STABLE SOLUTIONS OF $U(1)$ YANG-MILLS-HIGGS MODEL IN $\mathbb{R}^4$

YONG LIU, JUNCHENG WEI, AND ZIKAI YE

ABSTRACT. We give a positive answer to the conjecture of Liu-Ma-Wei-Wu in [32] that the family of entire solutions to the $U(1)$-Yang-Mills-Higgs equation constructed by the gluing method in that paper are stable. This is the first family of examples of nontrivial stable critical points to the $U(1)$-Yang-Mills-Higgs model in higher dimensional Euclidean space. Intuitively, the stability of these solutions corresponds to the fact that holomorphic curves are area-minimizing. We also show that these entire solutions are non-degenerate. Our proof is based on detailed analysis of the linearized operators around this family and the spectrum estimates of the Jacobi operator by Arezzo-Pacard [2].

1. INTRODUCTION AND STATEMENT OF MAIN RESULT

Yang-Mills-Higgs type functionals and their associated Euler-Lagrange equations are among the most important models in modern physics. The study of these models, for instance, the instanton and monopole equations[21, 30], has triggered many fascinating mathematical theories and results.

Here we are interested in a Yang-Mills-Higgs model with Ginzburg-Landau type potential and $U(1)$ gauge group (also called magnetic Ginzburg-Landau equation, in low dimensions):

$$\mathcal{E}(\psi, A) := \int_M \left\{ |\nabla_A \psi|^2 + |dA|^2 + \frac{\lambda}{4}\left(1 - |\psi|^2\right)^2 \right\}. \tag{1.1}$$

Recently there has been increasing interest in this action functional. Lin-Rivière, Parise-Pigati-Stern, Pigati-Stern [31, 38, 39] studied the asymptotic behavior of critical points of this functional in the self-dual case. Among other things, they showed that a sequence of solutions with uniformly bounded energies will converge in a suitable sense to a codimension two integral varifold. When the solutions are minimizers, the resulting varifold will also be area-minimizing. This extends the previous result of Hong-Jost-Struwe [29] for Riemann surfaces and Bradlow [8] for Kähler manifolds.

On the opposite side, using a variational argument, De Pilippis-Pigati [17] proved the existence of a family of solutions concentrating on given non-degenerate minimal submanifolds of codimension two. Presumably, the Morse indices of these solutions should be related to that of the minimal submanifolds. Note that in the self-dual case, classification results for entire solutions have obtained by Taubes [44, 45] for finite energy critical points in 2d and De Philippis-Halavati-Pigati in [16], which states that local minimizers satisfying suitable energy growing estimates have to be trivial. At this point, it is worth pointing out that in the non-self dual case, Rivière [41] proved in the 2d case that local minimizers are unique up to gauge transformation, provided that the coupling constant λ is large enough.

The aforementioned results and their proof are indeed partly inspired that of the Γ-convergence and related results for the Allen-Cahn functional:

$$\int \left\{ |\nabla u|^2 + \frac{1}{2}\left(1 - u^2\right)^2 \right\},$$

where u is a scalar function. The Allen-Cahn equation

$$\Delta u + u - u^3 = 0 \text{ in } \mathbb{R}^n, \tag{1.2}$$

which is the Euler-Langrange equation of this functional, can be used as a regularization of codimension one minimal submanifolds. This point of view turns out to be very useful and has some important applications in the minimal surface theory, see for instances [13, 14, 20, 24, 25]. It should also be emphasized that the codimension two case imposes more technical difficulties than the codimension one case.

To better state our main results, let us mention some other interesting nontrivial results obtained so far for the codimension one case. The celebrated De Giorgi conjecture [15] states that for any bounded entire solution u to the Allen-Cahn equation (1.2) which is strictly monotone in one direction (x_n-direction for example) should be one-dimensional. This is established in dimension 2 by Ghoussoub-Gui [22] and in dimension 3 by Ambrosio-Cabré [1]. Partial results are obtained by Ghoussoub-Gui [23] in dimension 4 and 5. Under the additional assumption that

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1, \text{ for all } x' \in \mathbb{R}^n,$$

Savin [43] proves the affirmative of the De Giorgi conjecture for $4 \leq n \leq 8$. On the other hand, Gluing techniques turn out to be powerful in the construction of entire solutions of the Allen-Cahn equation. As a matter of fact, counterexamples of the De Giorgi conjecture in dimension 9 have been constructed by Del Pino-Kowalczyk-Wei [18], using the Bombieri-De Giorgi-Giusti minimal graphs.

Later on, entire stable solutions to the Allen-Cahn equation in dimension $n \geq 8$ are also constructed in [37] based on the family of minimal submanifolds asymptotic to the famous Simons' cone. These solutions are shown to be global minimizers in Liu-Wang-Wei [33]. Other stable solutions are saddle solutions on Simons' cones (Cabré-Terra [10, 11]). They are stable in higher dimensions (Cabré [12], Liu-Wang-Wei [34]). The key ingredient in the proof of stability is the existence of a positive kernel $\phi > 0$ of the linearized operator $\Delta + 1 - 3u^2$ at the solution u. Once such ϕ exists, then the stability follows from testing the Allen-Cahn equation by $\phi^{-1}\psi^2$ to get

$$\int_{\mathbb{R}^n} |\nabla \psi|^2 - \psi^2 + 3u^2\psi^2 = \int_{\mathbb{R}^n} |\nabla \psi - \phi^{-1}\psi \nabla \phi|^2 \geq 0 \tag{1.3}$$

for any compactly supported test function ψ. However, a similar argument using (1.3) does not work for the stability of the Yang-Mills-Higgs equation that we will discuss later, since the solution is vector-valued and is not simply a scalar function. We also would like to mention that in $\mathbb{R}^3$, Del Pino-Kowalczyk-Wei [19] constructed a family of finite Morse index solutions to the Allen-Cahn equation that concentrated near complete, embedded, non-degenerate minimal surface with finite total curvature. The Morse indices of these solutions coincide with the concentrated minimal surfaces. In fact, all these solutions constructed by gluing techniques exhibit suitable concentration near codimension one minimal submanifolds, while in the normal direction, they look like the standard one-dimensional heteroclinic solution of the Allen-Cahn equation.

Let $\Delta_A := \nabla_A^* \nabla_A$ be the connection Laplacian. For the Yang-Mills-Higgs model (1.1) in the Euclidean space with a trivial Hermian bundle, the Euler-Lagrange equation has the form

$$\begin{cases} -\Delta_A \psi + \frac{\lambda}{2}(|\psi|^2 - 1)\psi = 0 \text{ in } \mathbb{R}^n, \\ d^* dA - Im(\nabla_A \psi \cdot \bar{\psi}) = 0 \text{ in } \mathbb{R}^n. \end{cases} \tag{1.4}$$

Naturally, in view of the developments for the Allen-Cahn function, one expects to be able to build solutions to (1.4) based on the standard vortex solutions (See Section 2 for more precise form of these solutions) in the two-dimensional plane, in replace of the one dimensional heteroclinic solution to the Allen-Cahn equation. The stability(or instability) of these 2d vortex solutions depends on the parameter λ and the degree of the solutions and is resolved in Gustafson-Sigal [26]. A quantitative stability for critical points is proved by Halavati [27, 28]. Detailed analysis, including the Γ-convergence theory, of solutions to the equation (1.4) in dimension two is discussed in Sandier-Serfaty [42]. Actually, the classical Ginzburg-Landau equation has also been discussed there. The main difference between the theory of the Ginzburg-Landau equation and (1.4) is their asymptotic behavior at far field. Vortex solutions to (1.4) decay exponentially fast, while that of the Ginzburg-Landau only decays at an algebraic rate, implying that they are much more difficult to deal with. Here we will not touch on the classical Ginzburg-Landau equation, only refer to [42] and the references therein for more discussion, see also [7, 36].

At this stage, we already know that entire solutions of the Yang-Mills-Higgs model with multiple vortex points can been constructed in [46] in the two dimensional plane. In higher dimensions, Brendle [9] and Badran-Del Pino [3, 4, 5] are able to use gluing construction to build manuy interesting solutions concentrated on codimension 2 minimal submanifolds based on the aforementioned 2d vortex solutions.

Since holomorphic curves are area-minimizing, many codimension two stable minimal submanifolds already exist in $\mathbb{R}^4$. In [32], a family of entire solutions is constructed using Lyapunov-Schmidt reduction arguments, they concentrated on suitable rescaling of the codimension 2 minimal submanifold studied by Arezzo-Pacard in [2], given by

$$\Gamma = \frac{e^{is}}{\sqrt{\sin 2s}}\Theta, \tag{1.5}$$

where $s \in (0, \frac{\pi}{2}), \Theta = (\cos\theta, \sin\theta) \in \mathbb{S}^1, \theta \in [0, 2\pi)$. Geometrically, this manifold has two planar ends and can be regarded as a desingularization of the union of two orthogonal planes. Let us denote these solutions by U_ϵ, where $\epsilon > 0$ is a sufficiently small rescaling parameter. Here we show that they are stable critical points of the Yang-Mills-Higgs model (1.1), with the manifold M being the four-dimensional Euclidean space $\mathbb{R}^4$. Our main result can be stated in the following

Theorem 1.1. *The solutions U_ϵ are stable and non-degenerate.*

The notion of stable means that the quadratic form associated to the Yang-Mills-Higgs functional is always nonnegative. In view of its counterpart in the Allen-Cahn case [19], intuitively, this should be true due to the fact that the concentrating minimal submanifold is area minimizing. However, the proof will involve many technical details. The main step is to get precise asymptotic behavior of the solutions. This will then enable us to reduce the stability of solutions to the analysis of the Jacobi operator. It is expected that this family of solutions are also minimizers of the functional, but this problem seems to be much more difficult. We believe that this strategy can be applied to prove that the Morse index of some critical points of $U(1)$-Yang-Mills-Higgs functional are determined by its concentrated minimal submanifolds, as in the Allen-Cahn case.

The non-degeneracy of U_ϵ means that any bounded kernel of linearized operator $L(\cdot; U_\epsilon)$ at U_ϵ is a linear combination of Z_j, $j = 1, \cdots, 6$(corresponding to translation or rotation invariance of the equation) and gauge kernels(which arise from the gauge invariance) to be introduced more precisely in Section 2.

The rest of the paper is devoted to proving Theorem 1.1. It is organized in the following way: In Section 2, we recall some preliminary results, including the properties of the minimal submanifold Γ, the vortex solution and related apriori estimates. In Section 3, we analyze the linearized operator

$\mathbb{L}(\cdot;\psi,A)$ in detail and build up its relationship between the Jacobi operator L_Γ of Γ. Section 4 finishes the proof of Theorem 1.1.

Acknowledgement. Y. Liu is supported by the National Key R&D Program of China 2022YFA1005400 and NSFC 12471204. J.C. Wei is supported by National Key R&D Program of China 2022YFA1005602, and Hong Kong General Research Fund "New frontiers in singular limits of nonlinear partial differential equations". The third author thanks Wangzhe Wu for useful discussion.

2. PRELIMINARIES

In this section, we collect some facts which will be used later, including the stability of the minimal submanifold Γ, Fermi coordinates, and related computations. We also list the properties of 2d vortex solutions and the vortex solutions constructed in [32]. Apriori estimates of the eigenvalue problems and linear theory for the linearized operator are also briefly discussed.

2.1. **The minimal submanifold Γ.** Recall that the minimal submanifold Γ defined by (1.5) is of codimension two. It is a holomorphic curve in $\mathbb{C}^2$. Hence, it is area-minimizing, in particular, it is stable. That is,

Proposition 2.1. *Γ is a stable minimal submanifold.*

After a rescaling by a small parameter ϵ, we get a family of minimal submanifolds, all of them being far away from the origin, with the form

$$\Gamma_\epsilon = \frac{e^{i\epsilon\tilde{s}}}{\epsilon\sqrt{\sin 2\epsilon\tilde{s}}}\tilde{\Theta},$$

where $\tilde{s} = \epsilon^{-1}s \in (0, \frac{\pi}{2\epsilon})$, $\tilde{\theta} = \epsilon^{-1}\theta \in [0, \frac{2\pi}{\epsilon})$ and $\tilde{\Theta} = (\cos\epsilon\tilde{\theta}, \sin\epsilon\tilde{\theta})$. [32] constructed a solution $(\psi_\epsilon, A_\epsilon)$ near Γ_ϵ. To describe the solutions in a more precise way, it will be necessary to use the Fermi coordinate, to be recalled in sequel.

Let $\mathbf{m} = ie^{-i\epsilon\tilde{s}}\tilde{\Theta}$, $\mathbf{n} = ie^{i\epsilon\tilde{s}}\tilde{\Theta}^\perp$ be the two unit normal vectors of Γ_ϵ, where $\tilde{\Theta}^\perp = (-\sin\epsilon\tilde{\theta}, \cos\epsilon\tilde{\theta})$. Define a map $T : (\tilde{s}, \tilde{\theta}, a, b) \to (z_1, z_2) \in \mathbb{C}^2$:

$$\begin{aligned} Y &= \Gamma_\epsilon + a\mathbf{m} + b\mathbf{n} \\ &= \left(\frac{\cos\epsilon\tilde{s}}{\epsilon\sqrt{\sin 2\epsilon\tilde{s}}}\tilde{\Theta} + a\sin\epsilon\tilde{s}\cdot\tilde{\Theta} - b\sin\epsilon\tilde{s}\cdot\tilde{\Theta}^\perp, \frac{\sin\epsilon\tilde{s}}{\epsilon\sqrt{\sin 2\epsilon\tilde{s}}}\tilde{\Theta} + a\cos\epsilon\tilde{s}\cdot\tilde{\Theta} + b\cos\epsilon\tilde{s}\cdot\tilde{\Theta}^\perp \right). \end{aligned}$$

Write $\Sigma_\epsilon := \{(\tilde{s}, \tilde{\theta}, a, b) : a^2 + b^2 < \frac{1}{\epsilon\sin 2\epsilon\tilde{s}} =: r_\epsilon^2\}$. These provide a local coordinate system for the tubular neighborhood Σ_ϵ, called the Fermi coordinates. We remark that actually, this local coordinate system is well-defined in $\{(\tilde{s}, \tilde{\theta}, a, b) : a^2 + b^2 < \epsilon^{-1}r_\epsilon^2\}$. But Σ_ϵ is sufficient for later purposes.

In the Fermi coordinates, the metric is given by

$$(g_{ij}) = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & 0 & 0 \\ \tilde{a}_{21} & \tilde{a}_{22} & -b\epsilon\cos(2\epsilon\tilde{s}) & a\epsilon\cos(2\epsilon\tilde{s}) \\ 0 & -b\epsilon\cos(2\epsilon\tilde{s}) & 1 & 0 \\ 0 & a\epsilon\cos(2\epsilon\tilde{s}) & 0 & 1 \end{pmatrix},$$

where

$$\tilde{A} := (\tilde{a}_{ij}) = \begin{pmatrix} \left(\epsilon a - \frac{1}{(\sin 2\epsilon\tilde{s})^{\frac{3}{2}}}\right)^2 + \epsilon^2 b^2 & \frac{-2b\epsilon}{\sqrt{\sin 2\epsilon\tilde{s}}} \\ \frac{-2b\epsilon}{\sqrt{\sin 2\epsilon\tilde{s}}} & \epsilon^2(a^2+b^2) + \frac{1}{\sin 2\epsilon\tilde{s}} + 2a\epsilon\sqrt{\sin 2\epsilon\tilde{s}} \end{pmatrix}$$

is a metric on Γ_ϵ. Its determinant will be denoted by $G = \det(g_{ij})$. One of the reasons that we will do computations in these coordinates, instead of some moving frames, is that all these functions are completely explicit.

For simplicity, we define ϱ such that $\varrho(x) := (\sin 2s)^{-1}$ for $x \in \Sigma_1$ and extend it outside Σ_1 to be a smooth function on $\mathbb{R}^4$ with

$$c_1(1+|x|) \le \varrho(x) \le c_2(1+|x|) \text{ in } \mathbb{R}^4$$

for some constants $c_1, c_2 > 0$. Then the inverse of the metric matrix can be expanded as

$$(g^{ij}) = \begin{pmatrix} \rho^{-6} + 2a\epsilon\rho^{-9} & 2b\epsilon\rho^{-7} & 0 & 0 \\ 2b\epsilon\rho^{-7} & \rho^{-2} - 2a\epsilon\rho^{-5} & b\epsilon\rho^{-2}\cos 2\epsilon\tilde{s} & -a\epsilon\rho^{-2}\cos 2\epsilon\tilde{s} \\ 0 & b\epsilon\rho^{-2}\cos 2\epsilon\tilde{s} & 1 & 0 \\ 0 & -a\epsilon\rho^{-2}\cos 2\epsilon\tilde{s} & 0 & 1 \end{pmatrix}$$
$$+ \begin{pmatrix} 3r^2\epsilon^2\rho^{-12} & 0 & b^2\epsilon^2\rho^{-7}\cos 2\epsilon\tilde{s} & -ab\epsilon^2\rho^{-7}\cos 2\epsilon\tilde{s} \\ 0 & 3r^2\epsilon^2\rho^{-8} & -2ab\epsilon^2\rho^{-5}\cos 2\epsilon\tilde{s} & 2a^2\epsilon^2\rho^{-5}\cos 2\epsilon\tilde{s} \\ b^2\epsilon^2\rho^{-7}\cos 2\epsilon\tilde{s} & -2ab\epsilon^2\rho^{-5}\cos 2\epsilon\tilde{s} & b^2\epsilon^2\rho^{-2}\cos^2 2s & -ab\epsilon^2\rho^{-2}\cos^2 2s \\ -ab\epsilon^2\rho^{-7}\cos 2\epsilon\tilde{s} & 2a^2\epsilon^2\rho^{-5}\cos 2\epsilon\tilde{s} & -ab\epsilon^2\rho^{-2}\cos^2 2s & a^2\epsilon^2\rho^{-2}\cos^2 2s \end{pmatrix} + O(\epsilon^3\rho^{-4}),$$

where $\rho(x) := \varrho(\epsilon x)$.

With these notations at hand, we can define the following inner products in normal space:

Definition 1. *Let $(y_1, y_2, y_3, y_4) = (\tilde{s}, \tilde{\theta}, a, b)$. For any complex-valued functions $\zeta_1(\tilde{s}, \tilde{\theta}, a, b)$ and $\zeta_2(\tilde{s}, \tilde{\theta}, a, b)$, one-forms $C_1 = C_{1i}dy_i$ and $C_2 = C_{2j}dy_j$, we define the following inner products in normal space:*

$$\langle \zeta_1, \zeta_2 \rangle := \Re(\zeta_1\overline{\zeta_2}),\ \langle C_1, C_2 \rangle := g^{ij}C_{1i}C_{2j},$$

and

$$\langle (\zeta_1, C_1), (\zeta_2, C_2) \rangle := \langle \zeta_1, \zeta_2 \rangle + \langle C_1, C_2 \rangle = \Re(\zeta_1\overline{\zeta_2}) + g^{ij}C_{1i}C_{2j}.$$

Now let us turn to the Jacobi operator L_Γ of the minimal submanifold Γ. For a normal vector field $N = ie^{-is}k_1\Theta + ie^{is}k_2\Theta^\perp = k_1\mathbf{m} + k_2\mathbf{n}$, we have

$$\begin{aligned} L_\Gamma N &= \Delta_\Gamma^\nu N + 2\rho^{-6}N \\ &= [\rho^{-6}k_{1ss} + \rho^{-2}k_{1\theta\theta} + 2\rho^{-4}\cos 2s k_{1s} - 2\rho^{-2}\cos 2s k_{2\theta} + \rho^{-2}(2\rho^{-4} - \cos^2 2s)k_1]\mathbf{m} \\ &\quad + [\rho^{-6}k_{2ss} + \rho^{-2}k_{2\theta\theta} + 2\rho^{-4}\cos 2s k_{2s} + 2\rho^{-2}\cos 2s k_{1\theta} + \rho^{-2}(2\rho^{-4} - \cos^2 2s)k_2]\mathbf{n}, \end{aligned}$$

where Δ_Γ^ν is the connection Laplacian on the normal bundle $\mathcal{N}\Gamma$ of Γ in $\mathbb{R}^4$. See [2] for details.

A normal vector field $N = k_1\mathbf{m} + k_2\mathbf{n}$ is called a Jacobi field on Γ if $L_\Gamma N = 0$. We have the following bounded Jacobi fields generated by rigid motions:

$$\begin{aligned} &N_1 = \cos\theta\sin s\mathbf{m} + \sin\theta\sin s\mathbf{n},\ N_2 = \sin\theta\sin s\mathbf{m} - \cos\theta\sin s\mathbf{n}, \\ &N_3 = \cos\theta\cos s\mathbf{m} - \sin\theta\cos s\mathbf{n},\ N_4 = \sin\theta\cos s\mathbf{m} + \cos\theta\cos s\mathbf{n}, \\ &N_5 = (\sin 2s)^{\frac{1}{2}}\mathbf{m},\ N_6 = (\sin 2s)^{\frac{1}{2}}\mathbf{n}. \end{aligned} \tag{2.1}$$

Here N_j, $j = 1, 2, 3, 4$ are generated by the translation in x_j-direction. N_5 is generated by dilation and N_6 is generated by the action $x \mapsto Jx$ in $O(4)/SU(2)$, where

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

It is shown in [32] that the submanifold Γ is non-degenerate.

Lemma 2.2. *Γ is non-degenerate in the sense that any bounded Jacobi field on Γ is a linear combination of N_j, $j = 1, \ldots, 6$.*

We also define the translated coordinate $t_1 = a - \epsilon f_1, t_2 = b - \epsilon f_2$ and the corresponding perturbed polar coordinate $(\tilde{r}, \tilde{\phi})$, where $f_1(s,\theta)$ and $f_2(s,\theta)$ are perturbations given in Theorem 8.1 of [32] with a different notation $f_1(s,\theta)$ and $f_2(s,\theta)$. Moreover, for $j = 1, 2$, f_j satisfy

$$\|\nabla_\Gamma^2 f_j\|_{4,p} + \|\nabla_\Gamma f_j\|_{3,p} + \|f_j\|_{2,p} \leq C.$$

for some $p > 1$. The norms will be introduced in (2.7).

With the notation above, the connection Laplacian $\Delta_{\Gamma_\epsilon}^\nu$ and

$$\Delta_A \xi = \Delta\xi + i d^* A\xi - 2i\langle A, d\xi\rangle - |A|^2\xi$$

can be computed.

Lemma 2.3. *For a smooth normal vector field $N = k_1(s,\theta)m + k_2(s,\theta)n$ and smooth functions $\eta(s,\theta)$ on Γ and $\xi(\tilde{s}, \tilde{\theta}, t_1, t_2)$ on Σ_ϵ, we have*

$$\begin{aligned}\Delta_\Gamma^\nu N =& [\rho^{-6}k_{1ss} + \rho^{-2}k_{1\theta\theta} + 2\rho^{-4}\cos 2s k_{1s} - 2\rho^{-2}\cos 2s k_{2\theta} - \rho^{-2}\cos^2 2s k_1]\mathbf{m}\\ &+ [\rho^{-6}k_{2ss} + \rho^{-2}k_{2\theta\theta} + 2\rho^{-4}\cos 2s k_{2s} + 2\rho^{-2}\cos 2s k_{1\theta} - \rho^{-2}\cos^2 2s k_2]\mathbf{n},\end{aligned}$$

$$\Delta_\Gamma \eta = \rho^{-6}\eta_{ss} + \rho^{-2}\eta_{\theta\theta} + 2\rho^{-4}\cos 2s\eta_s$$

and

$$\begin{aligned}\Delta\xi =& \xi_{t_1t_1} + \xi_{t_2t_2} + \Delta_{\Gamma_\epsilon}\xi + 2\epsilon(t_2\xi_{\tilde{\theta}t_1} - t_1\xi_{\tilde{\theta}t_2})\rho^{-2}\cos(2\epsilon\tilde{s})(1 - 2t_1\rho^{-3}\epsilon)\\ &+ \epsilon^2(t_2^2\xi_{t_1t_1} - 2t_1t_2\xi_{t_1t_2} + t_1^2\xi_{t_2t_2})\rho^{-2}\cos^2(2\epsilon\tilde{s}) - \epsilon^2\rho^{-2}(\rho^{-4} + 1)(t_1\xi_{t_1} + t_2\xi_{t_2})\\ &+ 2\epsilon^2\rho^{-2}\cos(2\epsilon\tilde{s})(f_2\xi_{t_1} - f_1\xi_{t_2}) + 4t_2\epsilon^2\rho^{-7}\cos 2\epsilon\tilde{s}(t_2\xi_{\tilde{s}t_1} - t_1\xi_{\tilde{s}t_2}) - 2\epsilon^2\rho^{-6}(f_{1s}\xi_{\tilde{s}t_1} + f_{2s}\xi_{\tilde{s}t_2})\\ &- 2\epsilon^2\rho^{-2}(f_{1\theta}\xi_{\tilde{\theta}t_1} + f_{2\theta}\xi_{\tilde{\theta}t_2}) + (2(t_1 + \epsilon f_1)\rho^{-9} + 3\tilde{r}^2\rho^{-12}\epsilon)\epsilon\xi_{\tilde{s}\tilde{s}} + 4(t_2 + \epsilon f_2)\epsilon\rho^{-7}\xi_{\tilde{s}\tilde{\theta}}\\ &+ (-2(t_2 + \epsilon f_2)\rho^{-5} + 3\tilde{r}^2\epsilon\rho^{-8})\epsilon\xi_{\tilde{\theta}\tilde{\theta}} + 8t_1\epsilon^2\rho^{-7}\cos^2(2\epsilon\tilde{s})\xi_{\tilde{s}} + 4t_2\epsilon^2\rho^{-5}\cos(2\epsilon\tilde{s})\xi_{\tilde{\theta}} + O(\epsilon^3),\end{aligned}$$

$$d^*A\xi = -\frac{1}{\sqrt{G}}\partial_j(\sqrt{G}g^{ij}A_j)\xi + d^*B_0\xi + O(\epsilon^3) = d^*B_0\xi + O(\epsilon^3), \tag{2.2}$$

(2.3)

$$\begin{aligned}\langle A, d\xi\rangle =& g^{ij}A_i\xi_j + \langle B_0, d\xi\rangle + O(\epsilon^3)\\ =& A_3\xi_a + A_4\xi_b - \epsilon^2\rho^{-2}\cos^2 2s(b\xi_a - a\xi_b) - \epsilon^2\rho^{-2}\cos 2s\xi_\theta\\ &- \epsilon^2\rho^{-6}(2t_2\rho^{-1}\cos(2\epsilon\tilde{s}) + A_3f_{1s} + A_4f_{2s})\xi_{\tilde{s}} + \epsilon^2\rho^{-2}(2t_1\rho^{-1}\cos(2\epsilon\tilde{s}) - A_3f_{1s} - A_4f_{2s})\xi_{\tilde{\theta}}\\ &+ \langle B_0, d\xi\rangle + O(\epsilon^3),\end{aligned}$$

$$\begin{aligned}|A|^2\xi &= g^{ij}A_iA_j\xi + (|A_0 + B_0|^2 - |A_0|^2)\xi + O(\epsilon^3)\\ &= (A_3^2 + A_4^2)\xi + \epsilon^2\rho^{-2}\cos^2 2s\xi + (|A_0 + B_0|^2 - |A_0|^2)\xi + O(\epsilon^3).\end{aligned} \tag{2.4}$$

The following formulas can be found in [3, 4, 5].

Lemma 2.4. *For a smooth one-form $\omega = \omega_i dy^i$, we have*

$$d^*\omega = -\frac{1}{\sqrt{G}}\partial_j(\sqrt{G}g^{ij}\omega_i)$$

and

$$d^*d\omega = -\frac{1}{\sqrt{G}}g_{ml}\partial_j(\sqrt{G}g^{ij}g^{kl}\omega_{ik})dy^m,$$

where $\omega_{ik} = \partial_i\omega_k - \partial_k\omega_i$.

2.2. **Vortex solutions.** In the two-dimensional plane, we have fundamental vortex solutions for each fixed integer j, called the topological degree of the solution. In the polar coordinate (r, ϕ), the j-vortex solutions $u_j = (\psi^{(j)}, A^{(j)})$ in $\mathbb{R}^2$ has the form

$$\psi^{(j)}(x) = U_j(r)e^{\sqrt{-1}j\phi} \text{ and } A^{(j)}(x) = V_j(r)d(j\phi),$$

where U_j, V_j satisfy the following ODE system

$$\begin{cases} -U_j'' - \frac{1}{r}U_j' + j^2\frac{(1-V_j)^2}{r^2}U_j - \frac{\lambda}{2}(1-U_j^2)U_j = 0, \\ -V_j'' + \frac{1}{r}V_j' - U_j^2(1-V_j) = 0. \end{cases} \tag{2.5}$$

Moreover, they have the following properties:

- $0 < U_j, V_j < 1$ on $(0, +\infty)$.
- $U_j', V_j' > 0$.
- $U_j \sim c_1 r$, $V_j \sim c_2 r^2$ as $r \to 0$ for some constants $c_1, c_2 > 0$.
- $1 - U_j, 1 - V_j \to 0$ as $r \to \infty$ with an exponential rate of decay. In particular, $1 - U_1 = O(e^{-m_\lambda r})$, $1 - V_1 = O(e^{-r})$ as $r \to \infty$, where $m_\lambda = \min\{\sqrt{\lambda}, 2\}$.

The existence and above properties of these functions are proved in [6, 40]. From results of [26], we know that for $|j| = 1$, the vortex solution is always stable for any positive λ; while for $|j| > 1$, it is stable when $\lambda \le 1$ and unstable for $\lambda > 1$. Since we are interested in stable solutions, here in this paper we will only use solutions with $|j| = 1$. To simplify the notations, we set $U = U_1, V = V_1$, although they are actually also depending on the coupling parameter λ.

Roughly speaking, the family of vortex solutions $(\psi_\epsilon, A_\epsilon)$ constructed in [32] can be written into the form:

$$\psi_\epsilon = \psi_{0\epsilon}(s, \theta, a, b) + \eta_{0\epsilon}(s, \theta, a, b), A_\epsilon = A_{0\epsilon}(s, \theta, a, b) + B_{0\epsilon}(s, \theta, a, b),$$

where $(\psi_{0\epsilon}, A_{0\epsilon})$ is suitable approximate solution, and $(\eta_{0\epsilon}, B_{0\epsilon})$ is small perturbation. For simplicity, we will omit the subscript ϵ later on. More precisely, (ψ_0, A_0) can be written as

$$\begin{aligned} &\psi_0(s,\theta,a,b) = \psi^{(1)}(a-\epsilon f_1, b-\epsilon f_2) = U(\tilde{r})e^{i\tilde{\phi}}, \\ &A_0(s,\theta,a,b) = A_2(s,\theta,a,b)d\theta + A_3(t_1,t_2)dt_1 + A_4(t_1,t_2)dt_2 \\ =&(-\epsilon f_{1s}A_3 - \epsilon f_{2s}A_4)ds + (A_2(s,\theta,a,b) - \epsilon f_{1\theta}A_3 - \epsilon f_{2\theta}A_4)d\theta + A_3 da + A_4 db \\ =&(\epsilon f_{1s}\frac{V}{\tilde{r}}\sin\tilde{\phi} - \epsilon f_{2s}\frac{V}{\tilde{r}}\cos\tilde{\phi})ds + [-(1-V)\cos 2s + \epsilon f_{1\theta}\frac{V}{\tilde{r}}\sin\tilde{\phi} - \epsilon f_{2\theta}\frac{V}{\tilde{r}}\cos\tilde{\phi}]d\theta - \frac{V}{\tilde{r}}\sin\tilde{\phi}da + \frac{V}{\tilde{r}}\cos\tilde{\phi}db. \end{aligned}$$

inside Σ_ϵ, where $A_2(s,\theta,a,b) = -(1-V)\cos 2s$, $A_3(t_1,t_2) = -\frac{V}{\tilde{r}}\sin\tilde{\phi}$ and $A_4(t_1,t_2) = \frac{V}{\tilde{r}}\cos\tilde{\phi}$. Outside Σ_ϵ, $(\psi_0, A_0) = (We^{i\varphi}, Zd\varphi)$, where $W, Z = 1 + O(e^{-\delta r})$ and φ is an extension of the angular function ϕ.

Remark 2.4.1. *We remark here that our approximate solution is slightly different from that of [32]. We add $(1-V)d\theta$ so that B_0 is of $O(\epsilon^2)$. The existence of such a solution is ensured with similar arguments as that of [32].*

To analyze the stability of such a solution, we need to study the linearized operator at (ψ, A), defined by

$$L(\xi, B, \psi, A) := \begin{pmatrix} -\Delta_A \xi - id^* B\psi + 2i\langle B, \nabla_A \psi\rangle + \frac{\lambda}{2}(\psi^2\overline{\xi} + 2|\psi|^2\xi - \xi) \\ d^*dB + Im(\overline{\nabla_A\psi}\cdot\xi + \overline{\nabla_A\xi}\cdot\psi) + B|\psi|^2 \end{pmatrix}.$$

For any $\gamma \in C^1(\mathbb{R}^4)$, we define the Gauge transformations

$$G_\gamma(\xi, B) := (e^{i\gamma}\xi, B + d\gamma) \text{ and } \tilde{G}_\gamma(\xi, B) := (e^{i\gamma}\xi, B).$$

Definition 2. *$L(\cdot;\psi, A)$ is called to be stable if for any $v = (\xi, B)$(integrable in suitable sense),*

$$\delta^2\mathcal{E}[v, v] = \int_{\mathbb{R}^4} \langle L(v;\psi, A), v\rangle \geq 0.$$

$L(\cdot;\psi, A)$ is invariant under the following Gauge transformation

$$L(\tilde{G}_\gamma(\xi, B); G_\gamma(\psi, A)) = \tilde{G}_\gamma(L(\xi, B)).$$

Consequently, $L(\cdot;\psi, A)$ admits an infinite-dimensional subspace of bounded kernels

$$(i\gamma\psi, d\gamma) \text{ for any } \gamma \in C^1(\mathbb{R}^4).$$

Also, $L(\cdot;\psi, A)$ admits several bounded kernels as following:

$$\tilde{Z}_j := (\partial_j\psi, \partial_j A),\ j = 1, 2, 3, 4,\ \tilde{Z}_5 := (x\cdot\nabla\psi, x\cdot\nabla A + A),\ \tilde{Z}_6 := (Jx\cdot\nabla\psi, Jx\cdot\nabla A + JA),$$

where $\nabla A := \nabla A^j dx_j$ for $A = A^j dx_j$. Here $\tilde{Z}_j$, $j = 1, 2, 3, 4$ are generated by translations in x_j-direction. $\tilde{Z}_5$ is generated by dilation

$$(\psi(x), A^j(x)dx_j) \mapsto (\psi(tx), tA^j(tx)dx_j).$$

Finally, $\tilde{Z}_6$ is generated by the an action in $O(4)/SU(2)$

$$(\psi(x), A^j(x)dx_j) \mapsto (\psi(Jx), A^j(Jx)d((Jx)_j)).$$

Even though these $\tilde{Z}_j$'s are not in $L^2(\mathbb{R}^4)$, we can use the gauge kernels $(i\gamma\psi, d\gamma)$ to modify them so that they decay exponentially. Specifically, the modified kernels are

$$\begin{aligned} Z_j &:= \tilde{Z}_j - (iA^j\psi, dA^j) = (\partial_j\psi - iA^j\psi, \partial_j A - dA^j),\ j = 1, 2, 3, 4, \\ Z_5 &:= \tilde{Z}_5 - (i(x\cdot A)\psi, d(x\cdot A)) = (x\cdot\nabla_A\psi, x_j(\partial_j A - dA^j)), \\ Z_6 &:= \tilde{Z}_6 - (i(Jx\cdot A)\psi, d(Jx\cdot A)) = (Jx\cdot\nabla_A\psi, (Jx)_j(\partial_j - dA^j)). \end{aligned} \tag{2.6}$$

We define Z_{sym} to be the space spanned by the Z_j (or $\tilde{Z}_j$), $j = 1, \ldots, 6$ and the gauge kernels. We will show in Theorem 1.1 that $L(\cdot;\psi, A)$ is non-degenerate in the sense that Z_{sym} is actually the space of all bounded kernels of $L(\cdot;\psi, A)$. However, it is inconvenient to deal with the infinite-dimensional gauge kernels. Also, the operator $L(\cdot;\psi, A)$ is not uniformly elliptic. To solve these problems, we restrict the perturbation v into the space that is orthogonal to all the gauge kernels. To see this, we notice that

$$\int_{\mathbb{R}^4} \langle v, (i\gamma\psi, d\psi)\rangle = 0 \text{ for any } \gamma \in C^1(\mathbb{R}^4)$$

is equivalent to the gauge condition

$$\Im(\overline{\psi}\xi) = d^*B.$$

Hence, we introduce the modified quadratic form

$$\int_{\mathbb{R}^4} \langle \mathbb{L}(v;\psi,A),v\rangle := \int_{\mathbb{R}^4} \langle L(v;\psi,A),v\rangle + \int_{\mathbb{R}^4} (\Im(\overline{\psi}\xi) - d^*B)^2$$

and the (Gauge-fixed) linearized operator at (ψ, A) as follows:

$$\mathbb{L}(\xi,B;\psi,A) := \begin{pmatrix} -\Delta_A\xi + 2i\langle B,\nabla_A\psi\rangle + \frac{1}{2}(\lambda-1)\psi^2\bar{\xi} + (\lambda+\frac{1}{2})|\psi|^2\xi - \frac{\lambda}{2}\xi \\ \Delta_H B + 2Im(\overline{\nabla_A\psi}\cdot\xi) + B|\psi|^2 \end{pmatrix},$$

where $\Delta_H = d^*d + dd^*$ is the Hodge Laplacian. Now $\mathbb{L}(\cdot,\psi,A)$ is uniformly elliptic and the stability of $L(v;\psi,A)$ is established if we show that there exists a constant $c>0$ such that

$$\int_{\mathbb{R}^4} \langle \mathbb{L}(v;\psi,A),v\rangle \geq 0, \text{ for any } v \in L^2(\mathbb{R}^4).$$

Furthermore, the non-degeneracy of $L(\cdot;\psi,A)$ is reduced to the non-degeneracy of $L(\cdot;\psi,A)$ in the sense that any bouned kernels of $\mathbb{L}(\cdot;\psi,A)$ is a linear combination of Z_j, $j=1,\ldots,6$.

The following two orthogonal approximate kernels $\mathcal{T}_i := (T_i, T_{B_i})$ of $\mathbb{L}(\xi,B;\psi,A)$ play a crucial role in the later discussions.

$$\begin{aligned}
T_1(t_1,t_2) &= \partial_{t_1}\psi_0 - iA_3\psi_0 = (U'\cos\tilde{\phi} - i\frac{U}{\tilde{r}}(1-V)\sin\tilde{\phi})e^{i\tilde{\phi}},\\
T_2(t_1,t_2) &= \partial_{t_2}\psi_0 - iA_4\psi_0 = (U'\sin\tilde{\phi} + i\frac{U}{\tilde{r}}(1-V)\cos\tilde{\phi})e^{i\tilde{\phi}},\\
T_{B_1}(t_1,t_2) &= \partial_{t_1}A_0 - dA_3 = \partial_{t_1}A_0 d\theta - (\partial_{t_2}A_3 - \partial_{t_1}A_4)dt_2 = \epsilon\cos 2s\cos\tilde{\phi}V'd\tilde{\theta} + \frac{V'}{\tilde{r}}dt_2,\\
T_{B_2}(t_1,t_2) &= \partial_{t_2}A_0 - dA_4 = \partial_{t_2}A_0 d\theta + (\partial_{t_2}A_3 - \partial_{t_1}A_4)dt_1 = \epsilon\cos 2s\sin\tilde{\phi}V'd\tilde{\theta} - \frac{V'}{\tilde{r}}dt_1.
\end{aligned}$$

More precisely, they are actually the kernels of $\mathbb{L}(\xi,B;\psi_0,A_0)$, the linearized operator at (ψ_0,A_0). Their norms can be computed:

Lemma 2.5. *We have the following identities:*

(1)

$$\int_{\mathbb{R}^2} |T_1|^2 = \int_{\mathbb{R}^2} |T_2|^2 = \pi\int_0^{+\infty} [\tilde{r}(U')^2 + \frac{1}{\tilde{r}}U^2(1-V)^2]d\tilde{r},$$

(2)

$$\Re\int_{\mathbb{R}^2} \overline{T_1}T_2 = \Re\int_{\mathbb{R}^2} \overline{T_2}T_1 = 0,\ \Im\int_{\mathbb{R}^2} \overline{T_1}T_2 = 2\pi\int_0^{+\infty} UU'(1-V)d\tilde{r},$$

(3)

$$\int_{\mathbb{R}^2} |T_{B_1}|^2 = \int_{\mathbb{R}^2} |T_{B_2}|^2 = 2\pi\int_0^{+\infty} \frac{(V')^2}{\tilde{r}}d\tilde{r} + O(\epsilon^3),\ \int_{\mathbb{R}^2} \langle T_{B_1}, T_{B_2}\rangle = O(\epsilon^3).$$

2.3. **Eigenvalue problems and apriori estimates.** In this subsection, we consider the eigenvalue problem of the Jacobi operator associated to nonpositive eigenvalues and derive the corresponding apriori estimates. We first introduce some weighted Sobolev norms on Γ and $\Gamma_\epsilon \times \mathbb{R}^2$:

$$\|f\|_{p,\beta} := \sup_{P\in\Gamma}\left(\int_\Gamma |f(P)|^p \rho(P)^\beta dvol_\Gamma\right)^{\frac{1}{p}}, \tag{2.7}$$

$$\|g\|_{0,\beta,p,\sigma} := \sup_{(P,a,b)\in\Gamma_\epsilon\times\mathbb{R}^2} \rho(P)^\beta e^{\sigma r}\|g\|_{L^p(B_1(P,a,b))},$$

$$\|g\|_{2,\beta,p,\sigma} := \|D^2 g\|_{0,\beta+2,p,\sigma} + \|Dg\|_{0,\beta+1,p,\sigma} + \|g\|_{0,\beta,p,\sigma},$$

$$\|(\xi,B)\|_{0,\beta,p,\sigma} := \|\xi\|_{0,\beta,p,\sigma} + \|B\|_{0,\beta,p,\sigma},\ \|(\xi,B)\|_{2,\beta,p,\sigma} := \|\xi\|_{2,\beta,p,\sigma} + \|B\|_{2,\beta,p,\sigma}$$

where $f(P)$ is a function on Γ and $g(P,a,b)$ is a function or one-form, $\xi(P,a,b)$ is a function, $B(P,a,b)$ is an one-form on $\Gamma_\epsilon \times \mathbb{R}^2$.

The eigenvalue problem of the Jacobi operator L_Γ arises from the second variation of the area functional

$$\mathrm{Q}(N,N) := \int_\Gamma |\nabla^\nu_\Gamma N| - 2\rho^{-6}|N|^2 dvol_\Gamma,$$

where $N = k_1(s,\theta)\mathbf{m} + k_2(s,\theta)\mathbf{n}$ is a normal vector field on Γ and ∇^ν_Γ is the covariant derivative on the normal bundle $\mathcal{N}\Gamma$. It can be written explicitly as

$$\begin{aligned}\mathrm{Q}(N,N) = \int_\Gamma & \rho^{-6}(|k_{1s}|^2 + |k_{2s}|^2) + \rho^{-2}(|k_{1\theta}|^2 + |k_{2\theta}|^2) - 2\rho^{-2}\cos 2s(k_1 k_{2\theta} - k_2 k_{1\theta}) \\ & - \rho^{-2}(2\rho^{-4} - \cos^2 2s)(|k_1|^2 + |k_2|^2) dvol_\Gamma,\end{aligned}$$

The related eigenvalue problem is

$$\Delta^\nu_\Gamma N + 2\rho^{-6}N + \mu\rho^{-6}N = 0 \text{ in } \Gamma.$$

We also consider the region

$$\Gamma^R := \{(s,\theta)\in\Gamma :\ \rho(s,\theta) < R\}$$

for a large number R and the eigenvalue problem in Γ^R

$$\begin{cases} \Delta^\nu_\Gamma N + 2\rho^{-6}N + \mu\rho^{-6}N = F & \text{in } \Gamma^R, \\ N = 0 & \text{on } \partial\Gamma^R. \end{cases} \tag{2.8}$$

Then we have the following apriori estimates

Lemma 2.6. *Let $p > 1$, $\sigma > 0$. Then for any fixed $R_0 > 0$ large and $\mu_0 > 0$, there exists a constant $C > 0$ such that for all $R > R_0 + 1$, $-\mu_0 < \mu \le 0$, normal vector field F, if (2.8) admits a solution $N = k_1\mathbf{m} + k_2\mathbf{n}$, then for $i = 1, 2$, If $\|F\|_{p,\beta+2} < +\infty$, then*

$$\|N\|_{L^\infty} \le C[\|F\|_{p,\beta+2} + \|N\|_{L^\infty(|\rho|<3R_0)}] \tag{2.9}$$

and

$$\|\nabla^2_\Gamma N\|_{p,\beta+2} + \|\nabla_\Gamma N\|_{p,\beta+1} + \|N\|_{p,\beta} \le C[\|F\|_{p,\beta+2} + \|N\|_{L^\infty(|\rho|<3R_0)}]. \tag{2.10}$$

Proof. Since $-\mu_0 < \mu \le 0$, the proof (2.9) and (2.10) is almost the same as the proof of Lemma 10.2 in [37] and Proposition 8.2 in Liu-Ma-Wei-Wu [32] respectively. □

We also consider the following eigenvalue problem for $\mathbb{L}(\cdot;\psi,A)$

$$\begin{cases} \mathbb{L}(v;\psi,A)-\mu\rho^{-6}v=0 & \text{in } B_{\epsilon^{-1}R} \\ v=0 & \text{on } \partial B_{\epsilon^{-1}R}. \end{cases} \tag{2.11}$$

and the eigenvalue problem in $\mathbb{R}^4$

$$\mathbb{L}(v;\psi,A)-\mu\rho^{-6}v=0 \text{ in } \mathbb{R}^4. \tag{2.12}$$

For the eigenvalue problems, we will show that the corresponding eigenfunction decays exponentially away from Γ_ϵ.

Lemma 2.7. *Let v be a solution to (2.11) or (2.12) with $\mu\le 0$. Then for any fixed $0<\delta<\frac{\sqrt{2}}{2}\min\{\sqrt{\lambda},1\}$, there exists a constant C depends on μ and δ but independent of sufficiently small $\epsilon>0$ and large $R>0$, such that*

$$|v(\tilde{s},\tilde{\theta},t_1,t_2)|\le C\|v\|_{L^\infty}e^{-\delta\tilde{r}} \text{ in } \Sigma_\epsilon,$$

Outside Σ_ϵ, we have

$$|v(\tilde{s},\tilde{\theta},t_1,t_2)|<Ce^{-\delta r_\epsilon} \text{ in } \Sigma_\epsilon^c. \tag{2.13}$$

Proof. We first assume that $v=(\xi,B)$ is a solution to (2.11). Note that if we test it against v, we find that

$$\begin{aligned} 0=&\langle\mathbb{L}(v;\psi,A)-\mu\rho^{-6}v,v\rangle=\Re(\mathbb{L}(v;\psi,A)_1\bar{\xi})+\langle\mathbb{L}(v;\psi,A)_2,B\rangle-\mu\rho^{-6}|v|^2\\ =&-\frac{1}{2}\Delta|v|^2+|\nabla v|^2-2\Im[\langle A,d\xi\rangle\bar{\xi}]+|A|^2|\xi|^2-4\Im[\langle B,\nabla_A\psi\rangle\bar{\xi}]\\ &+\frac{\lambda-1}{2}\Re(\psi^2\bar{\xi}^2)+(\lambda+\frac{1}{2})|\psi|^2|\xi|^2-\frac{\lambda}{2}|\xi|^2+|B|^2|\psi|^2-\mu\rho^{-6}|v|^2. \end{aligned}$$

Note that $|A|=O(\tilde{r}^{-1}e^{-\tilde{r}})$, $1-|\psi|^2=O(e^{-m_\lambda\tilde{r}})$ and $|\nabla_A\psi|=O(e^{-\min\{1,\sqrt{\lambda}\}\tilde{r}})$ as $\tilde{r}\to\infty$. In the region $r_0<\tilde{r}<r_\epsilon$ for some sufficiently large constant $r_0>0$ depending on λ and δ, we have

$$0\ge-\Delta|v|^2+4\delta^2|v|^2-2\mu\rho^{-6}|v|^2\ge-\Delta|v|^2+4\delta^2|v|^2.$$

Note that $w=e^{-2\delta\tilde{r}}+e^{-2(2r_\epsilon-\delta)\tilde{r}}$ is a positive supersolution that satisfies

$$-\Delta|w|^2+4\delta^2|w|^2>0 \text{ in } r_0<r<r_\epsilon$$

if ϵ is chosen sufficiently small and $R>0$ is sufficiently large. Then by maximum principle, we have

$$|v|\le C\|v\|_{L^\infty}e^{-\delta\tilde{r}} \text{ for } r_0<r<r_\epsilon.$$

Then (2.13) follows from the maximum principle. Then we proved the lemma for a solution to (2.11). If v is a solution to (2.12), the proof is similar. □

We also list below the linear theory of $\mathbb{L}(\cdot;\psi_0,A_0)$ studied in Liu-Ma-Wei-Wu [32].

Lemma 2.8 (Theorem 6.3 in [32]). *If $\|(\mu,C)\|_{0,2,p,\delta}<+\infty$ for some $p>1$ and $\delta>0$. Then for $\epsilon>0$ sufficiently small, there exists a unique solution (ξ,B) to*

$$\mathbb{L}(\xi,B;\psi_0,A_0)=(\mu,C)+c_1(s,\theta)\mathcal{T}_1+c_2(s,\theta)\mathcal{T}_2 \text{ for all } (s,\theta,t_1,t_2)\in\Gamma_\epsilon\times\mathbb{R}^2,$$

$$\int_{\mathbb{R}^2}\langle(\xi,B)(s,\theta,t_1,t_2),\mathcal{T}_j(t_1,t_2)\rangle dt_1dt_2=0 \text{ for all } (s,\theta)\in\Gamma_\epsilon$$

with

$$\|(\xi,B)\|_{2,2,p,\delta}\le C_0\|(\mu,C)\|_{0,2,p,\delta}$$

for some $C_0 > 0$ independent of ϵ and (μ, C). Here $c_j(s,\theta)$, $j = 1, 2$, is given by

$$c_j(s,\theta) = -\frac{\int_{\mathbb{R}^2} \langle (\mu, C)(s,\theta,t_1,t_2), \mathcal{T}_j(t_1,t_2)\rangle dt_1 dt_2}{\int_{\mathbb{R}^2} |\mathcal{T}_j|^2}.$$

2.4. **Improvement of approximation.** Establishing the stability amounts to analyze the spectrum of the linearized operator around the solution. In our situation, it will be necessary to find a more accurate approximate solution. That is, a higher order expansion of (η_0, B_0).

Let

$$S(\psi_0, A_0) := \begin{pmatrix} -\Delta_{A_0}\psi_0 + \frac{\lambda}{2}(|\psi_0|^2 - 1)\psi_0 \\ d^* dA_0 - Im(\nabla_{A_0}\psi_0 \cdot \bar{\psi}_0) \end{pmatrix}.$$

be the error of the approximate solution. Then direct computation tells us that

$$-\Delta_{A_0}\psi_0 + \frac{\lambda}{2}(|\psi_0|^2 - 1)\psi_0 = 2\epsilon^2\rho^{-6}\tilde{r}U' e^{i\tilde{\phi}} + O(\epsilon^3\rho^{-4}e^{-\tilde{r}})$$

and

$$\begin{aligned} d^* dA_0 - Im(\nabla_{A_0}\psi_0 \cdot \bar{\psi}_0) &= O(\epsilon^3\rho^{-2}e^{-\tilde{r}})d\tilde{s} + O(\epsilon^3\rho^{-2}e^{-\tilde{r}})d\tilde{\theta} + [2\epsilon^2\rho^{-6}t_2(\partial_{t_2}A_3 - \partial_{t_1}A_4) + O(\epsilon^3\rho^{-4}e^{-\tilde{r}})]dt_1 \\ &\quad - [2\epsilon^2\rho^{-6}t_1(\partial_{t_2}A_3 - \partial_{t_1}A_4) + O(\epsilon^3\rho^{-4}e^{-\tilde{r}}))]dt_2 \\ &= O(\epsilon^3\rho^{-2}e^{-\tilde{r}}))d\tilde{s} + O(\epsilon^3\rho^{-2}e^{-\tilde{r}})d\tilde{\theta} - [2\epsilon^2\rho^{-6}\sin\tilde{\phi}V' + O(\epsilon^3\rho^{-4}e^{-\tilde{r}}))]dt_1 \\ &\quad + [2\epsilon^2\rho^{-6}\cos\tilde{\phi}V' + O(\epsilon^3\rho^{-4}e^{-\tilde{r}}))]dt_2. \end{aligned}$$

See also Lemma 4.1 to Lemma 4.5 in [32] for similar computation.

The $O(\epsilon^2)$-terms above are orthogonal to the approximate kernels $\mathcal{T}_1$ and $\mathcal{T}_2$ up to $O(\epsilon^2)$. However, they are still large terms. In order to cancel these terms, we introduce the following improved approximate solution. Let (η_1, B_1) be the solution to the equation

$$\mathbb{L}(\eta_1, B_1; \psi^{(1)}, A^{(1)}) = \begin{pmatrix} \tilde{r}U' e^{i\tilde{\phi}} \\ -\sin\tilde{\phi}V' dt_1 + \cos\tilde{\phi}V' dt_2 \end{pmatrix} \text{ in } \mathbb{R}^2.$$

Note that the right-hand side is perpendicular to $\widetilde{\mathcal{T}_1} := \mathcal{T}_1 - (0, \cos 2s \sin\tilde{\phi}V' d\theta)$ and $\widetilde{\mathcal{T}_2} := \mathcal{T}_2 - (0, \cos 2s \sin\tilde{\phi}V' d\theta)$, the kernels of $\mathbb{L}(\eta_1, B_1; \psi^{(1)}, A^{(1)})$ in $\mathbb{R}^2$. Thus the existence of such (η_1, B_1) is ensured. Then the similar argument as in [32] implies that the solution constructed in [32] can be written as

$$(\psi, A) = (\psi_0, A_0) + (\eta_0, B_0) = (\psi_0, A_0) - 2\epsilon^2\rho^{-6}(\eta_1, B_1) + O(\epsilon^3\rho^{-4}e^{-\delta\tilde{r}}).$$

3. Analysis of the linearized operator

The idea of proof of stability is as follows: If (ξ, B) is an eigenfunction of $\mathbb{L}(\cdot; \psi, A)$ w.r.t. some negative eigenvalue, then it can be written as $(\xi, B) \approx k_1(\tilde{s}, \tilde{\theta})\mathcal{T}_1(t_1,t_2) + k_2(\tilde{s}, \tilde{\theta})\mathcal{T}_2(t_1,t_2)$, where $N = k_1(\tilde{s}, \tilde{\theta})\mathbf{m} + k_2(\tilde{s}, \tilde{\theta})\mathbf{n}$ is a negative direction of the Jacobi operator L_Γ. Then the stability of L_Γ can be applied. As a consequence, a detailed analysis of the linearized operator $\mathbb{L}(\cdot; \psi, A)$ is required. To start with, we need the following proposition to build up the relationship between $\mathbb{L}(\cdot; \psi, A)$ and L_Γ.

Proposition 3.1. *Let $N = k_1(\tilde{s}, \tilde{\theta})\mathbf{m} + k_2(\tilde{s}, \tilde{\theta})\mathbf{n}$ be a normal vector field. Consider the vector-valued function defined for $x \in \Sigma_\epsilon$ by*

$$v(x) = v(\tilde{s}, \tilde{\theta}, t_1, t_2) = (\xi, B)(\tilde{s}, \tilde{\theta}, t_1, t_2) = k_1(\tilde{s}, \tilde{\theta})\mathcal{T}_1(t_1,t_2) + k_2(\tilde{s}, \tilde{\theta})\mathcal{T}_2(t_1,t_2).$$

Then we have

$$\int_{(a,b)\in\Sigma_\epsilon}\langle\mathbb{L}(v;\psi,A),\mathcal{T}_1\rangle dt_1dt_2\mathbf{m}+\int_{(a,b)\in\Sigma_\epsilon}\langle\mathbb{L}(v;\psi,A),\mathcal{T}_2\rangle dt_1dt_2\mathbf{n}$$
$$=-L_{\Gamma_\epsilon}N\int_{\mathbb{R}^2}|\mathcal{T}_1|^2+O(\epsilon^3\rho^{-4}(\partial_{ij}N+\partial_iN+N)).$$

To prove the proposition above, we calculate the integrals term by term. For simplicity, we only compute the first integral. The computation of the second integral is similar. The following lemma computes the inner product of the first components of $\mathbb{L}(v;\psi,A)$ and $\mathcal{T}_1$.

Lemma 3.2.

$$\Re\int_{(a,b)\in\Sigma_\epsilon}(\mathbb{L}(v;\psi,A))_1\cdot\overline{T_1}$$
$$=-[\Delta_{\Gamma_\epsilon}k_1-2\epsilon\rho^{-2}\cos 2sk_{2\tilde\theta}+\epsilon^2\rho^{-2}(2\rho^{-4}-\cos^2 2\epsilon\tilde s)k_1+O(\epsilon^2\nabla^2_{\Gamma_\epsilon}k_1)]\int_{\mathbb{R}^2}|T_1|^2$$
$$+k_2\Im\int_{\mathbb{R}^2}d^*B_0T_2\overline{T_1}-2\Im\int_{\mathbb{R}^2}\langle B_0,d\xi\rangle\overline{T_1}+\Re\int_{\mathbb{R}^2}(|A^{(1)}+B_0|^2-|A^{(1)}|^2)(k_1|T_1|^2+k_2T_2\overline{T_1})$$
$$+O(\epsilon^3\rho^{-4}(\partial_{ij}k_l+\partial_ik_l+k_l)).$$

Proof. From the definition of $\mathbb{L}$, we see that

$$(\mathbb{L}(v;\psi,A))_1=-\Delta_A\xi+2i\langle B,\nabla_A\psi\rangle+\frac{1}{2}(\lambda-1)\psi^2\bar\xi+(\lambda+\frac{1}{2})|\psi|^2\xi-\frac{\lambda}{2}\xi.$$

In the following, we test T_1 and calculate the integrals one by one.

Step 1: The integral of $-\Delta_A\xi$.

From Lemma 2.3, we have

$$\begin{aligned}\Delta\xi=&(T_{1t_1t_1}+T_{1t_2t_2})k_1+(T_{2t_1t_1}+T_{2t_2t_2})k_2+\epsilon^2\rho^{-2}\cos^2(2\epsilon\tilde s)k_1(t_1^2T_{1t_2t_2}+t_2^2T_{1t_1t_1}-2t_1t_2T_{1t_1t_2})\\&+\Delta_{\Gamma_\epsilon}k_1T_1-\epsilon^2\rho^{-2}(\rho^{-4}+1)k_1(t_1T_{1t_1}+t_2T_{1t_2})+2\epsilon\rho^{-2}\cos(2\epsilon\tilde s)k_{1\tilde\theta}(t_2T_{1t_1}-t_1T_{1t_2})\\&+\epsilon^2\rho^{-2}\cos^2 2\epsilon\tilde sk_2(t_1^2T_{2t_t2t_2}+t_2^2T_{2t_1t_1}-2t_1t_2T_{2t_1t_2})+\Delta_{\Gamma_\epsilon}k_2T_2-\epsilon^2\rho^{-2}(\rho^{-4}+1)k_2(t_1T_{2t_1}+t_2T_{2t_2})\\&+2\epsilon\rho^{-2}\cos(2\epsilon\tilde s)k_{2\tilde\theta}(t_2T_{2t_1}-t_1T_{2t_2})-4\epsilon^2t_1(t_2\xi_{\tilde\theta t_1}-t_1\xi_{\tilde\theta t_2})\rho^{-5}\cos(2\epsilon\tilde s)\\&+2\epsilon^2\rho^{-2}\cos(2\epsilon\tilde s)(f_2\xi_{t_1}-f_1\xi_{t_2})+4t_2\epsilon^2\rho^{-7}\cos 2\epsilon\tilde s(t_2\xi_{\tilde st_1}-t_1\xi_{\tilde st_2})-2\epsilon^2\rho^{-6}(f_{1s}\xi_{\tilde st_1}+f_{2s}\xi_{\tilde st_2})\\&-2\epsilon^2\rho^{-2}(f_{1\theta}\xi_{\tilde\theta t_1}+f_{2\theta}\xi_{\tilde\theta t_2})+(2(t_1+\epsilon f_1)\rho^{-9}+3\tilde r^2\rho^{-12}\epsilon)\epsilon\xi_{\tilde s\tilde s}+4(t_2+\epsilon f_2)\epsilon\rho^{-7}\xi_{\tilde s\tilde\theta}\\&+(-2(t_2+\epsilon f_2)\rho^{-5}+3\tilde r^2\epsilon\rho^{-8})\epsilon\xi_{\tilde\theta\tilde\theta}+8t_1\epsilon^2\rho^{-7}\cos^2(2\epsilon\tilde s)\xi_{\tilde s}+4t_2\epsilon^2\rho^{-5}\cos(2\epsilon\tilde s)\xi_{\tilde\theta}+O(\epsilon^3\rho^{-4}).\end{aligned}$$

Denote

$$\begin{aligned}Q_1:=&-4\epsilon^2t_1(t_2\xi_{\tilde\theta t_1}-t_1\xi_{\tilde\theta t_2})\rho^{-5}\cos(2\epsilon\tilde s)+2\epsilon^2\rho^{-2}\cos(2\epsilon\tilde s)(f_2\xi_{t_1}-f_1\xi_{t_2})\\&+4t_2\epsilon^2\rho^{-7}\cos 2\epsilon\tilde s(t_2\xi_{\tilde st_1}-t_1\xi_{\tilde st_2})-2\epsilon^2\rho^{-6}(f_{1s}\xi_{\tilde st_1}+f_{2s}\xi_{\tilde st_2})-2\epsilon^2\rho^{-2}(f_{1\theta}\xi_{\tilde\theta t_1}+f_{2\theta}\xi_{\tilde\theta t_2})\\&+2t_1\epsilon\rho^{-9}\xi_{\tilde s\tilde s}+4t_2\epsilon\rho^{-7}\xi_{\tilde s\tilde\theta}-2t_2\epsilon\rho^{-5}\xi_{\tilde\theta\tilde\theta}+8t_1\epsilon^2\rho^{-7}\cos^2(2\epsilon\tilde s)\xi_{\tilde s}+4t_2\epsilon^2\rho^{-5}\cos(2\epsilon\tilde s)\xi_{\tilde\theta}.\end{aligned}$$

We see that Q_1 is orthogonal to both T_1 and T_2. Then we test $T_1(t_1,t_2)$ and integrate by parts in (t_1,t_2). Note that

$$\Re\int_{\mathbb{R}^2}(t_1T_{1t_1}+t_2T_{1t_2})\overline{T_1}=\frac{1}{2}\int_{\mathbb{R}^2}(t_1,t_2)\cdot\nabla|T_1|^2=-\int_{\mathbb{R}^2}|T_1|^2,$$

$$\Re\int_{\mathbb{R}^2}(t_2^2T_{1t_2t_2}+t_1^2T_{1t_1t_1}-2t_1t_2T_{1t_1t_2})\overline{T_1}=\Re\int_{\mathbb{R}^2}(t_2\partial_{t_1}-t_1\partial_{t_2})^2T_1\cdot\overline{T_1}+(t_1\partial_{t_1}+t_2\partial_{t_2})T_1\cdot\overline{T_1}$$
$$=\Re\int_{\mathbb{R}^2}\partial_{\tilde\phi}^2T_1\cdot\overline{T_1}-|T_1|^2=-3\pi\int_0^{+\infty}\tilde r(U')^2+\frac{1}{r}U^2(1-V)^2dr+4\pi\int_0^{+\infty}UU'(1-V)dr,$$

$$\Re\int_{\mathbb{R}^2}(t_2T_{1t_1}-t_1T_{1t_2})\overline{T_1}=\frac12\int_{\mathbb{R}^2}(t_2,-t_1)\cdot\nabla|T_1|^2=0,$$

$$\Re\int_{\mathbb{R}^2}(t_1T_{2t_1}+t_2T_{2t_2})\overline{T_1}=\Re\int_{\mathbb{R}^2}\tilde r\partial_{\tilde r}T_2\cdot\overline{T_1}=0,$$

$$\Re\int_{\mathbb{R}^2}(t_2^2T_{2t_2t_2}+t_1^2T_{2t_1t_1}-2t_1t_2T_{2t_1t_2})\overline{T_1}=\Re\int_{\mathbb{R}^2}(t_2\partial_{t_1}-t_1\partial_{t_2})^2T_2\cdot\overline{T_1}+(t_1\partial_{t_1}+t_2\partial_{t_2})T_2\cdot\overline{T_1}$$
$$=\Re\int_{\mathbb{R}^2}\partial_{\tilde\phi}^2T_2\cdot\overline{T_1}=0,$$

$$\Re\int_{\mathbb{R}^2}(t_2T_{2t_1}-t_1T_{2t_2})\overline{T_1}=-\Re\int_{\mathbb{R}^2}\partial_{\tilde\phi}T_2\cdot\overline{T_1}=2\pi\int_0^{+\infty}[\tilde r(U')^2+\frac{1}{\tilde r}U^2(1-V)^2-2UU'(1-V)]d\tilde r.$$

Combining the integrals above, we obtain

(3.1)
$$\begin{aligned}\Re\int_{(a,b)\in\Sigma_\epsilon}-\Delta\xi\cdot\overline{T_1}=&\Re\int_{\mathbb{R}^2}-(T_{1t_1t_1}+T_{1t_2t_2})k_1\overline{T_1}-(T_{2t_1t_1}+T_{2t_2t_2})k_2\overline{T_1}\\&-[\Delta_{\Gamma_\epsilon}k_1-2\epsilon\rho^{-2}\cos(2\epsilon\tilde s)k_{2\tilde\theta}+2\epsilon^2\rho^{-2}(\rho^{-4}-\cos^2(2\epsilon\tilde s))k_1+O(\epsilon^2\nabla^2_{\Gamma_\epsilon}k_1)]\int_{\mathbb{R}^2}|T_1|^2\\&-4(\epsilon\rho^{-2}\cos(2\epsilon\tilde s)k_{2\tilde\theta}+\epsilon^2\rho^{-2}\cos^2(2\epsilon\tilde s)k_1)\pi\int_0^{+\infty}UU'(1-V)d\tilde r+O(\epsilon^3\rho^{-4}).\end{aligned}$$

By (2.2), we have

$$d^*A\xi=d^*B_0(k_1T_1+k_2T_2)+O(\epsilon^3).$$

Likewise, we test $T_1(t_1,t_2)$ and integrate by parts in (t_1,t_2) to get

(3.2)
$$\Re\int_{\mathbb{R}^2}-id^*A\xi\overline{T_1}=\Im\int_{\mathbb{R}^2}d^*B_0(k_1T_1+k_2T_2)\overline{T_1}+O(\epsilon^3\rho^{-4})=k_2\Im\int_{\mathbb{R}^2}d^*B_0T_2\overline{T_1}+O(\epsilon^3\rho^{-4}).$$

By (2.3), there holds

$$\begin{aligned}\langle A,d\xi\rangle=&A_3\xi_a+A_4\xi_b-\epsilon^2\rho^{-2}\cos^2(2\epsilon\tilde s)(b\xi_a-a\xi_b)-\epsilon\rho^{-2}\cos(2\epsilon\tilde s)\xi_{\tilde\theta}\\&-\epsilon^2\rho^{-6}(2t_2\rho^{-1}\cos(2\epsilon\tilde s)+A_3f_{1s}+A_4f_{2s})\xi_{\tilde s}+\epsilon^2\rho^{-2}(2t_1\rho^{-1}\cos(2\epsilon\tilde s)-A_3f_{1s}-A_4f_{2s})\xi_{\tilde\theta}\\&+\langle B_0,d\xi\rangle+O(\epsilon^3\rho^{-4})\\&=:A_3\xi_a+A_4\xi_b-\epsilon^2\rho^{-2}\cos^2(2\epsilon\tilde s)(b\xi_a-a\xi_b)-\epsilon\rho^{-2}\cos(2\epsilon\tilde s)\xi_{\tilde\theta}+Q_2+\langle B_0,d\xi\rangle+O(\epsilon^3\rho^{-4}).\end{aligned}$$

We find that Q_2 is orthogonal to both T_1 and T_2. Then we test $T_1(t_1,t_2)$ and integrate by parts in (t_1,t_2) and deduce that

(3.3)
$$\begin{aligned}\Re\int_{\mathbb{R}^2}2i\langle A,d\xi\rangle\overline{T_1}=&\Re\int_{\mathbb{R}^2}2i(A_3T_{1t_1}+A_4T_{1t_2})k_1\overline{T_1}+2i(A_3T_{2t_1}+A_4T_{2t_2})k_2\overline{T_1}\\&-2\epsilon^2\rho^{-2}\cos^2(2\epsilon\tilde{s})k_1\Im\int_{\mathbb{R}^2}(t_2T_{1t_1}-t_1T_{1t_2})\overline{T_1}\\&-2\epsilon^2\rho^{-2}\cos^2(2\epsilon\tilde{s})k_2\Im\int_{\mathbb{R}^2}(t_2T_{2t_1}-t_1T_{2t_2})\overline{T_1}\\&-2\epsilon\rho^{-2}\cos(2\epsilon\tilde{s})k_{2\tilde{\theta}}\Im\int_{\mathbb{R}^2}T_2\overline{T_1}-2\Im\int_{\mathbb{R}^2}\langle B_0,d\xi\rangle\overline{T_1}+O(\epsilon^3)\\=&\Re\int_{\mathbb{R}^2}2i(A_3T_{1t_1}+A_4T_{1t_2})k_1\overline{T_1}+2i(A_3T_{2t_1}+A_4T_{2t_2})k_2\overline{T_1}\\&-2\epsilon^2\rho^{-2}\cos^2(2\epsilon\tilde{s})k_1\pi\int_0^{+\infty}\tilde{r}(U')^2+\frac{1}{\tilde{r}}U^2(1-V)^2-2UU'(1-V)d\tilde{r}\\&+4\epsilon\rho^{-2}\cos(2\epsilon\tilde{s})k_{2\tilde{\theta}}\pi\int_0^{+\infty}UU'(1-V)d\tilde{r}-2\Im\int_{\mathbb{R}^2}\langle B_0,d\xi\rangle\overline{T_1}+O(\epsilon^3).\end{aligned}$$

Applying (2.4), we have

$$|A|^2\xi=(A_3^2+A_4^2)(k_1T_1+k_2T_2)+\epsilon^2\rho^{-2}\cos^2(2\epsilon\tilde{s})(k_1T_1+k_2T_2)+(|A_0+B_0|^2-|A_0|^2)\xi+O(\epsilon^3).$$

Then we test $T_1(t_1,t_2)$ and integrate by parts in (t_1,t_2):

(3.4)
$$\begin{aligned}\Re\int_{\mathbb{R}^2}|A|^2\xi\overline{T_1}=&\Re\int_{\mathbb{R}^2}(A_3^2+A_4^2)(k_1|T_1|^2+k_2T_2\overline{T_1})+\epsilon^2\rho^{-2}\cos^2(2\epsilon\tilde{s})\Re\int_{\mathbb{R}^2}(k_1|T_1|^2+k_2T_2\overline{T_1})\\&+\Re\int_{\mathbb{R}^2}(|A_0+B_0|^2-|A_0|^2)(k_1|T_1|^2+k_2T_2\overline{T_1})+O(\epsilon^3)\\=&k_1\Re\int_{\mathbb{R}^2}(A_3^2+A_4^2)|T_1|^2+\epsilon^2\rho^{-2}\cos^2(2\epsilon\tilde{s})\cdot\pi\int_0^{+\infty}\tilde{r}(U')^2+\frac{1}{\tilde{r}}U^2(1-V)^2d\tilde{r}\\&+k_1\Re\int_{\mathbb{R}^2}(|A_0+B_0|^2-|A_0|^2)|T_1|^2+O(\epsilon^2).\end{aligned}$$

Combining (3.1), (3.2), (3.3) and (3.4), we get

(3.5)
$$\begin{aligned}&\Re\int_{(a,b)\in\Sigma_\epsilon}-\Delta_A\xi\cdot\overline{T_1}=\Re\int_{\mathbb{R}^2}[-T_{1t_1t_1}-T_{1t_2t_2}+2i(A_3T_{1t_1}+A_4T_{1t_2})+(A_3^2+A_4^2)T_1]k_1\overline{T_1}\\&-[\Delta_{\Gamma_\epsilon}k_1-2\epsilon\rho^{-2}\cos(2\epsilon\tilde{s})k_{2\theta}+\epsilon^2\rho^{-2}(2\rho^{-4}-\cos^2(2\epsilon\tilde{s}))k_1+O(\epsilon^2\nabla^2_{\Gamma_\epsilon}k_1)]\int_{\mathbb{R}^2}|T_1|^2\\&+k_2\Im\int_{\mathbb{R}^2}d^*B_0T_2\overline{T_1}-2\Im\int_{\mathbb{R}^2}\langle B_0,d\xi\rangle\overline{T_1}+\Re\int_{\mathbb{R}^2}(|A^{(1)}+B_0|^2-|A^{(1)}|^2)(k_1|T_1|^2+k_2T_2\overline{T_1})+O(\epsilon^3).\end{aligned}$$

Step 2: The integral of $2i\langle B,\nabla_A\psi\rangle$.

Note that $\nabla_A\psi=d\psi-iA\psi=\nabla_{A_0}\psi_0+\nabla_{A_0}\eta_0-iB_0\psi_0-B_0\eta_0$. Then we have

$$\begin{aligned}\nabla_{A_0}\psi_0=d\psi_0-iA_0\psi_0=&(-\epsilon^2f_sT_1-\epsilon^2g_sT_2)d\tilde{s}+(i(1-V)\epsilon\cos(2\epsilon\tilde{s})\psi_0-\epsilon^2f_\theta T_1-\epsilon^2g_\theta T_2)d\tilde{\theta}\\&+T_1da+T_2db.\end{aligned}$$

From this we infer

$$\langle B, \nabla_{A_0}\psi_0\rangle = k_1\langle T_{B_1}, \nabla_{A_0}\psi_0\rangle + k_2\langle T_{B_2}, \nabla_{A_0}\psi_0\rangle = k_1\frac{V'}{\tilde r}T_2 + k_2\frac{V'}{\tilde r}T_1 + O(\epsilon^3\rho^{-4}e^{-\delta r}).$$

Then we test $T_1(t_1,t_2)$ and integrate in (t_1,t_2) to get

$$\Re\int_{\mathbb{R}^2} 2i\langle B, \nabla_{A_0}\psi_0\rangle\overline{T_1} = \Re\int_{\mathbb{R}^2} 2ik_1\frac{V'}{\tilde r}T_2\overline{T_1} + 2ik_2\frac{V'}{\tilde r}|T_1|^2 + O(\epsilon^3\rho^{-4}). \tag{3.6}$$

Combining the integrals above, we get

$$\Re\int_{\mathbb{R}^2} 2i\langle B, \nabla_A\psi\rangle\overline{T_1} = \Re\int_{\mathbb{R}^2} 2ik_1(\partial_{t_2}A_3 - \partial_{t_1}A_4)|T_1|^2 + \Re\int_{\mathbb{R}^2} 2i\langle B, \nabla\eta_0 - iA_0\eta_0 - iB_0\psi_0 - B_0\eta_0\rangle\overline{T_1} + O(\epsilon^3\rho^{-4})$$

Step 3: Conclusion.

Observe that T_1 satisfies

$$-\Delta_{\mathbb{R}^2}T_1 + 2iA^{(1)}\cdot T_1 + |A^{(1)}|^2T_1 + 2i(\partial_{t_2}A_3 - \partial_{t_1}A_4)T_1 + \frac{1}{2}(\lambda-1)(\psi^{(1)})^2\bar T_1 + (\lambda+\frac{1}{2})|\psi^{(1)}|^2T_1 - \frac{\lambda}{2}T_1 = 0 \text{ in } \mathbb{R}^2.$$

Combining it with (3.5) and (3.6), we obtain the desired result. □

The inner product of the second components of $\mathbb{L}(v;\psi,A)$ and $\mathcal{T}_2$ can also be computed as follows.

Lemma 3.3.

$$\int_{\mathbb{R}^2}\langle(\mathbb{L}(v;\psi,A))_2, T_{B_1}\rangle = -\left[\Delta_{\Gamma_\epsilon}k_1 - 2\epsilon\rho^{-2}\cos 2s k_{2\tilde\theta} - \epsilon^2\rho^{-2}\cos^2(2\epsilon\tilde s)k_1 + O(\epsilon^2\nabla^2_{\Gamma_\epsilon}k_1)\right]\int_{\mathbb{R}^2}|T_{B_1}|^2 + O(\epsilon^3\rho^{-4})$$

Proof. Recall that

$$\mathbb{L}(v;\psi,A))_2 = \Delta_H B + 2Im(\overline{\nabla_A\psi}\cdot\xi) + B|\psi|^2.$$

In the following, we will test T_{B1} and compute the integrals term by term.

Firstly, by Lemma 2.4, we have

$$\begin{aligned}
\Delta_H B &= d^*dB + dd^*B = -\frac{1}{\sqrt{G}}g_{ml}\partial_j(\sqrt{G}g^{ij}g^{kl}B_{ik})dy^m - d\left(-\frac{1}{\sqrt{G}}\partial_j(\sqrt{G}g^{ij}B_i)\right)\\
&= O(\epsilon^2)d\tilde s + \left[\frac{2}{r^2}\epsilon(k_1\cos\tilde\phi + k_2\sin\tilde\phi)\cos(2\epsilon\tilde s)\left(\frac{V''}{\tilde r} - \frac{V'}{\tilde r}\right) + O(\epsilon^2)\right]d\tilde\theta + O(\epsilon^3)da + O(\epsilon^3)db\\
&+ \left[(\Delta_{\Gamma_\epsilon}k_2 - 2\epsilon\rho^{-2}\cos(2\epsilon\tilde s)k_{1\tilde\theta})\frac{V'}{\tilde r} + k_2\left(\frac{V'''}{\tilde r} - \frac{V''}{\tilde r^2} + \frac{V'}{\tilde r} - \epsilon^2\rho^{-2}\cos^2(2\epsilon\tilde s)\frac{V'}{\tilde r} - 2\epsilon^2\rho^{-6}V''\right) + Q_3 + Q_4\right]da\\
&+ \left[(-\Delta_{\Gamma_\epsilon}k_1 + 2\epsilon\rho^{-2}\cos(2\epsilon\tilde s)k_{2\tilde\theta})\frac{V'}{\tilde r} - k_1\left(\frac{V'''}{\tilde r} - \frac{V''}{\tilde r^2} + \frac{V'}{\tilde r} - \epsilon^2\rho^{-2}\cos^2(2\epsilon\tilde s)\frac{V'}{\tilde r} - 2\epsilon^2\rho^{-6}V''\right) + Q_5 + Q_6\right]db,
\end{aligned}$$

where

$$\begin{aligned}
Q_3 :=& 2\epsilon^2\rho^{-2}\cos(2\epsilon\tilde s)(f_2k_2\cos\tilde\phi - f_1k_2\sin\tilde\phi)\left(\frac{V'}{\tilde r}\right)' - 2\epsilon^2\rho^{-6}(f_{1s}k_{2\tilde s}\cos + f_{2s}k_{2\tilde s}\sin\tilde\phi)\left(\frac{V'}{\tilde r}\right)'\\
&- 2\epsilon^2\rho^{-2}(f_{1\theta}k_{2\tilde\theta}\cos\tilde\phi + f_{2\theta}k_{2\tilde\theta}\sin\tilde\phi)\left(\frac{V'}{r}\right)'\\
&+ (2t_1\epsilon\rho^{-9}k_{2\tilde s\tilde s} + 4t_2\epsilon\rho^{-7}k_{2\tilde s\tilde\theta} - 2t_2\epsilon\rho^{-5}k_{2\tilde\theta\tilde\theta} + 8t_1\epsilon^2\rho^{-7}\cos^2(2\epsilon\tilde s)k_{2\tilde s} + 4t_2\epsilon^2\rho^{-5}\cos(2\epsilon\tilde s)k_{2\tilde\theta})\frac{V'}{\tilde r},\\
Q_4 :=& [(2f_1\rho^{-9} + 3\tilde r^2\rho^{-12})k_{2\tilde s\tilde s} + 4f_2\rho^{-7}k_{2\tilde s\tilde\theta} + (-2f_2\rho^{-5} + 3\tilde r^2\rho^{-8})k_{2\tilde\theta\tilde\theta}]\epsilon^2\frac{V'}{\tilde r},
\end{aligned}$$

$$
\begin{aligned}
Q_5 :=& -2\epsilon^2\rho^{-2}\cos(2\epsilon\tilde{s})(f_2k_1\cos\tilde{\phi} - f_1k_1\sin\tilde{\phi})\left(\frac{V'}{\tilde{r}}\right)' + 2\epsilon^2\rho^{-6}(f_{1s}k_{1\tilde{s}}\cos + f_{2s}k_{1\tilde{s}}\sin\tilde{\phi})\left(\frac{V'}{\tilde{r}}\right)' \\
&+ 2\epsilon^2\rho^{-2}(f_{1\theta}k_{1\tilde{\theta}}\cos\tilde{\phi} + f_{2\theta}k_{1\tilde{\theta}}\sin\tilde{\phi})\left(\frac{V'}{r}\right)' \\
&- (2t_1\epsilon\rho^{-9}k_{1\tilde{s}\tilde{s}} + 4t_2\epsilon\rho^{-7}k_{1\tilde{s}\tilde{\theta}} - 2t_2\epsilon\rho^{-5}k_{1\tilde{\theta}\tilde{\theta}} + 8t_1\epsilon^2\rho^{-7}\cos^2(2\epsilon\tilde{s})k_{1\tilde{s}} + 4t_2\epsilon^2\rho^{-5}\cos(2\epsilon\tilde{s})k_{1\tilde{\theta}})\frac{V'}{\tilde{r}}, \\
Q_6 :=& -[(2f_1\rho^{-9} + 3\tilde{r}^2\rho^{-12})k_{1\tilde{s}\tilde{s}} + 4f_2\rho^{-7}k_{1\tilde{s}\tilde{\theta}} + (-2f_2\rho^{-5} + 3\tilde{r}^2\rho^{-8})k_{1\tilde{\theta}\tilde{\theta}}]\epsilon^2\frac{V'}{\tilde{r}}.
\end{aligned}
$$

Note that Q_3da and Q_5db are orthogonal to both T_{B_1} and T_{B_2} up to $O(\epsilon^2)$. It follows that
(3.7)

$$
\begin{aligned}
&\int_{\mathbb{R}^2}\langle\Delta_H B, T_{B_1}\rangle = -k_1\int_0^{+\infty}\int_0^{2\pi}\frac{V'}{\tilde{r}}\left(\frac{V'''}{\tilde{r}^2} - \frac{V''}{\tilde{r}^3} + \frac{V'}{r^4}\right)d\tilde{\phi}d\tilde{r} + 4\pi\epsilon^2\rho^{-6}k_1\int_0^{+\infty}V'V''d\tilde{r} \\
&+2(-\Delta_{\Gamma_\epsilon}k_1 + 2\epsilon\rho^{-2}\cos 2sk_{2\tilde{\theta}} + \epsilon^2\rho^{-2}\cos^2 2sk_1 + O(\epsilon^2\nabla^2_{\Gamma_\epsilon}k_1))\pi\int_0^{+\infty}\frac{(V')^2}{\tilde{r}}d\tilde{r} + O(\epsilon^3) \\
&= -2k_1\pi\int_0^{+\infty}\frac{V'}{\tilde{r}}\left(\frac{V'''}{\tilde{r}^2} - \frac{V''}{\tilde{r}^3} + \frac{V'}{\tilde{r}^4}\right)d\tilde{r} \\
&+2(-\Delta_{\Gamma_\epsilon}k_1 + 2\epsilon\rho^{-2}\cos 2sk_{2\tilde{\theta}} + \epsilon^2\rho^{-2}\cos^2 2sk_1 + O(\epsilon^2\nabla^2_{\Gamma_\epsilon}k_1))\pi\int_0^{+\infty}\frac{(V')^2}{\tilde{r}}d\tilde{r} + O(\epsilon^3\rho^{-4}).
\end{aligned}
$$

To calculate the second term, we decompose $\nabla_A\psi = \nabla_{A_0}\psi_0 + \nabla_{A_0}\eta_0 - iB_0\psi_0 - B_0\eta_0$. And we have

$$\langle 2\Im(\overline{\nabla_{A_0}\psi_0}\cdot\xi), T_{B_1}\rangle = 2k_1\Im\langle T_1\overline{\nabla_{A_0}\psi_0}, T_{B_1}\rangle + 2k_2\Im\langle T_2\overline{\nabla_{A_0}\psi_0}, T_{B_1}\rangle = 2k_1\frac{V'}{\tilde{r}}\Im(T_1\overline{T_2}) + O(\epsilon^3).$$

Integrating in (t_1, t_2), we have

$$\int_{\mathbb{R}^2}\langle 2\Im(\overline{\nabla_{A_0}\psi_0}\cdot\xi), T_{B_1}\rangle = -4k_1\pi\int_0^{+\infty}\frac{1}{\tilde{r}}UU'V'(1-V)d\tilde{r} + O(\epsilon^3\rho^{-4})$$

and hence,
(3.8)

$$\int_{\mathbb{R}^2}\langle 2\Im(\overline{\nabla_A\psi}\cdot\xi), T_{B_1}\rangle = -4k_1\pi\int_0^{+\infty}\frac{1}{\tilde{r}}UU'V'(1-V)d\tilde{r} + \int_{\mathbb{R}^2}\Im\langle 2(\nabla_{A_0}\eta_0 - iB_0\psi_0 - B_0\eta_0)\xi, T_{B_1}\rangle.$$

Finally, direct computations give

$$\langle B|\psi_0|^2, T_{B_1}\rangle = k_1U^2|T_{B_1}|^2 + k_2U^2\langle T_{B_2}, T_{B_1}\rangle + O(\epsilon^3\rho^{-4}).$$

Integrating in (t_1, t_2) yields

$$\text{(3.9)}\qquad \int_{\mathbb{R}^2}\langle B|\psi|^2, T_{B_1}\rangle = 2\pi k_1\int_0^{+\infty}U^2\frac{V'^2}{\tilde{r}}d\tilde{r} + \int_{\mathbb{R}^2}\langle B(|\psi|^2 - |\psi_0|^2), T_{B_1}\rangle + O(\epsilon^3\rho^{-4}).$$

Taking derivative on the second equation in (2.5), we get

$$-V''' - \frac{V''}{\tilde{r}} + \frac{V'}{r^2} - 2UU'(1-V) + U^2V' = 0.$$

Combining it with (3.7), (3.8) and (3.9), we obtain the desired result. □

With Lemma 3.2 and Lemma 3.3, we have the following rough result.

Lemma 3.4.

(3.10)

$$\int_{(a,b)\in\Sigma_\epsilon}\langle\mathbb{L}(v;\psi,A),\mathcal{T}_1\rangle dt_1dt_2=\Re\int_{(a,b)\in\Sigma_\epsilon}\mathbb{L}(v;\psi,A)_1\overline{T_1}dt_1dt_2+\int_{(a,b)\in\Sigma_\epsilon}\langle\mathbb{L}(v;\psi,A)_2,T_{B_1}\rangle dt_1dt_2$$
$$=-\left[\Delta_{\Gamma_\epsilon}k_1-2\epsilon\rho^{-2}\cos 2sk_{2\tilde\theta}+\epsilon^2\rho^{-2}(2\rho^{-4}-\cos^2(2\epsilon\tilde s))k_1+O(\epsilon^2\nabla^2_{\Gamma_\epsilon}k_1)\right]\int_{\mathbb{R}^2}|T_1|^2+|T_{B_1}|^2$$
$$+4\epsilon^2\rho^{-6}k_1\pi\int_0^{+\infty}\frac{(V')^2}{\tilde r}d\tilde r+k_2\Im\int_{\mathbb{R}^2}d^*B_0T_2\overline{T_1}-2\Im\int_{\mathbb{R}^2}\langle B_0,d\xi\rangle\overline{T_1}+\Re\int_{\mathbb{R}^2}(|A^{(1)}+B_0|^2-|A^{(1)}|^2)k_1|T_1|^2$$
$$+\Re\int_{\mathbb{R}^2}2i\langle B,\nabla_{A_0}\eta_0-iB_0\psi_0-B_0\eta_0\rangle\overline{T_1}+\Re\int_{\mathbb{R}^2}[\frac{1}{2}(\lambda-1)(\psi^2-\psi_0^2)\bar\xi+(\lambda+\frac{1}{2})(|\psi|^2-|\psi_0|^2)\xi]\overline{T_1}$$
$$+\int_{\mathbb{R}^2}\Im\langle 2(\nabla_{A_0}\eta_0-iB_0\psi_0-B_0\eta_0)\cdot\xi,T_{B_1}\rangle+\int_{\mathbb{R}^2}\langle B(|\psi|^2-|\psi_0|^2),T_{B_1}\rangle+O(\epsilon^3(|\partial_{ij}k|+|\partial_ik|+|k|)).$$

It remains to compute the terms from the nonlinear terms. From Section 2, we see that the improvement of approximation has the form:

$$(\eta_0,B_0)=-2\epsilon^2\rho^{-6}(\eta_1,B_1)+O(\epsilon^3\rho^{-4}e^{-\delta r})$$

where (η_1,B_1) solves

$$\mathbb{L}(\eta_1,B_1;\psi^{(1)},A^{(1)})=\begin{pmatrix}\tilde rU'e^{i\tilde\phi}\\-\sin\tilde\phi V'dt_1+\cos\tilde\phi V'dt_2\end{pmatrix}\text{ in }\mathbb{R}^2.\tag{3.11}$$

Taking ∂_{t_i} on $\mathbb{L}$, we have for any (μ,C),

(3.12)

$$\partial_{t_i}\mathbb{L}(\mu,C;\psi_0,A_0)=\mathbb{L}(\partial_{t_i}\mu,\partial_{t_i}C;\psi_0,A_0)$$
$$+\begin{pmatrix}2i\langle\partial_{t_i}A_0,\nabla_{A_0}\mu\rangle+2i\langle C,\nabla_{A_0}\partial_{t_i}\psi_0-i\partial_{t_i}A_0\psi_0\rangle+(\lambda-1)\psi_0\partial_{t_i}\psi_0\overline{\mu}+(2\lambda+1)\Re(\overline{\psi_0}\partial_{t_i}\psi_0)\mu\\2\Im[\overline{\nabla_{A_0}\partial_{t_i}\psi_0-i\partial_{t_i}A_0\psi_0}\cdot\mu]+2\Re(\overline{\psi_0}\partial_{t_i}\psi_0)C\end{pmatrix}$$
$$+\begin{pmatrix}-\partial_{t_i}(\frac{1}{\sqrt G}\partial_k(\sqrt G g^{jk}))(\nabla_{A_0}\mu)_j-\partial_{t_i}g^{jk}\partial_{jk}\mu+2i\partial_{t_i}g^{jk}A_{0k}\partial_j\mu+\partial_{t_i}g^{jk}A_{0j}A_{0k}\mu+2i\partial_{t_i}g^{jk}C_j(\nabla_{A_0}\psi_0)_k\\{}[-\partial_{t_i,m}(\frac{1}{\sqrt G}\partial_k(\sqrt G g^{jk}))C_j-\partial_{t_i,k}g^{jk}\partial_jC_m-\partial_{t_i}g^{jk}\partial_{jk}C_m-\partial_{t_i}(g_{mn}g^{jk}\partial_kg^{ln})C_{jl}]dy^m\end{pmatrix}.$$

Let us recall that $\mathbb{L}(\cdot;\psi_0,A_0)$ is invariant under the following gauge transformation

$$\mathbb{L}(\tilde G_\gamma(\xi,B);G_\gamma(\psi_0,A_0))=\tilde G_\gamma(\mathbb{L}(\xi,B))\text{ for any }\gamma\in C^1(\mathbb{R}^4).$$

Then we have

$$\begin{aligned}&\frac{d}{dt}|_{t=0}\tilde G_{t\gamma}(\mathbb{L}(\mu,C))=\frac{d}{dt}|_{t=0}\mathbb{L}(\tilde G_{t\gamma}(\mu,C);G_{t\gamma}(\psi_0,A_0))=\mathbb{L}(i\gamma\mu,C;\psi_0,A_0)\\&+\begin{pmatrix}-id^*d\gamma\mu+2i\langle d\gamma,\nabla_{A_0}\mu\rangle+2i\langle C,\nabla_{A_0}(i\gamma\psi_0)-id\gamma\psi_0\rangle+(\lambda-1)i\gamma\psi_0^2\overline{\mu}\\2\Im[\overline{\nabla_{A_0}(i\gamma\psi_0)-id\gamma\psi_0}\cdot\mu]\end{pmatrix}.\end{aligned}\tag{3.13}$$

Combining (3.12), (3.13) and taking $\gamma = A_3$, for example, we have

$$\partial_{t_i}\mathbb{L}(\mu, C; \psi_0, A_0) - \frac{d}{dt}|_{t=0}\tilde{G}_{tA_3}(\mathbb{L}(\mu, C)) = \mathbb{L}(\partial_{t_i}\mu, \partial_{t_i}C) - \mathbb{L}(iA_3\mu, C; \psi_0, A_0)$$
$$+\begin{pmatrix} -id^*T_{B_1}\mu + 2i\langle T_{B_1}, \nabla_{A_0}\mu\rangle + 2i\langle C, \nabla_{A_0}T_1 - iT_{B_1}\psi_0\rangle + (\lambda-1)\psi_0 T_1\overline{\mu} + (2\lambda+1)\Re(\overline{\psi_0}T_1)\mu \\ 2\Im[\overline{\nabla_{A_0}T_1 - iT_{B_1}\psi_0}\cdot\mu] + 2\Re(\overline{\psi_0}T_1)C \end{pmatrix}$$
$$+\begin{pmatrix} -\partial_{t_i}(\frac{1}{\sqrt{G}}\partial_k(\sqrt{G}g^{jk}))(\nabla_{A_0}\mu)_j - \partial_{t_i}g^{jk}\partial_{jk}\mu + 2i\partial_{t_i}g^{jk}A_{0k}\partial_j\mu + \partial_{t_i}g^{jk}A_{0j}A_{0k}\mu + 2i\partial_{t_i}g^{jk}C_j(\nabla_{A_0}\psi_0)_k \\ [-\partial_{t_i,m}(\frac{1}{\sqrt{G}}\partial_k(\sqrt{G}g^{jk}))C_j - \partial_{t_i,k}g^{jk}\partial_j C_m - \partial_{t_i}g^{jk}\partial_{jk}C_m - \partial_{t_i}(g_{mn}g^{jk}\partial_k g^{ln})C_{jl}]dy^m \end{pmatrix}.$$

Plugging in $(\mu, C) = (\eta_1, B_1)$ and testing the above equation by (T_1, T_{B_1}), and integrating in (t_1, t_2), we have

$$-2\Im\int_{\mathbb{R}^2}\langle B_1, dT_1\rangle + 2\Re\int_{\mathbb{R}^2}\langle B_1, A_0\rangle|T_1|^2 + \Re\int_{\mathbb{R}^2}2i\langle T_{B_1}, \nabla_{A_0}\eta_1 - iB_1\psi_0\rangle\overline{T_1} + O(\epsilon)$$
$$+\Re\int_{\mathbb{R}^2}(\lambda-1)\psi_0\eta_1\overline{T_1} + (2\lambda+1)\psi_0\eta_1|T_1|^2 + \int_{\mathbb{R}^2}\Im\langle 2(\nabla_{A_0}\eta_1 - iB_i\psi_0)T_1, T_{B1}\rangle + 2\int_{\mathbb{R}^2}\langle 2\psi_0\eta_1 T_{B_1}, T_{B_1}\rangle$$
$$= \int_{\mathbb{R}^2}\langle\partial_{t_1}L_0(\mu, C; \psi_0, A_0) - \frac{d}{dt}|_{t=0}\tilde{G}_{tA_3}(L_0(\mu, C)), (T_1, T_{B_1})\rangle =: E.$$

Plugging (3.11) into the right-hand side above, we can compute E as

$$E = \pi\int_0^{+\infty}\tilde{r}(\tilde{r}U'' + U')U' + UU'(1-V)^2 + \frac{(V')^2}{\tilde{r}}d\tilde{r} + O(\epsilon) = 2\pi\int_0^{+\infty}\frac{(V')^2}{\tilde{r}}d\tilde{r} + O(\epsilon).$$

Then (3.10) can be represented as

$$\int_{(a,b)\in\Sigma_\epsilon}\langle\mathbb{L}(v;\psi, A), \mathcal{T}_1\rangle dt_1 dt_2 = \Re\int_{(a,b)\in\Sigma_\epsilon}\mathbb{L}(v;\psi, A)_1\overline{T_1}dt_1 dt_2 + \int_{(a,b)\in\Sigma_\epsilon}\langle\mathbb{L}(v;\psi, A)_2, T_{B_1}\rangle dt_1 dt_2$$
$$= -\left[\Delta_{\Gamma_\epsilon}k_1 - 2\epsilon^2\rho^{-2}\cos 2\epsilon\tilde{s}k_{2\theta} + \epsilon^2\rho^{-2}(2\rho^{-4} - \cos^2 2\epsilon\tilde{s})k_1 + O(\epsilon^2\nabla^2_{\Gamma_\epsilon}k_1)\right]\int_{\mathbb{R}^2}|\mathcal{T}_1|^2$$
$$+4\epsilon^2\rho^{-6}k_1\pi\int_0^{+\infty}\frac{(V')^2}{\tilde{r}}d\tilde{r} + 2\epsilon^2\rho^{-6}k_1 E - 2\Im\int_{\mathbb{R}^2}\langle B_0, (k_{1\theta}T_1 + k_{2\theta}T_2)d\theta\rangle\overline{T_1} + O(\epsilon^3\rho^{-4})$$
$$= -\left[\Delta_{\Gamma_\epsilon}k_1 - 2\epsilon^2\rho^{-2}\cos 2\epsilon\tilde{s}k_{2\theta} + \epsilon^2\rho^{-2}(2\rho^{-4} - \cos^2 2\epsilon\tilde{s})k_1 + O(\epsilon^2\nabla^2_{\Gamma_\epsilon}k_1)\right]\int_{\mathbb{R}^2}|\mathcal{T}_1|^2 + O(\epsilon^3\rho^{-4}),$$

which is exactly the first component appeared in Proposition 3.1. Similar computation gives the second component and we finish the proof of Proposition 3.1.

Before we state the next proposition, we define a cut-off function χ supported in Σ_ϵ:

$$\begin{cases} \chi = 1 & \text{if } a^2 + b^2 \le \frac{1}{2}r_\epsilon, \\ \chi = 0 & \text{if } a^2 + b^2 \ge r_\epsilon, \end{cases}$$

where we recall that $r_\epsilon(\tilde{s}, \tilde{\theta}) = \frac{1}{2}\epsilon^{-\frac{1}{2}}\rho(\tilde{s}, \tilde{\theta})^2$. We also define the region

$$\mathcal{W}^R := \{x \in \Sigma_\epsilon :\ \rho(\tilde{s}, \tilde{\theta}) < R\}$$

and

$$\Gamma_\epsilon^R := \{(\tilde{s}, \tilde{\theta}) \in \Gamma_\epsilon :\ \rho(\tilde{s}, \tilde{\theta}) < R\}$$

for some large constant $R > 0$.

Proposition 3.5. *Let* $N = k_1(\tilde{s},\tilde{\theta})\mathbf{m}+k_2(\tilde{s},\tilde{\theta})\mathbf{n}$ *be a normal vector field that vanishes when* $\rho(\tilde{s},\tilde{\theta}) = R$ *and set*

$$v(x) = v(\tilde{s},\tilde{\theta},t_1,t_2) = (\xi, B)(\tilde{s},\tilde{\theta},t_1,t_2) = k_1(\tilde{s},\tilde{\theta})\mathcal{T}_1(t_1,t_2) + k_2(\tilde{s},\tilde{\theta})\mathcal{T}_2(t_1,t_2).$$

Then we have the following estimate

$$\begin{aligned}\mathcal{Q}(\chi v,\chi v) &:= \int_{\mathcal{W}^R} |\nabla_A\xi|^2 + |dB|^2 + |d^*B|^2 + |B|^2|\psi|^2 + 4\langle \Im(\overline{\nabla_A\psi}\xi), B\rangle \\ &+ \frac{\lambda-1}{2}\Re(\overline{\psi}\xi)^2 + (\lambda+\frac{1}{2})|\psi|^2|\xi|^2 - \frac{\lambda}{2}|\xi|^2 dx \\ &= \int_{\Gamma_\epsilon^R} |\nabla^\nu_{\Gamma_\epsilon} N|^2 - 2\epsilon^2\rho^{-6}|N|^2 dvol_{\Gamma_\epsilon}\cdot \int_{\mathbb{R}^2}|\mathcal{T}_1|^2 + O\left(\epsilon\int_{\Gamma_\epsilon^R}|\nabla N|^2 + \epsilon^2\rho^{-6}|N|^2 dvol_{\Gamma_\epsilon}\right),\end{aligned} \tag{3.14}$$

where $\nabla^\nu_{\Gamma_\epsilon} N$ *is the covariant derivative on the normal bundle* $\mathcal{N}\Gamma_\epsilon$.

Proof. Direct computation gives that

$$dx = \sqrt{G} = \sqrt{\det \tilde{A}}(1-\epsilon^2\tilde{r}\rho^{-6}).$$

Hence, we have

$$\int_{\mathcal{W}^R}\langle \mathbb{L}(v;\psi,A), v\rangle dx = \int_{\Gamma_\epsilon^R}\int_{\tilde{r}<r_\epsilon}\langle \mathbb{L}(v;\psi,A), v\rangle(1-\epsilon^2\tilde{r}^2\rho^{-6}) da db dvol_{\Gamma_\epsilon}. \tag{3.15}$$

By Proposition 3.1, we find that

$$\begin{aligned}&\int_{\Gamma_\epsilon^R}\int_{r<r_\epsilon}\langle \mathbb{L}(v;\psi,A), v\rangle \\ &= -\int_{\Gamma_\epsilon^R}\langle L_{\Gamma_\epsilon}N, N\rangle dvol_{\Gamma_\epsilon}\cdot(1+O(\epsilon))\int_{\mathbb{R}^2}|\mathcal{T}_1|^2 + \int_{\Gamma_\epsilon^R} O(\epsilon(D^2k + Dk + k))\cdot k \\ &= -\int_{\Gamma_\epsilon^R}\langle \Delta^\nu_{\Gamma_\epsilon}N + 2\epsilon^2\rho^{-6}N, N\rangle dvol_{\Gamma_\epsilon}\cdot(1+O(\epsilon))\int_{\mathbb{R}^2}|\mathcal{T}_1|^2 + \int_{\Gamma_\epsilon^R} O(\epsilon(D^2k + Dk + k))\cdot k.\end{aligned} \tag{3.16}$$

Integrating by parts yields

$$\begin{aligned}&\int_{\Gamma_\epsilon^R}\int_{r<r_\epsilon}\langle \mathbb{L}(v;\psi,A), v\rangle \\ &= \int_{\Gamma_\epsilon^R} |\nabla^\nu_{\Gamma_\epsilon} N|^2 - 2\epsilon^2\rho^{-6}|N|^2 dvol_{\Gamma_\epsilon}\cdot \int_{\mathbb{R}^2}|\mathcal{T}_1|^2 + O\left(\epsilon\int_{\Gamma_\epsilon^R}|\nabla N|^2 + \epsilon^2\rho^{-6}|N|^2 dvol_{\Gamma_\epsilon}\right).\end{aligned}$$

Since $\mathcal{T}_1$ and $\mathcal{T}_2$ decay exponentially on $\mathbb{R}^2$, similar computation gives that

$$\int_{\Gamma_\epsilon^R}\int_{r<r_\epsilon}\langle \mathbb{L}(v;\psi,A), v\rangle \epsilon^2\tilde{r}^2\rho^{-6} da db dvol_{\Gamma_\epsilon} = O\left(\epsilon\int_{\Gamma_\epsilon^R}|\nabla N|^2 + \epsilon^2\rho^{-6}|N|^2 dvol_{\Gamma_\epsilon}\right). \tag{3.17}$$

Combining (3.16) and (3.17), we see that (3.15) can be written as

$$\mathcal{Q}(v,v)=\int_{\mathcal{W}^R}\langle \mathbb{L}(v;\psi,A),v\rangle dx=\int_{\Gamma_\epsilon^R}|\nabla^\nu_{\Gamma_\epsilon}N|^2-2\epsilon^2\rho^{-6}|N|^2dvol_{\Gamma_\epsilon}\int_{\mathbb{R}^2}|\mathcal{T}_1|^2$$
$$+O\left(\epsilon\int_{\Gamma_\epsilon^R}|\nabla N|^2+\epsilon^2\rho^{-6}|N|^2dvol_{\Gamma_\epsilon}\right).$$

Then the conclusion follows for $\mathcal{Q}(v,v)$. A similar computation holds for $\mathcal{Q}(\chi v,\chi v)$ since χv vanishes on $\partial\mathcal{W}^R$ and we obtain the desired result. □

The proposition above builds up the relationship between $\mathcal{Q}$ and the second variation of the area functional on Γ_ϵ. With the stability of Γ_ϵ and Proposition 3.5, it suffices to show that the eigenfunctions of $\mathbb{L}(\cdot;\psi,A)$ with respect to negative eigenvalues (if exist) are almost $k_1(\tilde{s},\tilde{\theta})\mathcal{T}_1+k_2(\tilde{s},\tilde{\theta})\mathcal{T}_2$. To show this, we need to derive several estimates. We first show that the negative eigenvalues of $\mathbb{L}(\cdot;\psi,A)$ (if exist) are $O(\epsilon^2)$.

Lemma 3.6. *There exists a constant $\mu_0>0$ independent of R and sufficiently small ϵ, such that if $\mu\le 0$ is an eigenvalue of problem (2.11), then*

$$\mu\ge-\mu_0\epsilon^2.$$

Proof. we dentoe $\mathcal{Q}_\Omega$ be the restriction of $\mathcal{Q}$ in the domain Ω:

$$\mathcal{Q}_\Omega(v,v):=\int_\Omega|\nabla_A\xi|^2+|dB|^2+|d^*B|^2+|B|^2|\psi|^2+4\langle\Im(\overline{\nabla_A\psi}\xi),B\rangle+\frac{\lambda-1}{2}\Re(\overline{\psi}\xi)^2+(\lambda+\frac{1}{2})|\psi|^2|\xi|^2-\frac{\lambda}{2}|\xi|^2$$

for $v=(\xi,B)$.

Recall that $(\psi,A)=(We^{i\tilde{\varphi}},Zd\tilde{\varphi})+O(\epsilon^2e^{-\delta\tilde{r}})$ in $\mathbb{R}^4\setminus\Sigma_\epsilon$, where $W,Z=1-O(e^{-\delta\tilde{r}})$ for some $0<\delta<1$ and φ is an extension of the angle function ϕ in $\mathbb{R}^4$. Then (ψ,A) is stable in $\mathbb{R}^4\setminus\Sigma_\epsilon$. Now we set $\Omega:=\Sigma_\epsilon\cap\{\rho<R\}$. Then for any v, we have

$$\mathcal{Q}(v,v)\ge\mathcal{Q}_\Omega(v,v)+\gamma\int_{\mathbb{R}^4\setminus\Sigma_\epsilon}|v|^2$$

for some $\gamma>0$ independent of ϵ and R. In the following, we will show that

$$\mathcal{Q}_\Omega(v,v)\ge-\mu_0\epsilon^2\int_\Omega\rho^{-6}|v|^2.$$

Its corresponding eigenvalue problem is

$$\begin{cases}\mathbb{L}(v;\psi,A)-\mu\rho^{-6}v=0 & \text{in }\Omega\\ v=0 & \text{on }\{\rho=R\}\\ \partial_\nu\xi-i\langle A,\nu\rangle\xi=0,\ d^*B=0,\ *dB=0 & \text{on }\partial\Sigma_\epsilon.\end{cases}\tag{3.18}$$

For an eigenfunction v solving (3.18), we decompose v as

$$v=\chi k_1(\tilde{s},\tilde{\theta})\mathcal{T}_1+\chi k_2(\tilde{s},\tilde{\theta})\mathcal{T}_2+v^\perp,$$

where for $j=1,2$,

$$k_j(\tilde{s},\tilde{\theta})=\frac{\int_{r<r_\epsilon}\langle v(\tilde{s},\tilde{\theta},t_1,t_2),\mathcal{T}_j(t_1,t_2)\rangle dt_1dt_2}{\int_{\mathbb{R}^2}\chi|\mathcal{T}_1(t_1,t_2)|^2dt_1dt_2}.$$

From our decomposition, we see that k_j vanishes on $\{\rho = R\}$, $v^\perp$ satisfies the same boundary condition as v and

$$\int_{r<r_\epsilon} \langle v^\perp(\tilde{s},\tilde{\theta},t_1,t_2), \mathcal{T}_j(t_1,t_2)\rangle dt_1 dt_2 = 0, \text{ for } j=1,2 \text{ and } (\tilde{s},\tilde{\theta}) \in \Gamma_\epsilon^R. \tag{3.19}$$

With the decomposition, we have

$$\mathcal{Q}_\Omega(v,v) = \mathcal{Q}_\Omega(v^\perp,v^\perp) + \mathcal{Q}_\Omega(\chi k_1\mathcal{T}_1 + k_2\mathcal{T}_2, \chi k_1\mathcal{T}_1 + k_2\mathcal{T}_2) + 2\mathcal{Q}_\Omega(v^\perp, \chi k_1\mathcal{T}_1 + \chi k_2\mathcal{T}_2))$$

and each term will be estimated in the following. For simplicity, we denote $d_{\mathbb{R}^2}$, $d^*_{\mathbb{R}^2}$, $\nabla_{A^{(1)},\mathbb{R}^2} := d_{\mathbb{R}^2} - iA^{(1)}$ to be the exterior derivative, codifferential and connection gradient on the normal space (t_1,t_2). And d_{Γ_ϵ}, $d^*_{\Gamma_\epsilon}$, ∇_{Γ_ϵ} be the ones on Γ_ϵ.

For $Q_\Omega(v^\perp,v^\perp)$, we write $v^\perp = (\xi^\perp, B^\perp)$ and compute

$$\begin{aligned}\int_\Omega |\nabla_A\xi^\perp|^2 &= \int_\Omega |\nabla_{A^{(1)},\mathbb{R}^2}\xi^\perp|^2 + (1+O(\epsilon^2))|\nabla_{\Gamma_\epsilon}\xi^\perp|^2 + O(\epsilon)|\nabla_{\Gamma_\epsilon}\xi^\perp||\nabla_{A^{(1)},\mathbb{R}^2}\xi^\perp| \\ &\quad + O(\epsilon^2)|\nabla_{A^{(1)},\mathbb{R}^2}\xi^\perp|^2 + O(\epsilon^2)|\xi^\perp|^2 dx \\ &\geq \int_{\rho<R}\int_{r<r_\epsilon} (1+O(\epsilon^2))(|\nabla_{A^{(1)},\mathbb{R}^2}\xi^\perp|^2 + |\nabla_{\Gamma_\epsilon}\xi^\perp|^2) + O(\epsilon^2)|\xi^\perp|^2 dt_1 dt_2 dVol_{\Gamma_\epsilon}.\end{aligned} \tag{3.20}$$

We decompose $B^\perp = (B_1^\perp d\tilde{s} + B_2^\perp d\tilde{\theta}) + (B_3^\perp dt_1 + B_4^\perp dt_2) =: B^\perp_{\Gamma_\epsilon} + B^\perp_{\mathbb{R}^2}$, then the second and third terms in $Q_\Omega(v^\perp,v^\perp)$ can also be computed as

$$\begin{aligned}&\int_\Omega |dB^\perp|^2 + |d^*B^\perp|^2 dx \\ =& \int_\Omega \frac{1}{2}\sum_{j,k,l,m=1}^4 \tilde{g}^{jk}\tilde{g}^{lm}B^\perp_{jl}B^\perp_{km} + \sum_{j,k=1}^4 \left|\frac{1}{\sqrt{\tilde{G}}}\partial_k(\tilde{g}^{jk}\sqrt{\tilde{G}}B_j^\perp)\right|^2 dx \\ =& \int_\Omega (1+O(\epsilon^2))(|d_{\mathbb{R}^2}B^\perp_{\mathbb{R}^2}|^2 + |d^*_{\mathbb{R}^2}B^\perp_{\mathbb{R}^2}|^2 + |d_{\Gamma_\epsilon}B^\perp_{\Gamma_\epsilon}|^2 + |d^*_{\Gamma_\epsilon}B^\perp_{\Gamma_\epsilon}|^2) \\ &+ \sum_{j=1}^2\sum_{k=3}^4 \tilde{g}^{jj}\tilde{g}^{kk}(B^\perp_{jk})^2 + 2(\rho^{-6}B^\perp_{1\tilde{s}} + \rho^{-2}B^\perp_{2\tilde{\theta}} + 2\epsilon\rho^{-4}\cos(2\epsilon\tilde{s})B_1^\perp)(B^\perp_{3t_1} + B^\perp_{4t_2}) + O(\epsilon)|\nabla_{\Gamma_\epsilon}B^\perp||\nabla_{\mathbb{R}^2}B^\perp| dx \\ =& \int_{\rho<R}\int_{r<r_\epsilon} (1+O(\epsilon^2))(|d_{\mathbb{R}^2}B^\perp_{\mathbb{R}^2}|^2 + |d^*_{\mathbb{R}^2}B^\perp_{\mathbb{R}^2}|^2 + |d_{\Gamma_\epsilon}B^\perp_{\Gamma_\epsilon}|^2 + |d^*_{\Gamma_\epsilon}B^\perp_{\Gamma_\epsilon}|^2)\rho^4 \\ &+ \rho^{-2}((B^\perp_{3\tilde{s}})^2 + (B^\perp_{4\tilde{s}})^2 + (B^\perp_{1t_1})^2 + (B^\perp_{1t_2})^2) + \rho^2((B^\perp_{3\tilde{\theta}})^2 + (B^\perp_{4\tilde{\theta}})^2 + (B^\perp_{2t_1})^2 + (B^\perp_{2t_2})^2) \\ &- 2\rho^{-2}(B^\perp_{1t_1}B^\perp_{3\tilde{s}} + B^\perp_{1t_2}B^\perp_{4\tilde{s}}) - 2\rho^2(B^\perp_{2t_1}B^\perp_{3\tilde{\theta}} + B^\perp_{2t_2}B^\perp_{4\tilde{\theta}}) \\ &+ 2(\rho^{-2}B^\perp_{1\tilde{s}} + \rho^2 B^\perp_{2\tilde{\theta}} + 2\epsilon\cos(2\epsilon\tilde{s})B_1^\perp)(B^\perp_{3t_1} + B^\perp_{4t_2}) + O(\epsilon)|\nabla_{\Gamma_\epsilon}B^\perp||\nabla_{\mathbb{R}^2}B^\perp| dt_1 dt_2 d\tilde{s} d\theta.\end{aligned}$$

Integrating by parts, we found that the cross terms are canceled. Hence,

$$\int_\Omega |dB^\perp|^2 + |d^*B^\perp|^2 dx \geq (1+O(\epsilon))\int_{\rho<R}\int_{r<r_\epsilon} |d_{\mathbb{R}^2}B^\perp_{\mathbb{R}^2}|^2 + |d^*_{\mathbb{R}^2}B^\perp_{\mathbb{R}^2}|^2 + |d_{\Gamma_\epsilon}B^\perp_{\Gamma_\epsilon}|^2 + |d^*_{\Gamma_\epsilon}B^\perp_{\Gamma_\epsilon}|^2 dt_1 dt_2 dVol_{\Gamma_\epsilon}. \tag{3.21}$$

Combining (3.20) and (3.21), we have

(3.22)

$$\begin{aligned}\mathcal{Q}_\Omega(v^\perp, v^\perp) \geq &(1+O(\epsilon))\int_{\rho<R}\int_{r<r_\epsilon}\Big[|\nabla_{A^{(1)},\mathbb{R}^2}\xi^\perp|^2 + |d_{\mathbb{R}^2}B^\perp_{\mathbb{R}^2}|^2 + |d^*_{\mathbb{R}^2}B^\perp_{\mathbb{R}^2}|^2 + |B^\perp_{\mathbb{R}^2}|^2|\psi^{(1)}|^2\\ &+4\Im(\overline{\nabla_{A^{(1)}}\psi^{(1)}}\xi^\perp)\cdot B^\perp_{\mathbb{R}^2} + \frac{\lambda-1}{2}\Re(\overline{\psi^{(1)}}\xi^\perp)^2 + (\lambda+\frac{1}{2})|\psi^{(1)}|^2|\xi^\perp|^2 - \frac{\lambda}{2}|\xi|^2\\ &+|\nabla_{\Gamma_\epsilon}\xi^\perp|^2 + |d_{\Gamma_\epsilon}B^\perp_{\Gamma_\epsilon}|^2 + |d^*_{\Gamma_\epsilon}B^\perp_{\Gamma_\epsilon}|^2 + O(\epsilon^2)|\xi^\perp|^2\Big]\,dt_1dt_2dVol_{\Gamma_\epsilon}.\end{aligned}$$

Recall that the $+1$-vortex solution $(\psi^{(1)}, A^{(1)})$ is stable, then one can show that for any $a>0$, if (μ, C) is perpendicular to (T_j, T_{B_j}) in B_a for $j=1,2$, then a standard contradiction argument (see [3, 4, 5] for example) gives that

$$\begin{aligned}&\int_{r<a}|\nabla_{A^{(1)},\mathbb{R}^2}\mu|^2 + |d_{\mathbb{R}^2}C|^2 + |d^*_{\mathbb{R}^2}C|^2 + |C|^2|\psi^{(1)}|^2 + 4\Im(\overline{\nabla_{A^{(1)}}\psi^{(1)}}\mu)\cdot C + \frac{\lambda-1}{2}\Re(\overline{\psi^{(1)}}\mu)^2\\ &+(\lambda+\frac{1}{2})|\psi^{(1)}|^2|\mu|^2 - \frac{\lambda}{2}|\mu|^2dt_1dt_2\\ \geq&3\gamma\int_{r<a}|\nabla_{A^{(1)},\mathbb{R}^2}\mu|^2 + |d_{\mathbb{R}^2}C|^2 + |d^*_{\mathbb{R}^2}C|^2 + |\mu|^2 + |C|^2\end{aligned}$$

for some $\gamma>0$. Plugging it into (3.22), we have

(3.23)

$$\mathcal{Q}_\Omega(v^\perp, v^\perp) \geq 2\gamma\int_\Omega|\nabla_{A^{(1)},\mathbb{R}^2}\xi^\perp|^2+|d_{\mathbb{R}^2}B^\perp_{\mathbb{R}^2}|^2+|d^*_{\mathbb{R}^2}B^\perp_{\mathbb{R}^2}|^2+|\xi^\perp|^2+|B^\perp_{\mathbb{R}^2}|^2+|\nabla_{\Gamma_\epsilon}\xi^\perp|^2+|d_{\Gamma_\epsilon}B^\perp_{\Gamma_\epsilon}|^2+|d^*_{\Gamma_\epsilon}B^\perp_{\Gamma_\epsilon}|^2dx.$$

Next, we will estimate the cross-term $\mathcal{Q}_\Omega(v^\perp, \chi k_1\mathcal{T}_1 + \chi k_2\mathcal{T}_2))$, which can be written as

$$\mathcal{Q}_\Omega(v^\perp, \chi k_1\mathcal{T}_1 + \chi k_2\mathcal{T}_2)) = \int_\Omega\langle -\mathbb{L}(\chi k_1\mathcal{T}_1 + \chi k_2\mathcal{T}_2;\psi, A), v^\perp\rangle = I_1 + I_2,$$

where

$$I_1 := \int_\Omega \chi\langle -\mathbb{L}(k_1\mathcal{T}_1 + k_2\mathcal{T}_2;\psi, A), v^\perp\rangle dx,$$

and

$$I_2 := \int_\Omega\langle\begin{pmatrix}2\nabla\chi\cdot\nabla(k_1T_1+k_2T_2) + (k_1T_1+k_2T_2)\Delta\chi - 2i\langle A, d\chi\rangle(k_1T_1+k_2T_2)\\ 2\nabla\chi\cdot\nabla(k_1T_{B_1}+k_2T_{B_2}) + (k_1T_{B_1}+k_2T_{B_2})\Delta\chi\end{pmatrix}, v^\perp\rangle dx.$$

Since k_j decays exponentially in the normal direction and $O(\rho^{-\delta})$ on Γ_ϵ for some $0<\delta<\frac{1}{4}$,

$$I_2 = o(1)\int_{\Gamma^R_\epsilon}|\nabla^\nu_{\Gamma_\epsilon}N|^2 + \epsilon^2\rho^{-6}|N|^2dVol_{\Gamma_\epsilon} + o(1)\int_\Omega|v^\perp|^2 + |\nabla v^\perp|^2dt_1dt_2 =: \mathbf{o}.$$

For I_1, similar computations as in the proof of Proposition 3.1 show that

$$I_1 = \int_\Omega\chi\langle\Delta_{\Gamma_\epsilon}k_1\mathcal{T}_1 + \Delta_{\Gamma_\epsilon}k_2\mathcal{T}_2 - (\mathbb{L}(k_1\mathcal{T}_1 + k_2\mathcal{T}_2;\psi, A) - \mathbb{L}(k_1\mathcal{T}_1 + k_2\mathcal{T}_2;\psi_0, A_0)), v^\perp\rangle + \mathbf{o}.$$

The orthogonality condition (3.19) of $v^\perp$ implies that

$$\begin{aligned}\int_\Omega \chi\langle \Delta_{\Gamma_\epsilon} k_1\mathcal{T}_1 + \Delta_{\Gamma_\epsilon} k_2\mathcal{T}_2, v^\perp\rangle &= -\int_\Omega (1-\chi)\langle \Delta_{\Gamma_\epsilon} k_1\mathcal{T}_1 + \Delta_{\Gamma_\epsilon} k_2\mathcal{T}_2, v^\perp\rangle dx\\ &= \int_\Omega \nabla_{\Gamma_\epsilon} k_1 \cdot [-\nabla_{\Gamma_\epsilon}\chi\langle \mathcal{T}_1, v^\perp\rangle + (1-\chi)\langle \nabla_{\Gamma_\epsilon}\mathcal{T}_1, v^\perp\rangle + (1-\chi)\langle \mathcal{T}_1, \nabla_{\Gamma_\epsilon} v^\perp\rangle]\\ &\quad + \nabla_{\Gamma_\epsilon} k_2 \cdot [-\nabla_{\Gamma_\epsilon}\chi\langle \mathcal{T}_2, v^\perp\rangle + (1-\chi)\langle \nabla_{\Gamma_\epsilon}\mathcal{T}_2, v^\perp\rangle + (1-\chi)\langle \mathcal{T}_2, \nabla_{\Gamma_\epsilon} v^\perp\rangle] dx\\ &= \mathbf{o}\end{aligned}$$

due to the exponential decay of $\mathcal{T}_1$ and $\mathcal{T}_2$. For the second term (nonlinear terms), since $\eta_0, B_0 = O(\epsilon^2\rho^{-4}e^{-\delta r})$, we have that

$$\begin{aligned}&\int_\Omega \chi\langle \mathbb{L}(k_1\mathcal{T}_1 + k_2\mathcal{T}_2; \psi, A) - \mathbb{L}(k_1\mathcal{T}_1 + k_2\mathcal{T}_2; \psi_0, A_0), v^\perp\rangle \geq -C\epsilon^2\int_\Omega \rho^{-6}|N||v^\perp| dx\\ \geq &- C\epsilon^2\nu\int_{\Gamma_\epsilon}\rho^{-6}|N|^2 dvol_{\Gamma_\epsilon} - C\epsilon^2\nu^{-1}\int_\Omega \rho^{-6}|v^\perp|^2 dx.\end{aligned}$$

Hence, we have

$$\mathcal{Q}_\Omega(v^\perp, \chi k_1\mathcal{T}_1 + \chi k_2\mathcal{T}_2)) \geq -C\epsilon^2\nu^{-1}\int_{\Gamma_\epsilon}\rho^{-6}|N|^2 dvol_{\Gamma_\epsilon} - C\epsilon^2\nu\int_\Omega \rho^{-6}|v^\perp|^2 dx. \tag{3.24}$$

Finally, by Proposition (3.5), we see that

$$Q_\Omega(\chi k_1\mathcal{T}_1 + k_2\mathcal{T}_2, \chi k_1\mathcal{T}_1 + k_2\mathcal{T}_2) = \int_{\rho<R} |\nabla^\nu_{\Gamma_\epsilon} N|^2 - 2\epsilon^2\rho^{-6}|N|^2 dvol_{\Gamma_\epsilon} + \mathbf{o}. \tag{3.25}$$

Combining (3.23), (3.24) and (3.25), we find that if ν is sufficiently small, then

$$Q_\Omega(v, v) \geq -C\epsilon^2\int_{\Gamma_\epsilon}\rho^{-6}|N|^2 dvol_{\Gamma_\epsilon} \geq -\mu_0\epsilon^2\int_\Omega \rho^{-6}|v|^2,$$

which is our desired estimate. □

4. Proof of Theorem 1.1

In this section, we finish the proof of the main theorem, based on the estimates established in the previous sections. The strategy is essentially a contradiction argument. We assume to the contrary that there existed a sequence $\epsilon_n \to 0$ and $R_n \to +\infty$ such that (2.11) admits a sequence of eigenfunctions v_{ϵ_n,R_n} associated to the corresponding negative eigenvalues $\mu_{\epsilon_n,R_n} < 0$. We will show that v_{ϵ_n,R_n} can be written as $v_{\epsilon_n,R_n} \approx k_{1,\epsilon_n,R_n}(\tilde{s},\tilde{\theta})\mathcal{T}_1 + k_{2,\epsilon_n,R_n}(\tilde{s},\tilde{\theta})\mathcal{T}_2$. Moreover, $N_{\epsilon_n,R_n} := k_{1,\epsilon_n,R_n}\mathbf{m} + k_{2,\epsilon_n,R_n}\mathbf{n}$ is almost an eigenfunction of the Jacobi operator $L_{\Gamma_{\epsilon_n}}$, associated to the negative eigenvalue μ_{ϵ_n,R_n}. This contradicts with the stability of Γ_{ϵ_n}.

For simplicity, we will omit the subscript n. We may further assume that $\|v_{\epsilon,R}\|_{L^\infty} = 1$. By the variational characterization of the eigenvalue, we can further assume $\mu_{\epsilon,R}$ is monotone nonincreasing, tending to some $\mu_\epsilon < 0$ as $R \to +\infty$. In view of Lemma 3.6, the eigenvalues $\mu_{\epsilon,R} = O(\epsilon^2)$ and we denote $\mu_{\epsilon,R} = \hat{\mu}_{\epsilon,R}\epsilon^2$ and $\mu_\epsilon = \hat{\mu}_\epsilon\epsilon^2 < 0$. In the following, we show that up to a subsequence, $v_{\epsilon,R} \to v_\epsilon$ for some eigenfunction v_ϵ of problem (2.12) associated to eigenvalue $\hat{\mu}_\epsilon\epsilon^2$.

Since Lemma 3.6 shows that $\mu_{\epsilon,R}$ is uniformly bounded, by Lemma 2.7, the eigenfunction $v_{\epsilon,R}$ satisfies

$$|v_{\epsilon,R}| \leq Ce^{-\delta r} \text{ in } \Sigma_\epsilon$$

for some constant $C > 0$ independent of small ϵ and large R. Then the local elliptic estimates give that

$$|D^2 v_{\epsilon,R}| + |Dv_{\epsilon,R}| + |v_{\epsilon,R}| \le Ce^{-\delta r} \text{ in } \Sigma_\epsilon, \tag{4.1}$$

where $C > 0$ is independent of ϵ and R.

In the following, we omit the subscripts ϵ and R. Denote $\tilde{v} := \chi v$ and decompose $\tilde{v}$ as

$$\tilde{v} = \chi k_1(\tilde{s}, \tilde{\theta})\mathcal{T}_1(t_1, t_2) + \chi k_2(\tilde{s}, \tilde{\theta})\mathcal{T}_2(t_1, t_2) + \tilde{v}^\perp,$$

where for $j = 1, 2$,

$$k_j(\tilde{s}, \tilde{\theta}) = \frac{\int_{\mathbb{R}^2} \langle \tilde{v}(\tilde{s}, \tilde{\theta}, t_1, t_2), \mathcal{T}_j(t_1, t_2)\rangle dt_1 dt_2}{\int_{\mathbb{R}^2} \chi |\mathcal{T}_1(t_1, t_2)|^2 dt_1 dt_2}$$

and satisfies

$$|\nabla^2_{\Gamma_\epsilon} k_j| + |\nabla_{\Gamma_\epsilon} k_j| + |k_j| \le C$$

for some $C > 0$ independent of R and ϵ, due to (4.1). Then by our decomposition,

$$\int_{\mathbb{R}^2} \langle \tilde{v}^\perp(\tilde{s}, \tilde{\theta}, t_1, t_2), \mathcal{T}_j(t_1, t_2)\rangle dt_1 dt_2 = 0, \text{ for } j = 1, 2 \text{ and } (\tilde{s}, \tilde{\theta}) \in \Gamma^R_\epsilon.$$

In the following, we will show that $\tilde{v}^\perp$ is small.

By definition, $\tilde{v}$ satisfies

$$\mathbb{L}(\tilde{v}; \psi, A) - \epsilon^2 \mu \tilde{v} = -v\Delta\chi - 2\nabla\chi \cdot \nabla v + \begin{pmatrix} 2i(A \cdot d\chi)v \\ 0 \end{pmatrix} =: E_\epsilon.$$

Then (4.1) implies that

$$|E_\epsilon| \le C\epsilon^3 e^{-\delta r}\rho^{-6}$$

for some $\delta > 0$. Write

$$\mathbb{L}(\tilde{v}; \psi, A) = \mathbb{L}_0(\tilde{v}) + \mathcal{G}(\tilde{v}),$$

where $\mathbb{L}_0(\tilde{v})$ is the main contribution of $\mathbb{L}(; \psi, A)$:

$$\mathbb{L}_0(\tilde{v}) := \begin{pmatrix} -\Delta_{\mathbb{R}^2, A^{(1)}}\tilde{\xi} + 2i\epsilon\rho^{-2}\cos(2\epsilon\tilde{s})\tilde{\xi}_{\tilde{\theta}} - \Delta_{\Gamma_\epsilon}\tilde{\xi} + 2i\langle \tilde{B}, \nabla_{A^{(1)}}\psi^{(1)}\rangle + \frac{\lambda-1}{2}(\psi^{(1)})^2\bar{\tilde{\xi}} + (\lambda + \frac{1}{2})|\psi^{(1)}|^2\tilde{\xi} - \frac{\lambda}{2}\tilde{\xi} \\ \Delta_{H,\mathbb{R}^2}\tilde{B} + \Delta_{H,\Gamma_\epsilon}\tilde{B} + 2Im(\overline{\nabla_{A^{(1)}}\psi^{(1)}} \cdot \tilde{\xi}) + \tilde{B}|\psi^{(1)}|^2 \end{pmatrix}$$

for $\tilde{v} = (\tilde{\xi}, \tilde{B})$ and

$$\mathcal{G}(\tilde{v}) := \mathbb{L}(\tilde{v}; \psi, A) - \mathbb{L}_0(\tilde{v}).$$

We see that $\tilde{v}$ satisfies

$$\mathbb{L}_0(\tilde{v}) + \mathcal{G}(\tilde{v}) - \mu\rho^{-6}\tilde{v} = E_\epsilon \text{ in } \Sigma_\epsilon.$$

We can extend $\tilde{v}$ and E_ϵ as zero outside Σ_ϵ to get an equation in the entire $\Gamma^R_\epsilon \times \mathbb{R}^2$:

$$\mathbb{L}_0(\tilde{v}) + \tilde{\chi}\mathcal{G}(\tilde{v}) - \hat{\mu}\epsilon^2\rho^{-6}\tilde{v} = E_\epsilon, \tag{4.2}$$

where

$$\begin{cases} \tilde{\chi} = 1 & \text{if } a^2 + b^2 \le r_\epsilon + 1, \\ \tilde{\chi} = 0 & \text{if } a^2 + b^2 \ge r_\epsilon + 2. \end{cases}$$

We can also derive the equation of $\tilde{v}^\perp$:

$$\mathbb{L}_0(\tilde{v}^\perp) - \hat{\mu}\rho^{-6}\epsilon^2\tilde{v}^\perp = -\mathbb{L}_0(k_1\mathcal{T}_1 + k_2\mathcal{T}_2) + E_\epsilon - \tilde{\chi}\mathcal{G}(\tilde{v}) + \hat{\mu}\epsilon^2\rho^{-6}(k_1\mathcal{T}_1 + k_2\mathcal{T}_2) \text{ in } \Gamma_\epsilon \times \mathbb{R}^2. \tag{4.3}$$

Similar computation as in Proposition 3.1 shows that

$$\int_{\mathbb{R}^2}\langle \mathbb{L}_0(k_1\mathcal{T}_1+k_2\mathcal{T}_2)+\tilde\chi\mathcal{G}(k_1\mathcal{T}_1+k_2\mathcal{T}_2),\mathcal{T}_1\rangle dt_1dt_2\mathbf{m}+\int_{\mathbb{R}^2}\langle \mathbb{L}_0(k_1\mathcal{T}_1+k_2\mathcal{T}_2)+\tilde\chi\mathcal{G}(k_1\mathcal{T}_1+k_2\mathcal{T}_2),\mathcal{T}_2\rangle dt_1dt_2\mathbf{n}$$
$$=-L_{\Gamma_\epsilon}N\int_{\mathbb{R}^2}|\mathcal{T}_1|^2+O(\epsilon\rho^{-4}|\nabla^2_{\Gamma_\epsilon}k|)+O(\epsilon^2\rho^{-4}|\nabla_{\Gamma_\epsilon}k|)+O(\epsilon^3\rho^{-4}|k|),$$

where $N=k_1\mathbf{m}+k_2\mathbf{n}$.

Testing (4.3) against $\mathcal{T}_1$ and $\mathcal{T}_2$, one can see that N satisfies

$$\begin{aligned}&L_{\Gamma_\epsilon}N+\hat\mu\epsilon^2\rho^{-6}N+O(\epsilon\rho^{-4}|\nabla^2_{\Gamma_\epsilon}k|)+O(\epsilon^2\rho^{-4}|\nabla_{\Gamma_\epsilon}k|)+O(\epsilon^3\rho^{-4}|k|)\\ =&O(\epsilon^3\rho^{-6})+\frac{\int_{\mathbb{R}^2}\langle\tilde\chi\mathcal{G}(v^\perp),\mathcal{T}_1\rangle}{\int_{\mathbb{R}^2}|\mathcal{T}_1|^2}\mathbf{m}+\frac{\int_{\mathbb{R}^2}\langle\tilde\chi\mathcal{G}(v^\perp),\mathcal{T}_2\rangle}{\int_{\mathbb{R}^2}|\mathcal{T}_1|^2}\mathbf{n}.\end{aligned}$$

If $N(\tilde s,\tilde\theta)$ is regarded as a normal vector field $\tilde N(s,\theta):=\tilde k_1(s,\theta)\mathbf{m}+\tilde k_2(s,\theta)\mathbf{n}$ on Γ, then $\tilde N$ satisfies

(4.4)
$$L_\Gamma\tilde N+\hat\mu\rho^{-6}\tilde N+O(\epsilon\rho^{-4}|)(\nabla^2_\Gamma\tilde N|+|\nabla_\Gamma\tilde N|+|\tilde N|)=O(\epsilon\rho^{-6})+\frac{\int_{\mathbb{R}^2}\langle\tilde\chi\mathcal{G}(v^\perp),\mathcal{T}_1\rangle}{\epsilon^2\int_{\mathbb{R}^2}|\mathcal{T}_1|^2}\mathbf{m}+\frac{\int_{\mathbb{R}^2}\langle\tilde\chi\mathcal{G}(v^\perp),\mathcal{T}_2\rangle}{\epsilon^2\int_{\mathbb{R}^2}|\mathcal{T}_1|^2}\mathbf{n}.$$

Once we have shown that the left-hand side is of $O(\epsilon)$, and up to a subsequence, $\tilde N_{\epsilon,R}$ converges to some nontrivial bounded solution $\tilde N_{0,\infty}$ of

$$L_\Gamma\tilde N_{0,+\infty}+\hat\mu_0\rho^{-6}\tilde N_{0,+\infty}=0\text{ in }\Gamma$$

for some $\mu_0\le 0$ and differs from the known bounded Jacobi fields, this contradicts with the stability and nondegeneracy of Γ. Hence, it suffices to show that for $j=1,2$,

$$\int_{\mathbb{R}^2}\langle\tilde\chi\mathcal{G}(v^\perp),\mathcal{T}_j\rangle=O(\epsilon^3).\tag{4.5}$$

To see this, we need a refined estimate of $v^\perp$. Let us decompose $\tilde v^\perp=\tilde v_1^\perp+\tilde v_2^\perp$, where $v_1^\perp$ satisfies

$$\begin{aligned}\mathbb{L}_0(\tilde v_1^\perp)-\hat\mu\epsilon^2\rho^{-6}\tilde v_1^\perp=&\,2\epsilon\rho^{-5}[t_1\rho^{-2}(k_{1\tilde s\tilde s}\mathcal{T}_1+k_{2\tilde s\tilde s}\mathcal{T}_2)-t_1(k_{1\tilde\theta\tilde\theta}\mathcal{T}_1+k_{2\tilde\theta\tilde\theta}\mathcal{T}_2)+2t_2(k_{1\tilde s\tilde\theta}\mathcal{T}_1+k_{2\tilde s\tilde\theta}\mathcal{T}_2)]\\&+O(\epsilon^2\rho^{-2}(k_{1\tilde s}\nabla_{\mathbb{R}^2}\mathcal{T}_1+k_{2\tilde s}\nabla_{\mathbb{R}^2}\mathcal{T}_2+k_{1\tilde\theta}\nabla_{\mathbb{R}^2}\mathcal{T}_1+k_{2\tilde\theta}\nabla_{\mathbb{R}^2}\mathcal{T}_2))=:\mathcal{H}_1,\end{aligned}$$

and the right-hand side $\mathcal{H}_1$ consists of large terms in the expansion of $\mathbb{L}_0(k_1\mathcal{T}_1+k_2\mathcal{T}_2)+\mathcal{G}(k_1\mathcal{T}_1+k_2\mathcal{T}_2)$ which are orthogonal to both $\mathcal{T}_1$ and $\mathcal{T}_2$. See Q_1, Q_2, Q_3 and Q_5 in the proof of Proposition 3.1 for exact expressions.

Lemma 2.8 for $\mathbb{L}_0$ implies the existence of such $\tilde v_1^\perp$ with

$$\int_{\mathbb{R}^2}\langle\tilde v_1^\perp,\mathcal{T}_j\rangle=0,\ j=1,2$$

and

$$\|\tilde v_1^\perp\|_{2,2,p,\delta}\le C\|\mathcal{H}_1\|_{0,4,p,\delta}\le C\epsilon.$$

As a consequence, we have

$$\|\tilde\chi\mathcal{G}(\tilde v_1^\perp)\|_{0,4,p,\delta}\le C\epsilon^2.$$

We see that $\tilde v_2^\perp$ is a solution of

$$\mathbb{L}_0(\tilde v_2^\perp)-\hat\mu\epsilon^2\rho^{-6}\tilde v_2^\perp+\tilde\chi\mathcal{G}(\tilde v_2^\perp)=\mathcal{H}_1-\mathbb{L}_0(k_1\mathcal{T}_1+k_2\mathcal{T}_2)+E_\epsilon-\tilde\chi\mathcal{G}(\tilde v_1^\perp)+\hat\mu\epsilon^2\rho^{-6}(k_1\mathcal{T}_1+k_2\mathcal{T}_2)$$

with

$$\int_{\mathbb{R}^2} \langle \tilde{v}_2^\perp, \mathcal{T}_j \rangle = 0, \ j = 1, 2.$$

Note that Proposition 3.1 implies that the right-hand side can be written as

$$(L_{\Gamma_\epsilon} N)_1 \mathcal{T}_1 + (L_{\Gamma_\epsilon} N)_2 \mathcal{T}_2 + F$$

for some

$$\|F\|_{0,4,p,\delta} \le C\epsilon^2.$$

Lemma 2.8 for $\mathbb{L}_0$ tells us that

$$\|\tilde{v}_2^\perp\|_{2,2,p,\delta} \le C\epsilon^2, \tag{4.6}$$

which implies that

$$\|\tilde{\chi}\mathcal{G}(\tilde{v}_2^\perp)\|_{0,4,p,\delta} \le C\epsilon^3.$$

Applying Lemma 2.6 to (4.4), we see that

$$\|\nabla_\Gamma^2 \tilde{N}\|_{p,4} + \|\nabla_\Gamma \tilde{N}\|_{p,3} \le C + C\|\tilde{N}\|_{L^\infty} \le C$$

and equivalently,

$$\|\nabla_{\Gamma_\epsilon}^2 N\|_{p,4} + \|\nabla_{\Gamma_\epsilon} N\|_{p,3} \le C\epsilon.$$

This further improves the estimates of $\mathcal{H}_1$ to

$$\|\mathcal{H}_1\|_{0,4,p,\delta} \le C\epsilon^2.$$

Consequently, the estimate of $\tilde{v}^\perp$ can also be improved to

$$\|\tilde{v}_1^\perp\|_{2,2,p,\delta} \le C\|\mathcal{H}_1\|_{0,4,p,\delta} \le C\epsilon^2 \tag{4.7}$$

and therefore we get the following estimate for $\chi\mathcal{G}(\tilde{v}_1^\perp)$:

$$\|\tilde{\chi}\mathcal{G}(\tilde{v}_1^\perp)\|_{0,4,p,\delta} \le C\epsilon^3.$$

From this we obtain (4.5).

Recall that $\tilde{v}_{\epsilon,R} = k_{1,\epsilon,R}\mathcal{T}_1 + k_{2,\epsilon,R}\mathcal{T}_2 + \tilde{v}_{\epsilon,R}^\perp$, where $\tilde{v}_{\epsilon,R}^\perp = O(\epsilon\rho^{-2}e^{-\delta r})$ and $\|v_{\epsilon,R}\|_{L^\infty} = 1$ by our assumption. Since $\tilde{v}_{\epsilon,R}$ decays exponentially, the supremum is attained in Σ_ϵ if ϵ is sufficiently small. This forces

$$\|\tilde{N}_{\epsilon,R}\|_{L^\infty(\Gamma)} \ge c_0$$

for some $c_0 > 0$ independent of R.

By the uniform C^1 bound (4.1), we can pass the limit $R \to \infty$ in (4.2) and (4.4). We see that $v_{\epsilon,R} \to v_\epsilon$ and $\tilde{N}_{\epsilon,R} \to \tilde{N}_\epsilon$ for some nontrivial v_ϵ and $\tilde{N}_\epsilon = \tilde{k}_{1,\epsilon}\mathbf{m} + \tilde{k}_{2,\epsilon}\mathbf{n}$ solving

$$\mathbb{L}_0(\tilde{v}_\epsilon) + \mathcal{G}(\tilde{v}_\epsilon) - \hat{\mu}_\epsilon \epsilon^2 \rho^{-6} \tilde{v}_\epsilon = E_\epsilon \text{ in } \mathbb{R}^4 \tag{4.8}$$

and

$$L_\Gamma \tilde{N}_\epsilon + \hat{\mu}_\epsilon \rho^{-6} \tilde{N}_\epsilon = O(\epsilon\rho^{-4}) \text{ in } \Gamma$$

respectively for some $\hat{\mu}_\epsilon < 0$. Moreover, they satisfy

$$\tilde{v}_\epsilon = \tilde{k}_{1,\epsilon}\mathcal{T}_1 + \tilde{k}_{2,\epsilon}\mathcal{T}_2 + \tilde{v}_\epsilon^\perp \text{ in } \Sigma_\epsilon \tag{4.9}$$

and

$$\|\nabla_\Gamma^2 \tilde{N}_\epsilon\|_{p,4} + \|\nabla_\Gamma \tilde{N}_\epsilon\|_{p,3} \le C[\|\tilde{N}_\epsilon\|_{L^\infty(\rho<3R_0)} + O(\epsilon)]. \tag{4.10}$$

Next, we test (4.8) against Z_j, $j = 1, \dots, 6$. Recall that Z_j's are bounded kernels of $\mathbb{L}(\cdot; \psi, A)$ introduced in (2.6). Then we have

$$\int_{\mathbb{R}^4} \langle v_\epsilon, Z_j \rangle \rho^{-6} dx = O(\epsilon) \text{ for } j = 1, \dots, 6.$$

and hence,

$$\int_{\Sigma_\epsilon} \langle \tilde{v}_\epsilon, Z_j \rangle \rho^{-6} dx = O(\epsilon) \text{ for } j = 1, \dots, 6.$$

Observe that Z_j can be written as

$$Z_j = (N_j \cdot \mathbf{m})\mathcal{T}_1 + (N_j \cdot \mathbf{n})\mathcal{T}_2 + O(\epsilon \rho^{-2} e^{-\delta r}).$$

Here we recall the bounded Jacobi fields are given in (2.1). Then in view of the decomposition (4.9), the integral above passes to

$$\int_\Gamma \langle \tilde{N}_\epsilon, N_j \rangle \rho^{-6} dvol_\Gamma = O(\epsilon), \ j = 1, \dots, 6. \tag{4.11}$$

Due to the estimate (4.10), up to a subsequence, $\tilde{N}_\epsilon$ converges locally uniformly to a nontrivial bounded solution $\tilde{N}$ to

$$L_\Gamma \tilde{N} + \hat{\mu} \rho^{-6} \tilde{N} = 0 \text{ in } \Gamma$$

with $\hat{\mu} \le 0$. Furthermore, by the stability of Γ, we see that $\hat{\mu} = 0$.

However, passing $\epsilon \to 0$ in (4.11), we deduce

$$\int_\Gamma \langle \tilde{N}, N_j \rangle \rho^{-6} dvol_\Gamma = 0, \ j = 1, \dots, 6,$$

a contradiction to the non-degeneracy of Γ. Thus we finish the proof of the stability part in Theorem 1.1.

We proceed to prove the non-degeneracy of $U_\epsilon = (\psi_\epsilon, A_\epsilon)$. Assume to the contrary that $\mathbb{L}(\cdot; \psi, A)$ admitted another bounded kernel Z_7 with

$$\int_{\mathbb{R}^4} \langle Z_7, Z_j \rangle \rho^{-6} dx = 0, \ j = 1, \dots, 6.$$

Note that the arguments above also apply to Z_7 so that

$$Z_7 = (N_{7,\epsilon} \cdot \mathbf{m})\mathcal{T}_1 + (N_{7,\epsilon} \cdot \mathbf{n})\mathcal{T}_2 + Z_7^\perp \text{ in } \Sigma_\epsilon$$

for some function $Z_7^\perp = O(\epsilon \rho^{-2} e^{-\delta r})$ which is orthogonal to both $\mathcal{T}_1$ and $\mathcal{T}_2$ and normal vector field $N_{7,\epsilon}$ converging locally uniformly to a bounded Jacobi field N_7 on Γ that is orthogonal to N_j, $j = 1, \dots, 7$. This is a contradiction to the non-degeneracy of Γ. The proof of non-degeneracy of U_ϵ and Theorem 1.1 is thus completed.

References

[1] L. Ambrosio, X. Cabré, *Entire solutions of semilinear elliptic equations in $\mathbb{R}^3$ and a conjecture of De Giorgi.* Journal Amer. Math. Soc. 13 (2000), 725–739.

[2] C. Arezzo, F. Pacard, *Complete, embedded, minimal n-dimensional submanifolds in $\mathbb{C}^n$.* Comm. Pure Appl. Math. 56 (2003) 283–327. https://doi.org/10.1002/cpa.10060.

[3] M. Badran, M. Del Pino, *Solutions to the magnetic Ginzburg-Landau equations concentrating on codimension-2 minimal submanifolds.* Vietnam J. Math. 52 (2024), no. 4, 967–984.

[4] M. Badran, M. Del Pino, *Solutions of the Ginzburg-Landau equations concentrating on codimension-2 minimal submanifolds.* J. Lond. Math. Soc. (2) 109 (2024), no. 1, Paper No. e12851, 31 pp.

[5] M. Badran, M. Del Pino, *Entire solutions to 4 dimensional Ginzburg-Landau equations and codimension 2 minimal submanifolds.* Adv. Math. 435 (2023), part A, Paper No. 109365, 73 pp.

[6] M. S. Berger, Y. Chen, *Symmetric vortices for the nonlinear Ginzburg–Landau equations of superconductivity,and the nonlinear desingularization phenomenon.* J. Funct. Anal. 82, 259–295 (1989).

[7] F. Bethuel, H. Brezis, F. Hélein, *Ginzburg–Landau vortices, Progress in nonlinear differential equations and their applications*. vol. 13, Birkhäuser Boston, Inc., Boston, MA, 1994.

[8] S. B. Bradlow, *Vortices in holomorphic line bundles over closed Kähler manifolds.* Commun. Math. Phys., vol. 135, no. 1 (1990), pp. 1–17.

[9] S. Brendle, *On solutions to the Ginzburg-Landau equations in higher dimensions.* arXiv: Math/0302070. (2003). http://arxiv.org/abs/math/0302070.

[10] X. Cabré, J. Terra, *Saddle-shaped solutions of bistable diffusion equations in all of R^{2m}*. J. Eur. Math. Soc. (JEMS) 11 (2009), no. 4, 819–843.

[11] X. Cabré, J. Terra, *Qualitative properties of saddle-shaped solutions to bistable diffusion equations.* Comm. Partial Differential Equations 35 (2010), no. 11, 1923–1957.

[12] X. Cabré, *Uniqueness and stability of saddle-shaped solutions to the Allen-Cahn equation.* J. Math. Pures Appl. (9) 98 (2012), no. 3, 239–256.

[13] O. Chodosh, C. Mantoulidis, *Minimal surfaces and the Allen-Cahn equation on 3-manifolds: index, multiplicity, and curvature estimates.* Ann. of Math. (2) 191 (2020). https://doi.org/10/gn7c3j.

[14] O. Chodosh, C. Mantoulidis, *The p-widths of a surface.* Publ. Math. IHES. 137 (2023) 245–342. https://doi.org/10.1007/s10240-023-00141-7.

[15] E. De Giorgi, *Convergence problems for functionals and operators.* Proc. Int. Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), 131–188, Pitagora, Bologna (1979).

[16] G. De Philippis, A. Halavati, A. Pigati, *Decay of excess for the abelian Higgs model.* arXiv:2405.13953.

[17] G. De Philippis, A. Pigati, *Non-degenerate minimal submanifolds as energy concentration sets: A variational approach.* Comm. Pure Appl. Math. 2024;77:3581–3627.

[18] M. Del Pino, M. Kowalczyk, J. Wei, *On De Giorgi's conjecture in dimension $N \geq 9$*. Ann. of Math. (2) 174 (2011), no. 3, 1485–1569.

[19] M. Del Pino, M. Kowalczyk, J. Wei, *Entire solutions of the Allen-Cahn equation and complete embedded minimal surfaces of finite total curvature in $\mathbb{R}^3$*. J. Diff. Geom. 93 (1) 67–131. https://doi.org/10.4310/jdg/1357141507.

[20] A. Dey, *A comparison of the Almgren-Pitts and the Allen-Cahn min-max theory.* Geom. Funct. Anal. 32 (2022), 980–1040.

[21] S. K. Donaldson, P. B. Kronheimer, *The geometry of four-manifolds. Oxford Mathematical Monographs.* Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1990.

[22] N. Ghoussoub, C. Gui, *On a conjecture of De Giorgi and some related problems.* Math. Ann. 311 (1998), 481–491.

[23] N. Ghoussoub, C. Gui, *On De Giorgi's conjecture in dimensions 4 and 5.* Ann. of Math. (2) 157 (2003), no. 1, 313–334.

[24] P. Gaspar, M.A.M. Guaraco, *The Allen-Cahn equation on closed manifolds*. Calc. Var. Partial Differential Equations, 57 (2018) Art. 101, 42.
[25] M.A.M. Guaraco, *Minmax for phase transitions and the existence of embedded minimal hypersurfaces*. J. Differential Geom., 108 (1) (2018), 91–133.
[26] S. Gustafson, I. M. Sigal, *The stability of magnetic vortices.* Comm. Math. Phys. 212 (2000), no. 2, 257–275.
[27] A. Halavati, *New weighted inequalities on two manifolds.* preprint. 2023.
[28] A. Halavati, *Quantitative stability of Yang–Mills–Higgs instantons in two dimensions.* Arch. Rational Mech. Anal. (2024) 248:88.
[29] M.-C. Hong, J. Jost, M. Struwe, *Asymptotic limits of a Ginzburg–Landau type functional.* Geometric analysis and the calculus of variations. Dedicated to Stefan Hilde-brandt on the occasion of his 60th birthday. Cambridge, MA: International Press, 1996, pp. 99–123.
[30] A. Jaffe, C. Taubes, *Vortices and monopoles. Structure of static gauge theories.* Progress in Physics, 2. Birkhäuser, Boston, MA, 1980.
[31] F. H. Lin, T. Rivière, *Complex Ginzburg-Landau equations in high dimensions and codimension two area minimizing currents.* Journal of the European Mathematical Society. 1 (1999) 237-311. https://doi.org/10.1007/s100970050008.
[32] Y. Liu, X. Ma, J. Wei, W. Wu, *Entire solutions of the magnetic Ginzburg-Landau equation in* $\mathbb{R}^4$. accepted by Annali Scuola Normale Superiore-Classe De Scienze.
[33] Y. Liu, K. Wang, J. Wei, *Global minimizers of the Allen-Cahn equation in dimension* $n \geq 8$. J. Math. Pures Appl. (9) 108 (2017), no. 6, 818–840.
[34] Y. Liu, K. Wang, J. Wei, *Stability of saddle solutions for the Allen Cahn equation*, preprint 2020.
[35] D. McDuff, D. Salamon, *Introduction to symplectic topology*. Third edition. Oxford Graduate Texts in Mathematics. Oxford University Press, Oxford, 2017.
[36] F. Pacard, T. Rivière, *Linear and nonlinear aspects of vortices. The Ginzburg-Landau model.* Progress in Nonlinear Differential Equations and their Applications, 39. Birkhäuser Boston, Inc., Boston, MA, 2000.
[37] F. Pacard, J. Wei, *Stable solutions of the Allen–Cahn equation in dimension 8 and minimal cones.* J. Funct. Anal. 264 (5) (2013) 1131–1167.
[38] D. Parise, A. Pigati, D. Stern, *Convergence of the self-dual U(1)-Yang-Mills-Higgs energies to the (n-2)-area functional.* Comm. Pure Appl. Math. 77(2024), no.1, 670–730.
[39] A. Pigati, D. Stern, *Minimal submanifolds from the abelian Higgs model.* Invent. Math. 223 (2021) 1027–1095. https://doi.org/10.1007/s00222-020-01000-6.
[40] B. J. Plohr, *The existence, regularity, and behavior of isotropic solutions of classical gauge field theories.* Princeton University, 1980.
[41] T. Rivière, *Towards Jaffe and Taubes conjectures in the strongly repulsive limit.* Manuscripta Math. 108, 217–273 (2002). https://doi.org/10.1007/s002290200266.
[42] E. Sandier, S. Serfaty, *Vortices in the magnetic Ginzburg-Landau model.* Progress in Nonlinear Differential Equations and their Applications, 70. Birkhäuser Boston, Inc., Boston, MA, 2007.
[43] O. Savin, *Regularity of flat level sets in phase transitions.* Ann. of Math.(2) 169(2009), no.1, 41–78.
[44] C. H. Taubes, *On the equivalence of the first and second order equations for gauge theories.* Commun. Math. Phys., vol. 75, no. 3 (1980), pp. 207–227.
[45] C. H. Taubes, *Arbitrary N-vortex solutions to the first order Ginzburg–Landau equations.* Commun. Math. Phys., vol. 72, no. 3 (1980), pp. 277–292.

[46] F. Ting, J. Wei, *Multi-vortex non-radial solutions to the magnetic Ginzburg-Landau equations.* Commun. Math. Phys. 317 (2013) 69–97.

YONG LIU, SCHOOL OF MATHEMATICS AND STATISTICS, BEIJING TECHNOLOGY AND BUSINESS UNIVERSITY, BEIJING, CHINA.
Email address: yliumath@btbu.edu.cn

JUNCHENG WEI, DEPARTMENT OF MATHEMATICS, CHINESE UNIVERSITY OF HONG KONG, SHATIN, NT, HONG KONG.
Email address: wei@math.cuhk.edu.hk

ZIKAI YE, DEPARTMENT OF MATHEMATICS, CHINESE UNIVERSITY OF HONG KONG, SHATIN, NT, HONG KONG.
Email address: zkye@math.cuhk.edu.hk