# 1 <br> ESTIMATES FOR LIOUVILLE EQUATION WITH QUANTIZED SINGULARITIES 

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#### Abstract

For Liouville equations with singular sources, it is well known that blowup solutions may exhibit non-simple blowup phenomenon if the blowup point happens to be the singular source and the strength of the singular source is a multiple of $4 \pi$. In this article we prove that even in this case some coefficient functions must vanish at the singular source and bubbling solutions can still be accurately approximated by global solutions.


## 1. Introduction

In this article we study bubbling solutions of the following singular Liouville equation

$$
\begin{equation*}
\Delta u+h(x) e^{u}=4 \pi \alpha \delta_{0} \quad \text { in } \quad \Omega \subset \mathbb{R}^{2} \tag{1.1}
\end{equation*}
$$

where $\Omega$ is an open, bounded subset of $\mathbb{R}^{2}$ that contains the origin, $\alpha>-1$ is a constant and $\delta_{0}$ is the Dirac mass at $0, h$ is a positive and smooth function. One of the main difficulties in the study of blowup solutions to (1.1) is when the blowup point happens to be the location of a singular source. It is known from the works of Kuo-Lin [16], Bartolucci-Tarantello [4] that if $\alpha \notin \mathbb{N}$ ( the set of natural numbers) blowup solutions satisfy spherical Harnack inequality around the singular source and the asymptotic behavior is relatively easy to understand. However, when the strength of the singular source is a multiple of $4 \pi(\alpha \in \mathbb{N})$, the so called "nonsimple blowup" phenomenon does occur, which means the bubbling solutions may not satisfy spherical Harnack inequality and multiple local maximums near the singular source could appear. In this article we prove new estimates for non-simple bubbling solutions. Since the analysis is carried out in a neighborhood of the singular source, we just require the domain to be a small neighborhood of the origin, so our assumption of bubbling solutions is as follows: Let $\tilde{u}_{k}$ be a sequence of solutions of

$$
\begin{equation*}
\Delta \tilde{u}_{k}(x)+\tilde{h}_{k}(x) e^{\tilde{u}_{k}}=4 \pi N \delta_{0}, \quad \text { in } \quad B_{\tau} \tag{1.2}
\end{equation*}
$$

for some $\tau>0$ independent of $k . B_{\tau}$ is the ball centered at the origin with radius $\tau$. In addition we postulate the usual assumptions on $\tilde{u}_{k}$ and $\tilde{h}_{k}$ : For a positive

[^0]constant $C$ independent of $k$, the following holds:
\[

\left\{$$
\begin{array}{l}
\left\|\tilde{h}_{k}\right\|_{C^{3}\left(\bar{B}_{\tau}\right)} \leq C, \quad \frac{1}{C} \leq \tilde{h}_{k}(x) \leq C, \quad x \in \bar{B}_{\tau},  \tag{1.3}\\
\int_{B_{\tau}} \tilde{h}_{k} e^{\tilde{u}_{k}} \leq C, \\
\left|\tilde{u}_{k}(x)-\tilde{u}_{k}(y)\right| \leq C, \quad \forall x, y \in \partial B_{\tau},
\end{array}
$$\right.
\]

and since we study the asymptotic behavior of blowup solutions around the singular source, we assume that there is no blowup point except at the origin:

$$
\begin{equation*}
\max _{K \subset \subset B_{\tau} \backslash\{0\}} \tilde{u}_{k} \leq C(K) \tag{1.4}
\end{equation*}
$$

Also, for the convenience of notation we assume $\tilde{h}_{k}(0)=1$ and use the value of $\tilde{u}_{k}$ on $\partial B_{\tau}$ to define a harmonic function $\phi_{k}(x)$ :

$$
\left\{\begin{array}{l}
\Delta \phi_{k}(x)=0, \quad \text { in } \quad B_{\tau}  \tag{1.5}\\
\phi_{k}(x)=u_{k}(x)-\frac{1}{2 \pi \tau} \int_{\partial B_{\tau}} \tilde{u}_{k} d S, \quad x \in \partial B_{\tau}
\end{array}\right.
$$

Using the fact that $\Delta\left(\frac{1}{2 \pi} \log |x|\right)=\delta_{0}$, we set

$$
\begin{equation*}
u_{k}(x)=\tilde{u}_{k}(x)-2 N \log |x|-\phi_{k}(x) \tag{1.6}
\end{equation*}
$$

which satisfies

$$
\begin{equation*}
\Delta u_{k}(x)+|x|^{2 N} h_{k}(x) e^{u_{k}}=0, \quad \text { in } \quad B_{\tau} \tag{1.7}
\end{equation*}
$$

for

$$
\begin{equation*}
h_{k}(x)=\tilde{h}_{k}(x) e^{\phi_{k}(x)} \tag{1.8}
\end{equation*}
$$

It is easy to see that $\phi_{k}(0)=0$ and $u_{k}$ is a constant on $\partial B_{\tau}$.
In this article we consider the case that:

$$
\begin{equation*}
\max _{x \in B_{1}} u_{k}(x)+2(1+N) \log |x| \rightarrow \infty \tag{1.9}
\end{equation*}
$$

which is equivalent to saying that the spherical Harnack inequality does not hold for $u_{k}$. It is well known [16] that $u_{k}$ exhibits a non-simple blowup profile. It is established in [16, 4] that there are $N+1$ local maximum points of $u_{k}: p_{0}^{k}, \ldots ., p_{N}^{k}$ and they are evenly distributed on $\mathbb{S}^{1}$ after scaling according to their magnitude: Suppose along a subsequence

$$
\lim _{k \rightarrow \infty} p_{0}^{k} /\left|p_{0}^{k}\right|=e^{i \theta_{0}}
$$

then

$$
\lim _{k \rightarrow \infty} \frac{p_{l}^{k}}{\left|p_{0}^{k}\right|}=e^{i\left(\theta_{0}+\frac{2 \pi l}{N+1}\right)}, \quad l=1, \ldots, N
$$

For many reasons it is convenient to denote $\left|p_{0}^{k}\right|$ as $\delta_{k}$ and define $\mu_{k}$ as follows:

$$
\begin{equation*}
\delta_{k}=\left|p_{0}^{k}\right| \quad \text { and } \quad \mu_{k}=u_{k}\left(p_{0}^{k}\right)+2(1+N) \log \delta_{k} \tag{1.10}
\end{equation*}
$$

Since $p_{l}^{k}$ s are evenly distributed around $\partial B_{\delta_{k}}$, standard results for Liouville equations around a regular blowup point can be applied to have $u_{k}\left(p_{l}^{k}\right)=u_{k}\left(p_{0}^{k}\right)+$
$o(1)$. Also, (1.9) gives $\mu_{k} \rightarrow \infty$. The interested readers may look into [16, 4] for more detailed information.

The first main theorem is about using a sequence of global solutions of

$$
\begin{equation*}
\Delta U+|x|^{2 N} e^{U}=0, \quad \text { in } \quad \mathbb{R}^{2}, \quad \int_{\mathbb{R}^{2}}|x|^{2 N} e^{U}<\infty \tag{1.11}
\end{equation*}
$$

to approximate $u_{k}$. For regular Liouville equation, this type of approximation, initiated by Y.Y.Li [17], and further extended and refined by a series of works [2, 4, 9, 13, 28, 29] played an important role in a number of applications such as degree counting theorems [9, 10], uniqueness results [3], etc. Our Theorem 1.1 below seems to be the first such result for quantized singular sources:

Theorem 1.1. Let $u_{k}, \phi_{k}, h_{k}, \delta_{k}, \mu_{k}$ be defined by (1.7), (1.5), (1.8), (1.10) respectively. If $\delta_{k}^{2} / e^{-\mu_{k}} \leq c_{0}$ for some $c_{0}>0$ independent of $k$, we have, for some $c_{1}>0$ independent of $k$ and a sequence of global solutions $U_{k}$ of (1.11) such that

$$
\left|u_{k}(x)-\phi_{k}(x)-U_{k}(x)\right| \leq c_{1}\left(\delta_{k} e^{\mu_{k} / 2}+\mu_{k}^{2} e^{-\mu_{k}}\right), \quad x \in B_{\tau}
$$

Remark 1.1. For $\operatorname{dist}(x, 0) \sim 1, u_{k}(x)=-u_{k}\left(p_{0}^{k}\right)+O(1)$. This is already established in [16].

If $\delta_{k}^{2} /\left(\mu_{k} e^{-\mu_{k}}\right) \rightarrow \infty$, we set

$$
\begin{equation*}
E_{k}=O\left(\delta_{k}^{-2} \mu_{k} e^{-\mu_{k}}\right)+O\left(\delta_{k}\right) . \tag{1.12}
\end{equation*}
$$

Then in the second main result we prove the following vanishing estimates for the second derivatives of $\log h_{k}$ :
Theorem 1.2. Under the same context of Theorem [1.1] if $\delta_{k}^{2} /\left(\mu_{k} e^{-\mu_{k}}\right) \rightarrow \infty$, we have,

$$
\begin{equation*}
\Delta\left(\log h_{k}\right)(0)=E_{k}, \quad \text { if } \quad N \geq 2 \tag{1.13}
\end{equation*}
$$

where $E_{k}$ is defined in (I.12), and for $N=1$, we have

$$
\begin{align*}
\left(\partial_{e_{k}}\left(\log h_{k}\right)(0)\right)^{2}-\left(\partial_{e_{k}^{\perp}}\left(\log h_{k}\right)(0)\right)^{2}-2 \partial_{e_{k} e_{k}}\left(\log h_{k}\right)(0) & =E_{k},  \tag{1.14}\\
\partial_{e_{k}}\left(\log h_{k}\right)(0) \partial_{e_{k}^{\perp}}\left(\log h_{k}\right)(0)+\partial_{e_{k} e_{k}^{\perp}}\left(\log h_{k}\right)(0) & =E_{k},
\end{align*}
$$

where $e_{k}=p_{0}^{k} /\left|p_{0}^{k}\right|, e_{k}^{\perp}$ is an unit vector orthogonal to $e_{k}$.
Remark 1.2. Theorem [1.2 is surprising because it is usually difficult to obtain vanishing estimates at a singular source. There are many cancellations for Pohozaev identities around the singular source. However we would like to point out that some vanishing estimates for bubbling solutions of Toda systems have been obtained exactly at singular sources [20], [30].

Remark 1.3. The dichotomy in Theorem 1.1 and Theorem 1.2 appears to be contradictory to Lemma 9 of [16], which asserts that $\delta_{k}^{2}=c \mu_{k} e^{-\mu_{k}}(1+o(1))$. However we found (4.18) of [16] incorrect. In fact there should not be any deterministic
relation between $\mu_{k}$ and $\delta_{k}$, because for any $\xi_{k} \in \mathbb{R}^{2}$ and any $\lambda_{k} \in \mathbb{R}$,

$$
U_{k}(x)=\log \frac{e^{\lambda_{k}}}{\left(1+\frac{e^{\lambda_{k}}}{8(1+N)^{2}}\left|x^{N+1}-\xi_{k}\right|^{2}\right)^{2}}
$$

is a sequence of solutions to $\Delta U_{k}+|x|^{2 N} e^{U_{k}}=0$.
The study of bubbling solutions of (1.7) near the quantized singular source represents a core difficulty in many related problems. For example the following mean field equation defined on a Riemann surface $M$ :

$$
\begin{equation*}
\Delta u(x)+\rho\left(\frac{h(x) e^{u(x)}}{\int_{M} h(x) e^{u} d x}-\frac{1}{\operatorname{vol}(M)}\right)=4 \pi \sum_{j=1}^{d} \alpha_{j}\left(\delta_{q_{j}}-\frac{1}{\operatorname{vol}(M)}\right), \tag{1.15}
\end{equation*}
$$

represents a metric on $M$ with conic singularity. Also it is derived from the mean field limit of point vortices in the Euler flow [6, 7] and serves as a model equation in the Chern-Simons-Higgs theory [15] and in the electroweak theory [1], etc. The rich geometric and physical background manifests the importance of the study in this article.

The phenomena of non-simple bubbling solutions not only occur in single equations, but also in systems. In a recent work of the second author and Gu [14], the non-simple blowup solutions are studied for singular Liouville systems.

To end the introduction we would like to briefly explain the idea of the proof. In [16] and [4] it is already established that there are exactly $N+1$ local maximum points evenly distributed around the origin. Kuo-Lin and Bartolucci-Tarantello independently obtained this important information by studying the Pohozaev identity around each local maximum. The main contribution of this article is to go further in this investigation. Roughly speaking, what is achieved in [16, 4] is information contained in the leading terms in those Pohozaev identities. By studying more terms in the expansion of these identities we found further important information on the location of these local maximums and corresponding geometric quantities. From our proof the interested readers will see the more precise information about the location of local maximum points, which should be very useful for constructing such solutions in related studies.

The organization of this article is as follows: In section two we establish some preliminary estimates, in section three we establish precise locations of local maximum points. The approximation by global solutions (Theorem 1.1) is proved in section four and the proof of vanishing theorem (Theorem [1.2) is arranged in section five. Finally in the appendix we prove a sharp estimate of bubbling solutions if the spherical Harnack inequality holds.

Notation: We will use $B\left(x_{0}, r\right)$ to denote a ball centered at $x_{0}$ with radius $r$. If $x_{0}$ is the origin we use $B_{r}$. $C$ represents a positive constant that may change from place to place.

## 2. Preliminary Discussions

Writing $p_{0}^{k}$ as $p_{0}^{k}=\delta_{k} e^{i \theta_{k}}$ we define $v_{k}$ as

$$
\begin{equation*}
v_{k}(y)=u_{k}\left(\delta_{k} y e^{i \theta_{k}}\right)+2(N+1) \log \delta_{k}, \quad|y|<\tau \delta_{k}^{-1} \tag{2.1}
\end{equation*}
$$

If we write out each component, (2.1) is

$$
v_{k}\left(y_{1}, y_{2}\right)=u_{k}\left(\delta_{k}\left(y_{1} \cos \theta_{k}-y_{2} \sin \theta_{k}\right), \delta_{k}\left(y_{1} \sin \theta_{k}+y_{2} \cos \theta_{k}\right)\right)+2(1+N) \log \delta_{k}
$$

Then it is standard to verify that $v_{k}$ solves

$$
\begin{equation*}
\Delta v_{k}(y)+|y|^{2 N} \mathfrak{h}_{k}\left(\delta_{k} y\right) e^{v_{k}(y)}=0, \quad|y|<\tau / \delta_{k} \tag{2.2}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathfrak{h}_{k}(x)=h_{k}\left(x e^{i \theta_{k}}\right), \quad|x|<\tau . \tag{2.3}
\end{equation*}
$$

Thus the image of $p_{0}^{k}$ after scaling is $Q_{1}^{k}=e_{1}=(1,0)$. Let $Q_{1}^{k}, Q_{2}^{k}, \ldots, Q_{N}^{k}$ be the images of $p_{i}^{k}(i=1, \ldots, N)$ after the scaling. It is established by Kuo-Lin in [16] and independently by Bartolucci-Tarantello in [4] that

$$
\begin{equation*}
\lim _{k \rightarrow \infty} Q_{l}^{k}=\lim _{k \rightarrow \infty} p_{l}^{k} / \delta_{k}=e^{\frac{2 l \pi i}{N+1}}, \quad l=0, \ldots, N \tag{2.4}
\end{equation*}
$$

Choosing $\varepsilon>0$ small and independent of $k$, we can make disks centered at $Q_{l}^{k}$ with radius $\varepsilon$ (denoted as $B\left(Q_{l}^{k}, \varepsilon\right)$ ) mutually disjoint. Let

$$
\begin{equation*}
\mu_{k}=\max _{B\left(Q_{0}^{k}, \varepsilon\right)} v_{k} \tag{2.5}
\end{equation*}
$$

Since $Q_{l}^{k}$ are evenly distributed around $\partial B_{1}$, it is easy to use standard estimates for single Liouville equations ([28, 13, 9]) to obtain

$$
\max _{B\left(Q_{l}^{k}, \varepsilon\right)} v_{k}=\mu_{k}+o(1), \quad l=1, \ldots, N
$$

Recall that $v_{k}$ satisfies (2.2) and $v_{k}$ is a constant on $\partial B\left(0, \tau \delta_{k}^{-1}\right)$. The Green's representation formula for $v_{k}$ gives,

$$
v_{k}(y)=\int_{\Omega_{k}} G(y, \eta)|\eta|^{2 N_{\mathfrak{h}}} \mathfrak{h}_{k}\left(\delta_{k} \eta\right) e^{v_{k}(\eta)} d \eta+\left.v_{k}\right|_{\partial \Omega_{k}}
$$

where $\Omega_{k}=B\left(0, \tau \delta_{k}^{-1}\right)$ and

$$
G(y, \eta)=-\frac{1}{2 \pi} \log |y-\eta|+H(y, \eta)
$$

where

$$
H(y, \eta)=\frac{1}{2 \pi} \log \left(\frac{|\eta|}{\tau \delta_{k}^{-1}}\left|\frac{\tau^{2} \delta_{k}^{-2} \eta}{|\eta|^{2}}-y\right|\right)
$$

Also for $r>2$, let $\bar{v}_{k}(r)$ be the spherical average of $v_{k}$ on $\partial B_{r}$, then we have

$$
\frac{d}{d r} \bar{v}_{k}(r)=\frac{d}{d r} \frac{1}{2 \pi r} \int_{B_{r}} \Delta v_{k}=-\frac{8(N+1) \pi+o(1)}{2 \pi r}
$$

Because of the fast decay of $\bar{v}_{k}(r)$ it is easy to use the Green's representation of $v_{k}$ to obtain the following stronger estimate of $v_{k}$ :

$$
\begin{equation*}
v_{k}(y)=-\mu_{k}-(4 N+4) \log |y|+O(1), \quad 2<|y|<\tau \delta_{k}^{-1} \tag{2.6}
\end{equation*}
$$

Now we consider $v_{k}$ around $Q_{l, k}$. Using the results in [9, 28, 13] we have, for $v_{k}$ in $B\left(Q_{l, k}, \varepsilon\right)$, the following gradient estimate:

$$
\begin{equation*}
\delta_{k} \nabla\left(\log \mathfrak{h}_{k}\right)\left(\delta_{k} \tilde{Q}_{l, k}\right)+2 N \frac{\tilde{Q}_{l, k}}{\left|\tilde{Q}_{l, k}\right|^{2}}+\nabla \phi_{l, k}\left(\tilde{Q}_{l, k}\right)=O\left(\mu_{k} e^{-\mu_{k}}\right) \tag{2.7}
\end{equation*}
$$

where $\phi_{l, k}$ is the harmonic function that eliminates the oscillation of $v_{k}$ on $\partial B\left(Q_{l}^{k}, \varepsilon\right)$ and $\tilde{Q}_{l, k}$ is the maximum of $v_{k}-\phi_{l, k}$ that satisfies

$$
\begin{equation*}
\tilde{Q}_{l, k}-Q_{l, k}=O\left(e^{-\mu_{k}}\right) \tag{2.8}
\end{equation*}
$$

Using (2.8) in (2.7) we have

$$
\begin{equation*}
\delta_{k} \nabla\left(\log \mathfrak{h}_{k}\right)\left(\delta_{k} Q_{l, k}\right)+2 N \frac{Q_{l, k}}{\left|Q_{l, k}\right|^{2}}+\nabla \phi_{l, k}\left(Q_{l, k}\right)=O\left(\mu_{k} e^{-\mu_{k}}\right) \tag{2.9}
\end{equation*}
$$

For the discussion in this section we use the following version of (2.9):

$$
\begin{equation*}
\delta_{k} \nabla\left(\log \mathfrak{h}_{k}\right)(0)+2 N \frac{Q_{l, k}}{\left|Q_{l, k}\right|^{2}}+\nabla \phi_{l, k}\left(Q_{l, k}\right)=O\left(\delta_{k}^{2}\right)+O\left(\mu_{k} e^{-\mu_{k}}\right) \tag{2.10}
\end{equation*}
$$

The following lemma provides the first estimate of $\nabla \phi_{l}^{k}\left(Q_{l}^{k}\right)$ :
Lemma 2.1. For $l=0, \ldots, N$,

$$
\begin{equation*}
\nabla \phi_{l}^{k}\left(Q_{l}^{k}\right)=-4 \sum_{m=0, m \neq l}^{N} \frac{Q_{l, k}-Q_{m, k}}{\left|Q_{l, k}-Q_{m, k}\right|^{2}}+E \tag{2.11}
\end{equation*}
$$

where

$$
E=O\left(\delta_{k}^{2}\right)+O\left(\mu_{k} e^{-\mu_{k}}\right)
$$

## Proof of Lemma 2.1:

From the expression of $v_{k}$ on $\Omega_{k}=B\left(0, \tau \delta_{k}^{-1}\right)$ we have, for $y$ away from bubbling disks,

$$
\begin{align*}
v_{k}(y) & =\left.v_{k}\right|_{\partial \Omega_{k}}+\int_{\Omega_{k}} G(y, \eta)|\eta|^{2 N} \mathfrak{h}_{k}\left(\delta_{k} \eta\right) e^{v_{k}(\eta)} d \eta  \tag{2.12}\\
& =\left.v_{k}\right|_{\partial \Omega_{k}}+\sum_{l=0}^{N} G\left(y, Q_{l}^{k}\right) \int_{B\left(Q_{l}, \varepsilon\right)}|\eta|^{2 N} \mathfrak{h}_{k}\left(\delta_{k} \eta\right) e^{v_{k}} d \eta \\
& +\sum_{l} \int_{B\left(Q_{l}, \varepsilon\right)}\left(G(y, \eta)-G\left(y, Q_{l}^{k}\right)\right)|\eta|^{2 N_{\mathfrak{h}}} \mathfrak{h}_{k}\left(\delta_{k} \eta\right) e^{v_{k}} d \eta+O\left(\mu_{k} e^{-\mu_{k}}\right)
\end{align*}
$$

Before we evaluate each term, we use a sample computation: Suppose $f$ is a smooth function defined on $B\left(Q_{0}^{k}, \varepsilon\right)$, then we evaluate

$$
\begin{equation*}
\int_{B\left(Q_{0}^{k}, \varepsilon\right)} f(\eta)|\eta|^{2 N} \mathfrak{h}_{k}\left(\delta_{k} \eta\right) e^{v_{k}(\eta)} d \eta \tag{2.13}
\end{equation*}
$$

Let $\tilde{Q}_{0}^{k}$ be the maximum of $v_{k}-\phi_{0}^{k}$, then it is known [28, 13] that

$$
\begin{equation*}
\tilde{Q}_{0}^{k}-Q_{0}^{k}=O\left(e^{-\mu_{k}}\right) . \tag{2.14}
\end{equation*}
$$

Moreover, it is derived that

$$
\begin{equation*}
v_{k}(y)-\phi_{0}^{k}(y)=\log \frac{e^{\mu_{k}}}{\left(1+e^{\mu_{k}} \frac{\mid \underline{Q}_{0}^{\left.\tilde{L}^{2 N}\right|^{\mathcal{H}}\left(\delta_{k} \tilde{Q}_{0}^{k}\right)}}{8}\left|y-\tilde{Q}_{0}^{k}\right|^{2}\right)^{2}}+O\left(\mu_{k}^{2} e^{-\mu_{k}}\right) . \tag{2.15}
\end{equation*}
$$

Setting $\hat{v}_{k}=v_{k}-\phi_{0}^{k}$ and $\hat{h}_{k}=e^{\phi_{0}^{k}}|y|^{2 N} \mathfrak{h}_{k}\left(\delta_{k} y\right)$, we can write (2.13) as

$$
\int_{B\left(Q_{0}^{k}, \varepsilon\right)} f(\eta) \hat{h}_{k}(\eta) e^{\hat{v}_{k}(\eta)} d \eta
$$

Using the Taylor expansions of $f$ and $\hat{h}_{k}$ around $\tilde{Q}_{0}^{k}$ and the symmetry of the global solution in (2.15) it is easy to see that

$$
\int_{B\left(Q_{0}^{k}, \varepsilon\right)} f(\eta)|\eta|^{2 N} \mathfrak{h}_{k}\left(\delta_{k} \eta\right) e^{v_{k}(\eta)} d \eta=8 \pi f\left(\tilde{Q}_{0}^{k}\right)+O\left(\mu_{k} e^{-\mu_{k}}\right)
$$

Since (2.14) holds we further have

$$
\begin{equation*}
\int_{B\left(Q_{0}^{k}, \varepsilon\right)} f(\eta)|\eta|^{2 N_{\mathfrak{h}}} \mathfrak{h}_{k}\left(\delta_{k} \eta\right) e^{v_{k}(\eta)} d \eta=8 \pi f\left(Q_{0}^{k}\right)+O\left(\mu_{k} e^{-\mu_{k}}\right) \tag{2.16}
\end{equation*}
$$

Using the method of (2.16) in the evaluation of each term in (2.12) we have,

$$
v_{k}(y)=\left.v_{k}\right|_{\partial \Omega_{k}}-4 \sum_{l=0}^{N} \log \left|y-Q_{l, k}\right|+8 \pi \sum_{l=0}^{N} H\left(y, Q_{l, k}\right)+O\left(\mu_{k} e^{-\mu_{k}}\right) .
$$

The harmonic function that kills the oscillation of $v_{k}$ around $Q_{m, k}$ is

$$
\begin{aligned}
& \phi_{m}^{k}=-4 \sum_{l=0, l \neq m}^{N}\left(\log \left|y-Q_{l}^{k}\right|-\log \left|Q_{m}^{k}-Q_{l}^{k}\right|\right) \\
& +8 \pi \sum_{l=0}^{N}\left(H\left(y, Q_{l}^{k}\right)-H\left(Q_{m}^{k}, Q_{l}^{k}\right)\right)+O\left(\mu_{k} e^{-\mu_{k}}\right) .
\end{aligned}
$$

The corresponding estimate for $\nabla \phi_{m}^{k}$ is

$$
\nabla \phi_{m}^{k}\left(Q_{m}^{k}\right)=-4 \sum_{l=0, l \neq m}^{N} \frac{Q_{m}^{k}-Q_{l}^{k}}{\left|Q_{m}^{k}-Q_{l}^{k}\right|^{2}}+8 \pi \sum_{l=0}^{N} \nabla_{1} H\left(Q_{m}^{k}, Q_{l}^{k}\right)+O\left(\mu_{k} e^{-\mu_{k}}\right) .
$$

where $\nabla_{1}$ stands for the differentiation with respect to the first component. From the expression of $H$, we have

$$
\begin{align*}
\nabla_{1} H\left(Q_{m}^{k}, Q_{l}^{k}\right) & =\frac{1}{2 \pi} \frac{Q_{m}^{k}-\tau^{2} \delta_{k}^{-2} Q_{l}^{k} /\left|Q_{l}^{k}\right|^{2}}{\left|Q_{m}^{k}-\tau^{2} \delta_{k}^{-2} Q_{l}^{k} /\left|Q_{l}^{k}\right|^{2}\right|^{2}}  \tag{2.17}\\
& =\frac{1}{2 \pi} \tau^{-2} \delta_{k}^{2} \frac{\tau^{-2} \delta_{k}^{2} Q_{m}^{k}-Q_{l}^{k} /\left|Q_{l}^{k}\right|^{2}}{\left|Q_{l}^{k} /\left|Q_{l}^{k}\right|^{2}-\tau^{-2} \delta_{k}^{2} Q_{m}^{k}\right|^{2}} \\
& =-\frac{1}{2 \pi} \tau^{-2} \delta_{k}^{2} e^{\frac{2 \pi i l}{N+1}}+O\left(\sigma_{k} \delta_{k}^{2}\right) .
\end{align*}
$$

where $\sigma_{k}=\max _{l}\left|Q_{l}^{k}-e^{\frac{2 \pi i l}{N+1}}\right|$. Later we shall obtain more specific estimate of $\sigma_{k}$.

Thus

$$
\begin{align*}
& \nabla \phi_{m}^{k}\left(Q_{m}^{k}\right)  \tag{2.18}\\
= & -4 \sum_{l=0, l \neq m}^{N} \frac{Q_{m}^{k}-Q_{l}^{k}}{\left|Q_{m}^{k}-Q_{l}^{k}\right|^{2}}-4 \tau^{-2} \delta_{k}^{2} \sum_{l=0}^{N} e^{\frac{2 \pi i l}{N+1}}+O\left(\sigma_{k} \delta_{k}^{2}\right)+O\left(\mu_{k} e^{-\mu_{k}}\right) \\
= & -4 \sum_{l=0, l \neq m}^{N} \frac{Q_{m}^{k}-Q_{l}^{k}}{\left|Q_{m}^{k}-Q_{l}^{k}\right|^{2}}+O\left(\sigma_{k} \delta_{k}^{2}\right)+O\left(\mu_{k} e^{-\mu_{k}}\right)
\end{align*}
$$

where we have used $\sum_{l=0}^{N} e^{2 \pi l i /(N+1)}=0$. Since we don't have the estimate of $\sigma_{k}$ now we have

$$
\nabla \phi_{m}^{k}\left(Q_{m}^{k}\right)=-4 \sum_{l=0, l \neq m}^{N} \frac{Q_{m}^{k}-Q_{l}^{k}}{\left|Q_{m}^{k}-Q_{l}^{k}\right|^{2}}+E
$$

Lemma 2.1 is established.

## 3. Location of blowup points

Let $E=O\left(\delta_{k}^{2}\right)+O\left(\mu_{k} e^{-\mu_{k}}\right)$. The Pohozaev identity around $Q_{l}^{k}$ now reads

$$
-4 \sum_{j=0, j \neq l}^{N} \frac{Q_{l}^{k}-Q_{j}^{k}}{\left|Q_{l}^{k}-Q_{j}^{k}\right|^{2}}+2 N \frac{Q_{l}^{k}}{\left|Q_{l}^{k}\right|^{2}}=-\nabla\left(\log \mathfrak{h}_{k}\right)(0) \delta_{k}+E
$$

Using $L$ to denote $\nabla\left(\log \mathfrak{h}_{k}\right)(0)$, we have, treating every term as a complex number,

$$
N \frac{1}{Q_{l}^{k}}=2 \sum_{j=0, j \neq l}^{N} \frac{1}{Q_{l}^{k}-Q_{j}^{k}}-\frac{\bar{L}}{2} \delta_{k}+E
$$

where $\bar{L}$ is the conjugate of $L$. Thus

$$
\begin{equation*}
N=2 \sum_{j=0, j \neq l}^{N} \frac{Q_{l}^{k}}{Q_{l}^{k}-Q_{j}^{k}}-\frac{\bar{L}}{2} \delta_{k} Q_{l}^{k}+E . \tag{3.1}
\end{equation*}
$$

Let $\beta_{l}=2 \pi l /(N+1)$, we write $Q_{l}^{k}=e^{i \beta_{l}}+p_{l}^{k}$ for $p_{l}^{k} \rightarrow 0$. Then we write the first term on the right hand side of (3.1) as

$$
\begin{aligned}
& \frac{Q_{l}^{k}}{Q_{l}^{k}-Q_{j}^{k}}=\frac{e^{i \beta_{l}}+p_{l}^{k}}{e^{i \beta_{l}}-e^{i \beta_{j}}+p_{l}^{k}-p_{j}^{k}} \\
= & \frac{e^{i \beta_{l}}+p_{l}^{k}}{\left(e^{i \beta_{l}}-e^{i \beta_{j}}\right)\left(1+\left(p_{l}^{k}-p_{j}^{k}\right) /\left(e^{i \beta_{l}}-e^{i \beta_{j}}\right)\right)} \\
= & \frac{e^{i \beta_{l}}}{e^{i \beta_{l}}-e^{i \beta_{j}}}+\frac{p_{l}^{k}}{e^{i \beta_{l}}-e^{i \beta_{j}}}-\frac{e^{i \beta_{l}}}{\left(e^{i \beta_{l}}-e^{i \beta_{j}}\right)^{2}}\left(p_{l}^{k}-p_{j}^{k}\right)+O\left(\sigma_{k}^{2}\right) \\
= & \frac{e^{i \beta_{l}}}{e^{i \beta_{l}}-e^{i \beta_{j}}}+\frac{e^{i \beta_{l}} p_{j}^{k}-e^{i \beta_{j}} p_{l}^{k}}{\left(e^{i \beta_{l}}-e^{i \beta_{j}}\right)^{2}}+O\left(\sigma_{k}^{2}\right) .
\end{aligned}
$$

Using

$$
\begin{equation*}
N=2 \sum_{j=0, j \neq l}^{N} \frac{e^{i \beta_{l}}}{e^{i \beta_{l}}-e^{i \beta_{j}}}, \tag{3.2}
\end{equation*}
$$

we write (3.1) as

$$
\begin{equation*}
\sum_{j=0, j \neq l}^{N} \frac{e^{i \beta_{l}} p_{j}^{k}-e^{i \beta_{j}} p_{l}^{k}}{\left(e^{i \beta_{l}}-e^{i \beta_{j}}\right)^{2}}-\frac{\bar{L}}{4} \delta_{k} e^{i \beta_{l}}=E+O\left(\sigma_{k}^{2}\right) \tag{3.3}
\end{equation*}
$$

for $l=0,1,2, \ldots, N$. Setting $p_{l}^{k}=e^{i \beta_{l}} m_{l}^{k}$ and $\beta_{j l}=\beta_{j}-\beta_{l}$ we reduce (3.3) to

$$
\begin{align*}
& \sum_{j=0, j \neq l}^{N} \frac{e^{i \beta_{j l}} m_{j}^{k}}{\left(1-e^{i \beta_{j l}}\right)^{2}}-\left(\sum_{j=0, j \neq l}^{N} \frac{e^{i \beta_{j l}}}{\left(1-e^{i \beta_{j l}}\right)^{2}}\right) m_{l}^{k}-\frac{\bar{L}}{4} \delta_{k} e^{i \beta_{l}}  \tag{3.4}\\
& =E+O\left(\sigma_{k}^{2}\right)+O\left(\delta_{k} \sigma_{k}\right)
\end{align*}
$$

for $l=0,1 \ldots . ., N$. It is easy to verify that

$$
\begin{equation*}
\frac{e^{i \theta}}{\left(1-e^{i \theta}\right)^{2}}=\frac{1}{2(\cos \theta-1)}=\left(-\frac{1}{4}\right) \frac{1}{\sin ^{2}(\theta / 2)} . \tag{3.5}
\end{equation*}
$$

To deal with coefficients of $m_{j}^{k}$ in (3.4) we set

$$
d_{j}=\frac{1}{\sin ^{2}\left(\frac{j \pi}{N+1}\right)}, \quad j=1, \ldots, N
$$

and

$$
D=\sum_{j=0, j \neq l}^{N} d_{|j-l|} .
$$

Since $d_{l}=d_{N+1-l}$ it is easy to check that $D$ does not depend on $l$ :

$$
D=\sum_{k=1}^{N} d_{k}=\sum_{k=1}^{N} \frac{1}{\sin ^{2}\left(\frac{k \pi}{N+1}\right)} .
$$

Now (3.4) can be written as

$$
\begin{equation*}
-\sum_{j \neq l, j=0}^{N} d_{|j-l|} m_{j}^{k}+D m_{l}^{k}-\bar{L} \delta_{k} e^{i \beta_{l}}=E+O\left(\sigma_{k}^{2}\right), \quad l=0, \ldots ., N . \tag{3.6}
\end{equation*}
$$

For $l=0$, we have $\beta_{0}=0$ and $m_{0}^{k}=0$. Thus from (3.6) we have

$$
\begin{equation*}
-\sum_{j=1}^{N} d_{j} m_{j}^{k}-\bar{L} \delta_{k}=E+O\left(\sigma_{k}^{2}\right) . \tag{3.7}
\end{equation*}
$$

If we take $\left(m_{1}^{k}, \ldots, m_{n}^{k}\right)$ as unknowns in (3.6), the last $N$ equations of (3.6) ( for $l=1, \ldots, N$ ) can be written as

$$
A\left(\begin{array}{c}
m_{1}^{k}  \tag{3.8}\\
m_{2}^{k} \\
\vdots \\
m_{N}^{k}
\end{array}\right)=\bar{L} \delta_{k}\left(\begin{array}{c}
e^{i \beta_{1}} \\
e^{i \beta_{2}} \\
\vdots \\
e^{i \beta_{N}}
\end{array}\right)+E+O\left(\sigma_{k}^{2}\right)
$$

where

$$
A=\left(\begin{array}{cccc}
D & -d_{1} & \ldots & -d_{N-1} \\
-d_{1} & D & \ldots & -d_{N-2} \\
\vdots & \vdots & \ldots & \vdots \\
-d_{N-1} & -d_{N-2} & \ldots & D
\end{array}\right)
$$

Since $D=\left|d_{1}\right|+\ldots+\left|d_{N}\right|$ and each $d_{i}>0$, we see that the matrix is invertible, thus $\left|m_{i}^{k}\right|=O\left(\delta_{k}\right)$ for all $i$. This is a standard fact and we include a short proof for completeness:

Lemma 3.1. Let $B=\left(b_{i j}\right)_{n \times n}$ be an $n \times n$ matrix that satisfies

$$
\left|b_{i i}\right|>\sum_{j \neq i}\left|b_{i j}\right|, \quad \text { for all } i
$$

Then $B$ is invertible.
Proof of Lemma 3.1: Apply row reduction to $B$ by eliminating all the entries in the first column except for $b_{11}$, it is easy to see that $B$ can be changed to

$$
\left(\begin{array}{cccc}
b_{11} & b_{12} & \ldots & b_{1 n} \\
0 & c_{22} & \ldots & c_{2 n} \\
0 & c_{32} & \ldots & c_{3 n} \\
\vdots & \vdots & \ldots & \vdots \\
0 & c_{n 2} & \ldots & c_{n n}
\end{array}\right)
$$

for

$$
c_{i j}=b_{i j}-\frac{b_{1 j}}{b_{11}} b_{i 1}, \quad i=2, \ldots, n, \quad j=2, \ldots, n
$$

Direct computation shows that

$$
\left|c_{i i}\right|>\sum_{j \neq i}\left|c_{i j}\right|, \quad i=2, \ldots, n
$$

Lemma3.1 is established.
Since the $O\left(\sigma_{k}^{2}\right)$ is only an infinitesimal perturbation of $A$, equation (3.8) can be written as

$$
\left(A+O\left(\sigma_{k}\right)\right)\left(m_{1}^{k}, \ldots, m_{N}^{k}\right)^{\prime}=\bar{L} \delta_{k}\left(e^{i \beta_{1}}, \ldots, e^{i \beta_{N}}\right)^{\prime}+E
$$

Thus we have, using Lemma3.1,

$$
\left(\begin{array}{c}
m_{1}^{k}  \tag{3.9}\\
m_{2}^{k} \\
\vdots \\
m_{N}^{k}
\end{array}\right)=A^{-1} \delta_{k} \bar{L}(0)\left(\begin{array}{c}
e^{i \beta_{1}} \\
e^{i \beta_{2}} \\
\vdots \\
e^{i \beta_{N}}
\end{array}\right)+E .
$$

With this fact we can further write $\nabla_{1} H\left(Q_{m}^{k}, Q_{l}^{k}\right)$ in (2.17) as

$$
\begin{equation*}
\nabla_{1} H\left(Q_{m}^{k}, Q_{l}^{k}\right)=-\frac{1}{2 \pi} \tau^{-2} \delta_{k}^{2} e^{\frac{2 \pi i l}{N+1}}+O\left(\delta_{k}^{3}\right) \tag{3.10}
\end{equation*}
$$

and $\nabla \phi_{l}^{k}\left(Q_{l}^{k}\right)$ in (2.11) and (2.18) as

$$
\begin{equation*}
\nabla \phi_{l}^{k}\left(Q_{l}^{k}\right)=-4 \sum_{m \neq l} \frac{Q_{l}^{k}-Q_{m}^{k}}{\left|Q_{l}^{k}-Q_{m}^{k}\right|^{2}}+O\left(\delta_{k}^{3}\right)+O\left(\mu_{k} e^{-\mu_{k}}\right) \tag{3.11}
\end{equation*}
$$

Using $\left(a^{i j}\right)_{n \times n}$ to denote $A^{-1}$, we rewrite (3.9) as

$$
\begin{equation*}
m_{l}^{k}=\delta_{k} \bar{L}(0) \sum_{s=1}^{n} a^{l s} e^{i \beta_{s}}+O\left(\delta_{k}^{2}\right)+O\left(\mu_{k} e^{-\mu_{k}}\right), \quad l=1, \ldots, N . \tag{3.12}
\end{equation*}
$$

## 4. Approximate $u_{k}$ By global solutions

First we note that for simple blowup solutions, the approximation of $u_{k}$ using global solutions is much easier. This part will be discussed in the appendix.

Proof of Theorem 1.1: The assumption is $\delta_{k} e^{\mu_{k} / 2} \leq c_{0}$. Fixing the neighborhood of one $Q_{m}^{k}$, the expansion of $v_{k}$ is, taking $Q_{m}^{k}$ as the origin,

$$
\begin{equation*}
v_{k}(y)=\log \frac{e^{\mu_{k, m}}}{\left(1+e^{\mu_{k, m}} \frac{\left|\tilde{Q}_{m}^{k}\right|^{2 N_{\mathfrak{h}}}\left(\delta_{k} \tilde{Q}_{m}^{k}\right)}{8}\left|y-\tilde{Q}_{m}^{k}\right|^{2}\right)^{2}}+\phi_{m}^{k}(y)+O\left(\mu_{k}^{2} e^{-\mu_{k}}\right) \tag{4.1}
\end{equation*}
$$

where $\mu_{k, m}=v_{k}\left(\tilde{Q}_{m}^{k}\right)$. First we claim that

$$
\begin{equation*}
\mu_{k, m}-\mu_{k}=O\left(\delta_{k}\right)+O\left(\mu_{k}^{2} e^{-\mu_{k}}\right) . \tag{4.2}
\end{equation*}
$$

From the Green's representation formula for $v_{k}$, we have, for $y$ away from bubbling areas and $|y| \sim 1$,

$$
\begin{aligned}
v_{k}(y) & =\left.v_{k}\right|_{\partial \Omega_{k}}+\int_{\Omega_{k}} G(y, \eta) \mathfrak{h}_{k}(\eta)|\eta|^{2 N} e^{v_{k}} d \eta, \\
& =\left.v_{k}\right|_{\partial \Omega_{k}}+\sum_{l=0}^{N} G\left(y, Q_{l}^{k}\right) \int_{B\left(Q_{l}^{k}, \varepsilon\right)}|\eta|^{2 N} \mathfrak{h}_{k}\left(\delta_{k} \eta\right) e^{v_{k}} d \eta \\
& +\sum_{l}\left(G(y, \eta)-G\left(y, Q_{l}^{k}\right)\right)|\eta|^{2 N} \mathfrak{h}_{k}\left(\delta_{k} \eta\right) e^{v_{k}} d \eta+O\left(e^{-\mu_{k}}\right), \\
& =\left.v_{k}\right|_{\partial \Omega_{k}}+8 \pi \sum_{l} G\left(y, Q_{l}^{k}\right)+O\left(\mu_{k} e^{-\mu_{k}}\right)
\end{aligned}
$$

where $\Omega_{k}=B\left(0, \tau \delta_{k}^{-1}\right)$. In particular if we consider $y$ located at $\left|y-Q_{m}^{k}\right|=\frac{\varepsilon}{2}$, the expression of $v_{k}$ can be written as

$$
\begin{align*}
v_{k}(y) & =\left.v_{k}\right|_{\partial \Omega_{k}}-4 \log \left|y-Q_{m}^{k}\right|+\phi_{m}^{k}  \tag{4.3}\\
& -4 \sum_{l=0, l \neq m}^{N} \log \left|Q_{m}^{k}-Q_{l}^{k}\right|+8 \pi \sum_{l=0}^{N} H\left(Q_{m}^{k}, Q_{l}^{k}\right)+O\left(\mu_{k} e^{-\mu_{k}}\right),
\end{align*}
$$

where

$$
\phi_{m}^{k}=\sum_{l=0, l \neq m}^{N}(-4) \log \frac{\left|y-Q_{l}^{k}\right|}{\left|Q_{m}^{k}-Q_{l}^{k}\right|}+8 \pi \sum_{l=0}^{N}\left(H\left(y, Q_{l}^{k}\right)-H\left(Q_{m}^{k}, Q_{l}^{k}\right)\right) .
$$

Comparing (4.3) and (4.1) we have

$$
\begin{align*}
& -\mu_{m, k}-\log \frac{\left|\tilde{Q}_{m}^{k}\right|^{2 N} \mathfrak{h}_{k}\left(\delta_{k} \tilde{Q}_{m}^{k}\right)}{8}  \tag{4.4}\\
= & -4 \sum_{l=0, l \neq m}^{N} \log \left|Q_{m}^{k}-Q_{l}^{k}\right|+8 \pi \sum_{l=0}^{N} H\left(Q_{m}^{k}, Q_{l}^{k}\right)+v_{k} \mid \partial \Omega_{k}+O\left(\mu_{k}^{2} e^{-\mu_{k}}\right) .
\end{align*}
$$

To evaluate terms in (4.4) we observe that

$$
\begin{array}{cc}
\left|\tilde{Q}_{m}^{k}\right|^{2 N}=1+O\left(\delta_{k}\right), & \mathfrak{h}_{k}\left(\delta_{k} \tilde{Q}_{m}^{k}\right)=1+O\left(\delta_{k}\right), \\
Q_{m}^{k}=e^{\frac{2 \pi m i}{N+1}}+O\left(\delta_{k}\right), & \tilde{Q}_{m}^{k}=Q_{m}^{k}+O\left(e^{-\mu_{k}}\right),
\end{array}
$$

and by the expression of $H(y, \eta)$ we have

$$
H\left(Q_{m}^{k}, Q_{l}^{k}\right)=\frac{1}{2 \pi} \log \left(\tau \delta_{k}^{-1}\right)+O\left(\delta_{k}^{2}\right)
$$

Thus two terms in (4.4) are

$$
\begin{align*}
8 \pi \sum_{l=0}^{N} H\left(Q_{m}^{k}, Q_{l}^{k}\right) & =4(N+1) \log \left(\tau \delta_{k}^{-1}\right)+O\left(\delta_{k}\right)  \tag{4.5}\\
\sum_{l=0, l \neq m}^{N} \log \left|Q_{m}^{k}-Q_{l}^{k}\right| & =\sum_{l=0, l \neq m}^{N} \log \left|e^{\frac{2 \pi m i}{N+1}}-e^{\frac{2 \pi l i}{N+1}}\right|+O\left(\delta_{k}\right)  \tag{4.6}\\
& =\log (N+1)+O\left(\delta_{k}\right) .
\end{align*}
$$

Using (4.5) and (4.6) in (4.4) we have

$$
\begin{align*}
\left.v_{k}\right|_{\partial \Omega_{k}}= & -\mu_{m, k}+\log 8+4 \log (1+N)-4(1+N) \log \left(\tau \delta_{k}^{-1}\right)  \tag{4.7}\\
& +O\left(\delta_{k}\right)+O\left(\mu_{k}^{2} e^{-\mu_{k}}\right), \quad m=0,1, \ldots, N .
\end{align*}
$$

Clearly from (4.7) we see that (4.2) holds. In order to approximate $v_{k}$ with a global solution we find $U_{k}$ which exactly has local maximums located at $e^{\frac{2 \pi}{N+1} i}$ and $U_{k}\left(e_{1}\right)=\mu_{k}$ :

$$
U_{k}(x)=\log \frac{e^{\mu_{k}}}{\left(1+\frac{e^{\mu_{k}}}{8(1+N)^{2}}\left|y^{N+1}-e_{1}\right|^{2}\right)^{2}},
$$

where $e_{1}=(1,0)$ on $\mathbb{R}^{2}$.
First in the region $B\left(Q_{l}^{k}, e^{-\mu_{k} / 2}\right)$, the comparison between $v_{k}$ and $U_{k}$ boils down to the evaluation of:

$$
\begin{equation*}
\log \frac{e^{\mu_{l, k}}}{\left(1+\frac{e^{\mu_{l}, k}}{8}\left|y-p_{k}\right|^{2}\right)^{2}}-\log \frac{e^{\mu_{k}}}{\left(1+\frac{e^{\mu_{k}}}{8}|y|^{2}\right)^{2}}, \tag{4.8}
\end{equation*}
$$

for $\left|p_{k}\right|=O\left(\delta_{k}\right)$. By elementary computation we see that the difference between the two terms in (4.8) is $O\left(\delta_{k} e^{\mu_{k}}\right)$ if $|y| \leq C e^{\mu_{k} / 2}$. On the other hand, for $C e^{-\mu_{k} / 2}<$ $|y|<\varepsilon / 2$, the comparison of expressions of $v_{k}$ and $U_{k}$ leads to the same conclusion. Moreover

$$
v_{k}-U_{k}=O\left(\delta_{k}\right)+O\left(\mu_{k}^{2} e^{-\mu_{k}}\right) \quad \text { on } \quad \partial B\left(Q_{l}^{k}, \varepsilon\right) .
$$

Also we observe from the expression of $U_{k}$ that

$$
v_{k}-U_{k}=O\left(\delta_{k}\right)+O\left(\mu_{k}^{2} e^{-\mu_{k}}\right) \quad \text { on } \quad \partial \Omega_{k} .
$$

Thus we obtain the the closeness of $v_{k}$ and $U_{k}$ on $\Omega_{k} \backslash\left(\cup_{l} B\left(Q_{l}^{k}, \varepsilon / 2\right)\right)$ by the smallness of $v_{k}-U_{k}$ on $\partial \Omega_{k}$ and standard estimates by Green's representation formula. Theorem [1.1] is established.

## 5. DISCUSSION OF $\delta_{k}^{2} \geq C \mu_{k} e^{-\mu_{k}}$

The main purpose of this section is to prove the vanishing rate of the second derivatives of $\log \mathfrak{h}_{k}(0)$.

The equation of Pohozaev identity now becomes

$$
\begin{equation*}
2 N \frac{Q_{l}^{k}}{\left|Q_{l}^{k}\right|^{2}}-4 \sum_{m \neq l} \frac{Q_{l}^{k}-Q_{m}^{k}}{\left|Q_{l}^{k}-Q_{m}^{k}\right|^{2}}+\delta_{k} \nabla\left(\log \mathfrak{h}_{k}\right)\left(\delta_{k} Q_{l}^{k}\right)=E_{3} \tag{5.1}
\end{equation*}
$$

with

$$
E_{3}=O\left(\delta_{k}^{3}\right)+O\left(\mu_{k} e^{-\mu_{k}}\right)
$$

After simplification (5.1) becomes

$$
2 N=4 \sum_{m \neq l} \frac{Q_{l}^{k}}{Q_{l}^{k}-Q_{m}^{k}}+\delta_{k} \bar{\nabla}\left(\log \mathfrak{h}_{k}\right)\left(\delta_{k} Q_{l}^{k}\right) Q_{l}^{k}=E_{3} .
$$

According to previous computation (for simplicity we use $m_{l}$ instead of $m_{l}^{k}$ in this section)

$$
\begin{aligned}
& \frac{Q_{l}^{k}}{Q_{l}^{k}-Q_{j}^{k}} \\
= & \frac{e^{i \beta_{l}}\left(1+m_{l}\right)}{e^{i \beta_{l}}\left(1-e^{i \beta_{j l}}+m_{l}-m_{j} e^{i \beta_{j l}}\right)} \\
= & \frac{1}{1-e^{i \beta_{j l}}}-\frac{m_{l}-m_{j} e^{i \beta_{j l}}}{\left(1-e^{\beta_{j l}}\right)^{2}}+\frac{m_{l}}{1-e^{i \beta_{j l}}} \\
& +\frac{\left(m_{l}-m_{j} e^{i \beta_{j l}}\right)^{2}}{\left(1-e^{i \beta_{j l}}\right)^{3}}-\frac{m_{l}\left(m_{l}-m_{j} e^{i \beta_{j l}}\right)}{\left(1-e^{i \beta_{j l}}\right)^{2}}+O\left(\delta_{k}^{3}\right) .
\end{aligned}
$$

After simplification we have

$$
\begin{array}{r}
\frac{Q_{l}^{k}}{Q_{l}^{k}-Q_{j}^{k}}=\frac{1}{1-e^{i \beta_{j l}}}+\frac{e^{i \beta_{j l}}}{\left(1-e^{i \beta_{j l}}\right)^{2}}\left(m_{j}-m_{l}\right) \\
+\frac{e^{i \beta_{j l}}}{\left(1-e^{i \beta_{j l}}\right)^{3}}\left(m_{l}-m_{j}\right)\left(m_{l}-m_{j} e^{i \beta_{j l}}\right)+O\left(\delta_{k}^{3}\right)+E .
\end{array}
$$

Using (3.2) for each $l$, we have

$$
\begin{array}{r}
\sum_{j=0, j \neq l}^{N} \frac{4 e^{i \beta_{j l}}}{\left(1-e^{i \beta_{j l}}\right)^{2}}\left(m_{j}-m_{l}\right)+\sum_{j=0, j \neq l}^{N} \frac{4 e^{i \beta_{j l}}}{\left(1-e^{i \beta_{j l}}\right)^{3}}\left(m_{l}-m_{j}\right)\left(m_{l}-m_{j} e^{i \beta_{j l}}\right)  \tag{5.2}\\
+\delta_{k} \bar{\nabla}\left(\log \mathfrak{h}_{k}\right)\left(\delta_{k} Q_{l}^{k}\right) e^{i \beta_{l}}\left(1+m_{l}\right)=E,
\end{array}
$$

Using $L$ to denote $\nabla \log \mathfrak{h}_{k}$ and (3.5), we write (5.2) as

$$
\begin{align*}
& -\sum_{j=0, j \neq l}^{N} d_{|j-l|} m_{j}+D m_{l}  \tag{5.3}\\
& +\left(\sum_{j=0, j \neq l}^{N} \frac{4 e^{i \beta_{j l}}}{\left(1-e^{i \beta_{j l}}\right)^{3}}\right) m_{l}^{2}-\left(\sum_{j=0, j \neq l}^{N} \frac{4 e^{i \beta_{j l}( }\left(1+e^{i \beta_{j l}}\right)}{\left(1-e^{i \beta_{j l}}\right)^{3}} m_{j}\right) m_{l}+\sum_{j=0, j \neq l}^{N} \frac{4 e^{2 i \beta_{j l}}}{\left(1-e^{i \beta_{j l}}\right)^{3}} m_{j}^{2} \\
& +\delta_{k} \bar{L}\left(\delta_{k} Q_{l}^{k}\right) e^{i \beta_{l}}+\delta_{k} \bar{L}(0) e^{i \beta_{l}} m_{l}=E .
\end{align*}
$$

for $l=1, \ldots, N$. Note that in the last term on the left hand side we used $m_{l}^{k}=O\left(\delta_{k}\right)$, so for this term there is no need to evaluate at $Q_{l}^{k}$.

For $l=0$, using $\beta_{0}=0$ and $m_{0}=0$, we have

$$
\begin{equation*}
-\sum_{j \neq 0}^{N} d_{j} m_{j}+\sum_{j \neq 0} \frac{4 e^{2 i \beta_{j}}}{\left(1-e^{i \beta_{j}}\right)^{3}} m_{j}^{2}+\delta \bar{L}\left(\delta Q_{0}^{k}\right)=E . \tag{5.4}
\end{equation*}
$$

For the case $N=1$ we have

$$
D=1, \quad d_{1}=1, \quad \beta_{01}=-\pi, \quad \beta_{1}=\pi, \quad \beta_{0}=0
$$

Thus (5.3) and (5.4) are reduced to

$$
\left\{\begin{array}{l}
m_{1}-\frac{1}{2} m_{1}^{2}-\delta_{k} \bar{L}\left(\delta_{k} Q_{1}^{k}\right)-\delta_{k} \bar{L}(0) m_{1}=E,  \tag{5.5}\\
-m_{1}+\frac{1}{2} m_{1}^{2}+\delta_{k} \bar{L}\left(\delta_{k} Q_{0}^{k}\right)=E .
\end{array}\right.
$$

From the first equation of (5.5) we have

$$
\begin{equation*}
m_{1}=\delta_{k} \bar{L}(0)+O\left(\delta_{k}^{2}\right) \tag{5.6}
\end{equation*}
$$

Adding the two equations in (5.5) and using (5.6), we have

$$
\begin{equation*}
\delta_{k} \bar{L}\left(\delta_{k} Q_{0}^{k}\right)-\delta_{k} \bar{L}\left(\delta_{k} Q_{1}^{k}\right)=\left(\delta_{k} \bar{L}(0)\right)^{2}+E . \tag{5.7}
\end{equation*}
$$

By $Q_{0}^{k}=1+O\left(\delta_{k}\right)$ and $Q_{1}^{k}=-1+O\left(\delta_{k}\right)$ we evaluate $L$ as

$$
L\left(\delta_{k} Q_{0}^{k}\right)=\binom{\partial_{1} \log \mathfrak{h}_{k}\left(\delta_{k} Q_{0}^{k}\right)}{\partial_{2} \log \mathfrak{h}_{k}\left(\delta_{k} Q_{0}^{k}\right)}=\binom{\partial_{1} \log \mathfrak{h}_{k}(0)+\delta_{k} \partial_{11} \log \mathfrak{h}_{k}(0)}{\partial_{2} \log \mathfrak{h}_{k}(0)+\delta_{k} \partial_{12} \log \mathfrak{h}_{k}(0)}+O\left(\delta_{k}^{2}\right) .
$$

and

$$
L\left(\delta_{k} Q_{1}^{k}\right)=\binom{\partial_{1} \log \mathfrak{h}_{k}\left(\delta_{k} Q_{1}^{k}\right)}{\partial_{2} \log \mathfrak{h}_{k}\left(\delta_{k} Q_{1}^{k}\right)}=\binom{\partial_{1} \log \mathfrak{h}_{k}(0)-\delta_{k} \partial_{11} \log \mathfrak{h}_{k}(0)}{\partial_{2} \log \mathfrak{h}_{k}(0)-\delta_{k} \partial_{12} \log \mathfrak{h}_{k}(0)}+O\left(\delta_{k}^{2}\right) .
$$

By comparing coefficients we have

$$
\begin{align*}
a_{1}^{2}-a_{2}^{2}-2 a_{11} & =O\left(\delta_{k}\right)+O\left(\delta_{k}^{-2} \mu_{k} e^{-\mu_{k}}\right),  \tag{5.8}\\
a_{1} a_{2}+a_{12} & =O\left(\delta_{k}\right)+O\left(\delta_{k}^{-2} \mu_{k} e^{-\mu_{k}}\right),
\end{align*}
$$

where $a_{11}=\partial_{11}\left(\log \mathfrak{h}_{k}\right)(0), a_{12}=\partial_{12}\left(\log \mathfrak{h}_{k}\right)(0), a_{1}=\partial_{1}\left(\log \mathfrak{h}_{k}\right)(0), a_{2}=\partial_{2}\left(\log \mathfrak{h}_{k}\right)(0)$. Direct computation from $\mathfrak{h}(x)=h_{k}\left(x e^{i \theta_{k}}\right)$ gives

$$
\begin{aligned}
& a_{1}=\partial_{1}\left(\log h_{k}\right)(0) \cos \theta_{k}+\partial_{2}\left(\log h_{k}\right)(0) \sin \theta_{k}=\partial_{e_{k}}\left(\log h_{k}\right)(0), \\
& a_{2}=\partial_{1}\left(\log h_{k}\right)(0)\left(-\sin \theta_{k}\right)+\partial_{2}\left(\log h_{k}\right)(0) \cos \theta_{k}=\partial_{e_{k}^{\perp}}\left(\log h_{k}\right)(0), \\
& a_{11}=\partial_{11}\left(\log h_{k}\right)(0) \cos ^{2}\left(\theta_{k}\right)+2 \partial_{12}\left(\log h_{k}\right)(0) \cos \theta_{k} \sin \theta_{k}+\partial_{22}\left(\log h_{k}\right)(0) \sin ^{2} \theta_{k} \\
& =\partial_{e_{k} e_{k}}\left(\log h_{k}\right)(0), \\
& a_{12}=\left(\partial_{22}-\partial_{11}\right)\left(\log h_{k}\right)(0) \sin \theta_{k} \cos \theta_{k}+\partial_{12}\left(\log h_{k}\right)(0)\left(\cos ^{2} \theta_{k}-\sin ^{2} \theta_{k}\right) \\
& =\partial_{e_{k} e_{k}^{\perp}}\left(\log h_{k}\right)(0) .
\end{aligned}
$$

Thus (1.14) in Theorem 1.2]is established.
For the case $N \geq 2$ we take the sum of all equations in (5.3), (5.4) and obtain

$$
\sum_{l=0}^{N} c_{l} m_{l}^{2}+\sum_{l \neq j} c_{j l} m_{l} m_{j}+\delta_{k} \sum_{l=0}^{N} \bar{L}\left(\delta_{k} Q_{l}^{k}\right) e^{i \beta_{l}}+\delta_{k} \sum_{l=0}^{N} \bar{L}(0) e^{i \beta_{l}} m_{l}=E,
$$

where the coefficients $c_{l}$ and $c_{j l}$ will be specified later. We first deal with

$$
\begin{equation*}
\sum_{l=0}^{N} \delta_{k} \bar{L}\left(\delta_{k} Q_{l}^{k}\right) e^{i \beta_{l}}=\sum_{l=0}^{N} \delta_{k}\left(\bar{L}\left(\delta_{k} Q_{l}^{k}\right)-\bar{L}(0)\right) e^{i \beta_{l}} \tag{5.9}
\end{equation*}
$$

where we have used $\sum_{l=0}^{N} e^{i \beta_{l}}=0$. If we use $a_{11}=\partial_{11}\left(\log \mathfrak{h}_{k}\right)(0), a_{12}=\partial_{12}\left(\log \mathfrak{h}_{k}\right)(0)$ and $a_{22}=\partial_{22}\left(\log \mathfrak{h}_{k}\right)(0)$, we see that

$$
\begin{aligned}
& \bar{L}\left(\delta_{k} Q_{l}^{k}\right)-\bar{L}(0) \\
= & \partial_{1} \log \mathfrak{h}\left(\delta_{k} Q_{l}^{k}\right)-i \partial_{2} \log \mathfrak{h}\left(\delta_{k} Q_{l}^{k}\right)-\left(\partial_{1} \log \mathfrak{h}(0)-i \partial_{2} \log \mathfrak{h}(0)\right) \\
= & \left(a_{11} \delta_{k} \cos \beta_{l}+a_{12} \sin \beta_{l}\right)-i\left(a_{12} \cos \beta_{l}+a_{22} \sin \beta_{l}\right)
\end{aligned}
$$

Thus the real part of (5.9) is

$$
\delta_{k}^{2}\left(\sum_{l=0}^{N} a_{11} \cos ^{2} \beta_{l}+2 a_{12} \cos \beta_{l} \sin \beta_{l}+a_{22} \sin ^{2} \beta_{l}^{2}\right)=\frac{N+1}{2}\left(a_{11}+a_{22}\right) \delta_{k}^{2}
$$

and the imaginary part of (5.9) is

$$
\delta_{k}^{2}\left(\sum_{l=0}^{N}\left(a_{11}-a_{22}\right) \sin \beta_{l} \cos \beta_{l}-a_{12} \cos \left(2 \beta_{l}\right)\right)=0
$$

where we have used

$$
\sum_{l=0}^{N} e^{2 i \beta_{l}}=0
$$

To compute the coefficients of $m_{l}^{2}$ and $m_{l} m_{t}$ for $l \neq t$, we use

$$
\frac{e^{i \theta}\left(1+e^{i \theta}\right)}{\left(1-e^{i \theta}\right)^{3}}=-\frac{i}{4} \frac{\cos (\theta / 2)}{\sin ^{3}(\theta / 2)}
$$

Since $\cos (\cdot)$ is even and $\sin (\cdot)$ is odd, we see that the summation of cross terms $m_{l} m_{j}$ is zero: $c_{j l}=0$.

The following two properties are useful for evaluating $c_{l}$ :

$$
\begin{gather*}
\frac{e^{i \theta}}{\left(1-e^{i \theta}\right)^{3}}=-\frac{1}{8} \frac{\sin (\theta / 2)+i \cos (\theta / 2)}{\sin ^{3}(\theta / 2)}  \tag{5.10}\\
\frac{e^{2 i \theta}}{\left(1-e^{i \theta}\right)^{3}}=-\frac{1}{8} \frac{\sin (-\theta / 2)+i \cos (-\theta / 2)}{\sin ^{3}(\theta / 2)}
\end{gather*}
$$

By the two identities in (5.10), we see that the coefficient of $m_{l}^{2}$ is

$$
\begin{aligned}
c_{l} & =\sum_{j=0, j \neq l}^{N} \frac{4 e^{i \beta_{j l}}}{\left(1-e^{i \beta_{j l}}\right)^{3}}+\frac{4 e^{2 i \beta_{l j}}}{\left(1-e^{i \beta_{l j}}\right)^{3}}, \\
& =\left(-\frac{1}{2}\right) \sum_{j=0, j \neq l}^{N}\left(\frac{\sin \left(\frac{(j-l) \pi}{N+1}\right)+i \cos \left(\frac{(j-l) \pi}{N+1}\right)}{\sin ^{3}\left(\frac{(j-l) \pi}{N+1}\right)}+\frac{\sin \left(\frac{(j-l) \pi}{N+1}\right)+i \cos \left(\frac{(j-l) \pi}{N+1}\right)}{\sin ^{3}\left(\frac{(l-j) \pi}{N+1}\right)}\right) \\
& =0 .
\end{aligned}
$$

Thus all the terms related to $m_{l}$ cancel out. By (3.12) we have

$$
\begin{equation*}
\sum_{l=0}^{N}\left(\bar{L}\left(\delta_{k} Q_{l}^{k}\right)-\bar{L}(0)\right) e^{i \beta_{l}}+\delta_{k}^{2}(\bar{L}(0))^{2} \sum_{s, t} e^{i \beta_{s}} a^{s t} e^{i \beta_{t}}=E \tag{5.11}
\end{equation*}
$$

The first term in (5.11) is

$$
\sum_{l=0}^{N}\left(\bar{L}\left(\delta_{k} Q_{l}^{k}\right)-\bar{L}(0)\right) e^{i \beta_{l}}=\binom{\delta_{k}^{2} \frac{N+1}{2} \Delta\left(\log \mathfrak{h}_{k}\right)(0)}{0}+E .
$$

For the second term in (5.11) we use the symbolic computation of matlab to obtain

$$
\begin{equation*}
\sum_{s, t=1}^{n} e^{i \beta_{s}} a^{s t} e^{i \beta_{t}}=0 \tag{5.12}
\end{equation*}
$$

Consequently the following holds:

$$
\Delta\left(\log \mathfrak{h}_{k}\right)(0)=O\left(\delta_{k}^{-2} \mu_{k} e^{-\mu_{k}}\right)+O\left(\delta_{k}\right)
$$

Since $\Delta\left(\log h_{k}\right)(0)=\Delta\left(\log \mathfrak{h}_{k}\right)(0)$, we obtain the conclusion stated in Theorem 1.2 for $N \geq 2$. Thus, Theorem 1.2 is established.

## 6. Appendix: Simple blowup solutions

In the section we approximate simple bubbling solutions using global solutions. Recall that $u_{k}$ satisfies

$$
\begin{gathered}
\Delta u_{k}+|x|^{2 N} h_{k}(x) e^{u_{k}}=0, \quad|x|<\tau, \\
\max _{\bar{B}_{\tau}} u_{k}=\lambda_{k} \rightarrow \infty
\end{gathered}
$$

0 is the only blowup point in $\quad B_{\tau}$,
and

$$
u_{k} \text { is a constant at } \partial B_{\tau} .
$$

Since we talk about simple blowup solutions, we have

$$
\begin{equation*}
u_{k}(x)+2(1+N) \log |x| \leq C \tag{6.1}
\end{equation*}
$$

for some $C>0$ independent of $k$.
Let

$$
\begin{equation*}
\varepsilon_{k}=e^{-\frac{\lambda_{k}}{2(1+N)}} \tag{6.2}
\end{equation*}
$$

and

$$
\begin{equation*}
v_{k}(y)=u_{k}\left(\varepsilon_{k} y\right)+2(1+N) \log \varepsilon_{k}, \quad|y| \leq 1 / \varepsilon_{k} . \tag{6.3}
\end{equation*}
$$

Then clearly $v_{k} \leq 0$ and satisfies

$$
\begin{equation*}
\Delta v_{k}+h_{k}\left(\varepsilon_{k} y\right)|y|^{2 N} e^{v_{k}(y)}=0, \quad \text { in } \quad|y|<\tau / \varepsilon_{k} . \tag{6.4}
\end{equation*}
$$

It is easy to see that a subsequence of $v_{k}$, which is denoted as $v_{k}$ as well, converges uniformly to $v$ over any fixed compact subset of $\mathbb{R}^{2}$. The limit function $v$, which solves

$$
\begin{equation*}
\Delta v(y)+|y|^{2 N} e^{v}=0, \quad \text { in } \quad \mathbb{R}^{2}, \quad \int_{\mathbb{R}^{2}}|y|^{2 N} e^{v}<\infty, \tag{6.5}
\end{equation*}
$$

also satisfies

$$
v(y) \leq 0
$$

and, by the classification theorem of Prajapat-Tarantello [23]

$$
\int_{\mathbb{R}^{2}}|y|^{2 N} e^{v}=8 \pi(1+N)
$$

The asymptotic behavior of $v$ is determined by the total integration of $\int_{\mathbb{R}^{2}}|y|^{2 N} e^{v}$ :

$$
v(y)=-4(1+N) \log |y|+O(1) \quad \text { for } \quad|y|>1
$$

So $v_{k} \rightarrow v$ over any fixed $B_{R}$ in $\mathbb{R}^{2}$. Next we consider the behavior of $v_{k}$ outside $B_{R}$.

For $r \in\left(2 \varepsilon_{k} R, \tau / 3\right)$, let

$$
\tilde{v}_{k}(y)=u_{k}(r y)+2(1+N) \log r, \quad \frac{1}{2}<|y|<2 .
$$

Then clearly $\tilde{v}_{k}$ satisfies

$$
\Delta \tilde{v}_{k}(y)+|y|^{2 N} h_{k}(r y) e^{\tilde{v}_{k}(y)}=0, \quad B_{2} \backslash B_{1 / 2} .
$$

Let $c_{0}$ be the bound for $\tilde{v}_{k}: \tilde{v}_{k} \leq c_{0}$ in $B_{2} \backslash B_{1 / 2}$ and we set $g_{k}=\tilde{v}_{k}-c_{0}-1$ which immediately satisfies $g_{k} \leq-1$ in $B_{2} \backslash B_{1 / 2}$. Thus the equation for $g_{k}$ can be written as

$$
\Delta g_{k}+\frac{|y|^{2 N} h_{k}(r y) e^{\tilde{v_{k}}}}{g_{k}} g_{k}(y)=0, \quad \text { in } \quad B_{2} \backslash B_{1 / 2}
$$

The coefficient of $g_{k}$ is clearly bounded. Thus standard Harnack inequality on $\partial B_{1}$ gives

$$
\max _{\partial B_{1}}\left(-g_{k}\right) \leq c_{1}\left(c_{0}\right) \min _{\partial B_{1}}\left(-g_{k}\right)
$$

where $c_{1}>1$ only depends on $c_{0}$. Going back to $\tilde{v}_{k}$ we have

$$
\max _{\partial B_{1}} \tilde{v}_{k} \leq \frac{1}{c_{1}} \min _{\partial B_{1}} \tilde{v}_{k}+\left(c_{0}+1\right)\left(1-\frac{1}{c_{1}}\right) .
$$

For $u_{k}$ it is

$$
\begin{equation*}
\max _{\partial B_{r}} u_{k} \leq \frac{1}{c_{1}} \min _{\partial B_{r}} u_{k}-2(1+N)\left(1-\frac{1}{c_{1}}\right) \log r+\left(c_{0}+1\right)\left(1-\frac{1}{c_{1}}\right) . \tag{6.6}
\end{equation*}
$$

Let $\bar{v}_{k}(r)$ be the spherical average of $v_{k}$ on $\partial B_{r}$. Then for $r>R$ for some $R$ large,

$$
\bar{v}_{k}(r) \leq\left(-4(1+N)+\delta_{1}\right) \log r+O(1)
$$

because

$$
\frac{d}{d r} \bar{v}_{k}(r)=-\frac{1}{2 \pi} \int_{B_{r}}|y|^{2 N} h_{k}\left(\varepsilon_{k} y\right) e^{v_{k}(y)} d y
$$

From the value of $\bar{v}_{k}(R)$ and the estimate above, it is immediate to see that

$$
\begin{equation*}
\bar{v}_{k}(r) \leq(-4(N+1)+\varepsilon(R)) \log r+c, \quad r \geq R \tag{6.7}
\end{equation*}
$$

For $r \geq R$, the spherical Harnack inequality for $u_{k}$ gives,

$$
\max _{\partial B_{r}} v_{k} \leq \frac{1}{c_{1}} \min _{\partial B_{r}} v_{k}-2(1+N)\left(1-\frac{1}{c_{1}}\right) \log r+c_{2}
$$

where $c_{2}=\left(c_{0}+1\right)\left(1-\frac{1}{c_{1}}\right)$. This inequality readily gives the following estimate of $v_{k}$ :

$$
v_{k}(y) \leq(-2(1+N)-\delta) \log |y|, \quad R<|y|<\tau \varepsilon_{k}^{-1}
$$

for some $\delta>0$. Then it is easy to use Green's representation formula to obtain

$$
v_{k}(y) \leq-4(1+N) \log (1+|y|)+c, \quad|y| \leq \tau / \varepsilon_{k} .
$$

The classification theorem of Prajapat-Tarantello [23] gives

$$
v(y)=\log \frac{\Lambda}{\left(1+\frac{\Lambda}{8(1+N)^{2}}\left|y^{N+1}-\xi\right|^{2}\right)^{2}}
$$

where parameters $\Lambda>0$ and $\xi \in \mathbb{C}$. By the argument in Lin-Wei-Zhang [19] there is a perturbation of $\Lambda_{k} \rightarrow \Lambda$ and $\xi_{k} \rightarrow \xi$ such that

$$
V_{k}(y):=\log \frac{\Lambda_{k}}{\left(1+\frac{\Lambda_{k}}{8(1+N)^{2}}\left|y^{N+1}-\xi_{k}\right|^{2}\right)^{2}}
$$

satisfies

$$
\begin{equation*}
\left|v_{k}(y)-V_{k}(y)\right| \leq C \varepsilon_{k}(1+|y|), \quad y \in B\left(0, \varepsilon_{k}^{-1}\right) \tag{6.8}
\end{equation*}
$$

The idea of the proof in [19] for this case is the following: Choose $1 \ll\left|p_{1}\right| \ll$ $\left|p_{2}\right| \ll\left|p_{3}\right|$ such that the following matrix invertible:

$$
\left(\begin{array}{lll}
\frac{\partial v}{\partial \Lambda}\left(p_{1}\right) & \frac{\partial v}{\partial \Lambda}\left(p_{2}\right) & \frac{\partial v}{\partial \Lambda}\left(p_{3}\right)  \tag{6.9}\\
\frac{\partial v}{\partial \xi_{1}}\left(p_{1}\right) & \frac{\partial v}{\partial \xi_{1}}\left(p_{2}\right) & \frac{\partial v}{\partial \xi_{1}}\left(p_{3}\right) \\
\frac{\partial v}{\partial \xi_{2}}\left(p_{1}\right) & \frac{\partial v}{\partial \xi_{2}}\left(p_{2}\right) & \frac{\partial v}{\partial \xi_{2}}\left(p_{3}\right)
\end{array}\right)
$$

where $\xi=\xi_{1}+i \xi_{2}$. Thus if a $o(1)$ perturbation is placed on $v$ (to make $v_{k}\left(p_{j}\right)=$ $V_{k}\left(p_{j}\right)$ for $j=1,2,3$ ), all we need to do is change the parameters $\Lambda, \xi$ by a comparable amount. So even though we have a sequence of parameters $\Lambda_{k}$, $\mu_{k}$, they are not tending to infinity.

Now we improve this estimate. Let

$$
w_{k}(y)=v_{k}(y)-V_{k}(y), \quad|y| \leq \tau / \varepsilon_{k}
$$

The equation for $w_{k}$ is

$$
\begin{equation*}
\Delta w_{k}+|y|^{2 N} e^{\xi_{k}} w_{k}=-|y|^{2 N}\left(\sum_{t=1}^{2} \varepsilon_{k} \partial_{t} h_{k}(0) y^{t}+O\left(\varepsilon_{k}^{2}|y|^{2}\right)\right) e^{v_{k}} \tag{6.10}
\end{equation*}
$$

for $y \in B\left(0, \tau \varepsilon_{k}^{-1}\right)$. In addition, we know that $w_{k}\left(p_{t}\right)=0$ for $t=1,2,3$ and $w_{k}(y) \leq$ $O\left(\varepsilon_{k}\right)(1+|y|)$ and the oscillation of $w_{k}$ on $\partial B\left(0, \tau \varepsilon_{k}^{-1}\right)$ is $O\left(\varepsilon_{k}^{N+1}\right)$.

Our next step is to improve the estimate of $w_{k}$. From the Green's representation formula for $w_{k}$ we have

$$
\begin{aligned}
w_{k}(y)= & \int_{\Omega_{k}} G(y, \eta)|\eta|^{2 N}\left(e^{\xi_{i}} w_{k}(\eta)+\varepsilon_{k} \sum_{t} \partial_{t} h_{k}(0) y^{t}\right. \\
& \left.+O\left(\varepsilon_{k}^{2}|\eta|^{2}\right) e^{v_{k}(\eta)}\right) d y+w_{k} \mid \partial_{\Omega_{k}}+O\left(\varepsilon_{k}^{N+1}\right)
\end{aligned}
$$

where $\Omega_{k}=B\left(0, \tau \varepsilon_{k}^{-1}\right)$ and $\left.w_{k}\right|_{\partial \Omega_{k}}$ is the average of $w_{k}$ on $\partial \Omega_{k}$. Using crude estimate of $w_{k}$ we rewrite the above as

$$
\begin{equation*}
w_{k}(y)=\int_{\Omega_{k}} G(y, \eta) O\left(\varepsilon_{k}\right)(1+|\eta|)^{-3-2 N} d y+\left.w_{k}\right|_{\partial \Omega_{k}}+O\left(\varepsilon_{k}^{N+1}\right) \tag{6.11}
\end{equation*}
$$

Since $w_{k}\left(p_{1}\right)=0$. Evaluating the above at $p_{1}$ we have

$$
\begin{equation*}
0=\int_{\Omega_{k}} G\left(p_{1}, \eta\right) O\left(\varepsilon_{k}\right)(1+|\eta|)^{-3-2 N} d \eta+\left.w_{k}\right|_{\partial \Omega_{k}}+O\left(\varepsilon_{k}^{N+1}\right) \tag{6.12}
\end{equation*}
$$

The difference of (6.11) and (6.12) gives

$$
\begin{equation*}
w_{k}(y)=\int_{\Omega_{k}}\left(G(y, \eta)-G\left(p_{1}, \eta\right)\right) O\left(\varepsilon_{k}\right)(1+|\eta|)^{-3-2 N} d y+O\left(\varepsilon_{k}^{N+1}\right) \tag{6.13}
\end{equation*}
$$

Then elementary estimate gives

$$
w_{k}(y)=O\left(\varepsilon_{k}\right) \log (2+|y|)
$$

Next we shall identify the $O\left(\varepsilon_{k}\right)$ term in the expansion of $v_{k}$. Let $f_{1}$ and $f_{2}$ be solutions of the following equations respectively:

$$
\begin{array}{ll}
\Delta f_{1}+|y|^{2 N} e^{V} f_{1}=-y_{1} e^{V}|y|^{2 N}, & \text { in } \quad \mathbb{R}^{2} \\
\Delta f_{2}+|y|^{2 N} e^{V} f_{2}=-y_{2} e^{V}|y|^{2 N}, & \text { in } \quad \mathbb{R}^{2}
\end{array}
$$

Here is why $f_{1}, f_{2}$ exist: Let $\phi_{0}=\frac{\partial V}{\partial \Lambda}, \phi_{1}=\frac{\partial V}{\partial \xi_{1}}$ and $\phi_{2}=\frac{\partial V}{\partial \xi_{2}}$ where $\xi_{k}=\xi_{1}+i \xi_{2}$. Direct computation gives

$$
\begin{gather*}
\phi_{0}=\frac{1}{\Lambda}-\frac{1}{4(1+N)^{2}} \frac{\left|z^{N+1}-\xi\right|^{2}}{1+\frac{\Lambda}{8(1+N)^{2}}\left|z^{N+1}-\xi\right|^{2}}  \tag{6.14}\\
\phi_{1}=\frac{\Lambda}{4(1+N)^{2}} \frac{\bar{z}^{N+1}-\bar{\xi}}{1+\frac{\Lambda}{8(1+N)^{2}}\left|z^{N+1}-\xi\right|^{2}} \\
\phi_{2}=\frac{\Lambda}{4(1+N)^{2}} \frac{z^{N+1}-\xi}{1+\frac{\Lambda}{8(1+N)^{2}}\left|z^{N+1}-\xi\right|^{2}}
\end{gather*}
$$

By Lin-Wei-Ye [18], the kernel in the linearized equation is spanned by $\phi_{0}, \phi_{1}, \phi_{2}$ if a less than linear growth condition is imposed. Using this fact and 6.14 , we observe that

$$
\int_{\mathbb{R}^{2}} y_{1} e^{V}|y|^{2 N} \phi_{j}=0, \quad j=0,1,2
$$

Indeed,

$$
\begin{aligned}
& \int_{\mathbb{R}^{2}} y_{1} e^{V}|y|^{2 N} \phi_{j} \\
= & \int_{\mathbb{R}^{2}} y_{1} \Delta \phi_{j} \\
= & \lim _{R \rightarrow \infty}\left(\int_{B_{R}} \partial_{\nu} \phi_{j} y_{1}-\int_{B_{R}} \partial_{1} \phi_{j}\right) \\
= & 0,
\end{aligned}
$$

where the last equality is due to the asymptotic behavior of $\phi_{j}$ and $\nabla \phi_{j}$ at infinity.
Thus $f_{1}$ exists. The existence of $f_{2}$ can be derived in a similar way. By standard elliptic theory, the estimate of $f_{i}$ at infinity is:

$$
\left|f_{i}(y)\right| \leq C(1+|y|)^{-2-2 N}, \quad y \in \mathbb{R}^{2}, \quad i=1,2
$$

Let

$$
\begin{equation*}
w_{1}^{k}=\varepsilon_{k} \partial_{1} h_{k}(0) f_{1}+\varepsilon_{k} \partial_{2} h_{k}(0) f_{2} \tag{6.15}
\end{equation*}
$$

By the estimate of $w_{k}$ we write the equation of $w_{k}$ as

$$
\Delta w_{k}+|y|^{2 N} e^{V_{k}} w_{k}=\left(h_{k}(0)-h_{k}\left(\varepsilon_{k} y\right)\right)|y|^{2 N} e^{V_{k}}+O\left(\varepsilon_{k}^{2}\right)(1+|y|)^{-2-2 N}
$$

Comparing with the equation for $w_{1}^{k}$, we now write the equation for $w_{k}-w_{1}^{k}$ as

$$
\Delta\left(w_{k}-w_{1}^{k}\right)+|y|^{2 N} e^{V_{k}}\left(w_{k}-w_{1}^{k}\right)=O\left(\varepsilon_{k}^{2}\right)(1+|y|)^{-2-2 N}, \quad y \in \mathbb{R}^{2} .
$$

Since the matrix in (6.9) is invertible, we adjust the parameters of $V_{k}$ by $O\left(\varepsilon_{k}\right)$ to make the new global functions $\tilde{V}_{k}$ satisfy

$$
\begin{equation*}
v_{k}-\tilde{V}_{k}-w_{1}^{k}=0, \quad \text { for } \quad y=p_{1}, p_{2}, p_{3} \tag{6.16}
\end{equation*}
$$

Note that the parameters in $\tilde{V}_{k}$ are $O\left(\varepsilon_{k}\right)$ different from those in $V_{k}$.
Let

$$
w_{2}^{k}=v_{k}-\tilde{V}_{k}-w_{1}^{k}
$$

because of the closeness of $\tilde{V}_{k}$ and $V_{k}$, the equation for $w_{2}^{k}$ is still

$$
\Delta w_{2}^{k}+|y|^{2 N} e^{\tilde{v}_{k}} w_{2}^{k}=O\left(\varepsilon_{k}^{2}\right)(1+|y|)^{-2-2 N}, \quad y \in \mathbb{R}^{2}
$$

Claim:

$$
\begin{equation*}
\left|w_{2}^{k}(y)\right| \leq C \varepsilon_{k}^{2} \log (2+|y|) . \tag{6.17}
\end{equation*}
$$

To prove (6.17), we assume

$$
\Lambda_{k}=\max _{y} \frac{\left|w_{2}(y)\right|}{\varepsilon_{k}^{2} \log (2+|y|)} \rightarrow \infty .
$$

Suppose $\Lambda_{k}$ is attained at $y_{k}$. Let

$$
\tilde{w}_{2}^{k}(y)=\frac{w_{2}(y)}{\Lambda_{k} \varepsilon_{k}^{2} \log \left(2+\left|y_{k}\right|\right)} .
$$

From this definition we immediately see that

$$
\left|\tilde{w}_{2}^{k}(y)\right|=\frac{\left|w_{2}^{k}(y)\right| \log (2+|y|)}{\Lambda_{k} \varepsilon_{k}^{2} \log (2+|y|) \log \left(2+\left|y_{k}\right|\right)} \leq \frac{\log (2+|y|)}{\log \left(2+\left|y_{k}\right|\right)} .
$$

On $|y| \leq \tau \varepsilon_{k}^{-1}, \tilde{w}_{2}^{k}$ satisfies

$$
\begin{equation*}
\Delta \tilde{w}_{2}^{k}+|y|^{2 N} e^{\tilde{v}_{k}} \tilde{w}_{2}^{k}=O(1) \frac{(1+|y|)^{-2-2 N}}{\Lambda_{k} \log \left(2+\left|y_{k}\right|\right)}, \quad|y|<\tau \varepsilon_{k}^{-1} \tag{6.18}
\end{equation*}
$$

Moreover, since $\tilde{V}_{k}$ has a perturbation of $O\left(\varepsilon_{k}^{N+1}\right)$ on $\partial B\left(0, \tau \varepsilon_{k}^{-1}\right)$, we have

$$
\tilde{w}_{2}^{k}(y)=o(1), \quad y \in \partial B\left(0, \tau \varepsilon_{k}^{-1}\right) .
$$

If $y_{k} \rightarrow y^{*}, \tilde{w}_{2}^{k}$ converges to a solution of

$$
\Delta \phi+|y|^{2 N} e^{V} \phi=0, \quad \mathbb{R}^{2}
$$

with mild growth:

$$
|\phi(y)| \leq C \log (2+|y|) .
$$

By the non-degeneracy of the linearized equation,

$$
\phi(y)=c_{1} \frac{\partial v}{\partial \Lambda}(y)+c_{2} \frac{\partial v}{\partial \xi_{1}}(y)+c_{3} \frac{\partial v}{\partial \xi_{2}}(y) .
$$

Using $\phi\left(p_{i}\right)=0$ for $i=1,2,3$, we have, by the invertibility of matrix (6.9), $c_{1}=$ $c_{2}=c_{3}=0$, thus $\phi \equiv 0$, a contradiction to $\tilde{w}_{2}^{k}\left(y_{k}\right)= \pm 1$.

So we only need to consider the case that $\left|y_{k}\right| \rightarrow \infty$. In this case the Green's representation formula of $\tilde{w}_{2}^{k}\left(y_{k}\right)$ gives

$$
\pm 1=\tilde{w}_{2}^{k}\left(y_{k}\right)=\int_{\Omega_{k}} G\left(y_{k}, \eta\right) \frac{\log (2+|\eta|)-o(1)(1+|\eta|)^{-2-2 N}}{\log \left(2+\left|y_{k}\right|\right)} d \eta+o(1) .
$$

Using $\tilde{w}_{2}^{k}\left(p_{1}\right)=0$ we can further write the above as

$$
\begin{align*}
& \pm 1=\tilde{w}_{2}^{k}\left(y_{k}\right)  \tag{6.19}\\
& =\int_{\Omega_{k}}\left(G\left(y_{k}, \eta\right)-G\left(p_{1}, \eta\right)\right) \frac{\log (2+|\eta|)-o(1)(1+|\eta|)^{-2-2 N}}{\log \left(2+\left|y_{k}\right|\right)} d \eta+o(1) .
\end{align*}
$$

However by standard evaluation of the Green's function, the right hand side of (6.19) is $o(1)$. This contradiction proves that $v_{k}$ can be accurate to $O\left(\varepsilon_{k}^{2}\right)$ by two terms. So the conclusion of this section is

Theorem 6.1. Let $v_{k}, \tilde{V}_{k}, w_{1}^{k}, \varepsilon_{k}$ be defined in (6.3), (6.16), (6.15) and (6.2), respectively, then

$$
\left|v_{k}(y)-\tilde{V}_{k}(y)-w_{1}^{k}(y)\right| \leq C \varepsilon_{k}^{2} \log (2+|y|), \quad|y| \leq \tau \varepsilon_{k}^{-1} .
$$

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