CO-EXISTENCE OF TYPE II BLOW-UPS WITH MULTIPLE BLOW-UP RATES FOR FIVE-DIMENSIONAL HEAT EQUATION WITH CRITICAL NONLINEAR BOUNDARY CONDITIONS

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ABSTRACT. We consider the following five-dimensional heat equation with critical boundary condition

$$\partial_t u = \Delta u \text{ in } \mathbb{R}^5_+ \times (0,T), \quad -\partial_{x_5} u = |u|^{\frac{2}{3}} u \text{ on } \partial\mathbb{R}^5_+ \times (0,T).$$

Given $\mathfrak{o}$ distinct boundary points $q^{[i]} \in \partial\mathbb{R}^5_+$, and $\mathfrak{o}$ integers $l_i \in \mathbb{N}$ (possibly duplicated), $i = 1, 2, \ldots, \mathfrak{o}$, for $T > 0$ sufficiently small, we construct a finite-time blow-up solution u with a type II blow-up rate $(T-t)^{-3l_i-3}$ for x near $q^{[i]}$. This seems to be the first rigorous result of the co-existence of type II blowups with different blow-up rates. To accommodate highly unstable blowups with different blowup rates, we first develop a unified linear theory for the inner problem with more time decay in the blow-up scheme through restriction on the spatial growth of the right-hand side, and then use vanishing adjustment functions for deriving multiple rates at distinct points. This paper is inspired by [25, 52, 60].

CONTENTS

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1. Introduction and main results

In this paper we consider the heat equation with the critical boundary conditions

$$\partial_t u = \Delta u \ \text{ in } \ \mathbb{R}^n_+ \times (0,T), \quad -\partial_{x_n} u = |u|^{\frac{2}{n-2}} u \ \text{ on } \ \partial\mathbb{R}^n_+ \times (0,T), \quad u(\cdot,0) = u_0 \ \text{ in } \ \mathbb{R}^n_+, \tag{1.1}$$

where the dimension $n \geq 3$, $\tilde{x} \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}_+$, and $x = (\tilde{x}, x_n) \in \mathbb{R}^n_+$.

The phenomena of finite-time singularity formation triggered by the superlinear right-hand side appear in many evolution equations. It is interesting to reveal how boundary conditions influence the behaviour of solutions for the evolution equations.

In this paper, we give a comprehensive study of the possible phenomenon of the finite-time blow-up of heat equations with the nonlinear boundary condition. The regularity and existence problems of elliptic and parabolic equations with linear and nonlinear oblique derivatives are extensively studied. We refer to [29, Chapter III-5, Chapter V-7], [35, 37, 38].

For an integer $n \geq 2$, $\alpha \in (0,1)$, $p > 0$, $-\infty \leq t_0 < t_1 \leq \infty$, the heat equation with algebraic power nonlinear boundary condition in $\mathbb{R}^n_+$ has the form

$$\begin{cases} \partial_t u = \Delta_{\tilde{x}} u + \dfrac{1-2\alpha}{x_n} \partial_{x_n} u + \partial_{x_n x_n} u = x_n^{-(1-2\alpha)} \nabla_x \cdot \left(x_n^{1-2\alpha} \nabla_x u \right) & \text{for } (x,t) \in \mathbb{R}^n_+ \times (t_0, t_1), \\ -\lim\limits_{x_n \downarrow 0} \dfrac{u(\tilde{x}, x_n, t) - u(\tilde{x}, 0, t)}{x_n^{2\alpha}} = \lim\limits_{x_n \downarrow 0} \dfrac{-x_n^{1-2\alpha} \partial_{x_n} u(\tilde{x}, x_n, t)}{2\alpha} = \left(|u|^{p-1} u \right)(\tilde{x}, 0, t) & \text{for } (\tilde{x}, t) \in \mathbb{R}^{n-1} \times (t_0, t_1), \\ u(x, t_0) = u_0(x) & \text{for } x \in \mathbb{R}^n_+, \end{cases} \tag{1.2}$$

where the initial value is vacuum if $t_0 = -\infty$. This equation appears in the extension form of

$$\frac{\Gamma(-\alpha)}{4^\alpha \Gamma(\alpha)} \left(\partial_t - \Delta_{\tilde{x}} \right)^\alpha u(\tilde{x}, 0, t) = (|u|^{p-1} u)(\tilde{x}, 0, t)$$

for $(\tilde{x}, t) \in \mathbb{R}^{n-1} \times (-\infty, t_1)$ with the Gamma function $\Gamma(\cdot)$ according to [54, Theorem 1.7, 1.8]. (1.1) is a special case of (1.2).

For $(t_0, t_1) = \mathbb{R}$, $p > 0$, (1.2) is dilation and translation invariant in the sense that for u solving (1.2),

$$\lambda u \left(\lambda^{\frac{p-1}{2\alpha}} (\tilde{x} - \xi), \lambda^{\frac{p-1}{2\alpha}} x_n, \lambda^{\frac{p-1}{\alpha}} (t - s) \right) \tag{1.3}$$

still satisfies (1.2) with $\lambda > 0$, $\xi \in \mathbb{R}^{n-1}$, $s \in \mathbb{R}$. The energy associated with (1.2) is

$$J[u] := \frac{1}{2} \int_{\mathbb{R}^n_+} x_n^{1-2\alpha} |\nabla_x u|^2 dx - \frac{2\alpha}{p+1} \int_{\mathbb{R}^{n-1}} |u|^{p+1} (\tilde{x}, 0, t) d\tilde{x}.$$

For u with sufficient smoothness and spatial decay, $\frac{d}{dt} J[u] = -\int_{\mathbb{R}^n_+} x_n^{1-2\alpha} |\partial_t u|^2 dx$. Note that

$$J\left[\lambda u \left(\lambda^{\frac{p-1}{2\alpha}} (\tilde{x} - \xi), \lambda^{\frac{p-1}{2\alpha}} x_n, t \right) \right] = (\lambda^{\frac{p-1}{2\alpha}})^{\frac{4\alpha}{p-1} + 2\alpha - (n-1)} J[u(\tilde{x}, x_n, t)].$$

For this reason, we define the energy critical exponent

$$p_{\alpha,S} := \begin{cases} \infty & \text{if } n \leq 1 + 2\alpha \\ \frac{n-1+2\alpha}{n-1-2\alpha} & \text{if } n > 1 + 2\alpha \end{cases}$$

and $p < (=, >) p_{\alpha,S}$ is called the energy subcritical (critical, supercritical) case.

(1.2) has a close relationship with the well-known Fujita-type equation

$$\partial_t u - \Delta u = |u|^{p-1} u \quad \text{ in } \Omega \times (0,T), \quad u(\cdot, 0) = u_0 \quad \text{ in } \Omega, \tag{1.4}$$

where Ω is domain in $\mathbb{R}^n$. There are three important exponents for (1.4): the Fujita exponent p_F, the Sobolev exponent p_S, and the Joseph-Lundgren exponent p_{JL}, which are defined respectively as

$$p_F := 1 + \frac{2}{n}, \quad p_S := \begin{cases} \infty & \text{if } n = 1, 2 \\ \frac{n+2}{n-2} & \text{if } n \geq 3, \end{cases} \quad p_{JL} := \begin{cases} \infty & \text{if } n \leq 10 \\ 1 + \frac{4}{n-4-2\sqrt{n-1}} & \text{if } n \geq 11. \end{cases}$$

The dynamics and colorful phenomena of (1.4) are sensitive to the power p.

Given an initial value $u_0 \in L^\infty(\Omega)$, there is a unique solution of (1.4) in $L^\infty(\Omega \times (0,t))$ for $t \in (0,T)$ with a maximum life time $T \leq \infty$. We say that u blows up in finite time if $T < \infty$, and in this case, $\limsup_{t \uparrow T} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty$. For $p > 1$, there are two types of blow-ups depending on the rates compared with the ODE solution of (1.4), $(p-1)^{-\frac{1}{p-1}} (T-t)^{-\frac{1}{p-1}}$,

$$\text{Type I}: \limsup_{t \uparrow T} (T-t)^{\frac{1}{p-1}} \|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty, \tag{1.5}$$

$$\text{Type II} : \limsup_{t\uparrow T}(T-t)^{\frac{1}{p-1}}\|u(\cdot,t)\|_{L^\infty(\Omega)}=\infty. \tag{1.6}$$

Indeed, once u blows up at T, by comparison theorem, $\|u(\cdot,t)\|_{L^\infty(\Omega)} \geq (p-1)^{-\frac{1}{p-1}}(T-t)^{-\frac{1}{p-1}}$ with $p>1$ in bounded domain with zero Dirichlet boundary condition by [48, Proposition 23.1]. From the celebrated work [19] of Fujita, (1.4) possesses a global nontrivial solution $u \geq 0$ if and only if $p > p_F$.

For the subcritical case $1<p<p_S$, under the assumption of (1.5), $1<p\leq p_S$, Giga and Kohn [20] gave the asymptotic behavior of solutions in the parabolic cylinder. Giga and Kohn [21] proved (1.5) in a bounded convex domain or $\mathbb{R}^n$, provided that either (i) $u_0\geq 0$ and p satisfies $1<p<p_S$ or $n\leq 2$, or (ii) $1<p<\frac{3n+8}{3n-4}$ or $n=1$. Finally, Giga, Matsui and Sasayama [22, 23] proved (1.5) in a bounded, convex domain or $\mathbb{R}^n$ without assumptions on the sign of solutions. Quittner [46] proved that $\partial_t u = \Delta u + u^p$ in $\mathbb{R}^n \times \mathbb{R}$ does not possess positive classical solutions and there is only type I blow-up for non-negative classical solutions of (1.4) in arbitrary domain of $\mathbb{R}^n$ without convex assumption. For $1<p\leq p_S$, Souplet [53, Theorem 1.1] gave a much simpler proof when the initial value is bounded, nonnegative, radially symmetric, and nonincreasing.

For the critical case $p=p_S$, Filippas, Herrero and Velázquez [18, Section 6] excluded the type II blow-up in the positive radial and monotonically decreasing class for $n\geq 3$. The same result was obtained when the monotone assumption was removed by Matano and Merle [39, Theorem 1.7], and in higher dimensions $n\geq 7$ for positive solutions in bounded convex domains or $\mathbb{R}^n$ without radially symmetric assumptions by Wang and Wei [57].

By formal asymptotic analysis in [18], it is conjectured that there exist sign-changing type II blow-up solutions in lower dimensions $n=3,4,5,6$ and rigorous construction are given in a series of work [49, 12, 14, 25, 26, 31]. For $n=5$, del Pino, Musso, and the first author [12] constructed the first rate. Harada [25] adopted a self-similar variable and then adjusted eigenfunctions with cut-off functions to achieve the fast time decay for the outer problem at the blow-up point for deriving other rates. Zhang and Zhao [60] first constructed finite-time blow-up solutions with multiple different rates by the gluing method. Zhang and Zhao first split the right-hand side of the outer problem into several parts. Second, they introduced the idea of using the solution of the heat equation to eliminate the derivatives for the influence between distinct blow-up points. Finally, applying Tylor expansion deduced the desired vanishing condition for the outer problem at the prescribed blow-up points.

In the case $p_S<p<p_{JL}$, Matano and Merle [39] excluded the type II blow-up in the radially symmetric class when Ω is a ball or $\mathbb{R}^n$ (under some additional requirement). In the non-radial case, del Pino, Musso, and the first author [13] constructed positive type II blow-up solutions in some special domain in $\mathbb{R}^n$ for $n\geq 7$, $p=\frac{n+1}{n-3}\in(p_S,p_{JL})$. Du [15, Theorem 1.3] gave counterpart classification results for finite-time blow-up solutions of rotational symmetric harmonic map heat flow in dimension $3\leq n<7$.

For $p>p_S$, various characterizations of type I and type II blow-ups are established in [40].

For $p=p_{JL}$, Seki [50] constructed a type II blow-up solution.

For $p>p_{JL}$, Herrero and Velázquez [27] first constructed a type II blow-up solution in the radial case and Mizoguchi [41] gave a simpler proof. Collot [4] constructed non-radial type II blow-up solutions under some restrictions of the exponent $p>p_{JL}$.

Mizoguchi and Souplet [42, Theorem 1] built the connection between type I blow-up and $L^{\frac{n(p-1)}{2}}$ norm blow-up. We refer to the monograph [48] for various developments about Fujita-type equations.

Compared with the Fujita-type equations, there are much fewer studies about (1.2) and variations of (1.2). By Proposition A.1, when $(t_0,t_1)=(-\infty,T)$, $p>1$, there exists a solution of (1.2) with the form

$$u_1(x,t)=\frac{x_n^{2\alpha}}{4^\alpha\Gamma(\alpha)}\int_0^\infty \frac{e^{-\frac{x_n^2}{4\tau}}}{\tau^{1+\alpha}}\left(\pm C_{\alpha,p}\right)(T-t+\tau)^{\frac{-\alpha}{p-1}}\,d\tau,\quad C_{\alpha,p}:=\left[\frac{1}{4^\alpha\Gamma(\alpha)}\int_1^\infty\frac{1-z^{\frac{-\alpha}{p-1}}}{(z-1)^{1+\alpha}}dz\right]^{\frac{1}{p-1}}>0 \tag{1.7}$$

independent of $\tilde{x}$ and satisfying

$$u_1(\tilde{x},0,t)=\pm C_{\alpha,p}(T-t)^{\frac{-\alpha}{p-1}},\quad u_1(x,t)\sim\pm\left(\max\left\{T-t,x_n^2\right\}\right)^{\frac{-\alpha}{p-1}}\ \text{ for }\ (x,t)\in\overline{\mathbb{R}^n_+}\times(-\infty,T), \tag{1.8}$$

where "$\sim$" only depends on α,p. Compared with the time rate of (1.7), we define the type I and type II blow-up for (1.2) as

$$\text{Type I} : \limsup_{t\uparrow T}(T-t)^{\frac{\alpha}{p-1}}\|u(\cdot,t)\|_{L^\infty(\Omega)}<\infty, \tag{1.9}$$

$$\text{Type II} : \limsup_{t\uparrow T}(T-t)^{\frac{\alpha}{p-1}}\|u(\cdot,t)\|_{L^\infty(\Omega)}=\infty. \tag{1.10}$$

Fila and Quittner [17, p.205] considered

$$\partial_t u=\Delta u\ \text{ in }\ B\times(0,T),\quad \partial_\nu u=u^p\ \text{ on }\ \partial B\times(0,T),\quad u(\cdot,0)=u_0\ \text{ in }\ \bar{B}$$

with an outer normal derivative ∂_ν and a unit ball $B \subset \mathbb{R}^n$ in the radial class and got the convergence result for blow-up solutions under some non-negative restrictions on u_0 and its derivatives.

As a counterpart of [46], Quittner [45] proved that there is no positive classical bounded solution of (1.2) with $\alpha = \frac{1}{2}$, $1 < p < p_{\frac{1}{2},S}$, and only type I blow-up (1.9) can occur for positive classical solutions in a bounded smooth domain. See Quittner and Souplet [47, Theorem 4.1] for type I blow-up results in the half space without the sign assumption about the solution.

We believe many results in Fujita-type equations can be realized in (1.2) with more general nonlinear boundary conditions.

Now, we state our main results. By classical results of Li and Zhu [33, Theorem 1.2] (see also [16, 3]), for $n \ge 3$, all nonzero nonnegative solutions to

$$\Delta U = 0 \quad \text{in } \mathbb{R}^n_+, \quad -\partial_{x_n} U = U^{\frac{n}{n-2}} \quad \text{on } \partial\mathbb{R}^n_+ \tag{1.11}$$

are given by

$$c^{\frac{n-2}{2}} \left\{ |\tilde{x} - \tilde{x}_0|^2 + \left[x_n + (n-2)^{-1} c \right]^2 \right\}^{-\frac{n-2}{2}}, \quad x = (\tilde{x}, x_n) \in \mathbb{R}^n_+,\ \tilde{x}_0 \in \mathbb{R}^{n-1},\ c > 0. \tag{1.12}$$

We refer to [30, 32] for Liouville theorems for more Yamabe-type equations. Let

$$U(x) = (n-2)^{\frac{n-2}{2}} \left[|\tilde{x}|^2 + (1 + x_n)^2 \right]^{-\frac{n-2}{2}}, \quad x = (\tilde{x}, x_n) \in \mathbb{R}^n_+. \tag{1.13}$$

As in [18, p.2977], we set

$$\Theta_l(x,t) = -(T-t)^l \left(L_l^{\frac{n-2}{2}}(0) \right)^{-1} L_l^{\frac{n-2}{2}} \left(\frac{|x|^2}{4(T-t)} \right) \quad \text{for } n, l \in \mathbb{N},\ n \ge 1 \tag{1.14}$$

with the modified Laguerre polynomials

$$L_l^{\frac{n-2}{2}}(r) := r^{-\frac{n-2}{2}} e^r \frac{d^l}{dr^l} \left(r^{\frac{n-2}{2}+l} e^{-r} \right).$$

Obviously, $L_l^{\frac{n-2}{2}}(r) = \sum_{i=0}^{l} c_i r^i$ with some constants $c_i \in \mathbb{R}$, and $c_0 = L_l^{\frac{n-2}{2}}(0) > 0$. $\Theta_l(x,t)$ satisfies $\Theta_l(0,t) = -(T-t)^l$, and for $n \ge 2$,

$$\partial_t \Theta_l = \Delta \Theta_l \ \text{ in } \mathbb{R}^n_+ \times (-\infty, T), \quad -\partial_{x_n} \Theta_l = 0 \ \text{ on } \partial\mathbb{R}^n_+ \times (-\infty, T), \tag{1.15}$$

where the first formula in (1.15) can be deduced by the second equation below [18, p.2977 (A1)].

Inspired by the work [25, 52, 60], we construct finite-time blow-up solutions at a finite number of prescribed blow-up points with multiple rates for (1.1) with $n = 5$. The main theorem is stated as follows.

Theorem 1.1. *Consider*

$$\partial_t u = \Delta u \ \text{ in } \mathbb{R}^5_+ \times (0,T), \quad -\partial_{x_5} u = |u|^{\frac{2}{3}} u \ \text{ on } \partial\mathbb{R}^5_+ \times (0,T). \tag{1.16}$$

Given an integer $\mathfrak{o} \ge 1$, $\mathfrak{o}$ distinct boundary points $q^{[i]} \in \partial\mathbb{R}^5_+$, and $\mathfrak{o}$ integers $l_i \in \mathbb{N}$ (possibly duplicated), $i = 1, 2, \ldots, \mathfrak{o}$, $\delta = \min_{1 \le i \ne j \le \mathfrak{o}} |q^{[i]} - q^{[j]}|/32$, then for $T > 0$ sufficiently small, there exists a finite-time blow-up solution of the form

$$\begin{aligned} u(x,t) = \sum_{i=1}^{\mathfrak{o}} \Bigg\{ & 3^{\frac{3}{2}} \mu_i^{-\frac{3}{2}} \left[\left| \frac{\tilde{x} - \xi^{[i]}}{\mu_i} \right|^2 + \left(1 + \frac{x_5}{\mu_i} \right)^2 \right]^{-\frac{3}{2}} \eta\left(\frac{x - q^{[i]}}{2\delta} \right) + \Theta_{l_i}(x - q^{[i]}, t) \eta\left(\frac{x - q^{[i]}}{\delta} \right) \\ & + O\left(|\ln T|^2 (T-t)^{l_i} \eta\left(\frac{|x - q^{[i]}|}{2\sqrt{T-t}} \right) \right) \Bigg\} + O\left(|\ln T|^{-\frac{1}{5}} \langle x \rangle^{-2} \left(1 - \sum_{i=1}^{\mathfrak{o}} \eta\left(\frac{|x - q^{[i]}|}{\sqrt{T-t}} \right) \right) \right), \end{aligned}$$

where $x = (\tilde{x}, x_5)$, Θ_{l_i} is defined in (1.14) with $n = 5$, and $\mu_i = \mu_i(t) \in C^1([0,T), \mathbb{R}_+)$, $\xi^{[i]} = \xi^{[i]}(t) \in C^1([0,T), \mathbb{R}^4)$ satisfy

$$\mu_i = \left(D_i + O(|\ln T|^{-\frac{1}{15}}) \right) (T-t)^{2l_i+2}, \quad \left| (\xi^{[i]}, 0) - q^{[i]} \right| \le |\ln T|^{-\frac{1}{15}} (T-t)^{2l_i+2}$$

with some constants $D_i > 0$ independent of T. The initial value $u(\cdot, 0) \in C_c^\infty(\overline{\mathbb{R}^5_+})$ with the support in $\cup_{i=1}^{\mathfrak{o}} \overline{B_5^+(q^{[i]}, 4\delta)}$. In particular, $\|u(\cdot,t)\|_{L^\infty(\overline{B_5^+(q^{[i]},4\delta)})} \sim (T-t)^{-3l_i-3}$ for $i = 1, 2, \ldots, \mathfrak{o}$.

Remark 1.1.1. *The cut-off functions $\eta(\frac{x-q^{[i]}}{2\delta})$, $\eta(\frac{x-q^{[i]}}{\delta})$ are for the purpose of avoiding the influence between distinct blow-up points and the multiple relationship between 2δ and δ is not essential. The radial property of $\eta(x)$ helps simplify the error on the boundary a lot. See (3.3).*

Remark 1.1.2. *It is possible to add the type I blow-up rate in a solution with multiple rates.*

The proof relied on the parabolic gluing method established in the pioneering work [5, 11]. This method has the powerful ability to analyze concentration phenomena in finite-time and infinite-time cases and localize the blow-up points to achieve various concentration phenomena. We refer to [8, 9, 51, 7].

We develop the linear theory for the inner problem with more time decay in the blow-up scheme through spatial restriction on the right-hand sides. We illustrate necessary norms before stating the next proposition.

Given a non-negative function $\ell(\tau)$, we define the following norms for the linear theory for the inner problem. For $-\infty < \tau_0 < \tau_1 \le \infty, \varsigma \in (0,1)$,

$$\begin{aligned}
&\|g\|_{\sigma,2+a,\ell(\tau),\mathbb{R}^n_+,\tau_0,\tau_1} := \inf\left\{C \mid |g(y,\tau)| \le C\tau^\sigma \langle y\rangle^{-2-a}\mathbf{1}_{|y|\le \ell(\tau)} \ \text{ for } \ y\in\mathbb{R}^n_+, \tau_0<\tau<\tau_1\right\},\\
&\|h\|_{\sigma,1+a,\ell(\tau),\mathbb{R}^{n-1},\tau_0,\tau_1} := \inf\left\{C \mid |h(\tilde y,\tau)| \le C\tau^\sigma \langle \tilde y\rangle^{-1-a}\mathbf{1}_{|\tilde y|\le \ell(\tau)} \ \text{ for } \ \tilde y\in\mathbb{R}^{n-1}, \tau_0<\tau<\tau_1\right\},\\
&Q((\tilde y,\tau),r) := \left\{(\tilde z,s)\in\mathbb{R}^{n-1}\times(\tau_0,\tau_1) \mid |\tilde z-\tilde y|<r, \tau-r^2<s<\tau\right\},\\
&[h]_{C^{\varsigma,\frac{\varsigma}{2}}(Q((\tilde y,\tau),|\tilde y|/2))} := \sup_{(\tilde y^{[1]},\tau_1),(\tilde y^{[2]},\tau_2)\in Q((\tilde y,\tau),|\tilde y|/2)} \frac{\left|h(\tilde y^{[1]},\tau_1)-h(\tilde y^{[2]},\tau_2)\right|}{\left(\max\{|\tilde y^{[1]}-\tilde y^{[2]}|, |\tau_1-\tau_2|^{1/2}\}\right)^\varsigma},\\
&[h]_{\sigma,1+a,\ell(\tau),\varsigma,\mathbb{R}^{n-1},\tau_0,\tau_1} := \inf\left\{C \mid [h]_{C^{\varsigma,\frac{\varsigma}{2}}(Q((\tilde y,\tau),|\tilde y|/2))} \le C\tau^\sigma\langle\tilde y\rangle^{-1-a-\varsigma}\mathbf{1}_{|\tilde y|\le 2\ell(\tau)} \ \text{ for } \ \tilde y\in\mathbb{R}^{n-1}, \tau_0<\tau<\tau_1\right\},\\
&\|h\|_{\sigma,1+a,\ell(\tau),\varsigma,\mathbb{R}^{n-1},\tau_0,\tau_1} := \|h\|_{\sigma,1+a,\ell(\tau),\mathbb{R}^{n-1},\tau_0,\tau_1} + [h]_{\sigma,1+a,\ell(\tau),\varsigma,\mathbb{R}^{n-1},\tau_0,\tau_1}.
\end{aligned} \tag{1.17}$$

We arrive at the following linear theory for the inner problem:

Proposition 1.2. *Given an integer $n\ge 5$, consider*

$$\partial_\tau\phi = \Delta\phi + g \ \text{ in } \ \mathbb{R}^n_+\times(\tau_0,\tau_1), \quad -\partial_{y_n}\phi = \frac{n}{n-2}U^{\frac{2}{n-2}}\phi + h \ \text{ on } \ \partial\mathbb{R}^n_+\times(\tau_0,\tau_1), \quad \phi(y,\tau_0) = C_\phi\tilde Z_0(y) \ \text{ in } \ \mathbb{R}^n_+. \tag{1.18}$$

Suppose that $1\le\tau_0<\tau_1\le\infty$, $\ell(\tau)$ satisfies $C_\ell^{-1}\tau^p \le \ell(\tau)\le C_\ell\tau^p$ with a constant $C_\ell\ge 1$,

$$2<a<n-2, \quad a^{-1}<p\le\frac{1}{2}, \quad \iota\in(0,\frac{1}{4}), \quad \sigma - pa + 2\iota n > 0, \quad \varsigma\in(0,1), \tag{1.19}$$

$\|g\|_{\sigma,2+a,\ell(\tau),\mathbb{R}^n_+,\tau_0,\tau_1}<\infty$, $\|h\|_{\sigma,1+a,\ell(\tau),\varsigma,\mathbb{R}^{n-1},\tau_0,\tau_1}<\infty$, and $g=g(y,\tau)$, $h=h(\tilde y,\tau)$ satisfy the orthogonality conditions

$$\int_{\mathbb{R}^n_+} g(y,\tau)Z_j(y)dy + \int_{\mathbb{R}^{n-1}} h(\tilde y,\tau)Z_j(\tilde y,0)d\tilde y = 0 \quad \text{for } \ \tau\in(\tau_0,\tau_1), \quad j=1,2,\dots,n \tag{1.20}$$

with Z_j given in (2.1), $\tilde Z_0(y)\in C^2(\mathbb{R}^n_+)\cap C^{1,\varsigma}(\overline{\mathbb{R}^n_+})$ satisfies

$$\tilde Z_0(y) = 0 \ \text{ for } \ |y|\ge C_0, \quad \int_{\mathbb{R}^n_+}\tilde Z_0 Z_j dy = 0 \quad \text{for } \ j=1,2,\dots,n, \quad \int_{\mathbb{R}^n_+}\tilde Z_0 Z_0 dy \ne 0 \tag{1.21}$$

with Z_0 given in (2.3) and a constant $C_0>0$, then there exist $\phi=\phi[g,h]$ and a constant $C_\phi = C_\phi[g,h]$ as linear mappings of g,h solving (1.18) and satisfying

$$\int_{\mathbb{R}^n_+}\phi(y,\tau)Z_j(y)dy = 0 \quad \text{for } \ \tau\in(\tau_0,\tau_1), \quad j=1,2,\dots,n, \tag{1.22}$$

and

$$\begin{aligned}
&|\phi| \lesssim \left(\tau^\sigma\langle y\rangle^{-a}\mathbf{1}_{|y|\le\ell(\tau)} + \tau^\sigma\ell^{-a}(\tau)e^{-\iota\frac{|y|^2}{\tau}}\mathbf{1}_{|y|>\ell(\tau)}\right)\left(\|g\|_{\sigma,2+a,\ell(\tau),\mathbb{R}^n_+,\tau_0,\tau_1} + \|h\|_{\sigma,1+a,\ell(\tau),\mathbb{R}^{n-1},\tau_0,\tau_1}\right),\\
&|\nabla\phi| \lesssim \left(\tau^\sigma\langle y\rangle^{-1-a}\mathbf{1}_{|y|\le\ell(\tau)} + \tau^\sigma\ell^{-a}(\tau)|y|^{-1}\mathbf{1}_{\ell(\tau)<|y|\le\tau^{\frac12}} + \tau^{\sigma-\frac12}\ell^{-a}(\tau)e^{-\tilde\iota\frac{|y|^2}{\tau}}\mathbf{1}_{|y|>\tau^{\frac12}}\right)\\
&\qquad\times\left(\|g\|_{\sigma,2+a,\ell(\tau),\mathbb{R}^n_+,\tau_0,\tau_1} + \|h\|_{\sigma,1+a,\ell(\tau),\varsigma,\mathbb{R}^{n-1},\tau_0,\tau_1}\right),\\
&|C_\phi| \lesssim \tau_0^\sigma\left(\|g\|_{\sigma,2+a,\ell(\tau),\mathbb{R}^n_+,\tau_0,\tau_1} + \|h\|_{\sigma,1+a,\ell(\tau),\mathbb{R}^{n-1},\tau_0,\tau_1}\right)
\end{aligned} \tag{1.23}$$

with a constant $\tilde\iota\in(0,\iota)$, where all "$\lesssim$" are independent of τ_0,τ_1,g,h.

Remark 1.2.1. *By Lemma 2.4, there exists $\tilde Z_0(y)\in C^\infty(\overline{\mathbb{R}^n_+})$ satisfying the assumption (1.21).*

Remark 1.2.2. *The orthogonality conditions (1.20) are initiated from [52, Proposition 4.1, 4.2].*

Main difficulties and novelties.

• Due to the lack of an ODE method for the linearized equations around steady state solution $U(x)$, the strategy for deriving the linear theory for the inner problem in [5, Section 7] does not work here. Instead, we resort to the parabolic blow-up argument to get a desired linear theory. The parabolic blow-up argument was first introduced in the linear theory of mode 1 in [11, Section 7.3], and this argument works for many heat flows of Schrödinger operator with nondegenerate property. See [44, Section 5.1] for instance. In the proof of [58, Proposition A.2], the authors modified the blow-up argument when the spatial variable goes to infinity.

For $\mathcal{T}_{\mathbb{R}^n}[\cdot]$ defined in (B.8) with $n>2$, if $b\le 2$, $\lim_{t\to\infty} t^{-a}\mathcal{T}_{\mathbb{R}^n}\left[t^a\langle x\rangle^{-b}\right](0,t)=\infty$ by the similar lower bound estimate in [59, Lemma A.1]; if $b\ge n$ or $a\le -1$, $\lim_{t\to\infty}\left\{\left[(t^a\langle x\rangle^{2-b})^{-1}\mathcal{T}_{\mathbb{R}^n}\left[t^a\langle x\rangle^{-b}\right](x,t)\right]\big|_{|x|=\sqrt{t}}\right\}=\infty$ by similar lower bound estimates in [59, Lemma A.1] and [59, Lemma A.2] respectively. Thus the parameter about the growth of time in [58, Proposition A.2] should be optimal in the algebraic power sense.

To obtain better time decay for the linear theory in a wider application, we impose restrictions on the spatial growth of the right-hand side. Additionally, we discover that the scaling argument when the spatial variable goes to infinity is not necessary in previous works. Instead, we utilize convolution estimates directly.

• For the outer problem, Harada [25, 26] used a linear combination of eigenfunctions corresponding $\Delta_z-\frac{z}{2}\cdot\nabla_z$ with cut-off functions as the modified functions to adjust the time vanishing at the blow-up point in the outer problem. However, the algebraic growth of these eigenfunctions and the rough cut-off functions cause complexity in calculation.

We realized that the modified functions need not be eigenfunctions corresponding $\Delta_z-\frac{z}{2}\cdot\nabla_z$. The key point is that the modified functions are not orthogonal with these eigenfunctions, so we have more flexibility in the choice of modified functions and simplify the proof for the outer problem. See Corollary 2.5 and the application in Section 6.

• In order to derive multiple rates at distinct points, we introduce vanishing adjustment functions to eliminate derivatives at arbitrarily prescribed finitely many points. This method works for more general parabolic equations. See Proposition 2.13.

The method of adjusting the initial value for improving the vanishing condition of the outer problem was implemented in [11, p.386] for the case of one bubble with the first rate. Zhang and Zhao [60, p.8] introduced the idea of using the heat equation with some special initial value to improve the vanishing at the blow-up points for multiple rates. We make this process clearer and summarize it as the vanishing adjustment functions. A counterpart for the Schrödinger equation is given in [1, Lemma 4.1].

• The compactness argument for the parameters in the initial value in the fixed-point argument in Subsection 7.4 deduces a smooth initial value $u(x,0)$ with compact support. This can not be derived from the parabolic regularity theory. Although for a fixed $t\in(0,T)$, $u(x,t)$ is smooth, the support of $u(x,t)$ is usually not compact.

• Convolution estimates in Appendix B are in the spirit of [59, Appendix A] (See also [56, Appendix B.1]). We combine the comparison theorem to approach the best constant in the power of the exponential term. Appendix B is prepared for the proof of Proposition 4.2 and also establishes the foundation for the long-time dynamics of (1.1).

Notations:

- Denote $\mathbb{R}_+=(0,\infty)$, $\overline{\mathbb{R}_+}=[0,\infty)$, $\mathbb{R}^n_+=\mathbb{R}^{n-1}\times\mathbb{R}_+$, $\overline{\mathbb{R}^n_+}=\mathbb{R}^{n-1}\times\overline{\mathbb{R}_+}$.
- Denote the natural number set $\mathbb{N}=\{0,1,\dots\}$, the set of natural numbers n-tuples $\mathbb{N}^n=\underbrace{\mathbb{N}\times\mathbb{N}\times\cdots\times\mathbb{N}}_{\text{n multiplicities}}$. For $C_1,C_2\in\mathbb{R}$, denote $C_1\mathbb{N}+C_2=\{C_1 i+C_2\mid i\in\mathbb{N}\}$. In particular, $2\mathbb{N}$ $(2\mathbb{N}+1)$ is the set of even (odd) numbers.
- For a $m\times n$ matrix $A=(A_{ij})_{m\times n}$, denote $\|A\|_{\ell_1}=\sum_{i=1}^m\sum_{j=1}^n|A_{ij}|$.
- For $x\in\mathbb{R}^n$, denote the Japanese bracket $\langle x\rangle=\sqrt{1+|x|^2}$.
- Given $q\in\mathbb{R}^n$, $r>0$, denote $B_n(q,r)=\{x\in\mathbb{R}^n\mid |x-q|<r\}$, $B_n^+(q,r)=B_n(q,r)\cap\mathbb{R}^n_+$.
- For any $x\in\mathbb{R}^n$, we use $\tilde{x}$, x_n to denote the first $n-1$ components and the nth component of x respectively.
- $C(a,b,\dots)$ denotes a constant only depending on parameters $a,b,\dots$.
- We write $a\lesssim b$ (resp. $a\gtrsim b$) if there exists a constant $C>0$ independent of T such that $a\le Cb$ (resp. $a\ge Cb$). Set $a\sim b$ if $b\lesssim a\lesssim b$. Given a non-negative function g, $O(g)$ denotes some function f satisfying $|f|\lesssim g$.
- We write $a\lesssim_{\alpha,\beta,\dots} b$ (resp. $a\gtrsim_{\alpha,\beta,\dots} b$) to emphasize that there exists a constant $C(\alpha,\beta,\dots)>0$ such that $a\le C(\alpha,\beta,\dots)b$ (resp. $a\ge C(\alpha,\beta,\dots)b$). Set $a\sim_{\alpha,\beta,\dots} b$ if $b\lesssim_{\alpha,\beta,\dots} a\lesssim_{\alpha,\beta,\dots} b$.
- For constants $C_1,C_2>0$, the symbol $C_1\ll C_2$ denotes that there exists a constant $c>0$ sufficiently small such that $C_1\le cC_2$.
- For $C\in\mathbb{R}$, the ceiling function $\lceil C\rceil$ denotes the smallest integer greater than or equal to C.
- Given a positive integer n, a multi-index $\mathbf{m}=(m_1,m_2,\dots,m_n)\in\mathbb{N}^n$, and a sufficiently smooth function f defined in a domain of $\mathbb{R}^n$, denote $D_x^{\mathbf{m}}f=\partial_{x_1}^{m_1}\partial_{x_2}^{m_2}\cdots\partial_{x_n}^{m_n}f$.
- $\eta(x)$ is a smooth radial cut-off function in $\mathbb{R}^n$ satisfying $\eta(x)=1$ for $|x|\le 1$, $\eta(x)=0$ for $|x|\ge 2$, and $0\le\eta(x)\le 1$ in $\mathbb{R}^n$.
- Denote $\mathbf{1}_\Omega(x)$ as the indicator function with $\mathbf{1}_\Omega(x)=1$ if $x\in\Omega$ and $\mathbf{1}_\Omega(x)=0$ if $x\notin\Omega$. We will use $\mathbf{1}_\Omega$ to denote $\mathbf{1}_\Omega(x)$ if no ambiguity.

The structure of this paper is as follows. Section 2 is the preliminary to prepare some backgrounds and useful estimates. In Section 3, we calculate the errors of the approximate solution and give the inner-outer gluing system. Section 4 builds the linear theory of inner problem by blow-up argument. Section 5 gives the formal calculation of μ_i and introduces the topology for the fixed-point argument. In Section 6, we study the outer problem and obtain apriori estimates. Finally, in Section 7, we use the Schauder fixed-point theorem to solve the inner-outer gluing system away from T and then deduce Theorem 1.1.

2. Preliminary

2.1. **Kernels and the eigenfunction with negative eigenvalue.** For $n \geq 3$, by dilation and translation invariance (1.3), the linearized operator of the steady equation of (1.1) around U has kernels

$$\begin{aligned} Z_i(x) &:= \partial_{x_i} U = -(n-2)^{\frac{n}{2}} \left[|\tilde{x}|^2 + (1+x_n)^2 \right]^{-\frac{n}{2}} x_i, \quad i = 1, 2, \ldots, n-1, \\ Z_n(x) &:= \frac{n-2}{2} U + x \cdot \nabla U = 2^{-1}(n-2)^{\frac{n}{2}} \left[|\tilde{x}|^2 + (1+x_n)^2 \right]^{-\frac{n}{2}} \left(1 - |x|^2\right), \end{aligned} \tag{2.1}$$

which satisfy

$$\Delta Z_i = 0 \quad \text{in } \mathbb{R}^n_+, \quad -\partial_{x_n} Z_i = \frac{n}{n-2} U^{\frac{2}{n-2}} Z_i \quad \text{on } \partial\mathbb{R}^n_+, \qquad i = 1, 2, \ldots, n. \tag{2.2}$$

By Proposition D.5, the corresponding eigenvalue problem

$$-\Delta Z_0 = \lambda_0 Z_0 \quad \text{in } \mathbb{R}^n_+, \quad -\partial_{x_n} Z_0 = \frac{n}{n-2} U^{\frac{2}{n-2}} Z_0 \quad \text{on } \partial\mathbb{R}^n_+ \tag{2.3}$$

has only one negative eigenvalue λ_0 and λ_0 is simple with an eigenfunction $Z_0(x) \in C^\infty(\overline{\mathbb{R}^n_+}) \cap H^1(\mathbb{R}^n_+)$ satisfying $\|Z_0\|_{L^2(\mathbb{R}^n_+)} = 1$, $0 < Z_0(x) \leq C e^{-\nu|x|}$ in $\overline{\mathbb{R}^n_+}$ for all $\nu \in \left[0, \sqrt{-\lambda_0}\right)$ with a constant C depending on n, λ_0, ν.

2.2. **Properties of the operator $A_z = \Delta_z - \frac{z}{2} \cdot \nabla_z$.** Given a domain $\Omega \subset \mathbb{R}^n$, define the weighted $L^2(\Omega)$ space by

$$L^2_\rho(\Omega) := \left\{ f \mid \|f\|_{L^2_\rho(\Omega)} < \infty \right\} \tag{2.4}$$

equipped with the inner product and the norm

$$(f_1, f_2)_{L^2_\rho(\Omega)} := \int_\Omega f_1(z) f_2(z) \rho(z) dz, \quad \rho(z) := e^{-\frac{|z|^2}{4}}, \quad \|f\|_{L^2_\rho(\Omega)} := (f, f)^{1/2}_{L^2_\rho(\Omega)}.$$

Define the weighted $H^1(\Omega)$ space by

$$H^1_\rho(\Omega) := \left\{ f \mid f, \nabla f \in L^2_\rho(\Omega) \right\} \tag{2.5}$$

equipped with the inner product and the norm

$$(f_1, f_2)_{H^1_\rho(\Omega)} := (\nabla f_1, \nabla f_2)_{L^2_\rho(\Omega)} + (f_1, f_2)_{L^2_\rho(\Omega)}, \quad \|f\|_{H^1_\rho(\Omega)} := (f, f)^{1/2}_{H^1_\rho(\Omega)}.$$

Denote $A_z = \Delta_z - \frac{z}{2} \cdot \nabla_z$. For $f, g \in C^2(\mathbb{R}^n_+) \cap C^1(\overline{\mathbb{R}^n_+})$,

$$\begin{aligned} (A_z f, g)_{L^2_\rho(\mathbb{R}^n_+)} &= -(\nabla f, \nabla g)_{L^2_\rho(\mathbb{R}^n_+)} + \int_{\partial\mathbb{R}^n_+} g(z) e^{-\frac{|z|^2}{4}} (-\partial_{z_n} f)\, dS \\ &= (A_z g, f)_{L^2_\rho(\mathbb{R}^n_+)} + \int_{\partial\mathbb{R}^n_+} g(z) e^{-\frac{|z|^2}{4}} (-\partial_{z_n} f)(z) dS - \int_{\partial\mathbb{R}^n_+} f(z) e^{-\frac{|z|^2}{4}} (-\partial_{z_n} g)(z) dS. \end{aligned} \tag{2.6}$$

Denote $\tilde{H}_\alpha(s) = H_\alpha(\frac{s}{2})$, $\alpha \in \mathbb{N}$, $s \in \mathbb{R}$, where $H_\alpha(r) = (-1)^\alpha e^{r^2} \frac{d^\alpha}{dr^\alpha}(e^{-r^2})$ is the Hermite polynomial. Two basic properties of $\tilde{H}_\alpha(x)$ are given in the following lemma without proof.

Lemma 2.1. (1) *$\tilde{H}_\alpha(x)$ is an even (odd) polynomial of order α provided that α is an even (odd) number. And $\tilde{H}_\alpha(0) \neq 0$ if α is even, and $\frac{d}{dx}\tilde{H}_\alpha(0) \neq 0$ if α is odd.*

(2) *$(\tilde{H}_\alpha)_{\alpha\in\mathbb{N}}$ is the basis of the eigenfunctions of $-\left(\partial_{ss} - \frac{s}{2}\partial_s\right)$ in $L^2_\rho(\mathbb{R})$ satisfying $-\left(\partial_{ss} - \frac{s}{2}\partial_s\right) \tilde{H}_\alpha(s) = \frac{\alpha}{2}\tilde{H}_\alpha(s)$, and is an orthogonal basis in $L^2_\rho(\mathbb{R})$.*

Denote a multi-index $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with each $\alpha_i \in \mathbb{N}$, and $\tilde{\mathbf{H}}_{\boldsymbol{\alpha}}(z) := \prod_{i=1}^n \tilde{H}_{\alpha_i}(z_i)$. By separation of variables, $(\tilde{\mathbf{H}}_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{N}^n}$ is the basis of the eigenfunctions of $-A_z$ in $L^2_\rho(\mathbb{R}^n)$ satisfying $-A_z \tilde{\mathbf{H}}_{\boldsymbol{\alpha}}(z) = 2^{-1}\|\boldsymbol{\alpha}\|_{\ell_1} \tilde{\mathbf{H}}_{\boldsymbol{\alpha}}(z)$, and is an orthogonal basis in $L^2_\rho(\mathbb{R}^n)$.

Given a domain $\Omega \subset \mathbb{R}^n$, denote

$$\begin{aligned} &\mathcal{E}_{i/2}(\Omega) := \left\{ \tilde{\mathbf{H}}_{\boldsymbol{\alpha}}(z), z \in \Omega \mid |\boldsymbol{\alpha}| = i \right\}, \\ &\mathcal{E}_{i/2,\text{even}}(\Omega) := \left\{ \tilde{\mathbf{H}}_{\boldsymbol{\alpha}}(z) \in \mathcal{E}_{i/2}(\Omega) \mid \alpha_n \text{ is even} \right\}, \quad \mathcal{E}_{i/2,\text{odd}}(\Omega) := \left\{ \tilde{\mathbf{H}}_{\boldsymbol{\alpha}}(z) \in \mathcal{E}_{i/2}(\Omega) \mid \alpha_n \text{ is odd} \right\}. \end{aligned} \tag{2.7}$$

Some basic properties of Hermite polynomials are listed in the next lemma without proof.

Lemma 2.2. (1) *The eigenvalue problem $-A_z h = \lambda h$ in $\mathbb{R}^n$, $h \in L^2_\rho(\mathbb{R}^n)$ has the eigenvalue $\lambda = \frac{i}{2}$, $i \in \mathbb{N}$ with $\mathcal{E}_{i/2}(\mathbb{R}^n)$ as the orthogonal basis of the associated eigenspace.*

(2) *For $f_1, f_2 \in \cup_{j=0}^{\infty} \mathcal{E}_{j/2}(\mathbb{R}^n)$ and $f_1 \neq f_2$, we have $(f_1, f_2)_{L^2_\rho(\mathbb{R}^n)} = 0$.*

(3) *For $i \in \mathbb{N}$, denote $S_1 = \left\{ f \in H^1_\rho(\mathbb{R}^n) \mid (f, g)_{L^2_\rho(\mathbb{R}^n)} = 0 \text{ for all } g \in \cup_{j=0}^{i} \mathcal{E}_{j/2}(\mathbb{R}^n) \right\}$, then*

$$\inf_{0 \not\equiv f \in S_1} \|f\|^{-2}_{L^2_\rho(\mathbb{R}^n)} \|\nabla f\|^2_{L^2_\rho(\mathbb{R}^n)} = \frac{i+1}{2}. \tag{2.8}$$

A counterpart in the half space $\mathbb{R}^n_+$ is the following.

Lemma 2.3. (1) *The eigenvalue problem with the Neumann (Dirichlet) boundary condition*

$$-A_z h = \lambda h \quad \text{in } \mathbb{R}^n_+, \quad -\partial_{z_n} h = 0 \ (h = 0) \quad \text{on } \partial\mathbb{R}^n_+ \tag{2.9}$$

in $L^2_\rho(\mathbb{R}^n_+)$ has the eigenvalue $\lambda = \frac{i}{2}$ with $i \in \mathbb{N}$, and $\mathcal{E}_{i/2,\mathrm{even}}(\mathbb{R}^n_+)$ $(\mathcal{E}_{i/2,\mathrm{odd}}(\mathbb{R}^n_+))$ is the orthogonal basis of the associated eigenspace.

(2) *For $f_1, f_2 \in \cup_{j=0}^{\infty} \mathcal{E}_{j/2,\mathrm{even}}(\mathbb{R}^n_+)$ $(\mathcal{E}_{j/2,\mathrm{odd}}(\mathbb{R}^n_+))$ and $f_1 \neq f_2$, we have $(f_1, f_2)_{L^2_\rho(\mathbb{R}^n_+)} = 0$.*

(3) *For any $i \in \mathbb{N}$, denote $S_2 = \left\{ f \in H^1_\rho(\mathbb{R}^n_+) \mid (f, g)_{L^2_\rho(\mathbb{R}^n_+)} = 0 \text{ for all } g \in \cup_{j=0}^{i} \mathcal{E}_{j/2,\mathrm{even}}(\mathbb{R}^n_+) \right\}$,*

$$\inf_{0 \not\equiv f \in S_2} \|f\|^{-2}_{L^2_\rho(\mathbb{R}^n_+)} \|\nabla f\|^2_{L^2_\rho(\mathbb{R}^n_+)} = \frac{i+1}{2}. \tag{2.10}$$

Proof. Given any function f in $\overline{\mathbb{R}^n_+}$, denote

$$f_{\mathrm{e}}(z) := \begin{cases} f(z) & \text{if } z_n \geq 0 \\ f(\tilde{z}, -z_n) & \text{if } z_n < 0 \end{cases} \quad \text{and} \quad f_{\mathrm{o}}(z) := \begin{cases} f(z) & \text{if } z_n \geq 0 \\ -f(\tilde{z}, -z_n) & \text{if } z_n < 0. \end{cases}$$

(1). For the Neumann boundary condition, obviously $h_{\mathrm{e}}(z) \in L^2_\rho(\mathbb{R}^n)$ and it is straightforward to derive $-A_z h_{\mathrm{e}}(z) = \lambda h_{\mathrm{e}}(z)$ in $\mathbb{R}^n$. By Lemma 2.2 (1), $\partial_{z_n} h_{\mathrm{e}}|_{z_n=0} = 0$, and Lemma 2.1 (1), then $\lambda \in \frac{1}{2}\mathbb{N}$ and $h \in \mathcal{E}_{\lambda,\mathrm{even}}(\mathbb{R}^n_+)$. On the other hand, all elements in $\mathcal{E}_{i/2,\mathrm{even}}(\mathbb{R}^n_+)$, $i \in \mathbb{N}$ satisfy (2.9) with $\lambda = i/2$.

For the Dirichlet boundary condition, the conclusion can be deduced by analyzing $h_{\mathrm{o}}(z)$ similarly.

(2). For any g_1, g_2 in $\overline{\mathbb{R}^n_+}$, $(g_{1\mathrm{e}}, g_{2\mathrm{e}})_{L^2_\rho(\mathbb{R}^n)} = 2\,(g_1, g_2)_{L^2_\rho(\mathbb{R}^n_+)} = (g_{1\mathrm{o}}, g_{2\mathrm{o}})_{L^2_\rho(\mathbb{R}^n)}$. Then the result holds.

(3). For any $f \in S_2$, we have $f_{\mathrm{e}}(z) \in H^1_\rho(\mathbb{R}^n)$; $(f_{\mathrm{e}}, g)_{L^2_\rho(\mathbb{R}^n)} = 0$ for all $g \in \mathcal{E}_{j/2,\mathrm{odd}}(\mathbb{R}^n), j \in \mathbb{N}$; $(f_{\mathrm{e}}, g)_{L^2_\rho(\mathbb{R}^n)} = 2(f, g)_{L^2_\rho(\mathbb{R}^n_+)} = 0$ for all $g \in \cup_{j=0}^{i} \mathcal{E}_{j/2,\mathrm{even}}(\mathbb{R}^n)$. Applying Lemma 2.2 (3) to f_{e}, we get $\inf_{0 \not\equiv f \in S_2} \|f\|^{-2}_{L^2_\rho(\mathbb{R}^n_+)} \|\nabla f\|^2_{L^2_\rho(\mathbb{R}^n_+)} \geq \frac{i+1}{2}$. The equality sign can be attained by $\mathcal{E}_{(i+1)/2,\mathrm{even}}(\mathbb{R}^n_+)$. □

The next lemma gives a general localized modification method for approximate orthogonal functions.

Lemma 2.4. *Given a domain $\Omega \subset \mathbb{R}^n$ (possibly unbounded), a weight $w(z)$ (possibly sign-changing), and an integer $m \geq 1$, suppose that for $i, j = 1, 2, \ldots, m$, $\vartheta_i(z)\vartheta_j(z)w(z) \in L^1_{\mathrm{loc}}(\Omega)$, a function $\chi(z) \in L^\infty(\Omega)$ with compact support,*

$$\sum_{l=1, l\neq i}^{m} \left| \int_\Omega \vartheta_i(z)\vartheta_l(z)w(z)\chi(z)dz \right| < \left| \int_\Omega \vartheta_i^2(z)w(z)\chi(z)dz \right| < \infty \quad \text{for } i = 1, 2, \ldots, m,$$

then there exist functions $\tilde{\vartheta}_i(z)$, $i = 1, 2, \ldots, m$ of the form $\tilde{\vartheta}_i(z) = \sum_{l=1}^{m} a_{il}\vartheta_l(z)\chi(z)$, where $(a_{il})_{m\times m}$ is the inverse matrix of $\left(\int_\Omega \vartheta_i(z)\vartheta_l(z)w(z)\chi(z)dz\right)_{m\times m}$, such that

$$\int_\Omega \tilde{\vartheta}_i(z)\vartheta_j(z)w(z)dz = \delta_{ij} \quad \text{for } i, j = 1, 2, \ldots, m.$$

Proof. The conclusion is deduced by the non-singularity of the strictly diagonally dominant matrix. □

We arrange the eigenfunctions of (2.9) with the Neumann boundary condition according to non-decreasing eigenvalues and label them as $(e_i(z))_i$, $i = 0, 1, \ldots$. Due to the multiplicity of eigenspaces, the arrangement appears to be non-unique. We only fix one sequence $(e_i(z))_i$ and $e_i(z)$ satisfies

$$-A_z e_i(z) = \lambda_i e_i(z) \quad \text{in } \mathbb{R}^n_+, \quad -\partial_{z_n} e_i(z) = 0 \text{ on } \partial\mathbb{R}^n_+, \tag{2.11}$$

where the eigenvalues λ_i are non-decreasing about i and $\lambda_i \to \infty$ as $i \to \infty$. Define the eigenvalue counting function of $-A_z$ by

$$N(C) := \#\left\{ i \in \mathbb{N} \mid \lambda_i \leq C \right\}, \quad C \in \mathbb{R}. \tag{2.12}$$

Corollary 2.5. *Given $m \in \mathbb{N}$, there exists a constant $M > 0$ sufficiently large and a non-singular symmetric constant real-valued matrix $(a_{il})_{(m+1)\times(m+1)}$ to make*

$$\tilde{e}_i(z) := \sum_{l=0}^{m} a_{il} e_l(z) \eta(\frac{z}{M}), \quad i = 0, 1, \ldots, m \tag{2.13}$$

satisfy

$$\partial_{z_n} \tilde{e}_i = 0 \quad \text{on } \partial\mathbb{R}^n_+ \quad \text{and} \quad (\tilde{e}_i, e_j)_{L^2_\rho(\mathbb{R}^n_+)} = \delta_{ij} \quad \text{for } i, j = 0, 1, \ldots, m. \tag{2.14}$$

Proof. By Lemma 2.3 (2), and Lemma 2.4 in the case $\Omega = \mathbb{R}^n_+$, $w(z) = \rho(z)$, $(e_i(z))_{0\le i\le m}$, $\chi(z) = \eta(\frac{z}{M})$ with $M > 0$ sufficiently large, we get the desired $\tilde{e}_i(z)$ satisfying the second property in (2.14). Since $\eta(x)$ is a smooth radial cut-off function and $\partial_{z_n} e_i = 0$, $i \in \mathbb{N}$ on $\partial\mathbb{R}^n_+$, we have $\partial_{z_n}\tilde{e}_i = 0$ on $\partial\mathbb{R}^n_+$. □

2.3. **Green's functions in $\mathbb{R}^n_+$.** The fundamental solution with the Neumann boundary condition in $\mathbb{R}^n_+$ has the form

$$G_n(x, t, z, s) = [4\pi(t-s)]^{-\frac{n}{2}} \left[e^{-\frac{|\tilde{x}-\tilde{z}|^2+|x_n-z_n|^2}{4(t-s)}} + e^{-\frac{|\tilde{x}-\tilde{z}|^2+|x_n+z_n|^2}{4(t-s)}} \right] \quad \text{for } t > s. \tag{2.15}$$

Obviously,

$$\begin{aligned} &\partial_t^\iota D_x^{\mathbf{m}} G_n(x,t,z,s), \iota \in \mathbb{N}, \mathbf{m} \in \mathbb{N}^n \text{ is odd (even) about } x_n \text{ if } m_n \text{ is odd (even) for } t > s. \\ &\partial_t^\iota D_x^{\mathbf{m}} G_n((\tilde{x}, 0), t, z, s) = 0 \text{ if } m_n \text{ is odd for } t > s. \end{aligned} \tag{2.16}$$

By Green's identities, for f with sufficient smoothness and decay, we have

$$\begin{aligned} f(x,t) = &\int_{t_0}^{t} \int_{\mathbb{R}^n_+} G_n(x,t,z,s) \left(\partial_s f - \Delta_z f\right)(z,s) dz ds \\ &+ \int_{t_0}^{t} \int_{\mathbb{R}^{n-1}} G_n(x,t,(\tilde{z},0),s) \left[\left(-\partial_{z_n} f\right)((\tilde{z},0),s)\right] d\tilde{z} ds + \int_{\mathbb{R}^n_+} G_n(x,t,z,t_0) f(z,t_0) dz. \end{aligned} \tag{2.17}$$

(2.17) can be used for representing solutions in some weak sense with right-hand sides, boundary value and initial value in some Sobolev spaces.

Lemma 2.6. *Let $n \ge 2$ be an integer, $-\infty < t_1 < t_2 < \infty$,*

$$\partial_t \bar{\theta} = \Delta_x \bar{\theta} + g_1(x,t) \ \text{ in } \mathbb{R}^n_+ \times (t_1, t_2), \quad -\partial_{x_n} \bar{\theta} = g_2(\tilde{x}, t) \ \text{ on } \partial\mathbb{R}^n_+ \times (t_1, t_2), \quad \bar{\theta}(x, t_1) = g_3(x) \ \text{ in } \mathbb{R}^n_+. \tag{2.18}$$

Given $t_3 \in [t_2, \infty)$, $q \in \partial\mathbb{R}^n_+$, set

$$\begin{aligned} &s = s(t) = -\ln(t_3 - t) \in (s_1, s_2), \quad s_i = s(t_i) = -\ln(t_3 - t_i), \ i = 1, 2, \quad z = z(x,t) = (t_3 - t)^{-\frac{1}{2}} (x - q), \\ &\text{that is,} \quad t = t(s) = t_3 - e^{-s}, \quad t_i = t(s_i) = t_3 - e^{-s_i}, \ i = 1, 2, \quad x = x(z,s) = e^{-\frac{s}{2}} z + q; \\ &\bar{\theta}(x,t) := \theta\big((t_3 - t)^{-\frac{1}{2}} (x - q), -\ln(t_3 - t)\big), \quad \text{that is,} \quad \theta(z,s) = \bar{\theta}\big(e^{-\frac{s}{2}} z + q, t_3 - e^{-s}\big). \end{aligned}$$

Then (2.18) is equivalent to

$$\begin{cases} \partial_s \theta = \Delta_z \theta - \frac{z}{2} \cdot \nabla_z \theta + e^{-s} g_1(e^{-\frac{s}{2}} z + q, t_3 - e^{-s}) & \text{for } (z,s) \in \mathbb{R}^n_+ \times (s_1, s_2), \\ -\partial_{z_n} \theta = e^{-\frac{s}{2}} g_2(e^{-\frac{s}{2}} \tilde{z} + \tilde{q}, t_3 - e^{-s}) & \text{for } (z,s) \in \partial\mathbb{R}^n_+ \times (s_1, s_2), \\ \theta(z, s_1) = g_3\big((t_3 - t_1)^{\frac{1}{2}} z + q\big) & \text{for } z \in \mathbb{R}^n_+. \end{cases}$$

Proof. Plug $\bar{\theta}(x,t) := \theta\big((t_3 - t)^{-\frac{1}{2}} (x - q), -\ln(t_3 - t)\big)$ into (2.18), then

$$\begin{aligned} &2^{-1} (t_3 - t)^{-\frac{3}{2}} (x - q) \cdot (\nabla_z \theta) \big((t_3 - t)^{-\frac{1}{2}} (x - q), -\ln(t_3 - t)\big) \\ &+ (t_3 - t)^{-1} (\partial_s \theta) \big((t_3 - t)^{-\frac{1}{2}} (x - q), -\ln(t_3 - t)\big) \\ = \ &(t_3 - t)^{-1} (\Delta_z \theta) \big((t_3 - t)^{-\frac{1}{2}} (x - q), -\ln(t_3 - t)\big) + g_1(x,t) \ \text{ in } \mathbb{R}^n_+ \times (t_1, t_2), \\ &(t_3 - t)^{-\frac{1}{2}} (-\partial_{z_n} \theta) \big((t_3 - t)^{-\frac{1}{2}} (x - q), -\ln(t_3 - t)\big) = g_2(\tilde{x}, t) \ \text{ on } \partial\mathbb{R}^n_+ \times (t_1, t_2). \end{aligned}$$

Changing the variables deduces the conclusion. □

Lemma 2.7. *Let $n \ge 2$ be an integer, $-\infty < s_1 < s_2 \le \infty$,*

$$\partial_s \theta = \Delta_z \theta - \frac{z}{2} \cdot \nabla_z \theta + f_1(z,s) \ \text{ in } \mathbb{R}^n_+ \times (s_1, s_2), \quad -\partial_{z_n} \theta = f_2(\tilde{z}, s) \ \text{ on } \partial\mathbb{R}^n_+ \times (s_1, s_2), \quad \theta(z, s_1) = f_3(z) \ \text{ in } \mathbb{R}^n_+. \tag{2.19}$$

Given $t_3 \in \mathbb{R}$, $q \in \partial\mathbb{R}^n_+$, *set*

$$t = t(s) := t_3 - e^{-s} \in (t_1, t_2), \quad t_i = t_i(s_i) := t_3 - e^{-s_i},\ i = 1, 2, \quad x = x(z, s) := e^{-\frac{s}{2}} z + q,$$

$$\textit{that is, } s = s(t) = -\ln(t_3 - t), \quad s_i = s(t_i) = -\ln(t_3 - t_i),\ i = 1, 2, \quad z = z(x, t) = (t_3 - t)^{-\frac{1}{2}}(x - q);$$

$$\bar\theta(x, t) := \theta\big((t_3 - t)^{-\frac{1}{2}}(x - q), -\ln(t_3 - t)\big), \quad \textit{that is,} \quad \theta(z, s) = \bar\theta\big(e^{-\frac{s}{2}} z + q, t_3 - e^{-s}\big).$$

Then (2.19) *is equivalent to*

$$\begin{cases} \partial_t \bar\theta = \Delta_x \bar\theta + (t_3 - t)^{-1} f_1\big((t_3 - t)^{-\frac{1}{2}}(x - q), -\ln(t_3 - t)\big) & \textit{for } (x, t) \in \mathbb{R}^n_+ \times (t_1, t_2), \\ -\partial_{x_n} \bar\theta = (t_3 - t)^{-\frac{1}{2}} f_2\big((t_3 - t)^{-\frac{1}{2}}(\tilde x - \tilde q), -\ln(t_3 - t)\big) & \textit{for } (x, t) \in \partial\mathbb{R}^n_+ \times (t_1, t_2), \\ \bar\theta(x, t_1) = f_3(e^{\frac{s_1}{2}}(x - q)) & \textit{for } x \in \mathbb{R}^n_+. \end{cases} \tag{2.20}$$

Moreover, the solution of (2.19) *is given by*

$$\begin{aligned} \theta(z, s) &= \int_{s_1}^{s} \int_{\mathbb{R}^n_+} H_n(z, s, w, \sigma) f_1(w, \sigma) dw d\sigma \\ &+ \int_{s_1}^{s} \int_{\mathbb{R}^{n-1}} H_n(z, s, (\tilde w, 0), \sigma) f_2(\tilde w, \sigma) d\tilde w d\sigma + \int_{\mathbb{R}^n_+} H_n(z, s, w, s_1) f_3(w) dw, \end{aligned} \tag{2.21}$$

where

$$H_n(z, s, w, \sigma) := e^{-\sigma\frac{n}{2}} G_n(e^{-\frac{s}{2}} z, t_3 - e^{-s}, e^{-\frac{\sigma}{2}} w, t_3 - e^{-\sigma}) = \left[4\pi\left(1 - e^{\sigma - s}\right)\right]^{-\frac{n}{2}} \times$$
$$\left[\exp\left(-\frac{\left|e^{\frac{\sigma - s}{2}} \tilde z - \tilde w\right|^2 + \left|e^{\frac{\sigma - s}{2}} z_n - w_n\right|^2}{4\left(1 - e^{\sigma - s}\right)}\right) + \exp\left(-\frac{\left|e^{\frac{\sigma - s}{2}} \tilde z - \tilde w\right|^2 + \left|e^{\frac{\sigma - s}{2}} z_n + w_n\right|^2}{4\left(1 - e^{\sigma - s}\right)}\right)\right].$$

Proof. By direct calculation,

$$\bar\theta(x, t_1) = f_3(e^{\frac{s_1}{2}}(x - q)), \quad \big(-\partial_{x_n}\bar\theta\big)\left((\tilde x, 0), t\right) = (t_3 - t)^{-\frac{1}{2}} f_2\big((t_3 - t)^{-\frac{1}{2}}(\tilde x - \tilde q), -\ln(t_3 - t)\big),$$
$$\partial_s \theta = \nabla_x \bar\theta \cdot z e^{-\frac{s}{2}}(-\frac{1}{2}) + \partial_t \bar\theta e^{-s}, \quad \partial_{z_i}\theta = \partial_{x_i}\bar\theta e^{-\frac{s}{2}}, \quad \partial_{z_i z_i}\theta = \partial_{x_i x_i}\bar\theta e^{-s}.$$

Then (2.19) is equivalent to (2.20). By (2.17), the solution of (2.20) is given by

$$\begin{aligned} \bar\theta(x, t) &= \int_{t_1}^{t} \int_{\mathbb{R}^n_+} G_n(x, t, v, \vartheta)(t_3 - \vartheta)^{-1} f_1((t_3 - \vartheta)^{-\frac{1}{2}}(v - q), -\ln(t_3 - \vartheta)) dv d\vartheta \\ &+ \int_{t_1}^{t} \int_{\mathbb{R}^{n-1}} G_n(x, t, (\tilde v, 0), \vartheta)(t_3 - \vartheta)^{-\frac{1}{2}} f_2\big((t_3 - \vartheta)^{-\frac{1}{2}}(\tilde v - \tilde q), -\ln(t_3 - \vartheta)\big) d\tilde v d\vartheta \\ &+ \int_{\mathbb{R}^n_+} G_n(x, t, v, t_1) f_3(e^{\frac{s_1}{2}}(v - q)) dv. \end{aligned}$$

Then

$$\begin{aligned} &\theta(z, s) = \bar\theta\big(e^{-\frac{s}{2}} z + q, t_3 - e^{-s}\big) \\ = &\int_{t_3 - e^{-s_1}}^{t_3 - e^{-s}} \int_{\mathbb{R}^n_+} G_n(e^{-\frac{s}{2}} z + q, t_3 - e^{-s}, v, \vartheta)(t_3 - \vartheta)^{-1} f_1((t_3 - \vartheta)^{-\frac{1}{2}}(v - q), -\ln(t_3 - \vartheta)) dv d\vartheta \\ &+ \int_{t_3 - e^{-s_1}}^{t_3 - e^{-s}} \int_{\mathbb{R}^{n-1}} G_n(e^{-\frac{s}{2}} z + q, t_3 - e^{-s}, (\tilde v, 0), \vartheta)(t_3 - \vartheta)^{-\frac{1}{2}} f_2\big((t_3 - \vartheta)^{-\frac{1}{2}}(\tilde v - \tilde q), -\ln(t_3 - \vartheta)\big) d\tilde v d\vartheta \\ &+ \int_{\mathbb{R}^n_+} G_n(e^{-\frac{s}{2}} z + q, t_3 - e^{-s}, v, t_3 - e^{-s_1}) f_3(e^{\frac{s_1}{2}}(v - q)) dv \\ = &\int_{s_1}^{s} \int_{\mathbb{R}^n_+} e^{-\sigma\frac{n}{2}} G_n(e^{-\frac{s}{2}} z, t_3 - e^{-s}, e^{-\frac{\sigma}{2}} w, t_3 - e^{-\sigma}) f_1(w, \sigma) dw d\sigma \\ &+ \int_{s_1}^{s} \int_{\mathbb{R}^{n-1}} e^{-\sigma\frac{n}{2}} G_n(e^{-\frac{s}{2}} z, t_3 - e^{-s}, (e^{-\frac{\sigma}{2}} \tilde w, 0), t_3 - e^{-\sigma}) f_2(\tilde w, \sigma) d\tilde w d\sigma \\ &+ \int_{\mathbb{R}^n_+} e^{-s_1\frac{n}{2}} G_n(e^{-\frac{s}{2}} z, t_3 - e^{-s}, e^{-\frac{s_1}{2}} w, t_3 - e^{-s_1}) f_3(w) dw, \end{aligned}$$

where we used $G_n(x + q, t, z + q, s) = G_n(x, t, z, s)$ with $q \in \partial\mathbb{R}^n_+$ for the last step. $\square$

2.4. **Some barrier functions.**

Lemma 2.8. *Let $n \geq 2$ be an integer. For $\tilde{y} \in \mathbb{R}^{n-1}$, $y_n \geq 0$, $y = (\tilde{y}, y_n)$, $\vartheta > 0$, $a > 0$, denote $P(y) = \left|(\tilde{y}, y_n + 1 + \vartheta|\tilde{y}|)\right|^{-a}$, then $\partial_{y_n} P \sim_{a,\vartheta} -\langle y\rangle^{-a-1}$, $|y \cdot \nabla P| \lesssim_{a,\vartheta} \langle y\rangle^{-a}$. Moreover, if $a \in (0, n-2)$, $0 < \vartheta < C(n,a)$ with a sufficiently small constant $C(n,a) > 0$ only depending on n, a, then $\Delta P \lesssim_{n,a,\vartheta} -\langle y\rangle^{-a-2}$.*

Proof. Denote $g(y) = \left|(\tilde{y}, y_n + 1 + \vartheta|\tilde{y}|)\right|^2 = (1+\vartheta^2)|\tilde{y}|^2 + (y_n+1)^2 + 2\vartheta(y_n+1)|\tilde{y}|$. Obviously, $g(y) \sim_\vartheta \langle y\rangle^2$, $P(y) = g^{-\frac{a}{2}}(y)$. By direct calculation, for $i = 1, 2, \ldots, n-1$,

$$
\begin{aligned}
\partial_{y_i} P =& \, -a g^{-\frac{a}{2}-1}\left[(1+\vartheta^2)y_i + \vartheta(y_n+1)y_i|\tilde{y}|^{-1}\right], \\
\partial_{y_i y_i} P =& \, a(a+2)g^{-\frac{a}{2}-2}\left[1+\vartheta^2+\vartheta(y_n+1)|\tilde{y}|^{-1}\right]^2 y_i^2 \\
& - a g^{-\frac{a}{2}-1}\left[1+\vartheta^2+\vartheta(y_n+1)|\tilde{y}|^{-1} - \vartheta(y_n+1)y_i^2|\tilde{y}|^{-3}\right], \\
\partial_{y_n} P =& \, -a g^{-\frac{a}{2}-1}\left(\vartheta|\tilde{y}| + y_n + 1\right), \\
\partial_{y_n y_n} P =& \, a(a+2)g^{-\frac{a}{2}-2}\left(\vartheta|\tilde{y}| + y_n + 1\right)^2 - a g^{-\frac{a}{2}-1},
\end{aligned}
$$

which implies $\partial_{y_n} P \sim_{a,\vartheta} -\langle y\rangle^{-a-1}$, $|y \cdot \nabla P| \lesssim_{a,\vartheta} \langle y\rangle^{-a}$. Moreover,

$$
\begin{aligned}
\Delta P =& \, a(a+2)g^{-\frac{a}{2}-2}\left\{\left[(1+\vartheta^2)^2+\vartheta^2\right]|\tilde{y}|^2 + 2\vartheta(\vartheta^2+2)(y_n+1)|\tilde{y}| + (\vartheta^2+1)(y_n+1)^2\right\} \\
& - a g^{-\frac{a}{2}-1}\left[(\vartheta^2+1)(n-1) + 1 + \vartheta(n-2)(y_n+1)|\tilde{y}|^{-1}\right] \\
=& \, a g^{-\frac{a}{2}-1}\Big\{g^{-1}(a+2)\left[(\vartheta^4+3\vartheta^2+1)|\tilde{y}|^2 + 2\vartheta(\vartheta^2+2)(y_n+1)|\tilde{y}| + (\vartheta^2+1)(y_n+1)^2\right] \\
& - (n-1)\vartheta^2 - n - \vartheta(n-2)(y_n+1)|\tilde{y}|^{-1}\Big\} := a g^{-\frac{a}{2}-1} I(t),
\end{aligned}
$$

where we set $t = (y_n+1)|\tilde{y}|^{-1} \in (0, \infty)$ and

$$
\begin{aligned}
I(t) :=& \, (a+2)\frac{\vartheta^4+3\vartheta^2+1+2\vartheta\left(\vartheta^2+2\right)t + \left(\vartheta^2+1\right)t^2}{\vartheta^2+1+2\vartheta t+t^2} - (n-1)\,\vartheta^2 - n - (n-2)\,\vartheta t \\
=& \, -\frac{f(t)}{\vartheta^2+1+2\vartheta t+t^2},
\end{aligned}
$$

where

$$
\begin{aligned}
f(t) :=& \, t^3\vartheta(n-2) + t^2\left[\vartheta^2\left(3n-a-7\right) + n-a-2\right] + t\vartheta\left[\vartheta^2\left(3n-2a-8\right) + 3n-4a-10\right] \\
& + \vartheta^4\left(n-a-3\right) + \vartheta^2\left(2n-3a-7\right) + n-a-2.
\end{aligned}
$$

For $a < n-2$, by the discriminant of the quadratic polynomial, there exists a constant $C(n,a) > 0$ sufficiently small such that for any $0 < \vartheta < C(n,a)$, we have

$$
\vartheta^2\left(3n-a-7\right) + n-a-2 > 0 \quad \text{and} \quad f(t) - t^3\vartheta(n-2) > 0 \ \text{ for } \ t \geq 0,
$$

which implies $I(t) < 0$ for $t \geq 0$. Note that $\lim_{t\to\infty} I(t) < 0$. Thus, $\sup_{t\geq 0} I(t) \leq -C_1(n,a,\vartheta)$ with a constant $C_1(n,a,\vartheta) > 0$. By $a > 0$, we conclude the second result. □

Lemma 2.9. *Given $t_2 \geq t_1 > 0$, $a_1 > -1$, $a_2 \in \mathbb{R}$,*

$$
\int_0^{t_1} (t_1 - s)^{a_1}(t_2 - s)^{a_2}\, ds \lesssim_{a_1,a_2} \begin{cases} t_2^{a_1+a_2+1}, & \text{if } a_1+a_2 > -1 \\ 1 + \ln\left(\frac{t_2}{t_2 - [t_1-(t_2-t_1)]_+}\right), & \text{if } a_1+a_2 = -1 \\ (t_2-t_1)^{a_1+a_2+1}, & \text{if } a_1+a_2 < -1. \end{cases}
$$

Proof.

$$
\begin{aligned}
& \left(\int_0^{[t_1-(t_2-t_1)]_+} + \int_{[t_1-(t_2-t_1)]_+}^{t_1}\right)(t_1-s)^{a_1}(t_2-s)^{a_2}\, ds \\
& \sim_{a_1,a_2} \int_0^{[t_1-(t_2-t_1)]_+} (t_2-s)^{a_1+a_2}\, ds + (t_2-t_1)^{a_2}\int_{[t_1-(t_2-t_1)]_+}^{t_1} (t_1-s)^{a_1}\, ds.
\end{aligned}
$$

Using $a_1 > -1$, we conclude this lemma. □

Lemma 2.10. *Given an integer $n \geq 2$ and constants $s_1 \in \mathbb{R}$, $C_f \geq 0$, consider*

$$
\begin{cases} \partial_s \Phi = \Delta_z \Phi - \dfrac{z}{2}\cdot\nabla_z\Phi + f_1(z,s) \ \text{ in } \ \mathbb{R}^n_+ \times (s_1, \infty), \\ -\partial_{z_n}\Phi = f_2(\tilde{z}, s) \ \text{ on } \ \partial\mathbb{R}^n_+ \times (s_1,\infty), \quad \Phi(z, s_1) = f_3(z) \ \text{ in } \ \mathbb{R}^n_+, \end{cases} \tag{2.22}
$$

where Φ is given by the formula (2.21) formally.

(1) *If* $|f_1(z,s)| \le C_f e^{\alpha_2 s}\langle e^{\alpha_1 s} z\rangle^{-2-a}$, $|f_2(\tilde z,s)| \le C_f e^{(\alpha_2-\alpha_1)s}\langle e^{\alpha_1 s}\tilde z\rangle^{-1-a}$, $f_3(z)\equiv 0$ *with* $(2+a)\alpha_1-\alpha_2>0$, $a\in(0,n-2)$, *then there exists a constant* $C(n,a,\alpha_1,\alpha_2)>0$ *such that*

$$|\Phi(z,s)| \le C_f C(n,a,\alpha_1,\alpha_2)\left\{e^{(\alpha_2-2\alpha_1)s}\langle e^{\alpha_1 s}z\rangle^{-a} + e^{-[(2+a)\alpha_1-\alpha_2]s_1}\right\}.$$

(2) *If* $f_1(z,s)\equiv 0$, $f_2(\tilde z,s)\equiv 0$, *then there exists a constant* $C(n)>0$ *such that*

$$|\Phi(z,s)| \le C(n)\left(1-e^{s_1-s}\right)^{-\frac{n}{4}} e^{\frac{|z|^2}{4\left(1+e^{s-s_1}\right)}}\|f_3\|_{L^2_\rho(\mathbb{R}^n_+)}.$$

Moreover, if $|f_3(z)|\le C_f|z|^a$ *with* $a\ge 0$, *then there exists a constant* $C(n,a)>0$ *such that*

$$|\Phi(z,s)| \le C_f C(n,a)\left[e^{a\frac{s_1-s}{2}}|z|^a + \left(1-e^{s_1-s}\right)^{\frac{a}{2}}\right].$$

(3) *If* $|f_1(z,s)|\le C_f e^{\alpha s}$, $f_2(\tilde z,s)\equiv 0$, $f_3(z)\equiv 0$ *with* $\alpha\in\mathbb{R}$, *then*

$$|\Phi(z,s)| \le C_f\begin{cases}\alpha^{-1}\left(e^{\alpha s}-e^{\alpha s_1}\right), & \text{if } \alpha\ne 0\\ s-s_1, & \text{if } \alpha=0.\end{cases}$$

(4) *If* $f_1(z,s)\equiv 0$, $|f_2(\tilde z,s)|\le C_f e^{\alpha s}|\tilde z|^a$, $f_3(z)\equiv 0$ *with* $\alpha\in\mathbb{R}$, $a\ge 0$, *then there exists a constant* $C(n,a,\alpha)>0$ *such that*

$$|\Phi(z,s)| \le C_f C(n,a,\alpha)\Bigg[e^{-\frac{a}{2}s}|\tilde z|^a\begin{cases}e^{(\alpha+\frac{a}{2})s_1}, & \text{if } \alpha+\frac{a}{2}<0\\ \left\langle \ln\left(1-(1-2e^{s_1-s})_+\right)\right\rangle, & \text{if } \alpha+\frac{a}{2}=0\\ e^{(\alpha+\frac{a}{2})s}, & \text{if } \alpha+\frac{a}{2}>0\end{cases}$$
$$+\begin{cases}e^{\alpha s_1}, & \text{if } \alpha<0\\ \left\langle \ln\left(1-(1-2e^{s_1-s})_+\right)\right\rangle, & \text{if } \alpha=0\\ e^{\alpha s}, & \text{if } \alpha>0\end{cases}\Bigg].$$

Proof. Denote

$$T_1=e^{-s_1},\quad \psi(x,t):=\Phi\big((T_1-t)^{-\frac12}x,-\ln(T_1-t)\big),\quad \text{that is,}\quad \Phi(z,s)=\psi(e^{-\frac{s}{2}}z,T_1-e^{-s}).$$

Then by Lemma 2.7, (2.22) is equivalent to

$$\begin{cases}\partial_t\psi=\Delta_x\psi+(T_1-t)^{-1}f_1\big((T_1-t)^{-\frac12}x,-\ln(T_1-t)\big) & \text{for } (x,t)\in\mathbb{R}^n_+\times(0,T_1),\\ (-\partial_{x_n}\psi)((\tilde x,0),t)=(T_1-t)^{-\frac12}f_2\big((T_1-t)^{-\frac12}\tilde x,-\ln(T_1-t)\big) & \text{for } (\tilde x,t)\in\mathbb{R}^{n-1}\times(0,T_1),\\ \psi(x,0)=f_3(e^{\frac{s_1}{2}}x) & \text{for } x\in\mathbb{R}^n_+.\end{cases}\tag{2.23}$$

(1). In this case, we have $f_3(e^{\frac{s_1}{2}}x)\equiv 0$,

$$(T_1-t)^{-1}\left|f_1\big((T_1-t)^{-\frac12}x,-\ln(T_1-t)\big)\right| \le C_f(T_1-t)^{-\alpha_2-1}\langle (T_1-t)^{-\alpha_1-\frac12}x\rangle^{-2-a},$$
$$(T_1-t)^{-\frac12}\left|f_2\big((T_1-t)^{-\frac12}\tilde x,-\ln(T_1-t)\big)\right| \le C_f(T_1-t)^{-(\alpha_2-\alpha_1)-\frac12}\langle (T_1-t)^{-\alpha_1-\frac12}\tilde x\rangle^{-1-a}.$$

By Lemma 2.8, for $a\in(0,n-2)$, we set

$$P(y)=|(\tilde y,y_n+1+\vartheta|\tilde y|)|^{-a},\quad y=(T_1-t)^{-(\alpha_1+\frac12)}x,$$

where $\vartheta=\vartheta(n,a)>0$ is a small constant to make $\Delta_y P(y)\lesssim_{n,a,\vartheta}-\langle y\rangle^{-a-2}$.

Set $\psi_1(x,t)=2(T_1-t)^{2\alpha_1-\alpha_2}P(y)$ with $y=(T_1-t)^{-\alpha_1-\frac12}x$, then

$$-\partial_{x_n}\psi_1\big|_{x_n=0}=-2(T_1-t)^{\alpha_1-\alpha_2-\frac12}\partial_{y_n}P(y)\big|_{y_n=0}\sim_{a,\vartheta}(T_1-t)^{\alpha_1-\alpha_2-\frac12}\langle\tilde y\rangle^{-a-1},$$

and

$$\partial_t\psi_1-\Delta_x\psi_1=-(T_1-t)^{-\alpha_2-1}\Delta_y P(y)+g_1(x,t),$$

where

$$g_1(x,t):=(T_1-t)^{-\alpha_2-1}\left[-\Delta_y P(y)-(4\alpha_1-2\alpha_2)(T_1-t)^{2\alpha_1}P(y)+(2\alpha_1+1)(T_1-t)^{2\alpha_1}y\cdot\nabla_y P(y)\right].$$

By the estimates of $P(y)$ in Lemma 2.8, we have

$$\begin{aligned}&-(T_1-t)^{-\alpha_2-1}\Delta_y P(y)\gtrsim_{n,a,\vartheta}(T_1-t)^{-\alpha_2-1}\langle y\rangle^{-a-2},\\ g_1(x,t)&\geq(T_1-t)^{-\alpha_2-1}C_1(n,a,\vartheta)\left[\langle y\rangle^{-a-2}-C_2(n,a,\vartheta,\alpha_1,\alpha_2)(T_1-t)^{2\alpha_1}\langle y\rangle^{-a}\right]\\ &\geq -C_3(n,a,\vartheta,\alpha_1,\alpha_2)(T_1-t)^{\alpha_1 a+2\alpha_1-\alpha_2-1}\end{aligned}$$

with some positive constants $C_i(*,*,\dots)$, $i=1,2,3$. In order to control $g_1(x,t)$, we set $\psi_2(x,t)=T_1^{\alpha_1 a+2\alpha_1-\alpha_2}-(T_1-t)^{\alpha_1 a+2\alpha_1-\alpha_2}$, which satisfies $\partial_{x_n}\psi_2=0$, $(\partial_t-\Delta_x)\psi_2=(\alpha_1 a+2\alpha_1-\alpha_2)(T_1-t)^{\alpha_1 a+2\alpha_1-\alpha_2-1}$. Since $\alpha_1 a+2\alpha_1-\alpha_2>0$, $C_f C_5(n,a,\vartheta,\alpha_1,\alpha_2)\big(\psi_1+C_4(n,a,\vartheta,\alpha_1,\alpha_2)\psi_2\big)$ is a supersolution of (2.23) for some large constants $C_i(n,a,\vartheta,\alpha_1,\alpha_2)>0$, $i=4,5$. Hence, (1) is deduced by the comparison theorem and plugging $\Phi(z,s)=\psi(e^{-\frac{s}{2}}z,T_1-e^{-s})$.

(2). Using the even extension of f_3 in z_n variable, we can deduce the first result by the following calculation, which is similar to the proof of [24, Lemma 2.2]. For $g\in L^2_\rho(\mathbb{R}^n)$,

$$\begin{aligned}&\left\{4\pi\left[1-e^{-(s-s_1)}\right]\right\}^{-\frac{n}{2}}\int_{\mathbb{R}^n}\exp\left(-\frac{\left|e^{-\frac{s-s_1}{2}}z-w\right|^2}{4\left[1-e^{-(s-s_1)}\right]}\right)g(w)dw\\ \leq&\left\{4\pi\left[1-e^{-(s-s_1)}\right]\right\}^{-\frac{n}{2}}\left[\int_{\mathbb{R}^n}\exp\left(-\frac{\left|e^{-\frac{s-s_1}{2}}z-w\right|^2}{2\left[1-e^{-(s-s_1)}\right]}\right)e^{\frac{|w|^2}{4}}dw\right]^{\frac12}\|g\|_{L^2_\rho(\mathbb{R}^n)}\\ =&\,2^{-\frac{n}{2}}\pi^{-\frac{n}{4}}\left[1+e^{-(s-s_1)}\right]^{-\frac{n}{4}}\left[1-e^{-(s-s_1)}\right]^{-\frac{n}{4}}e^{\frac{|z|^2}{4\left(1+e^{s-s_1}\right)}}\|g\|_{L^2_\rho(\mathbb{R}^n)}.\end{aligned}$$

For the second result, without loss of generality, one taking $C_f=1$, then

$$\begin{aligned}|\psi(x,t)|&\leq\int_{\mathbb{R}^n_+}G_n(x,t,v,0)\left|f_3(e^{\frac{s_1}{2}}v)\right|dv\leq e^{a\frac{s_1}{2}}\int_{\mathbb{R}^n}(4\pi t)^{-\frac{n}{2}}e^{-\frac{|x-v|^2}{4t}}|v|^a dv\\ &=e^{a\frac{s_1}{2}}(4\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}e^{-\frac{|w|^2}{4}}|x-\sqrt{t}w|^a dw\leq C(n,a)e^{a\frac{s_1}{2}}\int_{\mathbb{R}^n}e^{-\frac{|w|^2}{4}}\left(|x|^a+t^{\frac{a}{2}}|w|^a\right)dw\leq C(n,a)e^{a\frac{s_1}{2}}\left(|x|^a+t^{\frac{a}{2}}\right),\end{aligned}$$

where we used $a\geq0$ for the third "$\leq$". Then we get the second result.

(3). Take $C_f=1$.

$$\begin{aligned}|\psi(x,t)|&\leq\int_0^t\int_{\mathbb{R}^n_+}G_n(x,t,v,\vartheta)(T_1-\vartheta)^{-1}\left|f_1((T_1-\vartheta)^{-\frac12}v,-\ln(T_1-\vartheta))\right|dvd\vartheta\\ &\leq\int_0^t\int_{\mathbb{R}^n}[4\pi(t-\vartheta)]^{-\frac{n}{2}}e^{-\frac{|x-v|^2}{4(t-\vartheta)}}(T_1-\vartheta)^{-1-\alpha}dvd\vartheta\\ &=\int_0^t(T_1-\vartheta)^{-1-\alpha}d\vartheta=\begin{cases}\alpha^{-1}\left[(T_1-t)^{-\alpha}-T_1^{-\alpha}\right], & \text{if }\ \alpha\neq0\\ \ln(\frac{T_1}{T_1-t}), & \text{if }\ \alpha=0.\end{cases}\end{aligned}$$

Using the relationship $\Phi(z,s)=\psi(e^{-\frac{s}{2}}z,T_1-e^{-s})$, we get the conclusion.

(4). Set $C_f=1$.

$$\begin{aligned}|\psi(x,t)|&\leq\int_0^t\int_{\mathbb{R}^{n-1}}G_n(x,t,(\tilde v,0),\vartheta)(T_1-\vartheta)^{-\frac12}\left|f_2\big((T_1-\vartheta)^{-\frac12}\tilde v,-\ln(T_1-\vartheta)\big)\right|d\tilde v d\vartheta\\ &\leq\int_0^t\int_{\mathbb{R}^{n-1}}2\left[4\pi(t-\vartheta)\right]^{-\frac{n}{2}}e^{-\frac{|\tilde x-\tilde v|^2+x_n^2}{4(t-\vartheta)}}(T_1-\vartheta)^{-\frac12-\alpha-\frac{a}{2}}|\tilde v|^a d\tilde v d\vartheta\\ &=\pi^{-\frac{n}{2}}\int_0^t(t-\vartheta)^{-\frac12}(T_1-\vartheta)^{-\frac12-\alpha-\frac{a}{2}}e^{-\frac{x_n^2}{4(t-\vartheta)}}\int_{\mathbb{R}^{n-1}}e^{-|\tilde w|^2}\left|\tilde x-2\sqrt{t-\vartheta}\tilde w\right|^a d\tilde w d\vartheta\\ &\leq C(n,a)\int_0^t(t-\vartheta)^{-\frac12}(T_1-\vartheta)^{-\frac12-\alpha-\frac{a}{2}}\left[|\tilde x|^a+(t-\vartheta)^{\frac{a}{2}}\right]d\vartheta\\ &\leq C(n,a,\alpha)\left[|\tilde x|^a\begin{cases}T_1^{-\alpha-\frac{a}{2}}, & \text{if }\ \alpha+\frac{a}{2}<0\\ 1+\ln\left(\frac{T_1}{T_1-[t-(T_1-t)]_+}\right), & \text{if }\ \alpha+\frac{a}{2}=0\\ (T_1-t)^{-\alpha-\frac{a}{2}}, & \text{if }\ \alpha+\frac{a}{2}>0\end{cases}+\begin{cases}T_1^{-\alpha}, & \text{if }\ \alpha<0\\ 1+\ln\left(\frac{T_1}{T_1-[t-(T_1-t)]_+}\right), & \text{if }\ \alpha=0\\ (T_1-t)^{-\alpha}, & \text{if }\ \alpha>0\end{cases}\right],\end{aligned}$$

where we used $a\geq0$ for the third "$\leq$" and Lemma 2.9 for the fourth "$\leq$". Then the conclusion holds.

□

2.5. **Vanishing adjustment functions.** In order to derive multiple rates at distinct points, we introduce vanishing adjustment functions to eliminate derivatives at arbitrarily prescribed finitely many points. This method works for more general parabolic equations.

Lemma 2.11. *Suppose that $n \geq 1$, $l \in \mathbb{N}$, $u_0(x) \in C^l(\mathbb{R}^n)$ satisfies $D_x^{\mathbf{k}} u_0(x) \to 0$ as $|x| \to \infty$ for all $\mathbf{k} \in \mathbb{N}^n$ with $\|\mathbf{k}\|_{\ell_1} \leq l$, $t_0 \in \mathbb{R}$, denote $u(x,t) = [4\pi(t-t_0)]^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-t_0)}} u_0(y)dy$, then for $\iota \in \mathbb{N}$, $\mathbf{m} \in \mathbb{N}^n$ with $2\iota + \|\mathbf{m}\|_{\ell_1} \leq l$, we have $\partial_t^\iota D_x^{\mathbf{m}} u \in C(\mathbb{R}^n \times [t_0, \infty))$,*

$$\partial_t^\iota D_x^{\mathbf{m}} u(x,t) \to \Delta_x^\iota D_x^{\mathbf{m}} u_0(x) \ \text{ in } \ L^\infty(\mathbb{R}^n) \ \text{ as } \ t \downarrow t_0, \quad \|\partial_t^\iota D_x^{\mathbf{m}} u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq \|\Delta_x^\iota D_x^{\mathbf{m}} u_0\|_{L^\infty(\mathbb{R}^n)} \ \text{ for } \ t \geq t_0.$$

Proof. For $t > t_0$,

$$\begin{aligned}
D_x^{\mathbf{m}} u(x,t) &= [4\pi(t-t_0)]^{-\frac{n}{2}} \int_{\mathbb{R}^n} D_x^{\mathbf{m}} \left[e^{-\frac{|x-y|^2}{4(t-t_0)}} \right] u_0(y)dy \\
= \ & [4\pi(t-t_0)]^{-\frac{n}{2}} (-1)^{\|\mathbf{m}\|_{\ell_1}} \int_{\mathbb{R}^n} D_y^{\mathbf{m}} \left[e^{-\frac{|x-y|^2}{4(t-t_0)}} \right] u_0(y)dy = [4\pi(t-t_0)]^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-t_0)}} D_y^{\mathbf{m}} u_0(y)dy, \\
\partial_t^\iota D_x^{\mathbf{m}} u(x,t) &= \int_{\mathbb{R}^n} \Delta_x^\iota \left\{ [4\pi(t-t_0)]^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4(t-t_0)}} \right\} D_y^{\mathbf{m}} u_0(y)dy = \int_{\mathbb{R}^n} [4\pi(t-t_0)]^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4(t-t_0)}} \Delta_y^\iota D_y^{\mathbf{m}} u_0(y)dy.
\end{aligned}$$

We conclude this lemma by the property of the heat kernel. □

Lemma 2.12. *Given $n \geq 2$, $u_0(x) \in L^\infty(\mathbb{R}^n_+)$, $t_0 \in \mathbb{R}$, denote*

$$u(x,t) := \int_{\mathbb{R}^n_+} G_n(x,t,z,t_0) u_0(z)dz = [4\pi(t-t_0)]^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|\tilde{x}-\tilde{z}|^2 + |x_n - z_n|^2}{4(t-t_0)}} \left(u_0(z)\mathbf{1}_{z_n \geq 0} + u_0(\tilde{z}, -z_n)\mathbf{1}_{z_n < 0} \right) dz$$

with G_n given in (2.15), then for $t > t_0$, $\iota \in \mathbb{N}$, $\mathbf{m} \in \mathbb{N}^n$, $\partial_t^\iota D_x^{\mathbf{m}} u(x,t)$ is odd (even) about $x_n \in \mathbb{R}$ if m_n is odd (even). In particular, $\partial_t^\iota D_x^{\mathbf{m}} u((\tilde{x},0),t) = 0$ if m_n is odd.

Under the additional assumptions that $u_0(x) \in C^l(\overline{\mathbb{R}^n_+})$, $l \in \mathbb{N}$ satisfies that for $\mathbf{k} \in \mathbb{N}^n$ with $\|\mathbf{k}\|_{\ell_1} \leq l$, $D_x^{\mathbf{k}} u_0(x) \to 0$ as $|x| \to \infty$ in $\overline{\mathbb{R}^n_+}$ and $D_x^{\mathbf{k}} u_0(x)|_{x_n=0} = 0$ if k_n is odd, then for $\iota \in \mathbb{N}$, $\mathbf{m} \in \mathbb{N}^n$ with $2\iota + \|\mathbf{m}\|_{\ell_1} \leq l$, we have $\partial_t^\iota D_x^{\mathbf{m}} u \in C(\overline{\mathbb{R}^n_+} \times [t_0, \infty))$,

$$\partial_t^\iota D_x^{\mathbf{m}} u(x,t) \to \Delta_x^\iota D_x^{\mathbf{m}} u_0(x) \ \text{ in } \ L^\infty(\mathbb{R}^n_+) \ \text{ as } \ t \downarrow t_0, \quad \|\partial_t^\iota D_x^{\mathbf{m}} u(\cdot,t)\|_{L^\infty(\mathbb{R}^n_+)} \leq \|\Delta_x^\iota D_x^{\mathbf{m}} u_0\|_{L^\infty(\mathbb{R}^n_+)} \ \text{ for } \ t \geq t_0.$$

Proof. (2.16) deduces the first result. Under the additional assumptions of u_0, $u_0(x)\mathbf{1}_{x_n \geq 0} + u_0(\tilde{x}, -x_n)\mathbf{1}_{x_n < 0}$ satisfies the assumption of Lemma 2.11, which implies the second result. □

We will give vanishing adjustment functions in the next proposition. First, we will find a basis for derivatives at the prescribed finite number of points and then use the continuity in a short time to derive a basis for derivatives at these points at $t = T \ll 1$.

Proposition 2.13. *Suppose that $\mathfrak{l}$ is a positive integer, $p^{[1]}, p^{[2]}, \ldots, p^{[\mathfrak{l}]}$ be arbitrary distinct points on $\partial\mathbb{R}^n_+$, $n \geq 2$, and a constant d satisfies $0 < d < \min_{1 \leq i \neq j \leq \mathfrak{l}} |p^{[i]} - p^{[j]}|/4$, $N_0 \in \mathbb{N}$, then for $T \ll 1$, there exist $V_{p^{[i]},\mathbf{m}}(x,t)$, $i = 1, 2, \ldots, \mathfrak{l}$ with $\mathbf{m} \in \mathbb{N}^n$, $\|\mathbf{m}\|_{\ell_1} \leq N_0$, $m_n \in 2\mathbb{N}$, solving*

$$\partial_t V_{p^{[i]},\mathbf{m}} = \Delta V_{p^{[i]},\mathbf{m}} \ \text{ in } \ \mathbb{R}^n_+ \times (0,T], \quad -\partial_{x_n} V_{p^{[i]},\mathbf{m}} = 0 \ \text{ on } \ \partial\mathbb{R}^n_+ \times (0,T], \quad V_{p^{[i]},\mathbf{m}}(x,0) = V_{p^{[i]},\mathbf{m},0}(x) \ \text{ in } \ \mathbb{R}^n_+,$$

and the following properties hold:

1. *$V_{p^{[i]},\mathbf{m},0}(x)$ is smooth in $\overline{\mathbb{R}^n_+}$ and $V_{p^{[i]},\mathbf{m},0}(x) = 0$ in $\overline{\mathbb{R}^n_+ \backslash B_n^+(p^{[i]}, 2d)}$.*
2. *$\partial_t^\iota D_x^{\mathbf{k}} V_{p^{[i]},\mathbf{m}}((\tilde{x},0),t) = 0$ for $\tilde{x} \in \mathbb{R}^{n-1}$, $t \in [0,T]$, and $\iota \in \mathbb{N}$, $\mathbf{k} \in \mathbb{N}^n$, $k_n \in 2\mathbb{N}+1$.*
3. *$D_x^{\mathbf{k}} V_{p^{[i]},\mathbf{m}}(p^{[j]},T) = \delta_{\mathbf{m},\mathbf{k}} \delta_{p^{[i]},p^{[j]}}$ for $\mathbf{k} \in \mathbb{N}^n$, $\|\mathbf{k}\|_{\ell_1} \leq N_0$, $j = 1, 2, \ldots, \mathfrak{l}$.*
4. *$\|\partial_t^\iota D_x^{\mathbf{k}} V_{p^{[i]},\mathbf{m}}\|_{L^\infty(\mathbb{R}^n_+ \times [0,T])} \leq C$ for $\iota \in \mathbb{N}$, $\mathbf{k} \in \mathbb{N}^n$, $2\iota + \|\mathbf{k}\|_{\ell_1} \leq N_0$ with a constant $C > 0$ only depending on $\mathfrak{l}$, d, N_0, $p^{[1]}, p^{[2]}, \ldots, p^{[\mathfrak{l}]}$.*

Proof. Denote $\phi_{\mathbf{m}}(x) = \eta(x/d) \prod_{j=1}^n (m_j!)^{-1} x_j^{m_j}$, and $g_{p^{[i]},\mathbf{m},0}(x) = \phi_{\mathbf{m}}(x - p^{[i]})$. Since η is radial, for $\mathbf{m} \in \mathbb{N}^n$ with $m_n \in 2\mathbb{N}$, $\mathbf{k} \in \mathbb{N}^n$ with $k_n \in 2\mathbb{N}+1$, $D_x^{\mathbf{k}} g_{p^{[i]},\mathbf{m},0}$ is odd about x_n. In particular, $D_x^{\mathbf{k}} g_{p^{[i]},\mathbf{m},0}(\tilde{x},0) \equiv 0$. For $\mathbf{m}, \mathbf{k} \in \mathbb{N}^n$, $D_x^{\mathbf{k}} g_{p^{[i]},\mathbf{m},0}(p^{[j]}) = \delta_{\mathbf{m},\mathbf{k}} \delta_{p^{[i]},p^{[j]}}$.

Denote $g_{p^{[i]},\mathbf{m}}(x,t) = \int_{\mathbb{R}^n_+} G_n(x,t,z,0) g_{p^{[i]},\mathbf{m},0}(z)dz$. By Lemma 2.12, $\|\partial_t^\iota D_x^{\mathbf{k}} g_{p^{[i]},\mathbf{m}}(\cdot,t)\|_{L^\infty(\mathbb{R}^n_+)} \leq \|\Delta_x^\iota D_x^{\mathbf{k}} g_{p^{[i]},\mathbf{m},0}\|_{L^\infty(\mathbb{R}^n_+)} \lesssim_{\iota,\mathbf{k},\mathbf{m}} 1$; $\partial_t^\iota D_x^{\mathbf{k}} g_{p^{[i]},\mathbf{m}}((\tilde{x},0),t) = 0$ for $\tilde{x} \in \mathbb{R}^{n-1}$, $t \in [0,\infty)$ if $k_n \in 2\mathbb{N}+1$, $m_n \in 2\mathbb{N}$; and given $\epsilon > 0$, for all $\mathbf{m},\mathbf{k}\in\mathbb{N}^n$ satisfying $\|\mathbf{m}\|_{\ell_1},\|\mathbf{k}\|_{\ell_1}\leq N_0$, $m_n\in 2\mathbb{N}$, and all $i,j=1,2,\dots,\mathfrak{l}$, there exists $T\ll 1$ such that $\left|D_x^{\mathbf{k}}g_{p^{[i]},\mathbf{m}}(p^{[j]},T)-\delta_{\mathbf{m},\mathbf{k}}\delta_{p^{[i]},p^{[j]}}\right|<\epsilon$.

One taking ϵ sufficiently small, by the non-singularity of the strictly diagonally dominant matrix, then for $i=1,2,\dots,\mathfrak{l}$, there exist $V_{p^{[i]},\mathbf{m}}(x,t)$ as a linear combination of $g_{p^{[i]},\mathbf{k}}$, $\|\mathbf{k}\|_{\ell_1}\leq N_0$ satisfying the conclusion. ◻

2.6. **Derivative estimate in the vanishing region.** In the next lemma, we only require T to have a fixed upper bound, but it does not need to be small.

Lemma 2.14. *Suppose that $n\geq 2$ is an integer, $0<T\leq C_T$, g_1,g_2,g_3 satisfy*

$$|g_1(x,t)|\leq C_g(T-t)^{a_1}|x-q|^{b_1}\mathbf{1}_{|x-q|\geq r},\quad |g_2(\tilde{x},t)|\leq C_g(T-t)^{a_2}|\tilde{x}-\tilde{q}|^{b_2}\mathbf{1}_{|\tilde{x}-\tilde{q}|\geq r},\quad |g_3(x)|\leq C_g|x-q|^{b_3}\mathbf{1}_{|x-q|\geq r}$$

for all $x=(\tilde{x},x_n)\in\mathbb{R}^n_+$, $t\in(0,T)$, where $C_T,C_g,r\in\mathbb{R}_+$, $a_1,a_2,b_1,b_2,b_3\in\mathbb{R}$, $q=(\tilde{q},0)\in\partial\mathbb{R}^n_+$, denote

$$f_1(x,t)=\int_0^t\int_{\mathbb{R}^n_+}G_n(x,t,z,s)g_1(z,s)dzds,\quad f_2(x,t)=\int_0^t\int_{\mathbb{R}^{n-1}}G_n(x,t,(\tilde{z},0),s)g_2(\tilde{z},s)d\tilde{z}ds,$$
$$f_3(x,t)=\int_{\mathbb{R}^n_+}G_n(x,t,z,0)g_3(z)dz$$

with G_n given in (2.15), then $f_1,f_2,f_3\in C^\infty\left(\overline{B_n^+(q,r/2)}\times[0,T]\right)$, and for $\iota\in\mathbb{N}$, $\mathbf{m}\in\mathbb{N}^n$, $(x,t)\in\overline{B_n^+(q,r/2)}\times[0,T]$, we have

$$|\partial_t^\iota D_x^{\mathbf{m}}f_i(x,t)|\lesssim_{n,\iota,r,a_i,b_i,C_T,\|\mathbf{m}\|_{\ell_1}}C_gt^3e^{-\frac{r^2}{22t}},\ i=1,2,\quad |\partial_t^\iota D_x^{\mathbf{m}}f_3(x,t)|\lesssim_{n,\iota,r,b_3,C_T,\|\mathbf{m}\|_{\ell_1}}C_gte^{-\frac{r^2}{21t}}.$$

Moreover, if m_n is odd, then for $(\tilde{x},t)\in\overline{B_{n-1}(\tilde{q},r/2)}\times[0,T]$, $\partial_t^\iota D_x^{\mathbf{m}}f_i((\tilde{x},0),t)=0$, $i=1,2,3$.

Proof. Due to the property of supports of g_1,g_2,g_3, by the parabolic regularity theory, then $f_1,f_2,f_3\in C^\infty\big(\overline{B_n^+(q,r/2)}\times[0,T]\big)$. Notice that

$$|x-z|\leq|x-(\tilde{z},-z_n)|\ \text{ for }\ x,z\in\overline{\mathbb{R}^n_+};\quad \text{ for }\ |x-q|\leq r/2,\ |z-q|\geq r,\ \text{ then }\ |x-z|\geq|z-q|/2,$$
$$\left|\partial_t^\iota D_x^{\mathbf{m}}\left[(t-s)^{-\frac{n}{2}}e^{-\frac{|\tilde{x}-\tilde{z}|^2+|x_n\pm z_n|^2}{4(t-s)}}\right]\right|\lesssim_{n,\iota,\|\mathbf{m}\|_{\ell_1}}(t-s)^{-\frac{n}{2}-\iota-\frac{\|\mathbf{m}\|_{\ell_1}}{2}}e^{-\frac{|\tilde{x}-\tilde{z}|^2+|x_n\pm z_n|^2}{5(t-s)}}\lesssim_{n,\iota,r,C_T,\|\mathbf{m}\|_{\ell_1}}e^{-\frac{|z-q|^2}{21(t-s)}}.\tag{2.24}$$

For an integer $n_1\geq 1$, $C_0\in\mathbb{R}_+$, $t_1\in(0,C_T]$, $b\in\mathbb{R}$,

$$\int_{\mathbb{R}^{n_1}}e^{-C_0\frac{|z|^2}{t_1}}|z|^b\mathbf{1}_{|z|\geq r}dz\sim_{C_0,n_1}t_1^{\frac{b}{2}+\frac{n_1}{2}}\int_{C_0\frac{r^2}{t_1}}^\infty e^{-y}y^{\frac{b}{2}+\frac{n_1}{2}-1}dy\sim_{C_0,n_1,r,b,C_T}t_1e^{-C_0\frac{r^2}{t_1}},\tag{2.25}$$

where we used Lemma B.1(1) for the last "$\sim$". Hereafter in this proof, we always assume $(x,t)\in\overline{B_n^+(q,r/2)}\times(0,T]$, and (2.24), (2.25) will be used repetitively in calculation.

$$|\partial_t^\iota D_x^{\mathbf{m}}f_3(x,t)|\lesssim_{n,\iota,r,C_T,\|\mathbf{m}\|_{\ell_1}}C_g\int_{\mathbb{R}^n_+}e^{-\frac{|z-q|^2}{21t}}|z-q|^{b_3}\mathbf{1}_{|z-q|\geq r}dz\lesssim_{n,\iota,r,b_3,C_T,\|\mathbf{m}\|_{\ell_1}}C_gte^{-\frac{r^2}{21t}}.$$

$$\left|\int_{\mathbb{R}^n_+}\partial_t^{\bar{\iota}}D_x^{\overline{\mathbf{m}}}G_n(x,t,z,s)g_1(z,s)dz\right|\lesssim_{n,\bar{\iota},r,C_T,\|\overline{\mathbf{m}}\|_{\ell_1}}C_g(T-s)^{a_1}\int_{\mathbb{R}^n_+}e^{-\frac{|z-q|^2}{21(t-s)}}|z-q|^{b_1}\mathbf{1}_{|z-q|\geq r}dz$$
$$\lesssim_{n,\bar{\iota},r,b_1,C_T,\|\overline{\mathbf{m}}\|_{\ell_1}}C_g(T-s)^{a_1}(t-s)e^{-\frac{r^2}{21(t-s)}}$$

for all $\bar{\iota}\in\mathbb{N},\overline{\mathbf{m}}\in\mathbb{N}^n$. In particular, $\lim\limits_{s\uparrow t}\int_{\mathbb{R}^n_+}\partial_t^{\bar{\iota}}D_x^{\overline{\mathbf{m}}}G_n(x,t,z,s)g_1(z,s)dz=0$. Thus,

$$\begin{aligned}|\partial_t^\iota D_x^{\mathbf{m}}f_1(x,t)|&=\left|\int_0^t\int_{\mathbb{R}^n_+}\partial_t^\iota D_x^{\mathbf{m}}\left(G_n(x,t,z,s)\right)g_1(z,s)dzds\right|\\
&\lesssim_{n,\iota,r,C_T,\|\mathbf{m}\|_{\ell_1}}C_g\int_0^t(T-s)^{a_1}\int_{\mathbb{R}^n_+}e^{-\frac{|z-q|^2}{21(t-s)}}|z-q|^{b_1}\mathbf{1}_{|z-q|\geq r}dzds\\
&\lesssim_{n,\iota,r,b_1,C_T,\|\mathbf{m}\|_{\ell_1}}C_g\int_0^t(T-s)^{a_1}(t-s)e^{-\frac{r^2}{21(t-s)}}ds\\
&\lesssim_{n,\iota,r,a_1,b_1,C_T,\|\mathbf{m}\|_{\ell_1}}C_g\int_0^t(t-s)e^{-\frac{r^2}{22(t-s)}}ds\sim_{n,\iota,r,a_1,b_1,C_T,\|\mathbf{m}\|_{\ell_1}}C_gt^3e^{-\frac{r^2}{22t}},\end{aligned}$$

where we used Lemma B.1(1) for the last "$\sim$".

$$\Big|\int_{\mathbb{R}^{n-1}}\partial_t^{\bar\iota}D_x^{\overline{\mathbf{m}}}G_n(x,t,(\tilde z,0),s)g_2(\tilde z,s)d\tilde z\Big|\lesssim_{n,\bar\iota,r,C_T,\|\overline{\mathbf{m}}\|_{\ell_1}}C_g(T-s)^{a_2}\int_{\mathbb{R}^{n-1}}e^{-\frac{|\tilde z-\tilde q|^2}{21(t-s)}}|\tilde z-\tilde q|^{b_2}\mathbf{1}_{|\tilde z-\tilde q|\geq r}d\tilde z$$
$$\lesssim_{n,\bar\iota,r,b_2,C_T,\|\overline{\mathbf{m}}\|_{\ell_1}}C_g(T-s)^{a_2}(t-s)e^{-\frac{r^2}{21(t-s)}}$$

for all $\bar\iota\in\mathbb{N},\overline{\mathbf{m}}\in\mathbb{N}^n$. In particular, $\lim\limits_{s\uparrow t}\int_{\mathbb{R}^{n-1}}\partial_t^{\bar\iota}D_x^{\overline{\mathbf{m}}}G_n(x,t,(\tilde z,0),s)g_2(\tilde z,s)d\tilde z=0$. Thus,

$$\begin{aligned}|\partial_t^\iota D_x^{\mathbf{m}}f_2(x,t)|&=\Big|\int_0^t\int_{\mathbb{R}^{n-1}}\partial_t^\iota D_x^{\mathbf{m}}\left(G_n(x,t,(\tilde z,0),s)\right)g_2(\tilde z,s)d\tilde zds\Big|\\
&\lesssim_{n,\iota,r,C_T,\|\mathbf{m}\|_{\ell_1}}C_g\int_0^t(T-s)^{a_2}\int_{\mathbb{R}^{n-1}}e^{-\frac{|\tilde z-\tilde q|^2}{21(t-s)}}|\tilde z-\tilde q|^{b_2}\mathbf{1}_{|\tilde z-\tilde q|\geq r}d\tilde zds\\
&\lesssim_{n,\iota,r,b_2,C_T,\|\mathbf{m}\|_{\ell_1}}C_g\int_0^t(T-s)^{a_2}(t-s)e^{-\frac{r^2}{21(t-s)}}ds\lesssim_{n,\iota,r,a_2,b_2,C_T,\|\mathbf{m}\|_{\ell_1}}C_gt^3e^{-\frac{r^2}{22t}}.\end{aligned}$$

If m_n is odd, by (2.16), $\partial_t^\iota D_x^{\mathbf{m}}f_i((\tilde x,0),t)=0$ for $t\in(0,T]$, $i=1,2,3$. Combining $f_1,f_2,f_3\in C^\infty\big(\overline{B_n^+(q,r/2)}\times[0,T]\big)$, we complete the proof. $\square$

2.7. Scaling argument.

Lemma 2.15. *Let $n\geq 2$ be an integer, $-\infty<t_0<t_1\leq\infty$, v be a weak solution of*

$$\partial_tv=\Delta v+h_1(x,t)\ \text{ in }\ \mathbb{R}^n_+\times(t_0,t_1),\quad \partial_{x_n}v+\beta^0(\tilde x,t)v=h_2(\tilde x,t)\ \text{ on }\ \partial\mathbb{R}^n_+\times(t_0,t_1),\quad v(x,t_0)=v_0(x)\ \text{ in }\ \mathbb{R}^n_+. \tag{2.26}$$

Given $x_*\in\overline{\mathbb{R}^n_+},t_*\in(t_0,t_1),\rho\in\mathbb{R}_+$, suppose $\rho\|\beta^0\|_{L^\infty\left(B_{n-1}(\tilde x_*,2\rho)\times\left(\max\{t_0,t_*-4\rho^2\},t_*\right]\right)}\leq C_{\beta^0,1}$, $v,h_1\in L^\infty_{\rm loc}(\overline{\mathbb{R}^n_+}\times[t_0,t_1))$, $h_2\in L^\infty_{\rm loc}(\mathbb{R}^{n-1}\times[t_0,t_1))$, then there exist positive constants C_1,α_0 only depending on n such that if $\alpha\in(0,\alpha_0]$, $v_0\in C^\alpha_{\rm loc}(\overline{\mathbb{R}^n_+})$, we have

$$\begin{aligned}&[v]_{C^{\alpha,\alpha/2}\left(B_n^+(x_*,\rho)\times\left(\max\{t_0,t_*-\rho^2\},t_*\right]\right)}\\
&\leq C_1\langle C_{\beta^0,1}\rangle\rho^{-\alpha}\Big[\|v\|_{L^\infty\left(B_n^+(x_*,2\rho)\times\left(\max\{t_0,t_*-4\rho^2\},t_*\right]\right)}+\rho^2\|h_1\|_{L^\infty\left(B_n^+(x_*,2\rho)\times\left(\max\{t_0,t_*-4\rho^2\},t_*\right]\right)}\\
&\quad+\mathbf{1}_{x_{*n}\leq4\rho}\,\rho\|h_2\|_{L^\infty\left(B_{n-1}(\tilde x_*,2\rho)\times\left(\max\{t_0,t_*-4\rho^2\},t_*\right]\right)}+\mathbf{1}_{\sqrt{t_*-t_0}\leq4\rho}\Big(\|v_0\|_{L^\infty(B_n^+(x_*,2\rho))}+\rho^\alpha[v_0]_{C^\alpha(B_n^+(x_*,2\rho))}\Big)\Big].\end{aligned}\tag{2.27}$$

Under the additional assumption that for $\gamma\in(0,1)$,

$$\rho\|\beta^0\|_{L^\infty\left(B_{n-1}(\tilde x_*,2\rho)\times\left(\max\{t_0,t_*-4\rho^2\},t_*\right]\right)}+\rho^{1+\gamma}[\beta^0]_{C^{\gamma,\gamma/2}\left(B_{n-1}(\tilde x_*,2\rho)\times\left(\max\{t_0,t_*-4\rho^2\},t_*\right]\right)}\leq C_{\beta^0,2},\tag{2.28}$$

$h_2\in C^{\gamma,\gamma/2}_{\rm loc}(\mathbb{R}^{n-1}\times[t_0,t_1))$, $v_0\in C^{1+\gamma}_{\rm loc}(\overline{\mathbb{R}^n_+})$, then we have

$$\begin{aligned}&\rho\|\nabla_xv\|_{L^\infty\left(B_n^+(x_*,\rho)\times\left(\max\{t_0,t_*-\rho^2\},t_*\right]\right)}+\rho^{1+\gamma}[\nabla_xv]_{C^{\gamma,\gamma/2}\left(B_n^+(x_*,\rho)\times\left(\max\{t_0,t_*-\rho^2\},t_*\right]\right)}\\
&\quad+\rho^{1+\gamma}\sup_{x\in B_n^+(x_*,\rho),\ \tau_1,\tau_2\in(\max\{t_0,t_*-\rho^2\},t_*]}\frac{|v(x,\tau_1)-v(x,\tau_2)|}{|\tau_1-\tau_2|^{(1+\gamma)/2}}\\
&\leq C_2\Big[\|v\|_{L^\infty\left(B_n^+(x_*,2\rho)\times\left(\max\{t_0,t_*-4\rho^2\},t_*\right]\right)}+\rho^2\|h_1\|_{L^\infty\left(B_n^+(x_*,2\rho)\times\left(\max\{t_0,t_*-4\rho^2\},t_*\right]\right)}\\
&\quad+\mathbf{1}_{x_{*n}\leq4\rho}\Big(\rho\|h_2\|_{L^\infty\left(B_{n-1}(\tilde x_*,2\rho)\times\left(\max\{t_0,t_*-4\rho^2\},t_*\right]\right)}+\rho^{1+\gamma}[h_2]_{C^{\gamma,\gamma/2}\left(B_{n-1}(\tilde x_*,2\rho)\times\left(\max\{t_0,t_*-4\rho^2\},t_*\right]\right)}\Big)\\
&\quad+\mathbf{1}_{\sqrt{t_*-t_0}\leq4\rho}\Big(\|v_0\|_{L^\infty(B_n^+(x_*,2\rho))}+\rho^{1+\gamma}[\nabla_xv_0]_{C^\gamma(B_n^+(x_*,2\rho))}\Big)\Big]\end{aligned}\tag{2.29}$$

with a positive constant C_2 only depending on $n,\gamma,C_{\beta^0,2}$.

Proof. Set $\tilde v(z,s)=v(x_*+\rho z,t_*+\rho^2s)$, that is, $v(x,t)=\tilde v\big(\rho^{-1}(x-x_*),\rho^{-2}(t-t_*)\big)$. By direct calculation,

$$\partial_tv=\rho^{-2}\left(\partial_s\tilde v\right)(\rho^{-1}(x-x_*),\rho^{-2}(t-t_*)),\quad D_x^{\mathbf{m}}v=\rho^{-\|\mathbf{m}\|_{\ell_1}}(D_z^{\mathbf{m}}\tilde v)(\rho^{-1}(x-x_*),\rho^{-2}(t-t_*))$$

for $\mathbf{m}\in\mathbb{N}^n$, $\|\mathbf{m}\|_{\ell_1}\leq2$. Then (2.26) is equivalent to

$$\begin{cases}\partial_s\tilde v=\Delta_z\tilde v+\tilde h_1(z,s)\ \text{ for }\ (z,s)\in Q,\quad \partial_{z_n}\tilde v+\tilde\beta^0(\tilde z,s)\tilde v=\tilde h_2(\tilde z,s)\ \text{ for }\ (z,s)\in SQ,\\ \tilde v=v_0(x_*+\rho z)\ \text{ for }\ (z,s)\in BQ,\end{cases}$$

where

$$\tilde\beta^0(\tilde z,s):=\rho\beta^0(\tilde x_*+\rho\tilde z,t_*+\rho^2 s),\quad \tilde h_1(z,s):=\rho^2 h_1(x_*+\rho z,t_*+\rho^2 s),\quad \tilde h_2(\tilde z,s):=\rho h_2(\tilde x_*+\rho\tilde z,t_*+\rho^2 s),$$
$$Q:=\left\{(z,s)\mid z\in A_1,\ s\in\left(\rho^{-2}(t_0-t_*),\rho^{-2}(t_1-t_*)\right)\right\},\quad A_1:=\{z\mid \tilde z\in\mathbb{R}^{n-1},\ z_n>-\rho^{-1}x_{*n}\},$$
$$SQ:=\left\{\left((\tilde z,-\rho^{-1}x_{*n}),s\right)\mid(\tilde z,s)\in A_2\right\},\quad A_2:=\left\{(\tilde z,s)\mid\tilde z\in\mathbb{R}^{n-1},\ s\in\left(\rho^{-2}(t_0-t_*),\rho^{-2}(t_1-t_*)\right)\right\},$$
$$BQ:=\left\{\left(z,\rho^{-2}(t_0-t_*)\right)\mid z\in A_1\right\}.$$

Obviously,

$$\begin{aligned}&(z,s)\in Q\Leftrightarrow x_*+\rho z\in\mathbb{R}^n_+,t_*+\rho^2 s\in(t_0,t_1);\quad(\tilde z,s)\in A_2\Leftrightarrow\tilde x_*+\rho\tilde z\in\mathbb{R}^{n-1},t_*+\rho^2 s\in(t_0,t_1);\\&z\in A_1\Leftrightarrow x_*+\rho z\in\mathbb{R}^n_+.\end{aligned}\tag{2.30}$$

For any $r\in\mathbb{R}_+$, for $r>0$, denote $Q_n(r):=B_n(0,r)\times(-r^2,0]$. For brevity, denote $\|f_1\|_{L^\infty(Q_n(r)\cap SQ)}:=\|f_1\|_{L^\infty\left(\left\{(\tilde z,s)\ \mid\ \left((\tilde z,-\rho^{-1}x_{*n}),s\right)\in Q_n(r)\cap SQ\right\}\right)}$, $\|f_2\|_{C^\alpha(Q_n(r)\cap BQ)}:=\|f_2\|_{C^\alpha\left(\left\{z\ \mid\ \left(z,\rho^{-2}(t_0-t_*)\right)\in Q_n(r)\cap BQ\right\}\right)}$, and $\|f_3\|_{C^{\gamma,\gamma/2}(Q_n(r)\cap SQ)}$, $\|f_4\|_{C^{1+\gamma}(Q_n(r)\cap BQ)}$ are defined similarly. Since

$$\|\tilde\beta^0\|_{L^\infty(Q_n(2)\cap SQ)}\le\rho\|\beta^0(\tilde x_*+\rho\tilde z,t_*+\rho^2 s)\|_{L^\infty(Q_{n-1}(2)\cap A_2)}=\rho\|\beta^0\|_{L^\infty\left(B_{n-1}(\tilde x_*,2\rho)\times\left(\max\{t_0,t_*-4\rho^2\},t_*\right]\right)}\le C_{\beta^0,1},\tag{2.31}$$

by [35, p.140 Theorem 6.44] (see also [36, Corollary 7.6]), there exist positive constants C_1,α_0 determined only by n such that if $\alpha\in(0,\alpha_0]$, $v_0\in C^\alpha_{\rm loc}(\overline{\mathbb{R}^n_+})$, we have

$$[\tilde v]_{C^{\alpha,\alpha/2}(Q_n(1)\cap Q)}\le C_1\langle C_{\beta^0,1}\rangle\Big(\|\tilde v\|_{L^\infty(Q_n(2)\cap Q)}+\|\tilde h_1\|_{L^\infty(Q_n(2)\cap Q)}+\|\tilde h_2\|_{L^\infty(Q_n(2)\cap SQ)}+\|v_0(x_*+\rho z)\|_{C^\alpha(Q_n(2)\cap BQ)}\Big).$$

We will handle the above term by term and (2.30) will be used repetitively.

$$\begin{aligned}&[\tilde v]_{C^{\alpha,\alpha/2}(Q_n(1)\cap Q)}=[v(x_*+\rho z,t_*+\rho^2 s)]_{C^{\alpha,\alpha/2}(Q_n(1)\cap Q)}\\&=\rho^\alpha\sup_{(z^{[1]},s_1),(z^{[2]},s_2)\in Q_n(1)\cap Q}\frac{\left|v(x_*+\rho z^{[1]},t_*+\rho^2 s_1)-v(x_*+\rho z^{[2]},t_*+\rho^2 s_2)\right|}{\left(\max\{|(x_*+\rho z^{[1]})-(x_*+\rho z^{[2]})|,|(t_*+\rho^2 s_1)-(t_*+\rho^2 s_2)|^{1/2}\}\right)^\alpha}\\&=\rho^\alpha[v]_{C^{\alpha,\alpha/2}\left(B_n^+(x_*,\rho)\times\left(\max\{t_0,t_*-\rho^2\},t_*\right]\right)},\end{aligned}\tag{2.32}$$

where we used (2.30) for the last step.

$$\begin{aligned}\|\tilde v\|_{L^\infty(Q_n(2)\cap Q)}&=\|v\|_{L^\infty\left(B_n^+(x_*,2\rho)\times\left(\max\{t_0,t_*-4\rho^2\},t_*\right]\right)},\\\|\tilde h_1\|_{L^\infty(Q_n(2)\cap Q)}&=\rho^2\|h_1\|_{L^\infty\left(B_n^+(x_*,2\rho)\times\left(\max\{t_0,t_*-4\rho^2\},t_*\right]\right)}.\end{aligned}\tag{2.33}$$

$\|\tilde h_2\|_{L^\infty(Q_n(2)\cap SQ)}$ is vacuum if $-\rho^{-1}x_{*n}<-4$. For $-\rho^{-1}x_{*n}\ge-4\Leftrightarrow x_{*n}\le4\rho$, similar to (2.31),

$$\|\tilde h_2\|_{L^\infty(Q_n(2)\cap SQ)}\le\rho\|h_2\|_{L^\infty\left(B_{n-1}(\tilde x_*,2\rho)\times\left(\max\{t_0,t_*-4\rho^2\},t_*\right]\right)}.$$

$\|v_0(x_*+\rho z)\|_{C^\alpha(Q_n(2)\cap BQ)}$ is vacuum if $\rho^{-2}(t_0-t_*)<-16$. For $\rho^{-2}(t_0-t_*)\ge-16\Leftrightarrow\sqrt{t_*-t_0}\le4\rho$,

$$\begin{aligned}&[v_0(x_*+\rho z)]_{C^\alpha(Q_n(2)\cap BQ)}\le\rho^\alpha\sup_{x_*+\rho z^{[1]},x_*+\rho z^{[2]}\in B_n^+(x_*,2\rho)}\frac{|v_0(x_*+\rho z^{[1]})-v_0(x_*+\rho z^{[2]})|}{|(x_*+\rho z^{[1]})-(x_*+\rho z^{[2]})|^\alpha}=\rho^\alpha[v_0]_{C^\alpha(B_n^+(x_*,2\rho))},\\&\|v_0(x_*+\rho z)\|_{L^\infty(Q_n(2)\cap BQ)}\le\|v_0\|_{L^\infty(B_n^+(x_*,2\rho))}.\end{aligned}\tag{2.34}$$

As a result, we get (2.27).

Under the additional assumption, we have $\|\tilde\beta^0\|_{C^{\gamma,\gamma/2}(Q_n(2)\cap SQ)}\le C_{\beta^0,2}$ since (2.28) holds and

$$\begin{aligned}&[\tilde\beta^0]_{C^{\gamma,\gamma/2}(Q_n(2)\cap SQ)}\le\rho\left[\beta^0(\tilde x_*+\rho\tilde z,t_*+\rho^2 s)\right]_{C^{\gamma,\gamma/2}(Q_{n-1}(2)\cap A_2)}\\&=\rho^{1+\gamma}\sup_{(\tilde z^{[1]},s_1),(\tilde z^{[2]},s_2)\in Q_{n-1}(2)\cap A_2}\frac{\left|\beta^0(\tilde x_*+\rho\tilde z^{[1]},t_*+\rho^2 s_1)-\beta^0(\tilde x_*+\rho\tilde z^{[2]},t_*+\rho^2 s_2)\right|}{\left(\max\{|(\tilde x_*+\rho\tilde z^{[1]})-(\tilde x_*+\rho\tilde z^{[2]})|,|(t_*+\rho^2 s_1)-(t_*+\rho^2 s_2)|^{1/2}\}\right)^\gamma}\\&=\rho^{1+\gamma}[\beta^0]_{C^{\gamma,\gamma/2}\left(B_{n-1}(\tilde x_*,2\rho)\times\left(\max\{t_0,t_*-4\rho^2\},t_*\right]\right)},\\&\|\tilde\beta^0\|_{L^\infty(Q_n(2)\cap SQ)}\le\rho\|\beta^0\|_{L^\infty\left(B_{n-1}(\tilde x_*,2\rho)\times\left(\max\{t_0,t_*-4\rho^2\},t_*\right]\right)}.\end{aligned}\tag{2.35}$$

Hence, by [35, p.69 Theorem 4.21, and p.79 Theorem 4.30], we have

$$\|\tilde{v}\|_{C^{1+\gamma,(1+\gamma)/2}(Q_n(1)\cap Q)} \le C_{21}\Big(\|\tilde{v}\|_{L^\infty(Q_n(2)\cap Q)} + \|\tilde{h}_1\|_{L^\infty(Q_n(2)\cap Q)} + \|\tilde{h}_2\|_{C^{\gamma,\gamma/2}(Q_n(2)\cap SQ)} + \|v_0(x_*+\rho z)\|_{C^{1+\gamma}(Q_n(2)\cap BQ)}\Big)$$

with a constant $C_{21}>0$ only depending on $n,\gamma,C_{\beta^0,2}$. We will handle the above term by term.

$$\|\tilde{v}\|_{C^{1+\gamma,(1+\gamma)/2}(Q_n(1)\cap Q)} = \|\tilde{v}\|_{L^\infty(Q_n(1)\cap Q)} + \|\nabla_z\tilde{v}\|_{L^\infty(Q_n(1)\cap Q)} + [\nabla_z\tilde{v}]_{C^{\gamma,\gamma/2}(Q_n(1)\cap Q)} + [\tilde{v}]_{C_t^{(1+\gamma)/2}(Q_n(1)\cap Q)}.$$

Therein, by (2.30),

$$\|\nabla_z\tilde{v}\|_{L^\infty(Q_n(1)\cap Q)} = \rho\|(\nabla_x v)(x_*+\rho z, t_*+\rho^2 s)\|_{L^\infty(Q_n(1)\cap Q)} = \rho\|\nabla_x v\|_{L^\infty\left(B_n^+(x_*,\rho)\times\left(\max\{t_0,t_*-\rho^2\},t_*\right]\right)}.$$

Similar to (2.32),

$$[\nabla_z\tilde{v}]_{C^{\gamma,\gamma/2}(Q_n(1)\cap Q)} = \rho[(\nabla_x v)(x_*+\rho z,t_*+\rho^2 s)]_{C^{\gamma,\gamma/2}(Q_n(1)\cap Q)} = \rho^{1+\gamma}[\nabla_x v]_{C^{\gamma,\gamma/2}\left(B_n^+(x_*,\rho)\times\left(\max\{t_0,t_*-\rho^2\},t_*\right]\right)}.$$

$$[\tilde{v}]_{C_t^{(1+\gamma)/2}(Q_n(1)\cap Q)} = \rho^{1+\gamma}\sup_{(z,s_1),(z,s_2)\in Q_n(1)\cap Q}\frac{|v(x_*+\rho z,t_*+\rho^2 s_1)-v(x_*+\rho z,t_*+\rho^2 s_2)|}{|(t_*+\rho^2 s_1)-(t_*+\rho^2 s_2)|^{(1+\gamma)/2}}$$
$$= \rho^{1+\gamma}\sup_{x\in B_n^+(x_*,\rho),\ \tau_1,\tau_2\in(\max\{t_0,t_*-\rho^2\},t_*]}\frac{|v(x,\tau_1)-v(x,\tau_2)|}{|\tau_1-\tau_2|^{(1+\gamma)/2}}.$$

$\|\tilde{v}\|_{L^\infty(Q_n(2)\cap Q)}, \|\tilde{h}_1\|_{L^\infty(Q_n(2)\cap Q)}$ have been handled in (2.33).
$\|\tilde{h}_2\|_{C^{\gamma,\gamma/2}(Q_n(2)\cap SQ)}$ is vacuum if $-\rho^{-1}x_{*n}<-4$. For $-\rho^{-1}x_{*n}\ge -4 \Leftrightarrow x_{*n}\le 4\rho$, similar to (2.35),

$$[\tilde{h}_2]_{C^{\gamma,\gamma/2}(Q_n(2)\cap SQ)} \le \rho^{1+\gamma}[h_2]_{C^{\gamma,\gamma/2}\left(B_{n-1}(\tilde{x}_*,2\rho)\times\left(\max\{t_0,t_*-4\rho^2\},t_*\right]\right)},$$
$$\|\tilde{h}_2\|_{L^\infty(Q_n(2)\cap SQ)} \le \rho\|h_2\|_{L^\infty\left(B_{n-1}(\tilde{x}_*,2\rho)\times\left(\max\{t_0,t_*-4\rho^2\},t_*\right]\right)}.$$

$\|v_0(x_*+\rho z)\|_{C^{1+\gamma}(Q_n(2)\cap BQ)}$ is vacuum if $\rho^{-2}(t_0-t_*)<-16$. For $\rho^{-2}(t_0-t_*)\ge -16 \Leftrightarrow \sqrt{t_*-t_0}\le 4\rho$, similar to (2.34), we have

$$\|v_0(x_*+\rho z)\|_{C^{1+\gamma}(Q_n(2)\cap BQ)} \sim_\gamma \|v_0(x_*+\rho z)\|_{L^\infty(Q_n(2)\cap BQ)} + \rho[(\nabla_x v_0)(x_*+\rho z)]_{C^\gamma(Q_n(2)\cap BQ)}$$
$$\le \|v_0\|_{L^\infty(B_n^+(x_*,2\rho))} + \rho^{1+\gamma}[\nabla_x v_0]_{C^\gamma(B_n^+(x_*,2\rho))}.$$

In sum, we obtain (2.29). □

3. Approximate solution and inner-outer gluing scheme

The finite-time blow-up solutions of heat equations with the critical boundary condition are expected to behave like the dilation of steady-state solutions. Given an integer $\mathfrak{o}\ge 1$, we define

$$U_{\mu_i,\xi^{[i]}}(x) := \mu_i^{-\frac{n-2}{2}}U\left(\frac{\tilde{x}-\xi^{[i]}}{\mu_i},\frac{x_n}{\mu_i}\right),\quad \tilde{x}\in\mathbb{R}^{n-1},\ x_n\in\mathbb{R}_+,\ x=(\tilde{x},x_n),\quad i=1,2,\dots,\mathfrak{o}$$

with $\mu_i=\mu_i(t)\in C^1([0,T),\mathbb{R}_+)$, $\xi^{[i]}=\xi^{[i]}(t)=\left(\xi_1^{[i]}(t),\xi_2^{[i]}(t),\dots,\xi_{n-1}^{[i]}(t)\right)\in C^1\left([0,T),\mathbb{R}^{n-1}\right)$ to be determined later. Given arbitrary $\mathfrak{o}$ distinct points $q^{[i]}$ on $\partial\mathbb{R}^n_+$, $i=1,2,\dots,\mathfrak{o}$, we define

$$x^{[i]} := x-q^{[i]},\quad y^{[i]} := \frac{x-(\xi^{[i]},0)}{\mu_i},\quad z^{[i]} := \frac{x-q^{[i]}}{\sqrt{T-t}},\quad \delta := \frac{1}{32}\min_{1\le i\ne j\le\mathfrak{o}}|q^{[i]}-q^{[j]}|,$$

and some cut-off functions

$$\eta_R(y^{[i]}) := \eta\Big(\frac{y^{[i]}}{R}\Big) \text{ with } R=|\ln T|,\quad \eta_{C\delta}(x^{[i]}) := \eta\Big(\frac{x^{[i]}}{C\delta}\Big) \text{ for } C>0. \tag{3.1}$$

Given $\mathfrak{o}$ integers $l_i\in\mathbb{N}$ (could be duplicated), $i=1,2,\dots,\mathfrak{o}$, denote $l_{\max} := \max_{i=1,2,\dots,\mathfrak{o}} l_i$. We look for the solution of the form

$$u=\sum_{i=1}^{\mathfrak{o}}\left(U_{\mu_i,\xi^{[i]}}(x)\eta_{2\delta}(x^{[i]})+\Theta_{l_i}(x^{[i]},t)\eta_\delta(x^{[i]})+\mu_i^{-\frac{n-2}{2}}\phi_i(y^{[i]},t)\eta_R(y^{[i]})\right)+\psi(x,t),$$

where Θ_{l_i} is given in (1.14) satisfying (1.15) and ϕ_i,ψ will be determined later. Denote

$$\boldsymbol{\mu}=(\mu_1,\mu_2,\dots,\mu_{\mathfrak{o}}),\quad \boldsymbol{\xi}=(\xi^{[1]},\xi^{[2]},\dots,\xi^{[\mathfrak{o}]}),\quad \boldsymbol{\phi}=(\phi_1,\phi_2,\dots,\phi_{\mathfrak{o}}).$$

We introduce the error operators as

$$\mathcal{E}_1[u] := -\partial_t u + \Delta_x u \ \text{ in } \ \mathbb{R}^n_+ \times (0,T), \qquad \mathcal{E}_2[u] := \partial_{x_n} u + |u|^{\frac{2}{n-2}} u \ \text{ on } \ \partial\mathbb{R}^n_+ \times (0,T).$$

Solving (1.1) is equivalent to making $\mathcal{E}_1[u] = 0$ and $\mathcal{E}_2[u] = 0$. By (1.15), direct calculations give that

$$\begin{aligned}\mathcal{E}_1[u] = &\left(-\partial_t + \Delta_x\right)\psi + \sum_{i=1}^{\mathfrak{o}} \Big[\mathcal{E}_1\left[U_{\mu_i,\xi^{[i]}}(x)\right]\eta_{2\delta}(x^{[i]}) + \mathcal{E}^{\rm cut}_{U,i} + \mathcal{E}^{\rm cut}_{\Theta,i} \\ &+ \mu_i^{-\frac{n+2}{2}}\eta_R(y^{[i]})\left(-\mu_i^2\partial_t\phi_i(y^{[i]},t) + \Delta_{y^{[i]}}\phi_i(y^{[i]},t)\right) + \Lambda_{1,i}[\phi_i,\mu_i,\xi^{[i]}] + \Lambda_{2,i}[\phi_i,\mu_i,\xi^{[i]}]\Big],\end{aligned}$$

where

$$\begin{aligned}&\mathcal{E}_1\left[U_{\mu_i,\xi^{[i]}}(x)\right] = \dot{\mu}_i\mu_i^{-\frac{n}{2}} Z_n(y^{[i]}) + \mu_i^{-\frac{n}{2}}\dot{\xi}^{[i]}\cdot\left(\nabla_{\tilde{y}^{[i]}}U\right)(\tilde{y}^{[i]}, y_n^{[i]}),\\ &\mathcal{E}^{\rm cut}_{U,i} := 2\nabla_x\left(U_{\mu_i,\xi^{[i]}}(x)\right)\cdot\nabla_x\left(\eta_{2\delta}(x^{[i]})\right) + U_{\mu_i,\xi^{[i]}}(x)\Delta_x\left(\eta_{2\delta}(x^{[i]})\right),\\ &\mathcal{E}^{\rm cut}_{\Theta,i} := 2\nabla_x\left(\Theta_{l_i}(x^{[i]},t)\right)\cdot\nabla_x\left(\eta_\delta(x^{[i]})\right) + \Theta_{l_i}(x^{[i]},t)\Delta_x\left(\eta_\delta(x^{[i]})\right),\\ &\Lambda_{1,i}[\phi_i,\mu_i,\xi^{[i]}] := \mu_i^{-\frac{n-2}{2}}\left[\frac{\dot{\xi}^{[i]}}{\mu_i R}\cdot(\nabla_{\tilde{x}}\eta)\,(\frac{\tilde{y}^{[i]}}{R},\frac{y_n^{[i]}}{R}) + \frac{(\mu_i R)'}{\mu_i R}\frac{y^{[i]}}{R}\cdot(\nabla_x\eta)\,(\frac{y^{[i]}}{R})\right]\phi_i(y^{[i]},t)\\ &\qquad + \mu_i^{-\frac{n+2}{2}}R^{-2}\phi_i(y^{[i]},t)\,(\Delta_x\eta)\,(\frac{y^{[i]}}{R}) + 2\mu_i^{-\frac{n+2}{2}}R^{-1}\nabla_{y^{[i]}}\phi_i(y^{[i]},t)\cdot(\nabla_x\eta)\,(\frac{y^{[i]}}{R}),\\ &\Lambda_{2,i}[\phi_i,\mu_i,\xi^{[i]}] := \mu_i^{-\frac{n-2}{2}}\eta_R(y^{[i]})\left[\mu_i^{-1}\dot{\xi}^{[i]}\cdot\nabla_{\tilde{y}^{[i]}}\phi_i(y^{[i]},t) + \dot{\mu}_i\mu_i^{-1}\left(\frac{n-2}{2}\phi_i(y^{[i]},t) + y^{[i]}\cdot\nabla_{y^{[i]}}\phi_i(y^{[i]},t)\right)\right].\end{aligned} \tag{3.2}$$

Since $\eta(x)$ is a smooth radial cut-off function, we have $\partial_{x_n} f\big|_{x_n=0} = 0$ on $\partial\mathbb{R}^n_+\times(0,T)$ for $f = \eta_R(y^{[i]}), \eta_\delta(x^{[i]}), \eta_{2\delta}(x^{[i]})$. Combining (1.11) for $U_{\mu_i,\xi^{[i]}}$ and (1.15) for Θ_{l_i}, we have

$$\begin{aligned}\mathcal{E}_2[u] &= \sum_{i=1}^{\mathfrak{o}}\Big[-\left(U_{\mu_i,\xi^{[i]}}(x)\right)^{\frac{n}{n-2}}\eta_{2\delta}(x^{[i]}) + \mu_i^{-\frac{n}{2}}\eta_R(y^{[i]})\partial_{y_n^{[i]}}\phi_i(y^{[i]},t)\Big] + \partial_{x_n}\psi + |u|^{\frac{2}{n-2}}u\\ &= \sum_{i=1}^{\mathfrak{o}}\Big[\left(U_{\mu_i,\xi^{[i]}}(x)\right)^{\frac{n}{n-2}}\left(\eta_{2\delta}^{\frac{n}{n-2}}(x^{[i]}) - \eta_{2\delta}(x^{[i]})\right) + \mu_i^{-\frac{n}{2}}\eta_R(y^{[i]})\partial_{y_n^{[i]}}\phi_i(y^{[i]},t)\Big] + \partial_{x_n}\psi\\ &\quad + \sum_{i=1}^{\mathfrak{o}}\frac{n}{n-2}\left(U_{\mu_i,\xi^{[i]}}(x)\right)^{\frac{2}{n-2}}\eta_{2\delta}^{\frac{2}{n-2}}(x^{[i]})\left(\Theta_{l_i}(x^{[i]},t)\eta_\delta(x^{[i]}) + \mu_i^{-\frac{n-2}{2}}\phi_i(y^{[i]},t)\eta_R(y^{[i]}) + \psi(x,t)\right) + \mathcal{N}\left[\psi,\boldsymbol{\phi},\boldsymbol{\mu},\boldsymbol{\xi}\right]\end{aligned} \tag{3.3}$$

with the nonlinear term

$$\begin{aligned}\mathcal{N}\left[\psi,\boldsymbol{\phi},\boldsymbol{\mu},\boldsymbol{\xi}\right] := &|u|^{\frac{2}{n-2}}u - \sum_{i=1}^{\mathfrak{o}}\left(U_{\mu_i,\xi^{[i]}}(x)\right)^{\frac{n}{n-2}}\eta_{2\delta}^{\frac{n}{n-2}}(x^{[i]})\\ &- \sum_{i=1}^{\mathfrak{o}}\frac{n}{n-2}\left(U_{\mu_i,\xi^{[i]}}(x)\right)^{\frac{2}{n-2}}\eta_{2\delta}^{\frac{2}{n-2}}(x^{[i]})\left(\Theta_{l_i}(x^{[i]},t)\eta_\delta(x^{[i]}) + \mu_i^{-\frac{n-2}{2}}\phi_i(y^{[i]},t)\eta_R(y^{[i]}) + \psi(x,t)\right).\end{aligned}$$

In order to make $\mathcal{E}_1[u] = 0$ in $\mathbb{R}^n_+\times(0,T)$ and $\mathcal{E}_2[u] = 0$ on $\partial\mathbb{R}^n_+\times(0,T)$, it suffices to solve the following inner-outer gluing system.

The inner problems: For $i = 1, 2, \ldots, \mathfrak{o}$,

$$\begin{cases}\mu_i^2\partial_t\phi_i = \Delta_{y^{[i]}}\phi_i + \mathcal{H}_{1,i}[\mu_i,\xi^{[i]}] & \text{for } \ t\in(0,T)\,, y^{[i]}\in B^+_n(0,2R),\\ -\partial_{y_n^{[i]}}\phi_i = \frac{n}{n-2}U^{\frac{2}{n-2}}(y^{[i]})\phi_i + \mathcal{H}_{2,i}[\psi,\mu_i,\xi^{[i]}] & \text{for } \ t\in(0,T)\,, y^{[i]}\in B_{n-1}(0,2R)\times\{0\},\end{cases} \tag{3.4}$$

where

$$\begin{aligned}&\mathcal{H}_{1,i}[\mu_i,\xi^{[i]}] := \eta\Big(\frac{y^{[i]}}{4R}\Big)\mu_i^{\frac{n+2}{2}}\mathcal{E}_1\left[U_{\mu_i,\xi^{[i]}}(x)\right] = \eta\Big(\frac{y^{[i]}}{4R}\Big)\Big(\dot{\mu}_i\mu_i Z_n(y^{[i]}) + \mu_i\dot{\xi}^{[i]}\cdot\left(\nabla_{\tilde{y}^{[i]}}U\right)(\tilde{y}^{[i]},y_n^{[i]})\Big),\\ &\mathcal{H}_{2,i}[\psi,\mu_i,\xi^{[i]}] := \frac{n}{n-2}\mu_i^{\frac{n}{2}-1}U^{\frac{2}{n-2}}(\tilde{y}^{[i]},0)\eta\Big(\frac{\tilde{y}^{[i]}}{4R},0\Big)\Big(\Theta_{l_i}\big((\mu_i\tilde{y}^{[i]}+\xi^{[i]}-\tilde{q}^{[i]},0),t\big) + \psi\big((\mu_i\tilde{y}^{[i]}+\xi^{[i]},0),t\big)\Big).\end{aligned} \tag{3.5}$$

The outer problem:

$$\partial_t\psi = \Delta_x\psi + \mathcal{G}_1[\boldsymbol{\phi},\boldsymbol{\mu},\boldsymbol{\xi}] \ \text{ in } \ \mathbb{R}^n_+\times(0,T)\,, \quad -\partial_{x_n}\psi = \mathcal{G}_2[\psi,\boldsymbol{\phi},\boldsymbol{\mu},\boldsymbol{\xi}] \ \text{ on } \ \partial\mathbb{R}^n_+\times(0,T)\,, \tag{3.6}$$

where

$$
\begin{aligned}
&\mathcal{G}_1[\boldsymbol{\phi},\boldsymbol{\mu},\boldsymbol{\xi}] := \sum_{i=1}^{\mathfrak{o}} \Big[\mathcal{E}_1\left[U_{\mu_i,\xi^{[i]}}(x)\right]\left(\eta_{2\delta}(x^{[i]})-\eta_R(y^{[i]})\right)+\mathcal{E}_{U,i}^{\mathrm{cut}}+\mathcal{E}_{\Theta,i}^{\mathrm{cut}}+\Lambda_{1,i}[\phi_i,\mu_i,\xi^{[i]}]+\Lambda_{2,i}[\phi_i,\mu_i,\xi^{[i]}]\Big],\\
&\mathcal{G}_2[\psi,\boldsymbol{\phi},\boldsymbol{\mu},\boldsymbol{\xi}] := \Bigg[\mathcal{N}[\psi,\boldsymbol{\phi},\boldsymbol{\mu},\boldsymbol{\xi}]+\sum_{i=1}^{\mathfrak{o}}\Big\{\left(U_{\mu_i,\xi^{[i]}}(x)\right)^{\frac{n}{n-2}}\left(\eta_{2\delta}^{\frac{n}{n-2}}(x^{[i]})-\eta_{2\delta}(x^{[i]})\right)\\
&+\frac{n}{n-2}\left(U_{\mu_i,\xi^{[i]}}(x)\right)^{\frac{2}{n-2}}\Big[\left(\eta_\delta(x^{[i]})-\eta_R(y^{[i]})\right)\Theta_{l_i}(x^{[i]},t)+\left(\eta_{2\delta}^{\frac{2}{n-2}}(x^{[i]})-\eta_R(y^{[i]})\right)\psi(x,t)\\
&+\left(\eta_{2\delta}^{\frac{2}{n-2}}(x^{[i]})-1\right)\eta_R(y^{[i]})\mu_i^{-\frac{n-2}{2}}\phi_i(y^{[i]},t)\Big]\Big\}\Bigg]\Bigg|_{x_n=0}.
\end{aligned}
\tag{3.7}
$$

In order to avoid the difficulty of the compactness argument due to the singularity of right-hand sides as $t\uparrow T$, we will solve (3.6), (3.4) for $t\in(0,T_{\sigma_0})$ with $T_{\sigma_0}:=T-\sigma_0,\sigma_0\in(0,T)$ instead of $(0,T)$. Details will be given in Section 7. For applying Proposition 1.2 in $\mathbb{R}^n_+$ for solving (3.4), we impose the cut-off functions $\eta(\frac{y^{[i]}}{4R}),\eta(\frac{\tilde{y}^{[i]}}{4R},0)$ to restrict the spatial growth of $\mathcal{H}_{1,i},\mathcal{H}_{2,i}$ respectively.

4. Proof of Proposition 1.2

In this section, we will prove Proposition 1.2, which will be used for solving the inner problems. The following nondegeneracy lemma is prepared for the forthcoming blow-up argument.

Lemma 4.1. *For an integer $n\ge 3$, all bounded solutions of the equation $\Delta\phi=0$ in $\mathbb{R}^n_+$, $-\partial_{x_n}\phi=\frac{n}{n-2}U^{\frac{2}{n-2}}\phi$ on $\partial\mathbb{R}^n_+$ are the linear combination of $Z_j(x)$, $j=1,2,\dots,n$, given in (2.1).*

Proof. Since $(-\Delta_{\tilde{x}})^{\frac12}\phi(\tilde{x},0)-\frac{n}{n-2}\left(U^{\frac{2}{n-2}}\phi\right)(\tilde{x},0)=0$ in $\mathbb{R}^{n-1}$ and [10, Theorem 1.1], we have $\phi(\tilde{x},0)=\sum_{j=1}^{n}c_jZ_j(\tilde{x},0)$ for some $c_j\in\mathbb{R}$, and thus $\phi(x)=\sum_{j=1}^{n}c_jZ_j(x)$ in $\mathbb{R}^n_+$, where we used the fact that the bounded solution of the equation $\Delta u=0$ in $\mathbb{R}^n_+$, $u=0$ on $\partial\mathbb{R}^n_+$ is zero. □

Recall the norms defined in (1.17).

Proposition 4.2. *Given an integer $n\ge 5$, consider*

$$
\partial_\tau\phi=\Delta\phi+g \ \text{ in } \ \mathbb{R}^n_+\times(\tau_0,\tau_1),\quad -\partial_{y_n}\phi=\frac{n}{n-2}U^{\frac{2}{n-2}}\phi+h \ \text{ on } \ \partial\mathbb{R}^n_+\times(\tau_0,\tau_1),\quad \phi(y,\tau_0)=0 \ \text{ in } \ \mathbb{R}^n_+, \tag{4.1}
$$

where ϕ is given by

$$
\phi(y,\tau)=\int_{\tau_0}^{\tau}\int_{\mathbb{R}^n_+}G_n(y,\tau,z,s)g(z,s)dzds+\int_{\tau_0}^{\tau}\int_{\mathbb{R}^{n-1}}G_n(y,\tau,(\tilde{z},0),s)\left[\frac{n}{n-2}\left(U^{\frac{2}{n-2}}\phi\right)((\tilde{z},0),s)+h(\tilde{z},s)\right]d\tilde{z}ds \tag{4.2}
$$

with G_n given in (2.15). Suppose that $1\le\tau_0<\tau_1\le\infty$, $\ell(\tau)$ satisfies $C_\ell^{-1}\tau^p\le\ell(\tau)\le C_\ell\tau^p$ with a constant $C_\ell\ge1$,

$$
2<a<n-2,\quad a^{-1}<p\le\frac12,\quad \iota\in(0,\frac14),\quad \sigma-pa+2\iota n>0,\quad \varsigma\in(0,1), \tag{4.3}
$$

$\|g\|_{\sigma,2+a,\ell(\tau),\mathbb{R}^n_+,\tau_0,\tau_1}<\infty$, $\|h\|_{\sigma,1+a,\ell(\tau),\varsigma,\mathbb{R}^{n-1},\tau_0,\tau_1}<\infty$, and $g=g(y,\tau)$, $h=h(\tilde{y},\tau)$ satisfy the orthogonality conditions

$$
\int_{\mathbb{R}^n_+}g(y,\tau)Z_j(y)dy+\int_{\mathbb{R}^{n-1}}h(\tilde{y},\tau)Z_j(\tilde{y},0)d\tilde{y}=0\quad \text{for } \ \tau\in(\tau_0,\tau_1),\quad j=0,1,\dots,n \tag{4.4}
$$

with Z_j given in (2.1) and (2.3), then there exists a unique solution ϕ in $L^\infty\left(\mathbb{R}^n_+\times(\tau_0,\tilde{\tau})\right)$ for all $\tilde{\tau}\in(\tau_0,\tau_1]\cap(\tau_0,\infty)$ and satisfying

$$
\int_{\mathbb{R}^n_+}\phi(y,\tau)Z_j(y)dy=0\quad \text{for } \ \tau\in(\tau_0,\tau_1),\quad j=0,1,\dots,n, \tag{4.5}
$$

and

$$
\begin{aligned}
|\phi|&\lesssim\left(\tau^\sigma\langle y\rangle^{-a}\mathbf{1}_{|y|\le\ell(\tau)}+\tau^\sigma\ell^{-a}(\tau)e^{-\iota\frac{|y|^2}{\tau}}\mathbf{1}_{|y|>\ell(\tau)}\right)\left(\|g\|_{\sigma,2+a,\ell(\tau),\mathbb{R}^n_+,\tau_0,\tau_1}+\|h\|_{\sigma,1+a,\ell(\tau),\mathbb{R}^{n-1},\tau_0,\tau_1}\right),\\
|\nabla\phi|&\lesssim\left(\tau^\sigma\langle y\rangle^{-1-a}\mathbf{1}_{|y|\le\ell(\tau)}+\tau^\sigma\ell^{-a}(\tau)|y|^{-1}\mathbf{1}_{\ell(\tau)<|y|\le\tau^{\frac12}}+\tau^{\sigma-\frac12}\ell^{-a}(\tau)e^{-\tilde{\iota}\frac{|y|^2}{\tau}}\mathbf{1}_{|y|>\tau^{\frac12}}\right)\\
&\quad\times\left(\|g\|_{\sigma,2+a,\ell(\tau),\mathbb{R}^n_+,\tau_0,\tau_1}+\|h\|_{\sigma,1+a,\ell(\tau),\varsigma,\mathbb{R}^{n-1},\tau_0,\tau_1}\right)
\end{aligned}
\tag{4.6}
$$

with a constant $\tilde{\iota} \in (0, \iota)$, where both "$\lesssim$" are independent of τ_0, τ_1, g, h.

We first use Proposition 4.2 to deduce Proposition 1.2.

Proof of Proposition 1.2. Set $\phi(y,\tau) = \phi_1(y,\tau) + c(\tau)\tilde{Z}_0(y)$. It suffices to consider

$$\partial_\tau \phi_1 = \Delta \phi_1 + g_1 \text{ in } \mathbb{R}^n_+ \times (\tau_0, \tau_1), \quad -\partial_{y_n}\phi_1 = \frac{n}{n-2} U^{\frac{2}{n-2}} \phi_1 + h_1 \text{ on } \partial\mathbb{R}^n_+ \times (\tau_0,\tau_1), \quad \phi_1(y,\tau_0) = 0 \text{ in } \mathbb{R}^n_+,$$

where

$$\begin{aligned} g_1 &= g_1(y,\tau) := g + c(\tau)\Delta \tilde{Z}_0(y) - c'(\tau)\tilde{Z}_0(y), \\ h_1 &= h_1(\tilde{y},\tau) := h + c(\tau)\frac{n}{n-2}\left(U^{\frac{2}{n-2}}\tilde{Z}_0\right)(\tilde{y},0) + c(\tau)\left(\partial_{y_n}\tilde{Z}_0\right)(\tilde{y},0). \end{aligned}$$

For $j = 0, 1, \dots, n$, by (2.2), (2.3), (1.20), orthogonality conditions in (1.21), we have

$$\begin{aligned} &\int_{\mathbb{R}^n_+} g_1(y,\tau) Z_j(y)dy + \int_{\mathbb{R}^{n-1}} h_1(\tilde{y},\tau)Z_j(\tilde{y},0)d\tilde{y} \\ =& \int_{\mathbb{R}^n_+} g(y,\tau)Z_j(y)dy + \int_{\mathbb{R}^{n-1}} h(\tilde{y},\tau)Z_j(\tilde{y},0)d\tilde{y} + c(\tau)\int_{\mathbb{R}^{n-1}} \tilde{Z}_0(\tilde{y},0)\left(\partial_{y_n}Z_j + \frac{n}{n-2}U^{\frac{2}{n-2}}Z_j\right)(\tilde{y},0)d\tilde{y} \\ &+ c(\tau)\int_{\mathbb{R}^n_+} \tilde{Z}_0 \Delta Z_j dy - c'(\tau)\int_{\mathbb{R}^n_+}\tilde{Z}_0 Z_j dy \\ =& \delta_{j0}\left(\int_{\mathbb{R}^n_+} g(y,\tau)Z_0(y)dy + \int_{\mathbb{R}^{n-1}} h(\tilde{y},\tau)Z_0(\tilde{y},0)d\tilde{y} - \lambda_0 c(\tau)\int_{\mathbb{R}^n_+}\tilde{Z}_0 Z_0 dy - c'(\tau)\int_{\mathbb{R}^n_+}\tilde{Z}_0 Z_0 dy\right) \end{aligned}$$

with $\lambda_0 < 0$. Since $\int_{\mathbb{R}^n_+}\tilde{Z}_0 Z_0 dy \neq 0$, we can take

$$c(\tau) = -\left(\int_{\mathbb{R}^n_+}\tilde{Z}_0 Z_0 dy\right)^{-1} e^{-\lambda_0 \tau}\int_\tau^{\tau_1} e^{\lambda_0 s}\left(\int_{\mathbb{R}^n_+} g(y,s)Z_0(y)dy + \int_{\mathbb{R}^{n-1}} h(\tilde{y},s)Z_0(\tilde{y},0)d\tilde{y}\right)ds,$$

which makes g_1, h_1 satisfy (4.4). Since $Z_0(y)$ decays exponentially, by (B.1), we have

$$|c(\tau)| + |c'(\tau)| \lesssim \tau^\sigma \left(\|g\|_{\sigma,2+a,\ell(\tau),\mathbb{R}^n_+,\tau_0,\tau_1} + \|h\|_{\sigma,1+a,\ell(\tau),\mathbb{R}^{n-1},\tau_0,\tau_1}\right).$$

Combining $\tilde{Z}_0(y) \in C^2(\mathbb{R}^n_+) \cap C^{1,\varsigma}(\overline{\mathbb{R}^n_+})$, $\tilde{Z}_0(y) = 0$ for $|y| \geq C_0$, for $C_1 = C_\ell \max\{C_0, 1\}$, we have

$$\begin{aligned} &\|g_1\|_{\sigma,2+a,C_1\ell(\tau),\mathbb{R}^n_+,\tau_0,\tau_1} + \|h_1\|_{\sigma,1+a,C_1\ell(\tau),\mathbb{R}^{n-1},\tau_0,\tau_1} \lesssim \|g\|_{\sigma,2+a,\ell(\tau),\mathbb{R}^n_+,\tau_0,\tau_1} + \|h\|_{\sigma,1+a,\ell(\tau),\mathbb{R}^{n-1},\tau_0,\tau_1}, \\ &\|h_1\|_{\sigma,1+a,C_1\ell(\tau),\varsigma,\mathbb{R}^{n-1},\tau_0,\tau_1} \lesssim \|g\|_{\sigma,2+a,\ell(\tau),\mathbb{R}^n_+,\tau_0,\tau_1} + \|h\|_{\sigma,1+a,\ell(\tau),\varsigma,\mathbb{R}^{n-1},\tau_0,\tau_1}. \end{aligned}$$

Thus, we can use Proposition 4.2 to find ϕ_1 satisfying (4.5), (4.6). Set $C_\phi = c(\tau_0)$. Combining (1.21), we complete the proof. □

Proof of Proposition 4.2. By the contraction mapping theorem, it is easy to get the existence and uniqueness for (4.2) in $L^\infty(\mathbb{R}^n \times (\tau_0, \tilde{\tau}))$ for all $\tilde{\tau} \in (\tau_0, \tau_1] \cap (\tau_0, \infty)$ and (4.1) holds in the weak sense. By the parabolic regularity theory, $\phi \in C(\overline{\mathbb{R}^n_+} \times [\tau_0, \tilde{\tau}))$.

We will use the comparison theorem for parabolic equations with the oblique derivative. See [6, p.122, 13.5 Theorem] for instance. Set $\tilde{\phi}(y,\tau) = e^{-\kappa\frac{|\tilde{y}|^2 + (y_n+1)^2}{\tau}}$. Using the supports of g, h, we have

$$\begin{aligned} (\partial_\tau - \Delta)\tilde{\phi} - g &= \tau^{-2}\left\{2\kappa n\tau + \kappa(1-4\kappa)\left[|\tilde{y}|^2 + (y_n+1)^2\right]\right\}\tilde{\phi} \quad \text{ in } |y| > \ell(\tau), \\ \left(-\partial_{y_n} - \frac{n}{n-2}U^{\frac{2}{n-2}}\right)\tilde{\phi} - h &= \left(2\kappa\tau^{-1} - \frac{n}{n-2}U^{\frac{2}{n-2}}\right)\tilde{\phi} \quad \text{ on } |\tilde{y}| > \ell(\tau), y_n = 0. \end{aligned}$$

$\kappa \in (0, \frac{1}{4}]$ is a necessary condition to make $C_1\tilde{\phi}$ be a barrier function. We take $\kappa = \frac{1}{4}$. For any $\tilde{\tau} \in (\tau_0, \tau_1] \cap (\tau_0, \infty)$, $C_1\tilde{\phi}$ is a barrier function in $\left\{(y,\tau) \mid |y| \geq C_2\tau^{\max\{\frac{1}{2},p\}},\ \tau \in (\tau_0, \tilde{\tau})\right\}$ with sufficiently large constants C_1 depending on $\tilde{\tau}$ and $C_2 > C_\ell$. By $\|g\|_{\sigma,2+a,\ell(\tau),\mathbb{R}^n_+,\tau_0,\tau_1} < \infty$, $\|h\|_{\sigma,1+a,\ell(\tau),\varsigma,\mathbb{R}^{n-1},\tau_0,\tau_1} < \infty$, and (2.29) in Lemma 2.15, $\nabla\phi(\cdot,\tau)$ has exponential decay in space.

For $j=0,1,\ldots,n$, we test (4.1) with $Z_j(y)\eta(M^{-1}y)$ and integrate by parts,

$$\begin{aligned}
&\int_{\mathbb{R}^n_+}\phi(y,\tau)Z_j(y)\eta(M^{-1}y)dy-\int_{\mathbb{R}^n_+}\phi(y,\tau_0)Z_j(y)\eta(M^{-1}y)dy\\
=&\int_{\tau_0}^{\tau}\int_{\mathbb{R}^{n-1}}\left(-\partial_{y_n}\phi Z_j\eta(M^{-1}\cdot)\right)((\tilde{y},0),s)d\tilde{y}ds+\int_{\tau_0}^{\tau}\int_{\mathbb{R}^{n-1}}\left[\phi\partial_{y_n}\left(Z_j\eta(M^{-1}\cdot)\right)\right]((\tilde{y},0),s)d\tilde{y}ds\\
&+\int_{\tau_0}^{\tau}\int_{\mathbb{R}^n_+}\phi\Delta\left(Z_j(y)\eta(M^{-1}y)\right)dyds+\int_{\tau_0}^{\tau}\int_{\mathbb{R}^n_+}gZ_j(y)\eta(M^{-1}y)dyds\\
=&\int_{\tau_0}^{\tau}\int_{\mathbb{R}^{n-1}}\left[\left(\frac{n}{n-2}U^{\frac{2}{n-2}}\phi+h\right)Z_j\eta(M^{-1}\cdot)\right]((\tilde{y},0),s)d\tilde{y}ds\\
&+\int_{\tau_0}^{\tau}\int_{\mathbb{R}^{n-1}}\left[\phi Z_j\partial_{y_n}\left(\eta(M^{-1}\cdot)\right)+\phi\eta(M^{-1}\cdot)\partial_{y_n}Z_j\right]((\tilde{y},0),s)d\tilde{y}ds\\
&+\int_{\tau_0}^{\tau}\int_{\mathbb{R}^n_+}\phi\left[\eta(M^{-1}y)\Delta Z_j(y)+2\nabla Z_j(y)\cdot\nabla\left(\eta(M^{-1}y)\right)+Z_j(y)\Delta\left(\eta(M^{-1}y)\right)\right]dyds\\
&+\int_{\tau_0}^{\tau}\int_{\mathbb{R}^n_+}gZ_j(y)\eta(M^{-1}y)dyds.
\end{aligned}$$

Taking $M\to\infty$ gives

$$\begin{aligned}
&\int_{\mathbb{R}^n_+}\phi(y,\tau)Z_j(y)dy-\int_{\mathbb{R}^n_+}\phi(y,\tau_0)Z_j(y)dy\\
=&\int_{\tau_0}^{\tau}\int_{\mathbb{R}^{n-1}}\left(\frac{n}{n-2}U^{\frac{2}{n-2}}\phi Z_j+hZ_j+\phi\partial_{y_n}Z_j\right)((\tilde{y},0),s)d\tilde{y}ds+\int_{\tau_0}^{\tau}\int_{\mathbb{R}^n_+}\phi\Delta Z_j(y)dyds+\int_{\tau_0}^{\tau}\int_{\mathbb{R}^n_+}gZ_j(y)dyds.
\end{aligned}$$

Using (2.2), (2.3), (4.4), and $\int_{\mathbb{R}^n_+}\phi(y,\tau_0)Z_j(y)dy=0$, we have

$$\int_{\mathbb{R}^n_+}\phi(y,\tau)Z_j(y)dy=-\delta_{j0}\lambda_0\int_{\tau_0}^{\tau}\int_{\mathbb{R}^n_+}\phi Z_0dy,$$

which implies (4.5). Define

$$\|f\|^{\#}_{\sigma,a,p,\tau_0,\tau_1}:=\inf\left\{C\mid|f(y,\tau)|\le Cw(y,\tau)\ \text{ for }\ y\in\mathbb{R}^n_+,\tau_0<\tau<\tau_1\right\},$$

where

$$w(y,\tau):=\tau^{\sigma}\langle y\rangle^{-a}\mathbf{1}_{|y|\le\tau^p}+\tau^{\sigma-pa}e^{-\iota\frac{|y|^2}{\tau}}\mathbf{1}_{|y|>\tau^p},\quad \iota\in(0,\frac{1}{4}].$$

The weight $w(y,\tau)$ is partially determined by the forthcoming estimate (4.11), (4.13). By the barrier function $C_1\tilde{\phi}$, for $1\le\tau_0<\tau_1<\infty$, we have $e^{-\frac{|\tilde{y}|^2+(y_n+1)^2}{4\tau}}\le C_3w(y,\tau)$ for $y\in\mathbb{R}^n_+$, $\tau_0<\tau<\tau_1$ with a constant $C_3>0$ depending on τ_1, which implies

$$\|\phi\|^{\#}_{\sigma,a,p,\tau_0,\tau_1}<\infty.$$

Claim: For all $1\le\tau_0<\tau_1<\infty$, there exists a constant $C_5>0$ independent of τ_0,τ_1,g,h such that

$$\|\phi\|^{\#}_{\sigma,a,p,\tau_0,\tau_1}\le C_5\left(\|g\|_{\sigma,2+a,\ell(\tau),\mathbb{R}^n_+,\tau_0,\tau_1}+\|h\|_{\sigma,1+a,\ell(\tau),\mathbb{R}^{n-1},\tau_0,\tau_1}\right).\tag{4.7}$$

Indeed, for $p\in[0,\frac{1}{2}]$, one taking $\tau_1\to\infty$, (4.7) deduces the estimate of ϕ in (4.6) for the case $\tau_1=\infty$. Combining $\|g\|_{\sigma,2+a,\ell(\tau),\mathbb{R}^n_+,\tau_0,\tau_1}<\infty$, $\|h\|_{\sigma,1+a,\ell(\tau),\varsigma,\mathbb{R}^{n-1},\tau_0,\tau_1}<\infty$, and (2.29) in Lemma 2.15, we get the desired bound of $\nabla\phi$ in (4.6).

We prove (4.7) by contradiction argument. Suppose that there exists a sequence $(\phi_k,g_k,h_k,\tau_{0k},\tau_{1k})_k$ satisfying

$$\begin{cases}\partial_\tau\phi_k=\Delta\phi_k+g_k\ \text{ in }\ \mathbb{R}^n_+\times(\tau_{0k},\tau_{1k}),\quad -\partial_{y_n}\phi_k=\frac{n}{n-2}U^{\frac{2}{n-2}}\phi_k+h_k\ \text{ on }\ \partial\mathbb{R}^n_+\times(\tau_{0k},\tau_{1k}),\\ \phi_k(y,\tau_{0k})=0\ \text{ in }\ \mathbb{R}^n_+,\quad \int_{\mathbb{R}^n_+}\phi_k(y,\tau)Z_j(y)dy=0\ \text{ for all }\ \tau\in(\tau_{0k},\tau_{1k}),\quad j=0,1,\ldots,n,\end{cases}\tag{4.8}$$

and

$$\|\phi_k\|^{\#}_{\sigma,a,p,\tau_{0k},\tau_{1k}}=1,\quad \|g_k\|_{\sigma,2+a,\ell(\tau),\mathbb{R}^n_+,\tau_{0k},\tau_{1k}}=o_k(1),\quad \|h_k\|_{\sigma,1+a,\ell(\tau),\mathbb{R}^{n-1},\tau_{0k},\tau_{1k}}=o_k(1),\tag{4.9}$$

where $o_k(1)\to0$ as $k\to\infty$. Therein, $\tau_{1k}\to\infty$ otherwise $\|\phi_k\|^{\#}_{\sigma,a,p,\tau_{0k},\tau_{1k}}\to0$. Thus, there exists a sequence $(y_k,\tau_{2k})_k$, $y_k\in\overline{\mathbb{R}^n_+}$, $\tau_{0k}<\tau_{2k}<\tau_{1k}$ such that

$$\left(w\left(y_k,\tau_{2k}\right)\right)^{-1}|\phi_k(y_k,\tau_{2k})|\ge\delta_1>0\tag{4.10}$$

with a small constant $\delta_1>0$ independent of k, where $\tau_{2k}\to\infty$ otherwise (4.10) fails.

For $(y,\tau)\in\mathbb{R}^n\times[\tau_{0k},\tau_{1k})$, denote

$$\mathcal{T}_1\left[f_1\right]:=\int_{\tau_{0k}}^{\tau}\int_{\mathbb{R}^n_+}G_n(y,\tau,z,s)f_1(z,s)dzds,\quad \mathcal{T}_2\left[f_2\right]:=\int_{\tau_{0k}}^{\tau}\int_{\mathbb{R}^{n-1}}G_n(y,\tau,(\tilde z,0),s)f_2(\tilde z,s)d\tilde zds$$

for some admissible functions f_1,f_2. Then

$$\phi_k(y,\tau)=\mathcal{T}_1\left[g_k\right]+\mathcal{T}_2\left[\frac{n}{n-2}\left(U^{\frac{2}{n-2}}\phi_k\right)((\tilde y,0),\tau)+h_k\right].$$

Hereafter, all general constants, like C_i, $C(*,*,\dots)$, are independent of k.

By (4.9), Lemma B.5,

$$|\mathcal{T}_1\left[g_k\right]|\le C_6o_k(1)\mathcal{T}_1\left[\tau^\sigma\langle y\rangle^{-2-a}\mathbf{1}_{|y|\le\ell(\tau)}\right]\le C_6o_k(1)\tau^\sigma\left(\langle y\rangle^{-a}\mathbf{1}_{|y|\le\tau^p}+\tau^{-pa}e^{-\frac{|y|^2}{4\tau}}\mathbf{1}_{|y|>\tau^p}\right)\tag{4.11}$$

under the assumption

$$p\ge0,\quad 0<a<n-2,\quad \begin{cases}p\le\frac12,\sigma-pa+\frac n2\ge0, & \text{if } \sigma+p(n-2-a)\ne-1\\ p<\frac12\left(\Leftrightarrow\sigma-pa+\frac n2>0\right), & \text{if } \sigma+p(n-2-a)=-1,\end{cases}\tag{4.12}$$

where C_6 varies from line to line.

By (4.9), Lemma B.3,

$$|\mathcal{T}_2\left[h_k\right]|\le C_6o_k(1)\mathcal{T}_2\left[\tau^\sigma\langle\tilde y\rangle^{-1-a}\mathbf{1}_{|\tilde y|\le\ell(\tau)}\right]\le C_6o_k(1)\tau^\sigma\left(\langle y\rangle^{-a}\mathbf{1}_{|y|\le\tau^p}+\tau^{-pa}e^{-\frac{|y|^2}{4\tau}}\mathbf{1}_{|y|>\tau^p}\right)\tag{4.13}$$

under the assumption (4.12).

By (4.9), Lemma B.3, Lemma B.4,

$$\begin{aligned}&\left|\mathcal{T}_2\left[\left(U^{\frac{2}{n-2}}\phi_k\right)((\tilde y,0),\tau)\right]\right|\le C_6\mathcal{T}_2\left[\tau^\sigma\langle\tilde y\rangle^{-a-2}\mathbf{1}_{|\tilde y|\le\tau^p}+\tau^{\sigma-pa}\langle\tilde y\rangle^{-2}e^{-\iota\frac{|\tilde y|^2}{\tau}}\mathbf{1}_{|\tilde y|>\tau^p}\right]\\
&\le C_6\mathcal{T}_2\left[\tau^\sigma\langle\tilde y\rangle^{-a-1-\epsilon}\mathbf{1}_{|\tilde y|\le\tau^p}+\tau^{\sigma-pa}|\tilde y|^{-1-\epsilon}\mathbf{1}_{\tau^p<|\tilde y|\le\tau^{\frac12}}+\tau^{\sigma-pa}|\tilde y|^{-1-\epsilon}e^{-\iota\frac{|\tilde y|^2}{\tau}}\mathbf{1}_{|\tilde y|>\tau^{\frac12}}\right]\\
&\le C_6\Bigg[\tau^\sigma\left(\langle y\rangle^{-a-\epsilon}\mathbf{1}_{|y|\le\tau^p}+\tau^{-p(a+\epsilon)}e^{-\frac{|y|^2}{4\tau}}\mathbf{1}_{|y|>\tau^p}\right)\\
&\quad+\tau^{\sigma-pa}\left(\tau^{-p\epsilon}\mathbf{1}_{|y|\le\tau^p}+|y|^{-\epsilon}\mathbf{1}_{\tau^p<|y|\le\tau^{\frac12}}+\tau^{-\frac\epsilon2}e^{-\frac{|y|^2}{4\tau}}\mathbf{1}_{|y|>\tau^{\frac12}}\right)+\tau^{\sigma-pa-\frac\epsilon2}e^{-\iota\frac{|y|^2}{\tau}}\Bigg]\end{aligned}$$

for any constant $\epsilon\in(0,\min\{n-2-a,1\})$, under the assumption

$$\begin{aligned}&0<a<n-2,\quad p\ge0,\quad \begin{cases}p\le\frac12,\sigma-p(a+\epsilon)+\frac n2\ge0, & \text{if } \sigma+p(n-2-a-\epsilon)\ne-1\\ p<\frac12, & \text{if } \sigma+p(n-2-a-\epsilon)=-1,\end{cases}\\
&\iota\in(0,\frac14),\quad \sigma-pa-\frac{\epsilon}{2}+2\iota n\ge0.\end{aligned}$$

Thus,

$$\left|\mathcal{T}_2\left[\left(U^{\frac{2}{n-2}}\phi_k\right)((\tilde y,0),\tau)\right]\right|\le C_6\tau^\sigma\left(\langle y\rangle^{-a-\epsilon}\mathbf{1}_{|y|\le\tau^p}+\tau^{-p(a+\epsilon)}e^{-\iota\frac{|y|^2}{\tau}}\mathbf{1}_{|y|>\tau^p}\right).\tag{4.14}$$

(4.11), (4.13), (4.14) imply

$$|\phi_k(y,\tau)|\le C_6w(y,\tau)\left(o_k(1)+\langle y\rangle^{-\epsilon}\mathbf{1}_{|y|\le\tau^p}+\tau^{-p\epsilon}\mathbf{1}_{|y|>\tau^p}\right).$$

Using $p>0$, $\tau_{2k}\to\infty$, and (4.10), we conclude that there exists a constant $C_7>0$ such that $|y_k|\le C_7$ for all k. By (4.10), for k large, there exists a positive constant C_8 such that

$$\tau_{2k}^{-\sigma}\left|\phi_k(y_k,\tau_{2k})\right|\ge C_8>0.\tag{4.15}$$

Set

$$\tilde\phi_k(y,t)=\tau_{2k}^{-\sigma}\phi_k(y,\tau_{2k}+t),\quad \tilde g_k(y,t)=\tau_{2k}^{-\sigma}g_k(y,\tau_{2k}+t),\quad \tilde h_k(\tilde y,t)=\tau_{2k}^{-\sigma}h_k(\tilde y,\tau_{2k}+t).$$

Then,

$$|\tilde\phi_k(y_k,0)|\ge C_8>0.\tag{4.16}$$

The following argument is in the same spirit as the proof of [11, Lemma 7.7].

Case 1: There exists a subsequence, (without loss of generality, we still use k as the serial number), such that $\tau_{2k} \geq 9\tau_{0k}$. By (4.8), we have

$$\begin{cases} \partial_t \tilde{\phi}_k = \Delta \tilde{\phi}_k + \tilde{g}_k \quad \text{ in } \mathbb{R}^n_+ \times (\tau_{0k} - \frac{\tau_{2k}}{2}, 0], \quad -\partial_{y_n} \tilde{\phi}_k = \frac{n}{n-2} U^{\frac{2}{n-2}} \tilde{\phi}_k + \tilde{h}_k \quad \text{ on } \partial\mathbb{R}^n_+ \times (\tau_{0k} - \frac{\tau_{2k}}{2}, 0], \\ \int_{\mathbb{R}^n_+} \tilde{\phi}_k(y,t) Z_j(y) dy = 0 \quad \text{ for all } t \in (\tau_{0k} - \frac{\tau_{2k}}{2}, 0], \quad j = 0, 1, \ldots, n. \end{cases} \tag{4.17}$$

And (4.9) implies

$$\begin{aligned} &|\tilde{\phi}_k(y,t)| \leq C(\sigma, a, p, \iota) \left(\langle y \rangle^{-a} \mathbf{1}_{|y| < \tau_{2k}^p} + \tau_{2k}^{-pa} e^{-\iota \frac{|y|^2}{\tau_{2k}}} \mathbf{1}_{|y| \geq \tau_{2k}^p} \right), \\ &|\tilde{g}_k(y,t)| \leq o_k(1) C(\sigma) \langle y \rangle^{-2-a} \mathbf{1}_{|y| \leq C_\ell \tau_{2k}^p}, \quad |\tilde{h}_k(\tilde{y}, t)| \leq o_k(1) C(\sigma) \langle \tilde{y} \rangle^{-1-a} \mathbf{1}_{|\tilde{y}| \leq C_\ell \tau_{2k}^p} \end{aligned} \tag{4.18}$$

holding in $t \in (\tau_{0k} - \frac{\tau_{2k}}{2}, 0]$, $y \in \mathbb{R}^n_+$, $\tilde{y} \in \mathbb{R}^{n-1}$. Since $\tau_{2k} \geq 9\tau_{0k}$, $\tau_{2k} \to \infty$ implies $\tau_{0k} - \frac{\tau_{2k}}{2} \to -\infty$.

Case 2: $\tau_{0k} < \tau_{2k} < 9\tau_{0k}$ holds for all k. Similarly, by (4.8), we have

$$\begin{cases} \partial_t \tilde{\phi}_k = \Delta \tilde{\phi}_k + \tilde{g}_k \quad \text{ in } \mathbb{R}^n_+ \times (\tau_{0k} - \tau_{2k}, 0], \quad -\partial_{y_n} \tilde{\phi}_k = \frac{n}{n-2} U^{\frac{2}{n-2}} \tilde{\phi}_k + \tilde{h}_k \quad \text{ on } \partial\mathbb{R}^n_+ \times (\tau_{0k} - \tau_{2k}, 0], \\ \tilde{\phi}_k(y, \tau_{0k} - \tau_{2k}) = 0 \text{ in } \mathbb{R}^n_+, \quad \int_{\mathbb{R}^n_+} \tilde{\phi}_k(y,t) Z_j(y) dy = 0 \quad \text{ for all } t \in (\tau_{0k} - \tau_{2k}, 0], \quad j = 0, 1, \ldots, n \end{cases} \tag{4.19}$$

and (4.9) makes (4.18) hold for all $t \in (\tau_{0k} - \tau_{2k}, 0]$, $y \in \mathbb{R}^n_+$, $\tilde{y} \in \mathbb{R}^{n-1}$. (4.16) implies $\tau_{0k} - \tau_{2k} \to -\infty$.

We will handle the above two cases in a unified way. Denote $t_k := \begin{cases} \tau_{0k} - \frac{\tau_{2k}}{2}, & \text{for Case 1} \\ \tau_{0k} - \tau_{2k}, & \text{for Case 2} \end{cases}$. Since $t_k \to -\infty$, by (4.18) and the parabolic regularity theorem (see [37, p.2418]), up to a subsequence, we have

$$\tilde{\phi}_k \to \tilde{\phi} \text{ in } C^{\alpha, \frac{\alpha}{2}}_{\text{loc}} \left(\overline{\mathbb{R}^n_+} \times (-\infty, 0] \right) \text{ with a constant } \alpha \in (0,1). \tag{4.20}$$

Combining (4.16) with $|y_k| \leq C_7$, (4.18), we have

$$\tilde{\phi} \not\equiv 0, \text{ and } |\tilde{\phi}(y,t)| \leq C(\sigma, a, p, \iota) \langle y \rangle^{-a} \text{ for } (y,t) \in \mathbb{R}^n_+ \times (-\infty, 0]. \tag{4.21}$$

Given $t \in (-\infty, 0]$, for any $\epsilon_1 > 0$, taking a large constant R_1, then for all $j = 0, 1, \ldots, n$, when k is sufficiently large, we have

$$\begin{aligned} &\left| \int_{\mathbb{R}^n_+ \cap \{|y| \geq R_1\}} \tilde{\phi}_k(y,t) Z_j(y) dy \right| \leq C(\sigma, a, p, \iota, n) \int_{|y| \geq R_1} \left(|y|^{2-n-a} \mathbf{1}_{|y| < \tau_{2k}^p} + |y|^{2-n} \tau_{2k}^{-pa} e^{-\iota \frac{|y|^2}{\tau_{2k}}} \mathbf{1}_{|y| \geq \tau_{2k}^p} \right) dy \\ &\leq C(\sigma, a, p, \iota, n) \left(R_1^{2-a} + \tau_{2k}^{1-pa} \int_{\tau_{2k}^{2p-1}}^\infty e^{-\iota z} dz \right) < \epsilon_1, \end{aligned}$$

where in the last step, we require the assumption

$$a > 2, \quad pa > 1.$$

Similarly, for $a > 2$, we have $\left| \int_{\mathbb{R}^n_+ \cap \{|y| \geq R_1\}} \tilde{\phi}(y,t) Z_j(y) dy \right| < \epsilon_1$. By (4.20), we have

$$\lim_{k \to \infty} \int_{\mathbb{R}^n_+ \cap \{|y| < R_1\}} \tilde{\phi}_k(y,t) Z_j(y) dy = \int_{\mathbb{R}^n_+ \cap \{|y| < R_1\}} \tilde{\phi}(y,t) Z_j(y) dy.$$

Thus, the orthogonality conditions in (4.17), (4.19) yields

$$\int_{\mathbb{R}^n_+} \tilde{\phi}(y,t) Z_j(y) dy = 0 \quad \text{ for all } t \in (-\infty, 0], \quad j = 0, 1, \ldots, n.$$

By (4.17), (4.19), we have

$$\begin{aligned} \tilde{\phi}_k(y,t) = &\int_{\mathbb{R}^n_+} G_n(y,t,z,t_k) \tilde{\phi}_k(z,t_k) dz + \int_{t_k}^t \int_{\mathbb{R}^n_+} G_n(y,t,z,s) \tilde{g}_k(z,s) dz ds \\ &+ \int_{t_k}^t \int_{\mathbb{R}^{n-1}} G_n(y,t,(\tilde{z},0),s) \left[\frac{n}{n-2} \left(U^{\frac{2}{n-2}} \tilde{\phi}_k \right) ((\tilde{z},0),s) + \tilde{h}_k(\tilde{z},s) \right] d\tilde{z} ds. \end{aligned}$$

Taking $k \to \infty$, using (4.18), $a > 0$, [59, Lemma A.3] for the first part in Case 1, $\int_0^\infty (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}} dt = |x|^{2-n} \pi^{-\frac{n}{2}} 2^{-1} (n-2)^{-1} \Gamma(\frac{n}{2})$ with $n > 2$ for the second and third parts, we have

$$\tilde{\phi}(y,t) = \int_{-\infty}^t \int_{\mathbb{R}^{n-1}} G_n(y,t,(\tilde{z},0),s) \frac{n}{n-2} \left(U^{\frac{2}{n-2}} \tilde{\phi} \right) ((\tilde{z},0),s) d\tilde{z} ds, \tag{4.22}$$

which satisfies

$$\begin{cases} \partial_t \tilde\phi = \Delta \tilde\phi \quad \text{ in } \mathbb{R}^n_+ \times (-\infty, 0], \quad -\partial_{y_n} \tilde\phi = \frac{n}{n-2} U^{\frac{2}{n-2}} \tilde\phi \quad \text{ on } \partial\mathbb{R}^n_+ \times (-\infty, 0], \\ \int_{\mathbb{R}^n_+} \tilde\phi(y,t) Z_j(y) dy = 0 \quad \text{ for all } t \in (-\infty, 0], \quad j = 0, 1, \dots, n. \end{cases} \tag{4.23}$$

Using (4.22), there exists a constant C_9 varying from line to line , such that

$$|\tilde\phi| \le C_9 \langle y \rangle^{2-n}, \tag{4.24}$$

and $\tilde\phi$ is smooth by the parabolic regularity theory (See [34, 37]). By the scaling argument, we have

$$\langle y \rangle^{-1} |D\tilde\phi| + |\tilde\phi_t| + |D^2 \tilde\phi| \le C_9 \langle y \rangle^{-n}. \tag{4.25}$$

Differentiating (4.23), we get

$$\begin{cases} \partial_t \tilde\phi_t = \Delta \tilde\phi_t \quad \text{ in } \mathbb{R}^n_+ \times (-\infty, 0], \quad -\partial_{y_n} \tilde\phi_t = \frac{n}{n-2} U^{\frac{2}{n-2}} \tilde\phi_t \quad \text{ on } \partial\mathbb{R}^n_+ \times (-\infty, 0], \\ \int_{\mathbb{R}^n_+} \tilde\phi_t(y,t) Z_j(y) dy = 0 \quad \text{ for all } t \in (-\infty, 0], \quad j = 0, 1, \dots, n \end{cases} \tag{4.26}$$

and then the scaling argument gives

$$\langle y \rangle^{-1} |D\tilde\phi_t| + |\tilde\phi_{tt}| + |D^2 \tilde\phi_t| \le C_9 \langle y \rangle^{-n-2}.$$

Moreover, multiplying (4.26) by $\tilde\phi_t$ and integrating by parts, we get

$$\frac{1}{2} \partial_t \int_{\mathbb{R}^n_+} |\tilde\phi_t|^2 dy + B[\tilde\phi_t, \tilde\phi_t] = 0,$$

where $B[u,v] := \int_{\mathbb{R}^n_+} \nabla u \cdot \nabla v dy - \frac{n}{n-2} \int_{\mathbb{R}^{n-1}} \left(U^{\frac{2}{n-2}} uv \right) (\tilde y, 0) d\tilde y$.

Then $B[\tilde\phi_t, \tilde\phi_t] \ge 0$ by $\int_{\mathbb{R}^n_+} \tilde\phi_t(y,t) Z_0(y) dy = 0$ since Z_0 is the only eigenfucntion of (2.3) with negative eigenvalue. Thus, $\partial_t \int_{\mathbb{R}^n_+} |\tilde\phi_t|^2 dy \le 0$.

Multiplying (4.23) by $\tilde\phi_t$ and integrating by parts, we have

$$\int_{\mathbb{R}^n_+} |\tilde\phi_t|^2 dy = -\frac{1}{2} \partial_t B[\tilde\phi, \tilde\phi].$$

By (4.24), (4.25), for $n > 2$, $|B[\tilde\phi, \tilde\phi](t)|$ is uniformly bounded for $t \in (-\infty, 0]$. Thus, we have

$$\int_{-\infty}^{0} \int_{\mathbb{R}^n_+} |\tilde\phi_t|^2 dy dt < \infty.$$

Hence $\tilde\phi_t = 0$, that is, $\tilde\phi = \tilde\phi(y)$ is independent of t.

By (4.23) and Lemma 4.1, we have $\tilde\phi \equiv 0$, which contradicts $\tilde\phi \not\equiv 0$ in (4.21). As a result, (4.7) holds. Due to the arbitrarily small choice of ϵ, we conclude the parameters restriction (4.3). □

5. Leading term of μ_i and topology of the inner and outer problems

Hereafter, we focus on the case $n = 5$ unless otherwise specified.

In order to apply Proposition 1.2 to solve the inner problems (3.4), suitable $\boldsymbol{\mu}, \boldsymbol{\xi}$ will be taken to satisfy the orthogonality conditions (1.20).

Recall (1.14). $\Theta_{l_i}(0,t) = -(T-t)^{l_i}$ is a leading role in the orthogonal equations. Roughly speaking, ψ is a smaller term compared with $\Theta_{l_i}(0,t)$ in the inner problem. As the leading term of μ_i, $\mu_{i,0}$ is determined by

$$\begin{aligned} & \dot\mu_{i,0} \mu_{i,0} \int_{\mathbb{R}^n_+} Z_n^2(y) \eta\Big(\frac{y}{4R}\Big) dy + \int_{\mathbb{R}^{n-1}} \frac{n}{n-2} \mu_{i,0}^{\frac{n}{2}-1} U^{\frac{2}{n-2}}(\tilde y, 0) \eta\Big(\frac{\tilde y}{4R}, 0\Big) \Theta_{l_i}(0,t) Z_n(\tilde y, 0) d\tilde y = 0 \\ \Leftrightarrow\ & \dot\mu_{i,0} \mu_{i,0}^{2-\frac{n}{2}} = -(T-t)^{l_i} A_R \end{aligned} \tag{5.1}$$

with

$$\begin{aligned} A_R &:= -\Big(\int_{\mathbb{R}^n_+} Z_n^2(y) \eta\Big(\frac{y}{4R}\Big) dy \Big)^{-1} \int_{\mathbb{R}^{n-1}} \frac{n}{n-2} U^{\frac{2}{n-2}}(\tilde y, 0) \eta\Big(\frac{\tilde y}{4R}, 0\Big) Z_n(\tilde y, 0) d\tilde y \\ &= \frac{n-2}{2} \Big(\int_{\mathbb{R}^n_+} Z_n^2(y) dy \Big)^{-1} \int_{\mathbb{R}^{n-1}} U^{\frac{n}{n-2}}(\tilde y, 0) d\tilde y + O(R^{-1}), \end{aligned} \tag{5.2}$$

where we used (2.1) and $n=5$ for the last step of A_R. We take

$$\mu_{i,0}(t)=\left[A_R\frac{6-n}{2}\left(l_i+1\right)^{-1}\right]^{\frac{2}{6-n}}(T-t)^{\frac{2}{6-n}(l_i+1)}. \tag{5.3}$$

Let $\mu_{i,1}(t):=\mu_i(t)-\mu_{i,0}(t)$ be the minor term of $\mu_i(t)$. Denote

$$\boldsymbol{\mu}_{,0}=(\mu_{1,0},\mu_{2,0},\ldots,\mu_{\mathfrak{o},0}),\quad \boldsymbol{\mu}_{,1}=(\mu_{1,1},\mu_{2,1},\ldots,\mu_{\mathfrak{o},1}).$$

For $T\ll 1$, one plugging $n=5$, then there exists a constant $C_{\mu_{i,0}}>9$ sufficiently large such that

$$9C_{\mu_{i,0}}^{-1}(T-t)^{2l_i+2}\le\mu_{i,0}(t)\le 9^{-1}C_{\mu_{i,0}}(T-t)^{2l_i+2},\quad |\dot\mu_{i,0}(t)|\le 9^{-1}C_{\mu_{i,0}}(T-t)^{2l_i+1}. \tag{5.4}$$

We make the ansatz that

$$\begin{aligned}&C_{\mu_{i,0}}^{-1}(T-t)^{2l_i+2}\le\mu_i\le C_{\mu_{i,0}}(T-t)^{2l_i+2},\quad |\dot\mu_i|\le C_{\mu_{i,0}}(T-t)^{2l_i+1},\\ &\left|\xi^{[i]}-\tilde q^{[i]}\right|\le C_{\mu_{i,0}}(T-t)^{2l_i+2},\quad |\dot\xi^{[i]}|\le C_{\mu_{i,0}}(T-t)^{2l_i+1}.\end{aligned} \tag{5.5}$$

Under the ansatz (5.5), $\left|\dot\mu_i\mu_i Z_5(y^{[i]})\right|\eta(\frac{y^{[i]}}{4R})\lesssim(T-t)^{4l_i+3}\langle y^{[i]}\rangle^{-3}\eta(\frac{y^{[i]}}{4R})\lesssim R^{\frac32}(T-t)^{4l_i+3}\langle y^{[i]}\rangle^{-\frac92}\eta(\frac{y^{[i]}}{4R})$. Using Proposition 1.2 with $n=5$ formally, for $i=1,2,\ldots,\mathfrak{o}$, we define $T_{\sigma_0}:=T-\sigma_0$ with $\sigma_0\in[0,T)$, and the spaces

$$B_{\mathrm{in},\sigma_0}:=\left\{(f_1,f_2,\ldots,f_{\mathfrak{o}})\mid f_i,\nabla f_i\in L^\infty\left(B_5^+(0,2R)\times(0,T_{\sigma_0})\right),i=1,2,\ldots,\mathfrak{o},\quad\sup_{i=1,2,\ldots,\mathfrak{o}}\|f_i\|_{\mathrm{in},l_i,\sigma_0}\le 1\right\} \tag{5.6}$$

with the norm

$$\|f\|_{\mathrm{in},l_i,\sigma_0}:=\sup_{(y,t)\in B_5^+(0,2R)\times(0,T_{\sigma_0})}\left(R^2(T-t)^{4l_i+3}\langle y\rangle^{-\frac52}\right)^{-1}\left(|f(y,t)|+\langle y\rangle|\nabla f(y,t)|\right),$$

where we used R^2 instead of $R^{\frac32}$ in the norm $\|\cdot\|_{\mathrm{in},l_i,\sigma_0}$ for the final fixed point argument.

In order to get the vanishing property around the blow-up points for the outer problem, we adopt the ideas in [25, p.318], [60, p.13] to define the following space for the outer problem

$$\mathcal{X}_{\delta_0,\sigma_0}:=\left\{f\in L^\infty(\overline{B_5^+(0,\sigma_0^{-1})}\times(0,T_{\sigma_0}))\mid\|f\|_{\mathcal{X},\sigma_0}\le\delta_0\right\} \tag{5.7}$$

with the norm

$$\|f\|_{\mathcal{X},\sigma_0}:=\sup_{(x,t)\in B_5^+(0,\sigma_0^{-1})\times(0,T_{\sigma_0})}\left[\langle x\rangle^{-2}\mathbf{1}_{\cap_{i=1}^{\mathfrak{o}}\left\{|z^{[i]}|>e^{\frac{l_is}{2l_i+2}}\right\}}+\sum_{i=1}^{\mathfrak{o}}(T-t)^{l_i}\langle z^{[i]}\rangle^{2l_i+2}\mathbf{1}_{|z^{[i]}|\le e^{\frac{l_is}{2l_i+2}}}\right]^{-1}|f(x,t)|$$

with $z^{[i]}=\frac{x-q^{[i]}}{\sqrt{T-t}}$ and $s=-\ln(T-t)$, where we admit $B_5^+(0,\sigma_0^{-1})\big|_{\sigma_0=0}=\mathbb{R}^5_+$ and the constant $\delta_0\in(0,1)$ will be determined later. Note that $|z^{[i]}|\le e^{\frac{l_is}{2l_i+2}}\Leftrightarrow|x-q^{[i]}|\le(T-t)^{\frac{1}{2l_i+2}}$, which implies $\{|z^{[i]}|\le e^{\frac{l_is}{2l_i+2}}\}\cap\{|z^{[j]}|\le e^{\frac{l_js}{2l_j+2}}\}=\emptyset$ for $i\ne j$ with $T\ll1$. Given $q\in\mathbb{R}^5_+$, $\langle x\rangle\sim_q\langle x-q\rangle$.

6. PRIORI ESTIMATE OF THE OUTER PROBLEM

Set $N_{\max}=\max_{i=1,2,\ldots,\mathfrak{o}}N(\lceil 5l_i/3\rceil+1)$ with $N(\cdot)$ defined in (2.12). We set $\tilde e_j(z)$ by Corollary 2.5 satisfying the properties

$$\begin{aligned}&\tilde e_j(z)=\sum_{\iota=0}^{N_{\max}}a_{j\iota}e_\iota(z)\eta(\frac{z}{C_{\tilde e}})\ \text{ with a constant }\ C_{\tilde e}>0,\quad j=0,1,\ldots,N_{\max},\\ &\partial_{z_n}\tilde e_i=0\quad\text{on }\ \partial\mathbb{R}^n_+\quad\text{and}\quad(\tilde e_i,e_j)_{L^2_\rho(\mathbb{R}^n_+)}=\delta_{ij}\quad\text{for }\ i,j=0,1,\ldots,N_{\max}\end{aligned} \tag{6.1}$$

with a constant matrix $(a_{il})_{(N_{\max}+1)\times(N_{\max}+1)}$. Denote

$$\tilde{\boldsymbol{e}}_i(z):=\left(\tilde e_0(z),\tilde e_1(z),\ldots,\tilde e_{N(\lceil 5l_i/3\rceil+1)}(z)\right). \tag{6.2}$$

We emphasize the following remark before further analysis.

Remark 6.0.1. *The property of $\boldsymbol{\mu}(\boldsymbol{\xi})$ being the original function of $\dot{\boldsymbol{\mu}}(\dot{\boldsymbol{\xi}})$ plays no role in this section, for which we can regard $\boldsymbol{\mu},\dot{\boldsymbol{\mu}},\boldsymbol{\xi},\dot{\boldsymbol{\xi}}$ as four independent functions in this section.*

In order to find a solution for the outer problem (3.6) with fast time decay near the blow-up points, we consider the following equation with a suitable initial value

$$\begin{cases}\partial_t\psi=\Delta_x\psi+\mathcal{G}_1[\boldsymbol{\phi},\boldsymbol{\mu},\boldsymbol{\xi}]\ \text{ in }\ \mathbb{R}^5_+\times(0,T), \quad -\partial_{x_5}\psi=\mathcal{G}_2[\psi,\boldsymbol{\phi},\boldsymbol{\mu},\boldsymbol{\xi}]\ \text{ on }\ \partial\mathbb{R}^5_+\times(0,T),\\ \psi(x,0)=\sum_{i=1}^{\mathfrak{o}}\boldsymbol{b}_i\cdot\tilde{\boldsymbol{e}}_i\big(T^{-\frac12}(x-q^{[i]})\big)+\varphi_0(x)\ \text{ in }\ \mathbb{R}^5_+,\end{cases} \tag{6.3}$$

where $\varphi_0(x)\in C^\infty_{\rm c}(\overline{\mathbb{R}^5_+})$ and $\boldsymbol{b}_i=(b_{i,0},b_{i,1},\dots,b_{i,N(\lceil 5l_i/3\rceil+1)})\in\mathbb{R}^{N(\lceil 5l_i/3\rceil+1)}$ will be determined later. By (2.17), (6.3) is rewritten as

$$\psi(x,t)=\mathcal{T}^{\rm out}[\psi,\boldsymbol{\phi},\boldsymbol{\mu},\boldsymbol{\xi}]:=\int_0^t\int_{\mathbb{R}^5_+}G_5(x,t,z,s)\mathcal{G}_1[\boldsymbol{\phi},\boldsymbol{\mu},\boldsymbol{\xi}](z,s)dzds$$

$$+\int_0^t\int_{\mathbb{R}^4}G_5(x,t,(\tilde z,0),s)\mathcal{G}_2[\psi,\boldsymbol{\phi},\boldsymbol{\mu},\boldsymbol{\xi}]((\tilde z,0),s)d\tilde z ds+\int_{\mathbb{R}^5_+}G_5(x,t,z,0)\Big[\sum_{i=1}^{\mathfrak{o}}\boldsymbol{b}_i\cdot\tilde{\boldsymbol{e}}_i\big(T^{-\frac12}(x-q^{[i]})\big)+\varphi_0(z)\Big]dz. \tag{6.4}$$

Adopting the idea in [60, p.14] about the distribution of the right-hand side, we decompose $\psi=\sum_{i=1}^{\mathfrak{o}+1}\psi_i$, where ψ_i satisfy the following equations respectively. For $i=1,2,\dots,\mathfrak{o}$,

$$\partial_t\psi_i=\Delta_x\psi_i+\mathcal{G}_{1,i}\ \text{ in }\ \mathbb{R}^5_+\times(0,T),\quad -\partial_{x_5}\psi_i=\mathcal{G}_{2,i}\ \text{ on }\ \partial\mathbb{R}^5_+\times(0,T),\quad \psi_i(x,0)=\boldsymbol{b}_i\cdot\tilde{\boldsymbol{e}}_i\big(T^{-\frac12}(x-q^{[i]})\big)\ \text{ in }\ \mathbb{R}^5_+, \tag{6.5}$$

and for $i=\mathfrak{o}+1$,

$$\partial_t\psi_{\mathfrak{o}+1}=\Delta_x\psi_{\mathfrak{o}+1}\ \text{ in }\ \mathbb{R}^5_+\times(0,T),\quad -\partial_{x_5}\psi_{\mathfrak{o}+1}=\mathcal{G}_{2,\mathfrak{o}+1}\ \text{ on }\ \partial\mathbb{R}^5_+\times(0,T),\quad \psi_{\mathfrak{o}+1}(x,0)=\varphi_0(x)\ \text{ in }\ \mathbb{R}^5_+, \tag{6.6}$$

where for $i=1,2,\dots,\mathfrak{o}$,

$$\begin{aligned}&\mathcal{G}_{1,i}=\mathcal{G}_{1,i}[\phi_i,\mu_i,\xi^{[i]}]:=\Lambda_{1,i}[\phi_i,\mu_i,\xi^{[i]}]+\Lambda_{2,i}[\phi_i,\mu_i,\xi^{[i]}]+\mathcal{E}_1\left[U_{\mu_i,\xi^{[i]}}(x)\right]\left(\eta_{2\delta}(x^{[i]})-\eta_R(y^{[i]})\right)+\mathcal{E}^{\rm cut}_{U,i}+\mathcal{E}^{\rm cut}_{\Theta,i},\\ &\mathcal{G}_{2,i}=\mathcal{G}_{2,i}\left[\psi,\boldsymbol{\phi},\boldsymbol{\mu},\boldsymbol{\xi}\right]:=\Big\{\eta_{4\delta}(x^{[i]})\mathcal{N}\left[\psi,\boldsymbol{\phi},\boldsymbol{\mu},\boldsymbol{\xi}\right]+\left(U_{\mu_i,\xi^{[i]}}(x)\right)^{\frac53}\left(\eta_{2\delta}^{\frac53}(x^{[i]})-\eta_{2\delta}(x^{[i]})\right)\\ &\quad+\frac53\left(U_{\mu_i,\xi^{[i]}}(x)\right)^{\frac23}\Big[\left(\eta_\delta(x^{[i]})-\eta_R(y^{[i]})\right)\Theta_{l_i}(x^{[i]},t)+\left(\eta_{2\delta}^{\frac23}(x^{[i]})-\eta_R(y^{[i]})\right)\psi(x,t)\Big]\Big\}\Big|_{x_5=0},\\ &\mathcal{G}_{2,\mathfrak{o}+1}=\mathcal{G}_{2,\mathfrak{o}+1}[\psi]:=\mathcal{G}_2-\sum_{i=1}^{\mathfrak{o}}\mathcal{G}_{2,i}=\Big\{\Big(1-\sum_{i=1}^{\mathfrak{o}}\eta_{4\delta}(x^{[i]})\Big)\mathcal{N}\left[\psi,\boldsymbol{\phi},\boldsymbol{\mu},\boldsymbol{\xi}\right]\Big\}\Big|_{x_5=0}\\ &\quad=\Big\{\Big(1-\sum_{i=1}^{\mathfrak{o}}\eta_{4\delta}(x^{[i]})\Big)|\psi|^{\frac23}\psi\Big\}\Big|_{x_5=0}.\end{aligned} \tag{6.7}$$

Here we used (5.5) and $T\ll 1$ for $\mathcal{G}_{2,\mathfrak{o}+1}$ and $\left(\eta_{2\delta}^{\frac23}(x^{[i]})-1\right)\eta_R(y^{[i]})\mu_i^{-\frac32}\phi_i(y^{[i]},t)\equiv 0$ for $\mathcal{G}_{2,i}$.

For $i=1,2,\dots,\mathfrak{o}$, set

$$\psi_i(x,t)=\Psi_i\Big((T-t)^{-\frac12}\left(x-q^{[i]}\right),-\ln(T-t)\Big),\quad \text{ that is, }\quad \Psi_i(z^{[i]},s)=\psi_i\big(e^{-\frac s2}z^{[i]}+q^{[i]},T-e^{-s}\big). \tag{6.8}$$

By Lemma 2.6, (6.5) is rewritten as

$$\begin{cases}\partial_s\Psi_i=A_{z^{[i]}}\Psi_i+g_{1,i}(z^{[i]},s)\quad \text{ in }\ \mathbb{R}^5_+\times(s_0,\infty),\quad -\partial_{z_5^{[i]}}\Psi_i=g_{2,i}(\tilde z^{[i]},s)\quad \text{ on }\ \partial\mathbb{R}^5_+\times(s_0,\infty),\\ \Psi_i\big(z^{[i]},s_0\big)=\boldsymbol{b}_i\cdot\tilde{\boldsymbol{e}}_i(z^{[i]})\quad \text{ in }\ \mathbb{R}^5_+,\end{cases}$$

where

$$s_0:=-\ln T,\quad g_{1,i}(z^{[i]},s):=e^{-s}\mathcal{G}_{1,i}(e^{-\frac s2}z^{[i]}+q^{[i]},T-e^{-s}),\quad g_{2,i}(\tilde z^{[i]},s):=e^{-\frac s2}\mathcal{G}_{2,i}(e^{-\frac s2}\tilde z^{[i]}+\tilde q^{[i]},T-e^{-s}). \tag{6.9}$$

In order to find a solution $\Psi_i(z^{[i]},s)$ of the form

$$\Psi_i(z^{[i]},s)=\boldsymbol{d}_i(s)\cdot\tilde{\boldsymbol{e}}_i(z^{[i]})+\Phi_i(z^{[i]},s) \tag{6.10}$$

with $\boldsymbol{d}_i(s)=(d_{i,0}(s),d_{i,1}(s),\dots,d_{i,N(\lceil 5l_i/3\rceil+1)}(s))\in C^1([s_0,\infty))$ to be determined later, one using $\partial_{z_n}\tilde e_i=0$ on $\partial\mathbb{R}^n_+$ in (6.1), it suffices to solve

$$\partial_s\Phi_i=A_{z^{[i]}}\Phi_i+\tilde g_{1,i}(z^{[i]},s)\ \text{ in }\ \mathbb{R}^5_+\times(s_0,\infty),\quad -\partial_{z_5^{[i]}}\Phi_i=g_{2,i}(\tilde z^{[i]},s)\ \text{ on }\ \partial\mathbb{R}^5_+\times(s_0,\infty),\quad \Phi_i(\cdot,s_0)=0\ \text{ in }\ \mathbb{R}^5_+, \tag{6.11}$$

where we set

$$\boldsymbol{b}_i := \boldsymbol{d}_i(s_0), \quad \tilde{g}_{1,i}(z^{[i]},s) := g_{1,i}(z^{[i]},s) + \sum_{j=0}^{N(\lceil 5l_i/3\rceil+1)} \left(d_{i,j}(s)A_{z^{[i]}}\tilde{e}_j(z^{[i]}) - \dot{d}_{i,j}(s)\tilde{e}_j(z^{[i]})\right). \tag{6.12}$$

In order to recover the time decay of the right-hand sides of (6.11) for Φ_i, we impose the orthogonality conditions

$$\int_{\mathbb{R}^5_+} \tilde{g}_{1,i}(z^{[i]},s)e_j(z^{[i]})e^{-\frac{|z^{[i]}|^2}{4}}dz^{[i]} + \int_{\mathbb{R}^4} g_{2,i}(\tilde{z}^{[i]},s)e_j(\tilde{z}^{[i]},0)e^{-\frac{|\tilde{z}^{[i]}|^2}{4}}d\tilde{z}^{[i]} = 0, \quad j=0,1,\dots,N(\lceil 5l_i/3\rceil+1). \tag{6.13}$$

One using (2.6), (2.11), and (6.1), then (6.13) is equivalent to

$$\dot{d}_{i,j}(s) + \lambda_j d_{i,j}(s) = \int_{\mathbb{R}^5_+} g_{1,i}(z^{[i]},s)e_j(z^{[i]})e^{-\frac{|z^{[i]}|^2}{4}}dz^{[i]} + \int_{\mathbb{R}^4} g_{2,i}(\tilde{z}^{[i]},s)e_j(\tilde{z}^{[i]},0)e^{-\frac{|\tilde{z}^{[i]}|^2}{4}}d\tilde{z}^{[i]} \tag{6.14}$$

for $j=0,1,\dots,N(\lceil 5l_i/3\rceil+1)$.

6.1. Estimates of $\mathcal{G}_{2,\mathfrak{o}+1}$, $\mathcal{G}_{1,i}$, $\mathcal{G}_{2,i}$, $i=1,2,\dots,\mathfrak{o}$ in (6.7), and $d_{i,j}(s)$.

Lemma 6.1. *Suppose that $\boldsymbol{\mu},\dot{\boldsymbol{\mu}},\boldsymbol{\xi},\dot{\boldsymbol{\xi}}$ satisfy (5.5), $\psi\in\mathcal{X}_{\delta_0,0}$, and $T\ll 1$, then*

$$|\mathcal{G}_{2,\mathfrak{o}+1}| \le \delta_0^{\frac{5}{3}}\langle\tilde{x}\rangle^{-\frac{10}{3}}\mathbf{1}_{\cap_{i=1}^{\mathfrak{o}}\{|\tilde{x}^{[i]}|\ge 4\delta\}}. \tag{6.15}$$

Under the additional assumption that $\boldsymbol{\phi}\in B_{\mathrm{in},0}$, then there exists a constant C independent of T,δ_0 such that for $i=1,2,\dots,\mathfrak{o}$, $\mathcal{G}_{1,i}$, $\mathcal{G}_{2,i}$ have the pointwise upper bounds

$$\begin{aligned}|\mathcal{G}_{1,i}| \le C\Big\{& R^{-\frac14}\mu_i^{-2}e^{-l_i s}\left(\langle y^{[i]}\rangle^{-\frac94}\mathbf{1}_{|y^{[i]}|\le 2R} + \langle y^{[i]}\rangle^{-\frac{11}{4}}\mathbf{1}_{|y^{[i]}|>R}\right)\mathbf{1}_{|z^{[i]}|\le 1}\\ &+\left[e^{-(3l_i+\frac12)s}|z^{[i]}|^{-3} + \mathbf{1}_{\delta e^{\frac{s}{2}}\le|z^{[i]}|\le 2\delta e^{\frac{s}{2}}}\right]\mathbf{1}_{1<|z^{[i]}|\le 4\delta e^{\frac{s}{2}}}\Big\},\end{aligned} \tag{6.16}$$

$$\begin{aligned}|\mathcal{G}_{2,i}| \le C\Big[&\left(R^{-\frac14}\mu_i^{-1}e^{-l_i s}\langle\tilde{y}^{[i]}\rangle^{-\frac74} + e^{-\frac53 l_i s}\right)\mathbf{1}_{|\tilde{z}^{[i]}|\le 1} + e^{-\frac53 l_i s}|\tilde{z}^{[i]}|^{\frac{10}{3}l_i+\frac{10}{3}}\mathbf{1}_{1<|\tilde{z}^{[i]}|\le e^{\frac{l_i s}{2l_i+2}}}\\ &+\left(e^{-\frac53 l_i s}|\tilde{z}^{[i]}|^{\frac{10}{3}l_i} + \delta_0^{\frac53}\right)\mathbf{1}_{e^{\frac{l_i s}{2l_i+2}}<|\tilde{z}^{[i]}|\le 8\delta e^{\frac{s}{2}}}\Big].\end{aligned} \tag{6.17}$$

Proof. The estimate of $\mathcal{G}_{2,\mathfrak{o}+1}$ is straightforward. By (5.5), we have

$$\langle y^{[i]}\rangle \sim 2C^2_{\mu_i,0} + |y^{[i]}| \sim \langle\mu_i^{-1}x^{[i]}\rangle = \langle\mu_i^{-1}(T-t)^{\frac12}z^{[i]}\rangle \sim \langle e^{(2l_i+\frac32)s}z^{[i]}\rangle. \tag{6.18}$$

Step 1: Estimate of $\mathcal{G}_{1,i}$. Recall (3.2). One using $R=|\ln T|$, (5.5), $\|\phi_i\|_{\mathrm{in},l_i,0}\le 1$ (by $\boldsymbol{\phi}\in B_{\mathrm{in},0}$), then

$$\begin{aligned}|\Lambda_{1,i}+\Lambda_{2,i}| &\lesssim \mu_i^{-2}\mu_i^{-\frac32}\left(R^{-2}|\phi_i| + R^{-1}|\nabla_{y^{[i]}}\phi_i| + \mu_i|\dot{\mu}_i||\phi_i| + \mu_i R^{-1}|\dot{\xi}^{[i]}||\phi_i|\right)\mathbf{1}_{R\le|y^{[i]}|\le 2R}\\ &\quad+\mu_i^{-2}\left[|\dot{\mu}_i|\mu_i^{-\frac12}\left(|\phi_i| + \left|y^{[i]}\cdot\nabla_{y^{[i]}}\phi_i\right|\right) + \mu_i^{-\frac12}|\dot{\xi}^{[i]}||\nabla_{\tilde{y}^{[i]}}\phi_i|\right]\mathbf{1}_{|y^{[i]}|\le 2R}\\ &\lesssim \mu_i^{-2}(T-t)^{l_i}R^{-\frac52}\mathbf{1}_{R\le|y^{[i]}|\le 2R} + \mu_i^{-2}R^2(T-t)^{5l_i+3}\langle y^{[i]}\rangle^{-\frac52}\mathbf{1}_{|y^{[i]}|\le 2R} \lesssim R^{-\frac14}\mu_i^{-2}e^{-l_i s}\langle y^{[i]}\rangle^{-\frac94}\mathbf{1}_{|y^{[i]}|\le 2R}.\end{aligned}$$

Note that $|y^{[i]}|\le 2R$ implies $|z^{[i]}|\le 1$.

$$\begin{aligned}&\left|\mathcal{E}_1\left[U_{\mu_i,\xi^{[i]}}(x)\right]\left(\eta_{2\delta}(x^{[i]}) - \eta_R(y^{[i]})\right)\right| \lesssim \left(\mathbf{1}_{|x^{[i]}|\le 4\delta} - \mathbf{1}_{|y^{[i]}|\le R}\right)(T-t)^{-3l_i-4}\langle y^{[i]}\rangle^{-3}\\ &= \mathbf{1}_{|x^{[i]}|\le 4\delta,\ |y^{[i]}|>R}(T-t)^{-3l_i-4}\langle y^{[i]}\rangle^{-3} \lesssim \mu_i^{-2}e^{-l_i s}\langle y^{[i]}\rangle^{-3}\mathbf{1}_{|z^{[i]}|\le 1,\ |y^{[i]}|>R} + e^{-(3l_i+\frac12)s}|z^{[i]}|^{-3}\mathbf{1}_{1<|z^{[i]}|\le 4\delta e^{\frac{s}{2}}},\end{aligned}$$

where we used (6.18) for the last "$\lesssim$".

$$\left|\mathcal{E}^{\mathrm{cut}}_{U,i}\right| \lesssim \left(\mu_i^{-\frac32}\langle y^{[i]}\rangle^{-3} + \mu_i^{-\frac52}\langle y^{[i]}\rangle^{-4}\right)\mathbf{1}_{2\delta\le|x^{[i]}|\le 4\delta} \sim e^{-(3l_i+\frac32)s}|z^{[i]}|^{-3}\mathbf{1}_{2\delta e^{\frac{s}{2}}\le|z^{[i]}|\le 4\delta e^{\frac{s}{2}}},$$

where we used (6.18) for the last "$\sim$". Using (1.14), then

$$\left|\mathcal{E}^{\mathrm{cut}}_{\Theta,i}\right| \lesssim \mathbf{1}_{\delta\le|x^{[i]}|\le 2\delta} = \mathbf{1}_{\delta e^{\frac{s}{2}}\le|z^{[i]}|\le 2\delta e^{\frac{s}{2}}}.$$

Summarizing all the estimates reaches the upper bound of $\mathcal{G}_{1,i}$.

Step2: Estimate of $\mathcal{G}_{2,i}$. By (6.18),

$$\left|\left(U_{\mu_i,\xi^{[i]}}(x)\right)^{\frac53}\left(\eta_{2\delta}^{\frac53}(x^{[i]}) - \eta_{2\delta}(x^{[i]})\right)\right| \lesssim \mu_i^{-\frac52}\langle y^{[i]}\rangle^{-5}\mathbf{1}_{2\delta\le|x^{[i]}|\le 4\delta} \sim e^{-(5l_i+\frac52)s}|z^{[i]}|^{-5}\mathbf{1}_{2\delta e^{\frac{s}{2}}\le|z^{[i]}|\le 4\delta e^{\frac{s}{2}}},$$

$$\begin{aligned}&\left|\left(U_{\mu_i,\xi^{[i]}}(x)\right)^{\frac23}\left(\eta_\delta(x^{[i]}) - \eta_R(y^{[i]})\right)\Theta_{l_i}(x^{[i]},t)\right| \lesssim \mu_i^{-1}\langle y^{[i]}\rangle^{-2}e^{-l_i s}\langle z^{[i]}\rangle^{2l_i}\mathbf{1}_{|z^{[i]}|\le 2\delta e^{\frac{s}{2}},\ |y^{[i]}|>R}\\ &\sim \mu_i^{-1}\langle y^{[i]}\rangle^{-2}e^{-l_i s}\mathbf{1}_{|z^{[i]}|\le 1,\ |y^{[i]}|>R} + e^{-(3l_i+1)s}|z^{[i]}|^{2l_i-2}\mathbf{1}_{1<|z^{[i]}|\le 2\delta e^{\frac{s}{2}}}.\end{aligned}$$

For $\psi \in \mathcal{X}_{\delta_0,0}$,

$$\begin{aligned}
&\left|\left(U_{\mu_i,\xi^{[i]}}(x)\right)^{\frac{2}{3}}\left(\eta_{2\delta}^{\frac{2}{3}}(x^{[i]})-\eta_R(y^{[i]})\right)\psi\right| \\
&\lesssim \mu_i^{-1}\langle y^{[i]}\rangle^{-2}\mathbf{1}_{|x^{[i]}|\le 4\delta,\ |y^{[i]}|>R}\,\delta_0\left[(T-t)^{l_i}\langle z^{[i]}\rangle^{2l_i+2}\mathbf{1}_{|z^{[i]}|\le e^{\frac{l_i s}{2l_i+2}}}+\langle x\rangle^{-2}\mathbf{1}_{|z^{[i]}|>e^{\frac{l_i s}{2l_i+2}},\ |x^{[i]}|\le 4\delta}\right] \\
&\lesssim \delta_0\left[\mu_i^{-1}\langle y^{[i]}\rangle^{-2}e^{-l_i s}\mathbf{1}_{|z^{[i]}|\le 1,|y^{[i]}|>R}+e^{-(3l_i+1)s}|z^{[i]}|^{2l_i}\mathbf{1}_{1<|z^{[i]}|\le e^{\frac{l_i s}{2l_i+2}}}+e^{-\left(2l_i+2-\frac{1}{l_i+1}\right)s}\langle x^{[i]}\rangle^{-2}\mathbf{1}_{e^{\frac{l_i s}{2l_i+2}}<|z^{[i]}|\le 4\delta e^{\frac{s}{2}}}\right],
\end{aligned}$$

where we used (6.18) and $\langle x\rangle \sim \langle x^{[i]}\rangle$ for the last "$\lesssim$".

For $|x^{[i]}|\le 8\delta$, $u = U_{\mu_i,\xi^{[i]}}(x)\eta_{2\delta}(x^{[i]})+\Theta_{l_i}(x^{[i]},t)\eta_\delta(x^{[i]})+\mu_i^{-\frac{n-2}{2}}\phi_i(y^{[i]},t)\eta_R(y^{[i]})+\psi$. By the elementary inequality $\left||a+b|^{p-1}(a+b)-|a|^{p-1}a-p|a|^{p-1}b\right|\le p|b|^p$ for $p\in(1,2]$, $a,b\in\mathbb{R}$, we have

$$\begin{aligned}
&\left|\eta_{4\delta}(x^{[i]})\mathcal{N}\left[\psi,\boldsymbol{\phi},\boldsymbol{\mu},\boldsymbol{\xi}\right]\right| = \eta_{4\delta}(x^{[i]})\Big||u|^{\frac{2}{3}}u-\left(U_{\mu_i,\xi^{[i]}}(x)\right)^{\frac{5}{3}}\eta_{2\delta}^{\frac{5}{3}}(x^{[i]}) \\
&\qquad -\frac{5}{3}\left(U_{\mu_i,\xi^{[i]}}(x)\right)^{\frac{2}{3}}\eta_{2\delta}^{\frac{2}{3}}(x^{[i]})\left(\Theta_{l_i}(x^{[i]},t)\eta_\delta(x^{[i]})+\mu_i^{-\frac{3}{2}}\phi_i(y^{[i]},t)\eta_R(y^{[i]})+\psi(x,t)\right)\Big| \\
&\lesssim \eta_{4\delta}(x^{[i]})\left|\Theta_{l_i}(x^{[i]},t)\eta_\delta(x^{[i]})+\mu_i^{-\frac{3}{2}}\phi_i(y^{[i]},t)\eta_R(y^{[i]})+\psi(x,t)\right|^{\frac{5}{3}} \\
&\lesssim e^{-\frac{5}{3}l_i s}\langle z^{[i]}\rangle^{\frac{10}{3}l_i}\mathbf{1}_{|z^{[i]}|\le 2\delta e^{\frac{s}{2}}}+R^{\frac{10}{3}}e^{-\frac{5}{3}l_i s}\langle y^{[i]}\rangle^{-\frac{25}{6}}\mathbf{1}_{|y^{[i]}|\le 2R} \\
&\quad +\delta_0^{\frac{5}{3}}e^{-\frac{5}{3}l_i s}\langle z^{[i]}\rangle^{\frac{10}{3}l_i+\frac{10}{3}}\mathbf{1}_{|z^{[i]}|\le e^{\frac{l_i s}{2l_i+2}}}+\delta_0^{\frac{5}{3}}\langle x^{[i]}\rangle^{-\frac{10}{3}}\mathbf{1}_{e^{\frac{l_i s}{2l_i+2}}<|z^{[i]}|\le 8\delta e^{\frac{s}{2}}}.
\end{aligned}$$

Combining the estimates above, we have the estimate of $\mathcal{G}_{2,i}$. □

To exploit the spectrum properties of the $-A_z$, we give the following estimates.

Lemma 6.2. *Under all assumptions in Lemma 6.1, then for $i=1,2,\dots,\mathfrak{o}$, $s\in[s_0,\infty)$, it holds that*

$$\|g_{1,i}(\cdot,s)\|_{L^2_\rho(\mathbb{R}^5_+)}\le Ce^{-(2l_i+\frac{3}{4})s},\quad \|g_{2,i}(\cdot,s)\|_{L^2_\rho(\mathbb{R}^4)}\le Ce^{-(\frac{5}{3}l_i+\frac{1}{2})s}$$

with a constant C independent of T,δ_0 and varying from line to line. For $j\in\mathbb{N}$,

$$d_{i,j}(s)=e^{-\lambda_j s}\int_{s_{0*}}^{s}e^{\lambda_j\sigma}\left[\int_{\mathbb{R}^5_+}g_{1,i}(z^{[i]},\sigma)e_j(z^{[i]})e^{-\frac{|z^{[i]}|^2}{4}}dz^{[i]}+\int_{\mathbb{R}^4}g_{2,i}(\tilde z^{[i]},\sigma)e_j(\tilde z^{[i]},0)e^{-\frac{|\tilde z^{[i]}|^2}{4}}d\tilde z^{[i]}\right]d\sigma \tag{6.19}$$

with $s_{0*}:=\begin{cases} s_0 & \text{if } \lambda_j\ge \frac{5}{3}l_i+\frac{1}{2}\\ \infty & \text{if } \lambda_j<\frac{5}{3}l_i+\frac{1}{2}\end{cases}$ are integrable and satisfy (6.14). Moreover, given $k\in\mathbb{N}$, for $j=0,1,\dots,k$,

$$|d_{i,j}(s)|+|\dot d_{i,j}(s)|\le C\begin{cases} e^{-(\frac{5}{3}l_i+\frac{1}{2})s} & \text{if } \lambda_j\ne\frac{5}{3}l_i+\frac{1}{2}\\ se^{-(\frac{5}{3}l_i+\frac{1}{2})s} & \text{if } \lambda_j=\frac{5}{3}l_i+\frac{1}{2},\end{cases} \tag{6.20}$$

$$\sum_{j=0}^{k}\left(|\dot d_{i,j}(s)|+|d_{i,j}(s)|\right)\left(\left|A_{z^{[i]}}\tilde e_j(z^{[i]})\right|+\left|\tilde e_j(z^{[i]})\right|\right)\le C\mathbf{1}_{|z^{[i]}|\le 2C_{\tilde e}}\begin{cases} e^{-(\frac{5}{3}l_i+\frac{1}{2})s} & \text{if } \lambda_k<\frac{5}{3}l_i+\frac{1}{2}\\ se^{-(\frac{5}{3}l_i+\frac{1}{2})s} & \text{if } \lambda_k\ge\frac{5}{3}l_i+\frac{1}{2}\end{cases} \tag{6.21}$$

with $\tilde e_j$ given in (6.1). In particular,

$$\|\tilde g_{1,i}(\cdot,s)\|_{L^2_\rho(\mathbb{R}^5_+)}\le Cse^{-(\frac{5}{3}l_i+\frac{1}{2})s}. \tag{6.22}$$

Proof. By (6.9), Lemme 6.1, (6.18),

$$\begin{aligned}
&\|g_{1,i}(\cdot,s)\|_{L^2_\rho(\mathbb{R}^5_+)}\lesssim e^{-s}R^{-\frac{1}{4}}\mu_i^{-2}e^{-l_i s}\Big\|\langle\mu_i^{-1}(T-t)^{\frac{1}{2}}z^{[i]}\rangle^{-\frac{9}{4}}\mathbf{1}_{|z^{[i]}|\le 4R\mu_i(T-t)^{-\frac{1}{2}}} \\
&\quad +\langle\mu_i^{-1}(T-t)^{\frac{1}{2}}z^{[i]}\rangle^{-\frac{11}{4}}\mathbf{1}_{2^{-1}R\mu_i(T-t)^{-\frac{1}{2}}<|z^{[i]}|\le 1}\Big\|_{L^2_\rho(\mathbb{R}^5_+)} \\
&\quad +e^{-s}\Big\|e^{-(3l_i+\frac{1}{2})s}|z^{[i]}|^{-3}\mathbf{1}_{1<|z^{[i]}|\le 4\delta e^{\frac{s}{2}}}+\mathbf{1}_{\delta e^{\frac{s}{2}}\le|z^{[i]}|\le 2\delta e^{\frac{s}{2}}}\Big\|_{L^2_\rho(\mathbb{R}^5_+)} \\
&\lesssim e^{-(2l_i+\frac{3}{4})s}+e^{-s}\left[e^{-(3l_i+\frac{1}{2})s}+e^{-\frac{1}{16}\delta^2e^s}\right]\sim e^{-(2l_i+\frac{3}{4})s},
\end{aligned}$$

where we used

$$
\begin{aligned}
&\left\|\langle\mu_i^{-1}(T-t)^{\frac12} z^{[i]}\rangle^{-\frac94}\mathbf{1}_{|z^{[i]}|\le 4R\mu_i(T-t)^{-\frac12}}\right\|_{L^2_\rho(\mathbb{R}^5_+)}\\
&\sim \left\|\mathbf{1}_{|z^{[i]}|\le \mu_i(T-t)^{-\frac12}}+\mu_i^{\frac94}(T-t)^{-\frac98}|z^{[i]}|^{-\frac94}\mathbf{1}_{\mu_i(T-t)^{-\frac12}<|z^{[i]}|\le 4R\mu_i(T-t)^{-\frac12}}\right\|_{L^2_\rho(\mathbb{R}^5_+)}\\
&\lesssim \mu_i^{\frac52}(T-t)^{-\frac54}+R^{\frac14}\mu_i^{\frac52}(T-t)^{-\frac54}\sim R^{\frac14}e^{-(5l_i+\frac{15}{4})s},\\
&\left\|\langle\mu_i^{-1}(T-t)^{\frac12} z^{[i]}\rangle^{-\frac{11}{4}}\mathbf{1}_{2^{-1}R\mu_i(T-t)^{-\frac12}<|z^{[i]}|\le 1}\right\|_{L^2_\rho(\mathbb{R}^5_+)}\\
&\sim \mu_i^{\frac{11}{4}}(T-t)^{-\frac{11}{8}}\left\||z^{[i]}|^{-\frac{11}{4}}\mathbf{1}_{2^{-1}R\mu_i(T-t)^{-\frac12}<|z^{[i]}|\le 1}\right\|_{L^2_\rho(\mathbb{R}^5_+)}\sim R^{-\frac14}\mu_i^{\frac52}(T-t)^{-\frac54}\sim R^{-\frac14}e^{-(5l_i+\frac{15}{4})s}.
\end{aligned}
$$

$$
\|g_{2,i}(\cdot,s)\|_{L^2_\rho(\mathbb{R}^4)}\lesssim e^{-\frac{s}{2}}\left[R^{-\frac14}\mu_i^{-1}e^{-l_is}e^{-(\frac72 l_i+\frac{21}{8})s}+e^{-\frac53 l_is}+\left(e^{-\frac53 l_is}+\delta_0^{\frac53}\right)e^{-\frac{1}{16}e^{\frac{l_i}{l_i+1}s}}\right]\sim e^{-(\frac53 l_i+\frac12)s},
$$

where we used (6.18) and

$$
\begin{aligned}
&\|\langle\tilde y^{[i]}\rangle^{-\frac74}\mathbf{1}_{|\tilde z^{[i]}|\le 1}\|_{L^2_\rho(\mathbb{R}^4)}\sim\|\langle\mu_i^{-1}(T-t)^{\frac12}\tilde z^{[i]}\rangle^{-\frac74}\mathbf{1}_{|\tilde z^{[i]}|\le 1}\|_{L^2(\mathbb{R}^4)}\\
&\sim\|\mathbf{1}_{|\tilde z^{[i]}|\le\mu_i(T-t)^{-\frac12}}+\mu_i^{\frac74}(T-t)^{-\frac78}|\tilde z^{[i]}|^{-\frac74}\mathbf{1}_{\mu_i(T-t)^{-\frac12}<|\tilde z^{[i]}|\le 1}\|_{L^2(\mathbb{R}^4)}\\
&\lesssim\mu_i^2(T-t)^{-1}+\mu_i^{\frac74}(T-t)^{-\frac78}\sim e^{-(\frac72 l_i+\frac{21}{8})s}.
\end{aligned}
$$

It follows that

$$
\left|\int_{\mathbb{R}^5_+}g_{1,i}(z^{[i]},s)e_j(z^{[i]})e^{-\frac{|z^{[i]}|^2}{4}}dz^{[i]}\right|+\left|\int_{\mathbb{R}^4}g_{2,i}(\tilde z^{[i]},s)e_j(\tilde z^{[i]},0)e^{-\frac{|\tilde z^{[i]}|^2}{4}}d\tilde z^{[i]}\right|\lesssim e^{-(\frac53 l_i+\frac12)s}.
$$

Now it is easy to get that $d_{i,j}(s)$ given in (6.19) are well-defined and satisfy (6.20). Due to the support of $\tilde e_j$, we deduce (6.21). The bound of $\|\tilde g_{1,i}(\cdot,s)\|_{L^2_\rho(\mathbb{R}^5_+)}$ is deduced by (6.12), the estimate of $\|g_{1,i}(\cdot,s)\|_{L^2_\rho(\mathbb{R}^5_+)}$, and (6.21). ☐

6.2. **Estimates of Φ_i satisfying (6.11), $i=1,2,\dots,\mathfrak{o}$.** This subsection is inspired by [25, 60]. We will derive the estimate of $\|\Phi_i(\cdot,s)\|_{L^2_\rho(\mathbb{R}^5_+)}$ and then give the pointwise estimate of Φ_i.

Lemma 6.3. *Under all assumptions in Lemma 6.1, then there exists a constant $C>0$ independent of T,δ_0 such that*

$$
\|\Phi_i(\cdot,s)\|_{L^2_\rho(\mathbb{R}^5_+)}\le Cse^{-(\frac53 l_i+\frac12)s}.
$$

Proof. We first assume $\mathcal{G}_{1,i},\mathcal{G}_{2,i}$ are smooth about $z^{[i]},s$ and $\tilde z^{[i]},s$ respectively and the derivatives of $\mathcal{G}_{1,i},\mathcal{G}_{2,i}$ belong to $L^\infty(\mathbb{R}^5_+\times(s_0,s))$ and $L^\infty(\mathbb{R}^4\times(s_0,s))$ respectively for all $s\in(s_0,\infty)$. Then Φ_i is smooth and the derivatives of Φ_i are bounded for any fixed $s\in(s_0,\infty)$. For $j=0,1,\dots,N(\lceil 5l_i/3\rceil+1)$, testing the equation (6.11) by $e_j(z^{[i]})\rho(z^{[i]})$, integrating by parts as (2.6), and using (2.11), (6.13), we obtain that

$$
\begin{aligned}
&\partial_s(\Phi_i(\cdot,s),e_j)_{L^2_\rho(\mathbb{R}^5_+)}=(\Phi_i(\cdot,s),A_{z^{[i]}}e_j)_{L^2_\rho(\mathbb{R}^5_+)}\\
&+\int_{\mathbb{R}^5_+}\tilde g_{1,i}(z^{[i]},s)e_j(z^{[i]})e^{-\frac{|z^{[i]}|^2}{4}}dz^{[i]}+\int_{\mathbb{R}^4}g_{2,i}(\tilde z^{[i]},s)e_j(\tilde z^{[i]},0)e^{-\frac{|\tilde z^{[i]}|^2}{4}}d\tilde z^{[i]}=-\lambda_j(\Phi_i(\cdot,s),e_j)_{L^2_\rho(\mathbb{R}^5_+)}.
\end{aligned}
$$

By $\Phi_i(\cdot,s_0)=0$, we have

$$
(\Phi_i(\cdot,s),e_j)_{L^2_\rho(\mathbb{R}^5_+)}=0,\quad j=0,1,\dots,N(\lceil 5l_i/3\rceil+1)\qquad\text{for all }\ s\ge s_0. \tag{6.23}
$$

Testing the equation (6.11) by $\Phi_i\rho(z^{[i]})$ and integrating by parts as (2.6), we obtain

$$
\begin{aligned}
&\frac12\partial_s\left(\|\Phi_i(\cdot,s)\|^2_{L^2_\rho(\mathbb{R}^5_+)}\right)=-\|\nabla\Phi_i(\cdot,s)\|^2_{L^2_\rho(\mathbb{R}^5_+)}+\int_{\mathbb{R}^5_+}\tilde g_{1,i}(z^{[i]},s)\Phi_i(z^{[i]},s)e^{-\frac{|z^{[i]}|^2}{4}}dz^{[i]}\\
&+\int_{\mathbb{R}^4}g_{2,i}(\tilde z^{[i]},s)\Phi_i((\tilde z^{[i]},0),s)e^{-\frac{|\tilde z^{[i]}|^2}{4}}d\tilde z^{[i]}\le-\|\nabla\Phi_i(\cdot,s)\|^2_{L^2_\rho(\mathbb{R}^5_+)}+Cf_1(s)\|\Phi_i(\cdot,s)\|_{H^1_\rho(\mathbb{R}^5_+)}
\end{aligned}
$$

with $f_1(s):=\|g_{1,i}(\cdot,s)\|_{L^2_\rho(\mathbb{R}^5_+)}+\|g_{2,i}(\cdot,s)\|_{L^2_\rho(\mathbb{R}^4)}+\sum_{j=0}^{N(\lceil 5l_i/3\rceil+1)}\left(|d_{i,j}(s)|+|\dot d_{i,j}(s)|\right)$, where we used (6.12) and Lemma C.3 for the last inequality. Then by (6.23), Lemma 2.3 (3), we have

$$
\frac12\partial_s\left(\|\Phi_i(\cdot,s)\|^2_{L^2_\rho(\mathbb{R}^5_+)}\right)\le-(1-\epsilon)\left(\lceil 5l_i/3\rceil+3/2\right)\|\Phi_i(\cdot,s)\|^2_{L^2_\rho(\mathbb{R}^5_+)}+\frac{C(\epsilon)}{2}(f_1(s))^2,
$$

where $\epsilon > 0$ can be close to 0 arbitrarily. By $\Phi_i(\cdot, s_0) = 0$,

$$\|\Phi_i(\cdot,s)\|^2_{L^2_\rho(\mathbb{R}^5_+)} \le e^{-(1-\epsilon)(2\lceil 5l_i/3\rceil+3)s}\int_{s_0}^s C(\epsilon)(f_1(a))^2 e^{(1-\epsilon)(2\lceil 5l_i/3\rceil+3)a}da. \tag{6.24}$$

Generally, when $\mathcal{G}_{1,i}, \mathcal{G}_{2,i}$ are not smooth, we apply the mollifier to $\mathcal{G}_{1,i}, \mathcal{G}_{2,i}$ in spatial and time variables. Then repeating the above process and taking the limitation will deduce (6.24). Using Lemma 6.2 and taking ϵ sufficiently small to make $(1-\epsilon)(2\lceil 5l_i/3\rceil + 3) > 10l_i/3 + 1$, we get the desired estimate. □

The representation formula of Φ_i in (6.11) is given by (2.21) of the form

$$\Phi_i(z^{[i]}, s) = \int_{s_0}^s\int_{\mathbb{R}^5_+} H_5\left(z^{[i]}, s, w, \sigma\right)\tilde{g}_{1,i}(w,\sigma)dwd\sigma + \int_{s_0}^s\int_{\mathbb{R}^4} H_5\left(z^{[i]}, s, (\tilde{w},0),\sigma\right) g_{2,i}(\tilde{w},\sigma)d\tilde{w}d\sigma.$$

Lemma 6.4. *Under all assumptions in Lemma 6.1, then there exists a constant $C > 0$ independent of T, δ_0 such that for any $z^{[i]} \in \overline{\mathbb{R}^5_+}$, $s > s_1 \ge s_0$, we have*

$$\begin{aligned}&\int_{s_1}^s\int_{\mathbb{R}^5_+} H_5\left(z^{[i]}, s, w, \sigma\right)|\tilde{g}_{1,i}(w,\sigma)|\, dwd\sigma + \int_{s_1}^s\int_{\mathbb{R}^4} H_5\left(z^{[i]}, s, (\tilde{w},0),\sigma\right)|g_{2,i}(\tilde{w},\sigma)|\, d\tilde{w}d\sigma\\ &\le C\Big[R^{-\frac14}e^{-l_i s}\langle e^{(2l_i+\frac32)s}z^{[i]}\rangle^{-\frac14} + R^{-\frac14}e^{-(l_i+\frac38)s_1} + \min\left\{e^{-s_1}, e^{-s_1}e^{-(l_i+1)s}|z^{[i]}|^{2l_i+2}\right\}\\ &\quad + \min\left\{e^{-\frac{s_1}{2}}, e^{-(\frac53 l_i+\frac12)s}|\tilde{z}^{[i]}|^{\frac{10}{3}l_i+\frac{10}{3}}\right\}\Big].\end{aligned}$$

Proof. By (6.12), (6.9), (6.21), Lemma 6.1,

$$\begin{aligned}&\left|\tilde{g}_{1,i}(z^{[i]},s)\right| \lesssim e^{-s}|\mathcal{G}_{1,i}| + se^{-(\frac53 l_i+\frac12)s}\mathbf{1}_{|z^{[i]}|\le 2C_{\tilde{e}}} \lesssim R^{-\frac14}\mu_i^{-2}e^{-(l_i+1)s}\langle y^{[i]}\rangle^{-\frac94} + se^{-(\frac53 l_i+\frac12)s} + e^{-s}\mathbf{1}_{\delta e^{\frac{s}{2}}\le|z^{[i]}|\le 2\delta e^{\frac{s}{2}}},\\ &\left|g_{2,i}(\tilde{z}^{[i]},s)\right| \lesssim e^{-\frac{s}{2}}|\mathcal{G}_{2,i}| \lesssim R^{-\frac14}\mu_i^{-1}e^{-(l_i+\frac12)s}\langle\tilde{y}^{[i]}\rangle^{-\frac74} + e^{-(\frac53 l_i+\frac12)s} + e^{-\frac{s}{2}}\min\left\{1, e^{-\frac53 l_i s}|\tilde{z}^{[i]}|^{\frac{10}{3}l_i+\frac{10}{3}}\right\},\end{aligned}$$

where we used $\delta_0 \in (0,1)$. By (5.5), (6.18),

$$R^{-\frac14}\mu_i^{-2}e^{-(l_i+1)s}\langle y^{[i]}\rangle^{-\frac94} \sim R^{-\frac14}e^{(3l_i+3)s}\langle e^{(2l_i+\frac32)s}z^{[i]}\rangle^{-\frac94},\quad R^{-\frac14}\mu_i^{-1}e^{-(l_i+\frac12)s}\langle\tilde{y}^{[i]}\rangle^{-\frac74} \sim R^{-\frac14}e^{(l_i+\frac32)s}\langle e^{(2l_i+\frac32)s}\tilde{z}^{[i]}\rangle^{-\frac74}.$$

By Lemma 2.10 (1),

$$\begin{aligned}&\int_{s_1}^s\int_{\mathbb{R}^5_+} H_5\left(z^{[i]}, s, w, \sigma\right)R^{-\frac14}e^{(3l_i+3)\sigma}\langle e^{(2l_i+\frac32)\sigma}w\rangle^{-\frac94}dwd\sigma\\ &+\int_{s_1}^s\int_{\mathbb{R}^4} H_5\left(z^{[i]}, s, (\tilde{w},0),\sigma\right)R^{-\frac14}e^{(l_i+\frac32)\sigma}\langle e^{(2l_i+\frac32)\sigma}\tilde{w}\rangle^{-\frac74}d\tilde{w}d\sigma\\ &\lesssim R^{-\frac14}\left[e^{-l_i s}\langle e^{(2l_i+\frac32)s}z^{[i]}\rangle^{-\frac14} + e^{-(\frac32 l_i+\frac38)s_1} + e^{-l_i s}\langle e^{(2l_i+\frac32)s}z^{[i]}\rangle^{-\frac34} + e^{-(\frac52 l_i+\frac98)s_1}\right]\\ &\sim R^{-\frac14}\left[e^{-l_i s}\langle e^{(2l_i+\frac32)s}z^{[i]}\rangle^{-\frac14} + e^{-(\frac32 l_i+\frac38)s_1}\right].\end{aligned}$$

By Lemma 2.10 (3), for an arbitrarily small constant $\epsilon \in (0, 1/4)$,

$$\int_{s_1}^s\int_{\mathbb{R}^5_+} H_5\left(z^{[i]}, s, w, \sigma\right)\sigma e^{-(\frac53 l_i+\frac12)\sigma}dwd\sigma \lesssim_\epsilon \int_{s_1}^s\int_{\mathbb{R}^5_+} H_5\left(z^{[i]}, s, w, \sigma\right)e^{-(\frac53 l_i+\frac12-\epsilon)\sigma}dwd\sigma \lesssim_\epsilon e^{-(\frac53 l_i+\frac12-\epsilon)s_1}.$$

By Lemma 2.10 (3) (2),

$$\begin{aligned}&\int_{s_1}^s\int_{\mathbb{R}^5_+} H_5\left(z^{[i]}, s, w, \sigma\right)\left(e^{-\sigma}\mathbf{1}_{\delta e^{\frac{\sigma}{2}}\le|w|\le 2\delta e^{\frac{\sigma}{2}}}\right)dwd\sigma\\ &\lesssim \int_{s_1}^s\int_{\mathbb{R}^5_+} H_5\left(z^{[i]}, s, w, \sigma\right)\min\left\{e^{-\sigma}, e^{-(l_i+2)\sigma}|w|^{2l_i+2}\right\}dwd\sigma\\ &\lesssim \min\left\{e^{-s_1}, \int_{s_1}^s\left[e^{-\sigma}e^{-(l_i+1)s}|z^{[i]}|^{2l_i+2} + e^{-(l_i+2)\sigma}\right]d\sigma\right\} \lesssim \min\left\{e^{-s_1}, e^{-s_1}e^{-(l_i+1)s}|z^{[i]}|^{2l_i+2} + e^{-(l_i+2)s_1}\right\}.\end{aligned}$$

By Lemma 2.10 (4),

$$\begin{aligned}&\int_{s_1}^s\int_{\mathbb{R}^4} H_5\left(z^{[i]}, s, (\tilde{w},0),\sigma\right)\left[e^{-(\frac53 l_i+\frac12)\sigma} + \min\left\{e^{-\frac{\sigma}{2}}, e^{-(\frac53 l_i+\frac12)\sigma}|\tilde{w}|^{\frac{10}{3}l_i+\frac{10}{3}}\right\}\right]d\tilde{w}d\sigma\\ &\lesssim \min\left\{e^{-\frac{s_1}{2}}, e^{-(\frac53 l_i+\frac12)s}|\tilde{z}^{[i]}|^{\frac{10}{3}l_i+\frac{10}{3}} + e^{-(\frac53 l_i+\frac12)s_1}\right\}.\end{aligned}$$

Combining the above estimates and using $C_1 + \min\{C_2, C_3\} = \min\{C_1 + C_2, C_1 + C_3\}$, then we get the desired estimate. $\square$

Proposition 6.5. *Under all assumptions in Lemma 6.1, then there exists a constant $C > 0$ independent of T, δ_0 such that for any $z^{[i]} \in \overline{\mathbb{R}^5_+}$, $s > s_0$, we have*

$$\begin{aligned}\left|\Phi_i(z^{[i]}, s)\right| \le C\Big[&\Big(R^{-\frac14}e^{-l_i s}\langle e^{(2l_i+\frac32)s}z^{[i]}\rangle^{-\frac14} + T^{\frac38}R^{-\frac14}e^{-l_i s}\langle|z^{[i]}|^{2l_i}\rangle + e^{-(l_i+\frac12)s}|z^{[i]}|^{2l_i+2}\Big)\mathbf{1}_{|z^{[i]}|\le e^{\frac{l_i s}{2l_i+2}}}\\ &+ R^{-\frac14}\mathbf{1}_{|z^{[i]}|> e^{\frac{l_i s}{2l_i+2}}}\Big].\end{aligned} \tag{6.25}$$

Proof. Applying Lemma 6.4 with $s_1 = s_0$, we get $|\Phi_i| \lesssim R^{-\frac14}$ in $\overline{\mathbb{R}^5_+} \times (s_0, \infty)$. For the more delicate estimate, we separate the domain into the following four parts and estimate separately.

Case 1: Fix $z^{[i]} \in \overline{\mathbb{R}^5_+}$ and $s \in (s_0, s_0 + 2]$. By Lemma 6.4 with $s_1 = s_0$,

$$\left|\Phi_i(z^{[i]}, s)\right| \lesssim R^{-\frac14}e^{-l_i s}\langle e^{(2l_i+\frac32)s}z^{[i]}\rangle^{-\frac14} + R^{-\frac14}e^{-(l_i+\frac38)s_0} + e^{-s_0}e^{-(l_i+1)s}|z^{[i]}|^{2l_i+2} + e^{-(\frac53 l_i+\frac12)s}|z^{[i]}|^{\frac{10}{3}l_i+\frac{10}{3}}.$$

Case 2: Fix $|z^{[i]}| \le 2$ and $s \in (s_0 + 2, \infty)$. By uniqueness, we split $\Phi_i = \Phi_i^{(1)} + \Phi_i^{(2)}$ and rewrite (6.11) as

$$\begin{cases}\partial_{s_*}\Phi_i^{(1)} = A_w\Phi_i^{(1)} + \tilde g_{1,i}(w, s_*) \ \text{ for } \ (w, s_*) \in \mathbb{R}^5_+ \times (s-1, \infty),\\ -\partial_{w_5}\Phi_i^{(1)}((\tilde w, 0), s_*) = g_{2,i}(\tilde w, s_*) \ \text{ for } \ (\tilde w, s_*) \in \mathbb{R}^4 \times (s-1, \infty), \quad \Phi_i^{(1)}(\cdot, s-1) = 0 \ \text{ in } \ \mathbb{R}^5_+,\end{cases}$$

$$\begin{cases}\partial_{s_*}\Phi_i^{(2)} = A_w\Phi_i^{(2)} \ \text{ for } \ (w, s_*) \in \mathbb{R}^5_+ \times (s-1, \infty),\\ -\partial_{w_5}\Phi_i^{(2)}((\tilde w, 0), s_*) = 0 \ \text{ for } \ (\tilde w, s_*) \in \mathbb{R}^4 \times (s-1, \infty), \quad \Phi_i^{(2)}(\cdot, s-1) = \Phi_i(\cdot, s-1) \ \text{ in } \ \mathbb{R}^5_+.\end{cases}$$

For $\Phi_i^{(1)}$, applying Lemma 6.4 with $s_1 = s - 1$ at the point $(z^{[i]}, s)$, we have

$$\left|\Phi_i^{(1)}(z^{[i]}, s)\right| \lesssim R^{-\frac14}e^{-l_i s}\langle e^{(2l_i+\frac32)s}z^{[i]}\rangle^{-\frac14} + R^{-\frac14}e^{-(l_i+\frac38)s} + e^{-s}e^{-(l_i+1)s}|z^{[i]}|^{2l_i+2} + e^{-(\frac53 l_i+\frac12)s}|z^{[i]}|^{\frac{10}{3}l_i+\frac{10}{3}}.$$

For $\Phi_i^{(2)}$, by Lemma 2.10 (2) with $s_1 = s - 1$, $|z^{[i]}| \le 2$, and Lemma 6.3, then

$$\left|\Phi_i^{(2)}(z^{[i]}, s)\right| \lesssim \|\Phi_i(\cdot, s-1)\|_{L^2_\rho(\mathbb{R}^5_+)} \lesssim s e^{-(\frac53 l_i+\frac12)s}.$$

Since $se^{-(\frac53 l_i+\frac12)s} \lesssim R^{-\frac14}e^{-(l_i+\frac38)s}$, we have

$$\left|\Phi_i(z^{[i]}, s)\right| \lesssim R^{-\frac14}e^{-l_i s}\langle e^{(2l_i+\frac32)s}z^{[i]}\rangle^{-\frac14} + R^{-\frac14}e^{-(l_i+\frac38)s} + e^{-s}e^{-(l_i+1)s}|z^{[i]}|^{2l_i+2} + e^{-(\frac53 l_i+\frac12)s}|z^{[i]}|^{\frac{10}{3}l_i+\frac{10}{3}}.$$

Case 3: Fix $2 < |z^{[i]}| < e^{\frac{s-s_0}{2}}$ and $s \in (s_0 + 2, \infty)$. We choose $s_1 \in (s_0, s-1)$ such that $|z^{[i]}| = e^{\frac{s-s_1}{2}}$. Similar to Case 2, we split $\Phi_i = \Phi_i^{(3)} + \Phi_i^{(4)}$ and the equation (6.11) is rewritten as

$$\begin{cases}\partial_{s_*}\Phi_i^{(3)} = A_w\Phi_i^{(3)} + \tilde g_{1,i}(w, s_*) \ \text{ for } \ (w, s_*) \in \mathbb{R}^5_+ \times (s_1, \infty),\\ -\partial_{w_5}\Phi_i^{(3)}((\tilde w, 0), s_*) = g_{2,i}(\tilde w, s_*) \ \text{ for } \ (\tilde w, s_*) \in \mathbb{R}^4 \times (s_1, \infty), \quad \Phi_i^{(3)}(\cdot, s_1) = 0 \ \text{ in } \ \mathbb{R}^5_+,\end{cases}$$

$$\begin{cases}\partial_{s_*}\Phi_i^{(4)} = A_w\Phi_i^{(4)} \ \text{ for } \ (w, s_*) \in \mathbb{R}^5_+ \times (s_1, \infty),\\ -\partial_{w_5}\Phi_i^{(4)}((\tilde w, 0), s_*) = 0 \ \text{ for } \ (\tilde w, s_*) \in \mathbb{R}^4 \times (s_1, \infty), \quad \Phi_i^{(4)}(\cdot, s_1) = \Phi_i(\cdot, s_1) \ \text{ in } \ \mathbb{R}^5_+.\end{cases}$$

By Lemma 6.4, we have

$$\left|\Phi_i^{(3)}(z^{[i]}, s)\right| \lesssim T^{\frac38}R^{-\frac14}e^{-l_i s}|z^{[i]}|^{2l_i} + Te^{-(l_i+1)s}|z^{[i]}|^{2l_i+2} + e^{-(\frac53 l_i+\frac12)s}|z^{[i]}|^{\frac{10}{3}l_i+\frac{10}{3}},$$

where we used

$$\begin{aligned}&e^{-s_1} = e^{-s}|z^{[i]}|^2, \quad R^{-\frac14}e^{-l_i s}\langle e^{(2l_i+\frac32)s}z^{[i]}\rangle^{-\frac14} \lesssim R^{-\frac14}e^{-l_i s_1}e^{-(\frac{l_i}{2}+\frac38)s}|z^{[i]}|^{-\frac14} \lesssim T^{\frac38}R^{-\frac14}e^{-l_i s}|z^{[i]}|^{2l_i},\\ &R^{-\frac14}e^{-(l_i+\frac38)s_1} = e^{-\frac38 s_1}R^{-\frac14}e^{-l_i s}|z^{[i]}|^{2l_i} \le T^{\frac38}R^{-\frac14}e^{-l_i s}|z^{[i]}|^{2l_i}.\end{aligned}$$

By Lemma 2.10 (2), $|z^{[i]}| = e^{\frac{s-s_1}{2}}$, $s_1 \le s - 1$, and Lemma 6.3, we obtain that

$$\left|\Phi_i^{(4)}(z^{[i]}, s)\right| \lesssim \|\Phi_i(\cdot, s_1)\|_{L^2_\rho(\mathbb{R}^5_+)} \lesssim s_1 e^{-(\frac53 l_i+\frac12)s_1} \lesssim (-\ln T)T^{\frac12}e^{-l_i s}|z^{[i]}|^{2l_i}.$$

Since $(-\ln T)T^{\frac12} \lesssim T^{\frac38}R^{-\frac14}$, we have

$$\left|\Phi_i(z^{[i]}, s)\right| \lesssim T^{\frac38}R^{-\frac14}e^{-l_i s}|z^{[i]}|^{2l_i} + Te^{-(l_i+1)s}|z^{[i]}|^{2l_i+2} + e^{-(\frac53 l_i+\frac12)s}|z^{[i]}|^{\frac{10}{3}l_i+\frac{10}{3}}.$$

Case 4: Fix $|z^{[i]}| \ge e^{\frac{s-s_0}{2}}$ and $s \in (s_0+2,\infty)$. By Lemma 6.4 with $s_1 = s_0$,

$$\left|\Phi_i(z^{[i]},s)\right| \lesssim T^{\frac38} R^{-\frac14} e^{-l_i s}|z^{[i]}|^{2l_i} + T e^{-(l_i+1)s}|z^{[i]}|^{2l_i+2} + e^{-(\frac53 l_i+\frac12)s}|z^{[i]}|^{\frac{10}{3}l_i+\frac{10}{3}},$$

where we used

$$e^{-s_0} \le e^{-s}|z^{[i]}|^2, \quad R^{-\frac14}e^{-l_i s}\langle e^{(2l_i+\frac32)s}z^{[i]}\rangle^{-\frac14} \lesssim R^{-\frac14}e^{-l_i s_0}e^{-(\frac{l_i}{2}+\frac38)s}|z^{[i]}|^{-\frac14} \lesssim T^{\frac38}R^{-\frac14}e^{-l_i s}|z^{[i]}|^{2l_i},$$
$$R^{-\frac14}e^{-(l_i+\frac38)s_0} \lesssim T^{\frac38}R^{-\frac14}e^{-l_i s}|z^{[i]}|^{2l_i}.$$

Combining the above four cases, we get

$$\left|\Phi_i(z^{[i]},s)\right| \lesssim R^{-\frac14}e^{-l_i s}\langle e^{(2l_i+\frac32)s}z^{[i]}\rangle^{-\frac14} + T^{\frac38}R^{-\frac14}e^{-l_i s}\langle |z^{[i]}|^{2l_i}\rangle + Te^{-(l_i+1)s}|z^{[i]}|^{2l_i+2} + e^{-(\frac53 l_i+\frac12)s}|z^{[i]}|^{\frac{10}{3}l_i+\frac{10}{3}}.$$

In particular,

$$\left|\Phi_i(z^{[i]},s)\right| \lesssim R^{-\frac14}e^{-l_i s}\langle e^{(2l_i+\frac32)s}z^{[i]}\rangle^{-\frac14} + T^{\frac38}R^{-\frac14}e^{-l_i s}\langle |z^{[i]}|^{2l_i}\rangle + e^{-(l_i+\frac12)s}|z^{[i]}|^{2l_i+2} \ \text{ for } \ |z^{[i]}| \le e^{\frac{l_i s}{2l_i+2}}.$$

Integrating $|\Phi_i| \lesssim R^{-\frac14}$ in $\overline{\mathbb{R}^5_+}\times(s_0,\infty)$, we attain the conclusion. □

Proposition 6.6. *Under all assumptions in Lemma 6.1, then for $\mathcal{T}^{\rm out}[\psi,\boldsymbol{\phi},\boldsymbol{\mu},\boldsymbol{\xi}]$ given in (6.4), there exist $\varphi_0(x) \in C_c^\infty(\overline{\mathbb{R}^5_+})$ in (6.3) and a constant $C>0$ independent of T,δ_0 such that*

$$|\mathcal{T}^{\rm out}[\psi,\boldsymbol{\phi},\boldsymbol{\mu},\boldsymbol{\xi}](x,t)| \le CR^{-\frac14}\Big[\mathbf{1}_{\cap_{i=1}^{\mathfrak{o}}\{|z^{[i]}|>e^{\frac{l_i s}{2l_i+2}}\}} + \sum_{i=1}^{\mathfrak{o}}(T-t)^{l_i}\langle |z^{[i]}|^{2l_i+2}\rangle \mathbf{1}_{|z^{[i]}|\le e^{\frac{l_i s}{2l_i+2}}}\Big],$$
$$\mathcal{T}^{\rm out}[\psi,\boldsymbol{\phi},\boldsymbol{\mu},\boldsymbol{\xi}](x,0) = \sum_{i=1}^{\mathfrak{o}}\boldsymbol{b}_i\cdot\tilde{\boldsymbol{e}}_i\big(T^{-\frac12}(x-q^{[i]})\big) + \sum_{i=1}^{\mathfrak{o}}\sum_{\mathbf{p}\in\mathbb{N}^5,\|\mathbf{p}\|_{\ell_1}\le 4l_{\max}+4,p_5\in 2\mathbb{N}} C_{q^{[i]},\mathbf{p}}\varphi_{q^{[i]},\mathbf{p},0}(x),$$

where $\boldsymbol{b}_i = \boldsymbol{b}_i[\psi,\boldsymbol{\phi},\boldsymbol{\mu},\boldsymbol{\xi}]$ are constant vectors and $|\boldsymbol{b}_i| \le C|\ln T|T^{\frac53 l_i+\frac12}$; $\tilde{\boldsymbol{e}}_i$ are given in (6.2); $C_{q^{[i]},\mathbf{p}} = C_{q^{[i]},\mathbf{p}}[\psi,\boldsymbol{\phi},\boldsymbol{\mu},\boldsymbol{\xi}]$ are constants and $|C_{q^{[i]},\mathbf{p}}| \le Ce^{-\frac{9\delta^2}{22T}}$; $\varphi_{q^{[i]},\mathbf{p},0}(x) \in C_c^\infty(\overline{\mathbb{R}^5_+})$ and $\varphi_{q^{[i]},\mathbf{p},0}(x)=0$ in $\overline{\mathbb{R}^5_+\backslash B_5^+(q^{[i]},2\delta)}$. In particular, for $\mathbf{m}\in\mathbb{N}^5$, there exists a constant $C_{\mathbf{m}}>0$ independent of T,δ_0 such that

$$|D_x^{\mathbf{m}}\mathcal{T}^{\rm out}[\psi,\boldsymbol{\phi},\boldsymbol{\mu},\boldsymbol{\xi}](x,0)| \le C_{\mathbf{m}}|\ln T|T^{\frac53 l_i+\frac12-\frac{\|\mathbf{m}\|_{\ell_1}}{2}}\sum_{i=1}^{\mathfrak{o}}\mathbf{1}_{|x-q^{[i]}|\le 2\delta}.$$

Proof. Recalling (6.8), (6.10), we have

$$\psi_i(x,t) = \boldsymbol{d}_i(s)\cdot\tilde{\boldsymbol{e}}_i(z^{[i]}) + \Phi_i(z^{[i]},s) \quad \text{ with } \ z^{[i]} = (T-t)^{-\frac12}\left(x-q^{[i]}\right), \quad s=-\ln(T-t),$$
$$\psi_i(x,0) = \boldsymbol{b}_i\cdot\tilde{\boldsymbol{e}}_i\big(T^{-\frac12}(x-q^{[i]})\big)$$

with $\boldsymbol{b}_i = \boldsymbol{b}_i[\psi,\boldsymbol{\phi},\boldsymbol{\mu},\boldsymbol{\xi}] = \boldsymbol{d}_i(s_0)$ given in (6.12). Lemma 6.2 deduces that $|\boldsymbol{b}_i| \lesssim |\ln T|T^{\frac53 l_i+\frac12}$ and $\left|\boldsymbol{d}_i(s)\cdot\tilde{\boldsymbol{e}}_i(z^{[i]})\right| \lesssim \mathbf{1}_{|z^{[i]}|\le 2C_{\tilde e}}se^{-(\frac53 l_i+\frac12)s}$. Combining (6.25) and $T^{\frac38} \ll R^{-\frac14}$ due to $T\ll 1$, we have

$$|\psi_i(x,t)| \lesssim R^{-\frac14}\Big[(T-t)^{l_i}\langle |z^{[i]}|^{2l_i+2}\rangle\mathbf{1}_{|z^{[i]}|\le e^{\frac{l_i s}{2l_i+2}}} + \mathbf{1}_{|z^{[i]}|>e^{\frac{l_i s}{2l_i+2}}}\Big], \quad i=1,2,\dots,\mathfrak{o}. \tag{6.26}$$

By uniqueness, we set $\psi_{\mathfrak{o}+1} = \psi^{(1)}_{\mathfrak{o}+1}(x,t) + \psi^{(2)}_{\mathfrak{o}+1}(x,t)$ and decompose the equation (6.6) of $\psi_{\mathfrak{o}+1}$ into the following two equations.

$$\begin{aligned}
&\partial_t\psi^{(1)}_{\mathfrak{o}+1} = \Delta_x\psi^{(1)}_{\mathfrak{o}+1} \ \text{ in } \ \mathbb{R}^5_+\times(0,T), \quad -\partial_{x_5}\psi^{(1)}_{\mathfrak{o}+1} = \mathcal{G}_{2,\mathfrak{o}+1} \ \text{ on } \ \partial\mathbb{R}^5_+\times(0,T), \quad \psi^{(1)}_{\mathfrak{o}+1}(x,0)=0 \ \text{ in } \ \mathbb{R}^5_+,\\
&\partial_t\psi^{(2)}_{\mathfrak{o}+1} = \Delta_x\psi^{(2)}_{\mathfrak{o}+1} \ \text{ in } \ \mathbb{R}^5_+\times(0,T), \quad -\partial_{x_5}\psi^{(2)}_{\mathfrak{o}+1} = 0 \ \text{ on } \ \partial\mathbb{R}^5_+\times(0,T), \quad \psi^{(2)}_{\mathfrak{o}+1}(x,0)=\varphi_0(x) \ \text{ in } \ \mathbb{R}^5_+.
\end{aligned} \tag{6.27}$$

Set

$$\hat{\psi}_i(x,t) := \psi^{(1)}_{\mathfrak{o}+1}(x,t) + \sum_{j=1,j\ne i}^{\mathfrak{o}}\psi_j(x,t) \quad \text{ for } \ i=1,2,\dots,\mathfrak{o}. \tag{6.28}$$

By (6.5), (6.27), $\hat{\psi}_i$ satisfies

$$\begin{cases}
\partial_t\hat{\psi}_i = \Delta_x\hat{\psi}_i + h_{1,i}(x,t) \quad \text{ in } \ \mathbb{R}^5_+\times(0,T), \quad -\partial_{x_5}\hat{\psi}_i = h_{2,i}(\tilde{x},t) \quad \text{ on } \ \partial\mathbb{R}^5_+\times(0,T),\\
\hat{\psi}_i(x,0) = \sum_{j=1,j\ne i}^{\mathfrak{o}}\boldsymbol{b}_j\cdot\tilde{\boldsymbol{e}}_j\big(T^{-\frac12}(x-q^{[j]})\big) \quad \text{ in } \ \mathbb{R}^5_+,
\end{cases} \tag{6.29}$$

where

$$h_{1,i}(x,t) := \sum_{j=1,j\neq i}^{\mathfrak{o}} \mathcal{G}_{1,j}(x,t), \quad h_{2,i}(\tilde{x},t) := \sum_{j=1,j\neq i}^{\mathfrak{o}+1} \mathcal{G}_{2,j}(\tilde{x},t).$$

By the properties of $\tilde{\boldsymbol{e}}_j$ given in (6.2), (6.1), $|\hat{\psi}_i(x,0)| \lesssim |\ln T| T^{\frac{1}{2}} \sum_{j=1,j\neq i}^{\mathfrak{o}} \mathbf{1}_{|x-q^{[j]}|\le 2C_{\tilde{e}}T^{\frac{1}{2}}}$, $\hat{\psi}_i(x,0) \in C_c^\infty\big(\cup_{j=1,j\neq i}^{\mathfrak{o}}\{x \in \overline{\mathbb{R}^5_+} \mid |x-q^{[j]}| \le 2C_{\tilde{e}}T^{\frac{1}{2}}\}\big)$. By Lemma 6.1, one sees that

$$|h_{1,i}(x,t)| \lesssim R^{-\frac{1}{4}} (T-t)^{-4-3l_{\max}} \mathbf{1}_{\cup_{j=1,j\neq i}^{\mathfrak{o}}\{|x-q^{[j]}|\le 4\delta\}},$$
$$|h_{2,i}(\tilde{x},t)| \lesssim \mathbf{1}_{\cap_{j=1}^{\mathfrak{o}}\{|\tilde{x}-\tilde{q}^{[j]}|\ge 4\delta\}} + R^{-\frac{1}{4}} (T-t)^{-2-l_{\max}} \mathbf{1}_{\cup_{j=1,j\neq i}^{\mathfrak{o}}\{|\tilde{x}-\tilde{q}^{[i]}|\le 8\delta\}}, \quad \text{with} \quad l_{\max} = \max_{j=1,2,\ldots,\mathfrak{o}} l_j.$$

$\hat{\psi}_i$ can be represented by the formula (2.17) with $n=5$ and $t_0=0$. Since $h_{1,i}(x,t) = \hat{\psi}_i(x,0) = 0$ for $|x-q^{[i]}| \le 3\delta$ and $h_{2,i}(\tilde{x},t) = 0$ for $|\tilde{x}-\tilde{q}^{[i]}| \le 3\delta$, applying Lemma 2.14 with $r = 3\delta$, we have

$$\hat{\psi}_i \in C^\infty(\overline{B_5^+(q^{[i]},3\delta/2)} \times [0,T]) \text{ and } |\partial_t^\iota D_x^{\mathbf{m}} \hat{\psi}_i| \lesssim_{\iota,\mathbf{m}} e^{-\frac{9\delta^2}{22T}} \text{ in } \overline{B_5^+(q^{[i]},3\delta/2)} \times [0,T] \text{ for all } \iota \in \mathbb{N}, \mathbf{m} \in \mathbb{N}^5,$$

where $\hat{\psi}_i$ can be defined by extension naturally for $t=T$, $x \in \overline{B_5^+(q^{[i]},3\delta/2)}$. Under the additional assumption $m_5 \in 2\mathbb{N}+1$, we have $\partial_t^\iota D_x^{\mathbf{m}} \hat{\psi}_i((\tilde{x},0),t) = 0$ in $\overline{B_4(\tilde{q}^{[i]},3\delta/2)} \times [0,T]$.

By Proposition 2.13, for $j=1,2,\ldots,\mathfrak{o}$, $\mathbf{p} \in \mathbb{N}^5$, $\|\mathbf{p}\|_{\ell_1} \le 4l_{\max}+4$, $p_5 \in 2\mathbb{N}$, we take the vanishing adjustment functions $\varphi_{q^{[j]},\mathbf{p}}(x,t)$ to satisfy

$$\partial_t \varphi_{q^{[j]},\mathbf{p}} = \Delta_x \varphi_{q^{[j]},\mathbf{p}} \text{ in } \mathbb{R}^5_+ \times (0,T], \quad -\partial_{x_5}\varphi_{q^{[j]},\mathbf{p}} = 0 \text{ on } \partial\mathbb{R}^5_+ \times (0,T], \quad \varphi_{q^{[j]},\mathbf{p}}(x,0) = \varphi_{q^{[j]},\mathbf{p},0}(x) \text{ in } \mathbb{R}^5_+,$$

and the following properties hold:

(1) $\varphi_{q^{[j]},\mathbf{p},0}(x)$ is smooth in $\overline{\mathbb{R}^5_+}$ and $\varphi_{q^{[j]},\mathbf{p},0}(x) = 0$ in $\overline{\mathbb{R}^5_+ \backslash B_5^+(q^{[j]},2\delta)}$.
(2) $\partial_t^\iota D_x^{\mathbf{m}} \varphi_{q^{[j]},\mathbf{p}}((\tilde{x},0),t) = 0$ for $\tilde{x} \in \mathbb{R}^4$, $t \in [0,T]$, and $\iota \in \mathbb{N}$, $\mathbf{m} \in \mathbb{N}^5$, $m_5 \in 2\mathbb{N}+1$.
(3) $D_x^{\mathbf{m}} \varphi_{q^{[j]},\mathbf{p}}(q^{[k]},T) = \delta_{\mathbf{p},\mathbf{m}} \delta_{q^{[j]},q^{[k]}}$ for $\mathbf{m} \in \mathbb{N}^5$, $\|\mathbf{m}\|_{\ell_1} \le 4l_{\max}+4$, $k=1,2,\ldots,\mathfrak{o}$.
(4) $\|\partial_t^\iota D_x^{\mathbf{m}} \varphi_{q^{[j]},\mathbf{p}}\|_{L^\infty(\mathbb{R}^5_+\times[0,T])} \lesssim 1$ for $\iota \in \mathbb{N}$, $\mathbf{m} \in \mathbb{N}^5$, $2\iota + \|\mathbf{m}\|_{\ell_1} \le 4l_{\max}+4$.

We take

$$\psi_{\mathfrak{o}+1}^{(2)} = \sum_{j=1}^{\mathfrak{o}} \sum_{\mathbf{p}\in\mathbb{N}^5, \|\mathbf{p}\|_{\ell_1} \le 4l_{\max}+4, p_5\in 2\mathbb{N}} -\left(D_x^{\mathbf{p}}\hat{\psi}_j\right)(q^{[j]},T)\varphi_{q^{[j]},\mathbf{p}}(x,t). \tag{6.30}$$

Then for all $i=1,2,\ldots,\mathfrak{o}$, $\mathbf{m} \in \mathbb{N}^5$, $\|\mathbf{m}\|_{\ell_1} \le 4l_{\max}+4$, $m_5 \in 2\mathbb{N}$, we have $D_x^{\mathbf{m}}\big(\hat{\psi}_i + \psi_{\mathfrak{o}+1}^{(2)}\big)(q^{[i]},T) = 0$. One using (6.27) and the support of $h_{1,i}$ in (6.29), it follows that

$$\partial_t^\iota D_x^{\mathbf{m}}\big(\hat{\psi}_i + \psi_{\mathfrak{o}+1}^{(2)}\big)(q^{[i]},T) = \Delta_x^\iota D_x^{\mathbf{m}}\big(\hat{\psi}_i + \psi_{\mathfrak{o}+1}^{(2)}\big)(q^{[i]},T) = 0 \text{ for } \iota \in \mathbb{N}, \mathbf{m} \in \mathbb{N}^5, 2\iota + \|\mathbf{m}\|_{\ell_1} \le 4l_{\max}+4, m_5 \in 2\mathbb{N}.$$

Besides, we have

$$\partial_t^\iota D_x^{\mathbf{m}}\big(\hat{\psi}_i + \psi_{\mathfrak{o}+1}^{(2)}\big)(q^{[i]},T) = 0 \text{ for } \iota \in \mathbb{N}, \mathbf{m} \in \mathbb{N}^5, m_5 \in 2\mathbb{N}+1;$$
$$\big|\partial_t^\iota D_x^{\mathbf{m}}\big(\hat{\psi}_i + \psi_{\mathfrak{o}+1}^{(2)}\big)\big| \lesssim e^{-\frac{9\delta^2}{22T}} \text{ for } \iota \in \mathbb{N}, \mathbf{m} \in \mathbb{N}^5, 2\iota + \|\mathbf{m}\|_{\ell_1} \le 4l_{\max}+4 \text{ in } \overline{B_5^+(q^{[i]},3\delta/2)} \times [0,T].$$

Applying Taylor expansion to $\hat{\psi}_i + \psi_{\mathfrak{o}+1}^{(2)}$ at the point $(q^{[i]},T)$, for $(x,t) \in \overline{B_5^+(q^{[i]},3\delta/2)} \times [0,T]$, we have

$$\big|\big(\hat{\psi}_i + \psi_{\mathfrak{o}+1}^{(2)}\big)(x,t)\big| \lesssim e^{-\frac{9\delta^2}{22T}} \left[(T-t)^{2l_{\max}+2} + |x-q^{[i]}|^{2l_{\max}+2}\right] \lesssim e^{-\frac{9\delta^2}{22T}} (T-t)^{l_i+1} \langle |z^{[i]}|^{2l_i+2}\rangle. \tag{6.31}$$

By (6.15), $|\mathcal{G}_{2,\mathfrak{o}+1}| \le 1$. Recall the equation of $\psi_{\mathfrak{o}+1}^{(1)}$ in (6.27). By the Green's formula (2.17), we have $|\psi_{\mathfrak{o}+1}^{(1)}| \lesssim t^{\frac{1}{2}}$. By (6.30), $|\psi_{\mathfrak{o}+1}^{(2)}| \lesssim e^{-\frac{9\delta^2}{22T}}$. By (6.26), $|\psi_k| \lesssim R^{-\frac{1}{4}}$, $k=1,2,\ldots,\mathfrak{o}$. Hence,

$$\big|\hat{\psi}_i + \psi_{\mathfrak{o}+1}^{(2)}\big| \lesssim R^{-\frac{1}{4}} \text{ in } \overline{\mathbb{R}^5_+} \times (0,T).$$

Combining (6.31), we get

$$\big|\hat{\psi}_i + \psi_{\mathfrak{o}+1}^{(2)}\big| \lesssim e^{-\frac{9\delta^2}{22T}} (T-t)^{l_i+1} \langle |z^{[i]}|^{2l_i+2}\rangle \mathbf{1}_{|x-q^{[i]}|\le 3\delta/2} + R^{-\frac{1}{4}} \mathbf{1}_{|x-q^{[i]}|>3\delta/2}. \tag{6.32}$$

Integrating (6.26), (6.32), and arbitrary choice of $i=1,2,\ldots,\mathfrak{o}$, we complete the proof. □

Lemma 6.7. *Under all assumptions in Lemma 6.1, then for $\mathcal{T}^{\text{out}}[\psi,\boldsymbol{\phi},\boldsymbol{\mu},\boldsymbol{\xi}]$ given in (6.4), there exists a constant $C>0$ independent of T, δ_0 such that*

$$|\mathcal{T}^{\text{out}}[\psi,\boldsymbol{\phi},\boldsymbol{\mu},\boldsymbol{\xi}](x,t)| \le C \max\{R^{-\frac{1}{4}}, \delta_0^{\frac{5}{3}}\} \langle x\rangle^{-\frac{7}{3}} \quad \text{for } |x| \ge 99 \max_{i=1,2,\ldots,\mathfrak{o}} |q^{[i]}|. \tag{6.33}$$

Proof. Recall the equation (6.3) of ψ, and $\mathcal{G}_1=\sum_{i=1}^{\mathfrak{o}}\mathcal{G}_{1,i}$, $\mathcal{G}_2=\sum_{i=1}^{\mathfrak{o}+1}\mathcal{G}_{2,i}$. Denote $r_0=99\max_{i=1,2,\ldots,\mathfrak{o}}|q^{[i]}|$. By Lemma 6.1 and Proposition 6.6,

$$\mathcal{G}_1,\mathcal{T}^{\rm out}[\psi,\boldsymbol{\phi},\boldsymbol{\mu},\boldsymbol{\xi}](x,0)\equiv 0 \ \text{ for } \ x\in\mathbb{R}^5_+\backslash B_5(0,r_0),\quad |\mathcal{G}_2|=|\mathcal{G}_{2,\mathfrak{o}+1}|\le \delta_0^{\frac53}\langle\tilde{x}\rangle^{-\frac{10}{3}} \ \text{ for } \ \tilde{x}\in\mathbb{R}^4\backslash B_4(0,r_0),$$
$$|\mathcal{T}^{\rm out}[\psi,\boldsymbol{\phi},\boldsymbol{\mu},\boldsymbol{\xi}](x,t)|\lesssim R^{-\frac14} \ \text{ for } \ x\in\mathbb{R}^5_+\cap\partial B_5(0,r_0).$$

By Lemma 2.8, set $P(x)=\left|(\tilde{x},x_5+1+\vartheta_1|\tilde{x}|)\right|^{-\frac73}$ with a constant $\vartheta_1>0$ sufficiently small. Then $P(x)$ satisfies $-\partial_{x_5}P\sim\langle x\rangle^{-\frac{10}{3}}$, $\Delta P\lesssim-\langle x\rangle^{-\frac{13}{3}}$. Thus, $C_1\max\{R^{-\frac14},\delta_0^{\frac53}\}P(x)$ with a large constant $C_1>0$ is a barrier function of (6.3) in $\mathbb{R}^5_+\backslash B_5(0,r_0)$. □

7. Completion of the proof of Theorem 1.1

Recall $T_{\sigma_0}=T-\sigma_0,\sigma_0\in(0,T)$. In order to avoid the difficulties of the singularity of the right-hand sides at $t=T$ (see (6.16) for example) and achieve the compactness for the mappings by the regularity theory, we solve the outer problem in $B_5^+(0,\sigma_0^{-1})\times(0,T_{\sigma_0})$, the orthogonal equations in $(0,T_{\sigma_0})$, and the inner problems in $B_5^+(0,2R)\times(0,T_{\sigma_0})$ in order. Then we get a solution u_{σ_0} of (1.16) in $B_5^+(0,\sigma_0^{-1})\times(0,T_{\sigma_0})$. Finally, we take $\sigma_0\downarrow 0$ and use the compactness argument to conclude Theorem 1.1.

7.1. Solving the outer problem in $B_5^+(0,\sigma_0^{-1})\times(0,T_{\sigma_0})$.

Lemma 7.1. *Suppose that $\delta_0=R^{-\frac15}$, $\sigma_0\in(0,T)$, $|\mu_{i,1}|\le(9C_{\mu_{i,0}})^{-1}(T-t)^{2l_i+2}$, $|\dot{\mu}_{i,1}|\le(9C_{\mu_{i,0}})^{-1}(T-t)^{2l_i+1}$ in $(0,T_{\sigma_0})$ for $i=1,2,\ldots,\mathfrak{o}$, $\boldsymbol{\xi},\dot{\boldsymbol{\xi}}$ satisfy (5.5) in $(0,T_{\sigma_0})$, $\boldsymbol{\phi}\in B_{{\rm in},\sigma_0}$, then for $T\ll1$, there exists $\psi_{\sigma_0}=\psi_{\sigma_0}[\boldsymbol{\phi},\boldsymbol{\mu}_{,1},\boldsymbol{\xi}]\in\mathcal{X}_{\delta_0,\sigma_0}$ solving the outer problem (3.6) in $B_5^+(0,\sigma_0^{-1})\times(0,T_{\sigma_0})$ with an initial value $\psi_{\sigma_0}(x,0)\in C_c^\infty(\overline{\mathbb{R}^5_+})$ satisfying*

$$\psi_{\sigma_0}(x,0)=\sum_{i=1}^{\mathfrak{o}}\boldsymbol{b}_{i,\sigma_0}\cdot\tilde{\boldsymbol{e}}_i\big(T^{-\frac12}(x-q^{[i]})\big)+\sum_{i=1}^{\mathfrak{o}}\sum_{\mathbf{p}\in\mathbb{N}^5,\|\mathbf{p}\|_{\ell_1}\le 4l_{\max}+4,p_5\in2\mathbb{N}}C_{q^{[i]},\mathbf{p},\sigma_0}\varphi_{q^{[i]},\mathbf{p},0}(x),$$

where $\boldsymbol{b}_{i,\sigma_0}=\boldsymbol{b}_{i,\sigma_0}[\psi_{\sigma_0},\boldsymbol{\phi},\boldsymbol{\mu}_{,1},\boldsymbol{\xi}]$ are constant vectors and $|\boldsymbol{b}_{i,\sigma_0}|\le C|\ln T|T^{\frac53l_i+\frac12}$ with a constant $C>0$ independent of T,σ_0; $\tilde{\boldsymbol{e}}_i$ are given in (6.2); $C_{q^{[i]},\mathbf{p},\sigma_0}=C_{q^{[i]},\mathbf{p},\sigma_0}[\psi_{\sigma_0},\boldsymbol{\phi},\boldsymbol{\mu}_{,1},\boldsymbol{\xi}]$ are constants and $|C_{q^{[i]},\mathbf{p},\sigma_0}|\le Ce^{-\frac{9\delta^2}{22T}}$; $\varphi_{q^{[i]},\mathbf{p},0}(x)\in C_c^\infty(\overline{\mathbb{R}^5_+})$ and $\varphi_{q^{[i]},\mathbf{p},0}(x)=0$ in $\mathbb{R}^5_+\backslash B_5^+(q^{[i]},2\delta)$.

For any compact set $K\subset\overline{\mathbb{R}^5_+}\times[0,T)$, there exists $\sigma_K\in(0,T)$ sufficiently small such that for all $\sigma_0\in(0,\sigma_K)$, ψ_{σ_0} are well-defined and uniformly Hölder continuous in K.

Moreover, there exists a universal constant $\varsigma_0\in(0,1)$ independent of any parameters such that if $\varsigma_1\in(0,\varsigma_0]$, there exists a constant $C_1>0$ independent of T,σ_0 such that for all

$$t_*\in(0,T_{\sigma_0}),\quad x_*\in\overline{B_5^+(q^{[j]},R^{-1}(T-t_*)^{1/2})},\quad j=1,2,\ldots,\mathfrak{o},\quad \rho\in(0,R^{-1}(T-t_*)^{1/2}],$$

we have

$$[\psi_{\sigma_0}]_{C^{\varsigma_1,\varsigma_1/2}\left(\overline{B_5^+(x_*,\rho)}\times\left(\max\{0,t_*-\rho^2\},t_*\right]\right)}\le C_1\rho^{-\varsigma_1}\left[\delta_0(T-t_*)^{l_j}+\rho^2R^{-\frac14}(T-t_*)^{-3l_j-4}\right].\tag{7.1}$$

Remark 7.1.1. *Recall $\mathcal{H}_{2,i}$ given in (3.5). The quantitative Hölder estimate of ψ_{σ_0} is required for the application of Proposition 1.2 with $n=5$ in solving the inner problems.*

Proof. Given a function f defined in a domain of $\mathbb{R}^5_+\times(0,T)$, $(0,T)$ or $B_5^+(0,2R)\times(0,T)$, denote $f_{\star1}$, $f_{\star2}$ or $f_{\star3}$ as the zero extension of f in $\mathbb{R}^5_+\times(0,T)$, $(0,T)$ or $B_5^+(0,2R)\times(0,T)$ respectively. Denote $\xi^{[i]}_{\star\tilde{q}^{[i]}}(t)=\xi^{[i]}(t)\mathbf{1}_{t\in(0,T_{\sigma_0})}+\tilde{q}^{[i]}\mathbf{1}_{t\in[T_{\sigma_0},T)}$ and $\boldsymbol{\xi}_{\star\tilde{\boldsymbol{q}}}=(\xi^{[1]}_{\star\tilde{q}^{[1]}},\xi^{[2]}_{\star\tilde{q}^{[2]}},\ldots,\xi^{[\mathfrak{o}]}_{\star\tilde{q}^{[\mathfrak{o}]}})$. Denote $\boldsymbol{A}=\big(\boldsymbol{\mu}_{,0}+(\boldsymbol{\mu}_{,1})_{\star2},\,\dot{\boldsymbol{\mu}}_{,0}+(\dot{\boldsymbol{\mu}}_{,1})_{\star2},\,\boldsymbol{\xi}_{\star\tilde{\boldsymbol{q}}},\,\dot{\boldsymbol{\xi}}_{*2}\big)$. It follows that $\boldsymbol{A}$ satisfies (5.5) in $(0,T)$.

By Remark 6.0.1, we make a distinction between $\boldsymbol{\mu}(\boldsymbol{\xi})$ and $\dot{\boldsymbol{\mu}}(\dot{\boldsymbol{\xi}})$ in Proposition 6.6 and Lemma 6.7. For $g\in\mathcal{X}_{\delta_0,\sigma_0}$, we denote $\mathcal{T}^{\rm out}[g_{\star1}]=\mathcal{T}^{\rm out}[g_{\star1},\boldsymbol{\phi}_{\star3},\boldsymbol{A}]$ for brevity in this proof and the following properties hold

$$|\mathcal{T}^{\rm out}[g_{\star1}](x,t)|\lesssim\max\{R^{-\frac14},\delta_0^{\frac53}\}\langle x\rangle^{-\frac73}\mathbf{1}_{\cap_{i=1}^{\mathfrak{o}}\left\{|z^{[i]}|>e^{\frac{l_is}{2l_i+2}}\right\}}+R^{-\frac14}\sum_{i=1}^{\mathfrak{o}}(T-t)^{l_i}\langle|z^{[i]}|^{2l_i+2}\rangle\mathbf{1}_{|z^{[i]}|\le e^{\frac{l_is}{2l_i+2}}},$$

$$|D_x^{\mathbf{m}}\mathcal{T}^{\rm out}[g_{\star1}](x,0)|\lesssim_{\mathbf{m}}|\ln T|T^{\frac53l_i+\frac12-\frac{\|\mathbf{m}\|_{\ell_1}}{2}}\sum_{i=1}^{\mathfrak{o}}\mathbf{1}_{|x-q^{[i]}|\le2\delta}\quad\text{for }\ \mathbf{m}\in\mathbb{N}^5,$$

$$\mathcal{T}^{\rm out}[g_{\star1}](x,0)=\sum_{i=1}^{\mathfrak{o}}\boldsymbol{b}_i\cdot\tilde{\boldsymbol{e}}_i\big(T^{-\frac12}(x-q^{[i]})\big)+\sum_{i=1}^{\mathfrak{o}}\sum_{\mathbf{p}\in\mathbb{N}^5,\|\mathbf{p}\|_{\ell_1}\le4l_{\max}+4,p_5\in2\mathbb{N}}C_{q^{[i]},\mathbf{p}}\varphi_{q^{[i]},\mathbf{p},0}(x),$$

where $\boldsymbol{b}_i = \boldsymbol{b}_i[g_{\star 1}, \boldsymbol{\phi}_{\star 3}, \boldsymbol{A}]$ are constant vectors and $C_{q^{[i]},\mathbf{p}} = C_{q^{[i]},\mathbf{p}}[g_{\star 1}, \boldsymbol{\phi}_{\star 3}, \boldsymbol{A}]$ are constants, which satisfy $|\boldsymbol{b}_i| \lesssim |\ln T| T^{\frac{5}{3}l_i+\frac{1}{2}}$ and $|C_{q^{[i]},\mathbf{p}}| \lesssim e^{-\frac{9\delta^2}{22T}}$. For $\delta_0 = R^{-\frac{1}{5}}$ and $T \ll 1$, then $\mathcal{T}^{\mathrm{out}}[g_{\star 1}] \in \mathcal{X}_{\delta_0,0} \cap \mathcal{X}_{\delta_0,\sigma_0}$.

By $\mathcal{G}_1 = \sum_{i=1}^{\mathfrak{o}} \mathcal{G}_{1,i}, \mathcal{G}_2 = \sum_{i=1}^{\mathfrak{o}+1} \mathcal{G}_{2,i}$, Lemma 6.1, $\mathcal{G}_1, \mathcal{G}_2$ are uniformly bounded for $t \in (0, T_{\sigma_0})$. One combining the uniform boundedness of $|D_x^{\mathbf{m}} \mathcal{T}^{\mathrm{out}}[g_{\star 1}](x, 0)|$ with $\mathbf{m} \in \mathbb{N}^5$, $\|\mathbf{m}\|_{\ell_1} \le 1$, by the parabolic regularity theory, $\{\mathcal{T}^{\mathrm{out}}[g_{\star 1}] \mid g \in \mathcal{X}_{\delta_0,\sigma_0}\}$ is uniformly Hölder continuous in $\overline{B_5^+(0, \sigma_0^{-1})} \times [0, T_{\sigma_0})$. Hence, $g \mapsto \mathcal{T}^{\mathrm{out}}[g_{\star 1}]$ is a compact operator from $\mathcal{X}_{\delta_0,\sigma_0}$ to $\mathcal{X}_{\delta_0,\sigma_0}$.The Schauder fixed-point theorem then yields the existence of a fixed point $\psi_{\sigma_0} = \mathcal{T}^{\mathrm{out}}[\psi_{\sigma_0,\star 1}]$ in $\overline{B_5^+(0, \sigma_0^{-1})} \times (0, T_{\sigma_0})$ and $\psi_{\sigma_0} \in \mathcal{X}_{\delta_0,\sigma_0}$, which implies that ψ_{σ_0} solves (3.6) in $B_5^+(0, \sigma_0^{-1}) \times (0, T_{\sigma_0})$. Set $\boldsymbol{b}_{i,\sigma_0} := \boldsymbol{b}_i[\psi_{\sigma_0,\star 1}, \boldsymbol{\phi}_{\star 3}, \boldsymbol{A}]$ and $C_{q^{[i]},\mathbf{p},\sigma_0} := C_{q^{[i]},\mathbf{p}}[\psi_{\sigma_0,\star 1}, \boldsymbol{\phi}_{\star 3}, \boldsymbol{A}]$. Then the initial value has the desired form.

The uniform Hölder continuity of ψ_{σ_0} in compact sets of $\overline{\mathbb{R}^5_+} \times [0, T)$ for σ_0 sufficiently small is straightforward by the parabolic estimate.

Finally, we will give the quantitative Hölder estimate of ψ_{σ_0} around the blow-up points. We always assume $T \ll 1$ to make $B_5^+(0, \sigma_0^{-1})$ sufficiently large to make the following estimates well-defined. By $\mathcal{G}_1 = \sum_{i=1}^{\mathfrak{o}} \mathcal{G}_{1,i}, \mathcal{G}_2 = \sum_{i=1}^{\mathfrak{o}+1} \mathcal{G}_{2,i}$, Lemma 6.1, and $\mathcal{T}^{\mathrm{out}}[\psi_{\sigma_0,\star 1}] \in \mathcal{X}_{\delta_0,0}$, for $j = 1, 2, \ldots, \mathfrak{o}$, we have

$$
\begin{aligned}
&|\mathcal{G}_1| \lesssim R^{-\frac{1}{4}}(T-t)^{-3l_j-4} && \text{in } Q_{\mathcal{G}_{1,j}} := \left\{(x,t) \mid t \in (0,T), x \in B_5^+(q^{[j]}, (T-t)^{1/2})\right\},\\
&|\mathcal{G}_2| \lesssim R^{-\frac{1}{4}}(T-t)^{-l_j-2} && \text{in } Q_{\mathcal{G}_{2,j}} := \left\{(\tilde{x},t) \mid t \in (0,T), \tilde{x} \in B_4(\tilde{q}^{[j]}, (T-t)^{1/2})\right\},\\
&|\mathcal{T}^{\mathrm{out}}[\psi_{\sigma_0,\star 1}]| \lesssim \delta_0 (T-t)^{l_j} && \text{in } Q_{\mathcal{G}_{1,j}}.
\end{aligned}
$$

Recall the definition of $\mathcal{T}^{\mathrm{out}}[\psi_{\sigma_0,\star 1}]$ in (6.4). Given

$$
t_* \in (0,T), \quad x_* \in \overline{B_5^+(q^{[j]}, R^{-1}(T-t_*)^{1/2})}, \quad \rho \in (0, R^{-1}(T-t_*)^{1/2}],
$$

notice that for any $t_1 \in \left(\max\{0, t_* - 4\rho^2\}, t_*\right]$, by $R^{-1} \ll 1$, we have $T - t_1 \in [T - t_*, 2(T-t_*)]$. Then for any $w \in B_5^+(x_*, 2\rho)$, it holds that $|w - q^{[j]}| \le |w - x_*| + |x_* - q^{[j]}| \le 3R^{-1}(T-t_*)^{1/2} \le (T-t_1)^{1/2}$, which implies $B_5^+(x_*, 2\rho) \times \left(\max\{0, t_* - 4\rho^2\}, t_*\right] \subset Q_{\mathcal{G}_{1,j}}$. Similarly, $B_4(\tilde{x}_*, 2\rho) \times \left(\max\{0, t_* - 4\rho^2\}, t_*\right] \subset Q_{\mathcal{G}_{2,j}}$ holds. Hence, for any $(w, t_1) \in B_5^+(x_*, 2\rho) \times \left(\max\{0, t_* - 4\rho^2\}, t_*\right]$, we have $|\mathcal{T}^{\mathrm{out}}[\psi_{\sigma_0,\star 1}](w, t_1)| \lesssim \delta_0 (T-t_1)^{l_j} \sim \delta_0 (T-t_*)^{l_j}$, that is,

$$
\|\mathcal{T}^{\mathrm{out}}[\psi_{\sigma_0,\star 1}]\|_{L^\infty\left(B_5^+(x_*,2\rho)\times\left(\max\{0,t_*-4\rho^2\},t_*\right]\right)} \lesssim \delta_0 (T-t_*)^{l_j}.
$$

Similarly, we can deduce that

$$
\begin{aligned}
&\rho^2 \|\mathcal{G}_1\|_{L^\infty\left(B_5^+(x_*,2\rho)\times\left(\max\{0,t_*-4\rho^2\},t_*\right]\right)} \lesssim \rho^2 R^{-\frac{1}{4}} (T-t_*)^{-3l_j-4},\\
&\rho \|\mathcal{G}_2\|_{L^\infty\left(B_4(\tilde{x}_*,2\rho)\times\left(\max\{0,t_*-4\rho^2\},t_*\right]\right)} \lesssim \rho R^{-\frac{1}{4}} (T-t_*)^{-l_j-2}.
\end{aligned}
$$

Since for any $w \in B_5^+(x_*, 2\rho)$, $|w - q^{[j]}| \le 3R^{-1}(T-t_*)^{1/2}$, then for any $\alpha \in (0,1)$,

$$
\begin{aligned}
&\|\mathcal{T}^{\mathrm{out}}[\psi_{\sigma_0,\star 1}](\cdot, 0)\|_{L^\infty(B_5^+(x_*,2\rho))} \lesssim |\ln T| T^{\frac{5}{3}l_j+\frac{1}{2}},\\
&\rho^\alpha [\mathcal{T}^{\mathrm{out}}[\psi_{\sigma_0,\star 1}](\cdot, 0)]_{C^\alpha(B_5^+(x_*,2\rho))}\\
= {}& \rho^\alpha \Big[\boldsymbol{b}_{j,\sigma_0} \cdot \tilde{\boldsymbol{e}}_j\big(T^{-\frac{1}{2}}(x - q^{[j]})\big) + \sum_{\mathbf{p}\in\mathbb{N}^5, \|\mathbf{p}\|_{\ell_1} \le 4l_{\max}+4, p_5 \in 2\mathbb{N}} C_{q^{[j]},\mathbf{p},\sigma_0} \varphi_{q^{[j]},\mathbf{p},0}(x)\Big]_{C^\alpha(B_5^+(x_*,2\rho))}\\
\lesssim {}& \rho^\alpha \big(|\ln T| T^{\frac{5}{3}l_j+\frac{1}{2}} T^{-\frac{1}{2}} \rho^{1-\alpha} + e^{-\frac{9\delta^2}{22T}} \rho^{1-\alpha}\big) \sim \rho |\ln T| T^{\frac{5}{3}l_j} \le R^{-1} |\ln T| T^{\frac{5}{3}l_j+\frac{1}{2}}.
\end{aligned}
$$

By Lemma 2.15, there exist positive universal constants $\varsigma_0 \in (0,1)$, K_1 independent of any parameters such that if $\varsigma_1 \in (0, \varsigma_0]$, we have

$$
\begin{aligned}
&\left[\mathcal{T}^{\mathrm{out}}[\psi_{\sigma_0,\star 1}]\right]_{C^{\varsigma_1,\varsigma_1/2}\left(\overline{B_5^+(x_*,\rho)}\times\left(\max\{0,t_*-\rho^2\},t_*\right]\right)}\\
\le {}& K_1 \rho^{-\varsigma_1} \Big[\delta_0 (T-t_*)^{l_j} + \rho^2 R^{-\frac{1}{4}} (T-t_*)^{-3l_j-4} + \mathbf{1}_{x_{*5} \le 4\rho}\, \rho R^{-\frac{1}{4}} (T-t_*)^{-l_j-2} + \mathbf{1}_{\sqrt{t_*} \le 4\rho}\, |\ln T| T^{\frac{5}{3}l_j+\frac{1}{2}}\Big].
\end{aligned}
$$

Therein, by the inequality $c_1 + c_2 \ge 2\sqrt{c_1 c_2}$ for $c_1, c_2 \ge 0$, it holds that

$$
\delta_0 (T-t_*)^{l_j} + \rho^2 R^{-\frac{1}{4}} (T-t_*)^{-3l_j-4} \ge 2(\delta_0 R^{-\frac{1}{4}})^{\frac{1}{2}} \rho (T-t_*)^{-l_j-2} \ge \rho R^{-\frac{1}{4}} (T-t_*)^{-l_j-2}.
$$

Besides, $\sqrt{t_*} \le 4\rho$, $\rho \in (0, R^{-1}(T-t_*)^{1/2}]$ deduce $t_* \le 16R^{-2}T$. Combining $R^{-1} \ll 1$, then $\mathbf{1}_{\sqrt{t_*} \le 4\rho}\, |\ln T| T^{\frac{5}{3}l_j+\frac{1}{2}} \lesssim \delta_0 (T-t_*)^{l_j}$. Thus, we get

$$
\left[\mathcal{T}^{\mathrm{out}}[\psi_{\sigma_0,\star 1}]\right]_{C^{\varsigma_1,\varsigma_1/2}\left(\overline{B_5^+(x_*,\rho)}\times\left(\max\{0,t_*-\rho^2\},t_*\right]\right)} \lesssim \rho^{-\varsigma_1} \left[\delta_0 (T-t_*)^{l_j} + \rho^2 R^{-\frac{1}{4}} (T-t_*)^{-3l_j-4}\right].
$$

Using $\psi_{\sigma_0} = \mathcal{T}^{\rm out}[\psi_{\sigma_0,\star 1}]$ in $B_5^+(0,\sigma_0^{-1})\times[0,T_{\sigma_0})$, we finally achieve (7.1). $\square$

7.2. **Solving the orthogonal equations in** $(0,T_{\sigma_0})$**.** In order to apply Proposition 1.2 with $n=5$ to solve the inner problems (3.4) in $B_5^+(0,2R)\times(0,T_{\sigma_0})$, we need to choose suitable $\boldsymbol{\mu},\boldsymbol{\xi}$ to attain the orthogonality conditions (1.20). In other words, we need to solve the following formal orthogonal equations.

$$
\begin{aligned}
&\int_{\mathbb{R}^n_+}\eta\Big(\frac{y}{4R}\Big)\Big(\dot\mu_i\mu_i Z_n(y)+\mu_i\dot\xi^{[i]}\cdot(\nabla_{\tilde y}U)(\tilde y,y_n)\Big)Z_j(y)dy\\
&+\int_{\mathbb{R}^{n-1}}\frac{n}{n-2}\mu_i^{\frac n2-1}U^{\frac{2}{n-2}}(\tilde y,0)\eta\Big(\frac{\tilde y}{4R},0\Big)\Big(\psi_{\sigma_0}\big((\mu_i\tilde y+\xi^{[i]},0),t\big)+\Theta_{l_i}\big((\mu_i\tilde y+\xi^{[i]}-\tilde q^{[i]},0),t\big)\Big)Z_j(\tilde y,0)d\tilde y=0
\end{aligned}\tag{7.2}
$$

for $j=1,2,\dots,n$, $t\in(0,T_{\sigma_0})$, where $\psi_{\sigma_0}=\psi_{\sigma_0}[\boldsymbol{\phi},\boldsymbol{\mu}_{,1},\boldsymbol{\xi}]\in\mathcal{X}_{\delta_0,\sigma_0}$ is given by Lemma 7.1 with $\boldsymbol{\mu}_{,1},\boldsymbol{\xi}$ in a suitable space to meet the assumption in Lemma 7.1. Here we used y instead of $y^{[i]}$ as the integral variable.

Lemma 7.2. *Suppose that $\delta_0=R^{-\frac15}$, $\sigma_0\in(0,T)$, $\boldsymbol{\phi}\in B_{\mathrm{in},\sigma_0}$, then for $T\ll 1$, there exist $\boldsymbol{\mu}_{,1,\sigma_0}=\boldsymbol{\mu}_{,1,\sigma_0}[\boldsymbol{\phi}]$, $\boldsymbol{\xi}_{\sigma_0}=\boldsymbol{\xi}_{\sigma_0}[\boldsymbol{\phi}]$ solving (7.2) with $n=5$ in $(0,T_{\sigma_0})$ and satisfying*

$$
|\mu_{i,1,\sigma_0}|+|\xi^{[i]}_{\sigma_0}-\tilde q^{[i]}|\le\delta_0^{\frac13}(T-t)^{2l_i+2},\quad |\dot\mu_{i,1,\sigma_0}|+|\dot\xi^{[i]}_{\sigma_0}|\le\delta_0^{\frac13}(T-t)^{2l_i+1}\tag{7.3}
$$

for $t\in[0,T_{\sigma_0})$, $i=1,2,\dots,\mathfrak{o}$. In particular, the ansatz (5.5) holds in $[0,T_{\sigma_0})$.

Moreover, for any compact set $K\subset[0,T)$, there exists $\sigma_K\in(0,T)$ sufficiently small such that for all $\sigma_0\in(0,\sigma_K)$, $\dot\mu_{i,1,\sigma_0}$ ($\dot\xi^{[i]}_{\sigma_0}$) are well-defined and uniformly Hölder continuous in K.

Proof. By the radial property of $\eta(x)$ and the parity of $U(y)$ and $Z_j(y)$ given in (1.13), (2.1) respectively, we have

$$
\int_{\mathbb{R}^n_+}\eta(\frac{y}{4R})Z_j(y)Z_k(y)dy=0\ \text{ for }\ j,k=1,2,\dots,n, j\ne k;\ \int_{\mathbb{R}^{n-1}}U^{\frac{2}{n-2}}(\tilde y,0)\eta(\frac{\tilde y}{4R},0)Z_j(\tilde y,0)d\tilde y=0\ \text{ for }\ j=1,2,\dots,n-1.
$$

Hence, (7.2) is equivalent to

$$
\begin{aligned}
\dot\mu_i=&-\frac{n}{n-2}\Big(\int_{\mathbb{R}^n_+}Z_n^2(y)\eta\Big(\frac{y}{4R}\Big)dy\Big)^{-1}\mu_i^{\frac{n-4}{2}}\times\\
&\int_{\mathbb{R}^{n-1}}U^{\frac{2}{n-2}}(\tilde y,0)\eta\Big(\frac{\tilde y}{4R},0\Big)\Big[\psi_{\sigma_0}\big((\mu_i\tilde y+\xi^{[i]},0),t\big)+\Theta_{l_i}\big((\mu_i\tilde y+\xi^{[i]}-\tilde q^{[i]},0),t\big)\Big]Z_n(\tilde y,0)d\tilde y,
\end{aligned}\tag{7.4}
$$

$$
\dot\xi^{[i]}=\mathcal{S}^{[i]}[\boldsymbol{\mu}_{,1},\boldsymbol{\xi}]:=\Big(\mathcal{S}^{[i]}_1[\boldsymbol{\mu}_{,1},\boldsymbol{\xi}],\mathcal{S}^{[i]}_2[\boldsymbol{\mu}_{,1},\boldsymbol{\xi}],\dots,\mathcal{S}^{[i]}_{n-1}[\boldsymbol{\mu}_{,1},\boldsymbol{\xi}]\Big)\tag{7.5}
$$

for $i=1,2,\dots,\mathfrak{o}$, where for $j=1,2,\dots,n-1$,

$$
\begin{aligned}
\mathcal{S}^{[i]}_j[\boldsymbol{\mu}_{,1},\boldsymbol{\xi}]:=&-\frac{n}{n-2}\Big(\int_{\mathbb{R}^n_+}Z_j^2(y)\eta\Big(\frac{y}{4R}\Big)dy\Big)^{-1}\mu_i^{\frac{n-4}{2}}\int_{\mathbb{R}^{n-1}}U^{\frac{2}{n-2}}(\tilde y,0)\eta\Big(\frac{\tilde y}{4R},0\Big)\Big[\psi_{\sigma_0}\big((\mu_i\tilde y+\xi^{[i]},0),t\big)\\
&+\Theta_{l_i}\big((\mu_i\tilde y+\xi^{[i]}-\tilde q^{[i]},0),t\big)-\Theta_{l_i}(0,t)\Big]Z_j(\tilde y,0)d\tilde y.
\end{aligned}\tag{7.6}
$$

Recalling (5.1) and $\Theta_{l_i}(0,t)=-(T-t)^{l_i}$, we have

$$
\dot\mu_{i,0}=-\frac{n}{n-2}\Big(\int_{\mathbb{R}^n_+}Z_n^2(y)\eta\Big(\frac{y}{4R}\Big)dy\Big)^{-1}\Theta_{l_i}(0,t)\mu_{i,0}^{\frac{n-4}{2}}\int_{\mathbb{R}^{n-1}}U^{\frac{2}{n-2}}(\tilde y,0)\eta\Big(\frac{\tilde y}{4R},0\Big)Z_n(\tilde y,0)d\tilde y.\tag{7.7}
$$

(7.4) minus (7.7) implies

$$
\dot\mu_{i,1}=\mathcal{F}_i[\boldsymbol{\mu}_{,1},\boldsymbol{\xi}](t)-\frac{n}{n-2}\Big(\int_{\mathbb{R}^n_+}Z_n^2(y)\eta\Big(\frac{y}{4R}\Big)dy\Big)^{-1}\Theta_{l_i}(0,t)\frac{n-4}{2}\mu_{i,0}^{\frac{n-6}{2}}\mu_{i,1}\int_{\mathbb{R}^{n-1}}U^{\frac{2}{n-2}}(\tilde y,0)\eta\Big(\frac{\tilde y}{4R},0\Big)Z_n(\tilde y,0)d\tilde y,\tag{7.8}
$$

where

$$
\begin{aligned}
\mathcal{F}_i[\boldsymbol{\mu}_{,1},\boldsymbol{\xi}](t):=&\frac{-n}{n-2}\Big(\int_{\mathbb{R}^n_+}Z_n^2(y)\eta\Big(\frac{y}{4R}\Big)dy\Big)^{-1}\Big\{\mu_i^{\frac{n-4}{2}}\int_{\mathbb{R}^{n-1}}U^{\frac{2}{n-2}}(\tilde y,0)\eta\Big(\frac{\tilde y}{4R},0\Big)\psi_{\sigma_0}\big((\mu_i\tilde y+\xi^{[i]},0),t\big)Z_n(\tilde y,0)d\tilde y\\
&+\mu_i^{\frac{n-4}{2}}\int_{\mathbb{R}^{n-1}}U^{\frac{2}{n-2}}(\tilde y,0)\eta\Big(\frac{\tilde y}{4R},0\Big)\big[\Theta_{l_i}\big((\mu_i\tilde y+\xi^{[i]}-\tilde q^{[i]},0),t\big)-\Theta_{l_i}(0,t)\big]Z_n(\tilde y,0)d\tilde y\\
&+\Theta_{l_i}(0,t)\Big(\mu_i^{\frac{n-4}{2}}-\mu_{i,0}^{\frac{n-4}{2}}-\frac{n-4}{2}\mu_{i,0}^{\frac{n-6}{2}}\mu_{i,1}\Big)\int_{\mathbb{R}^{n-1}}U^{\frac{2}{n-2}}(\tilde y,0)\eta\Big(\frac{\tilde y}{4R},0\Big)Z_n(\tilde y,0)d\tilde y\Big\}.
\end{aligned}\tag{7.9}
$$

By (5.2), (5.3), then (7.8) is equivalent to

$$\dot{\mu}_{i,1} + \beta(t)\mu_{i,1} = \mathcal{F}_i[\boldsymbol{\mu}_{,1}, \boldsymbol{\xi}](t) \qquad \text{with} \qquad \beta(t) := \frac{n-4}{6-n}(l_i+1)(T-t)^{-1}. \tag{7.10}$$

To obtain a solution of (7.5), (7.10), it suffices to solve the following fixed-point problem about $\dot{\boldsymbol{\mu}}_{,1}, \dot{\boldsymbol{\xi}}$.

$$\begin{aligned}
&(\dot{\boldsymbol{\mu}}_{,1}, \dot{\boldsymbol{\xi}}) = \mathcal{T}^{\mathrm{ort}}[\boldsymbol{\mu}_{,1}, \boldsymbol{\xi}] := \big((\mathcal{S}_1[\boldsymbol{\mu}_{,1}, \boldsymbol{\xi}], \mathcal{S}_2[\boldsymbol{\mu}_{,1}, \boldsymbol{\xi}], \dots, \mathcal{S}_{\mathfrak{o}}[\boldsymbol{\mu}_{,1}, \boldsymbol{\xi}]), (\mathcal{S}^{[1]}[\boldsymbol{\mu}_{,1}, \boldsymbol{\xi}], \mathcal{S}^{[2]}[\boldsymbol{\mu}_{,1}, \boldsymbol{\xi}], \dots, \mathcal{S}^{[\mathfrak{o}]}[\boldsymbol{\mu}_{,1}, \boldsymbol{\xi}])\big), \\
&\mathcal{S}_i[\boldsymbol{\mu}_{,1}, \boldsymbol{\xi}] := \frac{d}{dt}\Big(\int_{T_{\sigma_0}}^t e^{\int_t^s \beta(a)da}\mathcal{F}_i[\boldsymbol{\mu}_{,1}, \boldsymbol{\xi}](s)ds\Big) = -\beta(t)\int_{T_{\sigma_0}}^t e^{\int_t^s \beta(a)da}\mathcal{F}_i[\boldsymbol{\mu}_{,1}, \boldsymbol{\xi}](s)ds + \mathcal{F}_i[\boldsymbol{\mu}_{,1}, \boldsymbol{\xi}](t), \\
&\mu_{i,1} = \mu_{i,1}[\dot{\mu}_{i,1}](t) = \int_{T_{\sigma_0}}^t \dot{\mu}_{i,1}(a)da, \quad \xi^{[i]} = \xi^{[i]}[\dot{\xi}^{[i]}](t) = \int_{T_{\sigma_0}}^t \dot{\xi}^{[i]}(a)da + \tilde{q}^{[i]}, \quad i = 1, 2, \dots, \mathfrak{o}.
\end{aligned} \tag{7.11}$$

We will solve $\dot{\boldsymbol{\mu}}_{,1}, \dot{\boldsymbol{\xi}}$ in suitable space to make the above integrals and $\psi_{\sigma_0} = \psi_{\sigma_0}[\boldsymbol{\phi}, \boldsymbol{\mu}_{,1}, \boldsymbol{\xi}] \in \mathcal{X}_{\delta_0,\sigma_0}$ given by Lemma 7.1 well-defined. Set the norm

$$\|f\|_{l_i,\sigma_0} := \sup_{s\in(0,T_{\sigma_0})} (T-s)^{-(2l_i+1)}|f(s)|.$$

We will find a solution $(\dot{\boldsymbol{\mu}}_{,1}, \dot{\boldsymbol{\xi}})$ of (7.11) in the space $B_{*1,\sigma_0} \times B_{*2,\sigma_0}$, where

$$\begin{aligned}
B_{*1,\sigma_0} &:= \Big\{(f_1, f_2, \dots, f_{\mathfrak{o}}) \mid f_i \in C\big((0, T_{\sigma_0}), \mathbb{R}\big), i = 1, 2, \dots, \mathfrak{o}, \max_{i=1,2,\dots,\mathfrak{o}} \|f_i\|_{l_i,\sigma_0} \le \delta_0^{\frac{1}{2}}\Big\}, \\
B_{*2,\sigma_0} &:= \Big\{(\mathbf{f}^{[1]}, \mathbf{f}^{[2]}, \dots, \mathbf{f}^{[\mathfrak{o}]}) \mid \mathbf{f}^{[i]} \in C\big((0, T_{\sigma_0}), \mathbb{R}^{n-1}\big), i = 1, 2, \dots, \mathfrak{o}, \max_{i=1,2,\dots,\mathfrak{o}} \|\mathbf{f}^{[i]}\|_{l_i,\sigma_0} \le \delta_0^{\frac{1}{2}}\Big\}.
\end{aligned}$$

Hereafter, we plug in $n = 5$. Given any $(\dot{\boldsymbol{\mu}}_1, \dot{\boldsymbol{\xi}}) \in B_{*1,\sigma_0} \times B_{*2,\sigma_0}$, for $t \in (0, T_{\sigma_0})$, we have

$$|\mu_{i,1}| \le \int_t^{T_{\sigma_0}} |\dot{\mu}_{i,1}(a)|da \lesssim \delta_0^{\frac{1}{2}}(T-t)^{2l_i+2}, \quad |\xi^{[i]} - \tilde{q}^{[i]}| \le \int_t^{T_{\sigma_0}} |\dot{\xi}^{[i]}(a)|da \lesssim \delta_0^{\frac{1}{2}}(T-t)^{2l_i+2}. \tag{7.12}$$

In particular, the integrals about $\mu_{i,1}, \xi^{[i]}$ in (7.11) are well-defined. Due to the small quantity $\delta_0^{\frac{1}{2}}$ and $T \ll 1$, $\boldsymbol{\mu}, \dot{\boldsymbol{\mu}}, \boldsymbol{\xi}, \dot{\boldsymbol{\xi}}$ satisfy (5.5) in $(0, T_{\sigma_0})$ and the assumption in Lemma 7.1 holds, which implies that $\psi_{\sigma_0} = \psi_{\sigma_0}[\boldsymbol{\phi}, \boldsymbol{\mu}_{,1}, \boldsymbol{\xi}] \in \mathcal{X}_{\delta_0,\sigma_0}$ given by Lemma 7.1 is well-defined. For $|\tilde{y}| \le 8R$, $t \in (0, T_{\sigma_0})$,

$$|(\mu_i\tilde{y} + \xi^{[i]}, 0) - q^{[i]}| \le |\mu_i\tilde{y}| + |\xi^{[i]} - \tilde{q}^{[i]}| \lesssim R(T-t)^{2l_i+2} \ll (T-t)^{\frac{1}{2l_i+2}}.$$

Combining $\psi_{\sigma_0} \in \mathcal{X}_{\delta_0,\sigma_0}$ and Θ_{l_i} given in (1.14) with $n = 5$, we have

$$|\psi_{\sigma_0}\big((\mu_i\tilde{y} + \xi^{[i]}, 0), t\big)| \le \delta_0(T-t)^{l_i}\big\langle(T-t)^{-\frac{1}{2}}|\mu_i\tilde{y} + \xi^{[i]} - \tilde{q}^{[i]}|\big\rangle^{2l_i+2} \sim \delta_0(T-t)^{l_i}, \tag{7.13}$$

$$|\Theta_{l_i}\big((\mu_i\tilde{y} + \xi^{[i]} - \tilde{q}^{[i]}, 0), t\big) - \Theta_{l_i}(0, t)| \lesssim (T-t)^{l_i}\sum_{j=1}^{l_i}\Big[(T-t)^{-1}|\mu_i\tilde{y} + \xi^{[i]} - \tilde{q}^{[i]}|^2\Big]^j \lesssim R^2(T-t)^{5l_i+3}. \tag{7.14}$$

It then follows from (5.5), (7.13) and (7.14) that $\mathcal{S}_j^{[i]}[\boldsymbol{\mu}_{,1}, \boldsymbol{\xi}]$ given in (7.6) has the upper bound

$$|\mathcal{S}_j^{[i]}[\boldsymbol{\mu}_{,1}, \boldsymbol{\xi}]| \lesssim \delta_0(T-t)^{2l_i+1}. \tag{7.15}$$

By (5.4), (7.12), and $\delta_0 \ll 1$,

$$|\mu_i^{\frac{1}{2}} - \mu_{i,0}^{\frac{1}{2}} - 2^{-1}\mu_{i,0}^{-\frac{1}{2}}\mu_{i,1}| \lesssim \mu_{i,0}^{-\frac{3}{2}}\mu_{i,1}^2 \lesssim \delta_0(T-t)^{l_i+1}.$$

Combining (5.5), (7.13), (7.14), $\mathcal{F}_i[\boldsymbol{\mu}_{,1}, \boldsymbol{\xi}](t)$ given in (7.9) has the upper bound

$$|\mathcal{F}_i[\boldsymbol{\mu}_{,1}, \boldsymbol{\xi}](t)| \lesssim \delta_0(T-t)^{2l_i+1}. \tag{7.16}$$

This together with $\beta(t)$ given in (7.10) implies that for $t \in (0, T_{\sigma_0})$,

$$\Big|\beta(t)\int_{T_{\sigma_0}}^t e^{\int_t^s \beta(a)da}\mathcal{F}_i[\boldsymbol{\mu}_{,1}, \boldsymbol{\xi}](s)ds\Big| \lesssim (T-t)^{-1}\int_t^{T_{\sigma_0}} e^{(l_i+1)\int_t^s (T-a)^{-1}da}\delta_0(T-s)^{2l_i+1}ds \lesssim \delta_0(T-t)^{2l_i+1}. \tag{7.17}$$

By (7.15), (7.16), (7.17), for $\delta_0 \ll 1$, we deduce that $\mathcal{T}^{\mathrm{ort}}[\boldsymbol{\mu}_{,1}, \boldsymbol{\xi}] \in B_{*1,\sigma_0} \times B_{*2,\sigma_0}$.

For all $(\dot{\boldsymbol{\mu}}_{,1}, \dot{\boldsymbol{\xi}}) \in B_{*1,\sigma_0} \times B_{*2,\sigma_0}$, $\psi_{\sigma_0}[\boldsymbol{\phi}, \boldsymbol{\mu}_{,1}, \boldsymbol{\xi}]$ are uniformly Hölder continuous in $\overline{B_5^+(0, \sigma_0^{-1}/2)} \times [0, T_{\sigma_0})$. By (7.11), $\mathcal{T}^{\mathrm{ort}}[\boldsymbol{\mu}_{,1}, \boldsymbol{\xi}]$ is uniformly Hölder continuous in $[0, T_{\sigma_0})$. Hence, $(\boldsymbol{\mu}_{,1}, \boldsymbol{\xi}) \mapsto \mathcal{T}^{\mathrm{ort}}[\boldsymbol{\mu}_{,1}, \boldsymbol{\xi}]$ is a compact mapping from $B_{*1,\sigma_0} \times B_{*2,\sigma_0}$ to itself by the Arzelà-Ascoli theorem. By the Schauder fixed-point theorem, we find a solution $(\dot{\boldsymbol{\mu}}_{,1,\sigma_0}, \dot{\boldsymbol{\xi}}_{\sigma_0})$ of (7.11) in $B_{*1,\sigma_0} \times B_{*2,\sigma_0}$. Combining (7.12), $\delta_0 = R^{-\frac{1}{5}}$, $T \ll 1$, and uniform Hölder continuity of $\mathcal{T}^{\mathrm{ort}}[\boldsymbol{\mu}_{,1}, \boldsymbol{\xi}]$ in $[0, T_{\sigma_0})$, we conclude the final estimate. $\square$

7.3. **Solving the inner problems in $B_5^+(0,2R)\times(0,T_{\sigma_0})$.** Now the inner problems (3.4) for $t\in(0,T_{\sigma_0})$ with $n=5$ can be rewritten as the following form formally. For $i=1,2,\ldots,\mathfrak{o}$,

$$\begin{cases} (\mu_{i,\sigma_0}[\phi])^2\partial_t\phi_i=\Delta_{y^{[i]}_{\sigma_0}}\phi_i+\mathcal{H}_{1,i}[\phi](y^{[i]}_{\sigma_0},t) & \text{for } t\in(0,T_{\sigma_0}),\, y^{[i]}_{\sigma_0}\in B_5^+(0,2R),\\ -\partial_{y^{[i]}_{5,\sigma_0}}\phi_i=\frac{5}{3}U^{\frac{2}{3}}(y^{[i]}_{\sigma_0})\phi_i+\mathcal{H}_{2,i}[\phi](\tilde y^{[i]}_{\sigma_0},t) & \text{for } t\in(0,T_{\sigma_0}),\, y^{[i]}_{\sigma_0}\in B_4(0,2R)\times\{0\},\end{cases} \tag{7.18}$$

where $\boldsymbol{\mu}_{,1,\sigma_0}[\phi],\boldsymbol{\xi}_{\sigma_0}[\phi],\psi_{\sigma_0}[\phi]=\psi_{\sigma_0}[\phi,\boldsymbol{\mu}_{,1,\sigma_0}[\phi],\boldsymbol{\xi}_{\sigma_0}[\phi]]$ are given by Lemma 7.2 and Lemma 7.1 respectively, $\mu_{i,\sigma_0}[\phi]=\mu_{i,0}+\mu_{i,1,\sigma_0}[\phi]$, $y^{[i]}_{\sigma_0}=\frac{x-(\xi^{[i]}_{\sigma_0}[\phi],0)}{\mu_{i,\sigma_0}[\phi]}$, $\tilde y^{[i]}_{\sigma_0}$ is the first four components of $y^{[i]}_{\sigma_0}$, $y^{[i]}_{5,\sigma_0}$ is the fifth component of $y^{[i]}_{\sigma_0}$,

$$\begin{aligned}\mathcal{H}_{1,i}[\phi](y^{[i]}_{\sigma_0},t)&=\eta\Big(\frac{y^{[i]}_{\sigma_0}}{4R}\Big)\Big(\dot\mu_{i,\sigma_0}[\phi]\mu_{i,\sigma_0}[\phi]Z_5(y^{[i]}_{\sigma_0})+\mu_{i,\sigma_0}[\phi]\dot\xi^{[i]}_{i,\sigma_0}[\phi]\cdot\Big(\nabla_{\tilde y^{[i]}_{\sigma_0}}U\Big)(\tilde y^{[i]}_{\sigma_0},y^{[i]}_{5,\sigma_0})\Big),\\ \mathcal{H}_{2,i}[\phi](\tilde y^{[i]}_{\sigma_0},t)&=\frac{5}{3}\mu^{\frac{3}{2}}_{i,\sigma_0}[\phi]U^{\frac{2}{3}}(\tilde y^{[i]}_{\sigma_0},0)\eta\Big(\frac{\tilde y^{[i]}_{\sigma_0}}{4R},0\Big)\Big(\Theta_{l_i}\big((\mu_{i,\sigma_0}[\phi]\tilde y^{[i]}_{\sigma_0}+\xi^{[i]}_{\sigma_0}[\phi]-\tilde q^{[i]},0),t\big)\\ &\quad+\psi_{\sigma_0}[\phi]\big((\mu_{i,\sigma_0}[\phi]\tilde y^{[i]}_{\sigma_0}+\xi^{[i]}[\phi],0),t\big)\Big).\end{aligned} \tag{7.19}$$

Lemma 7.3. *Suppose that $\delta_0=R^{-\frac{1}{5}}$, $\sigma_0\in(0,T)$, then for $T\ll1$, there exists a solution $\phi_{\sigma_0}=(\phi_{1,\sigma_0},\phi_{2,\sigma_0},\ldots,\phi_{\mathfrak{o},\sigma_0})\in B_{\mathrm{in},\sigma_0}$ of (7.18). Moreover, for $i=1,2,\ldots,\mathfrak{o}$, the initial value $\phi_{i,\sigma_0}(\cdot,0)=C_{\mathrm{in},i,\sigma_0}\tilde Z_0$ in $B_5^+(0,2R)$, where $\tilde Z_0\in C^\infty(\overline{B_5^+(0,2R)})$ and a constant $C_{\mathrm{in},i,\sigma_0}$ satisfies $|C_{\mathrm{in},i,\sigma_0}|\le CT^{4l_i+3}R^{\frac{3}{2}}$ with a constant $C>0$ independent of T,σ_0.*

Furthermore, for any compact set $K\subset[0,T)$, there exists $\sigma_K\in(0,T)$ sufficiently small such that for all $\sigma_0\in(0,\sigma_K)$, $i=1,2,\ldots,\mathfrak{o}$, ϕ_{i,σ_0} are well-defined and uniformly $C^{1+\varsigma_1,(1+\varsigma_1)/2}$ bounded in $\overline{B_5^+(0,2R)}\times K$ with a constant $\varsigma_1\in(0,1/10)$ independent of T,σ_0.

Proof. For any $\phi\in B_{\mathrm{in},\sigma_0}$, $\boldsymbol{\mu}_{,1,\sigma_0}[\phi],\boldsymbol{\xi}_{\sigma_0}[\phi],\psi_{\sigma_0}[\phi]$ are well-defined. For $i=1,2,\ldots,\mathfrak{o}$, set the new time variable

$$\tau_{i,\sigma_0}=\tau_{i,\sigma_0}(t):=\int_0^t(\mu_{i,\sigma_0}[\phi](s))^{-2}\,ds+C^2_{\mu_{i,0}}T^{-4l_i-3}\sim(T-t)^{-4l_i-3}\ \text{ for } t\in(0,T_{\sigma_0}), \tag{7.20}$$

where we used (5.5) for the last step. For brevity, we use $y^{[i]},\tau_i$ to denote $y^{[i]}_{\sigma_0},\tau_{i,\sigma_0}$ in this proof. We set the corresponding inverse function about τ_i as

$$t=t_i(\tau_i)\ \text{ for }\ \tau_i\in(\tau_i(0),\tau_i(T_{\sigma_0})),$$

which is monotonically increasing in τ_i. Since $(\mu_{i,\sigma_0}[\phi])^2\partial_t\phi_i=\partial_{\tau_i}\phi_i$, we consider

$$\begin{cases}\partial_{\tau_i}\phi_i=\Delta_{y^{[i]}}\phi_i+\mathcal{H}_{1,i}[\phi](y^{[i]},t_i(\tau_i)) & \text{for } \tau_i\in(\tau_i(0),\tau_i(T_{\sigma_0})),\, y^{[i]}\in\mathbb{R}^5_+,\\ -\partial_{y^{[i]}_5}\phi_i=\frac{5}{3}U^{\frac{2}{3}}(y^{[i]})\phi_i+\mathcal{H}_{2,i}[\phi](\tilde y^{[i]},t_i(\tau_i)) & \text{for } \tau_i\in(\tau_i(0),\tau_i(T_{\sigma_0})),\, y^{[i]}\in\partial\mathbb{R}^5_+.\end{cases} \tag{7.21}$$

By (7.20), (5.4), (7.3), we make the following preparation for estimates about $\mathcal{H}_{1,i}[\phi],\mathcal{H}_{2,i}[\phi]$.

$$\begin{aligned}&T-t_i(\tau_i)\sim\tau_i^{-\frac{1}{4l_i+3}};\quad T-t_i(a_1)\sim T-t_i(a_2)\quad \text{for } a_1,a_2\in(\tau_i(0),\tau_i(T_{\sigma_0})),\ a_1\sim a_2;\\ &\frac{d}{d\tau_i}t_i(\tau_i)=\mu^2_{i,\sigma_0}[\phi](t_i(\tau_i));\quad \mu_{i,\sigma_0}[\phi](t_i(\tau_i))\sim(T-t_i(\tau_i))^{2l_i+2}\sim\tau_i^{-\frac{2l_i+2}{4l_i+3}};\\ &\Big|\frac{d}{d\tau_i}\Big(\mu_{i,\sigma_0}[\phi](t_i(\tau_i))\Big)\Big|=\Big|\frac{d}{dt}\Big(\mu_{i,\sigma_0}[\phi]\Big)(t_i(\tau_i))\frac{d}{d\tau_i}t_i(\tau_i)\Big|\lesssim(T-t_i(\tau_i))^{6l_i+5}\sim\tau_i^{-\frac{6l_i+5}{4l_i+3}};\\ &|\xi^{[i]}_{\sigma_0}[\phi](t_i(\tau_i))-\tilde q^{[i]}|\le\delta_0^{\frac{1}{3}}(T-t_i(\tau_i))^{2l_i+2}\sim\delta_0^{\frac{1}{3}}\tau_i^{-\frac{2l_i+2}{4l_i+3}};\\ &\Big|\frac{d}{d\tau_i}\Big(\xi^{[i]}_{\sigma_0}[\phi](t_i(\tau_i))\Big)\Big|=\Big|\frac{d}{dt}\Big(\xi^{[i]}_{\sigma_0}[\phi]\Big)(t_i(\tau_i))\frac{d}{d\tau_i}t_i(\tau_i)\Big|\lesssim\delta_0^{\frac{1}{3}}(T-t_i(\tau_i))^{6l_i+5}\sim\delta_0^{\frac{1}{3}}\tau_i^{-\frac{6l_i+5}{4l_i+3}}.\end{aligned} \tag{7.22}$$

For $\mathcal{H}_{1,i}[\phi]$,

$$\big|\mathcal{H}_{1,i}[\phi](y^{[i]},t_i(\tau_i))\big|\lesssim(T-t(\tau_i))^{4l_i+3}\langle y^{[i]}\rangle^{-3}\eta\Big(\frac{y^{[i]}}{4R}\Big)\lesssim R^{\frac{3}{2}}\tau_i^{-1}\langle y^{[i]}\rangle^{-\frac{9}{2}}\eta\Big(\frac{y^{[i]}}{4R}\Big), \tag{7.23}$$

where we used the trick in [2, p.19] for the last step.

For the application of Proposition 1.2, pointwise and Hölder estimates are required for $\mathcal{H}_{2,i}[\phi](\tilde y^{[i]},t_i(\tau_i))$. For $|\tilde y^{[i]}|\le\tau_i^{1/2}$, denote $Q_{\tilde y^{[i]},\tau_i}:=\Big\{(\tilde v,s)\in\mathbb{R}^4\times(\tau_i(0),\tau_i(T_{\sigma_0}))\ \Big|\ |\tilde v-\tilde y^{[i]}|\le\frac{|\tilde y^{[i]}|}{2},\tau_i-\frac{|\tilde y^{[i]}|^2}{4}\le s\le\tau_i\Big\}$. For any $(\tilde v^{[j]},s_j)\in Q_{\tilde y^{[i]},\tau_i}$, $j=1,2$ with $s_2\le s_1$, it holds that $t_i(s_2)\le t_i(s_1)$, $|\tilde v^{[j]}|\in\big[|\tilde y^{[i]}|/2,3|\tilde y^{[i]}|/2\big]$, and $s_j\in[\tau_i-\frac{|\tilde y^{[i]}|^2}{4},\tau_i]\subset[3\tau_i/4,\tau_i]$.

Due to $\eta(\frac{\tilde{y}^{[i]}}{4R},0)$, $[\mathcal{H}_{2,i}[\phi](\cdot,t_i(\cdot))]_{C^{\alpha,\alpha/2}(Q_{\tilde{y}^{[i]},\tau_i})}=0$ for $|\tilde{y}^{[i]}|>16R$ and $\mathcal{H}_{2,i}[\phi](\tilde{y}^{[i]},t_i(\tau_i))=0$ for $|\tilde{y}^{[i]}|>8R$. Hence, we always assume $|\tilde{y}^{[i]}|\le 16R$ in this proof. By (7.22),

$$
\begin{aligned}
&\mu_{i,\sigma_0}^{\frac32}[\phi](t_i(s_j))\sim \tau_i^{-\frac{3l_i+3}{4l_i+3}},\\
&\left|\mu_{i,\sigma_0}^{\frac32}[\phi](t_i(s_1))-\mu_{i,\sigma_0}^{\frac32}[\phi](t_i(s_2))\right|=\left|\frac{d}{ds}\left(\mu_{i,\sigma_0}^{\frac32}[\phi](t_i(s))\right)\big|_{s=\theta s_1+(1-\theta)s_2}(s_1-s_2)\right|\\
\lesssim\ &[\theta s_1+(1-\theta)s_2]^{-\frac{l_i+1}{4l_i+3}}[\theta s_1+(1-\theta)s_2]^{-\frac{6l_i+5}{4l_i+3}}|s_1-s_2|\sim\tau_i^{-\frac{7l_i+6}{4l_i+3}}|s_1-s_2|
\end{aligned}
\tag{7.24}
$$

with some $\theta\in[0,1]$. In this proof, $\theta\in[0,1]$ will be used repetitively and will vary from line to line.

Recall $U(x)$ given in (1.13) with $n=5$.

$$
\begin{aligned}
&U^{\frac23}(\tilde{v}^{[j]},0)\sim\langle\tilde{v}^{[j]}\rangle^{-2}\sim\langle\tilde{y}^{[i]}\rangle^{-2},\\
&\left|U^{\frac23}(\tilde{v}^{[1]},0)-U^{\frac23}(\tilde{v}^{[2]},0)\right|=3\left[\theta|\tilde{v}^{[1]}|^2+(1-\theta)|\tilde{v}^{[2]}|^2+1\right]^{-2}\left||\tilde{v}^{[1]}|^2-|\tilde{v}^{[2]}|^2\right|\lesssim\langle\tilde{y}^{[i]}\rangle^{-3}|\tilde{v}^{[1]}-\tilde{v}^{[2]}|.\\
&\left|\eta\Big(\frac{\tilde{v}^{[1]}}{4R},0\Big)-\eta\Big(\frac{\tilde{v}^{[2]}}{4R},0\Big)\right|\lesssim R^{-1}|\tilde{v}^{[1]}-\tilde{v}^{[2]}|\mathbf{1}_{|\tilde{y}^{[i]}|\le 16R}.
\end{aligned}
\tag{7.25}
$$

Denote $w^{[j]}=\left(\mu_{i,\sigma_0}[\phi](t_i(s_j))\tilde{v}^{[j]}+\xi_{\sigma_0}^{[i]}[\phi](t_i(s_j))-\tilde{q}^{[i]},0\right)$. By (7.22), for $|\tilde{y}^{[i]}|\le 16R$,

$$
\begin{aligned}
&|w^{[j]}|\lesssim(T-t_i(s_j))^{2l_i+2}\langle\tilde{y}^{[i]}\rangle\sim(T-t_i(\tau_i))^{2l_i+2}\langle\tilde{y}^{[i]}\rangle\ll R^{-1}(T-t_i(s_j))^{1/2},\\
&|w^{[1]}-w^{[2]}|\lesssim(T-t_i(\tau_i))^{6l_i+5}\langle\tilde{y}^{[i]}\rangle|s_1-s_2|+(T-t_i(\tau_i))^{2l_i+2}|\tilde{v}^{[1]}-\tilde{v}^{[2]}|,\\
&|t_i(s_1)-t_i(s_2)|\lesssim(T-t_i(\tau_i))^{4l_i+4}|s_1-s_2|,
\end{aligned}
\tag{7.26}
$$

where in the second line, $(T-t_i(\tau_i))^{6l_i+5}\langle\tilde{y}^{[i]}\rangle|s_1-s_2|\ll T(T-t_i(\tau_i))^{2l_i+2}$. Recall Θ_{l_i} given in (1.14) with $n=5$. By (7.26), (7.22),

$$
\begin{aligned}
&\left|\Theta_{l_i}\left(w^{[j]},t_i(s_j)\right)\right|\sim(T-t_i(s_j))^{l_i}\sim\tau_i^{-\frac{l_i}{4l_i+3}},\\
&\left|\Theta_{l_i}\left(w^{[1]},t_i(s_1)\right)-\Theta_{l_i}\left(w^{[2]},t_i(s_2)\right)\right|=\left(L_{l_i}^{\frac32}(0)\right)^{-1}\Bigg|\left[(T-t_i(s_1))^{l_i}-(T-t_i(s_2))^{l_i}\right]L_{l_i}^{\frac32}\left(\frac{|w^{[1]}|^2}{4(T-t_i(s_1))}\right)\\
&\quad+(T-t_i(s_2))^{l_i}\left[L_{l_i}^{\frac32}\left(\frac{|w^{[1]}|^2}{4(T-t_i(s_1))}\right)-L_{l_i}^{\frac32}\left(\frac{|w^{[2]}|^2}{4(T-t_i(s_2))}\right)\right]\Bigg|\\
\lesssim&\left|\frac{d}{ds}(T-t_i(s))^{l_i}\big|_{s=\theta s_1+(1-\theta)s_2}(s_1-s_2)\right|+(T-t_i(s_2))^{l_i}\left|\frac{|w^{[1]}|^2}{T-t_i(s_1)}-\frac{|w^{[2]}|^2}{T-t_i(s_2)}\right|\\
\lesssim&\left(T-t_i(\theta s_1+(1-\theta)s_2)\right)^{5l_i+3}|s_1-s_2|\\
&+(T-t_i(s_2))^{l_i}\left|\frac{|w^{[1]}|^2(t_i(s_1)-t_i(s_2))}{(T-t_i(s_1))(T-t_i(s_2))}+\frac{(|w^{[1]}|+|w^{[2]}|)(|w^{[1]}|-|w^{[2]}|)}{T-t_i(s_2)}\right|\\
\lesssim&\left(T-t_i(\tau_i)\right)^{5l_i+3}|s_1-s_2|+(T-t_i(\tau_i))^{l_i}\Big\{(T-t_i(\tau_i))^{8l_i+6}\langle\tilde{y}^{[i]}\rangle^2|s_1-s_2|\\
&+(T-t_i(\tau_i))^{2l_i+1}\langle\tilde{y}^{[i]}\rangle\left[(T-t_i(\tau_i))^{6l_i+5}\langle\tilde{y}^{[i]}\rangle|s_1-s_2|+(T-t_i(\tau_i))^{2l_i+2}|\tilde{v}^{[1]}-\tilde{v}^{[2]}|\right]\Big\}\\
\sim&\left(T-t_i(\tau_i)\right)^{5l_i+3}\left(|s_1-s_2|+\langle\tilde{y}^{[i]}\rangle|\tilde{v}^{[1]}-\tilde{v}^{[2]}|\right)\sim\tau_i^{-\frac{5l_i+3}{4l_i+3}}\left(|s_1-s_2|+\langle\tilde{y}^{[i]}\rangle|\tilde{v}^{[1]}-\tilde{v}^{[2]}|\right).
\end{aligned}
\tag{7.27}
$$

By (7.26), $T\ll 1$ makes $w^{[j]}+q^{[i]}\in\overline{B_5^+(0,\sigma_0^{-1})}$. Using $\psi_{\sigma_0}[\phi]\in\mathcal{X}_{\delta_0,\sigma_0}$ and $|w^{[j]}|\ll R^{-1}(T-t_i(s_j))^{1/2}$ in (7.26), we have

$$
\left|\psi_{\sigma_0}[\phi]\left(w^{[j]}+q^{[i]},t_i(s_j)\right)\right|\lesssim\delta_0(T-t_i(s_j))^{l_i}\sim\delta_0\tau_i^{-\frac{l_i}{4l_i+3}}.
\tag{7.28}
$$

If $\max\{|\tilde{v}^{[1]}-\tilde{v}^{[2]}|,|s_1-s_2|\}>1$, by (7.28), $\delta_0=R^{-\frac15}$, $|\tilde{y}^{[i]}|\le 16R$, for any $\varsigma_1\in(0,1/10)$, we have

$$
\frac{\left|\psi_{\sigma_0}[\phi]\left(w^{[1]}+q^{[i]},t_i(s_1)\right)-\psi_{\sigma_0}[\phi]\left(w^{[2]}+q^{[i]},t_i(s_2)\right)\right|}{\left(\max\left\{|\tilde{v}^{[1]}-\tilde{v}^{[2]}|,|s_1-s_2|^{1/2}\right\}\right)^{\varsigma_1}}\lesssim\delta_0\tau_i^{-\frac{l_i}{4l_i+3}}\lesssim R^{-\frac{1}{10}}\langle\tilde{y}^{[i]}\rangle^{-\varsigma_1}\tau_i^{-\frac{l_i}{4l_i+3}}.
\tag{7.29}
$$

If $\max\{|\tilde{v}^{[1]}-\tilde{v}^{[2]}|,|s_1-s_2|\}\le 1$, by (7.26) and $T-t_i(s_j)\sim T-t_i(\tau_i)$, we have $|w^{[1]}|\le R^{-1}(T-t_i(s_1))^{1/2}$, $|w^{[1]}-w^{[2]}|+|t_i(s_1)-t_i(s_2)|^{1/2}\lesssim(T-t_i(s_1))^{2l_i+2}$. Hence, by the Hölder estimate (7.1) with $(x_*,t_*)=(w^{[1]}+q^{[i]},t_i(s_1))$,

$\rho = (T - t_i(s_1))^{2l_i+2} \ln R$, where obviously $(T - t_i(s_1))^{2l_i+2} \ll \rho \le R^{-1}(T - t_i(s_1))^{1/2}$, then there exists a constant $\varsigma_1 \in (0, 1/10)$ independent of T, σ_0 such that

$$
\begin{aligned}
&\left|\psi_{\sigma_0}[\phi](w^{[1]} + q^{[i]}, t_i(s_1)) - \psi_{\sigma_0}[\phi](w^{[2]} + q^{[i]}, t_i(s_2))\right| \\
&\lesssim \rho^{-\varsigma_1}\left[\delta_0(T - t_i(s_1))^{l_i} + \rho^2 R^{-\frac14}(T - t_i(s_1))^{-3l_i-4}\right]\left(\max\left\{|w^{[1]} - w^{[2]}|, |t_i(s_1) - t_i(s_2)|^{1/2}\right\}\right)^{\varsigma_1} \\
&\lesssim \rho^{-\varsigma_1}\left[\delta_0(T - t_i(\tau_i))^{l_i} + \rho^2 R^{-\frac14}(T - t_i(\tau_i))^{-3l_i-4}\right]\left((T - t_i(\tau_i))^{2l_i+2}\max\left\{|\tilde v^{[1]} - \tilde v^{[2]}|, |s_1 - s_2|^{1/2}\right\}\right)^{\varsigma_1} \\
&\sim (\ln R)^{-\varsigma_1}\left(\delta_0 + (\ln R)^2 R^{-\frac14}\right)(T - t_i(\tau_i))^{l_i}\left(\max\left\{|\tilde v^{[1]} - \tilde v^{[2]}|, |s_1 - s_2|^{1/2}\right\}\right)^{\varsigma_1} \\
&\lesssim R^{-\frac{1}{10}}\langle \tilde y^{[i]}\rangle^{-\varsigma_1}\tau_i^{-\frac{l_i}{4l_i+3}}\left(\max\left\{|\tilde v^{[1]} - \tilde v^{[2]}|, |s_1 - s_2|^{1/2}\right\}\right)^{\varsigma_1},
\end{aligned}
\tag{7.30}
$$

where we used (7.26) for the second "$\lesssim$".

Combining (7.24), (7.25), (7.27), (7.28), (7.29), (7.30), we obtain

$$
\begin{aligned}
&\left|\mathcal{H}_{2,i}[\phi](\tilde y^{[i]}, t_i(\tau_i))\right| \lesssim \tau_i^{-1}\langle \tilde y^{[i]}\rangle^{-2}\eta\Big(\frac{\tilde y^{[i]}}{4R}, 0\Big) \lesssim R^{\frac32}\tau_i^{-1}\langle \tilde y^{[i]}\rangle^{-\frac72}\eta\Big(\frac{\tilde y^{[i]}}{4R}, 0\Big), \\
&\left[\mathcal{H}_{2,i}[\phi](\cdot, t_i(\cdot))\right]_{C^{\varsigma_1,\varsigma_1/2}(Q_{\tilde y^{[i]},\tau_i})} \lesssim \tau_i^{-1}\langle \tilde y^{[i]}\rangle^{-2}\mathbf{1}_{|\tilde y^{[i]}|\le 16R}\Big[\tau_i^{-1}|\tilde y^{[i]}|^{2-\varsigma_1} + \Big(\langle \tilde y^{[i]}\rangle^{-1} + \tau_i^{-1}\langle \tilde y^{[i]}\rangle\Big)|\tilde y^{[i]}|^{1-\varsigma_1} \\
&\quad + R^{-\frac{1}{10}}\langle \tilde y^{[i]}\rangle^{-\varsigma_1}\Big] \lesssim \tau_i^{-1}\langle \tilde y^{[i]}\rangle^{-2-\varsigma_1}\mathbf{1}_{|\tilde y^{[i]}|\le 16R} \lesssim R^{\frac32}\tau_i^{-1}\langle \tilde y^{[i]}\rangle^{-\frac72-\varsigma_1}\mathbf{1}_{|\tilde y^{[i]}|\le 16R}.
\end{aligned}
\tag{7.31}
$$

Recalling the norms given in (1.17), by (7.23), (7.31), we have $\|\mathcal{H}_{1,i}[\phi](\cdot, t_i(\cdot))\|_{-1,\frac92,\tau_i^p,\mathbb{R}^5_+,\tau_i(0),\tau_i(T_{\sigma_0})} \lesssim R^{\frac32}$, $\|\mathcal{H}_{2,i}[\phi](\cdot, t_i(\cdot))\|_{-1,\frac72,\tau_i^p,\varsigma_1,\mathbb{R}^4,\tau_i(0),\tau_i(T_{\sigma_0})} \lesssim R^{\frac32}$ provided $\tau_i^p > 16R$. The choice of $\boldsymbol{\mu}_{,1,\sigma_0}[\phi], \boldsymbol{\xi}_{\sigma_0}[\phi]$ in Lemma 7.2 meets the orthogonal equations (7.2). Namely, $\mathcal{H}_{1,i}[\phi](y^{[i]}, t_i(\tau_i)), \mathcal{H}_{2,i}[\phi](\tilde y^{[i]}, t_i(\tau_i))$ satisfy the orthogonality conditions (1.20) with $n = 5$ for $\tau_i \in (\tau_i(0), \tau_i(T_{\sigma_0}))$. Thus, for $T \ll 1$, Proposition 1.2 with $n = 5, \tau = \tau_i \in (\tau_i(0), \tau_i(T_{\sigma_0}))$, $\sigma = -1, a = \frac52, \ell(\tau) = \tau_i^p, p \in (\frac25, \frac35), \iota \in \left(\frac{1}{10}(\frac52 p + 1), \frac14\right), \varsigma = \varsigma_1, g = \mathcal{H}_{1,i}[\phi](y^{[i]}, t_i(\tau_i)), h = \mathcal{H}_{2,i}[\phi](\tilde y^{[i]}, t_i(\tau_i))$, and $\tilde Z_0 \in C_c^\infty(\overline{\mathbb{R}^5_+})$ satisfying (1.21) (see Remark 1.2.1 for the existence of $\tilde Z_0$) yields a mapping $\mathcal{T}_i^{\rm in}[\phi] = \mathcal{T}_i^{\rm in}[\phi](y^{[i]}, \tau_i)$ with the estimate

$$
|\mathcal{T}_i^{\rm in}[\phi]| + \langle y^{[i]}\rangle|\nabla_{y^{[i]}}\mathcal{T}_i^{\rm in}[\phi]| \lesssim R^{\frac32}\tau_i^{-1}\langle y^{[i]}\rangle^{-\frac52} \sim R^{-\frac12}R^2(T - t_i(\tau_i))^{4l_i+3}\langle y^{[i]}\rangle^{-\frac52}
$$

for $(y^{[i]}, t_i(\tau_i)) \in B_5^+(0, 9R) \times (0, T_{\sigma_0})$. For solving (7.18), it suffices to solve the fixed-point problem

$$
\phi = \mathcal{T}^{\rm in}[\phi] := \left(\mathcal{T}_1^{\rm in}[\phi](y^{[1]}, \tau_1(t)), \mathcal{T}_2^{\rm in}[\phi](y^{[2]}, \tau_2(t)), \dots, \mathcal{T}_{\mathfrak{o}}^{\rm in}[\phi](y^{[\mathfrak{o}]}, \tau_{\mathfrak{o}}(t))\right)
$$

for $t \in (0, T_{\sigma_0}), y^{[i]} \in B_5^+(0, 2R), i = 1, 2, \dots, \mathfrak{o}$. Due to the small quantity $R^{-\frac12}$, $\mathcal{T}^{\rm in}[\phi] \in B_{{\rm in},\sigma_0}$.

For all $\phi \in B_{{\rm in},\sigma_0}$, the uniform Hölder continuity of $\nabla_{y^{[i]}}\mathcal{T}_i^{\rm in}[\phi]$ in $t \in (0, T_{\sigma_0}), y^{[i]} \in \overline{B_5^+(0, 8R)}$, $i = 1, 2, \dots, \mathfrak{o}$ follows from the parabolic regularity theory. Hence $\mathcal{T}^{\rm in}[\cdot]$ is a compact mapping from $B_{{\rm in},\sigma_0}$ to itself. The Schauder fixed-point theorem gives a fixed point $\phi_{\sigma_0} = (\phi_{1,\sigma_0}, \phi_{2,\sigma_0}, \dots, \phi_{\mathfrak{o},\sigma_0}) \in B_{{\rm in},\sigma_0}$. Moreover, Proposition 1.2 gives $\phi_{i,\sigma_0}(\cdot, 0) = C_{{\rm in},i,\sigma_0}\tilde Z_0$ in $B_5^+(0, 2R)$ with a constant $C_{{\rm in},i,\sigma_0}$ satisfying $|C_{{\rm in},i,\sigma_0}| \lesssim (\tau_i(0))^{-1}R^{\frac32} \sim T^{4l_i+3}R^{\frac32}$. The last uniform $C^{1+\varsigma_1,(1+\varsigma_1)/2}$ boundedness in $\overline{B_5^+(0, 2R)} \times K$ follows from (7.23), (7.31), and the parabolic regularity theory. □

Remark 7.3.1. *We did not apply the Hölder estimate (7.1) to $\psi_{\sigma_0}[\phi]$ with $(x_*, t_*) = (q^{[i]}, t_i(\tau_i))$. Since due to (7.26), we require $\rho \gtrsim (T - t_i(\tau_i))^{2l_i+2}\langle \tilde y^{[i]}\rangle$. Then the term $\rho^{2-\varsigma_1}R^{-\frac14}(T - t_i(\tau_i))^{-3l_i-4}$ will lead to additional spatial growth $\langle \tilde y^{[i]}\rangle^2$, which can not be eliminated by $R^{-\frac14}$.*

7.4. **Proof of Theorem 1.1.**

Proof of Theorem 1.1. Combining Lemma 7.1, Lemma 7.2, Lemma 7.3, for $\sigma_0 \in (0, T)$, we set

$$
v_{\sigma_0}(x, t) := \psi_{\sigma_0}(x, t) + \sum_{i=1}^{\mathfrak{o}}\left(U_{\mu_{i,\sigma_0},\xi^{[i]}_{\sigma_0}}(x)\eta\Big(\frac{x^{[i]}}{2\delta}\Big) + \Theta_{l_i}(x^{[i]}, t)\eta\Big(\frac{x^{[i]}}{\delta}\Big) + \mu_{i,\sigma_0}^{-\frac32}\phi_{i,\sigma_0}\Big(\frac{x - \xi^{[i]}_{\sigma_0}}{\mu_{i,\sigma_0}}, t\Big)\eta\Big(\frac{x - \xi^{[i]}_{\sigma_0}}{\mu_{i,\sigma_0}R}\Big)\right)
\tag{7.32}
$$

with $\mu_{i,\sigma_0} = \mu_{i,0} + \mu_{i,1,\sigma_0}$, and then v_{σ_0} satisfies

$$\begin{cases} \partial_t v_{\sigma_0} = \Delta v_{\sigma_0} \ \text{ in } \ B_5^+(0, \sigma_0^{-1}) \times (0, T_{\sigma_0}), \quad (-\partial_{x_5} v_{\sigma_0})(\tilde{x}, 0) = \left(|v_{\sigma_0}|^{\frac{2}{3}} v_{\sigma_0}\right)(\tilde{x}, 0) \ \text{ on } \ B_4(0, \sigma_0^{-1}) \times (0, T_{\sigma_0}), \\ v_{\sigma_0}(x, 0) = \sum_{i=1}^{\mathfrak{o}} \Big[\boldsymbol{b}_{i,\sigma_0} \cdot \tilde{\boldsymbol{e}}_i \big(T^{-\frac{1}{2}} x^{[i]}\big) + \sum_{\mathbf{p} \in \mathbb{N}^5, \|\mathbf{p}\|_{\ell_1} \le 4 l_{\max} + 4, p_5 \in 2\mathbb{N}} C_{q^{[i]},\mathbf{p},\sigma_0} \varphi_{q^{[i]},\mathbf{p},0}(x) + U_{\mu_{i,\sigma_0}(0), \xi^{[i]}_{\sigma_0}(0)}(x) \eta\Big(\frac{x^{[i]}}{2\delta}\Big) \\ \quad + \Theta_{l_i}(x^{[i]}, 0) \eta\Big(\frac{x^{[i]}}{\delta}\Big) + \big(\mu_{i,\sigma_0}(0)\big)^{-\frac{3}{2}} C_{\mathrm{in},i,\sigma_0} \tilde{Z}_0\Big(\frac{x - \xi^{[i]}_{\sigma_0}(0)}{\mu_{i,\sigma_0}(0)}\Big) \eta\Big(\frac{x - \xi^{[i]}_{\sigma_0}(0)}{\mu_{i,\sigma_0}(0) R}\Big) \Big] \ \text{ in } \ B_5^+(0, \sigma_0^{-1}), \end{cases} \tag{7.33}$$

where $\boldsymbol{b}_{i,\sigma_0}$ are constant vectors and $C_{q^{[i]},\mathbf{p},\sigma_0}$, $C_{\mathrm{in},i,\sigma_0}$ are constants satisfying $|\boldsymbol{b}_{i,\sigma_0}| \le C |\ln T| T^{\frac{5}{3} l_i + \frac{1}{2}}$, $|C_{q^{[i]},\mathbf{p},\sigma_0}| \le C e^{-\frac{9\delta^2}{22T}}$, and $|C_{\mathrm{in},i,\sigma_0}| \le C T^{4 l_i + 3} R^{\frac{3}{2}}$ with a constant C independent of T, σ_0; $\tilde{\boldsymbol{e}}_i$ are given in (6.2); $\varphi_{q^{[i]},\mathbf{p},0} \in C_c^\infty(\overline{\mathbb{R}^5_+})$ and $\varphi_{q^{[i]},\mathbf{p},0} = 0$ in $\mathbb{R}^5_+ \backslash B_5^+(q^{[i]}, 2\delta)$; $\tilde{Z}_0 \in C^\infty(\overline{B_5^+(0, 2R)})$.

Up to a subsequence, there exist constant vectors $\boldsymbol{b}_{i,0}$ and constants $C_{q^{[i]},\mathbf{p},0}, C_{\mathrm{in},i,0}$ such that $\boldsymbol{b}_{i,\sigma_0} \to \boldsymbol{b}_{i,0}$, $C_{q^{[i]},\mathbf{p},\sigma_0} \to C_{q^{[i]},\mathbf{p},0}$, $C_{\mathrm{in},i,\sigma_0} \to C_{\mathrm{in},i,0}$ as $\sigma_0 \downarrow 0$. For simplicity of exposition, we will often take a subsequence with $\sigma_0 \downarrow 0$ but will not state it. By Lemma 7.1, by the diagonal method, there exists a function ψ_0 such that $\psi_{\sigma_0} \to \psi_0$ in $L^\infty_{\mathrm{loc}}(\overline{\mathbb{R}^5_+} \times [0, T))$, and then $\psi_0 \in \mathcal{X}_{\delta_0, 0}$. Similarly, by Lemma 7.2, $\mu_{i,1,\sigma_0} \to \mu_{i,1,0}, \xi^{[i]}_{\sigma_0} \to \xi^{[i]}_0$ in $C^1_{\mathrm{loc}}([0, T))$, and then $\mu_{i,1,0}, \xi^{[i]}_0$ satisfy (7.3) with $\sigma_0 = 0$. Denote $\mu_{i,0} = \mu_{i,0} + \mu_{i,1,0}$. By Lemma 7.3, $\phi_{i,\sigma_0} \to \phi_{i,0}$ and $\nabla \phi_{i,\sigma_0} \to \nabla \phi_{i,0}$ in $L^\infty\big(\overline{B_5^+(0, 2R)} \times K\big)$ for any compact set $K \subset [0, T)$, and then $\phi_{i,0} \in B_{\mathrm{in},0}$.

Now we can extend the definition (7.32) of v_{σ_0} at $\sigma_0 = 0$ naturally. Then $v_0(x, 0) \in C_c^\infty(\overline{\mathbb{R}^5_+})$. By the convergence argument above, $v_{\sigma_0} \to v_0$ in $L^\infty_{\mathrm{loc}}(\overline{\mathbb{R}^5_+} \times [0, T))$ and $v_{\sigma_0}(x, 0) \to v_0(x, 0)$ in $C^k_{\mathrm{loc}}(\overline{\mathbb{R}^5_+})$ for any $k \in \mathbb{N}$. One testing (7.33) with arbitrary functions in $C_c^\infty(\overline{\mathbb{R}^5_+} \times [0, T))$ with σ_0 sufficiently small, then taking $\sigma_0 \downarrow 0$ deduces that v_0 is a weak solution of (1.16). Set $u = v_0, \mu_i = \mu_{i,0}, \xi^{[i]} = \xi^{[i]}_0$, and then we get Theorem 1.1. $\square$

Remark 7.3.2. *We can not use contraction mapping like [5] to solve $\dot{\boldsymbol{\mu}}_{,1}, \dot{\boldsymbol{\xi}}$ under the current topology since we do not have gradient estimate of ψ and can not get the Lipschitz continuity about $\dot{\boldsymbol{\mu}}_{,1}, \dot{\boldsymbol{\xi}}$ in (7.11) with $T_{\sigma_0} = T$.*

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APPENDIX A. A TYPE I SOLUTION OF (1.2)

Proposition A.1. *Suppose $\alpha \in (0, 1)$, $p > 1$, then u_1 given in (1.7) satisfies (1.2) with $(t_0, t_1) = (-\infty, T)$ and (1.8).*

Proof. By [54, Corollary 1.4], for $t < T$,

$$\begin{aligned} &(\partial_t - \Delta)^\alpha (T - t)^a = (\partial_t)^\alpha (T - t)^a \\ = &\frac{1}{|\Gamma(-\alpha)|} \int_{-\infty}^t \frac{(T - t)^a - (T - \tau)^a}{(t - \tau)^{1+\alpha}} d\tau = \frac{1}{|\Gamma(-\alpha)|} (T - t)^{a - \alpha} \int_1^\infty \frac{1 - z^a}{(z - 1)^{1+\alpha}} dz, \end{aligned}$$

where we changed the variable $z = \frac{T - \tau}{T - t}$, and $\max\{0, a\} < \alpha < 1$ is required for the integrability.

Given $u(\tilde{x}, 0, t)$ of the form $C (T - t)^a$, by [54, Theorem 1.7, (1.7)], in order to make

$$\frac{|\Gamma(-\alpha)|}{4^\alpha \Gamma(\alpha)} (\partial_t - \Delta_{\tilde{x}})^\alpha [C (T - t)^a] = |C (T - t)^a|^{p-1} C (T - t)^a,$$

we take $a = \frac{-\alpha}{p-1}$ with $p > 1$, and $C = \pm C_{\alpha,p}$ with $C_{\alpha,p} > 0$ given in (1.7). Thus $u(\tilde{x}, 0, t) = \pm C_{\alpha,p} (T-t)^{\frac{-\alpha}{p-1}}$. By [54, Theorem 1.7, (1.5)], for $x_n > 0$,

$$\begin{aligned} u_1(\tilde{x}, x_n, t) &= \frac{x_n^{2\alpha}}{4^\alpha \Gamma(\alpha)} \int_0^\infty e^{-\frac{x_n^2}{4\tau}} \int_{\mathbb{R}^{n-1}} (4\pi\tau)^{-\frac{n-1}{2}} e^{-\frac{|z|^2}{4\tau}} (\pm C_{\alpha,p}) (T-t+\tau)^{\frac{-\alpha}{p-1}} dz \tau^{-1-\alpha} d\tau \\ &= \frac{x_n^{2\alpha}}{4^\alpha \Gamma(\alpha)} \int_0^\infty e^{-\frac{x_n^2}{4\tau}} (\pm C_{\alpha,p}) (T-t+\tau)^{\frac{-\alpha}{p-1}} \tau^{-1-\alpha} d\tau = \frac{\pm C_{\alpha,p}}{\Gamma(\alpha)} \int_0^\infty e^{-s} \left(T - t + \frac{x_n^2}{4s} \right)^{\frac{-\alpha}{p-1}} s^{\alpha-1} ds \\ &= \frac{\pm C_{\alpha,p}}{\Gamma(\alpha)} \left(\frac{x_n^2}{4} \right)^{\frac{-\alpha}{p-1}} \left(\int_0^{\frac{x_n^2}{T-t}} + \int_{\frac{x_n^2}{T-t}}^\infty \right) e^{-s} s^{\alpha-1} \left[\frac{4(T-t)}{x_n^2} + s^{-1} \right]^{\frac{-\alpha}{p-1}} ds \\ &\sim \frac{\pm C_{\alpha,p}}{\Gamma(\alpha)} \left(\frac{x_n^2}{4} \right)^{\frac{-\alpha}{p-1}} \left[\int_0^{\frac{x_n^2}{T-t}} e^{-s} s^{\alpha-1+\frac{\alpha}{p-1}} ds + \left(\frac{T-t}{x_n^2} \right)^{\frac{-\alpha}{p-1}} \int_{\frac{x_n^2}{T-t}}^\infty e^{-s} s^{\alpha-1} ds \right] \sim \pm \left(\max \left\{ T-t, x_n^2 \right\} \right)^{\frac{-\alpha}{p-1}}. \end{aligned}$$

□

Appendix B. Convolution estimates

B.1. Inequalities toolkit.

Lemma B.1. (1) *Given $a > 0$, $b > 0$, then for any $r > 0$, $r^a e^{-br} \le b \int_r^\infty x^a e^{-bx} dx$. Given $a < 0$, $b > 0$, then for any $r > 0$, $b \int_r^\infty x^a e^{-bx} dx \le r^a e^{-br}$. Given $a \in \mathbb{R}$, $b > 0$, $r_0 > 0$, then for any $r > r_0$,*

$$\int_r^\infty x^a e^{-bx} dx \sim C(a, b, r_0) r^a e^{-br}, \tag{B.1}$$

where $C(a, b, r_0) > 0$ is a constant depending on a, b, r_0, and "$\sim$" does not depend on any parameters.

(2) *Given $a \in \mathbb{R}$, $r_0 > 0$, then for any $A > B \ge r_0$,*

$$\int_B^A r^a (1 + \ln A - \ln r) e^{-r} dr \le \left(1 + \ln\left(B^{-1}A\right)\right) \int_B^A r^a e^{-r} dr \le C(a, r_0) B^a \left(1 + \ln\left(B^{-1}A\right)\right) e^{-B}. \tag{B.2}$$

(3) *Given $b > 0$, $a \ge 0$, then $r^a e^{br}$ is monotone increasing in $r > 0$. Given $b > 0$, $a < 0$, $r_0 > 0$, then for any $r_2 > r_1 \ge r_0$,*

$$r_1^a e^{br_1} \le C(a, b, r_0) r_2^a e^{br_2} \tag{B.3}$$

with a constant $C(a, b, r_0) > 0$ depending on a, b, r_0.

Given $b < 0$, $a \le 0$, then $r^a e^{br}$ is monotone decreasing in $r > 0$. Given $b < 0$, $a > 0$, $r_0 > 0$, then for any $r_2 > r_1 \ge r_0$,

$$r_1^a e^{br_1} \ge C(a, b, r_0) r_2^a e^{br_2} \tag{B.4}$$

with a constant $C(a, b, r_0) > 0$ depending on a, b, r_0.

(4) *Given $a < 0$, $r_0 > 0$, then for any $r_2 > r_1 \ge r_0$,*

$$r_1^a (1 + |\ln r_1|) \ge C(a, r_0) r_2^a (1 + |\ln r_2|), \tag{B.5}$$

$$r_1^a \left(1 + \ln\left(r_0^{-1} r_1\right)\right) \ge C(a) r_2^a \left(1 + \ln\left(r_0^{-1} r_2\right)\right) \tag{B.6}$$

with a constant $C(a, r_0) > 0$ depending on a, r_0 and a constant $C(a) > 0$ depending on a.

(5) *Given $a < -1$, $b \in \mathbb{R}$, $r_0 > 0$, then for any $r \ge r_0$,*

$$\int_r^\infty x^a (1 + |\ln x|)^b dx \le C(a, b, r_0) r^{a+1} (1 + |\ln r|)^b \tag{B.7}$$

with a constant $C(a, b, r_0) > 0$ depending on a, b, r_0.

Proof. (1). The first two inequalities are straightforward. Given $a \in \mathbb{R}$, $b = 1$, $r_0 > 0$, for $M > 1$,

$$\int_r^{Mr} x^a e^{-x} dx \sim C(M, a) r^a \int_r^{Mr} e^{-x} dx = C(M, a) r^a e^{-r} \left[1 - e^{-(M-1)r}\right].$$

One taking $M = M(r_0)$ large to make $(M-1)r_0 \ge 1$, then $\int_r^{Mr} x^a e^{-x} dx \sim C(M, a) r^a e^{-r}$. For the other part,

$$\int_{Mr}^\infty x^a e^{-x} dx = (Mr)^a e^{-Mr} + \int_{Mr}^\infty a x^{a-1} e^{-x} dx, \quad \text{and} \quad \left| \int_{Mr}^\infty a x^{a-1} e^{-x} dx \right| \le \frac{|a|}{M r_0} \int_{Mr}^\infty x^a e^{-x} dx.$$

One taking $M = M(|a|, r_0)$ large to make $\frac{|a|}{Mr_0} < 9^{-1}$, then

$$\int_{Mr}^{\infty} x^a e^{-x} dx \sim (Mr)^a e^{-Mr} \leq e^{(1-M)r_0} M^a r^a e^{-r}.$$

Thus, we conclude (B.1) for $b = 1$, which implies the general case $b > 0$ by

$$\int_r^{\infty} x^a e^{-bx} dx = b^{-a-1} \int_{br}^{\infty} z^a e^{-z} dz \sim C(a, b, r_0) r^a e^{-br}.$$

(2) is deduced by (1).

(3). For $b > 0$, $a < 0$, $r_0 > 0$, denote $f(r) = r^a e^{br}$, then $f'(r) = r^a e^{br}\left(b + r^{-1}a\right)$ and $f'(r) > 0$ for $r \geq 9|a|b^{-1}$. For $r_0 \leq r \leq \max\left\{9|a|b^{-1}, r_0\right\}$, we have $r^a e^{br} \sim C(a, b, r_0) r_0^a e^{br_0}$. Thus, (B.3) holds.

The inequality for the case $b < 0$, $a > 0$, $r_0 > 0$ is deduced by applying the above result to $r^{-a} e^{-br}$.

(4). Denote $g(r) = r^a(1 + |\ln r|)$. For $r > 1$, $g'(r) = r^{a-1}[a(1 + \ln r) + 1]$ and $g'(r) < 0$ for $r \geq C_1$ with a large constant $C_1 = C_1(a) > 1$. For $r \in [r_0, C_1]$, $g(r) \sim C(C_1, r_0)$. Thus, (B.5) holds.

(B.6) is deduced by changing the variable $z = r_0^{-1} r$ and the above result.

(5). For any $M > \max\{r_0, 1\}$, $r \in [r_0, M]$, we have

$$\int_r^{\infty} x^a (1 + |\ln x|)^b dx \leq C(a, b, r_0, M) r^{a+1} (1 + |\ln r|)^b.$$

For $r > M$,

$$\int_r^{\infty} x^a (1 + \ln x)^b dx = \frac{-1}{a+1} r^{a+1} (1 + \ln r)^b - \frac{b}{a+1} \int_r^{\infty} x^a (1 + \ln x)^{b-1} dx,$$

where

$$\left| \frac{b}{a+1} \int_r^{\infty} x^a (1 + \ln x)^{b-1} dx \right| \leq \frac{1}{9} \int_r^{\infty} x^a (1 + \ln x)^b dx$$

for $M = M(a, b)$ sufficiently large. Thus, we conclude (B.7).

□

B.2. **Beyond Neumann boundary value $v(t)|\tilde{x}|^{-b}\mathbf{1}_{l_1(t) \leq |\tilde{x}| \leq l_2(t)}$ and $v(t)\left(|\tilde{x}| + l_1(t)\right)^{-b}\mathbf{1}_{|\tilde{x}| \leq l_2(t)}$.**

Lemma B.2. *Let $n > 2$ be an integer, $t > t_0 \geq 0$, $b \in \mathbb{R}$. Suppose that $v(s) \geq 0$ for $s \in [t_0, t]$; $0 \leq l_1(s) \leq l_2(s) \leq C_* s^{\frac{1}{2}}$ for $s \in [t_0, t]$, $C_l^{-1} l_1(t) \leq l_1(s)$, and $l_2(s) \leq C_l l_2(t)$, for all $s \in [\max\{t_0, \frac{t}{2}\}, t]$, where $C_* > 0, C_l \geq 1$ are constants. Given $x = (\tilde{x}, x_n)$, $\tilde{x} \in \mathbb{R}^{n-1}$, $x_n \geq 0$, $C_0 > 0$, we define*

$$\mathcal{T}[f](x, t) := \int_{t_0}^{t} \int_{\mathbb{R}^{n-1}} (t-s)^{-\frac{n}{2}} e^{-C_0 \frac{|\tilde{x}-y|^2 + x_n^2}{t-s}} f(y, s) dy ds$$

for an admissible function f. Then for any $\epsilon \in (0, 1)$, we have

$$\mathcal{T}\left[v(t)|\tilde{x}|^{-b}\mathbf{1}_{l_1(t) \leq |\tilde{x}| \leq l_2(t)}\right](x, t) \leq C\Big(w_1 + \sup_{t_1 \in [\max\{t_0, t/2\}, t]} v(t_1) w_2\Big),$$

where C is a constant only depending on n, b, C_*, C_l, C_0,

$$w_1 := t^{-\frac{n}{2}} e^{-C_0 \frac{x_n^2}{t-t_0}} \left(\mathbf{1}_{|\tilde{x}| \leq C_*\left[1 + (1-\epsilon)^{\frac{1}{2}}\right]\epsilon^{-1} t^{\frac{1}{2}}} + e^{-C_0(1-\epsilon)\frac{|\tilde{x}|^2}{t-t_0}} \mathbf{1}_{|\tilde{x}| > C_*\left[1 + (1-\epsilon)^{\frac{1}{2}}\right]\epsilon^{-1} t^{\frac{1}{2}}} \right)$$

$$\times \int_{t_0}^{\max\{t_0, \frac{t}{2}\}} v(s) \begin{cases} l_2^{n-1-b}(s), & b < n-1 \\ \ln(\frac{l_2(s)}{l_1(s)}), & b = n-1 \\ l_1^{n-1-b}(s), & b > n-1 \end{cases} ds,$$

$$w_2 := \begin{cases} \begin{cases} l_2^{1-b}(t), & b<1 \\ \langle \ln(\frac{l_2(t)}{l_1(t)})\rangle, & b=1 \\ l_1^{1-b}(t), & b>1 \end{cases}, & |x| \le l_1(t) \\ \begin{cases} l_2^{1-b}(t), & b<1 \\ \langle \ln(\frac{l_2(t)}{|x|})\rangle, & b=1 \\ |x|^{1-b}, & 1<b<n-1 \\ |x|^{2-n}\langle \ln(\frac{|x|}{l_1(t)})\rangle, & b=n-1 \\ |x|^{2-n} l_1^{n-1-b}(t), & b>n-1 \end{cases}, & l_1(t) < |x| \le l_2(t) \\ \begin{cases} l_2^{n-1-b}(t), & b<n-1 \\ \langle \ln(\frac{l_2(t)}{l_1(t)})\rangle, & b=n-1 \\ l_1^{n-1-b}(t), & b>n-1 \end{cases} \begin{cases} |x|^{2-n}, & l_2(t) < |x| \le C_* t^{\frac12} \\ |x|^{-2} t^{2-\frac n2} e^{-2C_0 \frac{x_n^2 + \frac{64}{81}|\tilde x|^2}{t}}, & |x| > C_* t^{\frac12} \end{cases} \end{cases}$$

with the convention $\frac{l_2(s)}{l_1(s)} = 1$ $\left(\frac{l_2(t)}{l_1(t)} = 1\right)$ *if* $l_1(s) = l_2(s) = 0$ $(l_1(t) = l_2(t) = 0)$.

Under the additional assumption that $l_1(s) \le C_l l_1(t)$ *for all* $s \in [\max\{t_0, \frac t2\}, t]$, *one replacing* $\ln(\frac{l_2(s)}{l_1(s)})$ *by* $\langle \ln(\frac{l_2(s)}{l_1(s)})\rangle$ *in the definition of* w_1, *then the same upper bound is held for* $\mathcal{T}\left[v(t)\left(|\tilde x| + l_1(t)\right)^{-b} \mathbf{1}_{|\tilde x| \le l_2(t)}\right]$.

Proof. In this proof, we assume $\int_{t_1}^{t_2} \cdots ds = 0$ if $t_1 \ge t_2$ and $\int_{\mathbb{R}^{n-1}} \mathbf{1}_{c_1 \le |y| \le c_2} \cdots dy = 0$ if $c_1 \ge c_2$. We emphasize that all "$\lesssim$", "$\sim$" in this proof are independent of ϵ, t_0. Denote $t_* := \max\{t_0, \frac t2\}$.

$$\mathcal{T}\left[v(t)|\tilde x|^{-b}\mathbf{1}_{l_1(t)\le|\tilde x|\le l_2(t)}\right] \lesssim t^{-\frac n2} e^{-C_0\frac{x_n^2}{t-t_0}} \int_{t_0}^{t_*}\int_{\mathbb{R}^{n-1}} e^{-C_0\frac{|\tilde x - y|^2}{t-t_0}} v(s)|y|^{-b}\mathbf{1}_{l_1(s)\le|y|\le l_2(s)} dyds$$
$$+ \sup_{t_1\in[t_*,t]} v(t_1)\int_{t_*}^t (t-s)^{-\frac n2} e^{-C_0\frac{x_n^2}{t-s}}\int_{\mathbb{R}^{n-1}} e^{-C_0\frac{|\tilde x-y|^2}{t-s}}|y|^{-b}\mathbf{1}_{l_1(s)\le|y|\le l_2(s)} dyds := u_1 + \sup_{t_1\in[t_*,t]} v(t_1)\tilde u_2.$$

Obviously,

$$\tilde u_2 \le u_2 := \int_{t_*}^t (t-s)^{-\frac n2} e^{-C_0\frac{x_n^2}{t-s}}\int_{\mathbb{R}^{n-1}} e^{-C_0\frac{|\tilde x-y|^2}{t-s}}|y|^{-b}\mathbf{1}_{C_l^{-1}l_1(t)\le|y|\le C_l l_2(t)} dyds.$$

For u_1, notice $|y| \le C_* t^{\frac12}$. For $|\tilde x| \le M t^{\frac12}$, we use $e^{-C_0\frac{|\tilde x-y|^2}{t-t_0}} \le 1$; Given a constant $\epsilon \in (0,1)$, for $|\tilde x| > M t^{\frac12}$ with $M = C_*\left[1 + (1-\epsilon)^{\frac12}\right]\epsilon^{-1}$, one has $|\tilde x - y| \ge \left(1 - M^{-1}C_*\right)|\tilde x| \ge (1-\epsilon)^{\frac12}|\tilde x|$. Then, $u_1 \lesssim w_1$.

Let us estimate u_2 in different regions.

For $|\tilde x| \le 2^{-1}C_l^{-1} l_1(t)$, we have $|\tilde x - y|^2 \ge 4^{-1}|y|^2$, and then

$$u_2 \le \int_{t_*}^t (t-s)^{-\frac n2} e^{-C_0\frac{x_n^2}{t-s}}\int_{\mathbb{R}^{n-1}} e^{-4^{-1}C_0\frac{|y|^2}{t-s}}|y|^{-b}\mathbf{1}_{C_l^{-1}l_1(t)\le|y|\le C_l l_2(t)} dyds$$
$$\sim \left(\int_{t_*}^{t-\frac{l_2^2(t)}{(9C_*)^2}} + \int_{t-\frac{l_2^2(t)}{(9C_*)^2}}^{t-\frac{l_1^2(t)}{(9C_*)^2}} + \int_{t-\frac{l_1^2(t)}{(9C_*)^2}}^t\right) e^{-C_0\frac{x_n^2}{t-s}}(t-s)^{-\frac{b+1}{2}}\int_{4^{-1}C_0C_l^{-2}\frac{l_1^2(t)}{t-s}}^{4^{-1}C_0C_l^2\frac{l_2^2(t)}{t-s}} e^{-z} z^{\frac{n-b}{2}-\frac32} dzds := u_{21} + u_{22} + u_{23}.$$

For u_{21}, using $e^{-z} \sim 1$ and $t_* \ge \frac t2$, we have

$$u_{21} \lesssim x_n^{2-n}\int_{2C_0\frac{x_n^2}{t}}^{C_0\frac{(9C_*)^2x_n^2}{l_2^2(t)}} e^{-r} r^{\frac n2 - 2} dr \begin{cases} l_2^{n-b-1}(t), & b<n-1 \\ \ln\left(C_l^2\frac{l_2(t)}{l_1(t)}\right), & b=n-1 \\ l_1^{n-b-1}(t), & b>n-1, \end{cases}$$

where we always use the convention $\frac{l_2(t)}{l_1(t)} = 1$ if $l_1(t) = l_2(t) = 0$. Since $n > 2$, we have $\frac n2 - 2 > -1$. Then

$$u_{21} \lesssim \begin{cases} l_2^{n-b-1}(t), & b<n-1 \\ \ln\left(C_l^2\frac{l_2(t)}{l_1(t)}\right), & b=n-1 \\ l_1^{n-b-1}(t), & b>n-1 \end{cases} \begin{cases} l_2^{2-n}(t), & x_n \le l_2(t) \\ x_n^{2-n}, & l_2(t) < x_n \le C_* t^{\frac12} \\ x_n^{-2} t^{2-\frac n2} e^{-2C_0\frac{x_n^2}{t}}, & x_n > C_* t^{\frac12}, \end{cases}$$

where we used (B.1) for the case $x_n > C_* t^{\frac12}$.

For u_{22}, since $\frac{l_1^2(t)}{(9C_*)^2} \le t-s \le \frac{l_2^2(t)}{(9C_*)^2}$, then

$$u_{22} \lesssim \int_{t-\frac{l_2^2(t)}{(9C_*)^2}}^{t-\frac{l_1^2(t)}{(9C_*)^2}} e^{-C_0\frac{x_n^2}{t-s}}(t-s)^{-\frac{b+1}{2}} \begin{cases} 1, & b<n-1 \\ \langle \ln(\frac{t-s}{l_1^2(t)})\rangle, & b=n-1 \\ (\frac{l_1^2(t)}{t-s})^{\frac{n-b}{2}-\frac{1}{2}}, & b>n-1 \end{cases} ds$$

$$\sim \begin{cases} x_n^{1-b}\int_{C_0\frac{(9C_*)^2x_n^2}{l_2^2(t)}}^{C_0\frac{(9C_*)^2x_n^2}{l_1^2(t)}} e^{-r}r^{\frac{b}{2}-\frac{3}{2}}dr, & b<n-1 \\ x_n^{2-n}\int_{C_0\frac{(9C_*)^2x_n^2}{l_2^2(t)}}^{C_0\frac{(9C_*)^2x_n^2}{l_1^2(t)}} e^{-r}r^{\frac{n}{2}-2}\langle \ln(\frac{C_0x_n^2}{rl_1^2(t)})\rangle dr, & b=n-1 \\ l_1^{n-b-1}(t)x_n^{2-n}\int_{C_0\frac{(9C_*)^2x_n^2}{l_2^2(t)}}^{C_0\frac{(9C_*)^2x_n^2}{l_1^2(t)}} e^{-r}r^{\frac{n}{2}-2}dr, & b>n-1. \end{cases}$$

Since $n>2$, we have

$$u_{22} \lesssim \begin{cases} \begin{cases} l_2^{1-b}(t), & b<1 \\ \ln(\frac{l_2(t)}{l_1(t)}), & b=1 \\ l_1^{1-b}(t), & b>1 \end{cases}, & x_n \le l_1(t) \\ \begin{cases} l_2^{1-b}(t), & b<1 \\ \langle \ln(\frac{l_2(t)}{x_n})\rangle, & b=1 \\ x_n^{1-b}, & 1<b<n-1 \\ x_n^{2-n}\langle \ln(\frac{x_n}{l_1(t)})\rangle, & b=n-1 \\ x_n^{2-n}l_1^{n-b-1}(t), & b>n-1 \end{cases}, & l_1(t)<x_n\le l_2(t) \\ \begin{cases} x_n^{-2}l_2^{3-b}(t)e^{-C_0\frac{(9C_*)^2x_n^2}{l_2^2(t)}}, & b<n-1 \\ x_n^{-2}l_2^{4-n}(t)\langle \ln(\frac{l_2(t)}{l_1(t)})\rangle e^{-C_0\frac{(9C_*)^2x_n^2}{l_2^2(t)}}, & b=n-1 \\ x_n^{-2}l_2^{4-n}(t)l_1^{n-b-1}(t)e^{-C_0\frac{(9C_*)^2x_n^2}{l_2^2(t)}}, & b>n-1 \end{cases}, & x_n>l_2(t), \end{cases}$$

where we used (B.2), (B.7) for the case $l_1(t)<x_n\le l_2(t)$, $b=n-1$, and (B.1), (B.2) for the case $x_n>l_2(t)$.

For u_{23}, by (B.1), we have

$$u_{23} \lesssim l_1^{n-b-3}(t)\int_{t-\frac{l_1^2(t)}{(9C_*)^2}}^{t} e^{-C_0\left(x_n^2+4^{-1}C_l^{-2}l_1^2(t)\right)\frac{1}{t-s}}(t-s)^{1-\frac{n}{2}}ds$$

$$= l_1^{n-b-3}(t)\left[C_0\left(x_n^2+4^{-1}C_l^{-2}l_1^2(t)\right)\right]^{2-\frac{n}{2}}\int_{C_0(9C_*)^2\frac{x_n^2+4^{-1}C_l^{-2}l_1^2(t)}{l_1^2(t)}}^{\infty} e^{-r}r^{\frac{n}{2}-3}dr$$

$$\lesssim l_1^{3-b}(t)\left(x_n^2+4^{-1}C_l^{-2}l_1^2(t)\right)^{-1}e^{-C_0(9C_*)^2\frac{x_n^2+4^{-1}C_l^{-2}l_1^2(t)}{l_1^2(t)}} \sim l_1^{3-b}(t)\left(x_n^2+l_1^2(t)\right)^{-1}e^{-C_0\frac{(9C_*)^2x_n^2}{l_1^2(t)}}.$$

In sum, when $n>2$, for $|\tilde{x}|\le 2^{-1}C_l^{-1}l_1(t)$, we have

$$u_2 \lesssim w_{21} := \begin{cases} \begin{cases} l_2^{1-b}(t), & b<1 \\ \langle \ln(\frac{l_2(t)}{l_1(t)})\rangle, & b=1 \\ l_1^{1-b}(t), & b>1 \end{cases}, & x_n\le l_1(t) \\ \begin{cases} l_2^{1-b}(t), & b<1 \\ \langle \ln(\frac{l_2(t)}{x_n})\rangle, & b=1 \\ x_n^{1-b}, & 1<b<n-1 \\ x_n^{2-n}\langle \ln(\frac{x_n}{l_1(t)})\rangle, & b=n-1 \\ x_n^{2-n}l_1^{n-1-b}(t), & b>n-1 \end{cases}, & l_1(t)<x_n\le l_2(t) \\ \begin{cases} l_2^{n-1-b}(t), & b<n-1 \\ \langle \ln(\frac{l_2(t)}{l_1(t)})\rangle, & b=n-1 \\ l_1^{n-1-b}(t), & b>n-1 \end{cases} \begin{cases} x_n^{2-n}, & l_2(t)<x_n\le C_*t^{\frac{1}{2}} \\ x_n^{-2}t^{2-\frac{n}{2}}e^{-2C_0\frac{x_n^2}{t}}, & x_n>C_*t^{\frac{1}{2}}, \end{cases} \end{cases}$$

where we used (B.6) for the case $l_1(t) < x_n \le l_2(t)$, $b = n-1$, and Lemma B.1 (3) for the case $x_n > C_* t^{\frac{1}{2}}$.

For $2^{-1}C_l^{-1} l_1(t) < |\tilde{x}| \le \min\left\{9C_l l_2(t), 9C_* t^{\frac{1}{2}}\right\}$,

$$\begin{aligned}u_2 \le & \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{n}{2}} e^{-C_0 \frac{x_n^2}{t-s}} \int_{\mathbb{R}^{n-1}} e^{-C_0 \frac{|\tilde{x}-y|^2}{t-s}} |y|^{-b} \left(\mathbf{1}_{C_l^{-1} l_1(t) \le |y| < \frac{|\tilde{x}|}{2}} + \mathbf{1}_{\frac{|\tilde{x}|}{2} \le |y| \le 2|\tilde{x}|} + \mathbf{1}_{2|\tilde{x}| < |y| \le C_l l_2(t)}\right) dyds \\ := & u_{21} + u_{22} + u_{23}.\end{aligned}$$

For u_{21}, since $|\tilde{x} - y| \ge \frac{|\tilde{x}|}{2}$, we have

$$\begin{aligned}u_{21} &\lesssim \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{n}{2}} e^{-C_0\left(x_n^2 + \frac{|\tilde{x}|^2}{4}\right)\frac{1}{t-s}} ds \begin{cases} |\tilde{x}|^{n-1-b}, & b < n-1 \\ \langle \ln(\frac{|\tilde{x}|}{l_1(t)}) \rangle, & b = n-1 \\ l_1^{n-1-b}(t), & b > n-1 \end{cases} \\ &= \left[C_0\left(x_n^2 + \frac{|\tilde{x}|^2}{4}\right)\right]^{1-\frac{n}{2}} \int_{2C_0 \frac{x_n^2 + \frac{|\tilde{x}|^2}{4}}{t}}^{\infty} e^{-r} r^{\frac{n}{2}-2} dr \begin{cases} |\tilde{x}|^{n-1-b}, & b < n-1 \\ \langle \ln(\frac{|\tilde{x}|}{l_1(t)}) \rangle, & b = n-1 \\ l_1^{n-1-b}(t), & b > n-1. \end{cases}\end{aligned}$$

Since $n > 2$, using (B.1) for the case $x_n > t^{\frac{1}{2}}$, we have

$$u_{21} \lesssim \begin{cases} |\tilde{x}|^{n-1-b}, & b < n-1 \\ \langle \ln(\frac{|\tilde{x}|}{l_1(t)}) \rangle, & b = n-1 \\ l_1^{n-1-b}(t), & b > n-1 \end{cases} \begin{cases} |x|^{2-n}, & x_n \le t^{\frac{1}{2}} \\ |x|^{-2} t^{2-\frac{n}{2}} e^{-2C_0 \frac{x_n^2}{t}}, & x_n > t^{\frac{1}{2}}. \end{cases}$$

For u_{22}, using $n > 1$, we have

$$\begin{aligned}u_{22} &\lesssim |\tilde{x}|^{-b} \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{n}{2}} e^{-C_0 \frac{x_n^2}{t-s}} \int_{\mathbb{R}^{n-1}} e^{-C_0 \frac{|\tilde{x}-y|^2}{t-s}} \mathbf{1}_{|\tilde{x}-y| \le 3|\tilde{x}|} dyds \\ &\sim |\tilde{x}|^{-b} \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{1}{2}} e^{-C_0 \frac{x_n^2}{t-s}} \int_0^{9C_0 \frac{|\tilde{x}|^2}{t-s}} e^{-z} z^{\frac{n}{2}-\frac{3}{2}} dzds \\ &\lesssim |\tilde{x}|^{-b} \left[|\tilde{x}|^{n-1} \int_{\frac{t}{2}}^{t - \frac{|\tilde{x}|^2}{(99C_*)^2}} (t-s)^{-\frac{n}{2}} e^{-C_0 \frac{x_n^2}{t-s}} ds + \int_{t - \frac{|\tilde{x}|^2}{(99C_*)^2}}^{t} (t-s)^{-\frac{1}{2}} e^{-C_0 \frac{x_n^2}{t-s}} ds\right] \\ &= |\tilde{x}|^{-b} \left[|\tilde{x}|^{n-1} \left(C_0 x_n^2\right)^{1-\frac{n}{2}} \int_{2C_0 \frac{x_n^2}{t}}^{C_0 \frac{(99C_*)^2 x_n^2}{|\tilde{x}|^2}} e^{-r} r^{\frac{n}{2}-2} dr + \left(C_0 x_n^2\right)^{\frac{1}{2}} \int_{C_0 \frac{(99C_*)^2 x_n^2}{|\tilde{x}|^2}}^{\infty} e^{-r} r^{-\frac{3}{2}} dr\right].\end{aligned}$$

Since $n > 2$, using (B.1) to the case $|\tilde{x}| < x_n \le t^{\frac{1}{2}}$, and (B.1), (B.3) to the case $x_n > t^{\frac{1}{2}}$, we have

$$u_{22} \lesssim \begin{cases} |\tilde{x}|^{1-b}, & x_n \le |\tilde{x}| \\ x_n^{2-n} |\tilde{x}|^{n-1-b}, & |\tilde{x}| < x_n \le t^{\frac{1}{2}} \\ x_n^{-2} |\tilde{x}|^{n-1-b} t^{2-\frac{n}{2}} e^{-2C_0 \frac{x_n^2}{t}}, & x_n > t^{\frac{1}{2}}, \end{cases}$$

where we assume that if $|\tilde{x}| \ge t^{\frac{1}{2}}$, the case $|\tilde{x}| < x_n \le t^{\frac{1}{2}}$ is vacuum, and the cases $x_n \le |\tilde{x}|$ and $x_n > t^{\frac{1}{2}}$ have the common part $t^{\frac{1}{2}} < x_n \le |\tilde{x}|$ with the same upper bound up to a multiplicity of a constant. The same convention is used for the other similar conditions.

For u_{23},

$$\begin{aligned}u_{23} &\le \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{n}{2}} e^{-C_0 \frac{x_n^2}{t-s}} \int_{\mathbb{R}^{n-1}} e^{-\frac{C_0}{4} \frac{|y|^2}{t-s}} |y|^{-b} \mathbf{1}_{2|\tilde{x}| < |y| \le C_l l_2(t)} dyds \\ &\sim \left(\int_{\frac{t}{2}}^{t - \frac{l_2^2(t)}{(9C_*)^2}} + \int_{t - \frac{l_2^2(t)}{(9C_*)^2}}^{t - \frac{|\tilde{x}|^2}{(99C_l C_*)^2}} + \int_{t - \frac{|\tilde{x}|^2}{(99C_l C_*)^2}}^{t}\right) (t-s)^{-\frac{b}{2}-\frac{1}{2}} e^{-C_0 \frac{x_n^2}{t-s}} \int_{C_0 \frac{|\tilde{x}|^2}{t-s}}^{\frac{C_0 C_l^2}{4} \frac{l_2^2(t)}{t-s}} e^{-z} z^{\frac{n-b}{2}-\frac{3}{2}} dzds := u_{231} + u_{232} + u_{233},\end{aligned}$$

where $\frac{|\tilde{x}|^2}{(99C_l C_*)^2} \le \frac{l_2^2(t)}{(9C_*)^2} \le \frac{t}{2}$.

For u_{231}, using $e^{-z}\sim 1$ for this part, we have

$$
\begin{aligned}
u_{231} &\lesssim \int_{\frac{t}{2}}^{t-\frac{l_2^2(t)}{(9C_*)^2}} (t-s)^{-\frac{n}{2}} e^{-C_0\frac{x_n^2}{t-s}} ds \begin{cases} l_2^{n-1-b}(t), & b<n-1\\ \langle \ln\left(|\tilde{x}|^{-1} l_2(t)\right)\rangle, & b=n-1\\ |\tilde{x}|^{n-1-b}, & b>n-1\end{cases}\\
&= \left(C_0x_n^2\right)^{1-\frac{n}{2}} \int_{2C_0\frac{x_n^2}{t}}^{C_0\frac{(9C_*)^2x_n^2}{l_2^2(t)}} e^{-r} r^{\frac{n}{2}-2} dr \begin{cases} l_2^{n-1-b}(t), & b<n-1\\ \langle \ln\left(|\tilde{x}|^{-1} l_2(t)\right)\rangle, & b=n-1\\ |\tilde{x}|^{n-1-b}, & b>n-1.\end{cases}
\end{aligned}
$$

Since $n>2$, using (B.1) for the case $x_n > C_* t^{\frac{1}{2}}$, we have

$$
u_{231} \lesssim \begin{cases} l_2^{n-1-b}(t), & b<n-1\\ \langle \ln(\frac{l_2(t)}{|\tilde{x}|})\rangle, & b=n-1\\ |\tilde{x}|^{n-1-b}, & b>n-1\end{cases} \begin{cases} l_2^{2-n}(t), & x_n\le l_2(t)\\ x_n^{2-n}, & l_2(t)<x_n\le C_*t^{\frac{1}{2}}\\ x_n^{-2}t^{2-\frac{n}{2}}e^{-2C_0\frac{x_n^2}{t}}, & x_n>C_*t^{\frac{1}{2}}.\end{cases}
$$

For u_{232}, since $\frac{|\tilde{x}|^2}{(99C_lC_*)^2}\le t-s\le \frac{l_2^2(t)}{(9C_*)^2}$, we have

$$
\begin{aligned}
u_{232} &\lesssim \begin{cases} \int_{t-\frac{l_2^2(t)}{(9C_*)^2}}^{t-\frac{|\tilde{x}|^2}{(99C_lC_*)^2}} (t-s)^{-\frac{b}{2}-\frac{1}{2}} e^{-C_0\frac{x_n^2}{t-s}} ds, & b<n-1\\ \int_{t-\frac{l_2^2(t)}{(9C_*)^2}}^{t-\frac{|\tilde{x}|^2}{(99C_lC_*)^2}} (t-s)^{-\frac{n}{2}} e^{-C_0\frac{x_n^2}{t-s}} \langle \ln\left(|\tilde{x}|^{-2}(t-s)\right)\rangle ds, & b=n-1\\ |\tilde{x}|^{n-1-b}\int_{t-\frac{l_2^2(t)}{(9C_*)^2}}^{t-\frac{|\tilde{x}|^2}{(99C_lC_*)^2}} (t-s)^{-\frac{n}{2}} e^{-C_0\frac{x_n^2}{t-s}} ds, & b>n-1\end{cases}\\
&= \begin{cases} \left(C_0x_n^2\right)^{\frac{1-b}{2}} \int_{C_0\frac{(9C_*)^2x_n^2}{l_2^2(t)}}^{C_0\frac{(99C_lC_*)^2x_n^2}{|\tilde{x}|^2}} e^{-r} r^{\frac{b}{2}-\frac{3}{2}} dr, & b<n-1\\ (C_0x_n^2)^{1-\frac{n}{2}} \int_{C_0\frac{(9C_*)^2x_n^2}{l_2^2(t)}}^{C_0\frac{(99C_lC_*)^2x_n^2}{|\tilde{x}|^2}} e^{-r} r^{\frac{n}{2}-2} \langle \ln(\frac{C_0x_n^2}{r|\tilde{x}|^2})\rangle dr, & b=n-1\\ |\tilde{x}|^{n-1-b}(C_0x_n^2)^{1-\frac{n}{2}} \int_{C_0\frac{(9C_*)^2x_n^2}{l_2^2(t)}}^{C_0\frac{(99C_lC_*)^2x_n^2}{|\tilde{x}|^2}} e^{-r} r^{\frac{n}{2}-2} dr, & b>n-1.\end{cases}
\end{aligned}
$$

Since $n>2$, we have

$$
u_{232} \lesssim \begin{cases} \begin{cases} l_2^{1-b}(t), & b<1\\ \left|\ln(\frac{l_2(t)}{|\tilde{x}|})\right|, & b=1\\ |\tilde{x}|^{1-b}, & b>1\end{cases}, & x_n\le |\tilde{x}|\\ \begin{cases} l_2^{1-b}(t), & b<1\\ \langle \ln(\frac{l_2(t)}{x_n})\rangle, & b=1\\ x_n^{1-b}, & 1<b<n-1\\ x_n^{2-n}\langle \ln(\frac{x_n}{|\tilde{x}|})\rangle, & b=n-1\\ x_n^{2-n}|\tilde{x}|^{n-1-b}, & b>n-1\end{cases}, & |\tilde{x}|<x_n\le l_2(t)\\ \begin{cases} x_n^{-2}l_2^{3-b}(t)e^{-C_0\frac{(9C_*)^2x_n^2}{l_2^2(t)}}, & b<n-1\\ x_n^{-2}l_2^{4-n}(t)\langle \ln(\frac{l_2(t)}{|\tilde{x}|})\rangle e^{-C_0\frac{(9C_*)^2x_n^2}{l_2^2(t)}}, & b=n-1\\ x_n^{-2}|\tilde{x}|^{n-1-b}l_2^{4-n}(t)e^{-C_0\frac{(9C_*)^2x_n^2}{l_2^2(t)}}, & b>n-1\end{cases}, & x_n>l_2(t),\end{cases}
$$

where (B.1), (B.2) are utilized for the case $x_n > l_2(t)$; for the case $|\tilde{x}| < x_n \le l_2(t)$, $b = n-1$, we used $n > 2$, (B.7), (B.2) to get

$$\begin{aligned}
&\left(\int_{C_0 \frac{(9C_*)^2 x_n^2}{l_2^2(t)}}^{1} + \int_1^{C_0 \frac{(99C_l C_*)^2 x_n^2}{|\tilde{x}|^2}}\right) e^{-r} r^{\frac{n}{2}-2} \langle \ln(\frac{C_0 x_n^2}{r|\tilde{x}|^2})\rangle dr \\
\lesssim & \left(C_0 \frac{x_n^2}{|\tilde{x}|^2}\right)^{\frac{n}{2}-1} \int_{C_0 \frac{x_n^2}{|\tilde{x}|^2}}^{\frac{l_2^2(t)}{(9C_*)^2|\tilde{x}|^2}} z^{-\frac{n}{2}} \langle \ln z\rangle dz + \langle \ln(\frac{x_n}{|\tilde{x}|})\rangle \sim \langle \ln(\frac{x_n}{|\tilde{x}|})\rangle.
\end{aligned}$$

For u_{233}, by (B.1), then

$$\begin{aligned}
u_{233} \lesssim & |\tilde{x}|^{n-3-b} \int_{t-\frac{|\tilde{x}|^2}{(99C_l C_*)^2}}^{t} (t-s)^{1-\frac{n}{2}} e^{-C_0 \frac{|x|^2}{t-s}} ds = |\tilde{x}|^{n-3-b} \left(C_0 |x|^2\right)^{2-\frac{n}{2}} \int_{C_0 \frac{(99C_l C_*)^2 |x|^2}{|\tilde{x}|^2}}^{\infty} e^{-r} r^{\frac{n}{2}-3} dr \\
\lesssim & |x|^{-2} |\tilde{x}|^{3-b} e^{-C_0 \frac{(99C_l C_*)^2 |x|^2}{|\tilde{x}|^2}} \sim |x|^{-2} |\tilde{x}|^{3-b} e^{-C_0 \frac{(99C_l C_*)^2 x_n^2}{|\tilde{x}|^2}}.
\end{aligned}$$

In sum, when $n > 2$, for $2^{-1} C_l^{-1} l_1(t) < |\tilde{x}| \le \min\left\{9C_l l_2(t), 9C_* t^{\frac{1}{2}}\right\}$, through tedious comparison, we have

$$u_2 \lesssim w_{22} := \begin{cases}
\begin{cases} l_2^{1-b}(t), & b<1 \\ \langle \ln(\frac{l_2(t)}{|\tilde{x}|})\rangle, & b=1 \\ |\tilde{x}|^{1-b}, & 1<b<n-1 \\ |\tilde{x}|^{2-n} \langle \ln(\frac{|\tilde{x}|}{l_1(t)})\rangle, & b=n-1 \\ |\tilde{x}|^{2-n} l_1^{n-1-b}(t), & b>n-1 \end{cases}, & x_n \le |\tilde{x}| \\
\begin{cases} l_2^{1-b}(t), & b<1 \\ \langle \ln(\frac{l_2(t)}{x_n})\rangle, & b=1 \\ x_n^{1-b}, & 1<b<n-1 \\ x_n^{2-n} \langle \ln(\frac{x_n}{l_1(t)})\rangle, & b=n-1 \\ x_n^{2-n} l_1^{n-1-b}(t), & b>n-1 \end{cases}, & |\tilde{x}| < x_n \le l_2(t) \\
\begin{cases} l_2^{n-1-b}(t), & b<n-1 \\ \langle \ln(\frac{l_2(t)}{l_1(t)})\rangle, & b=n-1 \\ l_1^{n-1-b}(t), & b>n-1 \end{cases} \begin{cases} x_n^{2-n}, & l_2(t) < x_n \le C_* t^{\frac{1}{2}} \\ x_n^{-2} t^{2-\frac{n}{2}} e^{-2C_0 \frac{x_n^2}{t}}, & x_n > C_* t^{\frac{1}{2}}, \end{cases}
\end{cases}$$

where we used Lemma B.1 (3) for the case $x_n > C_* t^{\frac{1}{2}}$; (B.6) for the case $x_n \le |\tilde{x}|$, $b = n-1$, and the case $|\tilde{x}| < x_n \le l_2(t)$, $b = n-1$.

For $|\tilde{x}| > \min\left\{9C_l l_2(t), 9C_* t^{\frac{1}{2}}\right\}$, we estimate $\tilde{u}_2$ instead of u_2. For $\tilde{u}_2$, note that $|\tilde{x}-y|^2 \ge \frac{64}{81}|\tilde{x}|^2$, then

$$\begin{aligned}
\tilde{u}_2 \le & \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{n}{2}} e^{-C_0\left(x_n^2 + \frac{64}{81}|\tilde{x}|^2\right)\frac{1}{t-s}} \int_{\mathbb{R}^{n-1}} |y|^{-b} \mathbf{1}_{C_l^{-1} l_1(t) \le |y| \le C_l l_2(t)} dy ds \\
\lesssim & \left[C_0 \left(x_n^2 + \frac{64}{81}|\tilde{x}|^2\right)\right]^{1-\frac{n}{2}} \int_{2C_0 \frac{x_n^2 + \frac{64}{81}|\tilde{x}|^2}{t}}^{\infty} e^{-r} r^{\frac{n}{2}-2} dr \begin{cases} l_2^{n-1-b}(t), & b<n-1 \\ \ln(C_l^2 \frac{l_2(t)}{l_1(t)}), & b=n-1 \\ l_1^{n-1-b}(t), & b>n-1. \end{cases}
\end{aligned}$$

Since $n > 2$, using (B.1) for the case $|x| > t^{\frac{1}{2}}$, we have

$$\tilde{u}_2 \lesssim w_{23} := \begin{cases} l_2^{n-1-b}(t), & b<n-1 \\ \langle \ln(\frac{l_2(t)}{l_1(t)})\rangle, & b=n-1 \\ l_1^{n-1-b}(t), & b>n-1 \end{cases} \begin{cases} |x|^{2-n}, & |x| \le t^{\frac{1}{2}} \\ |x|^{-2} t^{2-\frac{n}{2}} e^{-2C_0 \frac{x_n^2 + \frac{64}{81}|\tilde{x}|^2}{t}}, & |x| > t^{\frac{1}{2}}. \end{cases}$$

Since $w_{21} \sim w_{22}$ when $|\tilde{x}| \sim l_1(t)$, $\min\left\{9C_l l_2(t), 9C_* t^{\frac{1}{2}}\right\} \sim l_2(t)$ due to the assumption $l_2(t) \le C_* t^{\frac{1}{2}}$, and $w_{22} \sim w_{23}$ when $|\tilde{x}| \sim l_2(t)$, we get the first conclusion

$$\mathcal{T}\left[v(t)|x|^{-b} \mathbf{1}_{l_1(t) \le |x| \le l_2(t)}\right] \lesssim w_1 + \sup_{t_1 \in [\max\{t_0, t/2\}, t]} v(t_1) \left(w_{21} \mathbf{1}_{|\tilde{x}| \le l_1(t)} + w_{22} \mathbf{1}_{l_1(t) < |\tilde{x}| \le l_2(t)} + w_{23} \mathbf{1}_{|\tilde{x}| > l_2(t)}\right),$$

where $w_{21} \mathbf{1}_{|\tilde{x}| \le l_1(t)} + w_{22} \mathbf{1}_{l_1(t) < |\tilde{x}| \le l_2(t)} + w_{23} \mathbf{1}_{|\tilde{x}| > l_2(t)} \sim w_2$.

The upper bound of $\mathcal{T}\left[v(t)\left(|\tilde{x}|+l_1(t)\right)^{-b}\mathbf{1}_{|\tilde{x}|\le l_2(t)}\right]$ is deduced by the first conclusion and

$$v(t)\left(|\tilde{x}|+l_1(t)\right)^{-b}\mathbf{1}_{|\tilde{x}|\le l_2(t)}\sim v(t)l_1^{-b}(t)\mathbf{1}_{|\tilde{x}|\le l_1(t)}+v(t)|\tilde{x}|^{-b}\mathbf{1}_{l_1(t)\le|\tilde{x}|\le l_2(t)}.$$

□

B.3. **Exponential decay estimates for Neumann boundary $t^a|\tilde{x}|^{-b}\mathbf{1}_{l_1(t)\le|\tilde{x}|\le l_2(t)}$ and $t^a\left(|\tilde{x}|+l_1(t)\right)^{-b}\mathbf{1}_{|\tilde{x}|\le l_2(t)}$.**

Lemma B.3. *Let $n>2$ be an integer, $t>t_0\ge 1$, $x=(\tilde{x},x_n)$, $\tilde{x}\in\mathbb{R}^{n-1}$, $x_n\ge 0$. For an admissible function f, we define*

$$\mathcal{T}_1\left[f\right](x,t):=2\int_{t_0}^{t}\int_{\mathbb{R}^{n-1}}\left[4\pi(t-s)\right]^{-\frac{n}{2}}e^{-\frac{|\tilde{x}-y|^2+x_n^2}{4(t-s)}}f(y,s)dyds,$$

which satisfies the equation

$$\partial_t u=\Delta u \ \text{ in } \mathbb{R}^n_+\times(t_0,\infty),\quad -\partial_{x_n}u=f(\tilde{x},t) \ \text{ on } \partial\mathbb{R}^n_+\times(t_0,\infty),\quad u(\cdot,t_0)=0 \ \text{ in } \mathbb{R}^n_+.$$

Suppose that $0\le l_1(s)\le l_2(s)\le C_* s^{\frac12}$ for $s\in[t_0,t]$, $C_l^{-1}l_1(t)\le l_1(s)$ for all $s\in[\max\{t_0,\frac{t}{2}\},t]$, $C_l^{-1}t^{p_2}\le l_2(t)\le C_l t^{p_2}$, where $C_*>0, C_l\ge 1$ are constants,

$$b<n-1,\quad \begin{cases} p_2\le\frac12, a+p_2(1-b)+\frac{n}{2}\ge 0, & \text{if } a+p_2(n-1-b)\neq -1\\ p_2<\frac12, & \text{if } a+p_2(n-1-b)=-1,\end{cases}$$

then

$$\mathcal{T}_1\left[t^a|\tilde{x}|^{-b}\mathbf{1}_{l_1(t)\le|\tilde{x}|\le l_2(t)}\right](x,t)\le Ct^a\begin{cases}\begin{cases} l_2^{1-b}(t), & b<1\\ \langle\ln(\frac{l_2(t)}{l_1(t)})\rangle, & b=1\\ l_1^{1-b}(t), & b>1\end{cases}, & |x|\le l_1(t)\\ \begin{cases} l_2^{1-b}(t), & b<1\\ \langle\ln(\frac{l_2(t)}{|x|})\rangle, & b=1\\ |x|^{1-b}, & b>1\end{cases}, & l_1(t)<|x|\le l_2(t)\\ l_2^{1-b}(t)e^{-\frac{|\tilde{x}|^2+|x_n+1|^2}{4t}}, & |x|>l_2(t)\end{cases}$$

with the convention $\frac{l_2(t)}{l_1(t)}=1$ if $l_1(t)=l_2(t)=0$, where C is a constant only depending on n,a,b,p_2,C_*,C_l.

Under the additional assumption that $l_1(s)\le C_l l_1(t)$ for all $s\in[\max\{t_0,\frac{t}{2}\},t]$, the same conclusion is held for $\mathcal{T}_1\left[t^a\left(|\tilde{x}|+l_1(t)\right)^{-b}\mathbf{1}_{|\tilde{x}|\le l_2(t)}\right]$.

Proof. By Lemma B.2, the upper bound estimate of $\mathcal{T}_1\left[f\right](x,t)$ has two parts. For $b<n-1$, $p_2\le\frac12$, in order to make the second part dominate the first part in the range $|x|\le l_2(t)$, it suffices to guarantee

$$t^{-\frac{n}{2}}\int_{t_0}^{\max\{t_0,\frac{t}{2}\}}s^{a+p_2(n-1-b)}ds\lesssim t^{a+p_2(1-b)},$$

which can be deduced under the assumption

$$\begin{cases} p_2\le\frac12, a+p_2(1-b)+\frac{n}{2}\ge 0, & a+p_2(n-1-b)\neq -1\\ p_2<\frac12, & a+p_2(n-1-b)=-1.\end{cases}$$

Then $\mathcal{T}_1\left[t^a|\tilde{x}|^{-b}\mathbf{1}_{l_1(t)\le|\tilde{x}|\le l_2(t)}\right]\lesssim t^{a+p_2(1-b)}$ on $|x|=l_2(t)$. Denote $f:=t^{a+p_2(1-b)}e^{-\kappa\frac{|\tilde{x}|^2+|x_n+1|^2}{t}}$. Then

$$\left(\partial_t-\Delta\right)f=e^{-\kappa\frac{|\tilde{x}|^2+|x_n+1|^2}{t}}t^{-1+a+p_2(1-b)}\left[\left(\kappa-4\kappa^2\right)t^{-1}\left(|\tilde{x}|^2+|x_n+1|^2\right)+2\kappa n+a+p_2(1-b)\right],$$

which is non-negative in $\mathbb{R}^n\times(t_0,\infty)$ if $\kappa\in[0,\frac14]$ and $2\kappa n+a+p_2(1-b)\ge 0$.

$$\left(-\partial_{x_n}f\right)(\tilde{x},0,t)=2\kappa t^{a+p_2(1-b)-1}e^{-\kappa\frac{|\tilde{x}|^2+1}{t}}.$$

Take $\kappa=\frac14$. As a result, Cf is a barrier function in the range $|x|>l_2(t)$ when C is sufficiently large and the first conclusion holds.

The pointwise upper bound of $\mathcal{T}_1\left[t^a\left(|\tilde{x}|+l_1(t)\right)^{-b}\mathbf{1}_{|\tilde{x}|\le l_2(t)}\right]$ is deduced similarly.

□

B.4. **Neumann boundary value** $t^a|\tilde{x}|^{-b}e^{-\ell\frac{|\tilde{x}|^2}{t}}\mathbf{1}_{|\tilde{x}|\geq t^{\frac12}}$.

Lemma B.4. *Let $n\geq 2$ be an integer, $\ell>0$, $\kappa\in(0,\frac14)$, $\frac12+a-\frac{b}{2}+2\kappa n\geq 0$, $t>t_0\geq 1$, $x=(\tilde{x},x_n)$, $\tilde{x}\in\mathbb{R}^{n-1}$, $x_n\geq 0$, $\mathcal{T}_1[\cdot]$ defined in Lemma B.3, then*

$$\mathcal{T}_1\left[t^a|\tilde{x}|^{-b}e^{-\ell\frac{|\tilde{x}|^2}{t}}\mathbf{1}_{|\tilde{x}|\geq t^{\frac12}}\right](x,t)\leq t^{a-\frac{b}{2}+\frac12}e^{-\kappa\frac{|x|^2}{t}}\begin{cases}C(\kappa), & \text{if } b\geq -1, \kappa\in(0,\ell]\cap(0,\frac14)\\ C(b,\ell,\kappa), & \text{if } b<-1, \kappa\in(0,\min\{\ell,\frac14\}),\end{cases}$$

where $C(\kappa)>0$ is a constant depending on κ and $C(b,\ell,\kappa)>0$ is a constant depending on b,ℓ,κ.

Proof. Set $f_1(x,t)=t^{a-\frac{b}{2}+\frac12}e^{-\kappa\frac{|\tilde{x}|^2+x_n^2}{t}}$, $f_2(x,t)=t^{a-\frac{b}{2}+\frac12}e^{-\kappa\frac{(|\tilde{x}|+x_n)^2}{t}}$ with a constant κ to be determined later. Direct calculation gives

$$\begin{aligned}
&(\partial_t-\Delta)f_1=e^{-\kappa\frac{|\tilde{x}|^2+x_n^2}{t}}t^{a-\frac{b}{2}-\frac32}\left[\left(\frac12+a-\frac{b}{2}+2\kappa n\right)t+\kappa(1-4\kappa)\left(|\tilde{x}|^2+x_n^2\right)\right],\\
&(-\partial_{x_n}f_1)(\tilde{x},0,t)=0,\\
&(\partial_t-\Delta)f_2=e^{-\kappa\frac{(|\tilde{x}|+x_n)^2}{t}}t^{a-\frac{b}{2}-\frac32}\left[\left(\frac12+a-\frac{b}{2}+2\kappa n\right)t+2\kappa(n-2)t\frac{x_n}{|\tilde{x}|}+\kappa(1-8\kappa)(|\tilde{x}|+x_n)^2\right],\\
&(-\partial_{x_n}f_2)(\tilde{x},0,t)=2\kappa t^{a-\frac{b}{2}-\frac12}|\tilde{x}|e^{-\kappa\frac{|\tilde{x}|^2}{t}}.
\end{aligned}$$

Since $n\geq 2$, $\kappa\in(0,\frac14)$, $\frac12+a-\frac{b}{2}+2\kappa n\geq 0$, $x_n\geq 0$, there exists $C_2(\kappa)>0$ small such that $(\partial_t-\Delta)(f_1+C_2(\kappa)f_2)\geq 0$. Additionally,

$$\begin{aligned}
&[-\partial_{x_n}(f_1+C_2(\kappa)f_2)](\tilde{x},0,t)\geq C_2(\kappa)2\kappa t^{a-\frac{b}{2}-\frac12}|\tilde{x}|e^{-\kappa\frac{|\tilde{x}|^2}{t}}\\
\geq{}& t^a|\tilde{x}|^{-b}e^{-\ell\frac{|\tilde{x}|^2}{t}}\mathbf{1}_{|\tilde{x}|\geq t^{\frac12}}\begin{cases}C(\kappa), & \text{if } b\geq -1, \kappa\in(0,\ell]\\ C(b,\ell,\kappa), & \text{if } b<-1, \kappa\in(0,\ell)\end{cases}
\end{aligned}$$

for some positive constants $C(\kappa), C(b,\ell,\kappa)$. Thus, $C(f_1+C_2(\kappa)f_2)$ is a barrier function with C sufficiently large. ∎

B.5. **Right-hand sides** $t^a|x|^{-b}\mathbf{1}_{l_1(t)\leq|x|\leq l_2(t)}$ **and** $t^a(|x|+l_1(t))^{-b}\mathbf{1}_{|x|\leq l_2(t)}$.

Lemma B.5. *Let $n>2$ be an integer, $t>t_0\geq 1$, $x\in\mathbb{R}^n$. For an admissible function f, we define*

$$\mathcal{T}_{\mathbb{R}^n}[f](x,t):=\int_{t_0}^{t}\int_{\mathbb{R}^n}[4\pi(t-s)]^{-\frac{n}{2}}e^{-\frac{|x-y|^2}{4(t-s)}}f(y,s)dyds, \tag{B.8}$$

which satisfies the equation $\partial_t u=\Delta u+f$ in $\mathbb{R}^n\times(t_0,\infty)$, $u(\cdot,t_0)=0$ in $\mathbb{R}^n$.

Suppose that $0\leq l_1(s)\leq l_2(s)\leq C_* s^{\frac12}$ for $s\in[t_0,t]$, $C_l^{-1}l_1(t)\leq l_1(s)$ for all $s\in[\max\{t_0,\frac{t}{2}\},t]$, $C_l^{-1}t^{p_2}\leq l_2(t)\leq C_l t^{p_2}$, where $C_*>0, C_l\geq 1$ are constants,

$$b<n,\quad \begin{cases}p_2\leq\frac12, a+p_2(2-b)+\frac{n}{2}\geq 0, & \text{if } a+p_2(n-b)\neq -1\\ p_2<\frac12, & \text{if } a+p_2(n-b)=-1,\end{cases}$$

then

$$\mathcal{T}_{\mathbb{R}^n}\left[t^a|x|^{-b}\mathbf{1}_{l_1(t)\leq|x|\leq l_2(t)}\right](x,t)\leq Ct^a\begin{cases}\begin{cases}l_2^{2-b}(t) & \text{if } b<2\\ \langle\ln(\frac{l_2(t)}{l_1(t)})\rangle & \text{if } b=2\\ l_1^{2-b}(t) & \text{if } b>2\end{cases}, & |x|\leq l_1(t)\\ \begin{cases}l_2^{2-b}(t) & \text{if } b<2\\ \langle\ln(\frac{l_2(t)}{|x|})\rangle & \text{if } b=2\\ |x|^{2-b} & \text{if } b>2\end{cases}, & l_1(t)<|x|\leq l_2(t)\\ l_2^{2-b}(t)e^{-\frac{|x|^2}{4t}}, & |x|>l_2(t)\end{cases}$$

with the convention $\frac{l_2(t)}{l_1(t)}=1$ if $l_1(t)=l_2(t)=0$, where C is a constant only depending on n,a,b,p_2,C_*,C_l.

Under the additional assumption that $l_1(s)\leq C_l l_1(t)$ for all $s\in[\max\{t_0,\frac{t}{2}\},t]$, the same conclusion is held for $\mathcal{T}_{\mathbb{R}^n}\left[t^a(|x|+l_1(t))^{-b}\mathbf{1}_{|x|\leq l_2(t)}\right]$.

Proof. Similar to the proof of Lemma B.2, the requirement "$C_l^{-1}l_i(t) \le l_i(s) \le C_l l_i(t)$, $i = 1, 2$, for all $\frac{t}{2} \le s \le t$, $t \ge t_0$" in [59, Lemma A.1] can be relaxed to "$C_l^{-1}l_1(t) \le l_1(s)$, and $l_2(s) \le C_l l_2(t)$, for all $s \in [\max\{t_0, \frac{t}{2}\}, t]$" and the "$\lesssim$" in [59, Lemma A.1] is indeed independent of t_0.

One recalling [59, Lemma A.1], the estimate of $\mathcal{T}_{\mathbb{R}^n}\left[t^a|x|^{-b}\mathbf{1}_{l_1(t)\le|x|\le l_2(t)}\right]$ is split into two parts. We call the term including $\int_{t_0/2}^{t/2}$ as the first part and the other term as the second part. For $b < n$, $p_2 \le \frac{1}{2}$, in order to make the second part dominate the first part in the range $|x| \le l_2(t)$, it suffices to make

$$t^{-\frac{n}{2}} \int_{t_0}^{t/2} s^{a+p_2(n-b)} ds \le C t^{a+p_2(2-b)}$$

with a constant C independent of t_0. This can be deduced under the assumption

$$\begin{cases} p_2 \le \frac{1}{2}, a + p_2(2-b) + \frac{n}{2} \ge 0, & \text{if } \ a + p_2(n-b) \neq -1 \\ p_2 < \frac{1}{2}, & \text{if } \ a + p_2(n-b) = -1. \end{cases}$$

Then $\mathcal{T}_{\mathbb{R}^n}\left[t^a|x|^{-b}\mathbf{1}_{l_1(t)\le|x|\le l_2(t)}\right] \lesssim t^{a+p_2(2-b)}$ on $|x| = l_2(t)$. Denote $f := t^{a+p_2(2-b)}e^{-\kappa\frac{|x|^2}{t}}$ with a constant κ to be determined later. Then

$$(\partial_t - \Delta) f = e^{-\kappa\frac{|x|^2}{t}} t^{-1+a+p_2(2-b)} \left[\left(\kappa - 4\kappa^2\right) t^{-1}|x|^2 + 2\kappa n + a + p_2(2-b)\right],$$

which is non-negative in $\mathbb{R}^n \times (t_0, \infty)$ if $\kappa \in [0, \frac{1}{4}]$ and $2\kappa n + a + p_2(2-b) \ge 0$. For $b < n$, $p_2 \le \frac{1}{2}$, we take $\kappa = \frac{1}{4}$ and Cf is a barrier function in the range $|x| > l_2(t)$ when C is sufficiently large.

The estimate about $\mathcal{T}_{\mathbb{R}^n}\left[t^a\left(|x| + l_1(t)\right)^{-b}\mathbf{1}_{|x|\le l_2(t)}\right]$ is similar. $\square$

Appendix C. Sobolev-type lemmas

Lemma C.1. *Given a domain $\Omega \subset \mathbb{R}^n$ (possibly unbounded), suppose that $2 \le p < \infty$ if $n = 1, 2$, $2 \le p \le \frac{2n}{n-2}$ if $n \ge 3$, and a sequence $(u_k)_{k\ge1} \subset H^1(\Omega)$ satisfies*

$$u_k \rightharpoonup v_1 \ \text{ in } \ H^1(\Omega), \quad \nabla u_k \rightharpoonup \boldsymbol{v_2} = (v_{21}, v_{22}, \ldots, v_{2n}) \ \text{ in } \ L^2(\Omega), \quad u_k \rightharpoonup v_3 \ \text{ in } \ L^p(\Omega),$$

then $\nabla v_1 = \boldsymbol{v_2}$ in $L^2(\Omega)$, $v_1 = v_3$ in $L^p(\Omega)$.

Proof. Given a function $f \in L^2(\Omega)$, $\int_\Omega f\partial_{x_i}u_k \to \int_\Omega f v_{2i}$, $i = 1, 2, \ldots, n$. For any $g \in H^1(\Omega)$, $g \mapsto \int_\Omega f\partial_{x_i}g$ is a bounded linear mapping on $H^1(\Omega)$. By $u_k \rightharpoonup v_1$ in $H^1(\Omega)$, we have $\int_\Omega f\partial_{x_i}u_k \to \int_\Omega f\partial_{x_i}v_1$. Due to the arbitrary choice of $f \in L^2(\Omega)$, then $\partial_{x_i}v_1 = v_{2i}$ in $L^2(\Omega)$.

For the other part, given a function $f_1 \in L^{p'}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$, $\int_\Omega f_1u_k \to \int_\Omega f_1v_3$. For any $g \in H^1(\Omega)$, $g \mapsto \int_\Omega f_1g$ is also a bounded linear mapping on $H^1(\Omega)$ by the Sobolev embedding theorem and the choice of p. By $u_k \rightharpoonup v_1$ in $H^1(\Omega)$, we have $\int_\Omega f_1u_k \to \int_\Omega f_1v_1$. Due to the arbitrary choice of $f_1 \in L^{p'}(\Omega)$, then $v_1 = v_3$ in $L^p(\Omega)$. $\square$

Lemma C.2. *Let $V(\tilde{x}) \in L^\infty(\mathbb{R}^{n-1})$, $V(\tilde{x}) \to 0$ as $|\tilde{x}| \to \infty$. If a sequence $(u_k)_{k\ge1}$ satisfies $u_k \rightharpoonup u_0$ in $H^1(\mathbb{R}^n_+)$ as $k \to \infty$, then up to a subsequence, $\int_{\mathbb{R}^{n-1}} |V(\tilde{x})|\,|(u_k - u_0)(\tilde{x}, 0)|^2\, d\tilde{x} \to 0$ and $\int_{\mathbb{R}^{n-1}} V(\tilde{x})\,|u_k(\tilde{x}, 0)|^2\, d\tilde{x} \to \int_{\mathbb{R}^{n-1}} V(\tilde{x})\,|u_0(\tilde{x}, 0)|^2\, d\tilde{x}$ as $k \to \infty$.*

Proof. Without loss of generality, assume $u_k \rightharpoonup 0$. Since the Sobolev embedding $H^1(\mathbb{R}^n_+) \hookrightarrow L^2(\partial\mathbb{R}^n_+)$ is continuous, we have $\sup_{k\ge0}\|u_k(\cdot, 0)\|_{L^2(\mathbb{R}^{n-1})} \le C$. Thus, for any $\epsilon > 0$, there exists R sufficiently large such that $\int_{|\tilde{x}|\ge R} |V(\tilde{x})|\,|u_k(\tilde{x}, 0)|^2\, d\tilde{x} < \epsilon$. Since for any compact set $\Omega \subset \partial\mathbb{R}^n_+$, the Sobolev embedding $H^1(\mathbb{R}^n_+) \hookrightarrow L^2(\Omega)$ is compact, up to a subsequence, $\lim\limits_{k\to\infty}\int_{|\tilde{x}|<R} V(\tilde{x})\,|u_k(\tilde{x}, 0)|^2\, d\tilde{x} = 0$. Thus, $\int_{\mathbb{R}^{n-1}} |V(\tilde{x})|\,|(u_k - u_0)(\tilde{x}, 0)|^2\, d\tilde{x} \to 0$ as $k \to \infty$. Combining $\sup_{k\ge0}\|u_k(\cdot, 0)\|_{L^2(\mathbb{R}^{n-1})} \le C$, we have the second convergence result. $\square$

Recall $L^2_\rho(\mathbb{R}^{n-1})$, $H^1_\rho(\mathbb{R}^n_+)$ defined in (2.4), (2.5) respectively.

Lemma C.3. *Given an integer $n \ge 2$, for $u \in H^1_\rho(\mathbb{R}^n_+)$, we have $\int_{\mathbb{R}^{n-1}} u^2(\tilde{x}, 0)e^{-\frac{|\tilde{x}|^2}{4}}\, d\tilde{x} \le \frac{n+4}{4}\|u\|^2_{H^1_\rho(\mathbb{R}^n_+)}$.*

Proof. Set the norm $\|f\|_{H^1_V(\mathbb{R}^n_+)} := \left[\int_{\mathbb{R}^n_+}(|\nabla f|^2 + V(x)f^2)dx\right]^{1/2}$ with $V(x) := \frac{|x|^2}{16} + 1$. For any $u \in C_c^\infty(\overline{\mathbb{R}^n_+})$, let $u(x) = e^{\frac{|x|^2}{8}}v(x)$, then direct calculation gives that

$$\int_{\mathbb{R}^n_+} |u|^2 e^{-\frac{|x|^2}{4}}dx = \int_{\mathbb{R}^n_+} v^2 dx, \quad \int_{\mathbb{R}^n_+} |\nabla u|^2 e^{-\frac{|x|^2}{4}}dx = \int_{\mathbb{R}^n_+}\left[|\nabla v|^2 + \left(\frac{|x|^2}{16} - \frac{n}{4}\right)v^2\right]dx,$$
$$\|v\|^2_{H^1_V(\mathbb{R}^n_+)} = \int_{\mathbb{R}^n_+}\left(|\nabla u|^2 + \frac{n+4}{4}u^2\right)e^{-\frac{|x|^2}{4}}dx,$$

which indicates $\|u\|^2_{H^1_\rho(\mathbb{R}^n_+)} \le \|v\|^2_{H^1_V(\mathbb{R}^n_+)} \le \frac{n+4}{4}\|u\|^2_{H^1_\rho(\mathbb{R}^n_+)}$. Thus,

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} u^2(\tilde{x},0)e^{-\frac{|\tilde{x}|^2}{4}}d\tilde{x} &= \int_{\mathbb{R}^{n-1}} v^2(\tilde{x},0)d\tilde{x} = -\int_{\mathbb{R}^n_+}\partial_{x_n}(v^2)dx = -2\int_{\mathbb{R}^n_+} v\partial_{x_n}v dx\\ &\le 2\|v\|_{L^2(\mathbb{R}^n_+)}\|\partial_{x_n}v\|_{L^2(\mathbb{R}^n_+)} \le \|v\|^2_{L^2(\mathbb{R}^n_+)} + \|\nabla v\|^2_{L^2(\mathbb{R}^n_+)} \le \frac{n+4}{4}\|u\|^2_{H^1_\rho(\mathbb{R}^n_+)},\end{aligned} \tag{C.1}$$

which implies the lemma by approximating general functions in $H^1_\rho(\mathbb{R}^n_+)$ by $C_c^\infty(\overline{\mathbb{R}^n_+})$. □

APPENDIX D. NEGATIVE EIGENVALUE AND EXPONENTIAL DECAY OF THE EIGENFUNCTION

This section is devoted to the eigenvalue problem about the linearized equation around the ground state $U(x)$. The main result is Proposition D.5.

Define $D^{1,2}(\mathbb{R}^n_+)$ as the completion of $C_c^\infty(\overline{\mathbb{R}^n_+})$ ($f(\tilde{x},0)$ can be nonzero) under the norm $\|\nabla f\|_{L^2(\mathbb{R}^n_+)}$. Obviously, $H^1(\mathbb{R}^n_+) \subsetneqq D^{1,2}(\mathbb{R}^n_+)$. Given an integer $n \ge 3$, $p = \frac{n}{n-2}$, for $v, f, g \in D^{1,2}(\mathbb{R}^n_+)$, set functionals

$$\begin{aligned}J[v] &:= \int_{\mathbb{R}^n_+}\frac{|\nabla v|^2}{2}dx - \int_{\mathbb{R}^{n-1}}\frac{|v|^{p+1}(\tilde{x},0)}{p+1}d\tilde{x}, \quad I[v] := \int_{\mathbb{R}^n_+}|\nabla v|^2 dx - \int_{\mathbb{R}^{n-1}}|v|^{p+1}(\tilde{x},0)d\tilde{x},\\ B_v[f,g] &:= \int_{\mathbb{R}^n_+}\nabla f\cdot\nabla g dx - p\int_{\mathbb{R}^{n-1}}\left(|v|^{p-1}fg\right)(\tilde{x},0)d\tilde{x}.\end{aligned} \tag{D.1}$$

Direct calculation deduces that $J[\cdot], I[\cdot] \in C^2\left(D^{1,2}(\mathbb{R}^n_+), \mathbb{R}\right)$ and $I[v] = \langle J'[v], v\rangle$, $B_v[f,g] := \langle J''[v]f, g\rangle$. Define

$$\begin{aligned}&D^{1,2}_*(\mathbb{R}^n_+) := \left\{f \in D^{1,2}(\mathbb{R}^n_+) \mid f(\tilde{x},0) \in L^{p+1}(\mathbb{R}^{n-1})\backslash\{0\}\right\},\\ &Q(f) := \left(\int_{\mathbb{R}^{n-1}}|f|^{p+1}(\tilde{x},0)d\tilde{x}\right)^{-\frac{2}{p+1}}\int_{\mathbb{R}^n_+}|\nabla f|^2 dx, \quad f \in D^{1,2}_*(\mathbb{R}^n_+).\end{aligned} \tag{D.2}$$

Define the Nehari manifold

$$\mathbf{Ne} := \left\{f \in D^{1,2}(\mathbb{R}^n_+)\backslash\{0\} \mid I[f] = 0\right\}. \tag{D.3}$$

By [16, Theorem 1], for all $f \in D^{1,2}(\mathbb{R}^n_+)$,

$$(n-2)^{\frac{1}{2}}2^{-\frac{1}{2}}|S^{n-1}|^{\frac{1}{2(n-1)}}\|f(\cdot,0)\|_{L^{p+1}(\mathbb{R}^{n-1})} \le \|\nabla f\|_{L^2(\mathbb{R}^n_+)} \quad \text{with } p+1 = \frac{2(n-1)}{n-2},\ n \ge 3, \tag{D.4}$$

where $|S^{n-1}|$ is the volume of the $n-1$ dimensional unit sphere. The equality sign is attained only by

$$\varphi(x) = c_1 c_2^{\frac{n-2}{2}}\left[|\tilde{x}-\tilde{v}|^2 + (x_n + c_2)^2\right]^{-\frac{n-2}{2}}, \quad x = (\tilde{x}, x_n) \in \mathbb{R}^n_+ \tag{D.5}$$

with constants $c_1 \ne 0$, $c_2 > 0$, and a constant vector $\tilde{v} \in \mathbb{R}^{n-1}$. It follows that (D.5) attains $\inf\limits_{f \in D^{1,2}_*(\mathbb{R}^n_+)} Q(f)$. By (D.4), we have

$$\int_{\mathbb{R}^n_+}|\nabla f|^2 dx = \int_{\mathbb{R}^{n-1}}|f|^{p+1}(\tilde{x},0)d\tilde{x} \ge \left[(n-2)^{\frac{1}{2}}2^{-\frac{1}{2}}|S^{n-1}|^{\frac{1}{2(n-1)}}\right]^{\frac{2(p+1)}{p-1}} \text{ for } f \in \mathbf{Ne}; \quad \mathbf{Ne} \subsetneqq D^{1,2}_*(\mathbb{R}^n_+). \tag{D.6}$$

Define the tangent space at $v \in \mathbf{Ne}$ as

$$T_{\mathbf{Ne}}v := \left\{f \in D^{1,2}(\mathbb{R}^n_+) \mid \langle I'[v], f\rangle = 2\int_{\mathbb{R}^n_+}\nabla v\cdot\nabla f dx - (p+1)\int_{\mathbb{R}^{n-1}}\left(|v|^{p-1}vf\right)(\tilde{x},0)d\tilde{x} = 0\right\}. \tag{D.7}$$

Lemma D.1. *Suppose that $n \ge 3$ is an integer, $p = \frac{n}{n-2}$, and $v \in \mathbf{Ne}$ attains $\inf\limits_{f\in\mathbf{Ne}} J[f] \in \mathbb{R}$, then*

$$\int_{\mathbb{R}^n_+}\nabla v\cdot\nabla g dx - \int_{\mathbb{R}^{n-1}}\left(|v|^{p-1}vg\right)(\tilde{x},0)d\tilde{x} = 0 \quad \textit{for } g \in D^{1,2}(\mathbb{R}^n_+); \quad B_v[\varphi,\varphi] \ge 0 \quad \textit{for } \varphi \in T_{\mathbf{Ne}}v.$$

Proof. The first result is deduced by the Lagrange multiplier method. For the second result, there exists $\psi \in D^{1,2}(\mathbb{R}^n_+)$ such that $\langle I'[v], \psi\rangle \neq 0$. Indeed, since $v \in \mathbf{Ne}$ and $\langle I'[v], v\rangle = (1-p)\int_{\mathbb{R}^{n-1}} |v|^{p+1}(\tilde{x},0)d\tilde{x} \neq 0$, we can take $\psi = v$. Given $\varphi \in T_{\mathbf{Ne}}v$, set $H(t,s) := I[v + t\varphi + s\psi]$. Then $H(0,0) = 0$ and $\partial_s H(t,s)\big|_{(t,s)=(0,0)} = \langle I'[v], \psi\rangle \neq 0$. For $|t|, |s|$ sufficiently small, $H(t,s)$ is smooth about t, s. By the implicit function theorem, there exists $\delta > 0$ small such that

$$H(t, s(t)) \equiv 0 \quad \forall t \in (-\delta, \delta); \quad s = s(t) \in C^2(-\delta,\delta); \quad s(0) = 0; \quad s'(0) = -\frac{\langle I'[v], \varphi\rangle}{\langle I'[v], \psi\rangle} = 0.$$

Set $h(t) := t\varphi + s(t)\psi$, and then $h(t)$ satisfies

$$h(0) = 0, \quad h'(0) = \varphi, \quad h(t) \in C^2((-\delta,\delta), D^{1,2}(\mathbb{R}^n_+)), \quad v + h(t) \in \mathbf{Ne} \text{ for } t \in (-\delta,\delta), 0 < \delta \ll 1,$$

where we take $\delta \ll 1$ to make $v + h(t) \neq 0$. The fact that $v \in \mathbf{Ne}$ attains $\inf_{f\in\mathbf{Ne}} J[f]$ implies $\frac{d^2}{dt^2}J(v+h(t))\big|_{t=0} \geq 0$. Therein,

$$\begin{aligned}\frac{d^2}{dt^2}J(v+h(t))\Big|_{t=0} &= \int_{\mathbb{R}^n_+} |\nabla h'(0)|^2 dx + \int_{\mathbb{R}^n_+} \nabla v \cdot \nabla h''(0) dx - p\int_{\mathbb{R}^{n-1}} \left(|v|^{p-1}(h'(0))^2\right)(\tilde{x},0)d\tilde{x} \\ &\quad - \int_{\mathbb{R}^{n-1}} \left(|v|^{p-1} v h''(0)\right)(\tilde{x},0)d\tilde{x} = B_v[\varphi,\varphi],\end{aligned}$$

where we used the first result for the last step. Hence the second result holds. □

Lemma D.2. *Assume that $n \geq 3$ is an integer, $p = \frac{n}{n-2}$. For $f \in \mathbf{Ne}$, we have*

$$J[f] = \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^n_+} |\nabla f|^2 dx = \left(\frac{1}{2} - \frac{1}{p+1}\right)(Q(f))^{\frac{p+1}{p-1}}. \tag{D.8}$$

Given $f \in D^{1,2}_*(\mathbb{R}^n_+)$, for a constant $c_f > 0$,

$$\begin{aligned} &I[c_f f] = c_f^2 \int_{\mathbb{R}^n_+} |\nabla f|^2 dx - c_f^{p+1}\int_{\mathbb{R}^{n-1}} |f|^{p+1}(\tilde{x},0)d\tilde{x} = 0 \Leftrightarrow c_f = \left(\frac{\int_{\mathbb{R}^n_+} |\nabla f|^2 dx}{\int_{\mathbb{R}^{n-1}} |f|^{p+1}(\tilde{x},0)d\tilde{x}}\right)^{\frac{1}{p-1}}, \\ &\text{and then } c_f f \in \mathbf{Ne}, \quad J[c_f f] = \left(\frac{1}{2} - \frac{1}{p+1}\right)(Q(f))^{\frac{p+1}{p-1}}. \end{aligned} \tag{D.9}$$

Moreover,

$$\inf_{f\in\mathbf{Ne}} J[f] = \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\inf_{f\in D^{1,2}_*(\mathbb{R}^n_+)} Q(f)\right)^{\frac{p+1}{p-1}}.$$

In particular, if $f_1 \in \mathbf{Ne}$ attains $\inf_{f\in\mathbf{Ne}} J[f]$, then f_1 attains $\inf_{f\in D^{1,2}_*(\mathbb{R}^n_+)} Q(f)$; conversely, if $f_2 \in D^{1,2}_*(\mathbb{R}^n_+)$ attains $\inf_{f\in D^{1,2}_*(\mathbb{R}^n_+)} Q(f)$, then $c_{f_2}f_2 \in \mathbf{Ne}$ and $c_{f_2}f_2$ attains $\inf_{f\in\mathbf{Ne}} J[f]$.

Proof. (D.8) and (D.6) imply

$$\inf_{f\in\mathbf{Ne}} J[f] = \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\inf_{f\in\mathbf{Ne}} Q(f)\right)^{\frac{p+1}{p-1}} \geq \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\inf_{f\in D^{1,2}_*(\mathbb{R}^n_+)} Q(f)\right)^{\frac{p+1}{p-1}}.$$

Given $f \in D^{1,2}_*(\mathbb{R}^n_+)$, the choice of c_f yields $c_f f \in \mathbf{Ne}$. One combining (D.8), then $J[c_f f] = \left(\frac{1}{2} - \frac{1}{p+1}\right)(Q(f))^{\frac{p+1}{p-1}}$, which implies

$$\left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\inf_{f\in D^{1,2}_*(\mathbb{R}^n_+)} Q(f)\right)^{\frac{p+1}{p-1}} = \inf_{f\in D^{1,2}_*(\mathbb{R}^n_+)} J[c_f f] \geq \inf_{c_f f\in\mathbf{Ne}} J[c_f f] \geq \inf_{f\in\mathbf{Ne}} J[f].$$

□

Lemma D.3. *Given an integer $n \geq 2$ and constants $\lambda = -m^2$, $m > 0$, then for $u \in C^2(\mathbb{R}^n_+) \cap C^1(\overline{\mathbb{R}^n_+})$ and $x \in \overline{\mathbb{R}^n_+}$, we have*

$$\begin{aligned} u(x) = &\int_{\mathbb{R}^n_+} (\Delta u + \lambda u)(y)\left[-E^m_n(|x-y|) - E^m_n(|x - (\tilde{y}, -y_n)|)\right] dy \\ &- \int_{\mathbb{R}^{n-1}} (-\partial_{y_n} u)(\tilde{y},0)\left[-2E^m_n(|x - (\tilde{y},0)|)\right] d\tilde{y}, \end{aligned} \tag{D.10}$$

where $E_n^m(r) = m^{\frac{n}{2}-1}(2\pi)^{-\frac{n}{2}}r^{1-\frac{n}{2}}K_{\frac{n}{2}-1}(mr)$ *for* $r > 0$ *and* $K_{\frac{n}{2}-1}$ *is the second kind modified Bessel function of order* $\frac{n}{2}-1$. *Moreover,*

$$|E_n^m(r)| \le C(m,n)\left(\mathbf{1}_{r<1}\begin{cases} r^{2-n}, & n>2 \\ \langle \ln r\rangle, & n=2 \end{cases} + \mathbf{1}_{r\ge 1}e^{-mr}\right) \quad \textit{for } r>0. \tag{D.11}$$

Remark D.3.1. *Similar to* (2.17)*,* (D.10) *can be used to represent solutions with some rough data.*

Proof. By [55, Theorem 2.5], the fundamental solution of $\Delta+\lambda$ in $\mathbb{R}^n$ is given by $-E_n^m(|x|)$, namely $(\Delta+\lambda)\left(-E_n^m(|x|)\right) = \delta(x)$ in $\mathbb{R}^n$. The properties of $-E_n^m$ are given in [55, pp.9-10].

Given $\tilde{x} \in \mathbb{R}^{n-1}$, $x_n > 0$, $x = (\tilde{x}, x_n)$, take $v(y) = -E_n^m(|x-y|) - E_n^m(|x-(\tilde{y},-y_n)|)$. Direct calculation yields $\partial_{y_n} v\big|_{y_n=0} = 0$, $(\Delta+\lambda)v = (\Delta_y+\lambda)\left[-E_n^m(|y-x|) - E_n^m\big(|y-(\tilde{x},-x_n)|\big)\right] = \delta_{y-x} + \delta_{y-(\tilde{x},-x_n)}$ in $\mathbb{R}^n$. Integration by parts yields

$$\begin{aligned}\int_{\mathbb{R}^n_+}(\Delta u+\lambda u)\,v dy &= \int_{\mathbb{R}^{n-1}}\left[(-\partial_{y_n}u)v - (-\partial_{y_n}v)u\right](\tilde{y},0)d\tilde{y} + \int_{\mathbb{R}^n_+} u\,(\Delta v+\lambda v)\,dy \\ = \int_{\mathbb{R}^{n-1}}\left[(-\partial_{y_n}u)v\right](\tilde{y},0)d\tilde{y} &+ \int_{\mathbb{R}^n}\left(u(y)\mathbf{1}_{y_n>0} + 0\mathbf{1}_{y_n\le 0}\right)\left(\delta_{y-x}+\delta_{y-(\tilde{x},-x_n)}\right)dy \\ = \int_{\mathbb{R}^{n-1}}\left[(-\partial_{y_n}u)v\right](\tilde{y},0)d\tilde{y} &+ u(x).\end{aligned}$$

Taking $x_n \downarrow 0$ deduces the case $x \in \partial\mathbb{R}^n_+$. Plugging $v(y)$, then we complete the proof. ☐

Given $V_1(x) \in L^\infty(\mathbb{R}^n_+)$ and $V_2(\tilde{x}) \in L^\infty(\mathbb{R}^{n-1})$, we say that $f \in H^1(\mathbb{R}^n_+)$ satisfies the equation

$$-\Delta f = V_1(x)f \ \text{ in } \mathbb{R}^n_+, \quad -\partial_{x_n} f = V_2(\tilde{x})f \ \text{ on } \partial\mathbb{R}^n_+$$

in the weak sense if

$$\int_{\mathbb{R}^n_+}\nabla f\cdot\nabla g dx - \int_{\mathbb{R}^{n-1}} V_2(\tilde{x})(fg)(\tilde{x},0)d\tilde{x} = \int_{\mathbb{R}^n_+} V_1(x) fg dx \quad \text{ holds for all } g \in H^1(\mathbb{R}^n_+).$$

Lemma D.4. *Suppose that* $n \ge 2$ *is an integer,* $\lambda < 0$*,* $V(\tilde{x})$ *satisfies* $V(\tilde{x}) \in L^\infty(\mathbb{R}^{n-1})$ *and* $\lim_{|\tilde{x}|\to\infty} V(\tilde{x}) = 0$*, let* $\phi \in H^1(\mathbb{R}^n_+)$ *satisfy*

$$-\Delta\phi = \lambda\phi \quad \textit{in } \mathbb{R}^n_+, \quad -\partial_{x_n}\phi = V(\tilde{x})\phi \quad \textit{on } \partial\mathbb{R}^n_+ \tag{D.12}$$

in the weak sense. Then for all $\nu \in [0, \sqrt{-\lambda})$*, we have* $|\phi(x)| \le Ce^{-\nu|x|}$ *in* $\overline{\mathbb{R}^n_+}$ *with a constant* C *depending on* $n, \lambda, \nu, \|\phi\|_{L^2(\mathbb{R}^n_+)}, V(\tilde{x})$*.*

Proof. Denote $m = \sqrt{-\lambda}$. Since $\|\phi\|_{L^2(\mathbb{R}^n_+)} < \infty$, by [38, Theorem 5.36, Theorem 5.45], $\phi \in C(\overline{\mathbb{R}^n_+})$ and $|\phi(x)| \to 0$ as $|x| \to \infty$. By the representation formula (D.10) in Lemma D.3 and uniqueness of the weak solution of (D.12) in $H^1(\mathbb{R}^n_+)$, ϕ can be written as

$$\phi(x) = 2\int_{\mathbb{R}^{n-1}} V(w)\phi(w,0)E_n^m(|x-(w,0)|)dw, \tag{D.13}$$

where E_n^m is given in Lemma D.3 and satisfies (D.11).

The following argument is in the same spirit of [28, Proof of Theorem 2.1]. Given $\nu \in [0,m)$, denote

$$A(x) := \sup_{w\in\mathbb{R}^{n-1}} |\phi(w,0)|\, e^{-\nu|x-(w,0)|}, \quad B(x) := \int_{\mathbb{R}^{n-1}} 2\,|E_n^m(|x-(w,0)|)|\, e^{\nu|x-(w,0)|}\,|V(w)|\,dw.$$

If $A(x) = 0$ for some $x \in \overline{\mathbb{R}^n_+}$, then $\phi(\cdot,0) \equiv 0$ in $\mathbb{R}^{n-1}$, which implies that $\phi \equiv 0$ in $\overline{\mathbb{R}^n_+}$ by (D.13) and the conclusion holds. Hereafter, we always assume $A(x) > 0$ in $\overline{\mathbb{R}^n_+}$. Obviously, $|\phi(x)| \le A(x)B(x)$. By (D.11), $\nu < m$, the properties of $V(x)$, and Lebesgue's dominated convergence theorem,

$$B(x) = \int_{\mathbb{R}^{n-1}} 2\,|E_n^m(|(z,x_n)|)|\, e^{\nu|(z,x_n)|}\,|V(\tilde{x}-z)|\,dz \to 0 \ \text{ as } \ |x|\to\infty,$$

which implies that there exists $R_1 = R_1(n,m,V(\tilde{x})) > 0$ sufficiently large such that

$$|\phi(x)| \le 2^{-1}A(x) \qquad \text{for } \ x \in \overline{\mathbb{R}^n_+},\ |x| \ge R_1. \tag{D.14}$$

Note that for $\nu \ge 0$,

$$\begin{aligned}A(x) &= \sup_{w\in\mathbb{R}^{n-1}}|\phi(w,0)| \sup_{z\in\mathbb{R}^{n-1}} e^{-\nu|x-(z,0)|}e^{-\nu|(z,0)-(w,0)|} \\ &= \sup_{z\in\mathbb{R}^{n-1}}\sup_{w\in\mathbb{R}^{n-1}} |\phi(w,0)|\, e^{-\nu|x-(z,0)|}e^{-\nu|(z,0)-(w,0)|} = \sup_{z\in\mathbb{R}^{n-1}} A(z,0)e^{-\nu|x-(z,0)|}.\end{aligned}$$

Combining (D.14), we have

$$\sup_{|w|>R_1} |\phi(w,0)|\, e^{-\nu|x-(w,0)|} \le 2^{-1} \sup_{|w|>R_1} A(w,0)e^{-\nu|x-(w,0)|} \le 2^{-1}A(x) < A(x).$$

By the definition of $A(x)$, it implies

$$A(x) = \sup_{|w|\le R_1} |\phi(w,0)|\, e^{-\nu|x-(w,0)|} \le e^{-\nu|x|} \sup_{|w|\le R_1} |\phi(w,0)|\, e^{\nu|w|}.$$

Combining (D.14) and the local L^∞ estimate [38, Theorem 5.36] in $\overline{B_n^+(0,2R_1)}$, we conclude this lemma. □

Proposition D.5. *Given an integer $n \ge 3$, $p = \frac{n}{n-2}$, and $U(x)$ given in (1.13), there exists only one negative eigenvalue λ_0 for the following eigenvalue problem in $H^1(\mathbb{R}^n_+)$,*

$$-\Delta f = \lambda_0 f \ \text{ in } \mathbb{R}^n_+, \quad -\partial_{x_n} f = pU^{p-1} f \ \text{ on } \partial\mathbb{R}^n_+. \tag{D.15}$$

The eigenvalue λ_0 is simple with an eigenfunction $Z_0(x) \in C^\infty(\overline{\mathbb{R}^n_+}) \cap H^1(\mathbb{R}^n_+)$ satisfying $\|Z_0\|_{L^2(\mathbb{R}^n_+)} = 1$, $0 < Z_0(x) \le Ce^{-\nu|x|}$ in $\overline{\mathbb{R}^n_+}$ for all $\nu \in \left[0, \sqrt{-\lambda_0}\right)$ with a constant C depending on n, λ_0, ν. Moreover, $\lambda_0 = \inf_{0\not\equiv f\in H^1(\mathbb{R}^n_+)} \|f\|^{-2}_{L^2(\mathbb{R}^n_+)} B_U[f,f]$.

Proof. **Step 1.** For any $f \in H^1(\mathbb{R}^n_+)$ satisfying (D.15) in the weak sense, by parabolic regularity theorem, $f \in C^\infty(\overline{\mathbb{R}^n_+})$. By Lemma D.4, for all $\nu \in \left[0, \sqrt{-\lambda_0}\right)$, we have $|f(x)| \le Ce^{-\nu|x|}$ in $\overline{\mathbb{R}^n_+}$ with a constant C depending on $n, \lambda_0, \nu, \|f\|_{L^2(\mathbb{R}^n_+)}$.

Step 2. Denote $\lambda_* = \inf_{0\not\equiv f\in H^1(\mathbb{R}^n_+)} \|f\|^{-2}_{L^2(\mathbb{R}^n_+)} B_U[f,f]$, where $B_U[f,f]$ is well-defined since $U(x) \in D^{1,2}(\mathbb{R}^n_+)$ for $n > 2$. Notice that $B_U[U,U] = (1-p)\int_{\mathbb{R}^{n-1}} U^{p+1}(\tilde{x},0)d\tilde{x} < 0$. Since $U \notin L^2(\mathbb{R}^n_+)$ when $n \le 4$, instead, applying $\|U\eta(x/R)\|^{-2}_{L^2(\mathbb{R}^n_+)} B_U[U\eta(x/R), U\eta(x/R)]$ with R sufficiently large implies $\lambda_* < 0$. For any $f \in H^1(\mathbb{R}^n_+)$, $\int_{\mathbb{R}^{n-1}} f^2(\tilde{x},0)d\tilde{x} \le 2\|f\|_{L^2(\mathbb{R}^n_+)}\|\partial_{x_n} f\|_{L^2(\mathbb{R}^n_+)}$, then for any $\epsilon > 0$,

$$p\int_{\mathbb{R}^{n-1}} \left(U^{p-1}f^2\right)(\tilde{x},0)d\tilde{x} \le \epsilon\|\nabla f\|^2_{L^2(\mathbb{R}^n_+)} + C(\epsilon)\|f\|^2_{L^2(\mathbb{R}^n_+)},$$

which implies $\lambda_* > -\infty$.

Step 3. Take a sequence $(f_k)_{k\ge1} \subset H^1(\mathbb{R}^n_+)$ such that $\|f_k\|_{L^2(\mathbb{R}^n_+)} = 1$ and $B_U[f_k,f_k] = \lambda_* + o(1)$ with $o(1) \to 0$ as $k \to \infty$. It implies

$$\epsilon\|\nabla f_k\|^2_{L^2(\mathbb{R}^n_+)} + C(\epsilon)\|f_k\|^2_{L^2(\mathbb{R}^n_+)} \ge p\int_{\mathbb{R}^{n-1}} \left(U^{p-1}f_k^2\right)(\tilde{x},0)d\tilde{x} = \int_{\mathbb{R}^n_+} |\nabla f_k|^2\, dx - \lambda_* + o(1) \ge -\frac{\lambda_*}{2} > 0$$

when k is sufficiently large. Thus

$$(1-\epsilon)\|\nabla f_k\|^2_{L^2(\mathbb{R}^n_+)} \le C(\epsilon) + \lambda_* + o(1), \quad p\int_{\mathbb{R}^{n-1}} \left(U^{p-1}f_k^2\right)(\tilde{x},0)d\tilde{x} \ge -\frac{\lambda_*}{2}.$$

It follows that $\sup_{k\ge1} \|f_k\|_{H^1(\mathbb{R}^n_+)} < \infty$. By Lemma C.1, C.2, up to a subsequence,

$$f_k \rightharpoonup f_* \ \text{ in } H^1(\mathbb{R}^n_+), \quad \nabla f_k \rightharpoonup \nabla f_* \ \text{ in } L^2(\mathbb{R}^n_+), \quad f_k \rightharpoonup f_* \ \text{ in } L^2(\mathbb{R}^n_+),$$
$$p\int_{\mathbb{R}^{n-1}} \left(U^{p-1}f_k^2\right)(\tilde{x},0)d\tilde{x} \to p\int_{\mathbb{R}^{n-1}} \left(U^{p-1}f_*^2\right)(\tilde{x},0)d\tilde{x} \ge -\frac{\lambda_*}{2},$$

which implies

$$B_U[f_*,f_*] \le \lambda_* < 0, \quad 0 < \|f_*\|_{L^2(\mathbb{R}^n_+)} \le 1$$

and then

$$\|f_*\|^{-2}_{L^2(\mathbb{R}^n_+)} B_U[f_*,f_*] \le B_U[f_*,f_*] \le \lambda_*.$$

By the definition of λ_*, we know that $\|f_*\|^{-2}_{L^2(\mathbb{R}^n_+)} B_U[f_*,f_*] \ge \lambda_*$. Thus $\|f_*\|_{L^2(\mathbb{R}^n_+)} = 1$ and $B_U[f_*,f_*] = \lambda_*$. Since $|\nabla|f|| \le |\nabla f|$ a.e., by the definition of λ_*, we also have $B_U[|f_*|,|f_*|] = \lambda_*$.

Denote $S_< := \left\{ f \in H^1(\mathbb{R}^n_+) \mid f \text{ satisfies (D.15) with some } \lambda_0 < 0 \text{ in the weak sense} \right\}$. By the Lagrange multiplier method, $f_*, |f_*| \in S_<$ with the eigenvalue λ_*. Step 1 shows that the elements in $S_<$ are smooth with exponential decay. By strong maximum principle and Hopf theorem, $|f_*| > 0$ in $\overline{\mathbb{R}^n_+}$.

Step 4. Claim: the dimension of $S_<$ is 1.

Assume the opposite that there exists two linearly independent functions $f_1, f_2 \in S_<$ with negative eigenvalues λ_1, λ_2 respectively. Note that $\lambda_1(f_1,f_2)_{L^2(\mathbb{R}^n_+)} = B_U[f_1,f_2] = \lambda_2(f_1,f_2)_{L^2(\mathbb{R}^n_+)}$. We can assume $(f_1,f_2)_{L^2(\mathbb{R}^n_+)} = 0$ since if

$\lambda_1 \neq \lambda_2$, it holds automatically, and if $\lambda_1 = \lambda_2$, we replace f_1 by $f_1 - \|f_2\|^{-2}_{L^2(\mathbb{R}^n_+)}(f_1, f_2)_{L^2(\mathbb{R}^n_+)} f_2 \not\equiv 0$ by the linear independence of f_1, f_2. Then $B_U[f_1, f_2] = 0$. For $i = 1, 2$, $f_i \neq 0$ deduces $B_U[f_i, f_i] = \lambda_i \|f_i\|^2_{L^2(\mathbb{R}^n_+)} < 0$.

Since $J[\cdot] \in C^2\left(D^{1,2}(\mathbb{R}^n_+), \mathbb{R}\right)$, $\langle J''[g]g, g\rangle = (1-p)\int_{\mathbb{R}^n_+} |\nabla g|^2 dx \neq 0$ for $g \in \mathbf{Ne}$, and (D.6) holds, by [43, Proposition 5.75], $\mathbf{Ne}$ is a complete C^1-Banach submanifold of $D^{1,2}(\mathbb{R}^n_+)$ of codimension 1. Since $U(x) \in \mathbf{Ne}$ for $n > 2$, in particular, $T_{\mathbf{Ne}}U$ is codimension 1 in $D^{1,2}(\mathbb{R}^n_+)$.

Thus, there exists $w_0 \in D^{1,2}(\mathbb{R}^n_+) \backslash T_{\mathbf{Ne}}U$, such that $f_1 = a_1 w_0 + a_2 w_1$, $f_2 = b_1 w_0 + b_2 w_2$ for some constants $a_1, b_1, a_2, b_2 \in \mathbb{R}$ and $w_1, w_2 \in T_{\mathbf{Ne}}U$. Taking $c_1 = c_2 = 1, \tilde{v} = 0$ in (D.5) and then using Lemma D.2, (D.9), we get that $U(x)$ given in (1.13) attains $\inf\limits_{f \in \mathbf{Ne}} J[f]$. By Lemma D.1 and $B_U[f_i, f_i] < 0$, $i = 1, 2$, we have $a_1 \neq 0, b_1 \neq 0$. So we can set $f_3 := f_1 - b_1^{-1} a_1 f_2 \in T_{\mathbf{Ne}}U$. One using $B_U[f_1, f_2] = 0$, then $B_U[f_3, f_3] = B_U[f_1, f_1] + (b_1^{-1}a_1)^2 B_U[f_2, f_2] < 0$, which contradicts with Lemma D.1.

Taking $\lambda_0 = \lambda_*$ and $Z_0 = |f_*|$, we complete the proof. □

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