ &= o(\beta_*^4+\|\rho_{**}^\perp\|^2)+\mathcal{O}\left(\beta_*\|f\|_{H^{-1}}\right).\end{aligned}$$

By Lemma 4.3 and (iii) of (1) of Proposition 4.1,

$$\left|\beta_*^2\left\langle \Psi_i^{2p-2}\mathcal{U}_i\chi_{\mathcal{B}_i}, \gamma_{\mathcal{N},led}\right\rangle_{L^2}\right| = o(\beta_*^4).$$

By Lemma 4.3 and Proposition 6.1,

$$\begin{aligned}\left|\beta_*^2\left\langle\Psi_i^{2p-2}\mathcal{U}_i\chi_{\mathcal{B}_i},\rho_{**}^\perp\right\rangle_{L^2}\right| &\lesssim \beta_*^2\|\rho_{**}^\perp\|\left(\left\langle\Psi_i^{4p-4},\mathcal{U}_i^2\right\rangle_{L^2(\mathcal{B}_i)}\right)^{\frac{1}{2}}\\ &= o(\beta_*^4+\|\rho_{**}^\perp\|^2)+\mathcal{O}\left(\beta_*\|f\|_{H^{-1}}\right).\end{aligned}$$

Summarizing the above estimates, we have

$$\sum_{i=1}^{\nu}\left\langle\beta_*\rho_*\Psi_i^{\frac{3p-5}{2}}\mathcal{U}_i\chi_{\mathcal{B}_i},\mathcal{V}_j\right\rangle_{L^2}=o(\beta_*^4+\|\rho_{**}^\perp\|^2)+\mathcal{O}\left(\beta_*\|f\|_{H^{-1}}\right).$$

Step. 5.8 The estimates of

$$\sum_{i=1}^{\nu}\left\langle\beta_*^2\Psi_i^{2p-2}\mathcal{U}_i\chi_{\mathcal{B}_i},\mathcal{V}_j\right\rangle_{L^2}\quad\text{and}\quad\left\langle\mathcal{U}^{p-2}\mathcal{V}^2\chi_{\mathcal{C}\backslash\mathcal{B}_*},\mathcal{V}_j\right\rangle_{L^2}.$$

By Lemma 4.3 and Proposition 6.1,

$$\begin{aligned}\left|\sum_{i=1}^{\nu}\left\langle\beta_*^2\Psi_i^{2p-2}\mathcal{U}_i\chi_{\mathcal{B}_i},\mathcal{V}_j\right\rangle_{L^2}\right| &\lesssim \beta_*^3\sum_{i=1}^{\nu}\left\langle\Psi_i^{\frac{5p-3}{2}},\mathcal{U}_i\right\rangle_{L^2(\mathcal{B}_i)}\\ &= o(\beta_*^4+\|\rho_{**}^\perp\|^2)+\mathcal{O}\left(\beta_*\|f\|_{H^{-1}}\right)\end{aligned}$$

and

$$\begin{aligned}\left|\left\langle\mathcal{U}^{p-2}\mathcal{V}^2\chi_{\mathcal{C}\backslash\mathcal{B}_*},\mathcal{V}_j\right\rangle_{L^2}\right| &\lesssim \beta_*^3\int_{\mathcal{C}\backslash\mathcal{B}_*}\mathcal{U}^{\frac{5p-1}{2}}d\mu\\ &= o(\beta_*^4+\|\rho_{**}^\perp\|^2)+\mathcal{O}\left(\beta_*\|f\|_{H^{-1}}\right).\end{aligned}$$

Step. 5.9 The estimate of $\left\langle\beta_*\rho_*\mathcal{U}^{\frac{3(p-1)}{2}}\chi_{\mathcal{C}\backslash\mathcal{B}_*},\mathcal{V}_j\right\rangle_{L^2}$.

By (1) of Proposition 4.1 and the oddness of $\mathcal{V}_j$ on $\mathbb{S}^{d-1}$,

$$\begin{aligned}\left|\left\langle\beta_*\rho_*\mathcal{U}^{\frac{3(p-1)}{2}}\chi_{\mathcal{C}\backslash\mathcal{B}_*},\mathcal{V}_j\right\rangle_{L^2}\right| &\lesssim \beta_*^2\left\langle\mathcal{U}^{2p-1}\chi_{\mathcal{C}\backslash\mathcal{B}_*},|\gamma_{2,ex}+\gamma_{\mathcal{N},led}|\right\rangle_{L^2}\\ &\quad+\beta_*^2\left\langle\mathcal{U}^{2p-1}\chi_{\mathcal{C}\backslash\mathcal{B}_*},|\rho_{**}^\perp|\right\rangle_{L^2}.\end{aligned}$$

By Lemma 4.3, (1) of Proposition 4.1 and Proposition 6.1,

$$\begin{aligned}\left|\left\langle\beta_*\rho_*\mathcal{U}^{\frac{3(p-1)}{2}}\chi_{\mathcal{C}\backslash\mathcal{B}_*},\mathcal{V}_j\right\rangle_{L^2}\right| &\lesssim \beta_*^4\left\|\mathcal{U}^{2p-\sigma}\right\|_{L^1(\mathcal{C}\backslash\mathcal{B}_*)}+\beta_*^2\|\rho_{**}^\perp\|\left\|\mathcal{U}^{2p-1}\right\|_{L^2(\mathcal{C}\backslash\mathcal{B}_*)}\\ &= o(\beta_*^4+\|\rho_{**}^\perp\|^2)+\mathcal{O}\left(\beta_*\|f\|_{H^{-1}}\right).\end{aligned}$$

Step. 5.10 The estimate of $\langle\mathcal{N}_0,\mathcal{V}_j\rangle_{L^2}$.

By (4.2), Lemmas 4.8 and 5.1 and the oddness of $\mathcal{V}_j$ on $\mathbb{S}^{d-1}$,

$$\begin{aligned}\langle\mathcal{N}_0,\mathcal{V}_j\rangle_{L^2} &= A_{p,1}\left\langle\mathcal{U}^{p-2}\left(\rho_0^2-(\gamma_{1,ex}+\gamma_*+\gamma_{\mathcal{N},led,j,*})^2\right),\mathcal{V}_j\right\rangle_{L^2}\\ &\quad+A_{p,2}\left\langle\mathcal{U}^{p-3}\left(\rho_0^3-(\gamma_{1,ex}+\gamma_*+\gamma_{\mathcal{N},led,j,*})^3\right),\mathcal{V}_j\right\rangle_{L^2}\\ &\quad+\mathcal{O}\left(\beta_*^2\left\langle\mathcal{U}^{p-3}\Psi_j^{p+1},\gamma_{ex}^2+|\gamma_{\mathcal{N},led}+\gamma_*|^2\right\rangle_{L^2}\right)\\ &\quad+\mathcal{O}\left(\beta_*\left\langle\mathcal{U}^{p-2-\sigma}\Psi_j^{\frac{p+1}{2}}\rho_0,\rho_{**}^\perp\right\rangle_{L^2}\right).\end{aligned}$$

By Lemmas 4.8 and 4.10 and (i) of (1) of Proposition 4.1,

$$\begin{aligned}&\left\langle\mathcal{U}^{p-2}\left(\rho_0^2-(\gamma_{1,ex}+\gamma_*+\gamma_{\mathcal{N},led,j,*})^2\right),\mathcal{V}_j\right\rangle_{L^2}\\ &= o\left(\beta_*\left\langle\mathcal{U}^{p-1-\sigma}\Psi_j^{\frac{p+1}{2}},\gamma_{2,ex}+\gamma_{\mathcal{N},led,rem,j,*}+\gamma_{\mathcal{N},led,j,**}+\rho_{**,2}^\perp\right\rangle_{L^2}\right).\end{aligned}$$

By Lemma 4.3, (i) of (1) of Proposition 4.1 and Propositions 6.1 and 7.1,

$$\beta_*\left\langle \mathcal{U}^{p-1-\sigma}\Psi_j^{\frac{p+1}{2}}, \gamma_{2,ex}\right\rangle_{L^2} = \mathcal{O}\left(\beta_*^2 Q^{1+\sigma}\right) = o(\beta_*^4) + \mathcal{O}\left(\beta_*\|f\|_{H^{-1}}\right).$$

By Lemmas 4.3 and 4.8 and Propositions 6.1 and 7.1,

$$\begin{aligned}\beta_*\left\langle \mathcal{U}^{p-1-\sigma}\Psi_j^{\frac{p+1}{2}}, \gamma_{\mathcal{N},led,rem,j,*}+\gamma_{\mathcal{N},led,j,**}\right\rangle_{L^2} &= \mathcal{O}\left(\beta_*^3 Q^{1-\sigma}+\beta_*^4\right)\\ &= \mathcal{O}\left(\beta_*^4+\beta_*\|f\|_{H^{-1}}\right).\end{aligned}$$

By Lemmas 4.3 and 4.10 and Propositions 6.1 and 7.1,

$$\beta_*\left\langle \mathcal{U}^{p-1-\sigma}\Psi_j^{\frac{p+1}{2}}, \rho_{**,2}^{\perp}\right\rangle_{L^2} = \mathcal{O}\left(\beta_*^4\right).$$

Thus,

$$\left\langle \mathcal{U}^{p-2}\left(\rho_0^2-(\gamma_{1,ex}+\gamma_*+\gamma_{\mathcal{N},led,j,*})^2\right), \mathcal{V}_j\right\rangle_{L^2} = o(\beta_*^4) + \mathcal{O}\left(\beta_*\|f\|_{H^{-1}}\right).$$

Similarly,

$$\begin{aligned}&\left\langle \mathcal{U}^{p-3}\left(\rho_0^3-(\gamma_{1,ex}+\gamma_*+\gamma_{\mathcal{N},led,j,*})^3\right), \mathcal{V}_j\right\rangle_{L^2}\\ =\;& o\left(\beta_*\left\langle \mathcal{U}^{p-1-\sigma}\Psi_j^{\frac{p+1}{2}}, \gamma_{2,ex}+\gamma_{\mathcal{N},led,rem,j,*}+\gamma_{\mathcal{N},led,j,**}+\rho_{**,2}^{\perp})\right\rangle_{L^2}\right)\\ =\;& o(\beta_*^4) + \mathcal{O}\left(\beta_*\|f\|_{H^{-1}}\right).\end{aligned}$$

By Lemma 4.3, (i) of (1) of Proposition 4.1 and Propositions 5.1 and 6.1,

$$\beta_*^2\left\langle \mathcal{U}^{p-3}\Psi_j^{p+1}, \gamma_{ex}^2\right\rangle_{L^2} \lesssim \beta_*^2 Q = o(\beta_*^4) + \mathcal{O}\left(\beta_*\|f\|_{H^{-1}}\right).$$

Similar to (5.10), by Lemma 4.3, (ii) and (iii) of (1) of Proposition 4.1 and Propositions 5.1 and 6.1,

$$\beta_*^2\left\langle \mathcal{U}^{2}\Psi_j^{p+1}, |\gamma_{\mathcal{N},led}+\gamma_*|^2\right\rangle_{L^2} = o(\beta_*^4+\|\rho_{**}^{\perp}\|^2) + \mathcal{O}\left(\beta_*\|f\|_{H^{-1}}\right).$$

By Lemma 4.3, (1) of Proposition 4.1 and Propositions 5.1 and 6.1,

$$\begin{aligned}\beta_*\left\langle \mathcal{U}^{p-2-\sigma}\Psi_j^{\frac{p+1}{2}}\rho_0, \rho_{**}^{\perp}\right\rangle_{L^2} &= \beta_*\mathcal{O}\left(\left\langle \mathcal{U}^{2p-4-2\sigma}\Psi_j^{p+1}, \gamma_{ex}^2\right\rangle_{L^2}^{\frac{1}{2}}\right)\|\rho_{**}^{\perp}\|\\ &\quad+\beta_*\mathcal{O}\left(\left\langle \mathcal{U}^{2p-4-2\sigma}\Psi_j^{p+1}, \gamma_*^2+\gamma_{\mathcal{N},led}^2\right\rangle_{L^2}^{\frac{1}{2}}\right)\|\rho_{**}^{\perp}\|\\ &= o(\beta_*^4+\|\rho_{**}^{\perp}\|^2) + \mathcal{O}\left(\beta_*\|f\|_{H^{-1}}\right).\end{aligned}$$

Summarizing the above estimates, we have

$$\langle \mathcal{N}_0, \mathcal{V}_j\rangle_{L^2} = o(\beta_*^4+\|\rho_{**}^{\perp}\|^2) + \mathcal{O}\left(\beta_*\|f\|_{H^{-1}}\right).$$

Step. 5.11 The estimate of $\langle \mathcal{N}_{rem}, \mathcal{V}_j\rangle_{L^2}$.
Similar to (5.14) and (5.15), by Lemma 5.1 and Propositions 5.1, 6.1 and 7.1,

$$\left|\langle \mathcal{N}_{rem}, \mathcal{V}_j\rangle_{L^2}\right| \lesssim \beta_*\left(\beta_*^4+\|\rho_{**}^{\perp}\|^{1+\sigma}+\|f\|_{H^{-1}}\right) = o(\beta_*^4) + \mathcal{O}\left(\beta_*\|f\|_{H^{-1}}\right).$$

By summarizing the estimates from Step. 5.1 to Step. 5.11, we have

$$\langle \mathcal{N}-\mathcal{N}_j, \mathcal{V}_j\rangle_{L^2} = o(\beta_*^4+\|\rho_{**}^{\perp}\|^2) + \mathcal{O}\left(\beta_*\|f\|_{H^{-1}}\right).$$

The conclusion then follows from the estimates in Step. 1 to Step. 5 and Proposition 7.1. □

With Proposition 8.1 in hands, we can finally estimate β_*.

⟨propn0005⟩ **Proposition 8.2.** *Let $d \geq 2$, $a < 0$ and $b = b_{FS}(a)$. Then we have $\beta_* \lesssim \|f\|_{H^{-1}}^{\frac{1}{3}}$.*

Proof. By Lemma 4.3, (1) of Proposition 4.1 and Propositions 5.1, 6.1 and 7.1,

$$\begin{aligned}\left\langle \left(\Psi_j^*\right)^{p-2}\mathcal{V}_j^2, \rho_*\right\rangle_{L^2} &= \left\langle \left(\Psi_j^*\right)^{p-2}\mathcal{V}_j^2, \gamma_{ex} + \gamma_* + \gamma_{\mathcal{N},led} + \rho_{**}^{\perp}\right\rangle_{L^2} \\ &= \left\langle \left(\Psi_j^*\right)^{p-2}\mathcal{V}_j^2, \sum_{i=1}^{\nu}(\gamma_{1,i} + \alpha_{i,0}^{**}\Psi_i) + \gamma_{\mathcal{N},led}\right\rangle_{L^2} \\ &\quad + o(\beta_*^4 + \beta_*\|f\|_{H^{-1}}). \end{aligned} \tag{8.4}$$ eqnewnew4443

We write $\sum_{i=1}^{\nu}\gamma_{1,i} + \gamma_{\mathcal{N},led} = \sum_{j=1}^{\nu}\overline{\alpha}_{j,*}\Psi_j + \overline{\gamma}_{**}^{\perp}$ such that $\langle \Psi_j, \overline{\gamma}_{**}^{\perp}\rangle = 0$ for all $1 \leq j \leq \nu$. Then by (4.40), (4.45) and (*iii*) of (1) of Proposition 4.1,

$$\begin{aligned}\left\langle \sum_{i=1}^{\nu}(\overline{\alpha}_{i,*} + \alpha_{i,0}^{**}\Psi_i), \Psi_j\right\rangle &= \left\langle \sum_{i=1}^{\nu}(\gamma_{1,i} + \alpha_{i,0}^{**}\Psi_i) + \gamma_{\mathcal{N},led}, \Psi_j\right\rangle \\ &= -\langle \gamma_{1,ex}, \Psi_j\rangle \\ &= \mathcal{O}\left(Q^{1+\sigma}\right). \end{aligned} \tag{8.5}$$ eqnewnew6678

It follows from (8.4) and Propositions 6.1 and 7.1 that

$$\left\langle \left(\Psi_j^*\right)^{p-2}\mathcal{V}_j^2, \rho_*\right\rangle_{L^2} = \left\langle \left(\Psi_j^*\right)^{p-2}\mathcal{V}_j^2, \overline{\gamma}_{**}^{\perp}\right\rangle_{L^2} + o(\beta_*^4 + \beta_*\|f\|_{H^{-1}}). \tag{8.6}$$ eqnewnew4444

By (4.29), (4.35) and Lemma 7.1, we know that $\overline{\gamma}_{**}^{\perp}$ satisfies

$$\begin{cases}\mathcal{L}(\overline{\gamma}_{**}^{\perp}) = \overline{\mathcal{R}}_{1,**}^{\perp}, & \text{in } \mathcal{C}, \\ \langle \Psi_j, \overline{\gamma}_{**}^{\perp}\rangle = \langle \partial_t\Psi_j, \overline{\gamma}_{**}^{\perp}\rangle = \langle w_{j,l}, \overline{\gamma}_{**}^{\perp}\rangle = 0 & \text{for all } 1 \leq j \leq \nu \text{ and all } 1 \leq l \leq d,\end{cases} \tag{8.7}$$ eqn5512

where

$$\overline{\mathcal{R}}_{1,**}^{\perp} = \overline{\mathcal{R}}_{1,**} - \sum_{i=1}^{\nu}\Psi_i^{p-1}\left((c_{1,j,i} + c_{\mathcal{N},led,i})\partial_t\Psi_i + \sum_{l=1}^{d}(\varsigma_{1,j,i,l} + \varsigma_{\mathcal{N},led,i,l})w_{i,l}\right)$$

and

$$\begin{aligned}\overline{\mathcal{R}}_{1,**} &= \mathcal{N}_{led} + \sum_{l=1}^{\nu}\left((\alpha_l^*)^p - \alpha_l^* - \overline{\alpha}_{l,*}\left(1 - p\,(\alpha_l^*)^{p-1}\right)\right)\Psi_l^p \\ &\quad + \sum_{l=1}^{\nu}p\overline{\alpha}_{l,*}\left(\mathcal{U}^{p-1} - (\Psi_l^*)^{p-1}\right)\Psi_l\end{aligned}$$

with $\mathcal{N}_{led}$ given by (4.36). Since by (4.41), (8.5) and Propositions 5.1, 6.1 and 7.1, we have $\sum_{i=1}^{\nu}|\overline{\alpha}_{i,*}| = \mathcal{O}\left(\beta_*^2 + \|f\|_{H^{-1}}\right)$. Thus, by the orthogonal conditions of $\overline{\gamma}_{**}^{\perp}$, multiplying (8.7) with $\overline{\gamma}_{**}^{\perp}$ on both sides and integrating by parts, Lemma 4.3 and Propositions 6.1 and 7.1, we have

$$\left\langle \mathcal{L}(\overline{\gamma}_{**}^{\perp}) - \mathcal{N}_{led}, \overline{\gamma}_{**}^{\perp}\right\rangle_{L^2} = o\left(\beta_*^4 + \beta_*\|f\|_{H^{-1}}\right) + \mathcal{O}\left(\|f\|_{H^{-1}}^2\right). \tag{8.8}$$ eqnewnew1220

Since $Q \to 0$ and $\beta_* \to 0$ as $\|f\|_{H^{-1}} \to 0$, by Lemma 4.3, Propositions 5.1, 6.1 and 7.1, it is easy to see that

$$\begin{aligned}\left\langle \mathcal{L}(\overline{\gamma}_{**}^{\perp}) - \mathcal{N}_{led}, \overline{\gamma}_{**}^{\perp}\right\rangle_{L^2} &= \|\overline{\gamma}_{**}^{\perp}\|^2 - \sum_{j=1}^{\nu}\left\langle p\Psi_j^{p-1} + \frac{p(p-1)}{2}\left(\Psi_j^*\right)^{p-2}\mathcal{V}_j^2, \overline{\gamma}_{**}^{\perp}\right\rangle_{L^2} \\ &\quad + o(\beta_*^4 + \|\overline{\gamma}_{**}^{\perp}\|) + \mathcal{O}(\beta_*\|f\|_{H^{-1}})\end{aligned}$$

which, together with (8.6), (8.8) and Propositions 5.1, 6.1, 7.1 and 8.1, implies that

$$\begin{aligned}&\sum_{j=1}^{\nu}\left(\langle f,\mathcal{V}_j\rangle_{H^1}-\frac{p\left\langle \Psi_j^{p-1},\mathcal{V}_j^2\right\rangle_{L^2}}{\alpha_j^*\|\Psi\|_{L^{p+1}}^{p+1}}\langle f,\Psi_j\rangle_{H^1}\right)\\ =\;& 2(1+o(1))\|\overline{\gamma}_{**}^{\perp}\|^2-\sum_{j=1}^{\nu}2\left\langle p\Psi_j^{p-1}+p(p-1)\left(\Psi_j^*\right)^{p-2}\mathcal{V}_j^2,\overline{\gamma}_{**}^{\perp}\right\rangle_{L^2}\\ &+\sum_{j=1}^{\nu}\frac{p(p-1)}{2}\left(\frac{p(\alpha_j^*)^{p-3}\left(\left\langle\Psi_j^{p-1},\mathcal{V}_j^2\right\rangle_{L^2}\right)^2}{\|\Psi\|_{L^{p+1}}^{p+1}}-\frac{p-2}{3}\left\langle\left(\Psi_j^*\right)^{p-3},\mathcal{V}_j^4\right\rangle_{L^2}\right)\\ &+o(\beta_*^4)+\mathcal{O}\left(\beta_*\|f\|_{H^{-1}}+\|f\|_{H^{-1}}^2\right). \end{aligned} \tag{8.9}$$

eqn0030

The conclusion then follows from applying the estimates in [54, Section 4.3] and the orthogonal conditions of $\overline{\gamma}_{**}^{\perp}$ given in (8.7) into (8.9). □

We are now ready to give the proof of (1) of Theorem 1.3.

Proof of (1) **of Theorem 1.3:** The conclusions for $\nu\geq 2$ follows immediately from Lemma 3.1 and Propositions 5.1, 6.1, 7.1, 8.1 and 8.2, since

$$dist_{D_a^{1,2}}(u,\mathcal{Z}_0^\nu)\leq\|\rho\|+\sum_{l=1}^{\nu}\left|(\alpha_l^*)^{p-1}-1\right|.$$

For $\nu=1$, there is no interaction between bubbles, that is, we have $Q=0$. Thus, the conclusion for $\nu=1$ follows from Lemma 3.1 and Propositions 5.1, 7.1, 8.1 and 8.2. □

9. Optimal example and proof of (2) of Theorem 1.3

Let $R>0$ be a sufficiently large parameter and $\beta>0$ is a sufficiently small parameter. We shall use the function, given by

$$v=\Psi+\Psi_R+\beta(w_d+w_{R,d}):=\Gamma_R+\beta\Phi_R,$$

to construct an optimal example of the stability stated in Theorem 1.3 and prove (2) of Theorem 1.3, where $\Psi_R=\Psi(t-R)$ and, as above, $w_d=\Psi^{\frac{p+1}{2}}\theta_d$ and $w_{R,d}=w_d(t-R)$. It is easy to see that

$$\frac{3}{2}\left(S_{FS}^{-1}\right)^{\frac{p+1}{p-1}}<\|v\|^2<\frac{5}{2}\left(S_{FS}^{-1}\right)^{\frac{p+1}{p-1}}.$$

Moreover, since $\Psi(t)$ is the unique positive solution of (2.5) for $d\geq 2$, $a<0$ and $b=b_{FS}(a)$, by Lemmas 2.1 and 4.2,

$$\begin{aligned}\Xi &:= -\Delta_\theta v-\partial_t^2 v+\Lambda_{FS}v-v^p\\ &= \Psi^p+\Psi_R^p+p\beta\left(\Psi^{p-1}w_d+\Psi_R^{p-1}w_{R,d}\right)-(\Gamma_R+\beta\Phi_R)^p\\ &= \Psi^p+\Psi_R^p-\Gamma_R^p+p\beta\left(\left(\Psi^{p-1}-\Gamma_R^{p-1}\right)w_d+\left(\Psi_R^{p-1}-\Gamma_R^{p-1}\right)w_{R,d}\right)\\ &\quad-\left(A_{p,1}\beta^2\Psi^{p-2}w_d^2+A_{p,2}\beta^3\Psi^{p-3}w_d^3\right)\chi_{\mathcal{B}}\\ &\quad-\left(A_{p,1}\beta^2\Psi_R^{p-2}w_{R,d}^2+A_{p,2}\beta^3\Psi_R^{p-3}w_{R,d}^3\right)\chi_{\mathcal{B}_R}\\ &\quad-\left(\beta^2\Gamma_R^{p-3}\Phi_R^2\left(A_{p,1}\Gamma_R+A_{p,2}\beta\Phi_R\right)\right)\chi_{\mathcal{C}\backslash(\mathcal{B}\cup\mathcal{B}_R)}+\Xi_{rem}\end{aligned}\tag{9.1}$$ eqqnew0001

where

$$\mathcal{B}=\left[-\frac{R}{2},\frac{R}{2}\right]\times\mathbb{S}^{d-1},\quad \mathcal{B}_R=\left[\frac{R}{2},\frac{3R}{2}\right]\times\mathbb{S}^{d-1}$$

and

$$\Xi_{rem}=\mathcal{O}\left(\beta^2\left(\Psi^{2(p-1)}\Psi_R\chi_{\mathcal{B}}+\Psi_R^{2(p-1)}\Psi\chi_{\mathcal{B}_R}\right)+\beta^4\Gamma_R^4\right).$$

We denote

$$\begin{aligned}\Xi_1 &= (\Gamma_R^p-\Psi^p-\Psi_R^p)+p\beta\left(\left(\Gamma_R^{p-1}-\Psi^{p-1}\right)w_d+\left(\Gamma_R^{p-1}-\Psi_R^{p-1}\right)w_{R,d}\right)\\ &:= \Xi_{1,1}+\beta\Xi_{1,2}\end{aligned}\tag{9.2}$$ eqqnew0005

and

$$\begin{aligned}\Xi_2 &= A_{p,1}\beta^2\left(\Psi^{p-2}w_d^2\chi_{\mathcal{B}}+\Psi_R^{p-2}w_{R,d}^2\chi_{\mathcal{B}_R}+\Gamma_R^{p-2}\Phi_R^2\chi_{\mathcal{C}\backslash(\mathcal{B}\cup\mathcal{B}_R)}\right)\\ &\quad+A_{p,2}\beta^3\left(\Psi^{p-3}w_d^3\chi_{\mathcal{B}}+\Psi_R^{p-3}w_{R,d}^3\chi_{\mathcal{B}_R}+\Gamma_R^{p-3}\Phi_R^3\chi_{\mathcal{C}\backslash(\mathcal{B}\cup\mathcal{B}_R)}\right)\\ &:= \beta^2\Xi_{2,1}+\beta^3\Xi_{2,2}.\end{aligned}\tag{9.3}$$ eqqnew0006

Applying Lemmas 4.4, 4.6 and 4.7, we immediately have the following.

⟨lemq1001⟩ **Lemma 9.1.** *Let $d\geq 2$, $a<0$ and $b=b_{FS}(a)$. Then the following equation*

$$\begin{cases}-\Delta_\theta\varrho_{i,j}-\partial_t^2\varrho_{i,j}+\Lambda_{FS}\varrho_{i,j}-p\Gamma_R^{p-1}\varrho_{i,j}=\Xi_{i,j}+\vartheta_{i,j}, & \text{in } \mathcal{C},\\ \langle\partial_t\Psi,\varrho_{i,j}\rangle=\langle\partial_t\Psi_R,\varrho_{i,j}\rangle=0,\\ \langle w_l,\varrho_{i,j}\rangle=\langle w_{R,l},\varrho_{i,j}\rangle=0 \text{ for all } 1\leq l\leq d,\end{cases}\tag{9.4}$$ eqqnew1002

is uniquely solvable, where $\Xi_{i,j}$ is given by (9.2) and (9.3), and

$$\vartheta_{i,j}=\Psi^{p-1}\left(c_{i,j}\partial_t\Psi+\sum_{l=1}^d\varsigma_{i,j,l}w_l\right)+\Psi_R^{p-1}\left(c_{R,i,j}\partial_t\Psi_R+\sum_{l=1}^d\varsigma_{R,i,j,l}w_{R,l}\right)\tag{9.5}$$ eqqnew0003

with $c_{i,j},c_{R,i,j}$ and $\{\varsigma_{i,j,l}\}$ and $\{\varsigma_{R,i,j,l}\}$ being chosen such that the right hand side of the equation (9.4) is orthogonal to $\partial_t\Psi$, $\partial_t\Psi_R$, $\{w_l\}$ and $\{w_{R,l}\}$ in $H^1(\mathcal{C})$. Moreover, $\varrho_{1,1}$ is even on $\mathbb{S}^{d-1}$ and $\varrho_{1,2}$ is odd on $\mathbb{S}^{d-1}$ with

$$\begin{cases}\|\varrho_{1,1}\|_\sharp\lesssim 1, & p\geq 3,\\ \|\varrho_{1,1}\|_{\natural,1}\lesssim 1, & 1<p<3,\end{cases}\quad \text{and}\quad \begin{cases}\|\varrho_{1,2}\|_\sharp\lesssim 1, & p\geq\frac{7}{3},\\ \|\varrho_{1,2}\|_{\natural,2}\lesssim 1, & 1<p<\frac{7}{3},\end{cases}$$

while, $\varrho_{2,1}$ is even on $\mathbb{S}^{d-1}$ and $\varrho_{2,2}$ is odd on $\mathbb{S}^{d-1}$ with

$$\sup_{(t,\theta)\in\mathcal{C}}\frac{|\varrho_{2,1}|+|\varrho_{2,2}|}{\Psi^{1-\sigma}+\Psi_R^{1-\sigma}}\lesssim 1.$$

The norms $\|\cdot\|_\sharp$, $\|\cdot\|_{\natural,1}$ and $\|\cdot\|_{\natural,2}$ are given in Lemmas 4.4 and 4.6.

Let $\varrho_*=\varrho_{1,1}+\beta\varrho_{1,2}+\beta^2\varrho_{2,1}+\beta^3\varrho_{2,2}$. Then we have the following.

⟨propq1001⟩ **Proposition 9.1.** *Let $d\geq 2$, $a<0$ and $b=b_{FS}(a)$. Then*

$$\|\varrho_*\|\sim\beta^2+\begin{cases}Q_R, & p>2,\\ Q_R\left|\log Q_R\right|, & p=2,\\ Q_R^{\frac{p}{2}}, & 1<p<2,\end{cases}\tag{9.6}$$ eqqnew0020

where $Q_R=e^{-\sqrt{\Lambda_{FS}}R}$.

Proof. By using the test functions

$$\widetilde{\varrho}_R(t)=\begin{cases}1, & \frac{R}{2}-3\leq t\leq\frac{R}{2}-2,\\ 0, & t\leq\frac{R}{2}-4\text{ or }t\geq\frac{R}{2}-1.\end{cases}$$

for $1<p<2$,

$$\widetilde{\varrho}_R(t)=\begin{cases}1, & \frac{R}{4}\leq t\leq\frac{R}{2}-2,\\ 0, & t\leq\frac{R}{4}-1\text{ or }t\geq\frac{R}{2}-1.\end{cases}$$

for $p=2$ and

$$\widehat{\varrho}_R(t)=\begin{cases}1, & T_*\leq t\leq T_*+1,\\ 0, & t\leq T_*-1\text{ or }t\geq T_*+2,\end{cases}$$

with $T_*>0$ sufficiently large for $p>2$ to (9.4), then as in the proof of [78, Proposition 6.2], we can show that

$$\|\varrho_{1,1}\|\gtrsim\begin{cases}Q_R, & p>2,\\ Q_R\left|\log Q_R\right|, & p=2,\\ Q_R^{\frac{p}{2}}, & 1<p<2,\end{cases}$$

which, together with (9.2), Lemma 9.1 and multiplying (9.4) of $\varrho_{1,1}$ with $\varrho_{1,1}$ on both sides and integrating by parts, implies that

$$\|\varrho_{1,1}\|\sim\begin{cases}Q_R, & p>2,\\ Q_R\left|\log Q_R\right|, & p=2,\\ Q_R^{\frac{p}{2}}, & 1<p<2.\end{cases}$$

Similar to (4.65), by (9.2), Lemma 9.1 and multiplying (9.4) of $\varrho_{1,2}$ with $\varrho_{1,2}$ on both sides and integrating by parts, we also have $\|\varrho_{1,2}\|\lesssim\|\varrho_{1,1}\|$. By (9.3), Lemma 9.1 and multiplying (9.4) of $\varrho_{2,2}$ with $\varrho_{2,2}$ on both sides and integrating by parts, it is also easy to see that $\|\varrho_{2,2}\|\lesssim 1$. It remains to estimate $\|\varrho_{2,1}\|$. By (9.3), Lemma 9.1 and multiplying (9.4) of $\varrho_{2,1}$ with $\varrho_{2,1}$ on both sides and integrating by parts, it is also easy to see that $\|\varrho_{2,1}\|\lesssim 1$. For the lower bound of $\|\varrho_{2,1}\|$, we recall that the spherical harmonics on $\mathbb{S}^{d-1}$, denoted by $\{\mathcal{Y}_{j,l}\}$ with $j=0,1,2,\cdots$ and $l=1,2,\cdots,l_j$ for some $l_j\in\mathbb{N}$, form an orthogonal basic of $L^2(\mathbb{S}^{d-1})$ with $\mathrm{span}_{1\leq l\leq l_j}\{\mathcal{Y}_{j,l}\}$ forming the eigenspace of the jth eigenvalue of $-\Delta_\theta$ on $L^2\left(\mathbb{S}^{d-1}\right)$, where Δ_θ is the Laplace-Beltrami operator on $\mathbb{S}^{d-1}$. Moreover, it is well known that the eigenvalues of $-\Delta_\theta$ on $L^2\left(\mathbb{S}^{d-1}\right)$ are given by $j(j+d-2)$. The first eigenvalue 0 is simple with the eigenfunction $\mathcal{Y}_{0,1}=1$, the eigenfunctions of the second eigenvalue $d-1$ are precisely $\mathcal{Y}_{1,l}=\theta_l$ for $1\leq l\leq d$. It is also well known that $\mathcal{Y}_{2,d}=\theta_d^2-\frac{1}{d}$ is a spherical harmonic on $\mathbb{S}^{d-1}$ with degree 2 (cf. [73, (2.6)] or [54, (4.9)]). Now, by (9.3), Lemma 9.1 and multiplying (9.4) of $\varrho_{2,1}$ with $\mathcal{Y}_{2,d}$ on both sides and integrating by parts,we have

$$\|\varrho_{2,1}\|\gtrsim\langle\varrho_{2,1},\mathcal{Y}_{2,d}\rangle-p\left\langle\Gamma_R\varrho_{2,1},\mathcal{Y}_{2,d}\right\rangle_{L^2}=\langle\Xi_{2,1},\mathcal{Y}_{2,d}\rangle_{L^2}\gtrsim\left\|\Psi^{2p-1}\mathcal{Y}_{2,d}^2\right\|_{L^1}.$$

Thus, by $\varrho_*=\varrho_{1,1}+\beta\varrho_{1,2}+\beta^2\varrho_{2,1}+\beta^3\varrho_{2,2}$, we have the desired estimate of $\|\varrho\|$ given by (9.7). □

We define

$$f_*:=-\Delta_\theta(\varrho_*+v)-\partial_t^2(\varrho_*+v)+\Lambda_{FS}(\varrho_*+v)-(v+\varrho_*)^p. \tag{9.7}$$ eqqnew0020

Then by (9.1) and Lemma 9.1,

$$\begin{aligned} f_* &= \left(-\Delta_\theta\varrho_*-\partial_t^2\varrho_*+\Lambda_{FS}\varrho_*-p\Gamma_R^{p-1}\varrho_*\right)+\Psi^p+\Psi_R^p+p\beta\left(\Psi^{p-1}w_d+\Psi_R^{p-1}w_{R,d}\right)\\ &\quad+p\Gamma_R^{p-1}\varrho_*-(\Gamma_R+\beta\Phi_R+\varrho_*)^p\\ &= \vartheta_{1,1}+\beta\vartheta_{1,2}+\beta^2\vartheta_{2,1}+\beta^3\vartheta_{2,2}+\Xi_{1,1}+\beta\Xi_{1,2}+\beta^2\Xi_{2,1}+\beta^3\Xi_{2,2}\\ &\quad+\Psi^p+\Psi_R^p-\Gamma_R^p+p\beta\left(\left(\Psi^{p-1}-\Gamma_R\right)w_d+\left(\Psi_R^{p-1}-\Gamma_R\right)w_{R,d}\right)\\ &\quad+p\Gamma_R^{p-1}(\varrho_*+\beta\Phi_R)+\Gamma_R^p-(\Gamma_R+\beta\Phi_R+\varrho_*)^p\\ &= \vartheta_{1,1}+\beta\vartheta_{1,2}+\beta^2\vartheta_{2,1}+\beta^3\vartheta_{2,2}+\beta^2\Xi_{2,1}+\beta^3\Xi_{2,2}-\mathcal{N}_{\varrho_*}, \end{aligned} \tag{9.8}$$ eqqnew1230

where $\vartheta_{i,j}$ is given by (9.5) and by Lemmas 4.1, 4.2, 5.1 and 7.2,

$$\begin{aligned} \mathcal{N}_{\varrho_*} &= \sum_{l=2}^{n_0}A_{p,l-1}\Gamma_R^{p-l}\left(\beta\Phi_R+\varrho_*\right)^l+\mathcal{O}\left(\Gamma_R^{p-4\sigma}\beta^4+Q_R^{1+\sigma}\Gamma_R^\sigma+\left(\beta^2\Gamma_R\right)^{1+\sigma}\chi_{\mathcal{C}\backslash\widetilde{\mathcal{B}}_{**}}\right)\\ &= \beta^2\Xi_{2,1}+\beta^3\Xi_{2,2}+\mathcal{N}_{\varrho_*,rem}, \end{aligned}$$

where $\widetilde{\mathcal{B}}_{**}=\{(\theta,t)\in\mathcal{C}\mid\left|\beta^2\varrho_{2,1}+\beta^3\varrho_{2,2}\right|\leq\frac{1}{2}\Gamma_R\}$ and

$$\begin{aligned} \mathcal{N}_{\varrho_*,rem} &= 2A_{p,1}\beta\left(\Psi^{p-2}w_d\chi_{\mathcal{B}}+\Psi_R^{p-2}w_{R,d}\chi_{\mathcal{B}_R}+\Gamma_R^{p-2}\Phi_R\chi_{\mathcal{C}\backslash(\mathcal{B}\cup\mathcal{B}_R)}\right)\varrho_*\\ &\quad+3A_{p,2}\beta^2\left(\Psi^{p-3}w_d^2\chi_{\mathcal{B}}+\Psi_R^{p-3}w_{R,d}^2\chi_{\mathcal{B}_R}\right)\varrho_*+\mathcal{O}\left(\beta^2\Gamma_R^{2(p-1)}\varrho_*\chi_{\mathcal{C}\backslash(\mathcal{B}\cup\mathcal{B}_R)}\right)\\ &\quad+2\beta A_{p,1}\left((\Gamma_R^{p-2}\Phi_R-\Psi^{p-2}w_d)\chi_{\mathcal{B}}+(\Gamma_R^{p-2}\Phi_R-\Psi_R^{p-2}w_{R,d}\chi_{\mathcal{B}_R})\right)\varrho_*\\ &\quad+\mathcal{O}\left(\beta^2\left(\left(\Psi^{2p-2}\Psi_R+\Psi^{2p-3}\Psi_R\varrho_*\right)\chi_{\mathcal{B}}+\left(\Psi_R^{2p-2}\Psi+\Psi_R^{2p-3}\Psi\varrho_*\right)\chi_{\mathcal{B}_R}\right)\right)\\ &\quad+\mathcal{O}\left(\Gamma_R^{p-4\sigma}\beta^4+Q_R^{1+\sigma}\Gamma_R^\sigma+\left(\beta^2\Gamma_R\right)^{1+\sigma}\chi_{\mathcal{C}\backslash\widetilde{\mathcal{B}}_{**}}\right)\\ &\quad+A_{p,1}\Gamma_R^{p-2}\varrho_*^2+A_{p,2}\Gamma_R^{p-3}(3\beta\Phi_R+\varrho_*)\varrho_*^2+\sum_{l=4}^{n_0}A_{p,l-1}\Gamma_R^{p-l}\left(\beta\Phi_R+\varrho_*\right)^l \end{aligned}$$

By Lemmas 4.11 and 9.1, we immediately have the following.

⟨lemq12001⟩ **Lemma 9.2.** *Let $d\geq 2$, $a<0$ and $b=b_{FS}(a)$. Then the following equation,*

$$\begin{cases} -\Delta_\theta\widetilde{\varrho}_{1,1,i}-\partial_t^2\widetilde{\varrho}_{1,1,i}+\Lambda_{FS}\widetilde{\varrho}_{1,1,i}-p\Gamma_R^{p-1}\widetilde{\varrho}_{1,1,i}=\widetilde{\widetilde{\Xi}}_{1,1,i}+\widetilde{\vartheta}_{1,1,i}, \quad \text{in } \mathcal{C},\\ \langle\partial_t\Psi,\widetilde{\varrho}_{1,1,i}\rangle=\langle\partial_t\Psi_R,\widetilde{\varrho}_{1,1,i}\rangle=0,\\ \langle w_l,\widetilde{\varrho}_{1,1,i}\rangle=\langle w_{R,l},\widetilde{\varrho}_{1,1,i}\rangle=0 \text{ for all } 1\leq l\leq d,\end{cases} \tag{9.9}$$ eqqnew111102

is uniquely solvable, where

$$\widetilde{\widetilde{\Xi}}_{1,1,i}=\begin{cases}\sum_{l=2}^{n_0}A_{p,l-1}\Gamma_R^{p-l}\varrho_{1,1}^l, \quad i=0,\\ \sum_{l=2}^{n_0}A_{p,l-1}\Gamma_R^{p-l}\left((\varrho_{1,1}+\widetilde{\varrho}_{1,1,0})^l-\varrho_{1,1}^l\right), \quad i=1,\\ \sum_{l=2}^{n_0}A_{p,l-1}\Gamma_R^{p-l}\left(\left(\varrho_{1,1}+\sum_{k=0}^{i-1}\widetilde{\varrho}_{1,1,k}\right)^l-\left(\varrho_{1,1}+\sum_{k=0}^{i-2}\widetilde{\varrho}_{1,1,k}\right)^l\right), \quad i\geq 2\end{cases}$$

and

$$\widetilde{\vartheta}_{1,1,i}=\Psi^{p-1}\left(\widetilde{c}_{1,1,i}\partial_t\Psi+\sum_{l=1}^d\widetilde{\varsigma}_{1,1,i,l}w_l\right)+\Psi_R^{p-1}\left(\widetilde{c}_{R,1,1,i}\partial_t\Psi_R+\sum_{l=1}^d\widetilde{\varsigma}_{R,1,1,i,l}w_{R,l}\right)$$

with $\widetilde{c}_{1,1,i},\widetilde{c}_{R,1,1,i}$ and $\{\widetilde{\varsigma}_{1,1,i,l}\}$ and $\{\widetilde{\varsigma}_{R,1,1,i,l}\}$ being chosen such that the right hand side of the equation (9.9) is orthogonal to $\partial_t\Psi$, $\partial_t\Psi_R$, $\{w_l\}$ and $\{w_{R,l}\}$ in $H^1(\mathcal{C})$. Moreover, $\widetilde{\varrho}_{1,1,i}$ is even on $\mathbb{S}^{d-1}$ with

$$\begin{cases}\|\widetilde{\varrho}_{1,1,i}\|_\sharp\lesssim Q_R^{((p-1)\wedge 1)(i+1)}, \quad p\geq 3,\\ \|\widetilde{\varrho}_{1,1,i}\|_{\natural,1}\lesssim Q_R^{((p-1)\wedge 1)(i+1)}, \quad 1<p<3\end{cases}$$

and the Lagrange multipliers satisfy

$$\sum_{l=1}^d(|\widetilde{\varsigma}_{1,1,i,l}|+|\widetilde{\varsigma}_{R,1,1,i,l}|)=0 \quad \text{and} \quad |\widetilde{c}_{1,1,i}|+|\widetilde{c}_{R,1,1,i}|\lesssim Q_R^{1+((p-1)\wedge 1)(i+1)}.$$

Next, by Lemmas 4.12 and 9.1, we immediately have the following.

⟨lemq1002⟩ **Lemma 9.3.** *Let $d\geq 2$, $a<0$ and $b=b_{FS}(a)$. Then the following equation,*

$$\begin{cases} -\Delta_\theta\widetilde{\varrho}_{1,2}-\partial_t^2\widetilde{\varrho}_{1,2}+\Lambda_{FS}\widetilde{\varrho}_{1,2}-p\Gamma_R^{p-1}\widetilde{\varrho}_{1,2}=\widetilde{\widetilde{\Xi}}_{1,2}+\widetilde{\vartheta}_{1,2}, \quad \text{in } \mathcal{C},\\ \langle\partial_t\Psi,\widetilde{\varrho}_{1,2}\rangle=\langle\partial_t\Psi_R,\widetilde{\varrho}_{1,2}\rangle=0,\\ \langle w_l,\widetilde{\varrho}_{1,2}\rangle=\langle w_{R,l},\widetilde{\varrho}_{1,2}\rangle=0 \text{ for all } 1\leq l\leq d,\end{cases} \tag{9.10}$$ eqqnew1102

is uniquely solvable, where $\widetilde{\widetilde{\Xi}}_{1,2}=2A_{p,1}\Gamma_R^{p-2}\Phi_R\left(\varrho_{1,1}+\sum_{k=0}^{n_0}\widetilde{\varrho}_{1,1,k}\right)$ and

$$\widetilde{\vartheta}_{1,2}=\Psi^{p-1}\left(\widetilde{c}_{1,2}\partial_t\Psi+\sum_{l=1}^d\widetilde{\varsigma}_{1,2,l}w_l\right)+\Psi_R^{p-1}\left(\widetilde{c}_{R,1,2}\partial_t\Psi_R+\sum_{l=1}^d\widetilde{\varsigma}_{R,1,2,l}w_{R,l}\right)$$

with $\widetilde{c}_{1,2},\widetilde{c}_{R,1,2}$ and $\{\widetilde{\varsigma}_{1,2,l}\}$ and $\{\widetilde{\varsigma}_{R,1,2,l}\}$ being chosen such that the right hand side of the equation (9.10) is orthogonal to $\partial_t\Psi$, $\partial_t\Psi_R$, $\{w_l\}$ and $\{w_{R,l}\}$ in $H^1(\mathcal{C})$.

Moreover, $\widetilde{\varrho}_{1,2}$ *is odd on* $\mathbb{S}^{d-1}$ *with*

$$\begin{cases} \|\widetilde{\varrho}_{1,2}\|_{\sharp} \lesssim 1, \quad p \geq \frac{7}{3}, \\ \|\widetilde{\varrho}_{1,2}\|_{\natural,2} \lesssim 1, \quad 1 < p < \frac{7}{3} \end{cases}$$

and the Lagrange multipliers satisfy

$$\sum_{l=1}^{d}(|\widetilde{\varsigma}_{1,2,l}| + |\widetilde{\varsigma}_{R,1,2,l}|) \lesssim Q_R \quad \text{and} \quad |\widetilde{c}_{1,2}| + |\widetilde{c}_{R,1,2}| = 0.$$

By Lemmas 4.9 and 9.1, we also immediately have the following.

⟨lemq11002⟩ **Lemma 9.4.** *Let* $d \geq 2$*,* $a < 0$ *and* $b = b_{FS}(a)$*. Then the following equation,*

$$\begin{cases} -\Delta_\theta \widetilde{\varrho}_{1,3} - \partial_t^2 \widetilde{\varrho}_{1,3} + \Lambda_{FS} \widetilde{\varrho}_{1,3} - p\Gamma_R^{p-1} \widetilde{\varrho}_{1,3} = \widetilde{\Xi}_{1,3} + \widetilde{\vartheta}_{1,3}, \quad \text{in } \mathcal{C}, \\ \langle \partial_t \Psi, \widetilde{\varrho}_{1,3} \rangle = \langle \partial_t \Psi_R, \widetilde{\varrho}_{1,3} \rangle = 0, \\ \langle w_l, \widetilde{\varrho}_{1,3} \rangle = \langle w_{R,l}, \widetilde{\varrho}_{1,3} \rangle = 0 \text{ for all } 1 \leq l \leq d, \end{cases} \tag{9.11}$$

eqqnew211102

is uniquely solvable, where $\widetilde{\Xi}_{1,3} = 2A_{p,1}\left(\Psi^{p-2} w_d + \Psi_R^{p-2} w_{R,d}\right) \varrho_{2,1}$ *and*

$$\widetilde{\vartheta}_{1,3} = \Psi^{p-1}\left(\widetilde{c}_{1,3}\partial_t \Psi + \sum_{l=1}^{d} \widetilde{\varsigma}_{1,3,l} w_l\right) + \Psi_R^{p-1}\left(\widetilde{c}_{R,1,3}\partial_t \Psi_R + \sum_{l=1}^{d} \widetilde{\varsigma}_{R,1,3,l} w_{R,l}\right)$$

with $\widetilde{c}_{1,3}, \widetilde{c}_{R,1,3}$ *and* $\{\widetilde{\varsigma}_{1,3,l}\}$ *and* $\{\widetilde{\varsigma}_{R,1,3,l}\}$ *being chosen such that the right hand side of the equation (9.11) is orthogonal to* $\partial_t \Psi$*,* $\partial_t \Psi_R$*,* $\{w_l\}$ *and* $\{w_{R,l}\}$ *in* $H^1(\mathcal{C})$*. Moreover,* $\widetilde{\varrho}_{1,3}$ *is odd on* $\mathbb{S}^{d-1}$ *with*

$$\sup_{(t,\theta)\in\mathcal{C}} \frac{|\widetilde{\varrho}_{1,3}|}{\Psi^{1-\sigma} + \Psi_R^{1-\sigma}} \lesssim 1.$$

Finally, by Lemmas 4.13 and 9.1, we immediately have the following.

⟨lemq22001⟩ **Lemma 9.5.** *Let* $d \geq 2$*,* $a < 0$ *and* $b = b_{FS}(a)$*. Then the following equation,*

$$\begin{cases} -\Delta_\theta \widetilde{\varrho}_{1,4,i} - \partial_t^2 \widetilde{\varrho}_{1,4,i} + \Lambda_{FS} \widetilde{\varrho}_{1,4,i} - p\Gamma_R^{p-1} \widetilde{\varrho}_{1,4,i} = \widetilde{\Xi}_{1,4,i} + \widetilde{\vartheta}_{1,4,i}, \quad \text{in } \mathcal{C}, \\ \langle \partial_t \Psi, \widetilde{\varrho}_{1,4,i} \rangle = \langle \partial_t \Psi_R, \widetilde{\varrho}_{1,4,i} \rangle = 0, \\ \langle w_l, \widetilde{\varrho}_{1,4,i} \rangle = \langle w_{R,l}, \widetilde{\varrho}_{1,4,i} \rangle = 0 \text{ for all } 1 \leq l \leq d, \end{cases} \tag{9.12}$$

eqqnew11102

is uniquely solvable, where

$$\widetilde{\Xi}_{1,4,i} = \begin{cases} \sum_{l=2}^{n_0} l A_{p,l-1} \Gamma_R^{p-l} \left(\Phi_R + \varrho_{1,2} + \widetilde{\varrho}_{1,2}\right) \left(\varrho_{1,1} + \sum_{k=0}^{n_0} \widetilde{\varrho}_{1,1,k}\right)^{l-1}, \quad i = 0, \\ \sum_{l=2}^{n_0} l A_{p,l-1} \Gamma_R^{p-l} \widetilde{\varrho}_{1,4,i-1} \left(\varrho_{1,1} + \sum_{k=0}^{n_0} \widetilde{\varrho}_{1,1,k}\right)^{l-1}, \quad i \geq 1 \end{cases}$$

and

$$\widetilde{\vartheta}_{1,4,i} = \Psi^{p-1}\left(\widetilde{c}_{1,4,i}\partial_t \Psi + \sum_{l=1}^{d} \widetilde{\varsigma}_{1,4,i,l} w_l\right) + \Psi_R^{p-1}\left(\widetilde{c}_{R,1,4,i}\partial_t \Psi_R + \sum_{l=1}^{d} \widetilde{\varsigma}_{R,1,4,i,l} w_{R,l}\right)$$

with $\widetilde{c}_{1,4,i}, \widetilde{c}_{R,1,4,i}$ and $\{\widetilde{\varsigma}_{1,4,i,l}\}$ and $\{\widetilde{\varsigma}_{R,1,4,i,l}\}$ being chosen such that the right hand side of the equation (9.12) is orthogonal to $\partial_t\Psi$, $\partial_t\Psi_R$, $\{w_l\}$ and $\{w_{R,l}\}$ in $H^1(\mathcal{C})$. Moreover, $\widetilde{\varrho}_{1,4,i}$ is odd on $\mathbb{S}^{d-1}$ with

$$\begin{cases}\|\widetilde{\varrho}_{1,4,i}\|_\sharp \lesssim Q_R^{((p-1)\wedge 1)i}, & p\geq \frac{7}{3},\\ \|\widetilde{\varrho}_{1,4,i}\|_{\natural,1} \lesssim Q_R^{((p-1)\wedge 1)i}, & 1<p<\frac{7}{3}\end{cases}$$

and the Lagrange multipliers satisfy

$$\sum_{l=1}^{d}(|\widetilde{\varsigma}_{1,4,i,l}|+|\widetilde{\varsigma}_{R,1,4,i,l}|)\lesssim Q_R^{1+((p-1)\wedge 1)i} \quad \textit{and} \quad |\widetilde{c}_{1,4,i}|+|\widetilde{c}_{R,1,4,i}|=0.$$

Let $\varrho=\varrho_*+\widetilde{\varrho}_{1,1}+\beta(\widetilde{\varrho}_{1,2}+\widetilde{\varrho}_{1,4})+\beta^3\widetilde{\varrho}_{1,3}$ with

$$\widetilde{\varrho}_{1,1}=\sum_{k=0}^{n_0}\widetilde{\varrho}_{1,1,k} \quad \text{and} \quad \widetilde{\varrho}_{1,4}=\sum_{k=0}^{n_0}\widetilde{\varrho}_{1,4,k}$$

and define

$$f:=-\Delta_\theta(\varrho+v)-\partial_t^2(\varrho+v)+\Lambda_{FS}(\varrho+v)-(v+\varrho)^p. \tag{9.13}$$ eqqnew1020

Then similar to (4.64), (7.4) and (9.8), by Lemmas 9.2, 9.3, 9.4 and 9.5,

$$f=\vartheta_{1,1}+\widetilde{\vartheta}_{1,1}+\beta(\vartheta_{1,2}+\widetilde{\vartheta}_{1,2}+\widetilde{\vartheta}_{1,4})+\beta^2\vartheta_{2,1}+\beta^3(\vartheta_{2,2}+\widetilde{\vartheta}_{1,3})-\mathcal{N}_{\varrho_*,rem,1},$$

where $\widetilde{\vartheta}_{1,1}=\sum_{k=0}^{n_0}\widetilde{\vartheta}_{1,1,k}$, $\widetilde{\vartheta}_{1,4}=\sum_{k=0}^{n_0}\widetilde{\vartheta}_{1,4,k}$ and

$$\begin{aligned}
&\mathcal{N}_{\varrho_*,rem,1}\\
=\;& 2A_{p,1}\beta\left(\Psi^{p-2}w_d\chi_{\mathcal{B}}+\Psi_R^{p-2}w_{R,d}\chi_{\mathcal{B}_R}\right)(\varrho-\varrho_{1,1}-\widetilde{\varrho}_{1,1}-\beta^2\varrho_{2,1})\\
&+3A_{p,2}\beta^2\left(\Psi^{p-3}w_d^2\chi_{\mathcal{B}}+\Psi_R^{p-3}w_{R,d}^2\chi_{\mathcal{B}_R}\right)\varrho_*+\mathcal{O}\left(\beta^2\Gamma_R^{2(p-1)}\varrho_*\chi_{\mathcal{C}\setminus(\mathcal{B}\cup\mathcal{B}_R)}\right)\\
&+\mathcal{O}\left(\beta\left(\Psi^{\frac{3p-5}{2}}\Psi_R\chi_{\mathcal{B}}+\Psi_R^{\frac{3p-5}{2}}\Psi\chi_{\mathcal{B}_R}+\Gamma_R^{\frac{3(p-1)}{2}}\chi_{\mathcal{B}\cup\mathcal{B}_R}\right)(\varrho-\varrho_{1,1}-\widetilde{\varrho}_{1,1})\right)\\
&+\mathcal{O}\left(\beta^2\left(\left(\Psi^{2p-2}\Psi_R+\Psi^{2p-3}\Psi_R\varrho_*\right)\chi_{\mathcal{B}}+\left(\Psi_R^{2p-2}\Psi+\Psi_R^{2p-3}\Psi\varrho_*\right)\chi_{\mathcal{B}_R}\right)\right)\\
&+o(\Gamma_R\beta^3)+\mathcal{O}\left(Q_R^{1+\sigma}\Gamma_R^\sigma+\left(\beta^2\Gamma_R\right)^{1+\sigma}\chi_{\mathcal{C}\setminus\widetilde{\mathcal{B}}_{**}}\right)\\
&+\sum_{l=2}^{n_0}A_{p,l-1}\Gamma_R^{p-l}\left(\left(\varrho_{1,1}+\widetilde{\varrho}_{1,1}\right)^l-\left(\varrho_{1,1}+\sum_{i=0}^{n_0-1}\widetilde{\varrho}_{1,1,i}\right)^l\right)\\
&+\sum_{l=2}^{n_0}lA_{p,l-1}\Gamma_R^{p-l}\left(\varrho_{1,1}+\widetilde{\varrho}_{1,1}\right)^{l-1}\left(\beta\widetilde{\varrho}_{1,4,n_0}+\beta^2\varrho_{2,1}+\beta^3(\varrho_{2,2}+\widetilde{\varrho}_{1,3})\right)\\
&+\sum_{l=2}^{n_0}\sum_{k=2}^{l}C_l^kA_{p,l-1}\Gamma_R^{p-l}\left(\varrho_{1,1}+\widetilde{\varrho}_{1,1}\right)^{l-k}\left(\varrho+\beta\Phi_R-\varrho_{1,1}-\widetilde{\varrho}_{1,1}\right)^k.
\end{aligned}$$

⟨propq0001⟩ **Proposition 9.2.** *Let $d\geq 2$, $a<0$ and $b=b_{FS}(a)$. Then*

$$\|f\|_{H^{-1}}\sim\beta^3+Q_R,$$

where $Q_R=e^{-\sqrt{\Lambda_{FS}}R}$.

Proof. As in the proof of Lemma 4.10, by Lemma 9.1, the orthogonality of $\Psi^{p-1}\partial_t\Psi$ and $\{\Psi^{p-1}w_l\}$ in $L^2(\mathcal{C})$ and the oddness of w_d on $\mathbb{S}^{d-1}$,

$$-\langle\Psi^{p-1}\partial_t\Psi,\Xi_{i,j}\rangle_{L^2}=\|\Psi^{p-1}\partial_t\Psi\|_{L^2}^2c_{i,j}+\left\langle\Psi^{p-1}\partial_t\Psi,\Psi_R^{p-1}\partial_t\Psi_R\right\rangle_{L^2}c_{R,i,j}$$

and

$$-\left\langle\Psi_R^{p-1}\partial_t\Psi_R,\Xi_{i,j}\right\rangle_{L^2}=\left\langle\Psi^{p-1}\partial_t\Psi,\Psi_R^{p-1}\partial_t\Psi_R\right\rangle_{L^2}c_{i,j}+\|\Psi^{p-1}\partial_t\Psi\|_{L^2}^2c_{R,i,j}$$

while for all $1\leq l\leq d$,

$$-\langle\Psi^{p-1}w_l,\Xi_{i,j}\rangle_{L^2}=\|\Psi^{p-1}w_l\|_{L^2}^2\varsigma_{i,j,l}+\left\langle\Psi^{p-1}w_l,\Psi_R^{p-1}w_{R,l}\right\rangle_{L^2}\varsigma_{R,i,j,l}$$

and

$$-\left\langle\Psi_R^{p-1}w_{R,l},\Xi_{i,j}\right\rangle_{L^2}=\left\langle\Psi_R^{p-1}w_{R,l},\Psi^{p-1}w_l\right\rangle_{L^2}\varsigma_{i,j,l}+\left\|\Psi_R^{p-1}w_{R,l}\right\|_{L^2}^2\varsigma_{R,i,j,l}.$$

It follows from Lemma 4.3 that

$$\begin{cases}c_{i,j}=-B_*\langle\Psi^{p-1}\partial_t\Psi,\Xi_{i,j}\rangle_{L^2}+\mathcal{O}\left(Q_R^p|\log Q_R|\left\langle\Psi_R^{p-1}\partial_t\Psi_R,\Xi_{i,j}\right\rangle_{L^2}\right),\\ c_{R,i,j}=-B_*\left\langle\Psi_R^{p-1}\partial_t\Psi_R,\Xi_{i,j}\right\rangle_{L^2}+\mathcal{O}\left(Q_R^p|\log Q_R|\langle\Psi^{p-1}\partial_t\Psi,\Xi_{i,j}\rangle_{L^2}\right)\end{cases}$$

and

$$\begin{cases}\varsigma_{i,j,l}=-B_{**}\langle\Psi^{p-1}w_l,\Xi_{i,j}\rangle_{L^2}+\mathcal{O}\left(Q_R^{\frac{3p-1}{2}}|\log Q_R|\left\langle\Psi_R^{p-1}w_{R,l},\Xi_{i,j}\right\rangle_{L^2}\right),\\ \varsigma_{R,i,j,l}=-B_{**}\left\langle\Psi_R^{p-1}w_{R,l},\Xi_{i,j}\right\rangle_{L^2}+\mathcal{O}\left(Q_R^{\frac{3p-1}{2}}|\log Q_R|\langle\Psi^{p-1}w_l,\Xi_{i,j}\rangle_{L^2}\right)\end{cases}$$

for all $1\leq l\leq d$, where $B_*=\|\Psi^{p-1}\partial_t\Psi\|_{L^2}^{-2}$ and $B_{**}=\|\Psi^{p-1}w_d\|_{L^2}^{-2}$. Thus, by (2.7) and Lemma 4.3, the oddness of $\partial_t\Psi$ in $\mathbb{R}$ and the oddness of w_d on $\mathbb{S}^{d-1}$,

$$c_{1,1}\sim c_{R,1,1}\sim Q_R\quad\text{and}\quad\sum_{l=1}^d(|\varsigma_{1,1,l}|+|\varsigma_{R,1,1,l}|)=0. \tag{9.14}$$ eqqnew0023

Similarly, we also have

$$|c_{1,2}|+|c_{R,1,2}|=0\quad\text{and}\quad\sum_{l=1}^d(|\varsigma_{1,2,l}|+|\varsigma_{R,1,2,l}|)\lesssim Q_R. \tag{9.15}$$ eqqnew0024

Again, by (2.7) and Lemma 4.3, the oddness of $\partial_t\Psi$ in $\mathbb{R}$ and the oddness of w_d on $\mathbb{S}^{d-1}$, we have

$$c_{2,1}\sim c_{R,2,1}\sim Q_R^p,\quad\sum_{l=1}^d(|\varsigma_{2,1,l}|+|\varsigma_{R,2,1,l}|)=0 \tag{9.16}$$ eqqnew0025

and

$$c_{2,2}=c_{R,2,2}=0,\quad\sum_{l=1}^d(|\varsigma_{2,2,l}|+|\varsigma_{R,2,2,l}|)\sim 1. \tag{9.17}$$ eqqnew0026

On the other hand, we write $\varrho_{2,1} = \alpha_{2,1}\Psi + \alpha_{2,1,R}\Psi_R + \varrho_{2,1}^{\perp}$ where $\alpha_{2,1}$ and $\alpha_{2,1,R}$ are chosen such that $\left\langle \varrho_{2,1}^{\perp}, \Psi \right\rangle = \left\langle \varrho_{2,1}^{\perp}, \Psi_R \right\rangle = 0$. By (9.4) for $\varrho_{2,1}$, we have

$$\begin{aligned}\|\Psi\|^2\alpha_{2,1} + \mathcal{O}(Q_R)\alpha_{2,1,R} &= \langle \Xi_{2,1}, \Psi\rangle_{L^2} + p\left\langle \Gamma_R^{p-1}\Psi, \alpha_{2,1}\Psi + \alpha_{2,1,R}\Psi_R + \varrho_{2,1}^{\perp}\right\rangle_{L^2}\\ &= \widetilde{B}_0 + \mathcal{O}\left(Q_R\right) + p\|\Psi\|^2\alpha_{2,1}\\ &\quad+\mathcal{O}(Q_R)(\alpha_{2,1} + \alpha_{2,1,R} + \|\varrho_{2,1}^{\perp}\|)\end{aligned}$$

and

$$\begin{aligned}\|\Psi\|^2\alpha_{2,1,R} + \mathcal{O}(Q_R)\alpha_{2,1} &= \langle \Xi_{2,1}, \Psi_R\rangle_{L^2} + p\left\langle \Gamma_R^{p-1}\Psi_R, \alpha_{2,1}\Psi + \alpha_{2,1,R}\Psi_R + \varrho_{2,1}^{\perp}\right\rangle_{L^2}\\ &= \widetilde{B}_0 + \mathcal{O}\left(Q_R\right) + p\|\Psi\|^2\alpha_{2,1,R}\\ &\quad+\mathcal{O}(Q_R)(\alpha_{2,1} + \alpha_{2,1,R} + \|\varrho_{2,1}^{\perp}\|),\end{aligned}$$

where $\widetilde{B}_0 > 0$ is a constant. Now, by similar estimates in the proof of Lemma 7.1, we have $\|\varrho_{2,1}^{\perp}\| \lesssim 1$ which implies that

$$\alpha_{2,1} = \left(1 + \mathcal{O}\left(Q_R\right)\right)\alpha_{2,1,R} = \widetilde{B}_0 + \mathcal{O}\left(Q_R\right) > 0. \tag{9.18}$$ eqnnewnew19990

We further write $\varrho_{2,1}^{\perp} = (\varrho_{2,1}^{\perp})_+ - (\varrho_{2,1}^{\perp})_-$ where $(\varrho_{2,1}^{\perp})_\pm = \max\{\pm\varrho_{2,1}^{\perp}, 0\}$. Since $\Xi_{2,1}$ is positive, by multiplying (9.4) for $\varrho_{2,1}$ with $-(\varrho_{2,1}^{\perp})_-$ and using (9.16) and (9.18), we know that $\|(\varrho_{2,1}^{\perp})_-\| \lesssim Q_R^{1+\sigma}$. It follows from Lemmas 9.2, 9.3 and 9.5 and (9.14), (9.15), (9.16) and (9.17) that

$$\left\|\vartheta_{1,1} + \widetilde{\vartheta}_{1,1} + \beta(\vartheta_{1,2} + \widetilde{\vartheta}_{1,2} + \widetilde{\vartheta}_{1,4}) + \beta^2\vartheta_{2,1} + \beta^3(\vartheta_{2,2} + \widetilde{\vartheta}_{1,3})\right\|_{L^2}^2 \sim \beta^3 + Q_R.$$

By Lemmas 4.3, 9.1, 9.2, 9.3, 9.4 and 9.5, we can estimate as in the proofs of Propositions 5.1, 6.1 and 7.1 to show that

$$\|\mathcal{N}_{\varrho_*, rem,1}\|_{L^2} = o(\beta^3 + Q_R).$$

Thus, we must have $\|f\|_{H^{-1}} \sim \beta^3 + Q_R$. □

We decompose $\varrho = \widetilde{\alpha}\Psi + \widetilde{\alpha}_R\Psi_R + \widetilde{\varrho}^{\perp}$ where $\widetilde{\alpha}$ and $\widetilde{\alpha}_R$ are chosen such that $\left\langle \Psi, \widetilde{\varrho}^{\perp}\right\rangle = 0$ and $\left\langle \Psi_R, \widetilde{\varrho}^{\perp}\right\rangle = 0$. It follows from Lemmas 9.1, 9.2, 9.3, 9.4 and 9.5 that

$$\|\Psi\|^2\widetilde{\alpha} + \mathcal{O}\left(Q_R\right)\widetilde{\alpha}_R = \langle \Xi, \Psi\rangle_{L^2} + \left\langle p\Gamma_R^{p-1}\rho, \Psi\right\rangle_{L^2}$$

and

$$\|\Psi\|^2\widetilde{\alpha}_R + \mathcal{O}\left(Q_R\right)\widetilde{\alpha} = \langle \Xi, \Psi\rangle_{L^2} + \left\langle p\Gamma_R^{p-1}\rho, \Psi\right\rangle_{L^2},$$

where

$$\Xi = \Xi_{1,1} + \widetilde{\Xi}_{1,1} + \beta(\Xi_{1,2} + \widetilde{\Xi}_{1,2} + \widetilde{\Xi}_{1,4}) + \beta^2\Xi_{2,1} + \beta^3(\Xi_{2,2} + \widetilde{\Xi}_{1,3}) \tag{9.19}$$ eqnnewnew19989

with $\widetilde{\Xi}_{1,1} = \sum_{i=0}^{n_0}\widetilde{\Xi}_{1,1,i}$ and $\widetilde{\Xi}_{1,4} = \sum_{i=0}^{n_0}\widetilde{\Xi}_{1,4,i}$.

⟨propq0003⟩ **Proposition 9.3.** *Let $d \geq 2$, $a < 0$ and $b = b_{FS}(a)$. Then $\left\|\beta\Phi_R + \widetilde{\varrho}^{\perp}\right\| \sim \|f\|_{H^{-1}}^{\frac{1}{3}}$ as $\beta \to 0$, provided $Q_R \lesssim \beta^3$.*

Proof. By the symmetry of Γ_R about $s=\frac{R}{2}$, (9.2) and (9.3),

$$\begin{aligned}
&\left\langle \Xi_{1,1}+\beta\Xi_{1,2}+\beta^2\Xi_{2,1}+\beta^3\Xi_{2,2},\Psi\right\rangle_{L^2}\\
=&\left\langle \Xi_{1,1}+\beta\Xi_{1,2}+\beta^2\Xi_{2,1}+\beta^3\Xi_{2,2},\Psi_R\right\rangle_{L^2}\\
=&\left(p\|\Psi\|_{L^{p-1}}^{p-1}+o(1)\right)Q_R+\left(\frac{A_{p,1}\left|\mathcal{S}^{d-1}\right|}{d}\|\Psi\|_{L^{2p}}^{2p}+o(1)\right)\beta^2. \qquad (9.20)
\end{aligned}$$

eqqnew0021

By Lemmas 4.3, 9.1, 9.2, 9.3, 9.4 and 9.5,

$$\begin{aligned}
&\left\langle \widetilde{\Xi}_{1,1}+\beta(\widetilde{\Xi}_{1,2}+\widetilde{\Xi}_{1,4})++\beta^3\widetilde{\Xi}_{1,3},\Psi\right\rangle_{L^2}\\
=&(1+o(1))\left\langle \widetilde{\Xi}_{1,1}+\beta(\widetilde{\Xi}_{1,2}+\widetilde{\Xi}_{1,4})++\beta^3\widetilde{\Xi}_{1,3},\Psi_R\right\rangle_{L^2}\\
=&o(Q_R+\beta^2). \qquad (9.21)
\end{aligned}$$

eqqnew1021

Moreover,

$$\begin{aligned}
p\left\langle\Gamma_R^{p-1}\rho,\Psi\right\rangle_{L^2} &= p\widetilde{\alpha}\left\langle\Gamma_R^{p-1},\Psi^2\right\rangle_{L^2}+p\left\langle\left(\Gamma_R^{p-1}-\Psi^{p-1}\right)\Psi,\widetilde{\varrho}^{\perp}\right\rangle_{L^2}\\
&\quad+\mathcal{O}\left(Q_R\right)\widetilde{\alpha}_R\\
&= p\widetilde{\alpha}\|\Psi\|^2+\mathcal{O}\left(Q_R\right)\left(\widetilde{\alpha}_R+\widetilde{\alpha}+\|\widetilde{\varrho}^{\perp}\|\right) \qquad (9.22)
\end{aligned}$$

eqqnew2021

and similarly,

$$p\left\langle\Gamma_R^{p-1}\rho,\Psi_R\right\rangle_{L^2}=p\widetilde{\alpha}_R\|\Psi\|^2+\mathcal{O}\left(Q_R\right)\left(\widetilde{\alpha}_R+\widetilde{\alpha}+\|\widetilde{\varrho}^{\perp}\|\right). \qquad (9.23)$$

eqqnew3021

It follows from (9.20), (9.21), (9.22) and (9.23) that

$$\widetilde{\alpha}=(1+o(1))\widetilde{\alpha}_R=(B+o(1))Q_R+(C+o(1))\beta^2+\mathcal{O}(Q_R\|\widetilde{\varrho}^{\perp}\|), \qquad (9.24)$$

eqqnew4021

where B and C are two positive constants. Since by Lemmas 9.1, 9.2, 9.3, 9.4 and 9.5, $\widetilde{\varrho}^{\perp}$ satisfies

$$\begin{cases}
-\Delta_\theta\widetilde{\varrho}^{\perp}-\partial_t^2\widetilde{\varrho}^{\perp}+\Lambda_{FS}\widetilde{\varrho}^{\perp}-p\Gamma_R^{p-1}\widetilde{\varrho}^{\perp}=\Xi+\vartheta, \quad \text{in } \mathcal{C},\\
\langle\Psi,\widetilde{\varrho}^{\perp}\rangle=\langle\Psi_R,\widetilde{\varrho}^{\perp}\rangle=\langle\partial_t\Psi,\widetilde{\varrho}^{\perp}\rangle=\langle\partial_t\Psi_R,\widetilde{\varrho}^{\perp}\rangle=0,\\
\langle w_l,\widetilde{\varrho}^{\perp}\rangle=\langle w_{R,l},\widetilde{\varrho}^{\perp}\rangle=0, \text{ for all } 1\leq l\leq d,
\end{cases} \qquad (9.25)$$

eqqnew41102

where Ξ is given by (9.19) and

$$\vartheta=\vartheta_{1,1}+\widetilde{\vartheta}_{1,1}+\beta(\vartheta_{1,2}+\widetilde{\vartheta}_{1,2}+\widetilde{\vartheta}_{1,4})+\beta^2\vartheta_{2,1}+\beta^3(\vartheta_{2,2}+\widetilde{\vartheta}_{1,3}). \qquad (9.26)$$

?eqnnewnew19988?

By Lemmas 9.1, 9.2, 9.3, 9.4 and 9.5, we can use the same test functions in the proof of Proposition 9.1 to (9.25) show that

$$\|\widetilde{\varrho}^{\perp}\|\sim\beta^2+\begin{cases}
Q_R, \quad p>2,\\
Q_R\left|\log Q_R\right|, \quad p=2,\\
Q_R^{\frac{p}{2}}, \quad 1<p<2.
\end{cases}$$

Thus, by the classical elliptic estimates and (9.24), we have

$$\|\varrho\|_{L^\infty}=\mathcal{O}\left(Q_R^{\frac{p}{2}\wedge 1}\left|\log Q_R\right|+\beta^2\right). \qquad (9.27)$$

eqqnew9999

Now, if $Q_R\lesssim\beta^3$ then we have $\left\|\beta\Phi_R+\widetilde{\varrho}^{\perp}\right\|\sim\beta\sim\|f\|_{H^{-1}}^{\frac{1}{3}}$ by Proposition 9.2. $\square$

For the sake of simplicity, we denote $\widetilde{v} = v + \varrho$. We shall decompose $\widetilde{v}$ as in lemma 3.1 by considering the following variational problem:

$$\inf_{\vec{\alpha}_2 \in (\mathbb{R}_+)^2, \vec{s}_2 \in \mathbb{R}^\nu} \left\| \widetilde{v} - \sum_{j=1}^{2} \alpha_j \Psi_{s_j} \right\|^2 . \tag{9.28}$$ eqqnew0011

Clearly, as (3.3), the variational problem (9.28) has minimizers, say $(\widetilde{\alpha}_1^*, \widetilde{\alpha}_2^*, \widetilde{s}_1^*, \widetilde{s}_2^*)$, satisfying

$$\sum_{j=1}^{2} \left|\widetilde{\alpha}_j^* - 1\right| \to 0 \quad \text{and} \quad |\widetilde{s}_1^* - \widetilde{s}_2^*| \to +\infty \tag{9.29}$$ eqqnew1005

as $R \to +\infty$ and $\beta \to 0$.

⟨propq0002⟩ **Proposition 9.4.** *Let $d \geq 2$, $a < 0$ and $b = b_{FS}(a)$. Then for $R > 0$ sufficiently large and $\beta > 0$ sufficiently small, the variational problem (9.28) has a unique minimizer, say $(\widetilde{\alpha}_1^*, \widetilde{\alpha}_2^*, \widetilde{s}_1^*, \widetilde{s}_2^*)$, satisfying*

$$\widetilde{s}_1^* = \mathcal{O}\left(\beta^4 + Q_R^p\right), \quad \widetilde{s}_2^* = R + \mathcal{O}\left(\beta^4 + Q_R^p\right)$$

and

$$\widetilde{\alpha}_1^* - 1 = \frac{\langle \rho, \Psi \rangle}{\|\Psi\|^2} + \mathcal{O}\left(\beta^4 + Q_R^p\right), \quad \widetilde{\alpha}_2^* - 1 = \frac{\langle \rho, \Psi_R \rangle}{\|\Psi\|^2} + \mathcal{O}\left(\beta^4 + Q_R^p\right).$$

Proof. Since $(\widetilde{\alpha}_1^*, \widetilde{\alpha}_2^*, \widetilde{s}_1^*, \widetilde{s}_2^*)$ is a minimizer of the variational problem (9.28) and Ψ and $\partial_t \Psi$ are solutions of (2.5) and (2.6), respectively, we have

$$0 = \left\langle \widetilde{v} - \sum_{i=1}^{2} \widetilde{\alpha}_i^* \Psi_{\widetilde{s}_i^*}, \Psi_{\widetilde{s}_j^*} \right\rangle = \left\langle \widetilde{v} - \sum_{i=1}^{2} \widetilde{\alpha}_i^* \Psi_{\widetilde{s}_i^*}, \Psi_{\widetilde{s}_j^*}^p \right\rangle_{L^2} \tag{9.30}$$ eqqnew0017

and

$$0 = \left\langle \widetilde{v} - \sum_{i=1}^{2} \widetilde{\alpha}_i^* \Psi_{\widetilde{s}_i^*}, \partial_t \Psi_{\widetilde{s}_j^*} \right\rangle = \left\langle \widetilde{v} - \sum_{i=1}^{2} \widetilde{\alpha}_i^* \Psi_{\widetilde{s}_i^*}, p\Psi_{\widetilde{s}_j^*}^{p-1} \partial_t \Psi_{\widetilde{s}_j^*} \right\rangle_{L^2} \tag{9.31}$$ eqqnew0016

for all $j = 1, 2$. By the oddness of w_d on $\mathbb{S}^{d-1}$, the oddness of $\partial_t \Psi$ in $\mathbb{R}$, (9.29) and (9.31), we have $\sum_{i=1}^{2} |\widetilde{\alpha}_i^*| \lesssim 1$ and

$$\left\langle \widetilde{v}, p\Psi_{\widetilde{s}_j^*}^{p-1} \partial_t \Psi_{\widetilde{s}_j^*} \right\rangle_{L^2} = \mathcal{O}\left(\left\langle \Psi_{\widetilde{s}_i^*}, \Psi_{\widetilde{s}_j^*}^{p-1} \partial_t \Psi_{\widetilde{s}_j^*} \right\rangle_{L^2} \right) = \mathcal{O}\left(Q_R^{\frac{|\widetilde{s}_1^* - \widetilde{s}_2^*|}{R}} \right).$$

Recall that $\varrho = \varrho_{1,1} + \widetilde{\varrho}_{1,1} + \beta(\varrho_{1,2} + \widetilde{\varrho}_{1,2} + \widetilde{\varrho}_{1,4}) + \beta^2 \varrho_{2,1} + \beta^3(\varrho_{2,2} + \widetilde{\varrho}_{1,3})$. Thus, by Lemmas 9.1, 9.2, 9.3, 9.4 and 9.5,

$$\left| \left\langle \varrho, \Psi_{\widetilde{s}_j^*}^{p-1} \partial_t \Psi_{\widetilde{s}_j^*} \right\rangle_{L^2} \right| \lesssim \beta^2 + Q_R. \tag{9.32}$$ eqqnew0018

On the other hand, for every $s_j \leq \frac{R}{2}$, by Lemma 4.3,

$$\left\langle \Gamma_R, \Psi_{s_j}^{p-1} \partial_t \Psi_{s_j} \right\rangle_{L^2} = \left\langle \Psi, \Psi_{s_j}^{p-1} \partial_t \Psi_{s_j} \right\rangle_{L^2} + \mathcal{O}\left(Q_R^{\frac{R - s_j}{R}} \right). \tag{9.33}$$ eqqnew0019

Note that by the evenness of Ψ in $\mathbb{R}$, $\left\langle \Psi, \Psi_{s_j}^{p-1} \partial_t \Psi_{s_j} \right\rangle_{L^2} = 0$ has a uniquely nondegenerate solution $s_j = 0$ on $(-\infty, \frac{R}{2}]$. Thus, by (9.32), (9.33), the symmetry of Γ_R

about $s=\frac{R}{2}$, for $R>0$ sufficiently large, the solution of (9.30) and (9.31) must satify

$$\widetilde{s}_1^*=\mathcal{O}\left(\beta^2+Q_R\right)\quad\text{and}\quad\widetilde{s}_2^*=R+\mathcal{O}\left(\beta^2+Q_R\right). \tag{9.34}$$

eqqnew0031

which, together with (9.30), implies that

$$\left\langle\Gamma_R+\varrho,\Psi^p_{\widetilde{s}_j^*}\right\rangle_{L^2}=\widetilde{\alpha}_j^*\|\Psi\|^2+\mathcal{O}\left(\left\langle\Psi_{\widetilde{s}_i^*},\Psi^p_{\widetilde{s}_j^*}\right\rangle_{L^2}\right)=\widetilde{\alpha}_j^*\|\Psi\|^2+\mathcal{O}\left(Q_R\right)$$

for all $j=1,2$. Similar to (9.32) and (9.33), we have

$$\left\langle\Gamma_R+\varrho,\Psi^p_{\widetilde{s}_j^*}\right\rangle_{L^2}=\|\Psi\|^2+\mathcal{O}\left(\beta^2+Q_R\right).$$

Thus, we also have

$$\widetilde{\alpha}_j^*=1+\mathcal{O}\left(\beta^2+Q_R\right). \tag{9.35}$$

eqqnew0030

Now, by (9.31) once more, the oddness of $\partial_t^3\Psi$, the Taylor expansion and the orthogonal conditions of ρ given by Lemmas 9.1, 9.2, 9.3, 9.4 and 9.5, we have

$$\begin{aligned}
0 &= \left\langle\widetilde{v}-\sum_{i=1}^{2}\widetilde{\alpha}_i^*\Psi_{\widetilde{s}_i^*},\partial_t\Psi_{\widetilde{s}_1^*}\right\rangle\\
&= \left\langle\Psi^p,\partial_t\Psi_{\widetilde{s}_1^*}\right\rangle_{L^2}+\left\langle\Psi_R-\widetilde{\alpha}_2^*\Psi_{\widetilde{s}_2^*},p\Psi^{p-1}_{\widetilde{s}_1^*}\partial_t\Psi_{\widetilde{s}_1^*}\right\rangle_{L^2}+\left\langle\varrho,p\Psi^{p-1}_{\widetilde{s}_1^*}\partial_t\Psi_{\widetilde{s}_1^*}\right\rangle_{L^2}\\
&= -\left\langle\Psi^p,\partial_t^2\Psi\right\rangle_{L^2}\widetilde{s}_1^*+\mathcal{O}\left((\widetilde{s}_1^*)^3\right)+(1-\widetilde{\alpha}_2^*)\left\langle\Psi_{\widetilde{s}_2^*},p\Psi^{p-1}_{\widetilde{s}_1^*}\partial_t\Psi_{\widetilde{s}_1^*}\right\rangle_{L^2}\\
&\quad+\left\langle\partial_t\Psi_{\widetilde{s}_2^*},p\Psi^{p-1}_{\widetilde{s}_1^*}\partial_t\Psi_{\widetilde{s}_1^*}\right\rangle_{L^2}(\widetilde{s}_2^*-R)+\mathcal{O}\left((\widetilde{s}_2^*-R)^2+\|\varrho\|_{L^\infty}\widetilde{s}_1^*\right),
\end{aligned}$$

which, together with (9.27), (9.34) and (9.35), implies that $\widetilde{s}_1^*=\mathcal{O}\left(\beta^4+Q_R^p\right)$. Similarly, we also have $\widetilde{s}_2^*=R+\mathcal{O}\left(\beta^4+Q_R^p\right)$. Again, by (9.30), (9.35) and the Taylor expansion,

$$\begin{aligned}
\left\langle\Gamma_R+\varrho,\Psi^p_{\widetilde{s}_1^*}\right\rangle_{L^2} &= \widetilde{\alpha}_1^*\|\Psi\|^2+\left\langle\Psi_{\widetilde{s}_2^*},\Psi^p_{\widetilde{s}_1^*}\right\rangle_{L^2}+\mathcal{O}\left(\left(\beta^2+Q_R\right)^2\right)\\
&= \widetilde{\alpha}_1^*\|\Psi\|^2+\left\langle\Psi_R,\Psi^p_{\widetilde{s}_1^*}\right\rangle_{L^2}+\mathcal{O}\left(\beta^4+Q_R^p\right),
\end{aligned}$$

which, together with

$$\left\langle\Gamma_R+\varrho,\Psi^p_{\widetilde{s}_1^*}\right\rangle_{L^2}=\|\Psi\|^2+\left\langle\Psi_R,\Psi^p_{\widetilde{s}_1^*}\right\rangle_{L^2}+\left\langle\rho,\Psi^p_{\widetilde{s}_1^*}\right\rangle_{L^2},$$

implies that $\widetilde{\alpha}_1^*-1=\frac{\langle\varrho,\Psi\rangle}{\|\Psi\|^2}+\mathcal{O}\left(\beta^4+Q_R^p\right)$. Similarly, we also have $\widetilde{\alpha}_2^*-1=\frac{\langle\varrho,\Psi_R\rangle}{\|\Psi\|^2}+\mathcal{O}\left(\beta^4+Q_R^p\right)$. □

Let $\widetilde{v}_\pm=\max\{\pm\widetilde{v},0\}$. Then $\widetilde{v}=\widetilde{v}_+-\widetilde{v}_-$ and by (9.7),

$$-\Delta_\theta\widetilde{v}_+-\partial_t^2\widetilde{v}_++\Lambda_{FS}\widetilde{v}_+-\widetilde{v}_+^p=f+\mathcal{G}(\widetilde{v}_-):=f_{\widetilde{v}_+}, \tag{9.36}$$

eqqnew1020

where $\mathcal{G}(\widetilde{v}_-)=-\Delta_\theta\widetilde{v}_--\partial_t^2\widetilde{v}_-+\Lambda_{FS}\widetilde{v}_--\widetilde{v}_-^p$.

Proof of (2) **of Theorem 1.3:** Recall that we have the decomposition

$$\widetilde{v}=v+\widetilde{\alpha}\Psi+\widetilde{\alpha}_R\Psi_R+\widetilde{\varrho}^\perp \tag{9.37}$$

eqqnew0028

in $H^1(\mathcal{C})$, where by the orthogonal conditions of $\widetilde{\varrho}^\perp$ and (9.24),

$$\langle\varrho,\Psi\rangle=\widetilde{\alpha}\|\Psi\|^2+\mathcal{O}\left(\left(\beta^2+Q_R\right)^2\right)\quad\text{and}\quad\langle\varrho,\Psi_R\rangle=\widetilde{\alpha}_R\|\Psi\|^2+\mathcal{O}\left(\left(\beta^2+Q_R\right)^2\right).$$

It follows from Proposition 9.4 that

$$\widetilde{\alpha}_1^* = 1 + \widetilde{\alpha} + \mathcal{O}\left(\beta^4 + Q_R^p\right) \quad \text{and} \quad \widetilde{\alpha}_2^* = 1 + \widetilde{\alpha}_R + \mathcal{O}\left(\beta^4 + Q_R^p\right), \tag{9.38}$$ eqqnew9998

which, together with Proposition 9.4 once more and the Taylor expansion, implies that

$$\begin{aligned} \widetilde{v} &= \sum_{j=1}^{2} \widetilde{\alpha}_j^* \Psi_{\widetilde{s}_j^*} + \widetilde{\varrho} \\ &= \Gamma_R + \widetilde{\alpha}\Psi + \widetilde{\alpha}_R \Psi_R + \widetilde{\varrho} + \mathcal{O}\left(\beta^4 + Q_R^p\right) \end{aligned} \tag{9.39}$$ eqqnew0029

in $H^1(\mathcal{C})$. By (9.37) and (9.39), we have

$$\widetilde{\varrho} = \beta\Phi_R + \widetilde{\varrho}^\perp + \mathcal{O}\left(\beta^4 + Q_R^p\right).$$

Thus, by (9.24), (9.38) and Proposition 9.3, we have

$$\begin{aligned} \left\| \widetilde{v} - \sum_{j=1}^{2} \Psi_{\widetilde{s}_j^*} \right\| &= \left\| \widetilde{v} - \sum_{j=1}^{2} \widetilde{\alpha}_j^* \Psi_{\widetilde{s}_j^*} \right\| + \mathcal{O}\left(\sum_{j=1}^{2} |\widetilde{\alpha}_j^* - 1| \right) \\ &\sim \|f\|_{H^{-1}}^{\frac{1}{3}}, \end{aligned} \tag{9.40}$$ eqqnew0032

provided $Q_R \lesssim \beta^3$. By Lemmas 9.1, 9.2, 9.3 and 9.5, we know that

$$|\beta\Phi_R + \varrho_{1,1} + \widetilde{\varrho}_{1,1} + \beta(\varrho_{1,2} + \widetilde{\varrho}_{1,2} + \widetilde{\varrho}_{1,4})| \lesssim \Gamma_R$$

in $\mathcal{C}$ for sufficiently small β and sufficiently large R. Thus, $0 \le \widetilde{v}_- \le |\beta^2 \varrho_{2,1} + \beta^3 \varrho_{2,2}|$ in $\mathcal{C}$. It follows from (9.7), Lemmas 9.1 and 9.4 and Proposition 9.2 that

$$\|\widetilde{v}_-\|^2 \lesssim \langle f, \widetilde{v}_- \rangle_{L^2} = \mathcal{O}\left(\left(\beta^2 + Q_R\right)^2 \right),$$

which, together with (9.36) and (9.40), implies that $\widetilde{v}_+$ is the desired function. □

⟨rmkn0001⟩ **Remark 9.1.** *The optimal example of Theorem 1.3 in this section, given by $\widetilde{v} = v + \varrho$, precisely describes the relation between $\|f\|_{H^{-1}}$ and $dist_{H^1}(\widetilde{v}, \mathcal{M}^2)$ where*

$$dist_{H^1}(\widetilde{v}, \mathcal{M}^2) = \inf_{\overrightarrow{\alpha}_\nu \in (\mathbb{R}_+)^2, \overrightarrow{s}_2 \in \mathbb{R}^\nu} \left\| \widetilde{v} - \sum_{j=1}^{2} \alpha_j \Psi_{s_j} \right\|_{H^1}.$$

Indeed, we have $\|f\|_{H^{-1}} \sim \beta^3 + Q_R$ and

$$dist_{H^1}(\widetilde{v}, \mathcal{M}^2) \sim \beta + \begin{cases} Q_R, & p > 2, \\ Q_R |\log Q_R|, & p = 2, \\ Q_R^{\frac{p}{2}}, & 1 < p < 2. \end{cases}$$

If the interaction of two bubbles is much smaller than the projections on nontrivial kernel, that is, $\beta^3 \gtrsim Q_R$, then we have $dist_{H^1}(\widetilde{v}, \mathcal{M}^2) \sim \|f\|_{H^{-1}}^{\frac{1}{3}}$. If the interaction of two bubbles is much large than the projections on nontrivial kernel, that is,

$$\beta \lesssim \begin{cases} Q_R, & p > 2, \\ Q_R |\log Q_R|, & p = 2, \\ Q_R^{\frac{p}{2}}, & 1 < p < 2, \end{cases}$$

then we have

$$dist_{H^1}(\widetilde{v},\mathcal{M}^2) \sim \begin{cases} \|f\|_{H^{-1}}, \quad p>2, \\ \|f\|_{H^{-1}} \left|\log \|f\|_{H^{-1}}\right|, \quad p=2, \\ \|f\|_{H^{-1}}^{\frac{p}{2}}, \quad 1<p<2. \end{cases} \tag{9.41}$$ eqnnewnew19987

If the interaction of two bubbles is somehow comparable with their projections on nontrivial kernel, that is $\beta^3 \lesssim Q_R$ and

$$\beta \gtrsim \begin{cases} Q_R, \quad p>2, \\ Q_R \left|\log Q_R\right|, \quad p=2, \\ Q_R^{\frac{p}{2}}, \quad 1<p<2, \end{cases}$$

then $dist_{H^1}(\widetilde{v},\mathcal{M}^2) \sim \|f\|_{H^{-1}}^t$ for some t between the values in (9.41) and $\frac{1}{3}$.

10. Acknowledgments

The research of J. Wei is partially supported by GRF grant entitled "New frontiers in singular limits of elliptic and parabolic equations" and the research of Y. Wu is supported by NSFC (No. 12171470). Part of the work was done when Y. Wu was visiting CUHK. He wants to express gratitude for its hospitality.

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