ESTIMATES FOR LIOUVILLE EQUATION WITH QUANTIZED SINGULARITIES

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ABSTRACT. For Liouville equations with singular sources, it is well known that blowup solutions may exhibit non-simple blowup phenomenon if the blowup point happens to be the singular source and the strength of the singular source is a multiple of 4π . In this article we prove that even in this case some coefficient functions must vanish at the singular source and bubbling solutions can still be accurately approximated by global solutions.

1. INTRODUCTION

In this article we study bubbling solutions of the following singular Liouville equation

(1.1)
$$\Delta u + h(x)e^u = 4\pi\alpha\delta_0 \quad \text{in} \quad \Omega \subset \mathbb{R}^2$$

where Ω is an open, bounded subset of \mathbb{R}^2 that contains the origin, $\alpha > -1$ is a constant and δ_0 is the Dirac mass at 0, *h* is a positive and smooth function. One of the main difficulties in the study of blowup solutions to (1.1) is when the blowup point happens to be the location of a singular source. It is known from the works of Kuo-Lin [16], Bartolucci-Tarantello [4] that if $\alpha \notin \mathbb{N}$ (the set of natural numbers) blowup solutions satisfy spherical Harnack inequality around the singular source and the asymptotic behavior is relatively easy to understand. However, when the strength of the singular source is a multiple of 4π ($\alpha \in \mathbb{N}$), the so called "non-simple blowup" phenomenon does occur, which means the bubbling solutions may not satisfy spherical Harnack inequality and multiple local maximums near the singular source could appear. In this article we prove new estimates for non-simple bubbling solutions. Since the analysis is carried out in a neighborhood of the origin, so our assumption of bubbling solutions is as follows: Let \tilde{u}_k be a sequence of solutions of

(1.2)
$$\Delta \tilde{u}_k(x) + \tilde{h}_k(x)e^{u_k} = 4\pi N\delta_0, \quad \text{in} \quad B_\tau$$

for some $\tau > 0$ independent of k. B_{τ} is the ball centered at the origin with radius τ . In addition we postulate the usual assumptions on \tilde{u}_k and \tilde{h}_k : For a positive

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constant *C* independent of *k*, the following holds:

(1.3)
$$\begin{cases} \|\tilde{h}_k\|_{C^3(\bar{B}_{\tau})} \leq C, & \frac{1}{C} \leq \tilde{h}_k(x) \leq C, \quad x \in \bar{B}_{\tau}, \\\\ \int_{B_{\tau}} \tilde{h}_k e^{\tilde{u}_k} \leq C, \\\\ |\tilde{u}_k(x) - \tilde{u}_k(y)| \leq C, \quad \forall x, y \in \partial B_{\tau}, \end{cases}$$

and since we study the asymptotic behavior of blowup solutions around the singular source, we assume that there is no blowup point except at the origin:

(1.4)
$$\max_{K \subset \subset B_{\tau} \setminus \{0\}} \tilde{u}_k \leq C(K).$$

Also, for the convenience of notation we assume $\tilde{h}_k(0) = 1$ and use the value of \tilde{u}_k on ∂B_{τ} to define a harmonic function $\phi_k(x)$:

(1.5)
$$\begin{cases} \Delta \phi_k(x) = 0, & \text{in } B_{\tau}, \\ \phi_k(x) = u_k(x) - \frac{1}{2\pi\tau} \int_{\partial B_{\tau}} \tilde{u}_k dS, & x \in \partial B_{\tau} \end{cases}$$

Using the fact that $\Delta(\frac{1}{2\pi}\log|x|) = \delta_0$, we set

(1.6)
$$u_k(x) = \tilde{u}_k(x) - 2N \log |x| - \phi_k(x),$$

which satisfies

(1.7)
$$\Delta u_k(x) + |x|^{2N} h_k(x) e^{u_k} = 0, \quad \text{in} \quad B_n$$

for

(1.8)
$$h_k(x) = \tilde{h}_k(x)e^{\phi_k(x)}$$

It is easy to see that $\phi_k(0) = 0$ and u_k is a constant on ∂B_{τ} .

In this article we consider the case that:

(1.9)
$$\max_{x \in B_1} u_k(x) + 2(1+N)\log|x| \to \infty$$

which is equivalent to saying that the spherical Harnack inequality does not hold for u_k . It is well known [16] that u_k exhibits a non-simple blowup profile. It is established in [16, 4] that there are N + 1 local maximum points of u_k : p_0^k, \dots, p_N^k and they are evenly distributed on \mathbb{S}^1 after scaling according to their magnitude: Suppose along a subsequence

$$\lim_{k\to\infty}p_0^k/|p_0^k|=e^{i\theta_0},$$

then

$$\lim_{k \to \infty} \frac{p_l^k}{|p_0^k|} = e^{i(\theta_0 + \frac{2\pi l}{N+1})}, \quad l = 1, ..., N.$$

For many reasons it is convenient to denote $|p_0^k|$ as δ_k and define μ_k as follows:

(1.10)
$$\delta_k = |p_0^k|$$
 and $\mu_k = u_k(p_0^k) + 2(1+N)\log\delta_k$,

Since p_l^k s are evenly distributed around ∂B_{δ_k} , standard results for Liouville equations around a regular blowup point can be applied to have $u_k(p_l^k) = u_k(p_0^k) + u_k(p_0^k)$

o(1). Also, (1.9) gives $\mu_k \to \infty$. The interested readers may look into [16, 4] for more detailed information.

The first main theorem is about using a sequence of global solutions of

(1.11)
$$\Delta U + |x|^{2N} e^U = 0$$
, in \mathbb{R}^2 , $\int_{\mathbb{R}^2} |x|^{2N} e^U < \infty$.

to approximate u_k . For regular Liouville equation, this type of approximation, initiated by Y.Y.Li [17], and further extended and refined by a series of works [2, 4, 9, 13, 28, 29] played an important role in a number of applications such as degree counting theorems [9, 10], uniqueness results [3], etc. Our Theorem 1.1 below seems to be the first such result for quantized singular sources:

Theorem 1.1. Let u_k , ϕ_k , h_k , δ_k , μ_k be defined by (1.7), (1.5), (1.8), (1.10) respectively. If $\delta_k^2/e^{-\mu_k} \leq c_0$ for some $c_0 > 0$ independent of k, we have, for some $c_1 > 0$ independent of k and a sequence of global solutions U_k of (1.11) such that

$$|u_k(x) - \phi_k(x) - U_k(x)| \le c_1(\delta_k e^{\mu_k/2} + \mu_k^2 e^{-\mu_k}), \quad x \in B_{\tau}.$$

Remark 1.1. For $dist(x,0) \sim 1$, $u_k(x) = -u_k(p_0^k) + O(1)$. This is already established in [16].

If
$$\delta_k^2/(\mu_k e^{-\mu_k}) \to \infty$$
, we set

(1.12)
$$E_k = O(\delta_k^{-2} \mu_k e^{-\mu_k}) + O(\delta_k).$$

Then in the second main result we prove the following vanishing estimates for the second derivatives of $\log h_k$:

Theorem 1.2. Under the same context of Theorem 1.1, if $\delta_k^2/(\mu_k e^{-\mu_k}) \to \infty$, we have,

(1.13)
$$\Delta(\log h_k)(0) = E_k, \quad if \quad N \ge 2,$$

where E_k is defined in (1.12), and for N = 1, we have

(1.14)
$$(\partial_{e_k} (\log h_k)(0))^2 - (\partial_{e_k^{\perp}} (\log h_k)(0))^2 - 2\partial_{e_k e_k} (\log h_k)(0) = E_k, \\ \partial_{e_k} (\log h_k)(0)\partial_{e_k^{\perp}} (\log h_k)(0) + \partial_{e_k e_k^{\perp}} (\log h_k)(0) = E_k,$$

where $e_k = p_0^k / |p_0^k|$, e_k^{\perp} is an unit vector orthogonal to e_k .

Remark 1.2. Theorem 1.2 is surprising because it is usually difficult to obtain vanishing estimates at a singular source. There are many cancellations for Pohozaev identities around the singular source. However we would like to point out that some vanishing estimates for bubbling solutions of Toda systems have been obtained exactly at singular sources [20], [30].

Remark 1.3. The dichotomy in Theorem 1.1 and Theorem 1.2 appears to be contradictory to Lemma 9 of [16], which asserts that $\delta_k^2 = c\mu_k e^{-\mu_k}(1+o(1))$. However we found (4.18) of [16] incorrect. In fact there should not be any deterministic

relation between μ_k and δ_k , because for any $\xi_k \in \mathbb{R}^2$ and any $\lambda_k \in \mathbb{R}$,

$$U_k(x) = \log \frac{e^{\lambda_k}}{(1 + \frac{e^{\lambda_k}}{8(1+N)^2} |x^{N+1} - \xi_k|^2)^2}$$

is a sequence of solutions to $\Delta U_k + |x|^{2N} e^{U_k} = 0.$

The study of bubbling solutions of (1.7) near the quantized singular source represents a core difficulty in many related problems. For example the following mean field equation defined on a Riemann surface M:

(1.15)
$$\Delta u(x) + \rho \left(\frac{h(x)e^{u(x)}}{\int_M h(x)e^u dx} - \frac{1}{vol(M)}\right) = 4\pi \sum_{j=1}^d \alpha_j \left(\delta_{q_j} - \frac{1}{vol(M)}\right),$$

represents a metric on M with conic singularity. Also it is derived from the mean field limit of point vortices in the Euler flow [6, 7] and serves as a model equation in the Chern-Simons-Higgs theory [15] and in the electroweak theory [1], etc. The rich geometric and physical background manifests the importance of the study in this article.

The phenomena of non-simple bubbling solutions not only occur in single equations, but also in systems. In a recent work of the second author and Gu [14], the non-simple blowup solutions are studied for singular Liouville systems.

To end the introduction we would like to briefly explain the idea of the proof. In [16] and [4] it is already established that there are exactly N + 1 local maximum points evenly distributed around the origin. Kuo-Lin and Bartolucci-Tarantello independently obtained this important information by studying the Pohozaev identity around each local maximum. The main contribution of this article is to go further in this investigation. Roughly speaking, what is achieved in [16, 4] is information contained in the leading terms in those Pohozaev identities. By studying more terms in the expansion of these identities we found further important information on the location of these local maximums and corresponding geometric quantities. From our proof the interested readers will see the more precise information about the location of local maximum points, which should be very useful for constructing such solutions in related studies.

The organization of this article is as follows: In section two we establish some preliminary estimates, in section three we establish precise locations of local maximum points. The approximation by global solutions (Theorem 1.1) is proved in section four and the proof of vanishing theorem (Theorem 1.2) is arranged in section five. Finally in the appendix we prove a sharp estimate of bubbling solutions if the spherical Harnack inequality holds.

Notation: We will use $B(x_0, r)$ to denote a ball centered at x_0 with radius r. If x_0 is the origin we use B_r . C represents a positive constant that may change from place to place.

2. PRELIMINARY DISCUSSIONS

Writing p_0^k as $p_0^k = \delta_k e^{i\theta_k}$ we define v_k as

(2.1)
$$v_k(y) = u_k(\delta_k y e^{i\theta_k}) + 2(N+1)\log \delta_k, \quad |y| < \tau \delta_k^{-1}$$

If we write out each component, (2.1) is

$$v_k(y_1, y_2) = u_k(\delta_k(y_1 \cos \theta_k - y_2 \sin \theta_k), \delta_k(y_1 \sin \theta_k + y_2 \cos \theta_k)) + 2(1+N)\log \delta_k$$

Then it is standard to verify that v_k solves

(2.2)
$$\Delta v_k(y) + |y|^{2N} \mathfrak{h}_k(\delta_k y) e^{v_k(y)} = 0, \quad |y| < \tau/\delta_k,$$

where

(2.3)
$$\mathfrak{h}_k(x) = h_k(xe^{i\theta_k}), \quad |x| < \tau$$

Thus the image of p_0^k after scaling is $Q_1^k = e_1 = (1,0)$. Let Q_1^k , Q_2^k ,..., Q_N^k be the images of p_i^k (i = 1, ..., N) after the scaling. It is established by Kuo-Lin in [16] and independently by Bartolucci-Tarantello in [4] that

(2.4)
$$\lim_{k \to \infty} Q_l^k = \lim_{k \to \infty} p_l^k / \delta_k = e^{\frac{2l\pi i}{N+1}}, \quad l = 0,, N.$$

Choosing $\varepsilon > 0$ small and independent of k, we can make disks centered at Q_l^k with radius ε (denoted as $B(Q_l^k, \varepsilon)$) mutually disjoint. Let

(2.5)
$$\mu_k = \max_{B(Q_0^k, \varepsilon)} v_k.$$

Since Q_l^k are evenly distributed around ∂B_1 , it is easy to use standard estimates for single Liouville equations ([28, 13, 9]) to obtain

$$\max_{B(Q_l^k,\varepsilon)} v_k = \mu_k + o(1), \quad l = 1, \dots, N.$$

Recall that v_k satisfies (2.2) and v_k is a constant on $\partial B(0, \tau \delta_k^{-1})$. The Green's representation formula for v_k gives,

$$v_k(y) = \int_{\Omega_k} G(y,\eta) |\eta|^{2N} \mathfrak{h}_k(\delta_k \eta) e^{v_k(\eta)} d\eta + v_k|_{\partial \Omega_k}$$

where $\Omega_k = B(0, \tau \delta_k^{-1})$ and

$$G(y,\eta) = -\frac{1}{2\pi} \log |y-\eta| + H(y,\eta)$$

where

$$H(y,\eta) = \frac{1}{2\pi} \log\left(\frac{|\eta|}{\tau \delta_k^{-1}} |\frac{\tau^2 \delta_k^{-2} \eta}{|\eta|^2} - y|\right).$$

Also for r > 2, let $\bar{v}_k(r)$ be the spherical average of v_k on ∂B_r , then we have

$$\frac{d}{dr}\bar{v}_{k}(r) = \frac{d}{dr}\frac{1}{2\pi r}\int_{B_{r}}\Delta v_{k} = -\frac{8(N+1)\pi + o(1)}{2\pi r}$$

Because of the fast decay of $\bar{v}_k(r)$ it is easy to use the Green's representation of v_k to obtain the following stronger estimate of v_k :

(2.6)
$$v_k(y) = -\mu_k - (4N+4)\log|y| + O(1), \quad 2 < |y| < \tau \delta_k^{-1}$$

Now we consider v_k around $Q_{l,k}$. Using the results in [9, 28, 13] we have, for v_k in $B(Q_{l,k},\varepsilon)$, the following gradient estimate:

(2.7)
$$\delta_k \nabla(\log \mathfrak{h}_k) (\delta_k \tilde{Q}_{l,k}) + 2N \frac{\tilde{Q}_{l,k}}{|\tilde{Q}_{l,k}|^2} + \nabla \phi_{l,k} (\tilde{Q}_{l,k}) = O(\mu_k e^{-\mu_k}),$$

where $\phi_{l,k}$ is the harmonic function that eliminates the oscillation of v_k on $\partial B(Q_l^k, \varepsilon)$ and $\tilde{Q}_{l,k}$ is the maximum of $v_k - \phi_{l,k}$ that satisfies

(2.8)
$$\tilde{Q}_{l,k} - Q_{l,k} = O(e^{-\mu_k}).$$

Using (2.8) in (2.7) we have

(2.9)
$$\delta_k \nabla (\log \mathfrak{h}_k) (\delta_k Q_{l,k}) + 2N \frac{Q_{l,k}}{|Q_{l,k}|^2} + \nabla \phi_{l,k} (Q_{l,k}) = O(\mu_k e^{-\mu_k}).$$

For the discussion in this section we use the following version of (2.9):

(2.10)
$$\delta_k \nabla (\log \mathfrak{h}_k)(0) + 2N \frac{Q_{l,k}}{|Q_{l,k}|^2} + \nabla \phi_{l,k}(Q_{l,k}) = O(\delta_k^2) + O(\mu_k e^{-\mu_k}).$$

The following lemma provides the first estimate of $\nabla \phi_l^k(Q_l^k)$:

Lemma 2.1. For l = 0, ..., N,

(2.11)
$$\nabla \phi_l^k(Q_l^k) = -4 \sum_{m=0, m \neq l}^N \frac{Q_{l,k} - Q_{m,k}}{|Q_{l,k} - Q_{m,k}|^2} + E$$

where

$$E = O(\delta_k^2) + O(\mu_k e^{-\mu_k}).$$

Proof of Lemma 2.1:

From the expression of v_k on $\Omega_k = B(0, \tau \delta_k^{-1})$ we have, for y away from bubbling disks,

$$(2.12) \quad v_k(y) = v_k|_{\partial\Omega_k} + \int_{\Omega_k} G(y,\eta) |\eta|^{2N} \mathfrak{h}_k(\delta_k\eta) e^{v_k(\eta)} d\eta$$
$$= v_k|_{\partial\Omega_k} + \sum_{l=0}^N G(y,Q_l^k) \int_{B(Q_l,\varepsilon)} |\eta|^{2N} \mathfrak{h}_k(\delta_k\eta) e^{v_k} d\eta$$
$$+ \sum_l \int_{B(Q_l,\varepsilon)} (G(y,\eta) - G(y,Q_l^k)) |\eta|^{2N} \mathfrak{h}_k(\delta_k\eta) e^{v_k} d\eta + O(\mu_k e^{-\mu_k})$$

Before we evaluate each term, we use a sample computation: Suppose f is a smooth function defined on $B(Q_0^k, \varepsilon)$, then we evaluate

(2.13)
$$\int_{B(Q_0^k,\varepsilon)} f(\eta) |\eta|^{2N} \mathfrak{h}_k(\delta_k \eta) e^{\nu_k(\eta)} d\eta.$$

Let \tilde{Q}_0^k be the maximum of $v_k - \phi_0^k$, then it is known [28, 13] that

(2.14)
$$\tilde{Q}_0^k - Q_0^k = O(e^{-\mu_k}).$$

Moreover, it is derived that

(2.15)
$$v_k(y) - \phi_0^k(y) = \log \frac{e^{\mu_k}}{(1 + e^{\mu_k} \frac{|\tilde{Q}_0^k|^{2N} \mathfrak{h}(\delta_k \tilde{Q}_0^k)|}{8} |y - \tilde{Q}_0^k|^2)^2} + O(\mu_k^2 e^{-\mu_k}).$$

Setting $\hat{v}_k = v_k - \phi_0^k$ and $\hat{h}_k = e^{\phi_0^k} |y|^{2N} \mathfrak{h}_k(\delta_k y)$, we can write (2.13) as

$$\int_{B(\mathcal{Q}_{0}^{k},arepsilon)}f(\eta)\hat{h}_{k}(\eta)e^{\hat{arepsilon}_{k}(\eta)}d\eta$$

Using the Taylor expansions of f and \hat{h}_k around \tilde{Q}_0^k and the symmetry of the global solution in (2.15) it is easy to see that

$$\int_{B(\mathcal{Q}_0^k,\varepsilon)} f(\eta) |\eta|^{2N} \mathfrak{h}_k(\delta_k \eta) e^{\nu_k(\eta)} d\eta = 8\pi f(\tilde{\mathcal{Q}}_0^k) + O(\mu_k e^{-\mu_k}).$$

Since (2.14) holds we further have

(2.16)
$$\int_{B(Q_0^k,\varepsilon)} f(\eta) |\eta|^{2N} \mathfrak{h}_k(\delta_k \eta) e^{\nu_k(\eta)} d\eta = 8\pi f(Q_0^k) + O(\mu_k e^{-\mu_k})$$

Using the method of (2.16) in the evaluation of each term in (2.12) we have,

$$v_k(y) = v_k|_{\partial\Omega_k} - 4\sum_{l=0}^N \log|y - Q_{l,k}| + 8\pi \sum_{l=0}^N H(y, Q_{l,k}) + O(\mu_k e^{-\mu_k}).$$

The harmonic function that kills the oscillation of v_k around $Q_{m,k}$ is

$$\phi_m^k = -4 \sum_{l=0, l \neq m}^N (\log |y - Q_l^k| - \log |Q_m^k - Q_l^k|) + 8\pi \sum_{l=0}^N (H(y, Q_l^k) - H(Q_m^k, Q_l^k)) + O(\mu_k e^{-\mu_k}).$$

The corresponding estimate for $\nabla \phi_m^k$ is

$$\nabla \phi_m^k(Q_m^k) = -4 \sum_{l=0, l \neq m}^N \frac{Q_m^k - Q_l^k}{|Q_m^k - Q_l^k|^2} + 8\pi \sum_{l=0}^N \nabla_1 H(Q_m^k, Q_l^k) + O(\mu_k e^{-\mu_k}).$$

where ∇_1 stands for the differentiation with respect to the first component. From the expression of *H*, we have

(2.17)
$$\nabla_{1}H(Q_{m}^{k},Q_{l}^{k}) = \frac{1}{2\pi} \frac{Q_{m}^{k} - \tau^{2}\delta_{k}^{-2}Q_{l}^{k}/|Q_{l}^{k}|^{2}}{|Q_{m}^{k} - \tau^{2}\delta_{k}^{-2}Q_{l}^{k}/|Q_{l}^{k}|^{2}|^{2}} \\ = \frac{1}{2\pi}\tau^{-2}\delta_{k}^{2}\frac{\tau^{-2}\delta_{k}^{2}Q_{m}^{k} - Q_{l}^{k}/|Q_{l}^{k}|^{2}}{|Q_{l}^{k}||^{2} - \tau^{-2}\delta_{k}^{2}Q_{m}^{k}|^{2}} \\ = -\frac{1}{2\pi}\tau^{-2}\delta_{k}^{2}e^{\frac{2\pi i l}{N+1}} + O(\sigma_{k}\delta_{k}^{2}).$$

where $\sigma_k = \max_l |Q_l^k - e^{\frac{2\pi i l}{N+1}}|$. Later we shall obtain more specific estimate of σ_k .

Thus

(2.18)
$$\nabla \phi_m^k(Q_m^k) = -4 \sum_{l=0, l \neq m}^N \frac{Q_m^k - Q_l^k}{|Q_m^k - Q_l^k|^2} - 4\tau^{-2}\delta_k^2 \sum_{l=0}^N e^{\frac{2\pi i l}{N+1}} + O(\sigma_k \delta_k^2) + O(\mu_k e^{-\mu_k})$$
$$= -4 \sum_{l=0, l \neq m}^N \frac{Q_m^k - Q_l^k}{|Q_m^k - Q_l^k|^2} + O(\sigma_k \delta_k^2) + O(\mu_k e^{-\mu_k})$$

where we have used $\sum_{l=0}^{N} e^{2\pi l i/(N+1)} = 0$. Since we don't have the estimate of σ_k now we have

$$\nabla \phi_m^k(Q_m^k) = -4 \sum_{l=0, l \neq m}^N \frac{Q_m^k - Q_l^k}{|Q_m^k - Q_l^k|^2} + E$$

Lemma 2.1 is established. \Box

3. LOCATION OF BLOWUP POINTS

Let $E = O(\delta_k^2) + O(\mu_k e^{-\mu_k})$. The Pohozaev identity around Q_l^k now reads

$$-4\sum_{j=0,j\neq l}^{N}\frac{Q_{l}^{k}-Q_{j}^{k}}{|Q_{l}^{k}-Q_{j}^{k}|^{2}}+2N\frac{Q_{l}^{k}}{|Q_{l}^{k}|^{2}}=-\nabla(\log\mathfrak{h}_{k})(0)\delta_{k}+E.$$

Using *L* to denote $\nabla(\log \mathfrak{h}_k)(0)$, we have, treating every term as a complex number,

$$N\frac{1}{Q_{l}^{k}} = 2\sum_{j=0, j\neq l}^{N} \frac{1}{Q_{l}^{k} - Q_{j}^{k}} - \frac{\bar{L}}{2}\delta_{k} + E,$$

where \overline{L} is the conjugate of L. Thus

(3.1)
$$N = 2 \sum_{j=0, j \neq l}^{N} \frac{Q_l^k}{Q_l^k - Q_j^k} - \frac{\bar{L}}{2} \delta_k Q_l^k + E.$$

Let $\beta_l = 2\pi l/(N+1)$, we write $Q_l^k = e^{i\beta_l} + p_l^k$ for $p_l^k \to 0$. Then we write the first term on the right hand side of (3.1) as

$$\begin{split} & \frac{Q_{l}^{k}}{Q_{l}^{k}-Q_{j}^{k}} = \frac{e^{i\beta_{l}}+p_{l}^{k}}{e^{i\beta_{l}}-e^{i\beta_{j}}+p_{l}^{k}-p_{j}^{k}} \\ = & \frac{e^{i\beta_{l}}+p_{l}^{k}}{(e^{i\beta_{l}}-e^{i\beta_{j}})(1+(p_{l}^{k}-p_{j}^{k})/(e^{i\beta_{l}}-e^{i\beta_{j}})))} \\ = & \frac{e^{i\beta_{l}}}{e^{i\beta_{l}}-e^{i\beta_{j}}} + \frac{p_{l}^{k}}{e^{i\beta_{l}}-e^{i\beta_{j}}} - \frac{e^{i\beta_{l}}}{(e^{i\beta_{l}}-e^{i\beta_{j}})^{2}}(p_{l}^{k}-p_{j}^{k}) + O(\sigma_{k}^{2}) \\ = & \frac{e^{i\beta_{l}}}{e^{i\beta_{l}}-e^{i\beta_{j}}} + \frac{e^{i\beta_{l}}p_{j}^{k}-e^{i\beta_{j}}p_{l}^{k}}{(e^{i\beta_{l}}-e^{i\beta_{j}})^{2}} + O(\sigma_{k}^{2}). \end{split}$$

Using

(3.2)
$$N = 2 \sum_{j=0, j \neq l}^{N} \frac{e^{i\beta_l}}{e^{i\beta_l} - e^{i\beta_j}},$$

we write (3.1) as

(3.3)
$$\sum_{j=0, j\neq l}^{N} \frac{e^{i\beta_l} p_j^k - e^{i\beta_j} p_l^k}{(e^{i\beta_l} - e^{i\beta_j})^2} - \frac{\bar{L}}{4} \delta_k e^{i\beta_l} = E + O(\sigma_k^2)$$

for l = 0, 1, 2, ..., N. Setting $p_l^k = e^{i\beta_l}m_l^k$ and $\beta_{jl} = \beta_j - \beta_l$ we reduce (3.3) to

(3.4)
$$\sum_{j=0, j\neq l}^{N} \frac{e^{i\beta_{jl}} m_{j}^{k}}{(1-e^{i\beta_{jl}})^{2}} - \left(\sum_{j=0, j\neq l}^{N} \frac{e^{i\beta_{jl}}}{(1-e^{i\beta_{jl}})^{2}}\right) m_{l}^{k} - \frac{\bar{L}}{4} \delta_{k} e^{i\beta_{kl}}$$
$$= E + O(\sigma_{k}^{2}) + O(\delta_{k} \sigma_{k})$$

for $l = 0, 1, \dots, N$. It is easy to verify that

(3.5)
$$\frac{e^{i\theta}}{(1-e^{i\theta})^2} = \frac{1}{2(\cos\theta - 1)} = (-\frac{1}{4})\frac{1}{\sin^2(\theta/2)}.$$

To deal with coefficients of m_j^k in (3.4) we set

$$d_j = \frac{1}{\sin^2(\frac{j\pi}{N+1})}, \quad j = 1, ..., N$$

and

$$D = \sum_{j=0, j
eq l}^N d_{|j-l|}.$$

Since $d_l = d_{N+1-l}$ it is easy to check that *D* does not depend on *l*:

$$D = \sum_{k=1}^{N} d_k = \sum_{k=1}^{N} \frac{1}{\sin^2(\frac{k\pi}{N+1})}.$$

Now (3.4) can be written as

(3.6)
$$-\sum_{j\neq l,j=0}^{N} d_{|j-l|} m_{j}^{k} + D m_{l}^{k} - \bar{L} \delta_{k} e^{i\beta_{l}} = E + O(\sigma_{k}^{2}), \quad l = 0, \dots, N.$$

For l = 0, we have $\beta_0 = 0$ and $m_0^k = 0$. Thus from (3.6) we have

(3.7)
$$-\sum_{j=1}^{N} d_j m_j^k - \bar{L} \delta_k = E + O(\sigma_k^2).$$

If we take $(m_1^k, ..., m_n^k)$ as unknowns in (3.6), the last N equations of (3.6) (for l = 1, ..., N) can be written as

(3.8)
$$A\begin{pmatrix} m_1^k\\m_2^k\\\vdots\\m_N^k \end{pmatrix} = \bar{L}\delta_k \begin{pmatrix} e^{i\beta_1}\\e^{i\beta_2}\\\vdots\\e^{i\beta_N} \end{pmatrix} + E + O(\sigma_k^2)$$

where

$$A = \begin{pmatrix} D & -d_1 & \dots & -d_{N-1} \\ -d_1 & D & \dots & -d_{N-2} \\ \vdots & \vdots & \dots & \vdots \\ -d_{N-1} & -d_{N-2} & \dots & D \end{pmatrix}$$

Since $D = |d_1| + ... + |d_N|$ and each $d_i > 0$, we see that the matrix is invertible, thus $|m_i^k| = O(\delta_k)$ for all *i*. This is a standard fact and we include a short proof for completeness:

Lemma 3.1. Let $B = (b_{ij})_{n \times n}$ be an $n \times n$ matrix that satisfies

$$|b_{ii}| > \sum_{j \neq i} |b_{ij}|, \quad for \ all \ i.$$

Then B is invertible.

Proof of Lemma 3.1: Apply row reduction to *B* by eliminating all the entries in the first column except for b_{11} , it is easy to see that *B* can be changed to

$$\left(\begin{array}{ccccc} b_{11} & b_{12} & \dots & b_{1n} \\ 0 & c_{22} & \dots & c_{2n} \\ 0 & c_{32} & \dots & c_{3n} \\ \vdots & \vdots & \dots & \vdots \\ 0 & c_{n2} & \dots & c_{nn} \end{array}\right)$$

for

$$c_{ij} = b_{ij} - \frac{b_{1j}}{b_{11}} b_{i1}, \quad i = 2, ..., n, \quad j = 2, ..., n.$$

Direct computation shows that

$$|c_{ii}| > \sum_{j \neq i} |c_{ij}|, \quad i = 2, ..., n$$

Lemma 3.1 is established. \Box

Since the $O(\sigma_k^2)$ is only an infinitesimal perturbation of *A*, equation (3.8) can be written as

$$(A + O(\sigma_k))(m_1^k, ..., m_N^k)' = \bar{L}\delta_k(e^{i\beta_1}, ..., e^{i\beta_N})' + E.$$

Thus we have, using Lemma 3.1,

(3.9)
$$\begin{pmatrix} m_1^k \\ m_2^k \\ \vdots \\ m_N^k \end{pmatrix} = A^{-1} \delta_k \bar{L}(0) \begin{pmatrix} e^{i\beta_1} \\ e^{i\beta_2} \\ \vdots \\ e^{i\beta_N} \end{pmatrix} + E.$$

With this fact we can further write $\nabla_1 H(Q_m^k, Q_l^k)$ in (2.17) as

(3.10)
$$\nabla_1 H(Q_m^k, Q_l^k) = -\frac{1}{2\pi} \tau^{-2} \delta_k^2 e^{\frac{2\pi i l}{N+1}} + O(\delta_k^3),$$

and $\nabla \phi_l^k(Q_l^k)$ in (2.11) and (2.18) as

(3.11)
$$\nabla \phi_l^k(Q_l^k) = -4 \sum_{m \neq l} \frac{Q_l^k - Q_m^k}{|Q_l^k - Q_m^k|^2} + O(\delta_k^3) + O(\mu_k e^{-\mu_k}).$$

Using $(a^{ij})_{n \times n}$ to denote A^{-1} , we rewrite (3.9) as

(3.12)
$$m_l^k = \delta_k \bar{L}(0) \sum_{s=1}^n a^{ls} e^{i\beta_s} + O(\delta_k^2) + O(\mu_k e^{-\mu_k}), \quad l = 1, ..., N.$$

4. Approximate u_k by global solutions

First we note that for simple blowup solutions, the approximation of u_k using global solutions is much easier. This part will be discussed in the appendix.

Proof of Theorem 1.1: The assumption is $\delta_k e^{\mu_k/2} \leq c_0$. Fixing the neighborhood of one Q_m^k , the expansion of v_k is, taking Q_m^k as the origin,

(4.1)
$$v_k(y) = \log \frac{e^{\mu_{k,m}}}{(1 + e^{\mu_{k,m}} \frac{|\tilde{\mathcal{Q}}_m^k|^{2N} \mathfrak{h}_k(\delta_k \tilde{\mathcal{Q}}_m^k)|}{8} |y - \tilde{\mathcal{Q}}_m^k|^2)^2} + \phi_m^k(y) + O(\mu_k^2 e^{-\mu_k})$$

where $\mu_{k,m} = v_k(\tilde{Q}_m^k)$. First we claim that

(4.2)
$$\mu_{k,m} - \mu_k = O(\delta_k) + O(\mu_k^2 e^{-\mu_k}).$$

From the Green's representation formula for v_k , we have, for *y* away from bubbling areas and $|y| \sim 1$,

$$\begin{split} v_k(y) &= v_k|_{\partial\Omega_k} + \int_{\Omega_k} G(y,\eta)\mathfrak{h}_k(\eta)|\eta|^{2N} e^{v_k} d\eta, \\ &= v_k|_{\partial\Omega_k} + \sum_{l=0}^N G(y,\mathcal{Q}_l^k) \int_{B(\mathcal{Q}_l^k,\varepsilon)} |\eta|^{2N} \mathfrak{h}_k(\delta_k\eta) e^{v_k} d\eta \\ &+ \sum_l (G(y,\eta) - G(y,\mathcal{Q}_l^k))|\eta|^{2N} \mathfrak{h}_k(\delta_k\eta) e^{v_k} d\eta + O(e^{-\mu_k}), \\ &= v_k|_{\partial\Omega_k} + 8\pi \sum_l G(y,\mathcal{Q}_l^k) + O(\mu_k e^{-\mu_k}) \end{split}$$

where $\Omega_k = B(0, \tau \delta_k^{-1})$. In particular if we consider *y* located at $|y - Q_m^k| = \frac{\varepsilon}{2}$, the expression of v_k can be written as

(4.3)
$$v_{k}(y) = v_{k}|_{\partial\Omega_{k}} - 4\log|y - Q_{m}^{k}| + \phi_{m}^{k}$$
$$-4\sum_{l=0, l\neq m}^{N} \log|Q_{m}^{k} - Q_{l}^{k}| + 8\pi \sum_{l=0}^{N} H(Q_{m}^{k}, Q_{l}^{k}) + O(\mu_{k}e^{-\mu_{k}}),$$

where

$$\phi_m^k = \sum_{l=0, l \neq m}^N (-4) \log \frac{|y - Q_l^k|}{|Q_m^k - Q_l^k|} + 8\pi \sum_{l=0}^N (H(y, Q_l^k) - H(Q_m^k, Q_l^k)).$$

Comparing (4.3) and (4.1) we have

(4.4)
$$-\mu_{m,k} - \log \frac{|\tilde{Q}_m^k|^{2N} \mathfrak{h}_k(\delta_k \tilde{Q}_m^k)}{8} \\ = -4 \sum_{l=0, l \neq m}^N \log |Q_m^k - Q_l^k| + 8\pi \sum_{l=0}^N H(Q_m^k, Q_l^k) + v_k|_{\partial \Omega_k} + O(\mu_k^2 e^{-\mu_k}).$$

To evaluate terms in (4.4) we observe that

$$\begin{split} &|\tilde{\mathcal{Q}}_m^k|^{2N} = 1 + O(\delta_k), \qquad \mathfrak{h}_k(\delta_k \tilde{\mathcal{Q}}_m^k) = 1 + O(\delta_k), \\ &\mathcal{Q}_m^k = e^{\frac{2\pi m}{N+1}i} + O(\delta_k), \qquad \tilde{\mathcal{Q}}_m^k = \mathcal{Q}_m^k + O(e^{-\mu_k}), \end{split}$$

and by the expression of $H(y, \eta)$ we have

$$H(Q_m^k, Q_l^k) = \frac{1}{2\pi} \log(\tau \delta_k^{-1}) + O(\delta_k^2).$$

Thus two terms in (4.4) are

(4.5)
$$8\pi \sum_{l=0}^{N} H(Q_m^k, Q_l^k) = 4(N+1)\log(\tau \delta_k^{-1}) + O(\delta_k)$$

(4.6)
$$\sum_{l=0, l \neq m}^{N} \log |Q_m^k - Q_l^k| = \sum_{l=0, l \neq m}^{N} \log |e^{\frac{2\pi m i}{N+1}} - e^{\frac{2\pi l i}{N+1}}| + O(\delta_k)$$
$$= \log(N+1) + O(\delta_k).$$

Using (4.5) and (4.6) in (4.4) we have

(4.7)
$$v_k|_{\partial\Omega_k} = -\mu_{m,k} + \log 8 + 4\log(1+N) - 4(1+N)\log(\tau\delta_k^{-1}) + O(\delta_k) + O(\mu_k^2 e^{-\mu_k}), \quad m = 0, 1, ..., N.$$

Clearly from (4.7) we see that (4.2) holds. In order to approximate v_k with a global solution we find U_k which exactly has local maximums located at $e^{\frac{2\pi l}{N+1}i}$ and $U_k(e_1) = \mu_k$:

$$U_k(x) = \log \frac{e^{\mu_k}}{(1 + \frac{e^{\mu_k}}{8(1+N)^2}|y^{N+1} - e_1|^2)^2},$$

where $e_1 = (1, 0)$ on \mathbb{R}^2 .

First in the region $B(Q_l^k, e^{-\mu_k/2})$, the comparison between v_k and U_k boils down to the evaluation of:

(4.8)
$$\log \frac{e^{\mu_{l,k}}}{(1+\frac{e^{\mu_{l,k}}}{8}|y-p_k|^2)^2} - \log \frac{e^{\mu_k}}{(1+\frac{e^{\mu_k}}{8}|y|^2)^2},$$

for $|p_k| = O(\delta_k)$. By elementary computation we see that the difference between the two terms in (4.8) is $O(\delta_k e^{\mu_k})$ if $|y| \le C e^{\mu_k/2}$. On the other hand, for $C e^{-\mu_k/2} < |y| < \varepsilon/2$, the comparison of expressions of v_k and U_k leads to the same conclusion. Moreover

$$v_k - U_k = O(\delta_k) + O(\mu_k^2 e^{-\mu_k})$$
 on $\partial B(Q_l^k, \varepsilon)$.

Also we observe from the expression of U_k that

$$v_k - U_k = O(\delta_k) + O(\mu_k^2 e^{-\mu_k})$$
 on $\partial \Omega_k$.

Thus we obtain the the closeness of v_k and U_k on $\Omega_k \setminus (\bigcup_l B(Q_l^k, \varepsilon/2))$ by the smallness of $v_k - U_k$ on $\partial \Omega_k$ and standard estimates by Green's representation formula. Theorem 1.1 is established. \Box

5. Discussion of $\delta_k^2 \ge C \mu_k e^{-\mu_k}$

The main purpose of this section is to prove the vanishing rate of the second derivatives of $\log \mathfrak{h}_k(0)$.

The equation of Pohozaev identity now becomes

(5.1)
$$2N \frac{Q_l^k}{|Q_l^k|^2} - 4 \sum_{m \neq l} \frac{Q_l^k - Q_m^k}{|Q_l^k - Q_m^k|^2} + \delta_k \nabla (\log \mathfrak{h}_k) (\delta_k Q_l^k) = E_3$$

with

$$E_3 = O(\delta_k^3) + O(\mu_k e^{-\mu_k}).$$

After simplification (5.1) becomes

$$2N = 4\sum_{m\neq l} \frac{Q_l^k}{Q_l^k - Q_m^k} + \delta_k \bar{\nabla}(\log \mathfrak{h}_k)(\delta_k Q_l^k) Q_l^k = E_3.$$

According to previous computation (for simplicity we use m_l instead of m_l^k in this section)

$$\begin{split} & \frac{Q_l^k}{Q_l^k - Q_j^k} \\ = & \frac{e^{i\beta_l}(1 + m_l)}{e^{i\beta_l}(1 - e^{i\beta_{jl}} + m_l - m_j e^{i\beta_{jl}})} \\ = & \frac{1}{1 - e^{i\beta_{jl}}} - \frac{m_l - m_j e^{i\beta_{jl}}}{(1 - e^{i\beta_{jl}})^2} + \frac{m_l}{1 - e^{i\beta_{jl}}} \\ & + \frac{(m_l - m_j e^{i\beta_{jl}})^2}{(1 - e^{i\beta_{jl}})^3} - \frac{m_l(m_l - m_j e^{i\beta_{jl}})}{(1 - e^{i\beta_{jl}})^2} + O(\delta_k^3). \end{split}$$

After simplification we have

$$\begin{aligned} \frac{Q_l^k}{Q_l^k - Q_j^k} &= \frac{1}{1 - e^{i\beta_{jl}}} + \frac{e^{i\beta_{jl}}}{(1 - e^{i\beta_{jl}})^2} (m_j - m_l) \\ &+ \frac{e^{i\beta_{jl}}}{(1 - e^{i\beta_{jl}})^3} (m_l - m_j) (m_l - m_j e^{i\beta_{jl}}) + O(\delta_k^3) + E. \end{aligned}$$

Using (3.2) for each l, we have

(5.2)
$$\sum_{j=0,j\neq l}^{N} \frac{4e^{i\beta_{jl}}}{(1-e^{i\beta_{jl}})^2} (m_j - m_l) + \sum_{j=0,j\neq l}^{N} \frac{4e^{i\beta_{jl}}}{(1-e^{i\beta_{jl}})^3} (m_l - m_j) (m_l - m_j e^{i\beta_{jl}}) + \delta_k \bar{\nabla} (\log \mathfrak{h}_k) (\delta_k Q_l^k) e^{i\beta_l} (1+m_l) = E,$$

Using *L* to denote $\nabla \log \mathfrak{h}_k$ and (3.5), we write (5.2) as

$$(5.3) - \sum_{j=0, j\neq l}^{N} d_{|j-l|} m_j + Dm_l + \left(\sum_{j=0, j\neq l}^{N} \frac{4e^{i\beta_{jl}}}{(1-e^{i\beta_{jl}})^3}\right) m_l^2 - \left(\sum_{j=0, j\neq l}^{N} \frac{4e^{i\beta_{jl}}(1+e^{i\beta_{jl}})}{(1-e^{i\beta_{jl}})^3} m_j\right) m_l + \sum_{j=0, j\neq l}^{N} \frac{4e^{2i\beta_{jl}}}{(1-e^{i\beta_{jl}})^3} m_j^2 + \delta_k \bar{L}(\delta_k Q_l^k) e^{i\beta_l} + \delta_k \bar{L}(0) e^{i\beta_l} m_l = E.$$

for l = 1, ..., N. Note that in the last term on the left hand side we used $m_l^k = O(\delta_k)$, so for this term there is no need to evaluate at Q_l^k . For l = 0, using $\beta_0 = 0$ and $m_0 = 0$, we have

(5.4)
$$-\sum_{j\neq 0}^{N} d_j m_j + \sum_{j\neq 0} \frac{4e^{2i\beta_j}}{(1-e^{i\beta_j})^3} m_j^2 + \delta \bar{L}(\delta Q_0^k) = E.$$

For the case N = 1 we have

$$D = 1$$
, $d_1 = 1$, $\beta_{01} = -\pi$, $\beta_1 = \pi$, $\beta_0 = 0$.

Thus (5.3) and (5.4) are reduced to

(5.5)
$$\begin{cases} m_1 - \frac{1}{2}m_1^2 - \delta_k \bar{L}(\delta_k Q_1^k) - \delta_k \bar{L}(0)m_1 = E \\ -m_1 + \frac{1}{2}m_1^2 + \delta_k \bar{L}(\delta_k Q_0^k) = E. \end{cases}$$

From the first equation of (5.5) we have

(5.6)
$$m_1 = \delta_k \bar{L}(0) + O(\delta_k^2)$$

Adding the two equations in (5.5) and using (5.6), we have

(5.7)
$$\delta_k \bar{L}(\delta_k Q_0^k) - \delta_k \bar{L}(\delta_k Q_1^k) = (\delta_k \bar{L}(0))^2 + E.$$

By $Q_0^k = 1 + O(\delta_k)$ and $Q_1^k = -1 + O(\delta_k)$ we evaluate *L* as

$$L(\delta_k Q_0^k) = \begin{pmatrix} \partial_1 \log \mathfrak{h}_k(\delta_k Q_0^k) \\ \partial_2 \log \mathfrak{h}_k(\delta_k Q_0^k) \end{pmatrix} = \begin{pmatrix} \partial_1 \log \mathfrak{h}_k(0) + \delta_k \partial_{11} \log \mathfrak{h}_k(0) \\ \partial_2 \log \mathfrak{h}_k(0) + \delta_k \partial_{12} \log \mathfrak{h}_k(0) \end{pmatrix} + O(\delta_k^2).$$

and

$$L(\delta_k Q_1^k) = \begin{pmatrix} \partial_1 \log \mathfrak{h}_k(\delta_k Q_1^k) \\ \partial_2 \log \mathfrak{h}_k(\delta_k Q_1^k) \end{pmatrix} = \begin{pmatrix} \partial_1 \log \mathfrak{h}_k(0) - \delta_k \partial_{11} \log \mathfrak{h}_k(0) \\ \partial_2 \log \mathfrak{h}_k(0) - \delta_k \partial_{12} \log \mathfrak{h}_k(0) \end{pmatrix} + O(\delta_k^2).$$

By comparing coefficients we have

(5.8)
$$a_1^2 - a_2^2 - 2a_{11} = O(\delta_k) + O(\delta_k^{-2}\mu_k e^{-\mu_k}),$$
$$a_1a_2 + a_{12} = O(\delta_k) + O(\delta_k^{-2}\mu_k e^{-\mu_k}),$$

where $a_{11} = \partial_{11}(\log \mathfrak{h}_k)(0)$, $a_{12} = \partial_{12}(\log \mathfrak{h}_k)(0)$, $a_1 = \partial_1(\log \mathfrak{h}_k)(0)$, $a_2 = \partial_2(\log \mathfrak{h}_k)(0)$. Direct computation from $\mathfrak{h}(x) = h_k(xe^{i\theta_k})$ gives

$$a_1 = \partial_1(\log h_k)(0)\cos \theta_k + \partial_2(\log h_k)(0)\sin \theta_k = \partial_{e_k}(\log h_k)(0),$$

$$a_2 = \partial_1(\log h_k)(0)(-\sin \theta_k) + \partial_2(\log h_k)(0)\cos \theta_k = \partial_{e_k^{\perp}}(\log h_k)(0),$$

 $\begin{aligned} a_{11} &= \partial_{11}(\log h_k)(0)\cos^2(\theta_k) + 2\partial_{12}(\log h_k)(0)\cos\theta_k\sin\theta_k + \partial_{22}(\log h_k)(0)\sin^2\theta_k \\ &= \partial_{e_ke_k}(\log h_k)(0), \end{aligned}$

$$a_{12} = (\partial_{22} - \partial_{11})(\log h_k)(0)\sin\theta_k\cos\theta_k + \partial_{12}(\log h_k)(0)(\cos^2\theta_k - \sin^2\theta_k)$$
$$= \partial_{e_k e_k^{\perp}}(\log h_k)(0).$$

Thus (1.14) in Theorem 1.2 is established.

For the case $N \ge 2$ we take the sum of all equations in (5.3), (5.4) and obtain

$$\sum_{l=0}^{N} c_l m_l^2 + \sum_{l \neq j} c_{jl} m_l m_j + \delta_k \sum_{l=0}^{N} \bar{L}(\delta_k Q_l^k) e^{i\beta_l} + \delta_k \sum_{l=0}^{N} \bar{L}(0) e^{i\beta_l} m_l = E,$$

where the coefficients c_l and c_{jl} will be specified later. We first deal with

(5.9)
$$\sum_{l=0}^{N} \delta_k \bar{L}(\delta_k Q_l^k) e^{i\beta_l} = \sum_{l=0}^{N} \delta_k (\bar{L}(\delta_k Q_l^k) - \bar{L}(0)) e^{i\beta_l}$$

where we have used $\sum_{l=0}^{N} e^{i\beta_l} = 0$. If we use $a_{11} = \partial_{11}(\log \mathfrak{h}_k)(0)$, $a_{12} = \partial_{12}(\log \mathfrak{h}_k)(0)$ and $a_{22} = \partial_{22}(\log \mathfrak{h}_k)(0)$, we see that

$$\begin{split} \bar{L}(\delta_k Q_l^k) - \bar{L}(0) \\ = \partial_1 \log \mathfrak{h}(\delta_k Q_l^k) - i \partial_2 \log \mathfrak{h}(\delta_k Q_l^k) - (\partial_1 \log \mathfrak{h}(0) - i \partial_2 \log \mathfrak{h}(0)) \\ = (a_{11} \delta_k \cos \beta_l + a_{12} \sin \beta_l) - i (a_{12} \cos \beta_l + a_{22} \sin \beta_l) \end{split}$$

Thus the real part of (5.9) is

$$\delta_k^2 \left(\sum_{l=0}^N a_{11} \cos^2 \beta_l + 2a_{12} \cos \beta_l \sin \beta_l + a_{22} \sin^2 \beta_l^2\right) = \frac{N+1}{2} (a_{11} + a_{22}) \delta_k^2$$

and the imaginary part of (5.9) is

$$\delta_k^2 (\sum_{l=0}^N (a_{11} - a_{22}) \sin \beta_l \cos \beta_l - a_{12} \cos(2\beta_l)) = 0,$$

where we have used

$$\sum_{l=0}^{N} e^{2i\beta_l} = 0.$$

To compute the coefficients of m_l^2 and $m_l m_t$ for $l \neq t$, we use

$$\frac{e^{i\theta}(1+e^{i\theta})}{(1-e^{i\theta})^3} = -\frac{i}{4}\frac{\cos(\theta/2)}{\sin^3(\theta/2)}.$$

Since $cos(\cdot)$ is even and $sin(\cdot)$ is odd, we see that the summation of cross terms $m_l m_j$ is zero: $c_{jl} = 0$.

The following two properties are useful for evaluating c_l :

(5.10)
$$\frac{e^{i\theta}}{(1-e^{i\theta})^3} = -\frac{1}{8} \frac{\sin(\theta/2) + i\cos(\theta/2)}{\sin^3(\theta/2)},$$
$$\frac{e^{2i\theta}}{(1-e^{i\theta})^3} = -\frac{1}{8} \frac{\sin(-\theta/2) + i\cos(-\theta/2)}{\sin^3(\theta/2)}.$$

By the two identities in (5.10), we see that the coefficient of m_1^2 is

$$c_{l} = \sum_{j=0, j\neq l}^{N} \frac{4e^{i\beta_{jl}}}{(1-e^{i\beta_{jl}})^{3}} + \frac{4e^{2i\beta_{lj}}}{(1-e^{i\beta_{lj}})^{3}},$$

= $\left(-\frac{1}{2}\right) \sum_{j=0, j\neq l}^{N} \left(\frac{\sin(\frac{(j-l)\pi}{N+1}) + i\cos(\frac{(j-l)\pi}{N+1})}{\sin^{3}(\frac{(j-l)\pi}{N+1})} + \frac{\sin(\frac{(j-l)\pi}{N+1}) + i\cos(\frac{(j-l)\pi}{N+1})}{\sin^{3}(\frac{(l-j)\pi}{N+1})}\right)$
= 0.

Thus all the terms related to m_l cancel out. By (3.12) we have

(5.11)
$$\sum_{l=0}^{N} (\bar{L}(\delta_{k}Q_{l}^{k}) - \bar{L}(0))e^{i\beta_{l}} + \delta_{k}^{2}(\bar{L}(0))^{2}\sum_{s,t}e^{i\beta_{s}}a^{st}e^{i\beta_{t}} = E.$$

The first term in (5.11) is

3.7

$$\sum_{l=0}^{N} (\bar{L}(\delta_k Q_l^k) - \bar{L}(0)) e^{i\beta_l} = \begin{pmatrix} \delta_k^2 \frac{N+1}{2} \Delta(\log \mathfrak{h}_k)(0) \\ 0 \end{pmatrix} + E$$

For the second term in (5.11) we use the symbolic computation of matlab to obtain

(5.12)
$$\sum_{s,t=1}^{n} e^{i\beta_s} a^{st} e^{i\beta_t} = 0.$$

Consequently the following holds:

$$\Delta(\log\mathfrak{h}_k)(0)=O(\delta_k^{-2}\mu_ke^{-\mu_k})+O(\delta_k).$$

Since $\Delta(\log h_k)(0) = \Delta(\log \mathfrak{h}_k)(0)$, we obtain the conclusion stated in Theorem 1.2 for $N \ge 2$. Thus, Theorem 1.2 is established. \Box

6. APPENDIX: SIMPLE BLOWUP SOLUTIONS

In the section we approximate simple bubbling solutions using global solutions. Recall that u_k satisfies

$$\Delta u_k + |x|^{2N} h_k(x) e^{u_k} = 0, \quad |x| < \tau,$$
$$\max_{\bar{B}_{\tau}} u_k = \lambda_k \to \infty,$$
0 is the only blowup point in $B_{\tau},$

and

 u_k is a constant at ∂B_{τ} .

Since we talk about simple blowup solutions, we have

(6.1)
$$u_k(x) + 2(1+N)\log|x| \le C$$

for some C > 0 independent of k.

Let

(6.2)
$$\varepsilon_k = e^{-\frac{\lambda_k}{2(1+N)}}$$

and

(6.3)
$$v_k(y) = u_k(\varepsilon_k y) + 2(1+N)\log\varepsilon_k, \quad |y| \le 1/\varepsilon_k.$$

Then clearly $v_k \leq 0$ and satisfies

(6.4)
$$\Delta v_k + h_k(\varepsilon_k y)|y|^{2N}e^{v_k(y)} = 0, \quad \text{in} \quad |y| < \tau/\varepsilon_k$$

It is easy to see that a subsequence of v_k , which is denoted as v_k as well, converges uniformly to v over any fixed compact subset of \mathbb{R}^2 . The limit function v, which solves

(6.5)
$$\Delta v(y) + |y|^{2N} e^{v} = 0, \text{ in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} |y|^{2N} e^{v} < \infty,$$

also satisfies

$$v(y) \leq 0$$

and, by the classification theorem of Prajapat-Tarantello [23]

$$\int_{\mathbb{R}^2} |y|^{2N} e^{v} = 8\pi (1+N).$$

The asymptotic behavior of v is determined by the total integration of $\int_{\mathbb{R}^2} |y|^{2N} e^{v}$:

$$v(y) = -4(1+N)\log|y| + O(1)$$
 for $|y| > 1$.

So $v_k \to v$ over any fixed B_R in \mathbb{R}^2 . Next we consider the behavior of v_k outside B_R .

For $r \in (2\varepsilon_k R, \tau/3)$, let

$$\tilde{v}_k(y) = u_k(ry) + 2(1+N)\log r, \quad \frac{1}{2} < |y| < 2.$$

Then clearly \tilde{v}_k satisfies

$$\Delta \tilde{v}_k(y) + |y|^{2N} h_k(ry) e^{\tilde{v}_k(y)} = 0, \quad B_2 \setminus B_{1/2}.$$

Let c_0 be the bound for \tilde{v}_k : $\tilde{v}_k \leq c_0$ in $B_2 \setminus B_{1/2}$ and we set $g_k = \tilde{v}_k - c_0 - 1$ which immediately satisfies $g_k \leq -1$ in $B_2 \setminus B_{1/2}$. Thus the equation for g_k can be written as

$$\Delta g_k + \frac{|y|^{2N}h_k(ry)e^{\tilde{v}_k}}{g_k}g_k(y) = 0, \quad \text{in} \quad B_2 \setminus B_{1/2}.$$

The coefficient of g_k is clearly bounded. Thus standard Harnack inequality on ∂B_1 gives

$$\max_{\partial B_1}(-g_k) \le c_1(c_0)\min_{\partial B_1}(-g_k)$$

where $c_1 > 1$ only depends on c_0 . Going back to \tilde{v}_k we have

$$\max_{\partial B_1} \tilde{v}_k \leq \frac{1}{c_1} \min_{\partial B_1} \tilde{v}_k + (c_0 + 1)(1 - \frac{1}{c_1}).$$

For u_k it is

(6.6)
$$\max_{\partial B_r} u_k \leq \frac{1}{c_1} \min_{\partial B_r} u_k - 2(1+N)(1-\frac{1}{c_1})\log r + (c_0+1)(1-\frac{1}{c_1}).$$

Let $\bar{v}_k(r)$ be the spherical average of v_k on ∂B_r . Then for r > R for some R large,

 $\bar{v}_k(r) \le (-4(1+N)+\delta_1)\log r + O(1)$

because

$$\frac{d}{dr}\bar{v}_k(r) = -\frac{1}{2\pi}\int_{B_r} |y|^{2N}h_k(\varepsilon_k y)e^{v_k(y)}dy.$$

From the value of $\bar{v}_k(R)$ and the estimate above, it is immediate to see that

(6.7)
$$\bar{v}_k(r) \le (-4(N+1) + \varepsilon(R))\log r + c, \quad r \ge R.$$

For $r \ge R$, the spherical Harnack inequality for u_k gives,

$$\max_{\partial B_r} v_k \le \frac{1}{c_1} \min_{\partial B_r} v_k - 2(1+N)(1-\frac{1}{c_1})\log r + c_2$$

where $c_2 = (c_0 + 1)(1 - \frac{1}{c_1})$. This inequality readily gives the following estimate of v_k :

$$v_k(y) \leq (-2(1+N) - \delta) \log |y|, \quad R < |y| < \tau \varepsilon_k^{-1}$$

for some $\delta > 0$. Then it is easy to use Green's representation formula to obtain

$$\psi_k(\mathbf{y}) \leq -4(1+N)\log(1+|\mathbf{y}|) + c, \quad |\mathbf{y}| \leq \tau/\varepsilon_k.$$

The classification theorem of Prajapat-Tarantello [23] gives

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$$v(y) = \log \frac{\Lambda}{(1 + \frac{\Lambda}{8(1+N)^2} |y^{N+1} - \xi|^2)^2},$$

where parameters $\Lambda > 0$ and $\xi \in \mathbb{C}$. By the argument in Lin-Wei-Zhang [19] there is a perturbation of $\Lambda_k \to \Lambda$ and $\xi_k \to \xi$ such that

$$V_k(y) := \log rac{\Lambda_k}{(1 + rac{\Lambda_k}{8(1+N)^2}|y^{N+1} - \xi_k|^2)^2},$$

satisfies

(6.8)
$$|v_k(y) - V_k(y)| \le C\varepsilon_k(1+|y|), \quad y \in B(0,\varepsilon_k^{-1}).$$

The idea of the proof in [19] for this case is the following: Choose $1 \ll |p_1| \ll |p_2| \ll |p_3|$ such that the following matrix invertible:

(6.9)
$$\begin{pmatrix} \frac{\partial v}{\partial \Lambda}(p_1) & \frac{\partial v}{\partial \Lambda}(p_2) & \frac{\partial v}{\partial \Lambda}(p_3) \\ \frac{\partial v}{\partial \xi_1}(p_1) & \frac{\partial v}{\partial \xi_1}(p_2) & \frac{\partial v}{\partial \xi_1}(p_3) \\ \frac{\partial v}{\partial \xi_2}(p_1) & \frac{\partial v}{\partial \xi_2}(p_2) & \frac{\partial v}{\partial \xi_2}(p_3) \end{pmatrix}$$

where $\xi = \xi_1 + i\xi_2$. Thus if a o(1) perturbation is placed on v (to make $v_k(p_j) = V_k(p_j)$ for j = 1, 2, 3), all we need to do is change the parameters Λ , ξ by a comparable amount. So even though we have a sequence of parameters Λ_k , μ_k , they are not tending to infinity.

Now we improve this estimate. Let

$$w_k(y) = v_k(y) - V_k(y), \quad |y| \le \tau/\varepsilon_k.$$

The equation for w_k is

(6.10)
$$\Delta w_k + |y|^{2N} e^{\xi_k} w_k = -|y|^{2N} (\sum_{t=1}^2 \varepsilon_k \partial_t h_k(0) y^t + O(\varepsilon_k^2 |y|^2)) e^{v_k}$$

for $y \in B(0, \tau \varepsilon_k^{-1})$. In addition, we know that $w_k(p_t) = 0$ for t = 1, 2, 3 and $w_k(y) \le O(\varepsilon_k)(1+|y|)$ and the oscillation of w_k on $\partial B(0, \tau \varepsilon_k^{-1})$ is $O(\varepsilon_k^{N+1})$.

Our next step is to improve the estimate of w_k . From the Green's representation formula for w_k we have

$$w_k(y) = \int_{\Omega_k} G(y, \eta) |\eta|^{2N} (e^{\xi_i} w_k(\eta) + \varepsilon_k \sum_t \partial_t h_k(0) y^t + O(\varepsilon_k^2 |\eta|^2) e^{v_k(\eta)}) dy + w_k |_{\partial \Omega_k} + O(\varepsilon_k^{N+1}).$$

where $\Omega_k = B(0, \tau \varepsilon_k^{-1})$ and $w_k|_{\partial \Omega_k}$ is the average of w_k on $\partial \Omega_k$. Using crude estimate of w_k we rewrite the above as

(6.11)
$$w_k(y) = \int_{\Omega_k} G(y, \eta) O(\varepsilon_k) (1 + |\eta|)^{-3 - 2N} dy + w_k|_{\partial \Omega_k} + O(\varepsilon_k^{N+1}).$$

Since $w_k(p_1) = 0$. Evaluating the above at p_1 we have

(6.12)
$$0 = \int_{\Omega_k} G(p_1, \eta) O(\varepsilon_k) (1 + |\eta|)^{-3 - 2N} d\eta + w_k|_{\partial \Omega_k} + O(\varepsilon_k^{N+1}).$$

The difference of (6.11) and (6.12) gives

(6.13)
$$w_k(y) = \int_{\Omega_k} (G(y,\eta) - G(p_1,\eta)) O(\varepsilon_k) (1+|\eta|)^{-3-2N} dy + O(\varepsilon_k^{N+1}).$$

Then elementary estimate gives

$$w_k(y) = O(\varepsilon_k) \log(2 + |y|).$$

Next we shall identify the $O(\varepsilon_k)$ term in the expansion of v_k . Let f_1 and f_2 be solutions of the following equations respectively:

$$\Delta f_1 + |y|^{2N} e^V f_1 = -y_1 e^V |y|^{2N}, \quad \text{in} \quad \mathbb{R}^2,$$

$$\Delta f_2 + |y|^{2N} e^V f_2 = -y_2 e^V |y|^{2N}, \quad \text{in} \quad \mathbb{R}^2.$$

Here is why f_1, f_2 exist: Let $\phi_0 = \frac{\partial V}{\partial \Lambda}$, $\phi_1 = \frac{\partial V}{\partial \xi_1}$ and $\phi_2 = \frac{\partial V}{\partial \xi_2}$ where $\xi_k = \xi_1 + i\xi_2$. Direct computation gives

(6.14)

$$\phi_{0} = \frac{1}{\Lambda} - \frac{1}{4(1+N)^{2}} \frac{|z^{N+1} - \xi|^{2}}{1 + \frac{\Lambda}{8(1+N)^{2}} |z^{N+1} - \xi|^{2}},$$

$$\phi_{1} = \frac{\Lambda}{4(1+N)^{2}} \frac{\overline{z}^{N+1} - \overline{\xi}}{1 + \frac{\Lambda}{8(1+N)^{2}} |z^{N+1} - \xi|^{2}},$$

$$\phi_{2} = \frac{\Lambda}{4(1+N)^{2}} \frac{z^{N+1} - \xi}{1 + \frac{\Lambda}{8(1+N)^{2}} |z^{N+1} - \xi|^{2}},$$

By Lin-Wei-Ye [18], the kernel in the linearized equation is spanned by ϕ_0 , ϕ_1 , ϕ_2 if a less than linear growth condition is imposed. Using this fact and (6.14), we observe that

$$\int_{\mathbb{R}^2} y_1 e^V |y|^{2N} \phi_j = 0, \quad j = 0, 1, 2.$$

Indeed,

$$\begin{split} & \int_{\mathbb{R}^2} y_1 e^V |y|^{2N} \phi_j \\ &= \int_{\mathbb{R}^2} y_1 \Delta \phi_j \\ &= \lim_{R \to \infty} (\int_{B_R} \partial_V \phi_j y_1 - \int_{B_R} \partial_1 \phi_j) \\ &= 0, \end{split}$$

where the last equality is due to the asymptotic behavior of ϕ_i and $\nabla \phi_i$ at infinity.

Thus f_1 exists. The existence of f_2 can be derived in a similar way. By standard elliptic theory, the estimate of f_i at infinity is:

$$|f_i(y)| \le C(1+|y|)^{-2-2N}, \quad y \in \mathbb{R}^2, \quad i=1,2.$$

Let

(6.15)
$$w_1^k = \varepsilon_k \partial_1 h_k(0) f_1 + \varepsilon_k \partial_2 h_k(0) f_2.$$

By the estimate of w_k we write the equation of w_k as

$$\Delta w_k + |y|^{2N} e^{V_k} w_k = (h_k(0) - h_k(\varepsilon_k y))|y|^{2N} e^{V_k} + O(\varepsilon_k^2)(1 + |y|)^{-2-2N}$$

Comparing with the equation for w_1^k , we now write the equation for $w_k - w_1^k$ as

$$\Delta(w_k - w_1^k) + |y|^{2N} e^{V_k} (w_k - w_1^k) = O(\varepsilon_k^2) (1 + |y|)^{-2-2N}, \quad y \in \mathbb{R}^2$$

Since the matrix in (6.9) is invertible, we adjust the parameters of V_k by $O(\varepsilon_k)$ to make the new global functions \tilde{V}_k satisfy

(6.16)
$$v_k - \tilde{V}_k - w_1^k = 0$$
, for $y = p_1, p_2, p_3$.

Note that the parameters in \tilde{V}_k are $O(\varepsilon_k)$ different from those in V_k .

Let

$$w_2^k = v_k - \tilde{V}_k - w_1^k,$$

because of the closeness of \tilde{V}_k and V_k , the equation for w_2^k is still

$$\Delta w_2^k + |y|^{2N} e^{\tilde{V}_k} w_2^k = O(\mathcal{E}_k^2)(1+|y|)^{-2-2N}, \quad y \in \mathbb{R}^2.$$

Claim:

(6.17)
$$|w_2^k(y)| \le C\varepsilon_k^2 \log(2+|y|).$$

To prove (6.17), we assume

$$\Lambda_k = \max_{y} \frac{|w_2(y)|}{\varepsilon_k^2 \log(2+|y|)} \to \infty.$$

Suppose Λ_k is attained at y_k . Let

$$\tilde{w}_2^k(y) = \frac{w_2(y)}{\Lambda_k \varepsilon_k^2 \log(2 + |y_k|)}.$$

From this definition we immediately see that

$$|\tilde{w}_2^k(y)| = \frac{|w_2^k(y)|\log(2+|y|)}{\Lambda_k \varepsilon_k^2 \log(2+|y|)\log(2+|y_k|)} \le \frac{\log(2+|y|)}{\log(2+|y_k|)} \le \frac{\log(2+|y|)}{\log(2+|y_k|)}$$

On $|y| \leq \tau \varepsilon_k^{-1}$, \tilde{w}_2^k satisfies

(6.18)
$$\Delta \tilde{w}_2^k + |y|^{2N} e^{\tilde{V}_k} \tilde{w}_2^k = O(1) \frac{(1+|y|)^{-2-2N}}{\Lambda_k \log(2+|y_k|)}, \quad |y| < \tau \varepsilon_k^{-1}.$$

Moreover, since \tilde{V}_k has a perturbation of $O(\varepsilon_k^{N+1})$ on $\partial B(0, \tau \varepsilon_k^{-1})$, we have

$$\tilde{w}_2^k(y) = o(1), \quad y \in \partial B(0, \tau \varepsilon_k^{-1}).$$

If $y_k \to y^*$, \tilde{w}_2^k converges to a solution of

$$\Delta \phi + |y|^{2N} e^V \phi = 0, \quad \mathbb{R}^2,$$

with mild growth:

$$|\phi(y)| \le C\log(2+|y|).$$

By the non-degeneracy of the linearized equation,

$$\phi(\mathbf{y}) = c_1 \frac{\partial \mathbf{v}}{\partial \Lambda}(\mathbf{y}) + c_2 \frac{\partial \mathbf{v}}{\partial \xi_1}(\mathbf{y}) + c_3 \frac{\partial \mathbf{v}}{\partial \xi_2}(\mathbf{y}).$$

Using $\phi(p_i) = 0$ for i = 1, 2, 3, we have, by the invertibility of matrix (6.9), $c_1 =$ $c_2 = c_3 = 0$, thus $\phi \equiv 0$, a contradiction to $\tilde{w}_2^k(y_k) = \pm 1$. So we only need to consider the case that $|y_k| \to \infty$. In this case the Green's

representation formula of $\tilde{w}_2^k(y_k)$ gives

$$\pm 1 = \tilde{w}_2^k(y_k) = \int_{\Omega_k} G(y_k, \eta) \frac{\log(2 + |\eta|) - o(1)(1 + |\eta|)^{-2 - 2N}}{\log(2 + |y_k|)} d\eta + o(1).$$

Using $\tilde{w}_2^k(p_1) = 0$ we can further write the above as

(6.19)

$$\begin{split} &\pm 1 = \tilde{w}_2^k(y_k) \\ &= \int_{\Omega_k} (G(y_k, \eta) - G(p_1, \eta)) \frac{\log(2 + |\eta|) - o(1)(1 + |\eta|)^{-2 - 2N}}{\log(2 + |y_k|)} d\eta + o(1). \end{split}$$

However by standard evaluation of the Green's function, the right hand side of (6.19) is o(1). This contradiction proves that v_k can be accurate to $O(\varepsilon_k^2)$ by two terms. So the conclusion of this section is

Theorem 6.1. Let v_k , \tilde{V}_k , w_1^k , ε_k be defined in (6.3),(6.16),(6.15) and (6.2), respectively, then

$$|v_k(y) - \tilde{V}_k(y) - w_1^k(y)| \le C \varepsilon_k^2 \log(2+|y|), \quad |y| \le \tau \varepsilon_k^{-1}.$$

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