F-STABILITY, ENTROPY AND ENERGY GAP FOR SUPERCRITICAL FUJITA EQUATION

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Abstract. We study some problems on self similar solutions to the Fujita equation when $p > (n+2)/(n-2)$, especially, the characterization of constant solutions by its energy. Motivated by recent advances in mean curvature flows, we introduce the notion of $F-$functional, F-stability and entropy for solutions of supercritical Fujita equation. Using these tools, we prove that among bounded nonzero self similar solutions, the constant solutions have the lowest entropy. Furthermore, there is also a gap between the entropy of constant and non-constant solutions. As an application of these results, we prove that if $p > (n+2)/(n-2)$, then the blow up set of type I blow up solutions is the union of a $(n-1)-$ rectifiable set and a set of Hausdorff dimension at most $n-3$.

Contents

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1. Introduction

Consider the Cauchy problem for the Fujita equation

$$\partial_t u = \Delta u + |u|^{p-1}u, \tag{F}$$

where $p > 1$, and the initial value of u is $u_0 \in L^\infty(\mathbb{R}^n)$. It is well known (see Fujita [19], [34]) that solutions of this problem could blow up in a finite time. Here a solution $u(x,t)$ is said to blow up in a finite time T if $\|u(\cdot,t)\|_\infty < \infty$ for any $t < T$, while

$$\limsup_{t\to T} \|u(\cdot,t)\|_\infty = \infty.$$

The finite time blow up is type I if

$$\limsup_{t\to T} (T-t)^{\frac{1}{p-1}} \|u(\cdot,t)\|_\infty < +\infty,$$

and type II if

$$\limsup_{t\to T} (T-t)^{\frac{1}{p-1}} \|u(\cdot,t)\|_\infty = +\infty.$$

If u is a finite time blow up solution of (F), a point x_0 is called a blow-up point if there exist sequences $\{x_k\}$ and $\{t_k\}$ such that

$$\lim_{k\to\infty} x_k = x_0, \quad \lim_{k\to\infty} t_k = T, \quad \lim_{k\to\infty} |u(x_k, y_k)| = +\infty.$$

The set Σ consisting of all the blow-up points is termed the blow-up set.

In a series of papers, Giga and Kohn [21, 22, 23] studied the asymptotic behavior of blow-up solutions to (F) when $1 < p < p_s(n)$, where

$$p_s(n) = \begin{cases} +\infty, & \text{if } 1 \le n \le 2, \\ \frac{n+2}{n-2}, & \text{if } n \ge 3. \end{cases}$$

For this purpose, they considered the self similar transform

$$w(y,\tau) = (T-t)^{\frac{1}{p-1}} u(x,t), \quad x = (T-t)^{\frac{1}{2}} y, \quad T - t = e^{-\tau}.$$

If u satisfies (F), then w satisfies

$$\partial_\tau w - \Delta w + \frac{y}{2}\cdot\nabla w + \frac{1}{p-1} w - |w|^{p-1} w = 0. \tag{RF}$$

The asymptotic behavior of u near a blow up point is equivalent to the large time asymptotics of the solution to (RF).

The equation (RF) is the gradient flow of the energy functional

$$E(w)=\int_{\mathbb{R}^n}\left[\frac{1}{2}|\nabla w|^2+\frac{1}{2(p-1)}w^2-\frac{1}{p+1}|w|^{p+1}\right]\rho dy, \tag{E}$$

where $\rho=(4\pi)^{-n/2}e^{-|y|^2/4}$ is the standard Gaussian density. Thus it is monotonically decreasing along the flow of (RF). With the help of this property, Giga and Kohn showed that as $\tau\to+\infty$, w (up to a subseqeunce of τ) converges to a stationary solution of (RF), that is, solution to the elliptic equation

$$\Delta w-\frac{y}{2}\cdot\nabla w-\frac{1}{p-1}w+|w|^{p-1}w=0, \quad \text{in } \mathbb{R}^n. \tag{SS}$$

It is clear that (SS) has three constant solutions $0,\pm\kappa$, where

$$\kappa:=(p-1)^{-\frac{1}{p-1}}.$$

By establishing the Pohozaev identity for bounded solutions of (SS), Giga and Kohn [21] proved that if $1<p<p_s(n)$, then these are all the bounded solutions. As a consequence of this Liouville property, Giga and Kohn [21] further showed that if $1<p<p_s(n)$, u is a finite time, Type I blow up solution of (F), then for any $x_0\in\mathbb{R}^n$,

$$\lim_{t\to T}(T-t)^{\frac{1}{p-}}u(x_0+(T-t)y,t)=0 \text{ or } \pm\kappa \tag{1.1}$$

uniformly on any compact set of $\mathbb{R}^n$.

Leter on, Giga-Kohn [22] (in the case of $u_0\geq 0$) and Giga-Matsui-Sasayama [24] (in the general case of sign-changing u_0) proved that if $1<p<p_s(n)$, any finite time blow up to the Cauchy problem of (F) is Type I. In [23], Giga and Kohn proved the nondegeneracy of blow ups, which implies that when x_0 is a blow up point, 0 cannot arise as the limit in (1.1). In [32], Merle and Zaag classified all bounded global nonnegative solutions to (RF) defined on $\mathbb{R}^n\times\mathbb{R}$ in the case $1<p<p_s(n)$.

In view of the above mentioned results, the blow up phenomenon of (F) when $1<p<p_s(n)$ is well understood. If $p=(n+2)/(n-2)$ (the critical exponent), type II blow up solutions to (F) do exist. In [18], Filippas-Herrero-Velázquez predicted and proved formally the existence of type II solutions. The first rigorous proof was given by Schweyer [35] for $n=4$ in the radial setting. For the remaining dimensions, the construction of type II blow up solutions are established in [15] for $n=3$, [14, 26] for $n=5$ and [27] for $n=6$. On the other hand, Collot, Merle and Raphaël [10] proved that if the energy of the initial value u_0 is clsose to that of the Aubin-Talenti solution, then (F) can only have type I blow up solution when $n\geq 7$. Under the same assumption on the dimension, Wang and Wei [40] proved that any finite time blow up solution of (F) must be of type I provided the initial value u_0 is nonnegative.

In the remaining part of the paper, it is always assumed that $n\geq 3$ and $p>(n+2)/(n-2)$ is supercritical. In this case, we expect that there are many solutions to (SS) (see [12]), and in general it is impossible to give a complete classification for all solutions. Instead, we will try to find a characterization of the constant solutions $\pm\kappa$.

1.1. **Setting and main results.** Denote

$$\mathcal{B}_n = \{w : w \text{ is a nonzero bounded solution of (SS)}\}.$$

Since $\kappa \in \mathcal{B}_n$, $\mathcal{B}_n \neq \emptyset$. Our first main result says that the constant solutions have the lowest energy among all nonzero bounded solutions.

Theorem 1.1. *If $w \in \mathcal{B}_n$ and $w \neq \pm\kappa$, then*

$$E(w) > E(\kappa).$$

Remark 1.2. *Let*

$$\mathcal{E} = \{\textit{the set of bounded radially symmetric solutions of (SS)}\}.$$

It has been proved by Matano and Merle [31, Theorem 1.4] *that*

$$E(w) \geq E(\kappa), \quad \textit{for any } w \in \mathcal{E}\backslash\{0\}.$$

Furthermore, the equality holds if and only if $w = \pm\kappa$. Their proof is "parabolic" and is based on the zero-number argument.

For each positive constant $m \geq \kappa$, we set

$$\mathcal{B}_{n,m} = \{w \in \mathcal{B}_n \text{ and } \|w\|_{L^\infty(\mathbb{R}^n)} < m.\} \tag{1.2}$$

Our next result indicates that not only the constant solution has the lowest energy among functions in $\mathcal{B}_n$, but there is a gap to the second lowest.

Theorem 1.3. *There exists a positive constant ε depending only on n, m and p such that if $w \in \mathcal{B}_{n,m}$ and $w \neq \pm\kappa$, then*

$$E(w) > E(\kappa) + \varepsilon.$$

If $p \geq p_s(n)$, it is known (see [11, 13]) that (F) can have type II blow up solutions. Therefore, it is plausible to extend Theorem 1.1 to unbounded solutions of (SS). In order to clarify this problem, we first recall the definition of suitable weak solutions introduced in [40].

Definition 1.4 (Suitable weak ancient solution). *A function u, defined on the full parabolic cylinder $Q_1 := B_1 \times (-1, 1)$, is a suitable weak solution if $\partial_t u, \nabla u \in L^2(Q_1)$, $u \in L^{p+1}(Q_1)$, and*

- *u satisfies (F) in the weak sense, that is, for any $\eta \in C_0^\infty(Q_1)$,*

$$\int_{Q_1} \left[\partial_t u\eta + \nabla u \cdot \nabla\eta - |u|^{p-1}u\eta\right] = 0; \tag{1.3}$$

- *u satisfies the localized energy inequality: for any $\eta \in C_0^\infty(Q_1)$,*

$$\int_{Q_1} \left[\left(\frac{|\nabla u|^2}{2} - \frac{|u|^{p+1}}{p+1}\right)\partial_t\eta^2 - |\partial_t u|^2\eta^2 - 2\eta\partial_t u\nabla u \cdot \nabla\eta\right] \geq 0; \tag{1.4}$$

- *u satisfies the stationary condition: for any $Y \in C_0^\infty(Q_1, \mathbb{R}^n)$,*

$$\int_{Q_1} \left[\left(\frac{|\nabla u|^2}{2} - \frac{|u|^{p+1}}{p+1} \right) divY - DY(\nabla u, \nabla u) - \partial_t u \nabla u \cdot Y \right] = 0. \tag{1.5}$$

Definition 1.5. *A function u is a suitable weak ancient solution of the equation (F) if for each $Q_r^-(x_0,t_0) = B_r(x_0) \times (t_0 - r^2, t_0) \subset \mathbb{R}^n \times (-\infty, 0)$, u is a suitable weak solution of (F) in $Q_r^-((x_0,t_0))$.*

Given $M > 0$, $\mathcal{G}_M$ denotes the set of suitable weak ancient solutions of (F) satisfying the following Morrey estimate: for any $(x_0,t_0) \in \mathbb{R}^n \times (-\infty, 0)$,

$$r^{\frac{2(p+1)}{p-1}-2-n} \int_{Q_r^-(x_0,t_0-r^2)} (|\nabla u|^2 + |u|^{p+1}) dxdt + r^{\frac{2(p+1)}{p-1}-n} \int_{Q_r^-(x_0,t_0-r^2)} (\partial_t u)^2 dxdt \le M.$$

Definition 1.6. *We set*

$$\mathcal{F}_n = \{ w : (-t)^{-1/(p-1)} w(x/\sqrt{-t}) \in \mathcal{G}_M \text{ for some positive constant } M \}.$$

It is easy to check that $\mathcal{F}_n$ contains bounded smooth solutions of (SS). However, a function in $\mathcal{F}_n$ need not to be smooth everywhere. Indeed, if $n \ge 3, p > (n+2)/(n-2)$, then

$$w(y) = \left[\frac{2}{p-1} \left(n - 2 - \frac{2}{p-1} \right) \right]^{\frac{1}{p-1}} |y|^{-\frac{2}{p-1}}$$

is a function in $\mathcal{F}_n$ which is not smooth. For any function w in $\mathcal{F}_n$, let $Reg(w)$ be the regular part of w, then $Reg(w)$ is an open subset of $\mathbb{R}^n$. Let

$$\mathcal{F}_n^+ = \{ w : w \in \mathcal{F}_n \text{ and } w \ge 0 \text{ on } Reg(w) \}$$

Theorem 1.7. *If $n \le 3$ or $n \ge 4, (n+2)/(n-2) < p < (n+1)/(n-3)$, then*

$$E(w) \ge E(\kappa)$$

for any $w \in \mathcal{F}_n^+$.

Compared to Theorem 1.1, the main obstruction in this case comes from solutions to the elliptic counterpart of (F),

$$-\Delta w = |w|^{p-1} w \quad \text{in } \mathbb{R}^n. \tag{1.6}$$

The technical restriction on the range of p in this theorem arises from the use of some rigidity results on homogeneous solutions to (1.6). Here a solution w is homogeneous if

$$w(\lambda x) = \lambda^{-\frac{2}{p-1}} w(x).$$

In polar coordinates, w is homogeneous if and only if there exists a function Φ on the unit sphere such that $w(r,\theta) = r^{-\frac{2}{p-1}} \Phi(\theta)$, where Φ solves

$$-\Delta_{\mathbb{S}^{n-1}} \Phi + \beta \Phi = |\Phi|^{p-1} \Phi \tag{1.7}$$

with

$$\beta = \frac{2}{p-1} \left(n - 2 - \frac{2}{p-1} \right),$$

For finite time blow up solutions of (F), another question that has deserved great attention is the structure of the blow up set. In the subcritical case, Velázquez [37] proved that if

$$u(x,t) \neq \kappa(T-t)^{-\frac{1}{p-1}}.$$

Then for any fixed $R>0$, $\Sigma \cap B_R(0)$ is a $n-1$-rectifiable set and there holds:

$$\mathcal{H}^{n-1}(\Sigma \cap B_R(0)) < \infty,$$

where $\mathcal{H}^{n-1}$ is the standard $(n-1)$ dimensional Hausdorff measure and $B_R(0) = \{x \in \mathbb{R}^n : |x| < R\}$. As pointed out in [37], the dimension $(n-1)$ is optimal. For more results on this topic, we refer to [4, 6, 30, 23, 43, 44, 17]. When the exponent p is supercritical, we can apply Theorem 1.3 to study the structure of type I finite blow up solutions.

Definition 1.8. *For any x_0, a tangent function w_0 at x_0 is a suitable weak solution of* (SS) *such that by defining*

$$u_0(x,t) := (-t)^{-\frac{1}{p-1}} w_0\left(\frac{x}{\sqrt{-t}}\right),$$

there exists a sequence $\lambda_i \to 0^+$ such that

$$\lambda_i^{-\frac{2}{p-1}} u(x_i + \lambda_i x, T + \lambda_i^2 t) \to u_0(x,t) \text{ in } L^{p+1}_{loc}(\mathbb{R}^n \times (-\infty, 0)).$$

The set of tangent functions is denoted by $\mathcal{T}(x_0, u)$.

The scaling limit in this definition is equivalent to the large time limiting behavior of the self-similar equation (RF).

Proposition 1.9. *Let $n \geq 3, p > (n+2)/(n-2), u_0 \geq 0$ and let*

$$u(x,t) \neq \kappa(T-t)^{-\frac{1}{p-1}}$$

be a solution of the equation (F) *that blows up at T. Assume there is a positive constant $m > \kappa$ such that*

$$u(x,t) \leq m(T-t)^{-\frac{1}{p-1}}, \quad \text{in } \mathbb{R}^n \times (0,T). \tag{1.8}$$

For any $R>0$, we set $\Sigma_R = \Sigma \cap B_R(0)$. Then

1. *$\Sigma_R = \Sigma_{n-1} \cup \Sigma_{n-3}$;*
2. *Σ_{n-1} is relatively open in Σ, and it is countably $(n-1)$-rectifiable;*
3. *the Hausdorff dimension of Σ_{n-3} is at most $n-3$;*
4. *$x_0 \in \Sigma_{n-1}$ if and only if $\mathcal{T}(u, x_0) = \{\kappa\}$.*

1.2. **Idea of the proof: F-functional and entropy.** For a hypersurface Σ of Euclidean space $\mathbb{R}^{n+1}$, the entropy is defined by

$$\lambda(\Sigma) = \sup (4\pi t_0)^{-\frac{n}{2}} \int_\Sigma e^{-\frac{|x-x_0|^2}{4t_0}} dx.$$

Here the supremum is taken over all $t_0 > 0$ and $x_0 \in \mathbb{R}^{n+1}$. This quantity was introduced by Colding -Minicozzi [7]. As a consequence of Huisken's monotonicity formula, it is non-increasing along the mean curvature flow, thus giving a Lyapunov functional. In [8], Colding– Ilmanen–Minicozzi-White proved that within the closed smooth self-shrinking solutions of the mean curvature flow in $\mathbb{R}^{n+1}$, not only does the round sphere have the lowest entropy, but also there is a gap to the second lowest. Based on this result, they conjectured that, for $2 \leq n \leq 6$, the round sphere minimizes the entropy among all closed hypersurfaces. Using a cleverly constructed weak mean curvature flow that ensured the extinction time singularity was of a special type, this conjecture was verified by Bernstein and Wang [2].

As pointed by Velázquez in [38], the structure of singularities which arise in mean curvature flow is strikingly similar to those appearing in (F). Because of this reason, we will borrow some ideas from [7] and [8] to consider bounded solutions of (SS). Inspired by the program developed by Colding and Minicozzi [7, 8], we will introduce the notation of F-functional and entropy. In the setting of (F), for a bounded C^1 function w, the F-functional is defined by

$$\begin{aligned} F_{x_0,t_0}(w) =& \frac{1}{2}(-t_0)^{\frac{p+1}{p-1}} \int_{\mathbb{R}^n} |\nabla w|^2 G(y-x_0,t_0)dy \\ &- \frac{1}{p+1}(-t_0)^{\frac{p+1}{p-1}} \int_{\mathbb{R}^n} |w|^{p+1} G(y-x_0,t_0)dy \\ &+ \frac{1}{2(p-1)}(-t_0)^{\frac{2}{p-1}} \int_{\mathbb{R}^n} w^2 G(y-x_0,t_0)dy, \end{aligned} \tag{1.9}$$

where for any $(y,t) \in \mathbb{R}^n \times (-\infty, 0)$,

$$G(y,t) = (-4\pi t)^{-\frac{n}{2}} e^{\frac{|y|^2}{4t}}.$$

In particular,

$$G(y,-1) = \rho(y) = (4\pi)^{-\frac{n}{2}} e^{-\frac{|y|^2}{4}}.$$

The motivation of defining F-functional in this way comes from Giga-Kohn's monotonicity formula (see [21, Proposition 3]). The main property of these functionals is that a bounded function is a critical point of F_{x_0,t_0} if and only if it is the time $t = t_0$ slice of a self similar solution of (F). Using the $F-$functionals, we can also define the entropy $\lambda(w)$ of a bounded smooth function to be the supremum of the F_{x_0,t_0} functionals

$$\lambda(w) = \sup_{x_0 \in \mathbb{R}^n, t_0 \in (-\infty, 0)} F_{x_0,t_0}(w). \tag{1.10}$$

Similar to [7], we can define F-stable and entropy-stable solutions. Then under some technical (but reasonable) conditions, we show that these two definitions are equivalent. Using this fact, we can perturb a bounded non-constant solution of (SS), while reducing the entropy and making the solution of (RF) starting at this perturbed function blows up at a finite time. After that, we use an induction argument to show that the minimizer of $\lambda(w)$ among $\mathcal{B}_n$ is attained by the constant. Once this is established, we will then prove by contradiction to obtain the entropy gap (or energy gap).

Finally, we point out even though Theorem 1.1 and Theorem 1.3 suggest that the constant solutions of (SS) serve the same role as the round sphere in mean curvature flow, there are still some striking differences. For instance, if a mean curvature flow in $\mathbb{R}^{n+1}$ starting at a closed smooth embedded hypersurface has only generic singularities, then the round sphere is included in the lowest strata $\mathcal{S}_0$ [1] consisting of isolated points (see [9, Lemma 4.2]). However, for finite blow up solutions of (F), the constant solution κ is included in the top strata $\mathcal{S}_n$. This will lead to some essential difficulties in our analysis.

Notation. Define the weighted spaces

$$L^q_\omega(\mathbb{R}^n) = \{g \in L^q_{loc}(\mathbb{R}^n) : \int_{\mathbb{R}^n} |g(y)|^q e^{-\frac{|y|^2}{4}} dy < \infty\}$$

and

$$H^1_\omega(\mathbb{R}^n) = \{g \in L^2_\omega(\mathbb{R}^n) : |\nabla g| + |g| \in L^2_\omega(\mathbb{R}^n)\}.$$

The inner product on $L^2_\omega(\mathbb{R}^n)$ is given by

$$\langle \psi_1, \psi_2 \rangle_w = \int_{\mathbb{R}^n} \psi_1 \psi_2 e^{-\frac{|y|^2}{4}} dy. \tag{1.11}$$

Both $L^2_\omega(\mathbb{R}^n)$ and $H^1_\omega(\mathbb{R}^n)$ are Hilbert spaces.

2. Preliminaries

In this section, we recall several results which will be used later. The first one is [21, Proposition 1'].

Lemma 2.1. *For any bounded solution w of (SS), there exists a positive constant M' depending only on n, p and $\|w\|_{L^\infty(\mathbb{R}^n)}$ such that*

$$|\nabla w| + |\nabla^2 w| + |\nabla^3 w| \le M', \quad \text{in } \mathbb{R}^n$$

The next result is about the regularity of solutions to (SS) which satisfies a natural decay condition at infinity.

Lemma 2.2. *Assume w is a bounded solution of (SS) satisfying, for some positive constant C,*

$$|w(y)| \le C(1+|y|)^{-\frac{2}{p-1}}, \quad \text{in } \mathbb{R}^n. \tag{2.1}$$

Then there exists a positive constant C_1 such that

$$(1+|y|)^{1+\frac{2}{p-1}} |\nabla w(y)| + (1+|y|)^{2+\frac{2}{p-1}} |\nabla^2 w(y)| \le C_1, \quad \text{in } \mathbb{R}^n. \tag{2.2}$$

Proof. Let

$$u(x,t) = (-t)^{-\frac{1}{p-1}} w\left(\frac{x}{\sqrt{-t}}\right). \tag{2.3}$$

Then u is a smooth, ancient self similar solution of (F).

By (2.1), u is bounded on $(B_2(0)\setminus B_{1/4}(0)) \times [-1, 0)$. Thus we get from standard parabolic estimates (see the proof of [21, Proposition 1]) that

$$|\nabla u| + |\nabla^2 u| \le C_1, \quad \text{in } (B_{3/2}(0)\setminus B_{1/2}(0)) \times [-1/2, 0)$$

for some positive constant C_1. After scaling back to w, this is (2.2). □

[1]Here we have adopted the notations in [9, Section 4]

Next, we recall the monotonicity formula for (F), which is a reformulation of [21, Proposition 3].

Lemma 2.3 (Monotonicity formula)**.** *Assume w is a bounded solution of* (SS) *and u is defined by* (2.3)*. For any $(x,t) \in \mathbb{R}^n \times (-\infty, 0]$ and $T \geq t$, the function*

$$\begin{aligned} E(s;x,T,u) =& \frac{1}{2}(-s)^{\frac{p+1}{p-1}} \int_{\mathbb{R}^n} |\nabla u(y,T+s)|^2 G(y-x,s)dy \\ &- \frac{1}{p+1}(-s)^{\frac{p+1}{p-1}} \int_{\mathbb{R}^n} |u(y,T+s)|^{p+1} G(y-x,s)dy \\ &+ \frac{1}{2(p-1)}(-s)^{\frac{2}{p-1}} \int_{\mathbb{R}^n} u(y,T+s)^2 G(y-x,s)dy \end{aligned} \tag{2.4}$$

is nonincreasing with respect to s in $(-\infty, -(T-t))$. If $t < 0$ and $T > t$, then $E(s;x,T,u)$ is nonincreasing with respect to s in $(-\infty, -(T-t)]$.

For any $(x,t) \in \mathbb{R}^n \times (-\infty, 0]$, we take $T = t$ into (2.4). The monotonicity of E allows us to define the density function

$$\Theta(x,t;u) := \lim_{s \to 0} E(s;x,t,u). \tag{2.5}$$

The density function defined in (2.5) satisfies the following two properties, whose proof are standard.

Lemma 2.4. *The density function Θ is upper semi-continuous in the sense that if (x_i,t_i) is a sequence of points converging to (x_∞, t_∞), then*

$$\Theta(x_\infty, t_\infty; u) \geq \limsup_{i \to \infty} \Theta(x_i, t_i; u).$$

Proof. For any $\varepsilon > 0$, choose an $s < 0$ such that

$$E(s;x_\infty,t_\infty,u) \leq \Theta(x_\infty,t_\infty;u) + \varepsilon. \tag{2.6}$$

Because $(x_i,t_i) \to (x_\infty,t_\infty)$ and w is a bounded solution of (SS), we get from Lemma 2.1 that

$$\lim_{i\to\infty} E(s;x_i,t_i,u) = E(s;x_\infty,t_\infty,u).$$

By the monotocinity formula,

$$\Theta(x_i,t_i,u) \leq E(s;x_i,t_i,u).$$

Therefore

$$\limsup_{i\to\infty} \Theta(x_i,t_i,u) \leq E(s;x_\infty,t_\infty,u).$$

Combining this inequality with (2.6), we get

$$\limsup_{i\to\infty} \Theta(x_i,t_i,u) \leq \Theta(x_\infty,t_\infty;u) + 2\varepsilon$$

Letting $\varepsilon \to 0$, we get the desired claim. □

Lemma 2.5. *For any $(x_0,t_0) \in \mathbb{R}^n \times (-\infty, 0]$, we have $\Theta(x_0,t_0;u) \leq \Theta(0,0;u)$. If $\Theta(x_0,t_0;u) = \Theta(0,0;u)$, then u is backward self similar with respect to (x_0,t_0). Moreover, for any $a \in \mathbb{R}$ and $(x,t) \in \mathbb{R}^n \times (-\infty,0)$,*

$$u(ax_0 + x, a^2 t_0 + t) = u(x,t).$$

Proof. Define the blow down sequence of u at (x_0, t_0) by

$$u_\lambda(x,t) = \lambda^{\frac{2}{p-1}} u(x_0 + \lambda x, t_0 + \lambda^2 t),$$

where $\lambda \to +\infty$. Since w is a bounded solution of (SS) and u is backward self similar with respect to $(0,0)$,

$$u_\lambda(x,t) = u(\lambda^{-1}x_0 + x, \lambda^{-2}t_0 + t) \to u(x,t)$$

locally uniformly in $\mathbb{R}^n \times (-\infty, 0)$. This uniform convergence implies that for $s < 0$,

$$\begin{aligned} \Theta(0,0;u) \quad &\equiv E(s;0,0,u) = \lim_{\lambda\to+\infty} E(s;0,0,u_\lambda) = \lim_{\lambda\to+\infty} E(\lambda^2 s; x_0, t_0, u) \\ &= \lim_{\tau\to-\infty} E(\tau; x_0, t_0, u) \geq \Theta(x_0, t_0; u). \end{aligned} \tag{2.7}$$

Here in the last inequality we have used the monotonicity formula at (x_0, t_0).

Assume that $\Theta(x_0, t_0; u) = \Theta(0,0;u)$. For any $a > 0$, we have

$$\Theta(ax_0, a^2 t_0; u) = \Theta(x_0, t_0; u) = \Theta(0,0;u).$$

Thus u is backward self similar with respect to $(ax_0, a^2 t_0)$. Since u is also backward self similar with respect to $(0,0)$, for any $\lambda > 0$, we have

$$u(ax_0 + x, a^2 t_0 + t) = \lambda^{\frac{2}{p-1}} u(ax_0 + \lambda x, a^2 t_0 + \lambda^2 t) = u(\lambda^{-1} a x_0 + x, \lambda^{-2} a^2 t_0 + t).$$

Since u is smooth, for any $(x,t) \in \mathbb{R}^n \times (-\infty, 0)$,

$$\lim_{\lambda\to+\infty} u(\lambda^{-1} a x_0 + x, (a\lambda)^{-2} t_0 + t) = u(x,t).$$

Thus

$$u(ax_0 + x, a^2 t_0 + t) = u(x,t).$$

The case $a < 0$ can be obtained by a change of variables. □

Remark 2.6. *According to the definition in* [41, Section 8]*, the density function* $\Theta(x,t;u)$ *is a* backward conelike function.

We also need the following Liouville type result.

Theorem 2.7. *If $n \leq 2$ or $n \geq 3, 1 < p \leq (n+2)/(n-2)$, if w is a bounded solution or $w \in \mathcal{F}_n$, then $w = 0$ or $w = \pm\kappa$.*

Proof. As mentioned in the introduction, for bounded solutions, Giga and Kohn [21, Theorem 1] obtained the Pohozaev identity

$$0 = \left(\frac{n}{p+1} + \frac{2-n}{2}\right) \int_{\mathbb{R}^n} |\nabla w|^2 e^{-\frac{|y|^2}{4}} dy + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}^n} |y|^2 |\nabla w|^2 e^{-\frac{|y|^2}{4}} dy.$$

This identity follows from integration by parts. For functions in $\mathcal{F}_n$, we can obtain the same identity by plugging suitable vector fields into the stationary condition (1.5).

Because the coefficients in Pohozaev identity are all nonnegative, it implies $\nabla w = 0$. Hence w must be a constant solution. □

3. First variation of the F-functional

In this section, we will derive the first variation formula for the F-functional defined in (1.9). Then we will use the first variation formula to give a variational characterization of bounded solutions of (SS).

3.1. **The general first variation formula.**

The following first variation formula holds for any bounded smooth function.

Lemma 3.1 (The first variation formula)**.** *Let w be a bounded smooth function. Assume $\phi \in C_0^\infty(\mathbb{R}^n)$, $x(s)$ and $t(s)$ are variations such that*

$$x(0) = x_0, \quad t(0) = t_0, \quad x'(0) = y_0, \quad t'(0) = h.$$

Then

$$\begin{aligned}
&\frac{d}{ds} F_{x(s),t(s)}(w + s\phi)|_{s=0} \\
= &-\frac{p+1}{2(p-1)}(-t_0)^{\frac{2}{p-1}} h \int_{\mathbb{R}^n} |\nabla w|^2 G(y - x_0, t_0) dy \\
&+ \frac{1}{p-1}(-t_0)^{\frac{2}{p-1}} h \int_{\mathbb{R}^n} |w|^{p+1} G(y - x_0, t_0) dy \\
&- \frac{1}{(p-1)^2}(-t_0)^{\frac{2}{p-1}-1} h \int_{\mathbb{R}^n} w^2 G(y - x_0, t_0) dy \\
&+ (-t_0)^{\frac{p+1}{p-1}} \int_{\mathbb{R}^n} (\nabla w \cdot \nabla \phi) G(y - x_0, t_0) dy \\
&- (-t_0)^{\frac{p+1}{p-1}} \int_{\mathbb{R}^n} |w|^{p-1} w \phi G(y - x_0, t_0) dy \\
&+ \frac{1}{(p-1)}(-t_0)^{\frac{2}{p-1}} \int_{\mathbb{R}^n} w \phi G(y - x_0, t_0) dy \\
&+ \frac{1}{2}(-t_0)^{\frac{p+1}{p-1}} \int_{\mathbb{R}^n} |\nabla w|^2 G(y - x_0, t_0) \\
&\qquad \times \left[-\frac{nh}{2t_0} + \frac{(x_0 - y) \cdot y_0}{2t_0} - \frac{h|x_0 - y|^2}{4t_0^2} \right] dy \\
&- \frac{1}{p+1}(-t_0)^{\frac{p+1}{p-1}} \int_{\mathbb{R}^n} |w|^{p+1} G(y - x_0, t_0) \\
&\qquad \times \left[-\frac{nh}{2t_0} + \frac{(x_0 - y) \cdot y_0}{2t_0} - \frac{h|x_0 - y|^2}{4t_0^2} \right] dy \\
&+ \frac{1}{2(p-1)}(-t_0)^{\frac{2}{p-1}} \int_{\mathbb{R}^n} w^2 G(y - x_0, t_0) \\
&\qquad \times \left[-\frac{nh}{2t_0} + \frac{(x_0 - y) \cdot y_0}{2t_0} - \frac{h|x_0 - y|^2}{4t_0^2} \right] dy.
\end{aligned} \tag{3.1}$$

Proof. By the definition of the F-functional in (1.9), we have

$$\begin{aligned}F_{x(s),t(s)}(w+s\phi)=&\frac{1}{2}[-t(s)]^{\frac{p+1}{p-1}}\int_{\mathbb{R}^n}|\nabla(w+s\phi)|^2G(y-x(s),t(s))dy\\&-\frac{1}{p+1}[-t(s)]^{\frac{p+1}{p-1}}\int_{\mathbb{R}^n}|w+s\phi|^{p+1}G(y-x(s),t(s))dy\\&+\frac{1}{2(p-1)}[-t(s)]^{\frac{2}{p-1}}\int_{\mathbb{R}^n}(w+s\phi)^2G(y-x(s),t(s))dy.\end{aligned}$$

Because

$$\begin{aligned}&\frac{d}{ds}\left[(-4\pi t(s))^{-\frac{n}{2}}e^{\frac{|x(s)-y|^2}{4t(s)}}\right]\\=&[-4\pi t(s)]^{-\frac{n}{2}}e^{\frac{|x(s)-y|^2}{4t(s)}}\left[-\frac{nt'(s)}{2t(s)}+\frac{(x(s)-y)\cdot x'(s)}{2t(s)}-\frac{t'(s)|x(s)-y|^2}{4t^2(s)}\right],\end{aligned}$$

we get

$$\begin{aligned}&\frac{d}{ds}F_{x(s),t(s)}(w+s\phi)\\=&-\frac{p+1}{2(p-1)}[-t(s)]^{\frac{2}{p-1}}t'(s)\int_{\mathbb{R}^n}|\nabla w+s\nabla\phi|^2G(y-x(s),t(s))dy\\&+\frac{1}{p-1}[-t(s)]^{\frac{2}{p-1}}t'(s)\int_{\mathbb{R}^n}|w+s\phi|^{p+1}G(y-x(s),t(s))dy\\&-\frac{1}{(p-1)^2}[-t(s)]^{\frac{2}{p-1}-1}t'(s)\int_{\mathbb{R}^n}(w+s\phi)^2G(y-x(s),t(s))dy\\&+[-t(s)]^{\frac{p+1}{p-1}}\int_{\mathbb{R}^n}(\nabla w+s\nabla\phi)\cdot\nabla\phi G(y-x(s),t(s))dy\\&-[-t(s)]^{\frac{p+1}{p-1}}\int_{\mathbb{R}^n}|w+s\phi|^{p-1}(w+s\phi)\phi G(y-x(s),t(s))dy\\&+\frac{1}{p-1}[-t(s)]^{\frac{2}{p-1}}\int_{\mathbb{R}^n}(w+s\phi)\phi G(y-x(s),t(s))dy\\&+\frac{1}{2}[-t(s)]^{\frac{p+1}{p-1}}\int_{\mathbb{R}^n}|\nabla w+s\nabla\phi|^2G(y-x(s),t(s))\\&\qquad\times\left[-\frac{nt'(s)}{2t(s)}+\frac{(x(s)-y)\cdot x'(s)}{2t(s)}-\frac{t'(s)|x(s)-y|^2}{4t^2(s)}\right]dy\\&-\frac{1}{p+1}[-t(s)]^{\frac{p+1}{p-1}}\int_{\mathbb{R}^n}|w+s\phi|^{p+1}G(y-x(s),t(s))\\&\qquad\times\left[-\frac{nt'(s)}{2t(s)}+\frac{(x(s)-y)\cdot x'(s)}{2t(s)}-\frac{t'(s)|x(s)-y|^2}{4t^2(s)}\right]dy\\&+\frac{1}{2(p-1)}[-t(s)]^{\frac{2}{p-1}}\int_{\mathbb{R}^n}(w+s\phi)^2G(y-x(s),t(s))\\&\qquad\times\left[-\frac{nt'(s)}{2t(s)}+\frac{(x(s)-y)\cdot x'(s)}{2t(s)}-\frac{t'(s)|x(s)-y|^2}{4t^2(s)}\right]dy.\end{aligned}$$

Plugging $s=0$ into the above formula, we get the desired formula. $\square$

Definition 3.2. *Let w be a bounded smooth function. If for all variations $x(s), t(s)$ and $\phi \in C_0^\infty(\mathbb{R}^n)$, we have*

$$\frac{d}{ds} F_{x(s),t(s)}(w + s\phi)|_{s=0} = 0,$$

then we say that w is a critical point of the functional F_{x_0,t_0}.

3.2. **Critical points of F-functional are self similar solutions.** In this section, we will prove that w is a critical point of F_{x_0,t_0} if and only if it is the time $-t_0$ slice of a self similar solution of (F), which blows up at the point x_0 and time 0.

Proposition 3.3. *A bounded smooth function w is a critical point of F_{x_0,t_0} if and only if w is a solution to the equation*

$$\Delta w + \frac{y - x_0}{2t_0} \cdot \nabla w + \frac{1}{(p-1)t_0} w + |w|^{p-1} w = 0, \quad \text{in } \mathbb{R}^n. \tag{3.2}$$

Proof. By the definition of the F-functional, it is easy to check that

$$F_{x_0,t_0}(w) = F_{0,-1}(w_{x_0,t_0}),$$

where

$$w_{x_0,t_0}(y) = (-t_0)^{\frac{1}{p-1}} w(x_0 + \sqrt{-t_0} y).$$

Notice that w_{x_0,t_0} satisfies (3.2) if and only if w satisfies (SS). Therefore, it is sufficient to prove Proposition 3.3 for $x_0 = 0$ and $t_0 = -1$.

First assume w is a critical point of $F_{0,-1}$. Taking $x_0 = 0, t_0 = -1, y_0 = 0$ and $h = 0$ in (3.1), we see that for any $\phi \in C_0^\infty(\mathbb{R}^n)$,

$$0 = \int_{\mathbb{R}^n} \nabla w \cdot \nabla \phi \rho dy + \frac{1}{p-1} \int_{\mathbb{R}^n} w \phi \rho dy - \int_{\mathbb{R}^n} |w|^{p-1} w \phi \rho dy. \tag{3.3}$$

Thus

$$0 = \int_{\mathbb{R}^n} \left[\frac{1}{\rho} \mathrm{div}(\rho \nabla w) - \frac{1}{p-1} w + |w|^{p-1} w \right] \phi \rho dy.$$

It follows that w satisfies (3.2).

Next, assume w is a solution of (3.2) with $x_0 = 0, t_0 = -1$. We need to show that for any y_0, h and $\phi \in C_0^\infty(\mathbb{R}^n)$,

$$\frac{d}{ds} F_{x(s),t(s)}(w + s\phi)|_{s=0} = 0.$$

This is equivalent to the requirement that

$$\begin{aligned} 0 = &- \frac{p+1}{2(p-1)} h \int_{\mathbb{R}^n} |\nabla w|^2 \rho dy + \frac{h}{p-1} \int_{\mathbb{R}^n} |w|^{p+1} \rho dy \\ &- \frac{h}{(p-1)^2} \int_{\mathbb{R}^n} w^2 \rho dy + \int_{\mathbb{R}^n} (\nabla w \cdot \nabla \phi) \rho dy \\ &+ \frac{1}{p-1} \int_{\mathbb{R}^n} w \phi \rho dy - \int_{\mathbb{R}^n} |w|^{p-1} w \phi dy \end{aligned} \tag{3.4}$$

$$
\begin{aligned}
&+\frac{1}{2}\int_{\mathbb{R}^n}|\nabla w|^2\left[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]\rho dy\\
&-\frac{1}{p+1}\int_{\mathbb{R}^n}|w|^{p+1}\left[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]\rho dy\\
&+\frac{1}{2(p-1)}\int_{\mathbb{R}^n}w^2\left[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]\rho dy.
\end{aligned}
$$

Multiplying both sides of (SS) by ρw and integrating by parts, we get

$$
0=\int_{\mathbb{R}^n}|\nabla w|^2\rho dy-\int_{\mathbb{R}^n}|w|^{p+1}\rho dy+\frac{1}{p-1}\int_{\mathbb{R}^n}w^2\rho dy. \tag{3.5}
$$

By [21, Formula (3,17)], we have

$$
\begin{aligned}
0=&\frac{2-n}{2}\int_{\mathbb{R}^n}|\nabla w|^2\rho dy-\frac{n}{2(p-1)}\int_{\mathbb{R}^n}w^2\rho dy\\
&+\frac{n}{p+1}\int_{\mathbb{R}^n}|w|^{p+1}\rho dy+\frac{1}{4}\int_{\mathbb{R}^n}|y|^2|\nabla w|^2\rho dy\\
&+\frac{1}{4(p-1)}\int_{\mathbb{R}^n}|y|^2w^2\rho dy-\frac{1}{2(p+1)}\int_{\mathbb{R}^n}|y|^2|w|^{p+1}\rho dy.
\end{aligned} \tag{3.6}
$$

For $i=1,2,\cdots n$, multiplying both sides of (3.2) by ρw_i and integrating by parts, we get

$$
\begin{aligned}
0=&-\int_{\mathbb{R}^n}\nabla w\cdot\nabla w_i\rho dy-\frac{1}{p-1}\int_{\mathbb{R}^n}w_iw\rho dy+\int_{\mathbb{R}^n}|w|^{p-1}ww_i\rho dy\\
=&-\frac{1}{2}\int_{\mathbb{R}^n}(|\nabla w|^2)_i\rho dy-\frac{1}{2(p-1)}\int_{\mathbb{R}^n}(w^2)_i\rho dy+\frac{1}{p+1}\int_{\mathbb{R}^n}(|w|^{p+1})_i\rho dy\\
=&\frac{1}{4}\int_{\mathbb{R}^n}y_i|\nabla w|^2\rho dy+\frac{1}{4(p-1)}\int_{\mathbb{R}^n}y_iw^2\rho dy-\frac{1}{2(p+1)}\int_{\mathbb{R}^n}y_i|w|^{p+1}\rho dy.
\end{aligned}
$$

It follows that for any $y_0\in\mathbb{R}^n$,

$$
0=\int_{\mathbb{R}^n}\left[\frac{1}{2}|\nabla w|^2+\frac{1}{2(p-1)}w^2-\frac{1}{p+1}|w|^{p+1}\right](y\cdot y_0)\rho dy. \tag{3.7}
$$

Combining (3.3), (3.5), (3.6) and (3.7) in the way

$$
(3.3)-\frac{h}{p-1}(3.5)-\frac{h}{2}(3.6)+\frac{1}{2}(3.7),
$$

we see that (3.4) holds. So w is a critical point of F_{x_0,t_0}. □

4. Second variation of the F-functional

In this section, we will derive the second variation formula of the F-functional for simultaneous variations in all three parameters w, x_0 and t_0. In particular, when w is a bounded solution of (SS), we will use our calculation to formulate a notion of stability.

4.1. The general second variation formula.

Lemma 4.1. *Let w be a bounded smooth function. If $x(s)$ and $t(s)$ are variations such that*

$$x(0) = x_0, \quad t(0) = t_0, \quad x'(0) = y_0, \quad t'(0) = h$$

and $x''(0) = y_0', t''(0) = h'$, then

$$\begin{aligned}
&\frac{d^2}{ds^2} F_{x(s),t(s)}(w + s\phi)|_{s=0} \\
=&\frac{p+1}{2(p-1)}(-t_0)^{-1+\frac{2}{p-1}} \left[t_0 h' + \frac{2}{p-1}h^2\right] \int_{\mathbb{R}^n} |\nabla w|^2 G(y - x_0, t_0) dy \\
&- \frac{1}{p-1}(-t_0)^{-1+\frac{2}{p-1}} \left[t_0 h' + \frac{2}{p-1}h^2\right] \int_{\mathbb{R}^n} |w|^{p+1} G(y - x_0, t_0) dy \\
&- \frac{1}{(p-1)^2}(-t_0)^{-2+\frac{2}{p-1}} \left[-t_0 h' + \frac{p-3}{p-1}h^2\right] \int_{\mathbb{R}^n} w^2 G(y - x_0, t_0) dy \\
&- \frac{2(p+1)}{p-1}(-t_0)^{\frac{2}{p-1}} h \int_{\mathbb{R}^n} (\nabla w \cdot \nabla \phi) G(y - x_0, t_0) dy \\
&+ \frac{2(p+1)}{p-1}(-t_0)^{\frac{2}{p-1}} h \int_{\mathbb{R}^n} |w|^{p-1} w\phi G(y - x_0, t_0) dy \\
&- \frac{4}{(p-1)^2}(-t_0)^{\frac{2}{p-1}-1} h \int_{\mathbb{R}^n} w\phi G(y - x_0, t_0) dy \\
&- \frac{p+1}{p-1}(-t_0)^{\frac{2}{p-1}} h \int_{\mathbb{R}^n} |\nabla w|^2 G(y - x_0, t_0) \\
&\qquad \times \left[-\frac{nh}{2t_0} + \frac{(x_0 - y) \cdot y_0}{2t_0} - \frac{h|x_0 - y|^2}{4t_0^2}\right] dy \\
&+ \frac{2}{p-1}(-t_0)^{\frac{2}{p-1}} h \int_{\mathbb{R}^n} |w|^{p+1} G(y - x_0, t_0) \\
&\qquad \times \left[-\frac{nh}{2t_0} + \frac{(x_0 - y) \cdot y_0}{2t_0} - \frac{h|x_0 - y|^2}{4t_0^2}\right] dy \\
&- \frac{2}{(p-1)^2}(-t_0)^{\frac{2}{p-1}-1} h \int_{\mathbb{R}^n} w^2 G(y - x_0, t_0) \\
&\qquad \times \left[-\frac{nh}{2t_0} + \frac{(x_0 - y) \cdot y_0}{2t_0} - \frac{h|x_0 - y|^2}{4t_0^2}\right] dy \\
&+ (-t_0)^{\frac{p+1}{p-1}} \int_{\mathbb{R}^n} |\nabla \phi|^2 G(y - x_0, t_0) dy \\
&- p(-t_0)^{\frac{p+1}{p-1}} \int_{\mathbb{R}^n} |w|^{p-1} \phi^2 G(y - x_0, t_0) dy \\
&+ \frac{1}{p-1}(-t_0)^{\frac{2}{p-1}} \int_{\mathbb{R}^n} \phi^2 G(y - x_0, t_0) dy \\
&+ 2(-t_0)^{\frac{p+1}{p-1}} \int_{\mathbb{R}^n} (\nabla w \cdot \nabla \phi) G(y - x_0, t_0)
\end{aligned}$$

$$
\begin{aligned}
&\times \left[-\frac{nh}{2t_0}+\frac{(x_0-y)\cdot y_0}{2t_0}-\frac{h|x_0-y|^2}{4t_0^2}\right]dy\\
&-2(-t_0)^{\frac{p+1}{p-1}}\int_{\mathbb{R}^n}|w|^{p-1}w\phi G(y-x_0,t_0)\\
&\qquad\times \left[-\frac{nh}{2t_0}+\frac{(x_0-y)\cdot y_0}{2t_0}-\frac{h|x_0-y|^2}{4t_0^2}\right]dy\\
&+\frac{2}{p-1}(-t_0)^{\frac{2}{p-1}}\int_{\mathbb{R}^n}w\phi G(y-x_0,t_0)\\
&\qquad\times \left[-\frac{nh}{2t_0}+\frac{(x_0-y)\cdot y_0}{2t_0}-\frac{h|x_0-y|^2}{4t_0^2}\right]dy\\
&+\frac{1}{2}(-t_0)^{\frac{p+1}{p-1}}\int_{\mathbb{R}^n}|\nabla w|^2G(y-x_0,t_0)\\
&\qquad\times\Big\{\left[-\frac{nh}{2t_0}+\frac{(x_0-y)\cdot y_0}{2t_0}-\frac{h|x_0-y|^2}{4t_0^2}\right]^2-\frac{nh't_0-nh^2}{2t_0^2}\\
&\qquad\quad+\frac{[|y_0|^2+(x_0-y)\cdot y_0']t_0-h(x_0-y)\cdot y_0}{2t_0^2}\\
&\qquad\quad-\frac{[h'|x_0-y|^2+2h(x_0-y)\cdot y_0]t_0-2h^2|x_0-y|^2}{4t_0^3}\Big\}dy\\
&-\frac{1}{p+1}(-t_0)^{\frac{p+1}{p-1}}\int_{\mathbb{R}^n}|w|^{p+1}G(y-x_0,t_0)\\
&\qquad\times\Big\{\left[-\frac{nh}{2t_0}+\frac{(x_0-y)\cdot y_0}{2t_0}-\frac{h|x_0-y|^2}{4t_0^2}\right]^2-\frac{nh't_0-nh^2}{2t_0^2}\\
&\qquad\quad+\frac{[|y_0|^2+(x_0-y)\cdot y_0']t_0-h(x_0-y)\cdot y_0}{2t_0^2}\\
&\qquad\quad-\frac{[h'|x_0-y|^2+2h(x_0-y)\cdot y_0]t_0-2h^2|x_0-y|^2}{4t_0^3}\Big\}dy\\
&+\frac{1}{2(p-1)}(-t_0)^{\frac{2}{p-1}}\int_{\mathbb{R}^n}w^2G(y-x_0,t_0)\\
&\qquad\times\Big\{\left[-\frac{nh}{2t_0}+\frac{(x_0-y)\cdot y_0}{2t_0}-\frac{h|x_0-y|^2}{4t_0^2}\right]^2-\frac{nh't_0-nh^2}{2t_0^2}\\
&\qquad\quad+\frac{[|y_0|^2+(x_0-y)\cdot y_0']t_0-h(x_0-y)\cdot y_0}{2t_0^2}\\
&\qquad\quad-\frac{[h'|x_0-y|^2+2h(x_0-y)\cdot y_0]t_0-2h^2|x_0-y|^2}{4t_0^3}\Big\}dy.
\end{aligned}
$$

Proof. After some computations, we get

$$
\frac{d^2}{ds^2}F_{x(s),t(s)}(w+s\phi)
$$

$$
\begin{aligned}
=&\frac{p+1}{2(p-1)}[-t(s)]^{-1+\frac{2}{p-1}}\left[t(s)t''(s)+\frac{2}{p-1}(t'(s))^2\right]\\
&\qquad\times\int_{\mathbb{R}^n}|\nabla w+s\nabla\phi|^2G(y-x(s),t(s))dy\\
&-\frac{1}{p-1}[-t(s)]^{-1+\frac{2}{p-1}}\left[t(s)t''(s)+\frac{2}{p-1}(t'(s))^2\right]\\
&\qquad\times\int_{\mathbb{R}^n}|w+s\phi|^{p+1}G(y-x(s),t(s))dy\\
&-\frac{1}{(p-1)^2}[-t(s)]^{-2+\frac{2}{p-1}}\left[\frac{p-3}{p-1}(t'(s))^2-t(s)t''(s)\right]\\
&\qquad\times\int_{\mathbb{R}^n}(w+s\phi)^2G(y-x(s),t(s))dy\\
&-\frac{2(p+1)}{p-1}[-t(s)]^{\frac{2}{p-1}}t'(s)\int_{\mathbb{R}^n}(\nabla w+s\nabla\phi)\cdot\nabla\phi G(y-x(s),t(s))dy\\
&+\frac{2(p+1)}{p-1}[-t(s)]^{\frac{2}{p-1}}t'(s)\int_{\mathbb{R}^n}|w+s\phi|^{p-1}(w+s\phi)\phi G(y-x(s),t(s))dy\\
&-\frac{4}{(p-1)^2}[-t(s)]^{\frac{2}{p-1}-1}t'(s)\int_{\mathbb{R}^n}(w+s\phi)\phi G(y-x(s),t(s))dy\\
&-\frac{p+1}{p-1}[-t(s)]^{\frac{2}{p-1}}t'(s)\int_{\mathbb{R}^n}|\nabla w+s\nabla\phi|^2G(y-x(s),t(s))\\
&\qquad\times\left[-\frac{nt'(s)}{2t(s)}+\frac{(x(s)-y)\cdot x'(s)}{2t(s)}-\frac{t'(s)|x(s)-y|^2}{4t^2(s)}\right]dy\\
&+\frac{2}{p-1}[-t(s)]^{\frac{2}{p-1}}t'(s)\int_{\mathbb{R}^n}|w+s\phi|^{p+1}G(y-x(s),t(s))\\
&\qquad\times\left[-\frac{nt'(s)}{2t(s)}+\frac{(x(s)-y)\cdot x'(s)}{2t(s)}-\frac{t'(s)|x(s)-y|^2}{4t^2(s)}\right]dy\\
&-\frac{2}{(p-1)^2}[-t(s)]^{\frac{2}{p-1}-1}t'(s)\int_{\mathbb{R}^n}(w+s\phi)^2G(y-x(s),t(s))\\
&\qquad\times\left[-\frac{nt'(s)}{2t(s)}+\frac{(x(s)-y)\cdot x'(s)}{2t(s)}-\frac{t'(s)|x(s)-y|^2}{4t^2(s)}\right]dy\\
&+[-t(s)]^{\frac{p+1}{p-1}}\int_{\mathbb{R}^n}|\nabla\phi|^2G(y-x(s),t(s))dy\\
&-p[-t(s)]^{\frac{p+1}{p-1}}\int_{\mathbb{R}^n}|w+s\phi|^{p-1}\phi^2G(y-x(s),t(s))dy\\
&+\frac{1}{p-1}[-t(s)]^{\frac{2}{p-1}}\int_{\mathbb{R}^n}\phi^2G(y-x(s),t(s))dy\\
&+2[-t(s)]^{\frac{p+1}{p-1}}\int_{\mathbb{R}^n}(\nabla w+s\nabla\phi)\cdot\nabla\phi G(y-x(s),t(s))
\end{aligned}
$$

$$
\begin{aligned}
&\times\left[-\frac{nt'(s)}{2t(s)}+\frac{(x(s)-y)\cdot x'(s)}{2t(s)}-\frac{t'(s)|x(s)-y|^2}{4t^2(s)}\right]dy\\
&-2[-t(s)]^{\frac{p+1}{p-1}}\int_{\mathbb{R}^n}|w+s\phi|^{p-1}(w+s\phi)\phi G(y-x(s),t(s))\\
&\qquad\times\left[-\frac{nt'(s)}{2t(s)}+\frac{(x(s)-y)\cdot x'(s)}{2t(s)}-\frac{t'(s)|x(s)-y|^2}{4t^2(s)}\right]dy\\
&+\frac{2}{p-1}[-t(s)]^{\frac{2}{p-1}}\int_{\mathbb{R}^n}(w+s\phi)\phi G(y-x(s),t(s))\\
&\qquad\times\left[-\frac{nt'(s)}{2t(s)}+\frac{(x(s)-y)\cdot x'(s)}{2t(s)}-\frac{t'(s)|x(s)-y|^2}{4t^2(s)}\right]dy\\
&+\frac{1}{2}[-t(s)]^{\frac{p+1}{p-1}}\int_{\mathbb{R}^n}|\nabla w+s\nabla\phi|^2G(y-x(s),t(s))\\
&\qquad\times\Bigg\{\left[-\frac{nt'(s)}{2t(s)}+\frac{(x(s)-y)\cdot x'(s)}{2t(s)}-\frac{t'(s)|x(s)-y|^2}{4t^2(s)}\right]^2-\frac{nt''(s)t(s)-nt'(s)t'(s)}{2t^2(s)}\\
&\qquad\quad+\frac{[|x'(s)|^2+(x(s)-y)\cdot x''(s)]t(s)-t'(s)(x(s)-y)\cdot x'(s)}{2t^2(s)}\\
&\qquad\quad-\frac{[t''(s)|x(s)-y|^2+2t'(s)(x(s)-y)\cdot x'(s)]t(s)-2t'(s)t'(s)|x(s)-y|^2}{4t^3(s)}\Bigg\}dy\\
&-\frac{1}{p+1}[-t(s)]^{\frac{p+1}{p-1}}\int_{\mathbb{R}^n}|w+s\phi|^{p+1}G(y-x(s),t(s))\\
&\qquad\times\Bigg\{\left[-\frac{nt'(s)}{2t(s)}+\frac{(x(s)-y)\cdot x'(s)}{2t(s)}-\frac{t'(s)|x(s)-y|^2}{4t^2(s)}\right]^2-\frac{nt''(s)t(s)-nt'(s)t'(s)}{2t^2(s)}\\
&\qquad\quad+\frac{[|x'(s)|^2+(x(s)-y)\cdot x''(s)]t(s)-t'(s)(x(s)-y)\cdot x'(s)}{2t^2(s)}\\
&\qquad\quad-\frac{[t''(s)|x(s)-y|^2+2t'(s)(x(s)-y)\cdot x'(s)]t(s)-2t'(s)t'(s)|x(s)-y|^2}{4t^3(s)}\Bigg\}dy\\
&+\frac{1}{2(p-1)}[-t(s)]^{\frac{2}{p-1}}\int_{\mathbb{R}^n}(w+s\phi)^2G(y-x(s),t(s))\\
&\qquad\times\Bigg\{\left[-\frac{nt'(s)}{2t(s)}+\frac{(x(s)-y)\cdot x'(s)}{2t(s)}-\frac{t'(s)|x(s)-y|^2}{4t^2(s)}\right]^2-\frac{nt''(s)t(s)-nt'(s)t'(s)}{2t^2(s)}\\
&\qquad\quad+\frac{[|x'(s)|^2+(x(s)-y)\cdot x''(s)]t(s)-t'(s)(x(s)-y)\cdot x'(s)}{2t^2(s)}\\
&\qquad\quad-\frac{[t''(s)|x(s)-y|^2+2t'(s)(x(s)-y)\cdot x'(s)]t(s)-2t'(s)t'(s)|x(s)-y|^2}{4t^3(s)}\Bigg\}dy.
\end{aligned}
$$

The proof is finished by substituting $s=0$ into the above formula. □

If $x(0)=0, t(0)=-1$, then the second variation formula has a simpler appearance.

Lemma 4.2. *Let w be a bounded smooth function. If $x(s), t(s)$ are variations such that*

$$x(0)=0, \quad t(0)=-1, \quad x'(0)=y_0, \quad t'(0)=h$$

and $x''(0)=y_0'$, $t''(0)=h'$, then

$$\begin{aligned}
&\frac{d^2}{ds^2}F_{x(s),t(s)}(w+s\phi)|_{s=0}\\
=&\frac{p+1}{2(p-1)}\left(-h'+\frac{2}{p-1}h^2\right)\int_{\mathbb{R}^n}|\nabla w|^2\rho dy\\
&-\frac{1}{p-1}\left(-h'+\frac{2}{p-1}h^2\right)\int_{\mathbb{R}^n}|w|^{p+1}\rho dy\\
&-\frac{1}{(p-1)^2}\left(h'+\frac{p-3}{p-1}h^2\right)\int_{\mathbb{R}^n}w^2\rho dy\\
&-\frac{2(p+1)}{p-1}h\int_{\mathbb{R}^n}(\nabla w\cdot\nabla\phi)\rho dy\\
&+\frac{2(p+1)}{p-1}h\int_{\mathbb{R}^n}|w|^{p-1}w\phi\rho dy-\frac{4}{(p-1)^2}h\int_{\mathbb{R}^n}w\phi\rho dy\\
&-\frac{p+1}{p-1}h\int_{\mathbb{R}^n}|\nabla w|^2\rho\left[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]dy\\
&+\frac{2}{p-1}h\int_{\mathbb{R}^n}|w|^{p+1}\rho\left[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]dy\\
&-\frac{2}{(p-1)^2}h\int_{\mathbb{R}^n}w^2\rho\left[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]dy\\
&+\int_{\mathbb{R}^n}|\nabla\phi|^2\rho dy-p\int_{\mathbb{R}^n}|w|^{p-1}\phi^2\rho dy+\frac{1}{p-1}\int_{\mathbb{R}^n}\phi^2\rho dy\\
&+2\int_{\mathbb{R}^n}(\nabla w\cdot\nabla\phi)\rho\left[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]dy\\
&-2\int_{\mathbb{R}^n}|w|^{p-1}w\phi\rho\left[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]dy\\
&+\frac{2}{p-1}\int_{\mathbb{R}^n}w\phi\rho\left[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]dy\\
&+\frac{1}{2}\int_{\mathbb{R}^n}|\nabla w|^2\rho\Bigg\{\left[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]^2+\frac{nh'+nh^2}{2}\\
&+\frac{-|y_0|^2+y\cdot y_0'+hy\cdot y_0}{2}-\frac{h'|y|^2-2hy\cdot y_0+2h^2|y|^2}{4}\Bigg\}dy\\
&-\frac{1}{p+1}\int_{\mathbb{R}^n}|w|^{p+1}\rho\Bigg\{\left[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]^2+\frac{nh'+nh^2}{2}
\end{aligned} \tag{4.1}$$

$$+ \frac{-|y_0|^2 + y \cdot y_0' + hy \cdot y_0}{2} - \frac{h'|y|^2 - 2hy \cdot y_0 + 2h^2|y|^2}{4} \Bigg\} dy$$

$$+ \frac{1}{2(p-1)} \int_{\mathbb{R}^n} w^2 \rho \Bigg\{ \Big[\frac{nh}{2} + \frac{y \cdot y_0}{2} - \frac{h|y|^2}{4} \Big]^2 + \frac{nh' + nh^2}{2}$$

$$+ \frac{-|y_0|^2 + y \cdot y_0' + hy \cdot y_0}{2} - \frac{h'|y|^2 - 2hy \cdot y_0 + 2h^2|y|^2}{4} \Bigg\} dy.$$

4.2. **The second variation of self similar solutions.** In this subsection, we will specialize our calculations from the previous subsection to the case where w satisfies (SS). In this case, by using (SS), the second order variation formula can be simplified further.

Theorem 4.3. *Let $x(s), t(s)$ be variations of $0, -1$ with $x'(0) = y_0, t'(0) = h$ and $x''(0) = y_0', t''(0) = h'$. If w is a bounded solution of (SS), then for any $\phi \in C_0^\infty(\mathbb{R}^n)$,*

$$\begin{aligned} &\frac{d^2}{ds^2} F_{x(s),t(s)}(w + s\phi)|_{s=0} \\ =& \int_{\mathbb{R}^n} \left[|\nabla \phi|^2 + \frac{1}{p-1} \phi^2 \rho dy - p|w|^{p-1} \phi^2 \right] \rho dy \\ &+ h \int_{\mathbb{R}^n} \left(\frac{2}{p-1} w + y \cdot \nabla w \right) \phi \rho dy - \int_{\mathbb{R}^n} \phi (\nabla w \cdot y_0) \rho dy \\ &- \frac{1}{2} \int_{\mathbb{R}^n} |\nabla w \cdot y_0|^2 \rho dy - h^2 \int_{\mathbb{R}^n} \left(\frac{1}{p-1} w + \frac{y}{2} \cdot \nabla w \right)^2 \rho dy. \end{aligned} \tag{4.2}$$

In particular, we have

$$\begin{aligned} &\frac{d^2}{ds^2} F_{x(s),t(s)}(w + s\phi)|_{s=0} \\ \leq& \int_{\mathbb{R}^n} \left[|\nabla \phi|^2 + \frac{1}{p-1} \phi^2 \rho dy - p|w|^{p-1} \phi^2 \right] \rho dy \\ &+ h \int_{\mathbb{R}^n} \left(\frac{2}{p-1} w + y \cdot \nabla w \right) \phi \rho dy - \int_{\mathbb{R}^n} \phi (\nabla w \cdot y_0) \rho dy. \end{aligned} \tag{4.3}$$

The proof will be given in Appendix A.

The second variation formula above allows us to formulate a notion of stability.

Definition 4.4. *Let w be a bounded solution of (SS). If for every $\phi \in H^1_w(\mathbb{R}^n)$, there exists variations $x(s), t(s)$ with*

$$x(0) = 0,\ t(0) = -1,\ x'(0) = y_0,\ t'(0) = h,\ x''(0) = y_0',\ t''(0) = h'$$

such that

$$\frac{d^2}{ds^2} F_{x(s),t(s)}(w + s\phi)|_{s=0} \geq 0,$$

then we say that w is F-stable.

Roughly speaking, a bounded solution of (SS) is F-stable if modulo translations and dilations, the second derivative of the F-functional is non-negative for all variations at the given solution.

Remark 4.5. *From [21], we know that the energy functional of (SS) is $E(w)$ defined in (E). Thus at first it seems natural to define a notion of stability as follows: w is stable if for any $\phi \in C_0^\infty(\mathbb{R}^n)$, the quadratic form*

$$Q(\phi,\phi) = \int_{\mathbb{R}^n} \left(|\nabla\phi|^2 + \frac{1}{(p-1)}\phi^2 - p|w|^{p-1}\phi^2 \right) \rho dy$$

is nonnegative definite. However, it is easy to check that if we use this definition, then any nonzero self similar solution is unstable. Therefore, this kind of stability can not provide any useful information.

5. Characterization of the constant solution

Our next objective is to classify F-stable self similar solutions. Before doing this, we first prove a result on the constant solutions of (SS).

Proposition 5.1. *Let w be a bounded solution of (SS), then w is the constant solution of (SS) if and only if the function*

$$\Lambda(w)(y) = \frac{2}{p-1}w(y) + y\cdot\nabla w(y) \tag{5.1}$$

does not change sign in $\mathbb{R}^n$.

Remark 5.2. *In a recent paper [5], Choi and Huang proved a similar result for smooth linearly stable self-similar solutions [2] of the equation (F) under an integral condition for all $p > 1$.*

In order to prove Proposition 5.1, we need some results concerning the linear operator

$$\mathcal{L}\psi = -\Delta\psi + \frac{y}{2}\cdot\nabla\psi + \frac{1}{p-1}\psi - p|w|^{p-1}\psi, \tag{LO}$$

where w is a bounded solution of (SS). Recall that $\lambda \in \mathbb{R}$ is an eigenvalue of $\mathcal{L}$ if there is a non-zero function $f \in H^1_\omega(\mathbb{R}^n)$ such that $\mathcal{L}f = \lambda f$. The operator $\mathcal{L}$ is self adjoint in $L^2_\omega(\mathbb{R}^n)$ with domain $D(L) := H^1_\omega(\mathbb{R}^n)$. Since the natural embedding $\iota : H^1_\omega(\mathbb{R}^n) \hookrightarrow L^2_\omega(\mathbb{R}^n)$ is compact, standard spectral theory gives the following corollary.

Corollary 5.3. *Assume w is a bounded solution of (SS) and $\mathcal{L}$ is defined by (LO). Then*

1. *$\mathcal{L}$ has a sequence of eigenvalues $\lambda_1 < \lambda_2 \le \cdots \to +\infty$.*
2. *There is an orthogonal basis $\{f_k\}$ for $L^2_w(\mathbb{R}^n)$, where all of f_k are eigenfunctions of $\mathcal{L}$.*

[2] The precise definition of linearly stable self similar solution can be found in [5, Definition 2.1].

(3) *The smallest eigenvalue* λ_1 *is simple, which can be characterized by*

$$\lambda_1 = \inf_{\psi \in H^1_w(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n}\left(|\nabla\psi|^2 + \frac{1}{p-1}\psi^2 - p|w|^{p-1}\psi^2\right)\rho dy}{\int_{\mathbb{R}^n}\psi^2\rho dy}. \tag{5.2}$$

(4) *Any eigenfunction associated to* λ_1 *does not change sign.*

The next lemma shows that the operator $\mathcal{L}$ has two explicit eigenfunctions which are induced by scaling and translations.

Lemma 5.4. *If* w *is a bounded solution of the equation* (SS)*, then*

$$\mathcal{L}w_i = -\frac{1}{2}w_i, \quad i = 1, 2, \cdots, n, \tag{5.3}$$

$$\mathcal{L}\left(\frac{2}{p-1}w + y\cdot\nabla w\right) = -\left(\frac{2}{p-1}w + y\cdot\nabla w\right). \tag{5.4}$$

Proof. Since w satisfies (SS), for $i = 1, 2, \cdots, n$,

$$\Delta w_i - \frac{y}{2}\cdot\nabla w_i - \frac{1}{2}w_i - \frac{1}{p-1}w_i + p|w|^{p-1}w_i = 0.$$

This is just (5.3).

To show (5.4), consider the variation $w_\lambda(y) = \lambda^{2/(p-1)}w(\lambda y)$. Since w satisfies (SS), w_λ satisfies

$$\Delta w_\lambda - \frac{\lambda^2}{2}y\cdot\nabla w_\lambda - \frac{\lambda^2}{p-1}w_\lambda + |w_\lambda|^{p-1}w_\lambda = 0. \tag{5.5}$$

Taking derivative with respect to λ at $\lambda = 1$, we obtain

$$\begin{aligned} 0 =& \Delta\left(\frac{2}{p-1}w + y\cdot\nabla w\right) - \frac{y}{2}\cdot\nabla\left(\frac{2}{p-1}w + y\cdot\nabla w\right) \\ &- \frac{1}{p-1}\left(\frac{2}{p-1}w + y\cdot\nabla w\right) + p|w|^{p-1}\left(\frac{2}{p-1}w + y\cdot\nabla w\right) \\ &- \left(\frac{2}{p-1}w + y\cdot\nabla w\right), \end{aligned} \tag{5.6}$$

which is (5.4). □

With Corollary 5.3 and Lemma 5.4 in hand, we now have all of the tools to prove Proposition 5.1.

Proof of Proposition 5.1. If w is the constant solution of (SS), then it is clear that either $\Lambda(w) \equiv 0$ or

$$\Lambda(w)(y) = \pm 2\left(\frac{1}{p-1}\right)^{\frac{p}{p-1}}.$$

Thus $\Lambda(w)$ does not change sign in $\mathbb{R}^n$.

Next, we prove that if $\Lambda(w)$ does not change sign, then w is the constant solution of (SS).

Assume that $\Lambda(w)$ does not change sign. By Lemma 5.4 and the strong maximum principle, either $\Lambda(w) \equiv 0$ or up to a sign, $\Lambda(w)$ is positive in $\mathbb{R}^n$. If the first case holds ($\Lambda(w) \equiv 0$), then w is a $2/(p-1)-$ homogeneous solution of (1.6). Since we have assumed that w is bounded, then w is smooth in $\mathbb{R}^n$. The homogeneity then implies that $w \equiv 0$.

If the second case holds, then we know from Lemma 5.4 and the last statement in Corollary 5.3 that -1 is the smallest eigenvalue of the operator $\mathcal{L}$ defined in (LO). Therefore, for any $\phi \in H^1_w(\mathbb{R}^n)$,

$$0 \le \int_{\mathbb{R}^n} |\nabla\phi|^2 \rho dy + \frac{1}{p-1}\int_{\mathbb{R}^n} \phi^2 \rho dy - p\int_{\mathbb{R}^n} |w|^{p-1}\phi^2 \rho dy + \int_{\mathbb{R}^n} \phi^2 \rho dy. \tag{5.7}$$

By taking $\phi = w$ into (5.7), we get

$$0 \le \int_{\mathbb{R}^n} |\nabla w|^2 \rho dy + \frac{1}{p-1}\int_{\mathbb{R}^n} w^2 \rho dy - p\int_{\mathbb{R}^n} |w|^{p+1} \rho dy + \int_{\mathbb{R}^n} w^2 \rho dy. \tag{5.8}$$

Multiplying both sides of (SS) by $w\rho$ and integrating by parts, we get

$$0 = \int_{\mathbb{R}^n} |\nabla w|^2 \rho dy + \frac{1}{p-1}\int_{\mathbb{R}^n} w^2 \rho dy - \int_{\mathbb{R}^n} |w|^{p+1} \rho dy. \tag{5.9}$$

Combining (5.8) and (5.9), we obtain

$$(1-p)\int_{\mathbb{R}^n} |w|^{p+1} \rho dy + \int_{\mathbb{R}^n} w^2 \rho dy \ge 0. \tag{5.10}$$

On the other hand, we get from (5.9) that

$$(1-p)\int_{\mathbb{R}^n} |w|^{p+1} \rho dy + \int_{\mathbb{R}^n} w^2 \rho dy = -(p-1)\int_{\mathbb{R}^n} |\nabla w|^2 \rho dy \le 0. \tag{5.11}$$

Hence

$$(1-p)\int_{\mathbb{R}^n} |w|^{p+1} \rho dy + \int_{\mathbb{R}^n} w^2 \rho dy = 0. \tag{5.12}$$

Plugging (5.12) into (5.9), we get

$$\int_{\mathbb{R}^n} |\nabla w|^2 \rho dy = 0. \tag{5.13}$$

Thus w is a constant function. □

Remark 5.5. *The proof of Proposition 5.1 indicates that a much more suitable choice to define "Morse index" of solutions of (SS) is the number of eigenvalues of $\mathcal{L}$ less than 1.*

Remark 5.6. *If we do not assume w is a bounded solution of (SS), then*

$$w(y) = \left[\frac{2}{p-1}(n-2-\frac{2}{p-1})\right]^{\frac{1}{p-1}} |y|^{-\frac{2}{p-1}}$$

is a solution of (SS) with $\Lambda(w)(y) \equiv 0$ on $\mathbb{R}^n \backslash \{0\}$.

6. Classification of F-stable self similar solutions

In this section, we will combine the second variation formula and the characterization of the constant solution to give a complete classification of F-stable bounded self similar solutions.

Theorem 6.1. *If w is a bounded solution of (SS) and it is not the constant solution, then w is not F-stable.*

Proof. If w is not the constant solution of (SS), then the function $\Lambda(w)$ defined in (5.1) can not vanish identically. By Lemma 5.4, $\Lambda(w)$ is an eigenfunction of $\mathcal{L}$ with eigenvalue -1. By Proposition 5.1, $\Lambda(w)$ must changes sign. Then the last point in Corollary 5.3 implies that -1 is not the smallest eigenvalue of $\mathcal{L}$. Thus $\lambda_1 < -1$ and there exists a positive, first eigenfunction f. Since $\mathcal{L}$ is self-adjoint in the weighted space $H^1_w(\mathbb{R}^n)$, f is orthogonal to the eigenfunctions with different eigenvalues. In particular, we have

$$\int_{\mathbb{R}^n} \left(\frac{2}{p-1}w + y\cdot\nabla w\right) f\rho dy = 0 \tag{6.1}$$

and for any $y_0 \in \mathbb{R}^n$

$$\int_{\mathbb{R}^n} (\nabla w\cdot y_0) f\rho dy = 0. \tag{6.2}$$

Substituting (6.1) and (6.2) into (4.3) gives

$$\begin{aligned}
&\frac{d^2}{ds^2}F_{x(s),t(s)}(w+sf)|_{s=0}\\
=&\int_{\mathbb{R}^n}\left(|\nabla f|^2 + \frac{1}{p-1}f^2 - p|w|^{p-1}f^2\right)\rho dy\\
&+h\int_{\mathbb{R}^n}\left(\frac{2}{p-1}w + y\cdot\nabla w\right)f\rho dy - \int_{\mathbb{R}^n} f(\nabla w\cdot y_0)\rho dy\\
&-\frac{1}{2}\int_{\mathbb{R}^n}|\nabla w\cdot y_0|^2\rho dy - h^2\int_{\mathbb{R}^n}(\frac{1}{p-1}w + \frac{y}{2}\cdot\nabla w)^2\rho dy\\
\leq & -\lambda_1\int_{\mathbb{R}^n} f^2\rho dy.
\end{aligned} \tag{6.3}$$

It follows that

$$\frac{d^2}{ds^2}F_{x(s),t(s)}(w+sf)|_{s=0} < 0$$

for any choice of h and y_0. By the definition, w is not F-stable. □

Remark 6.2. *In the proof of Theorem 6.1, the variation we can choose is not unique. Indeed, assume f is the first positive eigenfunction of $\mathcal{L}$ and ξ is a small positive constant which will be determined later. Let us take $\phi = f+\xi$ into the second variation formula. Without loss of generality, we may normalize f so that $\int_{\mathbb{R}^n} f^2\rho dy = 1$. Then*

$$\frac{d^2}{ds^2}F_{x(s),t(s)}(w+s(f+\xi))|_{s=0}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^n} \left(|\nabla f|^2 + \frac{1}{p-1} f^2 - p|w|^{p-1} f^2 \right) \rho dy \\
&\quad + \frac{2\xi}{p-1} \int_{\mathbb{R}^n} f \rho dy + \frac{\xi^2}{p-1} \int_{\mathbb{R}^n} \rho dy \\
&\quad - 2p\xi \int_{\mathbb{R}^n} |w|^{p-1} f \rho dy - p\xi^2 \int_{\mathbb{R}^n} |w|^{p-1} \rho dy \\
&\quad - \xi \int_{\mathbb{R}^n} (\nabla w \cdot y_0) \rho dy - \frac{1}{2} \int_{\mathbb{R}^n} |\nabla w \cdot y_0|^2 \rho dy \\
&\quad + 2h\xi \int_{\mathbb{R}^n} \left(\frac{1}{p-1} w + \frac{y}{2} \cdot \nabla w \right) \rho dy - h^2 \int_{\mathbb{R}^n} \left(\frac{1}{p-1} w + \frac{y}{2} \cdot \nabla w \right)^2 \rho dy \\
&\le \lambda_1 \int_{\mathbb{R}^n} f^2 \rho dy + \frac{3(p+1)}{p-1} (\xi^2 + \xi).
\end{aligned} \tag{6.4}
$$

If we choose ξ so that

$$\frac{3(p+1)}{p-1}(\xi^2 + \xi) < \lambda_1,$$

then we still have

$$\frac{d^2}{ds^2} F_{x(s),t(s)}(w + s(f + \xi))|_{s=0} < 0$$

for any choice of h and y_0.

In view of Theorem 6.1, it is natural to ask whether the constant solution is F-stable. By the second variation formula, it is clear that the zero solution is F-stable. Next, we will show if $w = \pm\kappa$, then the only way to decrease the $F_{0,1}$ functional is to translate in space; this will not be used elsewhere.

Proposition 6.3. *If $w = \pm\kappa$, then for any function $\phi \in H^1_w(\mathbb{R}^n)$ satisfying*

$$\int_{\mathbb{R}^n} y_i \phi \rho dy = 0, \quad i = 1, 2, \cdots n,$$

there exist y_0, h such that

$$\frac{d^2}{ds^2} F_{x(s),t(s)}(w + s\phi)|_{s=0} \ge 0.$$

To prove this proposition, we need some information on the eigenvalues of the operator $\mathcal{L}_0$ defined by

$$\mathcal{L}_0 \psi = -\Delta\psi + \frac{y}{2} \cdot \nabla\psi.$$

Lemma 6.4. *The eigenvalues of the operator $\mathcal{L}_0$ are given by*

$$\lambda_k = |\alpha|/2, \quad \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n),$$

where α_i is a nonnegative integer for $1 \le i \le n$.

Proof of Proposition 6.3. We consider only the positive case, that is,

$$w = \kappa = \left(\frac{1}{p-1} \right)^{\frac{1}{p-1}}.$$

Plugging this into (4.2), we have

$$\begin{aligned}&\frac{d^2}{ds^2}F_{x(s),t(s)}(w+s\phi)|_{s=0}\\&=\int_{\mathbb{R}^n}|\nabla\phi|^2\rho dy-\int_{\mathbb{R}^n}\phi^2\rho dy+\frac{2hw}{p-1}\int_{\mathbb{R}^n}\phi\rho dy-\frac{h^2w^2}{(p-1)^2}\int_{\mathbb{R}^n}\rho dy\end{aligned}\tag{6.5}$$

Take a constant a so that

$$\int_{\mathbb{R}^n}(\phi-a)\rho dy=0\tag{6.6}$$

and let $\phi_0(y)=\phi(y)-a$. Then

$$\begin{aligned}&\frac{d^2}{ds^2}F_{x(s),t(s)}(w+s\phi)|_{s=0}\\&=\int_{\mathbb{R}^n}|\nabla\phi_0|^2\rho dy-\int_{\mathbb{R}^n}\phi_0^2\rho dy\\&\quad+2\frac{h}{p-1}\left(\frac{1}{p-1}\right)^{\frac{1}{p-1}}a-\frac{h^2}{(p-1)^2}\left(\frac{1}{p-1}\right)^{\frac{2}{p-1}}-a^2\\&=-\left[\frac{h}{p-1}\left(\frac{1}{p-1}\right)^{\frac{1}{p-1}}-a\right]^2+\int_{\mathbb{R}^n}|\nabla\phi_0|^2\rho dy-\int_{\mathbb{R}^n}\phi_0^2\rho dy.\end{aligned}$$

If we choose h so that

$$\frac{h}{p-1}\left(\frac{1}{p-1}\right)^{\frac{1}{p-1}}-a=0,$$

then Lemma 6.4 implies

$$\frac{d^2}{ds^2}F_{x(s),t(s)}(w+s\phi)|_{s=0}\geq 0.$$

The proof of Proposition 6.3 is thus complete. □

7. Entropy

The entropy $\lambda(w)$ of a bounded smooth function w is defined to be

$$\lambda(w)=\sup_{x_0\in\mathbb{R}^n,t_0\in(-\infty,0)}F_{x_0,t_0}(w).$$

Remark 7.1. *The notion of entropy is used by Colding and Minicozzi (see [7]) in mean curvature flow to classify generic singularities. This quantity can be used to measure the complexity of self shrinkers.*

As discussed in [7], the advantage of the entropy functional is that it is invariant under dilations, rotations, or translations of w. The main disadvantage of the entropy is that for a variation w_s, it need not depend smoothly on s. To deal with this, we define the definition of entropy stable as follows.

Definition 7.2. *A bounded function w is entropy stable if it is a local minimum for the entropy functional.*

7.1. **The entropy is achieved for bounded self similar solutions.** Although in the definition of entropy, the supremum is over a noncompact space-time domain, the next lemma shows that for a bounded solution of (SS), the function $(x_0, t_0) \to F_{x_0,t_0}(w)$ has a global maximum at $x_0 = 0, t_0 = -1$.

Lemma 7.3. *If w is a bounded solution of (SS), then $\lambda(w)$ is achieved at $(0, -1)$.*

Proof. For any bounded solution of (SS), set $u(x,t) = (-t)^{-1/(p-1)} w(x/\sqrt{-t})$. It follows from Lemma 2.3 that for any $(x,t) \in \mathbb{R}^n \times (-\infty, 0)$ and $T > t$,

$$\begin{aligned} E(s; x, T, u) =& \frac{1}{2}(-s)^{\frac{p+1}{p-1}} \int_{\mathbb{R}^n} |\nabla u(y, T+s)|^2 G(y-x, s) dy \\ &- \frac{1}{p+1}(-s)^{\frac{p+1}{p-1}} \int_{\mathbb{R}^n} |u(y, T+s)|^{p+1} G(y-x, s) dy \\ &+ \frac{1}{2(p-1)}(-s)^{\frac{2}{p-1}} \int_{\mathbb{R}^n} u(y, T+s)^2 G(y-x, s) dy \end{aligned}$$

is nonincreasing with respect to s in $(-\infty, -(T-t)]$. By the definition of u, we have

$$\begin{aligned} E(s; x, T, u) =& \frac{1}{2}\left(\frac{s}{T+s}\right)^{\frac{p+1}{p-1}} \int_{\mathbb{R}^n} |\nabla w(y)|^2 G\left(y - \frac{x}{\sqrt{-(T+s)}}, -\frac{s}{T+s}\right) dy \\ &- \frac{1}{p+1}\left(\frac{s}{T+s}\right)^{\frac{p+1}{p-1}} \int_{\mathbb{R}^n} |w(y)|^{p+1} G\left(y - \frac{x}{\sqrt{-(T+s)}}, -\frac{s}{T+s}\right) dy \\ &+ \frac{1}{2(p-1)}\left(\frac{s}{T+s}\right)^{\frac{2}{p-1}} \int_{\mathbb{R}^n} w(y)^2 G\left(y - \frac{x}{\sqrt{-(T+s)}}, -\frac{s}{T+s}\right) dy \\ =& F_{\frac{x}{\sqrt{-(T+s)}}, -\frac{s}{T+s}}(w). \end{aligned}$$

We take

$$T - t = 1, \quad x = x_0\sqrt{-\frac{1}{t_0}}, \quad T = 1 + \frac{1}{t_0}.$$

Since $E(s; x, t, u)$ is nonincreasing with respect to s in $(-\infty, -1]$, we have

$$E(-1; x, T, u) = F_{x_0,t_0}(w) \le \lim_{s \to -\infty} E(s; x, T, u) = F_{0,-1}(w).$$

Since $(x_0, t_0) \in \mathbb{R}^n \times (-\infty, 0)$ can be arbitrary, we conclude that $\lambda(w)$ achieves its maximum at $(0, -1)$. □

By the definition of the energy $E(w)$ and the F-functional, we have $E(w) = F_{0,-1}(w)$. Hence Lemma 7.3 implies the following corollary.

Corollary 7.4. *If w is a bounded solution of (SS), then $\lambda(w) = E(w)$.*

Lemma 7.3 only tells us the location where the F-functional achieves the maximum. For applications later, we also need to show that if w is a bounded solution of (SS) and

$$\nabla w \cdot y_0 \neq 0, \quad \text{for any } y_0 \in \mathbb{R}^n \backslash \{0\}, \tag{7.1}$$

then $(x_0, t_0) \to F_{x_0,t_0}(w)$ has a strict global maximum at $x_0 = 0, t_0 = -1$. The condition (7.1) says that w is not a solution of (SS) in low dimensions.

Lemma 7.5. *Suppose w is a bounded solution of (SS) satisfying (7.1). Then for every $\varepsilon > 0$ sufficiently small, there exists $\delta > 0$ such that*

$$\sup\{F_{x_0,t_0}(w) : |x_0| + |\log(-t_0)| > \varepsilon\} \leq \lambda(w) - \delta. \tag{7.2}$$

Proof. Since w satisfies (SS), we know from Proposition 3.3 that w is a critical point of the $F_{0,1}$-functional. Moreover, using the second variation formula with $\phi \equiv 0$, we see the second derivative of $F_{x(s),t(s)}(w)$ at $s = 0$ along the paths

$$x(s) = sy_0, \quad t(s) = -(1 + sh)$$

is given by

$$\begin{aligned} &\frac{d^2}{ds^2} F_{x(s),t(s)}(w + s\phi)|_{s=0} \\ = &-\frac{1}{2}\int_{\mathbb{R}^n} |\nabla w \cdot y_0|^2 \rho dy - h^2 \int_{\mathbb{R}^n} \left(\frac{1}{p-1} w + \frac{y}{2}\cdot \nabla w\right)^2 \rho dy. \end{aligned}$$

This quantity is clearly non-positive. Moreover, it equals 0 only if

$$h = 0, \ \nabla w \cdot y_0 \equiv 0$$

or

$$\nabla w \cdot y_0 \equiv 0, \quad \frac{1}{p-1} w + \frac{y}{2} \cdot \nabla w \equiv 0.$$

If $y_0 \neq 0$ and $\nabla w \cdot y_0 \equiv 0$, then we get a contradiction with the assumption (7.1). If $y_0 = 0$ and $h \neq 0$, then

$$\frac{d^2}{ds^2} F_{x(s),t(s)}(w + s\phi)|_{s=0} = 0$$

implies

$$\frac{1}{p-1} w + \frac{y}{2} \cdot \nabla w \equiv 0.$$

In particular, w is $2/(p-1)-$homogeneous. Since we have assumed that w is a bounded solution of (SS), it is only possible that $w \equiv 0$, which contradicts our assumption that w is not translation invariant in any direction again.

We conclude from the above analyses that the second derivative at $(0, 1)$ is strictly negative. As a consequence, the function $(x_0, t_0) \to F_{x_0,t_0}(w)$ has a strict local maximum at $(0, 1)$. Thus for every $\varepsilon > 0$ sufficiently small there exists $\delta > 0$ such that

$$\max_{(x,t)\in\{|x|+|\log(-t)|=\varepsilon\}} F_{x,t}(w) < \lambda(w) - \delta.$$

For every (x_0, t_0) satisfying $|x_0| + |\log(-t_0)| > \varepsilon$, it follows from the proof of Lemma 7.3 that for every $s < -1$,

$$F_{x_0,t_0}(w) \leq F_{\frac{x_0}{\sqrt{t_0+1+t_0 s}}, -\frac{st_0}{t_0+1+t_0 s}}(w). \tag{7.3}$$

Since

$$\lim_{s\to-\infty} \frac{x_0}{\sqrt{t_0+1+t_0 s}} = 0$$

and

$$\lim_{s\to-\infty} -\frac{st_0}{t_0+1+t_0 s} = -1,$$

there exists $\tilde{s}$ such that

$$\left(\frac{x_0}{\sqrt{t_0+1+t_0\tilde{s}}}, -\frac{st_0}{t_0+1+t_0\tilde{s}}\right) \in \{|x| + |\log(-t)| = \varepsilon\}.$$

By (7.3), we know that $F_{x_0,t_0}(w) \le \lambda(w) - \delta$. Hence (7.2) holds. □

7.2. **The equivalence of F-stability and entropy-stability.** Our next objective is to prove that if w is a bounded non-constant solution of (SS) and there exists a positive constant C such that

$$|w(y)| \le C(1+|y|)^{-\frac{2}{p-1}}, \quad \text{in } \mathbb{R}^n, \tag{7.4}$$

then F-stability and entropy stability are equivalent. In the course of the proof, we need a result concerning the first eigenfunction of the operator $\mathcal{L}$ defined in (LO).

Lemma 7.6. *Assume w is a bounded non-constant solution of (SS). Let λ_1 be the first eigenvalue of $\mathcal{L}$, and f be a positive eigenfunction associated to λ_1. Then there exists a positive constant C such that*

$$(1+|y|)^{\frac{2p}{p-1}}|f(y)| + (1+|y|)^{\frac{2p}{p-1}+1}|\nabla f(y)| \le C, \quad \text{in } \mathbb{R}^n. \tag{7.5}$$

Proof. Since w is a bounded non-constant solution of (SS), $\lambda_1 < -1$. Hence (7.4) implies there exists a positive constant R such that that

$$-\frac{2p}{p-1}\left(n-2-\frac{2p}{p-1}\right)|y|^{-2} + 1 + p|w|^{p-1} + \lambda_1 < 0, \quad \text{in } \mathbb{R}^n\backslash B_R(0). \tag{7.6}$$

By taking R large enough, we may also assume that

$$-\frac{1}{p-1} + p|w|^{p-1} + \lambda_1 < 0, \quad \text{in } \mathbb{R}^n\backslash B_R(0). \tag{7.7}$$

By standard elliptic regularity theory (see [25]), there exists a positive constant M such that

$$|f| < MR^{-\frac{2p}{p-1}}, \quad \text{in } B_R. \tag{7.8}$$

For any $k \ge 1$, consider the Dirichlet problem

$$\begin{cases} \Delta g_k - \frac{y}{2}\cdot\nabla g_k - \frac{1}{p-1}g_k + p|w|^{p-1}g_k + \lambda_1 g_k = 0, & \text{in } B_{R+k}\backslash B_R, \\ g_k = f, & \text{on } \partial B_R, \\ g_k = M|y|^{-\frac{2p}{p-1}}, & \text{on } \partial B_{R+k}, \end{cases} \tag{7.9}$$

Since we have assumed that (7.7) holds, the zeroth order term of the second order elliptic equation in (7.9) is negative. Observe that

$$(\mathcal{L}+\lambda_1)|y|^{-\frac{2p}{p-1}} = \left[-\frac{2p}{p-1}\left(n-2-\frac{2p}{p-1}\right)|y|^{-2} + 1 + p|w|^{p-1} + \lambda_1\right]|y|^{-\frac{2p}{p-1}}.$$

Thus, it follows from (7.6), [25, Theorem 8.3] and the maximum principle that for any $k \geq 1$, (7.9) has a unique smooth solution g_k, which is bounded above by $M|y|^{-(2p)/(p+1)}$. By using the maximum principle again, g_k is also bounded below by 0. Letting $k \to \infty$, we get from the Arzelá-Ascoli theorem that $\{g_k\}$ (after passing to a subsequence) converges to some function g_∞ in $C^2_{loc}(\mathbb{R}^n \backslash B_R)$, where

$$\begin{cases} \Delta g_\infty - \frac{y}{2} \cdot \nabla g_\infty - \frac{1}{p-1} g_\infty + p|w|^{p-1} g_\infty + \lambda_1 g_\infty = 0, & \text{in } \mathbb{R}^n \backslash B_R, \\ g_\infty = f, & \text{on } \partial B_R. \end{cases} \tag{7.10}$$

Moreover,

$$|g_\infty| \leq M|y|^{-\frac{2p}{p-1}}, \quad \text{in } \mathbb{R}^n \backslash B_R. \tag{7.11}$$

Multiplying both sides of (7.10) by $g_\infty \rho$ and integrating over $\mathbb{R}^n \backslash B_R$, we get

$$\int_{\mathbb{R}^n \backslash B_R} |\nabla g_\infty|^2 \rho dy < \infty. \tag{7.12}$$

We claim that $g_\infty = f$.

Indeed, by denoting $h := f - g_\infty$, then h satisfies

$$\begin{cases} \Delta h - \frac{y}{2} \cdot \nabla h - \frac{1}{p-1} h + p|w|^{p-1} h + \lambda_1 h = 0, & \text{in } \mathbb{R}^n \backslash B_R, \\ h = 0, & \text{on } \partial B_R. \end{cases} \tag{7.13}$$

Let ϕ_k be a smooth cutoff function such that $\phi_k = 1$ in B_k, $\phi_k = 0$ outsider B_{2k} and $|\nabla \phi_k| \leq C/k$. Multiplying both sides of (7.13) by $h\phi_k^2 \rho$ and integrating by parts, we get

$$\begin{aligned} 0 = & - \int_{\mathbb{R}^n \backslash B_R} |\nabla h|^2 \phi_k^2 \rho dy - \frac{1}{p-1} \int_{\mathbb{R}^n \backslash B_R} h^2 \phi_k^2 \rho dy \\ & + p \int_{\mathbb{R}^n \backslash B_R} |w|^{p-1} h^2 \phi_k^2 \rho dy + \lambda_1 \int_{\mathbb{R}^n \backslash B_R} h^2 \phi_k^2 \rho dy \\ & - 2 \int_{\mathbb{R}^n \backslash B_R} h \phi_k \nabla h \cdot \nabla \phi_k \rho dy. \end{aligned} \tag{7.14}$$

Recall that we have assumed that (7.7) holds, so

$$\int_{\mathbb{R}^n \backslash B_R} |\nabla h|^2 \phi_k^2 \rho dy \leq -2 \int_{\mathbb{R}^n \backslash B_R} h \phi_k \nabla h \cdot \nabla \phi_k \rho dy.$$

This yields

$$\int_{\mathbb{R}^n \backslash B_R} |\nabla h|^2 \phi_k^2 \rho dy \leq C \int_{\mathbb{R}^n} h^2 |\nabla \phi_k|^2 \rho dy,$$

where C is a positive constant independent of R. Letting $k \to \infty$, we conclude that

$$\int_{\mathbb{R}^n \backslash B_R} |\nabla h|^2 \rho dy = 0.$$

Hence $h = 0$ and the claim is proved.

By (7.8) and (7.11), there exists a positive constant C such that

$$|f| \leq C(1 + |y|)^{-\frac{2p}{p-1}}, \quad \text{in } \mathbb{R}^n. \tag{7.15}$$

Let

$$u(x,t) = (-t)^{-\frac{1}{p-1}} w\left(\frac{x}{\sqrt{-t}}\right), \ \tilde{f}(x,t) = (-t)^{\lambda_1 - \frac{1}{p-1}} f\left(\frac{x}{\sqrt{-t}}\right).$$

Then $\tilde{f}$ satisfies

$$\partial_t \tilde{f} = \Delta \tilde{f} + p|u|^{p-1}\tilde{f}, \quad \text{in } \mathbb{R}^n \times (-\infty, 0).$$

Combining (7.4), (7.15) and parabolic estimates, we see there exists a positive constant C depending only on n, p, w and f such that

$$|\nabla \tilde{f}(x,t)| + |\nabla^2 \tilde{f}(x,t)| \leq C(-t)^{\lambda_1+1}, \quad \text{in } (B_2(0)\backslash B_{1/4}(0)) \times (-2,0). \tag{7.16}$$

This implies

$$(1+|y|)^{\frac{2p}{p-1}+1}|\nabla f| + (1+|y|)^{\frac{2p}{p-1}+2}|\nabla^2 f| \leq C, \quad \text{in } \mathbb{R}^n \backslash B_1. \tag{7.17}$$

By (7.15) and (7.17), we see (7.5) holds. □

Now we can prove the main result in this section.

Theorem 7.7. *If w is a bounded non-constant solution of (SS) satisfying (7.4), then there is a variation w_s with $w_0 = w$ such that*

$$\lambda(w_s) < \lambda(w)$$

for all $s \neq 0$. In particular, w is not entropy stable.

Proof. Take the one-parameter variation $w_s = w + sf$ for $s \in [-2\varepsilon, 2\varepsilon]$, where f is the positive, first eigenfunction of the operator $\mathcal{L}$ defined in (LO).

By the proof of Theorem 6.1, we know that for any $x(s)$ and $t(s)$ with $x(0) = 0$ and $t(0) = -1$,

$$\frac{d^2}{ds^2} F_{x(s),t(s)}(w_s)|_{s=0} < 0. \tag{7.18}$$

We will use this to prove that w is not entropy stable. For this purpose, we define a function $G : \mathbb{R}^n \times \mathbb{R}^- \times [-2\varepsilon, 2\varepsilon] \to \mathbb{R}$ to be

$$G(x_0, t_0, s) = F_{x_0,t_0}(w_s). \tag{7.19}$$

We will show that there exists some $\varepsilon_1 > 0$ such that if $s \neq 0$ and $|s| \leq \varepsilon_1$, then

$$\lambda(w_s) := \sup_{x_0 \in \mathbb{R}^n, t_0 \in (-\infty,0)} G(x_0, t_0, s) < G(0,-1,0) = F_{0,-1}(w) = \lambda(w). \tag{7.20}$$

This will give the desired variation w_s.

The proof of (7.20) is divided into the following seven steps.

Step 1: G has a strict local maximum at $(0, -1, 0)$.

Since w is a bounded solution of the equation (SS), it follows from Proposition 3.3 that ∇G vanishes at $(0, -1, 0)$. Given $y_0 \in \mathbb{R}^n, a \in \mathbb{R}$ and $b \in \mathbb{R}\backslash\{0\}$, the second derivative of $G(sy_0, -(1+as), bs)$ at $s = 0$ is just

$$\frac{d^2}{ds^2} G(sy_0, -(1+as), bs)|_{s=0} = b^2 \frac{d^2}{ds^2} F_{x(s),t(s)}(w_s)|_{s=0}$$

with

$$x(s) = s\frac{y_0}{b}, \quad t(s) = -\left(1 + \frac{a}{b}s\right).$$

Thus

$$\begin{aligned}&\frac{d^2}{ds^2}G(sy_0,-(1+as),bs)|_{s=0}\\ =&-\frac{1}{2b^2}\int_{\mathbb{R}^n}|\nabla w\cdot y_0|^2\rho dy-\frac{a^2}{b^2}\int_{\mathbb{R}^n}\left(\frac{1}{p-1}w+\frac{y}{2}\cdot\nabla w\right)^2\rho dy.\end{aligned}$$

Similar to the reasoning used in the proof of Lemma 7.5, it is negative. In other words, the Hessian of G at $(0,-1,0)$ is negative definite. It follows that G has a strict local maximum at $(0,-1,0)$.

As a consequence, there exists $\varepsilon_2\in(0,\varepsilon)$ such that

$$G(x_0,t_0,s)<G(0,-1,0) \tag{7.21}$$

provided that $0<|x_0|^2+|\log(-t_0)|^2+s^2<\varepsilon_2^2$.

Step 2: $|\partial_s G|$ is bounded on $\mathbb{R}^n\times\mathbb{R}^-\times[-2\varepsilon,2\varepsilon]$.

By the definition of G and the first variation formula, we have

$$\begin{aligned}\partial_s G(x_0,t_0,s)=&(-t_0)^{\frac{p+1}{p-1}}\int_{\mathbb{R}^n}(\nabla w\cdot\nabla f)G(y-x_0,t_0)dy\\ &-(-t_0)^{\frac{p+1}{p-1}}\int_{\mathbb{R}^n}|w+sf|^{p-1}(w+sf)fG(y-x_0,t_0)dy\\ &+\frac{1}{(p-1)}(-t_0)^{\frac{2}{p-1}}\int_{\mathbb{R}^n}fwG(y-x_0,t_0)dy.\end{aligned} \tag{7.22}$$

The first two integrals can be estimated as follows. By Lemma 2.2 and Lemma 7.6,

$$|\nabla w(y)|\leq C(1+|y|)^{-\frac{p+1}{p-1}},\quad |\nabla f(y)|\leq C(1+|y|)^{-\frac{2p}{p-1}-1}.$$

Hence

$$\begin{aligned}|\nabla w(y)||\nabla f(y)|&\leq C(1+|y|)^{-\frac{4p}{p-1}}\\ &\leq C(1+|y|)^{-2\frac{p+1}{p-1}}\quad(\text{because }\frac{4p}{p-1}>2\frac{p+1}{p-1})\\ &\leq C|y|^{-2\frac{p+1}{p-1}}.\end{aligned}$$

Similarly, because

$$|w(y)|\leq C(1+|y|)^{-\frac{2}{p-1}},\quad 0\leq f(y)\leq C(1+|y|)^{-\frac{2p}{p-1}},$$

we also have

$$|w(y)+sf(y)|^pf(y)\leq C|y|^{-2\frac{p+1}{p-1}}.$$

Because p is supercritical, we have

$$2\frac{p+1}{p-1}<n,$$

which implies that $|y|^{-2\frac{p+1}{p-1}}$ is locally integrable on $\mathbb{R}^n$.

Then the integral we want to estimate is controlled by

$$(-t_0)^{\frac{p+1}{p-1}}\int_{\mathbb{R}^n}|y|^{-2\frac{p+1}{p-1}}(-4\pi t_0)^{-\frac{n}{2}}\exp\left(\frac{|x_0-y|^2}{4t_0}\right)dy$$

$$
\begin{aligned}
&=(4\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} |z|^{-2\frac{p+1}{p-1}} \exp\left(-\frac{1}{4}\left|\frac{x_0}{\sqrt{-t_0}}-z\right|^2\right) dz \quad (\text{by letting } z=y/\sqrt{-t_0})\\
&\le C(n)\left(1+\frac{|x_0|}{\sqrt{-t_0}}\right)^{-2\frac{p+1}{p-1}}\\
&\le C(n).
\end{aligned}
$$

In the above, to estimate the last integral, we consider two cases separately, $|x_0|/\sqrt{-t_0} \le 10$ and $|x_0|/\sqrt{-t_0} > 10$. For the first case, the integral is bounded by a dimensional constant $C(n)$, by using the local integrability of $|z|^{-2\frac{p+1}{p-1}}$ and the decay of the exponential function at infinity. For the second case, the integral can be estiamted by decomposing $\mathbb{R}^n$ into two domains, $B_{|x_0|/(2\sqrt{-t_0})}(x_0/\sqrt{-t_0})$ and $\mathbb{R}^n \setminus B_{|x_0|/(2\sqrt{-t_0})}(x_0/\sqrt{-t_0})$.

As above, the third integral in (7.22) is controlled by

$$
\begin{aligned}
&(-t_0)^{\frac{2}{p-1}} \int_{\mathbb{R}^n} (1+|y|)^{-2\frac{p+1}{p-1}} (-4\pi t_0)^{-\frac{n}{2}} \exp\left(\frac{|x_0-y|^2}{4t_0}\right) dy\\
&= (-t_0)^{\frac{2}{p-1}} \int_{\mathbb{R}^n} \left(1+\sqrt{-t_0}|z|\right)^{-2\frac{p+1}{p-1}} (4\pi)^{-\frac{n}{2}} \exp\left(-\frac{1}{4}\left|\frac{x_0}{\sqrt{-t_0}}-z\right|^2\right) dz.
\end{aligned}
$$

If $|t_0| \le 1$, this integral is bounded by

$$
\int_{\mathbb{R}^n} \exp\left(-\frac{1}{4}\left|\frac{x_0}{\sqrt{-t_0}}-z\right|^2\right) dz \le C(n).
$$

If $|t_0| \ge 1$, this integral is bounded by

$$
(-t_0)^{-1} \int_{\mathbb{R}^n} |z|^{-2\frac{p+1}{p-1}} \exp\left(-\frac{1}{4}\left|\frac{x_0}{\sqrt{-t_0}}-z\right|^2\right) dz.
$$

As in the first case, it is bounded by a dimensional constant $C(n)$.

Step 3: Because $G(x_0,t_0,0) = F_{x_0,t_0}(w)$ and $G(0,-1,0) = F_{0,-1}(w) = \lambda(w)$, by Lemma 7.5, $G(x_0,t_0,0)$ has a strict global maximum at $(0,-1)$. In fact, there exists a positive constant $\delta > 0$ such that

$$
G(x_0,t_0,0) < G(0,-1,0) - \delta \tag{7.23}
$$

for all x_0, t_0 satisfying $\varepsilon_2^2/4 < x_0^2 + (\log(-t_0))^2$.

By Step 2, we have

$$
G(x_0,t_0,s) \le G(x_0,t_0,0) + C(n)|s|.
$$

Combining this inequality with (7.23), we see for all x_0, t_0 satisfying $\varepsilon_2^2/4 < x_0^2 + (\log(-t_0))^2$, there exist $\varepsilon_3 > 0$ such that for any $|s| < \varepsilon_2$,

$$
\begin{aligned}
G(x_0,t_0,s) &\le G(x_0,t_0,0) + C(n)|s|\\
&\le G(0,-1,0) - \delta + C(n)\varepsilon_3\\
&< \lambda(w).
\end{aligned}
$$

Combining this fact with the result in Step 1, we finish the proof. □

8. The energy of self similar solutions with a natural decay

In this section, we turn our attention to the energy of functions in $\mathcal{B}_n$. We will prove that if $w \in \mathcal{B}_n$ and there exists a positive constant C such that (7.4) holds, i.e.

$$|w(y)| \le C(1+|y|)^{-\frac{2}{p-1}}, \quad \text{in } \mathbb{R}^n,$$

then $E(w) > E(\kappa)$ strictly.

Lemma 8.1 (Blow up criteria)**.** *Let w be a solution of (RF). Assume for any $M > 0$, there exists a positive constant $C(M)$ such that*

$$|w| \le C(M) \quad \text{in } \mathbb{R}^n \times [0, M].$$

Let

$$I(w(\tau)) = -2E(w(\cdot,\tau)) + \frac{p-1}{p+1}\left[\int_{\mathbb{R}^n} w^2(\cdot,\tau)\rho dy\right]^{\frac{p+1}{2}}.$$

If

$$I(w(\tau_0)) > 0$$

for some $\tau_0 \in (0, \infty)$, then w blows up at some finite time.

Proof. This is exactly [33, Proposition 2.1]. □

Lemma 8.2. *Suppose*

- *w is a bounded solution of the equation (SS) such that (7.4) holds;*
- *f is the positive, first eigenfunction of the linearized operator $\mathcal{L}$ at w.*

There exists a positive constant s_* such that if $0 < s < s_*$, then

$$\Delta(w+sf) - \frac{y}{2}\cdot\nabla(w+sf) - \frac{1}{p-1}(w+sf) + |w+sf|^{p-1}(w+sf) > 0, \quad \text{in } \mathbb{R}^n.$$

Proof. By the assumptions, we have

$$\begin{aligned} &\Delta(w+sf) - \frac{y}{2}\cdot\nabla(w+sf) - \frac{1}{p-1}(w+sf) + |w+sf|^{p-1}(w+sf) \\ =&|w+sf|^{p-1}(w+sf) - |w|^{p-1}w - sp|w|^{p-1}f - s\lambda_1 f \end{aligned}$$

with λ_1 being the first eigenvalue of $\mathcal{L}$. Take the decomposition

$$\mathbb{R}^n = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4,$$

where

$$\begin{aligned} \Omega_1 =& \{x \in \mathbb{R}^n : w(x) \ge 0\}, \\ \Omega_2 =& \{x \in \mathbb{R}^n : w(x) < 0, w(x) + sf(x) \ge 0\}, \\ \Omega_3 =& \left\{x \in \mathbb{R}^n : w(x) + sf(x) < 0, |w(x)| < (\frac{-\lambda_1}{p})^{\frac{1}{p-1}}\right\}, \\ \Omega_4 =& \left\{x \in \mathbb{R}^n : w(x) + sf(x) < 0, |w(x)| \ge (\frac{-\lambda_1}{p})^{\frac{1}{p-1}}\right\}. \end{aligned}$$

Case 1. For any $a \geq 0, b \geq 0$, we have $(a+b)^p \geq a^p + pa^{p-1}b$. Therefore,

$$|w+sf|^{p-1}(w+sf) - |w|^{p-1}w - sp|w|^{p-1}f - s\lambda_1 f > 0, \quad \text{in } \Omega_1.$$

Case 2. If $x \in \Omega_2$, then $|w(x)| \leq sf(x)$ and

$$|w(x)+sf(x)|^{p-1}(w(x)+sf(x)) - |w(x)|^{p-1}w(x) \geq 0. \tag{8.1}$$

By Lemma 7.6, f is bounded in $\mathbb{R}^n$. Hence there exists a positive constant $s_{*,1}$ such that

$$p|w(x)|^{p-1} < -\lambda_1$$

provided that $0 < s < s_{*,1}$. Combining this with (8.1), we have

$$|w+sf|^{p-1}(w+sf) - |w|^{p-1}w - sp|w|^{p-1}f - s\lambda_1 f > 0, \quad \text{in } \Omega_2$$

provided that $0 < s < s_{*,1}$.

Case 3. If $x \in \Omega_3$, it is easy to see that

$$|w(x)+sf(x)|^{p-1}(w(x)+sf(x)) - |w(x)|^{p-1}w(x) > 0$$

and

$$p|w(x)|^{p-1} + \lambda_1 < 0.$$

Hence

$$|w+sf|^{p-1}(w+sf) - |w|^{p-1}w - sp|w|^{p-1}f - s\lambda_1 f > 0, \quad \text{in } \Omega_3.$$

Case 4. Finally, for $x \in \Omega_4$,

$$\begin{aligned}
&|w(x)+sf(x)|^{p-1}(w(x)+sf(x)) - |w(x)|^{p-1}w(x) - sp|w(x)|^{p-1}f(x) - s\lambda_1 f(x)\\
=&|w(x)+sf(x)|^{p-1}(w(x)+sf(x)) - |w(x)|^{p-1}w(x) - sp|w(x)+sf(x)|^{p-1}f(x)\\
&+ sp|w(x)+sf(x)|^{p-1}f(x) - sp|w(x)|^{p-1}f(x) - s\lambda_1 f(x)\\
\geq& sp|w(x)+sf(x)|^{p-1}f(x) - sp|w(x)|^{p-1}f(x) - s\lambda_1 f(x).
\end{aligned}$$

Because $|w|$ is bounded from below and above in Ω_4, there exists a positive constant C depending only on n, p, λ_1 such that

$$sp|w+sf|^{p-1}f - sp|w|^{p-1}f - s\lambda_1 f \geq -Cs^2f^2 - s\lambda_1 f, \quad \text{in } \Omega_4.$$

Hence there exists a positive constant $s_{*,2}$ such that

$$|w+sf|^{p-1}(w+sf) - |w|^{p-1}w - sp|w|^{p-1}f - s\lambda_1 f > 0, \quad \text{in } \Omega_4$$

provided that $0 < s < s_{*,2}$.

The proof is finished by choosing $s_* = \min\{s_{*,1}, s_{*,2}\}$. □

Lemma 8.3. *Assume w and f satisfy the condition in Lemma 8.2. If $\widetilde{w}$ is the solution to the Cauchy problem*

$$\begin{cases} \partial_\tau \widetilde{w} = \Delta \widetilde{w} - \frac{y}{2} \cdot \nabla \widetilde{w} - \frac{1}{p-1}\widetilde{w} + |\widetilde{w}|^{p-1}\widetilde{w}, \\ \widetilde{w}(\cdot, 0) = w + sf, \end{cases} \tag{8.2}$$

where $s < s_*$ is a small positive constant, then $\widetilde{w}$ blows up in finite time.

Proof. Without loss of generality, we may assume that $\{w > 0\} \neq \emptyset$. Otherwise, we can consider $-w$ instead of w. Assume by the contrary that $\widetilde{w}$ exists in $\mathbb{R}^n \times (0, +\infty)$.

Step 1: We claim that

$$\partial_\tau \widetilde{w} > 0, \quad \text{in } \mathbb{R}^n \times (0, +\infty). \tag{8.3}$$

Since $\widetilde{w}$ satisfies (8.2), we have

$$\partial_\tau \widetilde{w}(\cdot, 0) = \Delta(w + sf) - \frac{y}{2} \cdot \nabla(w + sf) - \frac{1}{p-1}(w + sf) + |w + sf|^{p-1}(w + sf).$$

By Lemma 8.2, we have

$$\partial_\tau \widetilde{w}(\cdot, 0) > 0, \quad \text{in } \mathbb{R}^n.$$

We also have the equation for $\partial_\tau \widetilde{w}$,

$$\partial_\tau \partial_\tau \widetilde{w} = \Delta \partial_\tau \widetilde{w} - \frac{y}{2} \cdot \nabla \partial_\tau \widetilde{w} - \frac{1}{p-1} \partial_\tau \widetilde{w} + p|\widetilde{w}|^{p-1} \partial_\tau \widetilde{w}, \quad \text{in } \mathbb{R}^n \times (0, +\infty).$$

By the maximum principle, we get (8.3).

Since $\widetilde{w}$ is monotone increasing with respect to τ, there exists a function $\widetilde{w}_\infty$ such that $\widetilde{w}(y, \tau)$ tends to $\widetilde{w}_\infty$ as $\tau \to \infty$.

Step 2: $\widetilde{w}_\infty \neq +\infty$ is a weak solution of (SS).

Let $\{\tau_k\}$ be a sequence such that $\lim_{k\to\infty} \tau_k = +\infty$. Without loss of generality, we may assume that $\lim_{k\to\infty}(\tau_{k+1} - \tau_k) = +\infty$. By the monotonicity formula, the energy of $\widetilde{w}(\tau)$, $E(\widetilde{w}(\tau))$ is decreasing in τ. Thus for any $\tau > 0$,

$$E(\widetilde{w}(\tau)) \leq E(\widetilde{w}(0).$$

By Lemma 8.1, for any $\tau > 0$,

$$I(\widetilde{w}(\tau)) = -2E(\widetilde{w}(\tau)) + \frac{p-1}{p+1} \left[\int_{\mathbb{R}^n} \widetilde{w}(\tau)^2 \rho dy \right]^{\frac{p+1}{2}} \leq 0.$$

Hence

$$E(\widetilde{w}(\tau)) \geq \frac{p-1}{2(p+1)} \left[\int_{\mathbb{R}^n} \widetilde{w}(\tau)^2 \rho dy \right]^{\frac{p+1}{2}} \geq 0.$$

In particular, $E(\widetilde{w}(\tau))$ is bounded in $[0, +\infty)$. This in turn implies that $\|\widetilde{w}(\tau)\|_{L^2_w(\mathbb{R}^n)}$ is bounded.

Because

$$\frac{d}{d\tau} E(\widetilde{w}(\tau)) = -\int_{\mathbb{R}^n} (\partial_\tau \widetilde{w}(\tau))^2 \rho dy,$$

there exists a positive constant C independent of k such that

$$\int_{\tau_k}^{\tau_{k+1}} \int_{\mathbb{R}^n} (\partial_\tau \widetilde{w})^2 \rho dy d\tau \leq C.$$

Testing (8.2) with $\widetilde{w}$, we obtain

$$\int_{\mathbb{R}^n} \widetilde{w}(\tau) \partial_\tau \widetilde{w}(\tau) \rho dy = \int_{\mathbb{R}^n} \left[-|\nabla \widetilde{w}(\tau)|^2 - \frac{1}{p-1} \widetilde{w}(\tau)^2 + |\widetilde{w}(\tau)|^{p+1} \right] \rho dy.$$

Combing these two estimates with the boundedness of $\|\widetilde{w}(\tau)\|_{L^2_w(\mathbb{R}^n)}$ and $E(\widetilde{w}(\tau))$, an application of Cauchy-Schwarz inequality gives

$$\frac{p-1}{p+1}\int_{\tau_k}^{\tau_{k+1}}\int_{\mathbb{R}^n}|\widetilde{w}|^{p+1}\rho dyd\tau \le C.$$

Because $E(\widetilde{w}(\tau))$ is bounded in $[0,+\infty)$, this implies that

$$\int_{\tau_k}^{\tau_{k+1}}\int_{\mathbb{R}^n}\left[|\nabla\widetilde{w}|^2+\widetilde{w}^2\right]\rho dyd\tau \le C.$$

Therefore, there exists a function $\hat{w}$ such that $\widetilde{w}_k(\cdot,\tau)=\widetilde{w}(\cdot,\tau+\tau_k)\to\hat{w}_\infty$ weakly in $L^2_{loc}((0,\infty),H^1_w(\mathbb{R}^n))\cap L^2_{loc}((0,\infty),L^{p+1}_w(\mathbb{R}^n))$. Moreover, $\hat{w}_\infty$ is a weak solution of (8.2). Since we have proved that $\widetilde{w}(\cdot,\tau)$ is increasing with respect to τ, then $\hat{w}_\infty=\widetilde{w}_\infty$ and $\widetilde{w}_\infty\in H^1_w(\mathbb{R}^n)\cap L^{p+1}_w(\mathbb{R}^n)$ is a weak solution of (SS).

Step 3: $\widetilde{w}$ blows up at a finite time.

Let Ω_0 be a connected component of $\{w>0\}$, then w satisfies

$$\begin{cases}\Delta w-\frac{y}{2}\cdot\nabla w-\frac{1}{p-1}w+w^p=0, & \text{in } \Omega_0,\\ w=0, & \text{on } \partial\Omega_0.\end{cases}$$

Since $\widetilde{w}_\infty$ is a weak solution of(SS), we get from Step 1 that the restriction of $\widetilde{w}_\infty$ on Ω_0 is a weak solution of (SS) such that

$$\widetilde{w}_\infty > w \quad \text{in } \Omega_0,$$

For each $R\gg 1$ such that $B_R(0)\cap\Omega_0\neq\emptyset$. Let $\lambda_{1,R,D}$ be the first eigenvalue of the eigenvalue problem

$$\begin{cases}\Delta\phi-\frac{y}{2}\cdot\nabla\phi-\frac{1}{p-1}\phi+pw^{p-1}\phi+\lambda\phi=0, & \text{in } \Omega_0\cap B_R(0),\\ \phi=0, & \text{on } \partial(\Omega_0\cap B_R(0)),\end{cases}$$

then [1, Corollary 2.4] yields $\lambda_{1,R,D}$ is positive. Let $\lambda_{1,D}$ be the first eigenvalue of the eigenvalue problem

$$\begin{cases}\Delta\phi-\frac{y}{2}\cdot\nabla\phi-\frac{1}{p-1}\phi+pw^{p-1}\phi+\lambda\phi=0, & \text{in } \Omega_0,\\ \phi=0, & \text{on } \partial\Omega_0.\end{cases}$$

Since the first Dirichlet eigenvalue is decreasing with respect to the domain and w is bounded, then $\lambda_{1,D}$ is finite. Moreover, we have $\lim_{R\to\infty}\lambda_{1,R,D}=\lambda_{1,D}$ and $\lambda_{1,D}\ge 0$. On the other hand, by choosing w as a test function, it is easy to see that $\lambda_{1,D}<0$, this contradiction yields that $\widetilde{w}$ can not exist on $\mathbb{R}^n\times(0,+\infty)$. □

Lemma 8.4. *Let τ_1 be the first blow up time of $\widetilde{w}$. There exist $R>0$ and $C>0$ such that*

$$|\widetilde{w}|\le C, \quad \textit{in } (\mathbb{R}^n\backslash B_R(0))\times(0,\tau_1). \tag{8.4}$$

In particular, if we use Σ to denote the blow up set of $\widetilde{w}$, then Σ is a compact subset of $\mathbb{R}^n$.

Proof. Since w satisfies (7.4), by Lemma 2.2 and Lemma 7.6, there exists a positive constant C such that

$$(1+|y|)^{\frac{2}{p-1}}|w(y)|+(1+|y|)^{1+\frac{2}{p-1}}|\nabla w(y)|\le C, \quad \text{in } \mathbb{R}^n$$

and

$$(1+|y|)^{\frac{2p}{p-1}}|f(y)|+(1+|y|)^{1+\frac{2p}{p-1}}|\nabla f(y)|\leq C,\quad \text{in } \mathbb{R}^n.$$

Thus given $\delta_0>0,\varepsilon>0$, there exists R such that if $|y_0|>R$ and $(z_0,\tau_0)\in B_{\delta_0}(y_0)\times(\tau_1-\delta_0^2,\tau_1+\delta_0^2)$, then

$$\mathcal{E}_{(z_0,\tau_0)}(w+sf)<\varepsilon,$$

where

$$\begin{aligned}\mathcal{E}_{(z_0,\tau_0)}(w+sf)=&\frac{1}{2}\tau_0^{\frac{2}{p-1}-\frac{n}{2}+1}\int_{\mathbb{R}^n}|\nabla(w+sf)|^2e^{-\frac{|y-z_0|^2}{4\tau_0}}dy\\&-\frac{1}{p+1}\tau_0^{\frac{2}{p-1}-\frac{n}{2}+1}\int_{\mathbb{R}^n}|w+sf|^{p+1}e^{-\frac{|y-z_0|^2}{4\tau_0}}dy\\&+\frac{1}{2(p-1)}\tau_0^{\frac{2}{p-1}-\frac{n}{2}}\int_{\mathbb{R}^n}(w+sf)^2e^{-\frac{|y-z_0|^2}{4\tau_0}}dy.\end{aligned}$$

Following the arguments in the proof of [17, Theorem 3.1] (starting from formula (3.12)) or [28, Theorem 4.1], we obtain

$$|\widetilde{w}|\leq C_*\delta_0^{-\frac{2}{p-1}},\quad \text{in } B_{\delta_0/2}(y_0)\times(\tau_1-\delta_0^2,\tau_1),$$

where C_* is a universal positive constant. The estimate (8.4) follows from this estimate and the definition of τ_1. □

Proposition 8.5. *If $w\in\mathcal{B}_n$ and (7.4) holds, then*

$$E(w)\geq E(\kappa).$$

Proof. The proof will be divided into several steps.

Step 1: Let $\widetilde{w}$ be the solution of (8.2) and let τ_1 be the first blow up time. Then there exists a positive constant C such that

$$|\widetilde{w}|\leq C(\tau_1-\tau)^{-\frac{1}{p-1}},\quad \text{in } \mathbb{R}^n\times(0,\tau_1). \tag{8.5}$$

By Lemma 8.4, the singular set Σ is a compact subset of $\mathbb{R}^n$. Fix $R>0$ such that $\Sigma\subset B_{R/2}(0)$. By Lemma 8.2 and the maximum principle, there exist $0<\tau_0<\tau_1$ and a positive constant c_0 such that

$$\partial_\tau\widetilde{w}\geq c_0,\quad \text{on } \left(B_R(0)\times\{\tau=\tau_0\}\right)\cup\left(\partial B_R(0)\times[\tau_0,\tau_1]\right). \tag{8.6}$$

Moreover, by the choice of R, there exists a positive constant C_1 such that

$$|\widetilde{w}|^p\leq C_1,\quad \text{on } \left(B_R(0)\times\{\tau=\tau_0\}\right)\cup\left(\partial B_R(0)\times[\tau_0,\tau_1]\right). \tag{8.7}$$

Combining (8.6) and (8.7), we see there exists a small positive constant ε such that

$$\partial_\tau\widetilde{w}\geq\varepsilon|\widetilde{w}|^p,\quad \text{on } \left(B_R(0)\times\{\tau=\tau_0\}\right)\cup\left(\partial B_R(0)\times[\tau_0,\tau_1]\right). \tag{8.8}$$

Because $\widetilde{w}$ satisfies (8.2), we get from Kato inequality that

$$\partial_\tau|\widetilde{w}|\leq\Delta|\widetilde{w}|-\frac{y}{2}\cdot\nabla|\widetilde{w}|-\frac{1}{p-1}|\widetilde{w}|+p|\widetilde{w}|^p. \tag{8.9}$$

Then the chain rule gives

$$\partial_\tau|\widetilde{w}|^p\leq\Delta|\widetilde{w}|^p-\frac{y}{2}\cdot\nabla|\widetilde{w}|^p+p|\widetilde{w}|^{p-1}|\widetilde{w}|^p-[p(p-1)|\widetilde{w}|^{p-2}|\nabla|\widetilde{w}||^2+|\widetilde{w}|^p].$$

Let $v(y,\tau)=\partial_\tau\widetilde{w}(y,\tau)-\varepsilon|\widetilde{w}(y,\tau)|^p$, where ε is a small positive constant satisfying (8.8). Then v satisfies

$$\begin{cases} \partial_\tau v-\Delta v+\frac{y}{2}\cdot\nabla v+\frac{1}{p-1}v-p|\widetilde{w}|^{p-1}v\geq 0, & \text{in } B_R\times(\tau_0,\tau_1), \\ v>0 & \text{on } \left(B_R(0)\times\{\tau=\tau_0\}\right)\cup\left(\partial B_R(0)\times[\tau_0,\tau_1]\right). \end{cases} \tag{8.10}$$

By the maximum principle,

$$v>0, \quad \text{in } B_R\times(\tau_0,\tau_1).$$

This is equivalent to

$$\partial_\tau\widetilde{w}\geq\varepsilon|\widetilde{w}|^p, \quad \text{in } B_R\times(\tau_0,\tau_1). \tag{8.11}$$

For any $\tau_0<\tau<\tau_1-\delta$, integrating (8.11) from τ to $\tau_1-\delta$ gives

$$|\widetilde{w}(\tau_1-\delta)|^{1-p}-|\widetilde{w}(\tau)|^{1-p}\leq-(p-1)\varepsilon(\tau_1-\delta-\tau).$$

Hence

$$|\widetilde{w}(y,\tau)|\leq M(\tau_1-\delta-\tau)^{-\frac{1}{p-1}}, \quad \text{in } B_R\times(\tau,\tau_1-\delta),$$

where $M=((p-1)\varepsilon)^{-1/(p-1)}$.

Sending $\delta\to 0$, we conclude that

$$|\widetilde{w}(y,\tau)|\leq M(\tau_1-\tau)^{-\frac{1}{p-1}}, \quad \text{in } B_R\times(\tau_0,\tau_1),$$

Combining this with Lemma 8.4, we get (8.5).

Step 2: There exists a smooth solution $\widetilde{w}_*$ of (SS) such that $E(\widetilde{w}_*)<E(w)$.

Assume (y_1,τ_1) is a blow up point. Set

$$\hat{w}(z,\varsigma)=(1-e^{\tau-\tau_1})^{\frac{1}{p-1}}w\left(e^{\frac{\tau-\tau_1}{2}}y_1+(1-e^{\tau-\tau_1})^{1/2}z,\tau\right),$$

where

$$\varsigma=-\log(e^{-\tau}-e^{-\tau_1}).$$

Then $\hat{w}$ is a solution to (RF) with initial data

$$\hat{w}(z,-\log(1-e^{-\tau_1}))=(1-e^{-\tau_1})^{\frac{1}{p-1}}[w(\tilde{y})+sf(\tilde{y})],$$

where $\tilde{y}=e^{-\tau_1/2}y_1+\sqrt{1-e^{-\tau_1}}z$.

Let $\{\varsigma_k\}$ be a sequence such that $\lim_{k\to\infty}\varsigma_k=\infty$ and let $w_k(z,\varsigma)=w(z,\varsigma+\varsigma_k)$. Similar to the proof of [21, Proposiiton 4], $\lim_{k\to\infty}\widetilde{w}_k=\widetilde{w}_*$ for some solution of (SS) uniformly on compact subsets. By applying the monotonicity formula (see [21, Proposiiton 3]), we have

$$E(\hat{w}(z,-\log(1-e^{-\tau_1})))>E(\widetilde{w}_*). \tag{8.12}$$

We compute

$$\begin{aligned}
&E(\hat{w}(z,-\log(1-e^{-\tau_1})))\\
=&\int_{\mathbb{R}^n}\left[\frac{1}{2}|\nabla\hat{w}(z,-\log(1-e^{-\tau_1}))|^2-\frac{1}{p+1}|\hat{w}(z,-\log(1-e^{-\tau_1}))|^{p+1}\right]\rho(z)dz\\
&+\frac{1}{2(p-1)}\int_{\mathbb{R}^n}\hat{w}(z,-\log(1-e^{-\tau_1}))^2\rho(z)dz\\
=&\frac{1}{2}(1-e^{-\tau_1})^{\frac{p+1}{p-1}}\int_{\mathbb{R}^n}|\nabla w(y)+s\nabla f(y)|^2G(y-e^{-\tau_1/2}y_1,e^{-\tau_1}-1)dy\\
&-\frac{1}{p+1}(1-e^{-\tau_1})^{\frac{p+1}{p-1}}\int_{\mathbb{R}^n}|w(y)+sf(y)|^{p+1}]G(y-e^{-\tau_1/2}y_1,e^{-\tau_1}-1)dy\\
&+\frac{1}{2(p-1)}(1-e^{-\tau_1})^{\frac{2}{p-1}}\int_{\mathbb{R}^n}|w(y)+sf(y)|^2G(y-e^{-\tau_1/2}y_1,e^{-\tau_1}-1)dy\\
=&F_{e^{-\tau_1/2}y_1,e^{-\tau_1}-1}(w+sf).
\end{aligned}$$

By the proof of Theorem 7.7,

$$F_{e^{-\tau_1/2}y_1,e^{-\tau_1}-1}(w+sf)<E(w)$$

provided that s is small enough. Combining this with (8.12), we conclude the proof of this step.

Step 3: The function $\widetilde{w}_*$ in the previous step is κ.

Since $\widetilde{w}$ satisfies (8.11), a direct calculation using the definition of $\hat{w}$ shows that

$$\frac{1}{p-1}\hat{w}+\frac{z}{2}\cdot\nabla\hat{w}+\partial_\varsigma\hat{w}\geq\varepsilon|\hat{w}|^p>0,\quad\text{in }\mathbb{R}^n\times(-\log(1-e^{-\tau_0}),\infty).$$

By Step 1, there exists a positive constant C_3 such that

$$|\hat{w}|\leq C_3,\quad\text{in }\mathbb{R}^n\times(-\log(1-e^{-\tau_0}),\infty).$$

Let $\{w_k\}$ be the functions defined in Step 2. Then $\partial_\varsigma\hat{w}$ converges to 0 uniformly on compact subsets. By the above analysis, we conclude that $\widetilde{w}_*$ is a solution of (SS) satisfying

$$|\widetilde{w}_*|\leq C_3\quad\text{in }\mathbb{R}^n,$$

and

$$\frac{1}{p-1}\widetilde{w}_*+\frac{z}{2}\cdot\nabla\widetilde{w}_*\geq 0,\quad\text{in }\mathbb{R}^n.$$

By Proposition 5.1, $\widetilde{w}_*$ is a constant solution of (SS). Moreover, Theorem 3.1 in [17] implies $\widetilde{w}_*\neq 0$, so $\widetilde{w}_*\equiv\kappa$.

Combining Step 1, Step 2 and Step 3, the proof is complete. □

9. Constant solutions have the lowest energy

In this section, we will combine the Federer type dimension reduction arguments with the results obtained in the previous sections to prove that the positive constant solution of (SS) has the lowest energy among functions in $\mathcal{B}_n$.

Lemma 9.1. *Let $\mathcal{B}_{n,m}$ be the set defined in (1.2). There exists a positive constant C depending only on n, p, m such that if $w \in \mathcal{B}_{n,m}$, then*

$$C^{-1} \leq E(w) \leq C.$$

Proof. It follows from (3.5) and (E) that

$$E(w) = \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}^n} |w|^{p+1} \rho dy. \tag{9.1}$$

By Kato inequality (see [29]), we have

$$\Delta w \cdot sgn(w) \leq \Delta |w|, \quad \text{in } D'(\mathbb{R}^n),$$

here $sgn(w)$ is the sign function. Since w satisfies (SS), we have

$$\Delta |w| - \frac{y}{2} \cdot \nabla |w| - \frac{1}{p-1}|w| + |w|^p \geq 0.$$

Testing this inequality with ρ and integrating over $\mathbb{R}^n$, we get

$$-\frac{1}{p-1} \int_{\mathbb{R}^n} |w| \rho dy + \int_{\mathbb{R}^n} |w|^p \rho dy \geq 0.$$

Therefore, we have either

$$\sup_{\mathbb{R}^n} w \geq \kappa \geq \inf_{\mathbb{R}^n} w,$$

or

$$\inf_{\mathbb{R}^n} w \leq -\kappa \leq \sup_{\mathbb{R}^n} w.$$

In particular, there exists a point $y_0 \in \mathbb{R}^n$ such that $w(y_0) = \kappa$. Since $w \in \mathcal{B}_{n,m}$, it follows from Lemma 2.1 that there exists a constant $r_1 > 0$ depending only on n, p and m such that

$$w \geq \left(\frac{1}{2(p-1)}\right)^{\frac{1}{p-1}}, \quad \text{in } B_{r_1}(y_0).$$

Therefore, there exists a positive constant C depending only on n, p and m such that

$$\int_{\mathbb{R}^n} |w|^{p+1} \rho dy \geq \int_{B_{r_1}(y_0)} |w|^{p+1} \rho dy \geq C. \tag{9.2}$$

By (9.1) and (9.2), we have

$$E(w) \leq \left(\frac{1}{2} - \frac{1}{p+1}\right) C. \tag{9.3}$$

Next, since $w \in \mathcal{B}_{n.m}$, then

$$E(w) = \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}^n} |w|^{p+1} \rho dy \leq \left(\frac{1}{2} - \frac{1}{p+1}\right) m^{p+1}. \tag{9.4}$$

Combining (9.3) and (9.4), we finish the proof. □

Lemma 9.2. *There exists $w_0 \in \mathcal{B}_{n,m}$ such that*

$$E(w_0) = \inf_{w \in \mathcal{B}_{n,m}} E(w).$$

Proof. Let $\{w_i\}$ be a sequence such that

$$\lim_{i\to\infty} E(w_i) = \inf_{w\in\mathcal{B}_{n,m}} E(w).$$

It follows from Lemma 2.1 that $|w_i|+|\nabla w_i|+|\nabla^2 w_i|+|\nabla^3 w_i|$ are uniformly bounded. By the Arzelá-Ascoli theorem, we know that there exists a function w_0 such that $\lim_{i\to\infty} w_i = w_0$. Since the functions w_i converge to w_0 uniformly on compact subsets of $\mathbb{R}^n$, then w_0 is a bounded solution of (SS) such that $\|w_0\|_{l^\infty(\mathbb{R}^n)} \le m$. The convergence also implies that

$$E(w_0) = \lim_{i\to\infty} E(w_i),$$

Using the functions $\{w_i\}$ are uniformly bounded once again, we know that there exists a positive constant R_0 such that for each i,

$$\left(\frac{1}{2}-\frac{1}{p+1}\right)\int_{\mathbb{R}^n\setminus B_{R_0}(0)} |w_i|^{p+1}\rho dy \le (4C)^{-1},$$

here C is the constant in Lemma 9.1. Thus we have

$$\left(\frac{1}{2}-\frac{1}{p+1}\right)\int_{B_{R_0}} |w_i|^{p+1}\rho dy \ge (4C)^{-1},$$

so w_0 can not be zero and it attains the minima in $\mathcal{B}_{n,m}$. □

Proof of Theorem 1.1. By Lemma 9.2, there exists $w_0 \in \mathcal{B}_{n,m}$ such that

$$E(w_0) = \inf_{w\in\mathcal{B}_{n,m}} E(w). \tag{9.5}$$

Since $\kappa \in \mathcal{B}_{n,m}$, it is clear that

$$E(w_0) \le E(\kappa). \tag{9.6}$$

We will prove by induction that

$$E(w_0) = E(\kappa). \tag{9.7}$$

If $n = 1, 2$, by Theorem 2.7, $w_0 \equiv \kappa$ or $-\kappa$. Hence (9.7) trivially holds.

Let us assume that Theorem 1.1 holds for $n-1$ with $n \ge 3$. That is to say, for any $w \in \mathcal{B}_{n-1,m}$,

$$E(w) \ge E(\kappa), \tag{9.8}$$

and the inequality is strict unless $w \equiv \pm\kappa$. We want to show that Theorem 1.1 holds for dimension n.

If w_0 is the constant solution of (SS), then we are done. Therefore, we assume w_0 is not the constant solution of (SS). Set $u_0(x,t) = (-t)^{-1/(p-1)} w_0(x/\sqrt{-t})$. Since $w_0 \in \mathcal{B}_{n,m}$, u_0 is an ancient solution of (F) satisfying

$$|u_0| \le m(-t)^{-\frac{1}{p-1}}, \quad \text{in } \mathbb{R}^n \times (-\infty, 0).$$

Next we divide the discussion into two cases.

Case 1. u_0 blows up at some point $x_0 \ne 0$.

Blowing up u_0 at $(x_0, 0)$ gives a $w_1 \in \mathcal{B}_{n,m}$, which is translation invariant in the x_0 direction. Equivalently, $w_1 \in \mathcal{B}_{n-1,m}$. By the inductive assumption,

$$E(w_0) \geq E(\kappa).$$

By Lemma 2.5, $\Theta(x_0, 0; u_0) \leq \Theta(0, 0; u_0)$. Hence

$$E(w_0) = \Theta(0, 0; u_0) \geq E(w_1) \geq E(\kappa).$$

Case 2. u_0 does not blow up at any $x_0 \neq 0$.

By this assumption, there exist C_1, δ_0 such that

$$|u_0| \leq C_1, \quad \text{in } \{1/4 < |x| < 1/2\} \times (-\delta_0, 0).$$

This is equivalent to

$$|w_0| \leq C_1 |y|^{-\frac{2}{p-1}}, \quad \text{in } |y| \geq \frac{1}{4\sqrt{\delta_0}}.$$

Thus $w_0 \in \mathcal{B}_{n,m}$ satisfies (7.4). By Proposition 8.5,

$$E(w_0) > E(\kappa),$$

which contradicts (9.6) again.

Finally, let us show that $w_0 \equiv \kappa$ (or $-\kappa$). First by the analysis of the above Case 2, u_0 must blow up at some point $x_0 \neq 0$. Next, by the analysis of Case 1 and inductive assumption, $\Theta(x_0) = E(\kappa)$. Then by Lemma 2.5, u_0 is translation invariant in the x_0 direction. This implies $w_0 \in \mathcal{B}_{n-1,m}$. Using the inductive assumption again, we deduce that $w_0 \equiv \kappa$ or $-\kappa$. □

10. Proof of Theorem 1.3 and Proposition 1.9

In this section, we prove Theorem 1.3 and Proposition 1.9.

10.1. Energy gap. First, we prove that not only does the positive constant solution achieve the lowest weighted energy among functions in $\mathcal{B}_{n,m}$, but there is a gap to the second lowest.

Proof of Theorem 1.3. Suppose instead that there is a sequence of self-similar solutions $\{w_i\} \subset \mathcal{B}_{n,m}$, which is not equal to the positive constant solution with

$$E(w_i) < E(\kappa) + 2^{-i}. \tag{10.1}$$

Since $\{w_i\} \subset \mathcal{B}_{n,m}$, regularity theories imply there exists a positive constant $C(n, p, m)$ such that for any i

$$|w_i| + |\nabla w_i| + |\nabla^2 w_i| + |\nabla^3 w_i| \leq C(n, p, m).$$

By the Ascoli-Arzelà theorem, there exists a function $w_\infty \in C^2(\mathbb{R}^n)$ such that $w_i \to w_\infty$ uniformly on compact subset of $\mathbb{R}^n$. Moreover, w_∞ is a solution of (SS) such that

$$E(w_\infty) \leq E(\kappa).$$

Because Theorem 1.1 says that κ is the unique least energy solution, $w_\infty \equiv \kappa$ or $-\kappa$.

Let $\lambda_{1,i}$ be the first eigenfunction of

$$\mathcal{L}_i\psi = -\Delta\psi + \frac{y}{2}\cdot\nabla\psi + \frac{1}{p-1}\psi - p|w_i|^{p-1}\psi$$

and let f_i be the positive, first eigenfunction satisfying $\int_{\mathbb{R}^n} f_i^2\rho dy = 1$. That is to say, f_i satisfies

$$\Delta f_i - \frac{y}{2}\cdot\nabla f_i - \frac{1}{p-1}f_i + p|w_i|^{p-1}f_i + \lambda_{1,i}f_i = 0, \quad \text{in } \mathbb{R}^n. \tag{10.2}$$

Since $\{w_i\}\subset\mathcal{B}_{n,m}$, it follows from (5.2) that the eigenvalues $\lambda_{1,i}$ are uniformly bounded. By choosing a subsequence if necessary, we may assume that

$$\lambda_{1,i}\to\lambda_{1,\infty}, \quad \text{as } i\to\infty.$$

Multiplying both sides of (10.2) by $f_i\rho$ and integrating by parts, we see $\{f_i\}$ are uniformly bounded in $H^1_w(\mathbb{R}^n)$. Then standard elliptic regularity theory implies that $\{f_i\}$ are uniformly bounded in $C^{2,\alpha}_{loc}(\mathbb{R}^n)$. Using the Ascoli-Arzelà theorem again, we know that there exists a function f_∞ such that $f_i\to f_\infty$ uniformly on compact subsets of $\mathbb{R}^n$. Taking limit in (10.2), we deduce that f_∞ is a C^2 solution of the equation

$$\Delta f_\infty - \frac{y}{2}\cdot\nabla f_\infty + f_\infty + \lambda_{1,\infty}f_\infty = 0, \quad \text{in } \mathbb{R}^n. \tag{10.3}$$

Since we have assumed that f_i is positive and $\int_{\mathbb{R}^n} f_i^2\rho dy = 1$, f_∞ is positive and $\int_{\mathbb{R}^n} f_\infty^2\rho dy = 1$. Hence λ_∞ is the first eigenvalue of the linear operator

$$\mathcal{L}_\infty\psi = \Delta\psi - \frac{y}{2}\cdot\nabla\psi + \psi + \lambda\psi$$

and f_∞ is the associated eigenfunction. By Lemma 6.4, $\lambda_{1,\infty} = -1$ and $f_\infty\equiv 1$. By Lemma 5.4 and Proposition 5.1, we have

$$\int_{\mathbb{R}^n} f_i\left(\frac{2}{p-1}w_i + y\cdot\nabla w_i\right)\rho dy = 0.$$

Sending i to ∞ gives

$$\int_{\mathbb{R}^n} f_\infty\left(\frac{2}{p-1}w_\infty + y\cdot\nabla w_\infty\right)\rho dy = 0.$$

Since $f_\infty\equiv 1$ and $w_\infty\equiv\kappa$, this is a contradiction. □

10.2. **Proof of Proposition 1.9.** In this section, we combine Theorem 1.1 and Theorem 1.3 to prove Proposition 1.9.

Proof of Proposition 1.9. Define the stratification of the blow up set to be

$$\mathcal{S}_0\subset\mathcal{S}_1\cdots\subset\mathcal{S}_n = \Sigma,$$

where $\mathcal{S}_k$ consists of all blow up points whose tangent functions are at most translation invariant in k directions. By [41, Theorem 8.1], the Hausdorff dimension of $\mathcal{S}_k$ is at most k. Set

$$\Sigma_{n-1} = (\Sigma\cap B_R(0))\backslash\mathcal{S}_{n-3}(R), \quad \Sigma_{n-2} = (\Sigma\cap B_R(0))\backslash\Sigma_{n-1},$$

then $\Sigma_R = \Sigma \cap B_R(0) = \Sigma_{n-1} \cup \Sigma_{n-2}$. Moreover, the Hausdorff dimenion of Σ_{n-2} is at most $n-3$. Thus we have proved both (1) and (3).

If $(x_0, T) \in \Sigma_{n-1}$, then the tangent functions are at least translation invariant in $n-2$ directions. Hence any tangent function can be regarded as a bounded solution of (SS) in $\mathbb{R}^2$. By Theorem 2.7 and [17, Theorem 3.1], any tangent function is the positive constant solution of (SS). Hence we have proved (4).

In order to finish the proof of Proposition 1.9, it still remains to prove (2). First, we prove that Σ_{n-1} is relative open in Σ_R. Assume it is false, then there exists a point $(x_0, T) \in \Sigma_{n-1}$ and a sequence $\{(x_i, T)\} \subset \Sigma_{n-2}$ such that $\lim_{i\to\infty} x_i = x_0$. Since $\{(x_i, T)\} \subset \Sigma_{n-2}$, we get from Theorem 1.3 that

$$\Theta(x_i, T; u) \geq E(\kappa) + \varepsilon.$$

Since $(x_0, T) \in \Sigma_{n-1}$, then

$$\Theta(x_0, T; u) = E(\kappa).$$

Applying Lemma 2.4, we have

$$\Theta(x_0, T; u) = E(\kappa) \geq \limsup_{i\to\infty} \Theta(x_i, T; u) \geq E(\kappa) + \varepsilon,$$

which is a contradiction. Thus we have shown that Σ_{n-1} is relative open in Σ_R. Once this is established, the claim that Σ_{n-1} is $(n-1)-$ rectifiable follows from Velázquez [37]. Although in [37], only the subscritical case is considered. However, by checking the proof, the arguments there still work in the supercritical case under the assumption that u satisfies (F), (1.8) and for each $x_0 \in \Sigma$,

$$\lim_{t\to T} (T-t)^{\frac{1}{p-1}} u(x_0 + y(T-t)^{\frac{1}{2}}, t) = \kappa. \qquad \square$$

11. Proof of Theorem 1.7

In this section, we prove Theorem 1.7. First, we study the case when $1 \leq n \leq 3, 1 < p < \infty$ or $n \geq 3$ and $1 < p < (n+1)/(n-3)$. If we view any homogeneous positive solution w of (1.6) as a suitable weak solution of (SS), then w is not the lowest energy solution among self-similar solutions. The proof of this fact is based on the following classification result.

Proposition 11.1. *Assume $1 \leq n \leq 3, 1 < p < \infty$ or $n \geq 3$ and $1 < p < (n+1)/(n-3)$. If Φ is a bounded positive solution of (1.7), then $\Phi = \beta^{1/(p-1)}$.*

Proof. The proof of this rigidity result can be found in [3, Theorem 6.1], [20, Theorem B.2] and [16, Theorem 1]. $\square$

Lemma 11.2. *Assume $n \geq 3$ and $1 < p < (n+1)/(n-3)$. If w is a positive homogeneous of (1.6), then*

$$E(w) > E(\kappa) \tag{11.1}$$

Proof. Let Φ be a positive function defined on $\mathbb{S}^{n-1}$ such that $w(r, \theta) = r^{-2/(p-1)}\Phi(\theta)$. Then Φ satisfies (1.7). By Proposition 11.1, we have $\Phi = \beta^{1/(p-1)}$. It follows that

$w = \beta^{1/(p-1)} r^{-\frac{2}{p-1}}$. Then

$$
\begin{aligned}
E(w) &= \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}^n} w^{p+1} \rho dy \\
&= \left(\frac{1}{2} - \frac{1}{p+1}\right) \left[\frac{2}{p-1}\left(n-2-\frac{2}{p-1}\right)\right]^{\frac{p+1}{p-1}} \int_{\mathbb{R}^n} |y|^{\frac{-2(p+1)}{p-1}} \rho dy
\end{aligned}
$$

Thus in order to get (11.1), it suffices to show that

$$
\left[\frac{2}{p-1}\left(n-2-\frac{2}{p-1}\right)\right]^{\frac{p+1}{p-1}} \int_{\mathbb{R}^n} |y|^{\frac{-2(p+1)}{p-1}} \rho dy > (p-1)^{-\frac{p+1}{p-1}}. \tag{11.2}
$$

This has essentially been proved by Matano and Merle in [31]. In [31], Matano and Merle gave a "parabolic proof" without using the explicit formula. To be self contained, we will give a more direct proof in Appendix B. □

Proof of Theorem 1.7. Similar to the proof of Theorem 1.1, we will prove by induction. If $n = 1, 2$, we get from Theorem 2.7 that $\mathcal{F}_n^+$ consists only of the constant solutions, so Theorem 1.3 holds.

Assume that Theorem 1.3 holds for $n-1$ with $n \geq 3$. We want to show that Theorem 1.3 holds for dimension n.

Let $u(x,t) := (-t)^{-1/(p-1)} w(x/\sqrt{-t})$, which is a suitable weak solution of (F).

Assume u blows up at some point $x_0 \neq 0$, then we can apply the Federer dimension reduction to get a solution w_1 of (SS) such that w_1 is translation invariant in the x_0 direction. Moreover,

$$
E(w_1) = \Theta(x_0, 0; u) \leq \Theta(0, 0; u) \leq E(w).
$$

By the inductive assumption,

$$
E(w) \geq E(\kappa),
$$

and the proof is complete.

Next, assume u does not blow up at any point $x_0 \neq 0$. By standard parabolic regularity theory, there exist C_1, δ_0 such that

$$
u \leq C_1, \quad \text{in } \{1/4 < |x| < 1/2\} \times [-\delta_0, 0].
$$

This is equivalent to

$$
w \leq C_1 |y|^{-\frac{2}{p-1}}, \quad \text{in } |y| \geq \frac{1}{4\sqrt{\delta_0}}.
$$

There are two cases.

- Case 1: There exists a constant C_0 such that
$$
w \leq C_0, \quad \text{in } \mathbb{R}^n;
$$
- Case 2: There exists a point $y_0 \in \mathbb{R}^n$ such that w_0 blows up at y_0.

In Case 1, w_0 satisfies (7.4). Then we can apply Lemma 8.5 to obtain

$$
E(w) > E(\kappa).
$$

In Case 2, we choose a sequence $\{\lambda_k\}$ such that $\lim_{k\to\infty} \lambda_k = +\infty$. For any k, define $w_k(y) = \lambda_k^{-2/(p-1)} w(\lambda_k^{-1}(y - y_0))$. Then w_k satisfies the equation

$$\Delta w_k - \frac{\lambda_k^{-2}}{2}(y + y_0) \cdot \nabla w_k - \frac{\lambda_k^{-2}}{p-1} w_k + w_k^p = 0. \tag{11.3}$$

By the convergence theories obtained in [40], there exists a function w_∞ such that $\lim_{k\to\infty} w_k = w_\infty$ strongly, where w_∞ is a homogeneous suitable weak solution of (1.6). Therefore, there exists a function Φ defined on $\mathbb{S}^{n-1}$ such that $w_\infty(r,\theta) = r^{-\frac{2}{p-1}}\Phi(\theta)$, where (r,θ) is the polar coordinate. If Φ is not smooth on $\mathbb{S}^{n-1}$, we can apply the Federer dimension procedure (see [39, Section 4]) once again to reduce to the case that Φ is smooth on $\mathbb{S}^{n-2}$. By Lemma 11.2,

$$E(w) \geq E(w_\infty) > E(\kappa). \qquad \square$$

12. Extensions and some related questions

We have defined F-functional and entropy for bounded smooth functions. These definitions can be extended to a larger class of functions, for example, the class $\mathcal{G}_m$ of suitable weak ancient solutions as defined in Definition 1.5. The Morrey space bound therein ensures that the F_{x_0,t_0}-functional is well defined for any (x_0, t_0), and the monotonicity formula (in particular, Lemma 2.5) ensures that the entropy is also well defined. Moreover, if w is a self-similar, suitable weak ancient solution in $\mathcal{G}_M$, then Lemma 2.5 implies that

$$\lambda(w) = E(w).$$

The first variation formula and the second variation formula also hold for them. We just replace various integration by parts techniques used before by substituting suitable smooth vector fields into the stationary condition (1.5) and the localized energy inequality (1.4).

Our main results, Theorem 1.1 and Theorem 1.7, suggest the following natural problem.

Conjecture 12.1. *For any $w \in \mathcal{F}_n$, if $w \neq 0$, then*

$$E(w) \geq E(\kappa).$$

Moreover, there exists a constant ε such that if $w \neq \pm\kappa$, then

$$E(w) \geq E(\kappa) + \varepsilon.$$

It seems that the main obstruction to prove this conjecture lies in the class of "elliptic solutions" in $\mathcal{F}_n$, or more precisely, $2/(p-1)$ homogeneous solutions of the elliptic equation (1.6). If w is such a solution, then there exists a function Φ defined on $\mathbb{S}^{n-1}$, which is a solution of the equation (1.7), such that $w(r,\theta) = r^{-2/(p-1)}\Phi(\theta)$. The energy functional for (1.7) is

$$\mathcal{E}(\Phi) = \int_{\mathbb{S}^{n-1}} \left[\frac{1}{2}|\nabla\Phi|^2 + \frac{1}{2\beta}\Phi^2 - \frac{1}{p+1}|\Phi|^{p+1}\right] d\theta.$$

The above conjecture will follow if we can prove the following conjecture.

Conjecture 12.2. *For any bounded solution of* (1.7),

$$\mathcal{E}(\Phi) \geq \mathcal{E}\left(\beta^{\frac{1}{p-1}}\right).$$

This conjecture says once again, among all bounded solutions of (1.7), the constant solution $\beta^{1/(p-1)}$ has the lowest energy.

A. Proof of Theorem 4.3

In this appendix, we prove Theorem 4.3.

Proof of Theorem 4.3. Because w is a bounded solution of (SS), for any $\phi \in C_0^\infty(\mathbb{R}^n)$,

$$\frac{d}{ds}F_{x(s),t(s)}(w+s\phi)|_{s=0} = 0.$$

Substituting (3.3), (3.4) and (3.7) into (4.1), we have

$$\begin{aligned}
&\frac{d^2}{ds^2}F_{x(s),t(s)}(w+s\phi)|_{s=0}\\
=&\frac{p+1}{2(p-1)}\frac{2}{p-1}h^2\int_{\mathbb{R}^n}|\nabla w|^2\rho dy\\
&-\frac{1}{p-1}\frac{2}{p-1}h^2\int_{\mathbb{R}^n}|w|^{p+1}\rho dy\\
&-\frac{1}{(p-1)^2}\left(-\frac{2}{p-1}+1\right)h^2\int_{\mathbb{R}^n}w^2\rho dy\\
&+\frac{2}{p-1}h\int_{\mathbb{R}^n}w\phi\rho dy\\
&-\frac{p+1}{p-1}h\int_{\mathbb{R}^n}|\nabla w|^2\rho\left[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]dy\\
&+\frac{2}{p-1}h\int_{\mathbb{R}^n}|w|^{p+1}\rho\left[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]dy\\
&-\frac{2}{(p-1)^2}h\int_{\mathbb{R}^n}w^2\rho\left[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]dy\\
&+\int_{\mathbb{R}^n}\left(|\nabla\phi|^2+\frac{1}{p-1}\phi^2-p|w|^{p-1}\phi^2\right)\rho dy\\
&+2\int_{\mathbb{R}^n}(\nabla w\cdot\nabla\phi)\rho\left[\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]dy\\
&-2\int_{\mathbb{R}^n}|w|^{p-1}w\phi\rho\left[\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]dy\\
&+\frac{2}{p-1}\int_{\mathbb{R}^n}w\phi\rho\left[\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]dy\\
&+\frac{1}{2}\int_{\mathbb{R}^n}|\nabla w|^2\rho\left\{\left[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]^2+\frac{nh^2}{2}-\frac{|y_0|^2}{2}-\frac{h^2|y|^2}{2}\right\}dy
\end{aligned} \tag{A.1}$$

$$
\begin{aligned}
&-\frac{1}{p+1}\int_{\mathbb{R}^n}|w|^{p+1}\rho\bigg\{\Big[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\Big]^2+\frac{nh^2}{2}-\frac{|y_0|^2}{2}-\frac{h^2|y|^2}{2}\bigg\}dy\\
&+\frac{1}{2(p-1)}\int_{\mathbb{R}^n}w^2\rho\bigg\{\Big[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\Big]^2+\frac{nh^2}{2}-\frac{|y_0|^2}{2}-\frac{h^2|y|^2}{2}\bigg\}dy.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
&-\frac{p+1}{p-1}h\int_{\mathbb{R}^n}|\nabla w|^2\rho\left[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]dy\\
&+\frac{2}{p-1}h\int_{\mathbb{R}^n}|w|^{p+1}\rho\left[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]dy\\
&-\frac{2}{(p-1)^2}h\int_{\mathbb{R}^n}w^2\rho\left[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]dy\\
=&-2h\left(\frac{p}{p-1}+\frac{1}{p-1}\right)\frac{1}{2}\int_{\mathbb{R}^n}|\nabla w|^2\rho\left[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]dy \qquad \text{(A.2)}\\
&+2h\left(\frac{p}{p-1}+\frac{1}{p-1}\right)\frac{1}{p+1}\int_{\mathbb{R}^n}|w|^{p+1}\rho\left[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]dy\\
&-2h\left(\frac{p}{p-1}+\frac{1}{p-1}\right)\frac{1}{2(p-1)}\int_{\mathbb{R}^n}w^2\rho\left[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]dy\\
&+\frac{h}{p-1}\int_{\mathbb{R}^n}w^2\rho\left[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]dy.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\frac{d^2}{ds^2}F_{x(s),t(s)}(w+s\phi)|_{s=0}\\
=&-\frac{1}{(p-1)^2}h^2\int_{\mathbb{R}^n}w^2\rho dy+\frac{2}{p-1}h\int_{\mathbb{R}^n}w\phi\rho dy\\
&-\frac{2ph}{p-1}\frac{1}{2}\int_{\mathbb{R}^n}|\nabla w|^2\rho\left[\frac{nh}{2}-\frac{h|y|^2}{4}\right]dy\\
&+\frac{2ph}{p-1}\frac{1}{p+1}\int_{\mathbb{R}^n}|w|^{p+1}\rho\left[\frac{nh}{2}-\frac{h|y|^2}{4}\right]dy\\
&-\frac{2ph}{p-1}\frac{1}{2(p-1)}\int_{\mathbb{R}^n}w^2\rho\left[\frac{nh}{2}-\frac{h|y|^2}{4}\right]dy\\
&+\frac{h}{p-1}\int_{\mathbb{R}^n}w^2\rho\left[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]dy\\
&+\int_{\mathbb{R}^n}\left(|\nabla\phi|^2+\frac{1}{p-1}\phi^2-p|w|^{p-1}\phi^2\right)\rho dy \qquad \text{(A.3)}\\
&+2\int_{\mathbb{R}^n}(\nabla w\cdot\nabla\phi)\rho\left[\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]dy
\end{aligned}
$$

$$
\begin{aligned}
&-2\int_{\mathbb{R}^n}|w|^{p-1}w\phi\rho\left[\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]dy\\
&+\frac{2}{p-1}\int_{\mathbb{R}^n}w\phi\rho\left[\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]dy\\
&+\frac{1}{2}\int_{\mathbb{R}^n}|\nabla w|^2\rho\left\{\left[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]^2+\frac{nh^2}{2}-\frac{|y_0|^2}{2}-\frac{h^2|y|^2}{2}\right\}dy\\
&-\frac{1}{p+1}\int_{\mathbb{R}^n}|w|^{p+1}\rho\left\{\left[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]^2+\frac{nh^2}{2}-\frac{|y_0|^2}{2}-\frac{h^2|y|^2}{2}\right\}dy\\
&+\frac{1}{2(p-1)}\int_{\mathbb{R}^n}w^2\rho\left\{\left[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]^2+\frac{nh^2}{2}-\frac{|y_0|^2}{2}-\frac{h^2|y|^2}{2}\right\}dy.
\end{aligned}
$$

Multiplying both sides of (SS) by $(y\cdot y_0)\phi\rho$ and integrating by parts, we obtain

$$
\begin{aligned}
0=&\int_{\mathbb{R}^n}\left[\frac{1}{\rho}\mathrm{div}(\rho\nabla w)-\frac{1}{p-1}w+|w|^{p-1}w\right](y\cdot y_0)\phi\rho dy\\
=&-\int_{\mathbb{R}^n}\rho\nabla w\nabla\left((y\cdot y_0)\phi\right)dy+\int_{\mathbb{R}^n}|w|^{p-1}w(y\cdot y_0)\phi\rho dy\\
&-\frac{1}{p-1}\int_{\mathbb{R}^n}w(y\cdot y_0)\phi\rho dy\\
=&-\int_{\mathbb{R}^n}(\nabla w\cdot y_0)\phi\rho dy-\int_{\mathbb{R}^n}(\nabla w\cdot\nabla\phi)(y\cdot y_0)\rho dy\\
&+\int_{\mathbb{R}^n}|w|^{p-1}w(y\cdot y_0)\phi\rho dy-\frac{1}{p-1}\int_{\mathbb{R}^n}w(y\cdot y_0)\phi\rho dy.
\end{aligned}
\tag{A.4}
$$

Multiplying both sides of (SS) by $|y|^2\phi\rho$ and integrating by parts, we get

$$
\begin{aligned}
0=&\int_{\mathbb{R}^n}\left[\frac{1}{\rho}\mathrm{div}(\rho\nabla w)-\frac{1}{p-1}w+|w|^{p-1}w\right]|y|^2\phi\rho dy\\
=&-\int_{\mathbb{R}^n}\rho\nabla w\nabla(|y|^2\phi)dy+\int_{\mathbb{R}^n}|w|^{p-1}w|y|^2\phi\rho dy\\
&-\frac{1}{p-1}\int_{\mathbb{R}^n}w|y|^2\phi\rho dy\\
=&-2\int_{\mathbb{R}^n}(\nabla w\cdot y)\phi\rho dy-\int_{\mathbb{R}^n}(\nabla w\cdot\nabla\phi)|y|^2\rho dy\\
&+\int_{\mathbb{R}^n}|w|^{p-1}w|y|^2\phi\rho dy-\frac{1}{p-1}\int_{\mathbb{R}^n}w|y|^2\phi\rho dy.
\end{aligned}
\tag{A.5}
$$

Substituting (A.4) and (A.5) into (A.3), we get

$$
\begin{aligned}
&\frac{d^2}{ds^2}F_{x(s),t(s)}(w+s\phi)|_{s=0}\\
=&\int_{\mathbb{R}^n}\left(|\nabla\phi|^2+\frac{1}{p-1}\phi^2-p|w|^{p-1}\phi^2\right)\rho dy\rho dy
\end{aligned}
$$

$$
\begin{aligned}
&+ h\int_{\mathbb{R}^n}\left(\frac{2}{p-1}w+\nabla w\cdot y\right)\phi\rho dy-\int_{\mathbb{R}^n}(\nabla w\cdot y_0)\phi\rho dy\\
&-\frac{2ph}{p-1}\frac{1}{2}\int_{\mathbb{R}^n}|\nabla w|^2\rho\left[\frac{nh}{2}-\frac{h|y|^2}{4}\right]dy\\
&+\frac{2ph}{p-1}\frac{1}{p+1}\int_{\mathbb{R}^n}|w|^{p+1}\rho\left[\frac{nh}{2}-\frac{h|y|^2}{4}\right]dy\\
&-\frac{2ph}{p-1}\frac{1}{2(p-1)}\int_{\mathbb{R}^n}w^2\rho\left[\frac{nh}{2}-\frac{h|y|^2}{4}\right]dy \qquad\qquad (A.6)\\
&+\frac{h}{p-1}\int_{\mathbb{R}^n}w^2\rho\left[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]dy-\frac{h^2}{(p-1)^2}\int_{\mathbb{R}^n}w^2\rho dy\\
&+\frac{1}{2}\int_{\mathbb{R}^n}|\nabla w|^2\rho\left\{\left[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]^2+\frac{nh^2}{2}-\frac{|y_0|^2}{2}-\frac{h^2|y|^2}{2}\right\}dy\\
&-\frac{1}{p+1}\int_{\mathbb{R}^n}|w|^{p+1}\rho\left\{\left[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]^2+\frac{nh^2}{2}-\frac{|y_0|^2}{2}-\frac{h^2|y|^2}{2}\right\}dy\\
&+\frac{1}{2(p-1)}\int_{\mathbb{R}^n}w^2\rho\left\{\left[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]^2+\frac{nh^2}{2}-\frac{|y_0|^2}{2}-\frac{h^2|y|^2}{2}\right\}dy.
\end{aligned}
$$

To continue the proof, we need several more identities. Multiplying both sides of (SS) by $\left(\frac{n}{2}-\frac{|y|^2}{4}\right)(\nabla w\cdot y_0)\rho$ and integrating by parts, we get

$$
\begin{aligned}
0 &= \int_{\mathbb{R}^n}\left[\frac{1}{\rho}\mathrm{div}(\rho\nabla w)-\frac{1}{p-1}w+|w|^{p-1}w\right]\left(\frac{n}{2}-\frac{|y|^2}{4}\right)(\nabla w\cdot y_0)\rho dy\\
&= -\int_{\mathbb{R}^n}\rho\nabla w\cdot\nabla\left[\left(\frac{n}{2}-\frac{|y|^2}{4}\right)(\nabla w\cdot y_0)\right]dy\\
&\quad -\frac{1}{p-1}\int_{\mathbb{R}^n}w\left(\frac{n}{2}-\frac{|y|^2}{4}\right)(\nabla w\cdot y_0)\rho dy\\
&\quad +\int_{\mathbb{R}^n}|w|^{p-1}w\left(\frac{n}{2}-\frac{|y|^2}{4}\right)(\nabla w\cdot y_0)\rho dy\\
&= -\frac{1}{4(p-1)}\int_{\mathbb{R}^n}w^2(y\cdot y_0)\rho dy+\frac{1}{2(p+1)}\int_{\mathbb{R}^n}|w|^{p+1}(y\cdot y_0)\rho dy \qquad (A.7)\\
&\quad -\frac{1}{4(p-1)}\int_{\mathbb{R}^n}\left(\frac{n}{2}-\frac{|y|^2}{4}\right)w^2(y\cdot y_0)\rho dy\\
&\quad +\frac{1}{2(p+1)}\int_{\mathbb{R}^n}\left(\frac{n}{2}-\frac{|y|^2}{4}\right)|w|^{p+1}(y\cdot y_0)\rho dy\\
&\quad +\frac{1}{2}\int_{\mathbb{R}^n}(\nabla w\cdot y)(\nabla w\cdot y_0)\rho dy-\frac{1}{4}\int_{\mathbb{R}^n}|\nabla w|^2(y\cdot y_0)\rho dy\\
&\quad -\frac{1}{4}\int_{\mathbb{R}^n}|\nabla w|^2(y\cdot y_0)\left(\frac{n}{2}-\frac{|y|^2}{4}\right)\rho dy.
\end{aligned}
$$

Multiplying both sides of (SS) by $(\nabla w \cdot y_0)(y \cdot y_0)\rho$ and integrating by parts, we obtain

$$\begin{aligned}
0 =& \int_{\mathbb{R}^n} \left[\frac{1}{\rho}\mathrm{div}(\rho\nabla w) - \frac{1}{p-1}w + |w|^{p-1}w\right](\nabla w \cdot y_0)(y \cdot y_0)\rho dy \\
=& -\int_{\mathbb{R}^n} \rho\nabla w \cdot \nabla\left[(\nabla w \cdot y_0)(y \cdot y_0)\right] dy + \int_{\mathbb{R}^n} |w|^{p-1}w(\nabla w \cdot y_0)(y \cdot y_0)\rho dy \\
& - \frac{1}{p-1}\int_{\mathbb{R}^n} w(\nabla w \cdot y_0)(y \cdot y_0)\rho dy \\
=& -\frac{1}{4(p-1)}\int_{\mathbb{R}^n} |y \cdot y_0|^2 w^2 \rho dy + \frac{1}{2(p-1)}\int_{\mathbb{R}^n} w^2|y_0|^2\rho dy \\
& + \frac{1}{2(p+1)}\int_{\mathbb{R}^n} (y \cdot y_0)(y \cdot y_0)|w|^{p+1}\rho dy - \frac{1}{p+1}\int_{\mathbb{R}^n} |w|^{p+1}|y_0|^2\rho dy \\
& - \int_{\mathbb{R}^n} |\nabla w \cdot y_0|^2\rho dy - \frac{1}{4}\int_{\mathbb{R}^n} |\nabla w|^2|y \cdot y_0|^2\rho dy \\
& + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla w|^2|y_0|^2\rho dy.
\end{aligned} \tag{A.8}$$

Multiplying both sides of (SS) by $(\frac{n}{2} - \frac{|y|^2}{4})(\nabla w \cdot \nabla\rho)$ and integrating by parts, we get

$$\begin{aligned}
0 =& \int_{\mathbb{R}^n} \left[\frac{1}{\rho}\mathrm{div}(\rho\nabla w) - \frac{1}{p-1}w + |w|^{p-1}w\right]\left(\frac{n}{2} - \frac{|y|^2}{4}\right)(\nabla w \cdot \nabla\rho) dy \\
=& -\int_{\mathbb{R}^n} \rho\nabla w \cdot \nabla\left[\left(\frac{n}{2} - \frac{|y|^2}{4}\right)(\nabla \log\rho \cdot \nabla w)\right] dy \\
& - \frac{1}{2(p-1)}\int_{\mathbb{R}^n} \left(\frac{n}{2} - \frac{|y|^2}{4}\right)(\nabla w^2 \cdot \nabla\rho) dy \\
& + \frac{1}{p+1}\int_{\mathbb{R}^n} \left(\frac{n}{2} - \frac{|y|^2}{4}\right)(\nabla|w|^{p+1} \cdot \nabla\rho) dy \\
=& \frac{1}{8(p-1)}\int_{\mathbb{R}^n} w^2|y|^2\rho dy - \frac{1}{4(p+1)}\int_{\mathbb{R}^n} |w|^{p+1}|y|^2\rho dy \\
& - \frac{1}{2(p-1)}\int_{\mathbb{R}^n} \left(\frac{n}{2} - \frac{|y|^2}{4}\right)\left(\frac{n}{2} - \frac{|y|^2}{4}\right) w^2\rho dy \\
& + \frac{1}{p+1}\int_{\mathbb{R}^n} \left(\frac{n}{2} - \frac{|y|^2}{4}\right)\left(\frac{n}{2} - \frac{|y|^2}{4}\right)|w|^{p+1}\rho dy \\
& + \frac{n}{4}\int_{\mathbb{R}^n} |\nabla w|^2\rho dy - \frac{1}{4}\int_{\mathbb{R}^n} |\nabla w \cdot y|^2\rho dy \\
& - \frac{1}{2}\int_{\mathbb{R}^n} \left(\frac{n}{2} - \frac{|y|^2}{4}\right)^2|\nabla w|^2\rho dy.
\end{aligned} \tag{A.9}$$

Combining (A.7), (A.8) and (A.9), we have

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^n}|\nabla w|^2\rho\left[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]^2dy\\
&-\frac{1}{p+1}\int_{\mathbb{R}^n}|w|^{p+1}\rho\left[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]^2dy\\
&+\frac{1}{2(p-1)}\int_{\mathbb{R}^n}w^2\rho\left[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]^2dy\\
=&\frac{h^2}{8(p-1)}\int_{\mathbb{R}^n}w^2|y|^2\rho dy-\frac{h^2}{4(p+1)}\int_{\mathbb{R}^n}|w|^{p+1}|y|^2\rho dy\\
&+\frac{nh^2}{4}\int_{\mathbb{R}^n}|\nabla w|^2\rho dy-\frac{h^2}{4}\int_{\mathbb{R}^n}|\nabla w\cdot y|^2\rho dy\\
&+\frac{1}{4(p-1)}\int_{\mathbb{R}^n}w^2|y_0|^2\rho dy-\frac{1}{2(p+1)}\int_{\mathbb{R}^n}|w|^{p+1}|y_0|^2\rho dy\\
&-\frac{1}{2}\int_{\mathbb{R}^n}|\nabla w\cdot y_0|^2\rho dy+\frac{1}{4}\int_{\mathbb{R}^n}|\nabla w|^2|y_0|^2\rho dy\\
&-\frac{h}{2(p-1)}\int_{\mathbb{R}^n}w^2(y\cdot y_0)\rho dy+\frac{h}{p+1}\int_{\mathbb{R}^n}|w|^{p+1}(y\cdot y_0)\rho dy\\
&+h\int_{\mathbb{R}^n}(\nabla w\cdot y)(\nabla w\cdot y_0)\rho dy-\frac{h}{2}\int_{\mathbb{R}^n}|\nabla w|^2(y\cdot y_0)\rho dy\\
=&-\frac{1}{2}\int_{\mathbb{R}^n}|\nabla w\cdot y_0|^2\rho dy-\frac{h^2}{4}\int_{\mathbb{R}^n}|\nabla w\cdot y|^2\rho dy\\
&+h\int_{\mathbb{R}^n}(\nabla w\cdot y)(\nabla w\cdot y_0)\rho dy+\frac{h^2}{2}\int_{\mathbb{R}^n}|\nabla w|^2\left[\frac{n}{2}-\frac{|y|^2}{4}\right]\rho dy\\
&+\frac{1}{2}\int_{\mathbb{R}^n}|\nabla w|^2\left[\frac{|y_0|^2}{2}+\frac{|y|^2}{4}]\rho dy+\frac{1}{2(p-1)}\int_{\mathbb{R}^n}w^2[\frac{|y_0|^2}{2}+\frac{|y|^2}{4}\right]\rho dy\\
&-\frac{1}{p+1}\int_{\mathbb{R}^n}|w|^{p+1}\left[\frac{|y_0|^2}{2}+\frac{|y|^2}{4}\right]\rho dy,
\end{aligned}\tag{A.10}$$

where we have applied (3.7). Then

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^n}|\nabla w|^2\rho\left\{\left[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]^2+\frac{nh^2}{2}-\frac{|y_0|^2}{2}-\frac{h^2|y|^2}{2}\right\}dy\\
&+\frac{1}{2(p-1)}\int_{\mathbb{R}^n}w^2\rho\left\{\left[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]^2+\frac{nh^2}{2}-\frac{|y_0|^2}{2}-\frac{h^2|y|^2}{2}\right\}dy\\
&-\frac{1}{p+1}\int_{\mathbb{R}^n}|w|^{p+1}\rho\left\{\left[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]^2+\frac{nh^2}{2}-\frac{|y_0|^2}{2}-\frac{h^2|y|^2}{2}\right\}dy\\
=&h^2\int_{\mathbb{R}^n}|\nabla w|^2\left[\frac{n}{2}-\frac{|y|^2}{4}\right]\rho dy-\frac{h^2}{p+1}\int_{\mathbb{R}^n}|w|^{p+1}\left[\frac{n}{2}-\frac{|y|^2}{4}\right]\rho dy
\end{aligned}$$

$$+\frac{h^2}{2(p-1)}\int_{\mathbb{R}^n} w^2\left[\frac{n}{2}-\frac{|y|^2}{4}\right]\rho dy-\frac{h^2}{4}\int_{\mathbb{R}^n}|\nabla w\cdot y|^2\rho dy$$
$$-\frac{1}{2}\int_{\mathbb{R}^n}|\nabla w\cdot y_0|^2\rho dy+h\int_{\mathbb{R}^n}(\nabla w\cdot y)(\nabla w\cdot y_0)\rho dy.$$

Plugging this into (A.6), we get

$$\begin{aligned}
&\frac{d^2}{ds^2}F_{x(s),t(s)}(w+s\phi)|_{s=0}\\
&=\int_{\mathbb{R}^n}\left(|\nabla\phi|^2+\frac{1}{p-1}\phi^2-p|w|^{p-1}\phi^2\right)\rho dy\\
&+h\int_{\mathbb{R}^n}\left(\frac{2}{p-1}w+\nabla w\cdot y\right)\phi\rho dy-\int_{\mathbb{R}^n}(\nabla w\cdot y_0)\phi\rho dy\\
&-\frac{(p+1)h}{p-1}\frac{1}{2}\int_{\mathbb{R}^n}|\nabla w|^2\rho\left[\frac{nh}{2}-\frac{h|y|^2}{4}\right]dy\\
&+\frac{(p+1)h}{p-1}\frac{1}{p+1}\int_{\mathbb{R}^n}|w|^{p+1}\rho\left[\frac{nh}{2}-\frac{h|y|^2}{4}\right]dy\\
&-\frac{(p+1)h}{p-1}\frac{1}{2(p-1)}\int_{\mathbb{R}^n}w^2\rho\left[\frac{nh}{2}-\frac{h|y|^2}{4}\right]dy \qquad \text{(A.11)}\\
&+\frac{h}{p-1}\int_{\mathbb{R}^n}w^2\rho\left[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]dy-\frac{h^2}{(p-1)^2}\int_{\mathbb{R}^n}w^2\rho dy\\
&+\frac{h^2}{2}\int_{\mathbb{R}^n}|\nabla w|^2\left[\frac{n}{2}-\frac{|y|^2}{4}\right]\rho dy-\frac{h^2}{4}\int_{\mathbb{R}^n}|\nabla w\cdot y|^2\rho dy\\
&-\frac{1}{2}\int_{\mathbb{R}^n}|\nabla w\cdot y_0|^2\rho dy+h\int_{\mathbb{R}^n}(\nabla w\cdot y)(\nabla w\cdot y_0)\rho dy.
\end{aligned}$$

Multiplying both sides of (3.2) by $(\frac{n}{2}-\frac{|y|^2}{4})w\rho$ and integrating by parts, we get

$$\begin{aligned}
0&=\int_{\mathbb{R}^n}\left[\frac{1}{\rho}\mathrm{div}(\rho\nabla w)-\frac{1}{p-1}w+|w|^{p-1}w\right]\left(\frac{n}{2}-\frac{|y|^2}{4}\right)w\rho dy\\
&=-\int_{\mathbb{R}^n}\rho\nabla w\nabla\left[\left(\frac{n}{2}-\frac{|y|^2}{4}\right)w\right]-\frac{1}{p-1}\int_{\mathbb{R}^n}w^2\left(\frac{n}{2}-\frac{|y|^2}{4}\right)\rho dy\\
&\quad+\int_{\mathbb{R}^n}|w|^{p+1}\left(\frac{n}{2}-\frac{|y|^2}{4}\right)\rho dy\\
&=-\int_{\mathbb{R}^n}|\nabla w|^2\left(\frac{n}{2}-\frac{|y|^2}{4}\right)\rho dy+\frac{1}{2}\int_{\mathbb{R}^n}w(\nabla w\cdot y)\rho dy\\
&\quad-\frac{1}{p-1}\int_{\mathbb{R}^n}w^2\left(\frac{n}{2}-\frac{|y|^2}{4}\right)\rho dy+\int_{\mathbb{R}^n}|w|^{p+1}\left(\frac{n}{2}-\frac{|y|^2}{4}\right)\rho dy\\
&=-\int_{\mathbb{R}^n}|\nabla w|^2\left(\frac{n}{2}-\frac{|y|^2}{4}\right)\rho+\frac{1}{4}\int_{\mathbb{R}^n}\left(\nabla w^2\cdot y\right)\rho dy
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{p-1}\int_{\mathbb{R}^n} w^2\left(\frac{n}{2}-\frac{|y|^2}{4}\right)\rho dy+\int_{\mathbb{R}^n}|w|^{p+1}\left(\frac{n}{2}-\frac{|y|^2}{4}\right)\rho dy\\
=&-\int_{\mathbb{R}^n}|\nabla w|^2\left(\frac{n}{2}-\frac{|y|^2}{4}\right)\rho dy-\frac{1}{2}\int_{\mathbb{R}^n} w^2\left(\frac{n}{2}-\frac{|y|^2}{4}\right)\rho dy\\
&-\frac{1}{p-1}\int_{\mathbb{R}^n} w^2\left(\frac{n}{2}-\frac{|y|^2}{4}\right)\rho dy+\int_{\mathbb{R}^n}|w|^{p+1}\left(\frac{n}{2}-\frac{|y|^2}{4}\right)\rho dy.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
0=&-\frac{1}{p+1}\int_{\mathbb{R}^n}|\nabla w|^2\left(\frac{n}{2}-\frac{|y|^2}{4}\right)\rho dy-\frac{1}{2(p-1)}\int_{\mathbb{R}^n} w^2\left(\frac{n}{2}-\frac{|y|^2}{4}\right)\rho dy \qquad (A.12)\\
&+\frac{1}{p+1}\int_{\mathbb{R}^n}|w|^{p+1}\left(\frac{n}{2}-\frac{|y|^2}{4}\right)\rho dy.
\end{aligned}
$$

By (3.6) and (A.12), we have

$$
0=\frac{1}{p-1}\int_{\mathbb{R}^n}|\nabla w|^2\rho dy-\frac{1}{p+1}\int_{\mathbb{R}^n}|\nabla w|^2\left(\frac{n}{2}-\frac{|y|^2}{4}\right)\rho dy. \qquad (A.13)
$$

Substituting (A.13) and (3.6) into (A.11) gives

$$
\begin{aligned}
&\frac{d^2}{ds^2}F_{x(s),t(s)}(w+s\phi)|_{s=0}\\
=&\int_{\mathbb{R}^n}\left(|\nabla\phi|^2+\frac{1}{p-1}\phi^2-p|w|^{p-1}\phi^2\right)\rho dy\\
&+h\int_{\mathbb{R}^n}\left(\frac{2}{p-1}w+\nabla w\cdot y\right)\phi\rho dy-\int_{\mathbb{R}^n}(\nabla w\cdot y_0)\phi\rho dy \qquad (A.14)\\
&+\frac{h}{p-1}\int_{\mathbb{R}^n} w^2\rho\left[\frac{nh}{2}+\frac{y\cdot y_0}{2}-\frac{h|y|^2}{4}\right]dy-\frac{h^2}{(p-1)^2}\int_{\mathbb{R}^n} w^2\rho dy\\
&-\frac{h^2}{4}\int_{\mathbb{R}^n}|\nabla w\cdot y|^2\rho dy-\frac{1}{2}\int_{\mathbb{R}^n}|\nabla w\cdot y_0|^2\rho dy\\
&+h\int_{\mathbb{R}^n}(\nabla w\cdot y)(\nabla w\cdot y_0)\rho dy.
\end{aligned}
$$

Recall that $\mathcal{L}$ is the linearized operator defined by

$$
\mathcal{L}\psi=-\Delta\psi+\frac{y}{2}\cdot\nabla\psi+\frac{1}{p-1}\psi-p|w|^{p-1}\psi.
$$

By Lemma 5.4, $2/(p-1)w+y\cdot\nabla w$ is an eigenfunction of $\mathcal{L}$ associated to the eigenvalue -1 and $w_i, i=1,2,\cdots,n$ are eigenfunctions of $\mathcal{L}$ associated to the eigenvalue $-1/2$. Since $\mathcal{L}$ is self-adjoint, for any $y_0\in\mathbb{R}^n$,

$$
\int_{\mathbb{R}^n}\left(\frac{2}{p-1}w+y\cdot\nabla w\right)(\nabla w\cdot y_0)\rho dy=0. \qquad (A.15)
$$

By (A.15), we have

$$
\int_{\mathbb{R}^n}(\nabla w\cdot y)(\nabla w\cdot y_0)\rho dy
$$

$$\begin{aligned}
&= -\frac{2}{p-1}\int_{\mathbb{R}^n} w(\nabla w\cdot y_0)\rho dy \\
&= -\frac{1}{p-1}\int_{\mathbb{R}^n} (\nabla w^2\cdot y_0)\rho dy \\
&= -\frac{1}{2(p-1)}\int_{\mathbb{R}^n} w^2\rho(y\cdot y_0)dy.
\end{aligned} \tag{A.16}$$

Finally, we have

$$\begin{aligned}
&\frac{h^2}{p-1}\int_{\mathbb{R}^n} w^2\rho\left[\frac{n}{2}-\frac{|y|^2}{4}\right]dy \\
=& -\frac{h^2}{p-1}\int_{\mathbb{R}^n} w^2\Delta\rho \\
=&\frac{h^2}{p-1}\int_{\mathbb{R}^n} \nabla w^2\cdot\nabla\rho dy \\
=& -\frac{h^2}{p-1}\int_{\mathbb{R}^n} w(\nabla w\cdot y)\rho dy.
\end{aligned} \tag{A.17}$$

Substituting (A.16) and (A.17) into (A.14) gives

$$\begin{aligned}
&\frac{d^2}{ds^2}F_{x(s),t(s)}(w+s\phi)|_{s=0} \\
&= \int_{\mathbb{R}^n}\left(|\nabla\phi|^2+\frac{1}{p-1}\phi^2-p|w|^{p-1}\phi^2\right)\rho dy \\
&+ h\int_{\mathbb{R}^n}\left(\frac{2}{p-1}w+\nabla w\cdot y\right)\phi\rho dy-\int_{\mathbb{R}^n}(\nabla w\cdot y_0)\phi\rho dy \\
&- \frac{h^2}{(p-1)^2}\int_{\mathbb{R}^n} w^2\rho dy-\frac{h^2}{p-1}\int_{\mathbb{R}^n} w(\nabla w\cdot y)\rho dy \\
&- \frac{h^2}{4}\int_{\mathbb{R}^n}|\nabla w\cdot y|^2\rho dy-\frac{1}{2}\int_{\mathbb{R}^n}|\nabla w\cdot y_0|^2\rho dy.
\end{aligned} \tag{A.18}$$

Since

$$\begin{aligned}
&-\frac{h^2}{(p-1)^2}\int_{\mathbb{R}^n} w^2\rho dy-\frac{h^2}{p-1}\int_{\mathbb{R}^n} w(\nabla w\cdot y)\rho dy-\frac{h^2}{4}\int_{\mathbb{R}^n}|\nabla w\cdot y|^2\rho dy \\
=& -h^2\int_{\mathbb{R}^n}\left(\frac{1}{p-1}w+\frac{y}{2}\cdot\nabla w\right)^2\rho dy,
\end{aligned}$$

we get (4.2) with the help of (A.18). □

B. Proof of (11.2)

In this appendix, our main objective is to prove (11.2). The calculation in this appendix is inspired by Stone [36].

Proof of (11.2). By the definition of ρ, we have

$$\begin{aligned}&\int_{\mathbb{R}^n}|y|^{-\frac{2(p+1)}{p-1}}\rho dy\\=&(4\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}|y|^{-\frac{2(p+1)}{p-1}}e^{-\frac{|y|^2}{4}}dy\\=&(4\pi)^{-\frac{n}{2}}\omega_{n-1}\int_0^{+\infty}r^{n-1-\frac{2(p+1)}{p-1}}e^{-\frac{r^2}{4}}dr,\end{aligned}\tag{B.1}$$

where ω_{n-1} is the area of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$. Recall that we have assumed $p > (n+2)/(n-2)$, so the above integral is well defined. Let $r = 2\sqrt{s}$. Then $dr = s^{-1/2}ds$ and

$$\begin{aligned}E(w)=&c(n,p)\int_0^{+\infty}r^{n-1-\frac{2(p+1)}{p-1}}e^{-\frac{r^2}{4}}dr\\=&c(n,p)\int_0^{+\infty}s^{-\frac{1}{2}}(2\sqrt{s})^{n-1-\frac{2(p+1)}{p-1}}e^{-s}ds\\=&2^{n-3-\frac{4}{p-1}}c(n,p)\int_0^{\infty}s^{\frac{n-4-\frac{4}{p-1}}{2}}e^{-s}ds\\=&2^{n-3-\frac{4}{p-1}}c(n,p)\Gamma\left(\frac{n-2-\frac{4}{p-1}}{2}\right),\end{aligned}$$

where

$$c(n,p)=\left(\frac{1}{2}-\frac{1}{p+1}\right)\left[\frac{2}{p-1}\left(n-2-\frac{2}{p-1}\right)\right]^{\frac{p+1}{p-1}}(4\pi)^{-\frac{n}{2}}\omega_{n-1}$$

and

$$\Gamma(\tau)=\int_0^{+\infty}s^{\tau-1}e^{-s}ds$$

is the Γ-function. It is well known that the area of the unit sphere is

$$\omega_{n-1}=\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}.$$

Then

$$\begin{aligned}E(w)=&2^{n-3-\frac{4}{p-1}}c(n,p)\Gamma\left(\frac{n-2-\frac{4}{p-1}}{2}\right)\\=&2^{-2-\frac{4}{p-1}}\left(\frac{1}{2}-\frac{1}{p+1}\right)\left[\frac{2}{p-1}\left(n-2-\frac{2}{p-1}\right)\right]^{\frac{p+1}{p-1}}\frac{\Gamma\left(\frac{n-2-\frac{4}{p-1}}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}.\end{aligned}\tag{B.2}$$

In order that Lemma 11.2 holds, we need only to show

$$2^{-2-\frac{4}{p-1}}\left[\frac{2}{p-1}\left(n-2-\frac{2}{p-1}\right)\right]^{\frac{p+1}{p-1}}\frac{\Gamma\left(\frac{n-2-\frac{4}{p-1}}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}>\left(\frac{1}{p-1}\right)^{\frac{p+1}{p-1}}.$$

This is equivalent to

$$\left(\frac{n-2}{2}-\frac{1}{p-1}\right)^{\frac{p+1}{p-1}}\frac{\Gamma\left(\frac{n-2-\frac{4}{p-1}}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}>1. \tag{B.3}$$

To obtain (B.3), we set $\alpha=2/(p-1)$. Using $p>(n+2)/(n-2)$ again, we have $\alpha\in(0,(n-2)/2)$. Let

$$x=\frac{n}{2},\quad f(x)=\frac{\Gamma(x-1-\alpha)}{\Gamma(x)}\left(x-1-\frac{\alpha}{2}\right)^{1+\alpha}$$

and

$$\phi(x)=\log f(x)=\log\Gamma(x-1-\alpha)-\log\Gamma(x)+(1+\alpha)\log\left(x-1-\frac{\alpha}{2}\right).$$

Then

$$\phi'(x)=\frac{\Gamma'(x-1-\alpha)}{\Gamma(x-1-\alpha)}-\frac{\Gamma'(x)}{\Gamma(x)}+\frac{1+\alpha}{x-1-\frac{\alpha}{2}} \tag{B.4}$$

and

$$\begin{aligned}\phi''(x)=&\left(\frac{\Gamma'(x-1-\alpha)}{\Gamma(x-1-\alpha)}\right)'-\left(\frac{\Gamma'(x)}{\Gamma(x)}\right)'-\frac{1+\alpha}{\left(x-1-\frac{\alpha}{2}\right)^2}\\ =&\sum_{l=0}^{\infty}(x-1-\alpha+l)^{-2}-\sum_{l=0}^{\infty}(x+l)^{-2}-\frac{1+\alpha}{\left(x-1-\frac{\alpha}{2}\right)^2}\end{aligned}$$

Because for any $\tau>1$,

$$\left(\tau-1-\frac{\alpha}{2}\right)\left(\tau+\frac{\alpha}{2}\right)\leq\tau(\tau-1),$$

we have

$$\begin{aligned}&\frac{1}{1+\alpha}\left[\frac{1}{\left(\tau-1-\frac{\alpha}{2}\right)^2}-\frac{1}{\left(\tau+\frac{\alpha}{2}\right)^2}\right]-\left[\frac{1}{(\tau-1)^2}-\frac{1}{\tau^2}\right]\\ =&\frac{1}{1+\alpha}\frac{(1+\alpha)(2\tau-1)}{\left(\tau-1-\frac{\alpha}{2}\right)^2\left(\tau+\frac{\alpha}{2}\right)^2}-\frac{2\tau-1}{\tau^2(\tau-1)^2}\\ =&(2\tau-1)\left[\frac{1}{(\tau-1-\frac{\alpha}{2})^2(\tau+\frac{\alpha}{2})^2}-\frac{1}{\tau^2(\tau-1)^2}\right]\\ \geq&0.\end{aligned}$$

Taking $\tau=x+l-\alpha/2$ in the above inequality, it gives

$$\phi''(x)\geq(1+\alpha)\sum_{l=0}^{\infty}\left[\frac{1}{(x-1+l-\alpha/2)^2}-\frac{1}{(x+l-\alpha/2)^2}\right]-\frac{1+\alpha}{(x-1-\frac{\alpha}{2})^2}=0.$$

Thus ϕ is convex. By the mean value theorem, we have

$$(1+\alpha)\left[\frac{1}{x-1-\frac{\alpha}{2}}-\sum_{l=0}^{\infty}(x+l)^{-2}\right]\leq\phi'(x)\leq(1+\alpha)\left[\frac{1}{x-1-\frac{\alpha}{2}}-\sum_{l=0}^{\infty}(x-1-\alpha+l)^{-2}\right].$$

It follows that $\lim_{x\to\infty}\phi'(x)=0$; and for any $\alpha\in(0,(n-2)/2)$, ϕ is decreasing in $[3/2,+\infty)$. Moreover, we see from (B.4) that $\phi'(x)=O(|x|^{-2})$ as $x\to\infty$. Thus there exists a constant c_0 such that $\lim_{x\to\infty}\phi(x)=c_0$. By the Stirling's formula (see [42]), we have for $m=1,2,\cdots$,

$$\log\Gamma(x)=\left(x-\frac{1}{2}\right)\log x-x+\frac{1}{2}\log(2\pi)+\sum_{l=1}^{m}\frac{B_{2l}}{2l(2l-1)}x^{-2l+1}+O(x^{-2m-1}),$$

where B_{2l} are the Bernoulli numbers. Therefore, as $x\to\infty$,

$$\begin{aligned}\phi(x)=&\log\Gamma(x-1-\alpha)-\log\Gamma(x)+(1+\alpha)\log\left(x-1-\frac{\alpha}{2}\right)\\ =&\left(x-1-\alpha-\frac{1}{2}\right)\log(x-1-\alpha)-(x-1-\alpha)\\ &-\left(x-\frac{1}{2}\right)\log x+x+(1+\alpha)\log\left(x-1-\frac{\alpha}{2}\right)+O(|x|^{-1})\\ =&\left(x-\frac{1}{2}\right)\log\left(1-\frac{1+\alpha}{x}\right)+(1+\alpha)\left[\log\left(1+\frac{\alpha}{2(x-1-\alpha)}\right)+1\right]+O(|x|^{-1}).\end{aligned}$$

Because

$$\lim_{x\to\infty}\left(x-\frac{1}{2}\right)\log\left(1-\frac{1+\alpha}{x}\right)=-(1+\alpha),$$

we get

$$\phi(x)=O(|x|^{-1}),\quad \text{as } x\to+\infty.$$

This then implies that for any $\alpha\in(0,(n-2)/2)$, if $x>1$, then $\phi(x)>0$. ☐

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