**AARMS Summer School Lecture III: Extensions of Nonlocal Eignevalue Problem (NLEP)**

[1] Chapters 8 and 9 of Wei-Winter's book on "Mathematical Aspects of Pattern Formation in Biological Systems", Applied Mathematical Sciences Series, Vol. 189, Springer 2014 , ISBN: 978-4471-5525-6.

In this lecture, we discuss two extensions of the theory of NLEP.

# 1 Shadow system in finite domains

We consider monotone solutions for the shadow Gierer-Meinhardt system

$$\begin{cases} A_t = \epsilon^2 \Delta A - A + \frac{A^2}{\xi}, \quad x \in \Omega, \quad t > 0, \\ \tau \xi_t = -\xi + \frac{1}{|\Omega|} \int_\Omega A^2 \, dx, \\ A > 0, \quad \frac{\partial A}{\partial \nu} = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\epsilon > 0, \tau > 0$ are positive constants, $\Delta := \sum_{i=1}^N \frac{\partial^2}{\partial x^2}$ is the usual Laplace operator and $\Omega \subset \mathcal{R}^n$ is a bounded and smooth domain. Note that we that here $\epsilon > 0$ is a fixed positive number and we do not assume that $\epsilon$ is small which stands in marked contrast to all the previous chapters.

Problem (1.1) is derived, at least formally, by taking the limit $D \to +\infty$ in the Gierer-Meinhardt system (**??**). For further details concerning the derivation of (1.1) from (**??**), we refer to [**?**, **?**, **?**, **?**].

We first consider the one-dimensional case $N = 1$. In Subsection 1.3, we will study some extensions to higher dimensions. Due to rescaling and translation with respect to the spatial variable, we may assume that $\Omega = (0, 1)$. Thus we have

$$\begin{cases} A_t = \epsilon^2 A_{xx} - A + \frac{A^2}{\xi}, \quad 0 < x < 1, \quad t > 0, \\ \tau \xi_t = -\xi + \int_0^1 A^2 \, dx, \\ A > 0, \quad A_x(0,t) = A_x(1,t) = 0. \end{cases} \tag{1.2}$$

Setting $u(x) = \xi^{-1} A(x)$, then $(A, \xi)$ is a monotone decreasing steady-state of (1.2) if and only if :

$$\xi^{-1} = \int_0^1 u^2(x) dx$$

and

$$\epsilon^2 u_{xx} - u + u^2 = 0, \quad u_x(x) < 0, \quad 0 < x < 1, \quad u_x(0) = u_x(1) = 0. \tag{1.3}$$

We let

$$L := \frac{1}{\epsilon} \tag{1.4}$$

and rescale $u(x) = w_L(y)$, where $y = Lx$. Then $w_L$ solves

$$w_L^{''} - w_L + w_L^2 = 0, \quad w_L'(y) < 0, \quad 0 < y < L, \quad w_L'(0) = w_L'(L) = 0. \tag{1.5}$$

Now (1.5) has a a nontrivial solution if and only if

$$\epsilon < \frac{1}{\pi} \quad \text{which is equivalent to} \quad L > \pi). \tag{1.6}$$

On the other hand, if $\epsilon \geq \frac{1}{\pi}$ (or $L \leq \pi$), then $w_L = 1$. This follows for example from (1.34) below.

By Theorem 1.1 of [**?**] we know that any stable solution to (1.2) is asymptotically monotone. More precisely, if $(A(x,t), \xi(t)), t \geq 0$ is a linearly neutrally stable solution to (1.2), then there exists $t_0 > 0$ such that

$$A_x(x, t_0) \neq 0 \text{ for all } (x,t) \in (0,1) \times [t_0, +\infty). \tag{1.7}$$

This implies that all non-monotone steady-state solutions are linearly unstable. Hence we will concentrate on monotone solutions. Obliviously are two monotone solutions, the monotone increasing and the monotone decreasing one, and they are related by reflection. Without loss of generality, we will study the the monotone decreasing solution which we denote by $u_\epsilon$. By [**?**] it has the least energy among all positive solutions of (1.3). If $L \leq \pi$, then $w_L = 1$. For the solutions to (1.2) we set

$$A_L(x) = \xi_L w_L(Lx), \quad \xi_L^{-1} = \int_0^1 w_L^2(Lx)dx. \tag{1.8}$$

In [**?**] and [**?**], under the assumption that $L$ is sufficiently large, it has been shown that that $(A_L, \xi_L)$ is linearly stable for $\tau$ small enough by the SLEP (singular limit eigenvalue problem) approach. In [**?**], it has been proved that for $\epsilon$ sufficiently small $u_\epsilon$ is linearly stable for $\tau$ small enough, using the NLEP (nonlocal eigenvalue problem) method.

Then the question arises if these stability results can be extended to the case of finite $\epsilon$ (corresponding to finite $L$). This is of huge practical relevance since in real-life experiments the physical constants are fixed and it is often hard to justify that they are small in a suitable sense. Therefore the results in this chapter will be useful for experimentalists and inform the setting up of models, testing of hypotheses and prediction of results. In fact, we will derive results on the stability of steady states for all finite $\epsilon$ (or $L$).

We begin our analysis by introducing some notation. For $I = (0, L)$ and $\phi \in H^2(I)$ we set

$$\mathcal{L}[\phi] = \phi'' - \phi + 2w_L\phi. \tag{1.9}$$

In Subsection 1.1, we will show that the spectrum of $\mathcal{L}$ is given by

$$\lambda_1 > 0, \quad \lambda_j < 0, \, j = 2, 3, \ldots . \tag{1.10}$$

This implies for $\mathcal{L} : H^2(I) \to L^2(I)$ that its inverse

$$(H1c) \qquad \mathcal{L}^{-1} \quad \text{exists.}$$

Next we state

**Theorem 1** *Assume that $L > \pi$ and*

$$(H2c) \qquad \int_0^L w_L \mathcal{L}^{-1} w_L \, dy > 0.$$

*Then the steady state $(A_L, \xi_L)$ to (1.2) given in (1.8) is linearly stable for $\tau$ small enough.*

Thus to determine the stability we only have to compute the integral $\int_0^L w_L \mathcal{L}^{-1} w_L \, dy$. Whereas for general $L$ this is quite hard, in the limiting cases $L \to +\infty$ or $L \to \pi$ this can be achieved by asymptotic analysis (see Lemma 1.2 below). If $L$ is sufficiently large, we will see that (H2c) is valid. In particular, Theorem 1 recovers results of [**?**] and [**?**]. On the other hand, if $L$ is near $\pi$, then $w_L \sim 1$, $\mathcal{L}^{-1} w_L \sim 1$, and thus $\int_0^L w_L \mathcal{L}^{-1} w_L \, dy > 0$.

For finite $\tau$, we have the following result.

**Theorem 2** *Assume that (H2c) holds and let $L > \pi$. Then there is a unique $\tau_c > 0$ such that for $\tau < \tau_c$, $(A_L, \xi_L)$ is stable and for $\tau > \tau_c$ it is unstable. At $\tau = \tau_c$ there exists a unique Hopf bifurcation. The Hopf bifurcation is transversal, i.e.*

$$\frac{d\lambda_R}{d\tau}|_{\tau=\tau_c} > 0, \tag{1.11}$$

*where $\lambda_R$ is the real part of the eigenvalue.*

By the results of Subsection 1.1, will calculate that $\int_0^L w_L \mathcal{L}^{-1} w_L \, dy > 0$ for all $L > \pi$ using Weierstrass $p(z)$ functions and Jacobi elliptic integrals. Then we have

**Theorem 3** *Assume that $L > \pi$. Then there exists a unique $\tau_c > 0$ such that for $\tau < \tau_c$, $(A_L, \xi_L)$ is stable and for $\tau > \tau_c$, $(A_L, \xi_L)$ is unstable. At $\tau = \tau_c$, there exists a Hopf bifurcation. Furthermore, the Hopf bifurcation is transversal.*

Thus for the shadow Gierer-Meinhardt system we have given a complete picture of the stability of nontrivial monotone solutions for all $\tau > 0$ and $L > 0$. Note that in case $L \leq \pi$ we necessarily have $w_L \equiv 1$ and there are only trivial monotone solutions. We remark that singular perturbation which can be applied in case $\epsilon$ is small enough can not be used here.

In the previous chapters, we have considered the existence and stability of multiple spikes for small activator diffusivity $\epsilon^2$ and finite inhibitor diffusivity $D$. Now we study the complementary case of finite $\epsilon^2$ and infinite $D$.

## 1.1 Some properties of the function $w_L$

In this subsection, we consider the the unique solution of the boundary value problem

$$w_L^{''} - w_L + w_L^2 = 0,\ w_L'(0) = w_L'(L) = 0,\ w_L'(y) < 0 \text{ for } 0 < y < L. \tag{1.12}$$

Using Weierstrass functions and elliptic integrals we will derive some properties of $w_L$.

Recall that

$$\mathcal{L}[\phi] = \phi^{''} - \phi + 2w_L\phi.$$

Our first result is

**Lemma 1.1** *For the eigenvalue problem*

$$\begin{cases} \mathcal{L}\phi = \lambda\phi, & 0 < y < L, \\ \phi'(0) = \phi'(L) = 0. \end{cases} \tag{1.13}$$

*the eigenvalues satisfy*

$$\lambda_1 > 0, \quad \lambda_j < 0,\ j = 2, 3, \ldots . \tag{1.14}$$

*The eigenfunction $\Phi_1$ to the eigenvalue $\lambda_1$ can be chosen to be positive.*

**Proof:** Let $\lambda_1 \geq \lambda_2 \geq \ldots$ be the eigenvalues of $\mathcal{L}$. It is well-known that $\lambda_1 > \lambda_2$ and that the eigenfunction $\Phi_1$ to $\lambda_1$ can be made positive. Further, we have

$$-\lambda_1 = \min_{\int_0^L \phi^2\, dy = 1} \left( \int_0^L (|\phi'|^2 + \phi^2 - 2w_L\phi^2)\, dy \right) \tag{1.15}$$

$$\leq \left(\int_0^L w_L^2\,dy\right)^{-1}\left(\int_0^L (|w_L'|^2 + w_L^2 - 2w_L w_L^2)\,dy\right) < 0.$$

By a standard argument (see Theorem 2.11 of [?]) it follows that $\lambda_2 \leq 0$. We include a proof for the convenience of the reader. Using the variational characterisation of $\lambda_2$, we get

$$-\lambda_2 = \sup_{v\in H^1(I)} \inf_{\phi\in H^1(I),\phi\not\equiv 0}\left[\frac{\int_0^L (|\phi'|^2 + \phi^2 - 2w_L\phi^2)\,dy}{\int_0^L \phi^2\,dy} \;:\; v\not\equiv 0,\ \int_0^L \phi v\,dy = 0\right]. \tag{1.16}$$

Since $w_L$ has least energy, namely

$$E[w_L] = \inf_{u\not\equiv 0, u\in H^1(I)} E[u],$$

where

$$E[u] = \frac{\int_0^L (|u'|^2 + u^2)\,dy}{(\int_0^L u^3\,dy)^{\frac{2}{3}}}$$

and so for

$$h(t) = E[w_L + t\phi], \quad \phi\in H^1(I).$$

we know that $h(t)$ attains its minimum at $t = 0$. Thus we get

$$h''(0) = 2\left[\int_0^L (|\phi'|^2 + \phi^2)\,dy - 2\int_0^L w_L\phi^2\,dy + 2\frac{(\int_0^L w_L^2\phi\,dy)^2}{\int_0^L w_L^3\,dy}\right]$$

$$\times\frac{1}{\left(\int_0^L w_L^3\,dy\right)^{2/3}} \geq 0.$$

By (1.16), we see that

$$-\lambda_2 \geq \inf_{\int_0^L \phi w\,dy = 0}\left[\int_0^L (|\phi'|^2 + \phi^2)\,dy - 2\int_0^L w_L\phi^2\,dy + 2\frac{(\int_0^L w_L^2\phi\,dy)^2}{\int_0^L w_L^3\,dy}\right]$$

$$\times\frac{1}{\left(\int_0^L w_L^3\,dy\right)^{2/3}} \geq 0.$$

Now from the proof of uniqueness of $w_L$, Appendix B, we can conclude that $\lambda_2 < 0$.

By Lemma 1.1, we know that $\mathcal{L}^{-1}$ exists. In the next step we calculate the integral $\int_0^L w_L\mathcal{L}^{-1}w_L\,dy$. Using a perturbation argument, we get

**Lemma 1.2** *We have*

$$\lim_{L\to\pi}\int_0^L w_L\mathcal{L}^{-1}w_L\,dy=\pi, \tag{1.17}$$

$$\lim_{L\to+\infty}\int_0^L w_L\mathcal{L}^{-1}w_L\,dy=\frac{3}{4}\int_0^\infty w_\infty^2\,dy, \tag{1.18}$$

*where* $w_\infty(y)$ *is given by*

$$w''-w+w^2=0,\quad w'(0)=0,\quad w'(y)<0,\quad w(y)>0,\quad 0<y<+\infty. \tag{1.19}$$

We will compute $\int_0^L w_L\mathcal{L}^{-1}w_L\,dy$ by using elliptic integrals and derive the following result.

**Lemma 1.3** *We have*

$$\int_0^L w_L\mathcal{L}^{-1}w_L\,dy>0$$

*for all* $L>\pi$.

Before proving Lemma 1.3, we rewrite $w_L$ using Weierstrass functions. An introduction to Weierstrass functions can be found in [**?**].

Let $w_L(0)=M$, $w_L(L)=m$.

From (1.12), we have

$$(w_L')^2=w_L^2-\frac{2}{3}w_L^3-M^2+\frac{2}{3}M^3 \tag{1.20}$$

and

$$-m^2+\frac{2}{3}m^3=-M^2+\frac{2}{3}M^3. \tag{1.21}$$

From (1.21), we deduce that

$$\frac{Mm}{M+m}=M+m-\frac{3}{2}. \tag{1.22}$$

Now let

$$\hat{w}=-\frac{1}{6}w_L+\frac{1}{12}. \tag{1.23}$$

Elementary calculations give

$$(\hat{w}')^2=4\hat{w}^3-g_2\hat{w}-g_3=4(\hat{w}-e_1)(\hat{w}-e_2)(\hat{w}-e_3), \tag{1.24}$$

where

$$g_2=\frac{1}{12},\quad g_3=-\frac{1}{216}-\frac{1}{36}\left(-M^2+\frac{2}{3}M^3\right), \tag{1.25}$$

$$e_1 = \frac{1}{6}(M+m) - \frac{1}{6},\ e_2 = -\frac{1}{6}m + \frac{1}{12},\ e_3 = -\frac{1}{6}M + \frac{1}{12}. \tag{1.26}$$

For the Weierstrass function $p(z)$ we have [**?**]:

$$\hat{w}(x) = p(x+\alpha; g_2, g_3) \tag{1.27}$$

for some constant $\alpha$. From now on, we will avoid the arguments $g_2$ and $g_3$ of $p$.

We get

$$p(f_i) = e_i,\ p'(f_i) = 0,\ i = 1,2,3,\ f_1 + f_2 + f_3 = 0 \tag{1.28}$$

which implies that

$$\hat{w}(x) = p(f_3 + x), \quad L = f_1. \tag{1.29}$$

The Weierstrass function $\zeta(z)$ satisfies

$$\zeta(z) = \frac{1}{z} - \int_0^z \left(p(u) - \frac{1}{u^2}\right) du$$

and so we get

$$\zeta'(u) = -p(u),\ \zeta(f_i) = \eta_i,\ i = 1,2,3,\ \eta_1 + \eta_2 + \eta_3 = 0. \tag{1.30}$$

We calculate

$$\int_0^L \hat{w}(x)dx = \int_0^{f_1} p(f_3 + x)dx = -\zeta(u)|_{f_3}^{-f_2} = \zeta(f_3) + \zeta(f_2) \tag{1.31}$$

$$= -\zeta(f_1) = -\zeta(L).$$

This implies that

$$\int_0^L w_L^2\, dy = \int_0^L w_L\, dy = \int_0^L \left(-6\hat{w} + \frac{1}{2}\right) dy = 6\zeta(L) + \frac{L}{2}. \tag{1.32}$$

By the formulas on page 649 of [**?**], we get

$$\zeta(L) = \frac{K(k)}{3L}[3E(k) + (k-2)K(k)], \tag{1.33}$$

$$e_1 = \frac{(2-k)K^2(k)}{3L^2},$$

$$e_2 = \frac{(2k-1)K^2(k)}{3L^2},$$

$$e_3 = \frac{-(k+1)K^2(k)}{3L^2},$$

where $e_1, e_2$ and $e_3$ are given in (1.26) and

$$e_1e_2 + e_2e_3 + e_1e_3 = -\frac{1}{4}g_2 = -\frac{1}{48}.$$

Here $E(k)$ and $K(k)$ denote Jacobi elliptic integrals defined as

$$E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\varphi}\,d\varphi, \quad K(k) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-k^2\sin^2\varphi}}d\varphi.$$

We get

$$L = 2(k^2-k+1)^{\frac{1}{4}}K(k). \tag{1.34}$$

Now (1.34) implies

$$\frac{dL}{dk} = \frac{4K^2((2k-1)K^2 + 4KK'(k^2-k+1))}{L^3}, \tag{1.35}$$

where the argument $k$ of $K$ has been omitted. By (1.34), for every $L > \pi$ there is a unique $k$. Further, we have $\frac{dk}{dL} > 0$ and

$$(2k-1)K + 4K'(k^2-k+1) > 0. \tag{1.36}$$

Now we come to the **Proof of Lemma 1.3:**
We set $\phi_L = \mathcal{L}^{-1}w_L$ and so $\phi_L$ solves

$$\phi_L'' - \phi_L + 2w_L\phi_L = w_L, \quad \phi_L'(0) = \phi_L'(L) = 0.$$

Set

$$\phi_L = w_L + \frac{1}{2}yw_L'(y) + \Psi. \tag{1.37}$$

Then $\Psi(y)$ satisfies

$$\Psi'' - \Psi + 2w_L\Psi = 0,$$
$$\Psi'(0) = 0, \quad \Psi'(L) = -\frac{1}{2}Lw_L''(L). \tag{1.38}$$

Next we set $\Psi_0 = \frac{\partial w_L}{\partial M}$. Then $\Psi_0$ solves

$$\Psi_0'' - \Psi_0 + 2w_L\Psi_0 = 0, \tag{1.39}$$

$$\Psi_0(0) = 1, \quad \Psi_0'(0) = 0.$$

Integration of (1.39) gives

$$\Psi_0'(L) = \int_0^L \frac{\partial w_L}{\partial M}\, dy - 2\int_0^L w_L \frac{\partial w_L}{\partial M}\, dy$$

$$= \frac{d}{dM}\left(\int_0^L (w_L - w_L^2)\, dy\right) - \left(w_L(L) - w_L^2(L)\right)\frac{dL}{dM}.$$

Using the equation for $w_L$, we have $\int_0^L (w_L - w_L^2)\, dy = 0$. Thus we obtain

$$\Psi_0'(L) = -(w_L(L) - w_L^2(L))\frac{dL}{dM}. \tag{1.40}$$

Comparing (1.38) and (1.40), we have

$$\Psi(x) = \frac{L}{2}\left(\frac{dL}{dM}\right)^{-1}\Psi_0(x). \tag{1.41}$$

Thus we get

$$\int_0^L w_L\phi_L\, dy = \int_0^L \left(w_L + \frac{1}{2}yw_L' + \Psi\right) w_L\, dy$$

$$= \frac{3}{4}\int_0^L w_L^2\, dy + \frac{1}{4}Lw_L^2(L) + \frac{L}{2}\left(\frac{dL}{dM}\right)^{-1}\int_0^L w_L\Psi_0\, dy. \tag{1.42}$$

Further, we have

$$\int_0^L w_L\Psi_0\, dy = \int_0^L w_L\frac{\partial w_L}{\partial M}\, dy$$

$$= \frac{1}{2}\frac{d}{dM}\int_0^L w_L^2\, dy - \frac{1}{2}w_L^2(L)\frac{dL}{dM}$$

$$= \frac{1}{2}\left[\frac{d}{dL}\int_0^L w_L^2\, dy - w_L^2(L)\right]\frac{dL}{dM}. \tag{1.43}$$

Substituting (1.43) into (1.42), we obtain

$$\int_0^L w_L\phi_L\, dy = \frac{3}{4}\int_0^L w_L^2\, dy + \frac{1}{4}L\frac{d}{dL}\int_0^L w_L^2\, dy \tag{1.44}$$

$$= \frac{L^{-2}}{4}\frac{d}{dL}\left(L^3\int_0^L w_L^2\, dy\right).$$

By (1.32) and (1.34), we derive

$$L^3\int_0^L w_L^2\,dy = L^3\int_0^L w_L\,dy = 2L^2K[3E+(k-2)K]+\frac{L^4}{2}$$
$$= 8\sqrt{k^2-k+1}K^3\left[3E+(k-2+\sqrt{k^2-k+1})K\right]. \tag{1.45}$$

If $2k-1\geq 0$, we compute

$$\frac{1}{8}\frac{d}{dk}\left(L^3\int_0^L w_L^2\right)>0.$$

If $2k-1<0$, using (1.36) and

$$\frac{dK}{dk}=\frac{E-(k')^2K}{k(k')^2},\quad \frac{dE}{dk}=\frac{E-K}{k},$$

where $k'=\sqrt{1-k^2}$, we have

$$\frac{1}{8}\frac{d}{dk}\left(L^3\int_0^L w_L^2\right)=\frac{d}{dk}\left[\sqrt{k^2-k+1}K^3[3E+\rho_kK]\right]$$
$$=\sqrt{k^2-k+1}K^2\left[9\frac{dK}{dk}E+3K\frac{dE}{dk}+\frac{d\rho_k}{dk}K^2+4\rho_kK\frac{dK}{dk}+\frac{2k-1}{2(k^2-k+1)}K[3E+\rho_kK]\right]$$
$$=\sqrt{k^2-k+1}K^2\left[3\frac{d(EK)}{dk}+2E\left(\frac{dK}{dk}+\frac{2k-1}{4(k^2-k+1)}K\right)+4\frac{dK}{dk}(E+\rho_kK)\right]$$
$$+\sqrt{k^2-k+1}K^2\left[K\left(\frac{d\rho_k}{dk}K+\frac{2k-1}{2(k^2-k+1)}(2E+\rho_kK)\right)\right],$$

where $\rho_k=k-2+\sqrt{k^2-k+1}$. In the previous expression each term is positive which follows from basic calculations.

This completes the proof.

## 1.2 Nonlocal Eigenvalue Problems

Since the nonlocal eigenvalue problem in this problem is defined in a finite interval in contrast to all previous studies in the book we have to derive and study it afresh.

Linearising (1.2) around the steady state

$$A_L=\xi w_L(Lx),\quad \xi_L^{-1}=\int_0^1 w_L^2(Lx)dx, \tag{1.46}$$

we get the eigenvalue problem

$$\epsilon^2\phi_{xx} - \phi + 2w_L\phi - \eta w_L^2 = \lambda\phi, \tag{1.47}$$

$$-\eta + 2\xi_L \int_0^1 w_L\phi\,dx = \tau\lambda\eta.$$

We also rescale:

$$y = Lx. \tag{1.48}$$

Solving the second equation for $\eta$ and putting it into the first equation, we derive the following NLEP:

$$\phi'' - \phi + 2w_L\phi - \frac{2}{1+\tau\lambda}\frac{\int_0^L w_L\phi\,dy}{\int_0^L w_L^2\,dy}w_L^2 = \lambda\phi, \quad y \in (0, L), \tag{1.49}$$

with

$$\phi'(0) = \phi'(L) = 0$$

and

$$\lambda = \lambda_R + \sqrt{-1}\lambda_I \in \mathcal{C}. \tag{1.50}$$

In this subsection, we assume that $\tau = 0$. Thus (1.49) can be written as

$$L_\gamma[\phi] := \mathcal{L}[\phi] - \gamma\frac{\int_0^L w_L\phi\,dy}{\int_0^L w_L^2\,dy}w_L^2 = \lambda\phi, \quad \phi'(0) = \phi'(L) = 0. \tag{1.51}$$

Then we have

**Lemma 1.4** *Suppose that $\gamma \neq 1$. Then $\lambda = 0$ is not an eigenvalue of (1.49).*

**Proof:** Supposing $\lambda = 0$, we get

$$0 = \mathcal{L}[\phi] - \gamma\frac{\int_0^L w_L\phi\,dy}{\int_0^L w_L^2\,dy}w_L^2$$

$$= \mathcal{L}\left(\phi - \gamma\frac{\int_0^L w_L\phi\,dy}{\int_0^L w_L^2\,dy}w_L\right).$$

By Lemma 1.1,

$$\phi - \gamma\frac{\int_0^L w_L\phi\,dy}{\int_0^L w_L^2\,dy}w_L = 0.$$

Multiplying this equation by $w_L$ and integrating, we get

$$(1-\gamma)\int_0^L w_L\phi\,dy = 0.$$

Hence, since $\gamma \neq 1$, we have

$$\int_0^L w_L\phi\,dy = 0.$$

This implies

$$\mathcal{L}[\phi] = 0$$

and by Lemma 1.1 we get

$$\phi = 0.$$

Next we prove that the unstable eigenvalues are bounded uniformly in $\tau$.

**Lemma 1.5** *Let $\lambda$ be an eigenvalue of (1.49) with Re$(\lambda) \geq 0$. Then there is a constant $C$ independent of $\tau > 0$ which satisfies*

$$|\lambda| \leq C. \tag{1.52}$$

**Proof:** We multiply (1.49) by the complex conjugate $\bar{\phi}$ of $\phi$ and integrate. Then we get

$$\lambda\int_0^L |\phi|^2\,dy = -\int_0^L (|\phi'|^2 + |\phi|^2 - 2w_L|\phi|^2)\,dy$$

$$-\frac{2}{1+\tau\lambda}\,\frac{(\int_0^L w_L\phi\,dy)(\int_0^L w_L^2\bar{\phi}\,dy)}{\int_0^L w_L^2\,dy}, \tag{1.53}$$

where $|\phi|^2 = \phi\bar{\phi}$. Using

$$\left|\frac{1}{1+\tau\lambda}\right| \leq \quad \text{for } \mathrm{Re}(\lambda) \geq 0, \tag{1.54}$$

we have

$$\left|\frac{2}{1+\tau\lambda}\frac{(\int_0^L w_L\phi\,dy)(\int_0^L w_L^2\bar{\phi}\,dy)}{\int_0^L w_L^2\,dy}\right| \leq C\int_0^L |\phi|^2\,dy, \tag{1.55}$$

where $C$ is independent of $\tau$.

Now (1.52) follows from (1.53) and (1.55).

Next we study the eigenvalue problem (1.49) and complete the proof of Theorem 1. We remark that the operator $L_\gamma$ is not self-adjoint.

Assuming that $\tau = 0$, we have

**Lemma 1.6** *Assume that (H2c) holds, i.e.*

$$\int_0^L w_L \mathcal{L}^{-1} w_L \, dy > 0. \tag{1.56}$$

*Let $\lambda$ be an eigenvalue of (1.51). Then*

$$Re(\lambda) < 0.$$

The proof of Lemma 1.6 requires the following result:

**Lemma 1.7** *Assuming that (H2c) is valid, there is $a_1 > 0$ such that*

$$Q[\phi, \phi] := \int_0^L (|\phi'|^2 + \phi^2 - 2w_L\phi^2)\, dy + \frac{2\int_0^L w_L^2\phi\, dy \int_0^L w_L\phi\, dy}{\int_0^L w_L^2\, dy} \tag{1.57}$$

$$-\frac{\int_0^L w_L^3\, dy}{\left(\int_0^L w_L^2\, dy\right)^2}\left(\int_0^L w_L\phi\, dy\right)^2 \geq a_1 d_{L^2}^2(\phi, X_1) \quad \textit{for all } \phi \in H^1(0, L).$$

*Here $X_1 = span\ \{w\}$ and $d_{L^2}$ denotes distance in $L^2$-norm.*

Using Lemma 1.7, we have

**Lemma 1.8** *Let $(\lambda, \phi)$ satisfy (1.49) with $Re(\lambda) \geq 0$. Assuming that (H2c) is valid, we get*

$$Re[\bar{\lambda}\chi(\tau\lambda) - \lambda] + |\chi(\tau\lambda) - 1|^2 \left(\frac{\int_0^L w_L^3\, dy}{\int_0^L w_L^2\, dy}\right) \leq 0, \tag{1.58}$$

*where*

$$\chi(\tau\lambda) = \frac{2}{1 + \tau\lambda}. \tag{1.59}$$

*and $\bar{\lambda}$ denotes the conjugate of $\lambda$.*

**Proof of Lemma 1.8:** Let $(\lambda, \phi)$ solve (1.49) and set $\lambda = \lambda_R + \sqrt{-1}\lambda_I$ and $\phi = \phi_R + \sqrt{-1}\phi_I$. Let $\chi(\tau\lambda)$ be given in (1.59). By (1.49) and its complex conjugate, we have

$$\mathcal{L}\phi - \chi(\tau\lambda)\frac{\int_0^L w_L\phi\, dy}{\int_0^L w_L^2\, dy} w_L^2 = \lambda\phi, \tag{1.60}$$

$$\mathcal{L}\bar{\phi} - \bar{\chi}(\tau\lambda)\frac{\int_0^L w_L\bar{\phi}\, dy}{\int_0^L w_L^2\, dy} w_L^2 = \bar{\lambda}\bar{\phi}. \tag{1.61}$$

We multiply (1.60) by $\bar{\phi}$ and integrate by parts to get

$$-\lambda \int_0^L |\phi|^2 \, dy - \chi(\tau\lambda) \frac{(\int_0^L w_L \phi \, dy)(\int_0^L w_L^2 \bar{\phi} \, dy)}{\int_0^L w_L^2 \, dy} \tag{1.62}$$

$$= \int_0^L (|\phi'|^2 + |\phi|^2) \, dy - 2 \int_0^L w_L |\phi|^2 \, dy.$$

Multiplication of (1.61) by $w_L$ gives

$$\int_0^L w_L^2 \bar{\phi} \, dy - \bar{\chi}(\tau\lambda) \frac{\int_0^L w_L \bar{\phi} \, dy}{\int_0^L w_L^2 \, dy} \int_0^L w_L^3 \, dy = \bar{\lambda} \int_0^L w_L \bar{\phi} \, dy. \tag{1.63}$$

Multiplying (1.63) by $\int_0^L w_L \phi \, dy$ and substituting the result into (1.62), we have

$$\int_0^L (|\phi'|^2 + |\phi|^2 - 2 w_L |\phi|^2) \, dy + \lambda \int_0^L |\phi|^2 \, dy \tag{1.64}$$

$$= -\chi(\tau\lambda) \left[ \bar{\lambda} + \overline{\chi}(\tau\lambda) \left( \frac{\int_0^L w_L^3 \, dy}{\int_0^L w_L^2 \, dy} \right) \right] \frac{|\int_0^L w_L \phi \, dy|^2}{\int_0^L w_L^2 \, dy}.$$

We express (1.64) by the quadratic functional $Q$ defined in Lemma 1.7. Using (1.63), we have

$$\left[ \text{Re}[\bar{\lambda} \chi(\tau\lambda) - \lambda] + |\chi(\tau\lambda) - 1|^2 \left( \frac{\int_0^L w_L^3 \, dy}{\int_0^L w_L^2 \, dy} \right) \right] \frac{|\int_0^L w_L \phi \, dy|^2}{\int_0^L w_L^2 \, dy} \tag{1.65}$$

$$= -Q[\phi_R, \phi_R] - Q[\phi_I, \phi_I] - \text{Re}(\lambda) \left[ \int_0^L |\phi|^2 \, dy - \frac{|\int_0^L w_L \phi \, dy|^2}{\int_0^L w_L^2 \, dy} \right] \leq 0.$$

The lemma follows.

Finally, we prove **Proof of Lemma 1.6:**

Assuming $\tau = 0$, from (1.58) we get

$$\text{Re}[\bar{\lambda} \chi(\tau\lambda) - \lambda] + |\chi(\tau\lambda) - 1|^2 \left( \frac{\int_0^L w_L^3 \, dy}{\int_0^L w_L^2 \, dy} \right)$$

$$= (\gamma - 1) \text{Re}(\lambda) + |\gamma - 1|^2 \left( \frac{\int_0^L w_L^3 \, dy}{\int_0^L w_L^2 \, dy} \right) \leq 0$$

which implies

$$\text{Re}(\lambda) \leq -(\gamma - 1) \left( \frac{\int_0^L w_L^3 \, dy}{\int_0^L w_L^2 \, dy} \right) < 0$$

since $\gamma > 1$.

Finally, we prove **Proof of Lemma 1.7:**

The operator

$$\mathcal{L}_1\phi := \mathcal{L}\phi - \frac{\int_0^L w_L\phi\,dy}{\int_0^L w_L^2\,dy}w_L^2$$

$$-\frac{\int_0^L w_L^2\phi\,dy}{\int_0^L w_L^2\,dy}w_L + \frac{\int_0^L w_L^3\,dy\int_0^L w_L\phi\,dy}{(\int_0^L w_L^2\,dy)^2}w_L \tag{1.66}$$

is self-adjoint and

$$Q[\phi,\phi] \geq 0 \iff \mathcal{L}_1 \text{ has no positive eigenvalues.}$$

Simple computations give

$$\mathcal{L}_1 w_L = 0.$$

If $\mathcal{L}_1\phi = 0$, then we have

$$\mathcal{L}\phi = c_1(\phi)w_L + c_2(\phi)w_L^2,$$

where

$$c_1(\phi) = \frac{\int_0^L w_L^2\phi\,dy}{\int_0^L w_L^2\,dy} - \frac{\int_0^L w_L^3\,dy\int_0^L w_L\phi\,dy}{(\int_0^L w_L^2\,dy)^2}, \tag{1.67}$$

$$c_2(\phi) = \frac{\int_0^L w_L\phi\,dy}{\int_0^L w_L^2\,dy}. \tag{1.68}$$

Thus we get

$$\phi - c_1(\phi)(\mathcal{L}^{-1}w_L) - c_2(\phi)w_L = 0. \tag{1.69}$$

Substitution of (1.69) into (1.67) gives

$$c_1(\phi) = c_1(\phi)\frac{\int_0^L w_L^2\mathcal{L}^{-1}w_L\,dy}{\int_0^L w_L^2\,dy} - c_1(\phi)\frac{\int_0^L w_L^3\,dy\int_0^L w_L\mathcal{L}^{-1}w_L\,dy}{(\int_0^L w_L^2\,dy)^2}$$

$$= c_1(\phi) - c_1(\phi)\frac{\int_0^L w_L^3\,dy\int_0^L w_L\mathcal{L}^{-1}w_L\,dy}{(\int_0^L w_L^2\,dy)^2}.$$

Now (H2c) gives $c_1(\phi) = 0$. Thus we have $\phi = c_2(\phi)w_L$. This implies that $w_L$ is the only eigenfunction of $\mathcal{L}_1$ to eigenvalue zero.

Next we assume that the operator $\mathcal{L}_1$ has a positive eigenvalue $\lambda_0 > 0$ with eigenfunction $\phi_0$. Due to the self-adjointness of $\mathcal{L}_1$, we have

$$\int_0^L w_L \phi_0 \, dy = 0 \tag{1.70}$$

and so

$$(\mathcal{L} - \lambda_0)\phi_0 = \frac{\int_0^L w_L^2 \phi_0 \, dy}{\int_0^L w_L^2 \, dy} w_L. \tag{1.71}$$

Note that $\int_0^L w_L^2 \phi_0 \, dy \neq 0$. In fact, if $\int_0^L w_L^2 \phi_0 \, dy = 0$, then $\lambda_0 > 0$ is an eigenvalue of $\mathcal{L}$. By Lemma 1.1, $\lambda_0 = \lambda_1$ and $\phi_0$ does not change sign. This contradicts $\phi_0 \perp w_L$ and so $\lambda_0 \neq \lambda_1$. Thus $\mathcal{L} - \lambda_0$ is invertible. From (1.71), we get

$$\phi_0 = \frac{\int_0^L w_L^2 \phi_0 \, dy}{\int_0^L w_L^2 \, dy} (\mathcal{L} - \lambda_0)^{-1} w_L.$$

Thus

$$\int_0^L w_L^2 \phi_0 \, dy = \frac{\int_0^L w_L^2 \phi_0 \, dy}{\int_0^L w_L^2 \, dy} \int_0^L ((\mathcal{L} - \lambda_0)^{-1} w_L) w_L^2 \, dy.$$

Since $\int_0^L w_L^2 \phi_0 \, dy \neq 0$, we have

$$\int_0^L w_L^2 \, dy = \int_0^L ((\mathcal{L} - \lambda_0)^{-1} w_L) w_L^2 \, dy$$

and therefore

$$\int_0^L w_L^2 \, dy = \int_0^L ((\mathcal{L} - \lambda_0)^{-1} w_L)((\mathcal{L} - \lambda_0) w_L + \lambda_0 w_L) \, dy.$$

Using $\lambda_0 > 0$, we get

$$0 = \int_0^L ((\mathcal{L} - \lambda_0)^{-1} w_L) w_L \, dy. \tag{1.72}$$

For $\beta(t) = \int_0^L ((\mathcal{L} - t)^{-1} w_L) w_L \, dy$ for $t > 0, t \neq \lambda_1$ we compute

$$\beta(0) = \int_0^L (\mathcal{L}^{-1} w_L) w_L \, dy > 0$$

using assumption (H2c) and

$$\beta'(t) = \int_0^L ((\mathcal{L} - t)^{-2} w_L) w_L \, dy > 0.$$

Thus we have $\beta(t) > 0$ for all $t \in (0, \lambda_1)$. Further, we get

$$\beta(t) \to 0 \text{ as } t \to +\infty$$

which implies $\beta(t) < 0$ for $t > \lambda_1$.

To summerise, we have $\beta(t) \neq 0$ for $t > 0, t \neq \lambda_1$. Therefore (1.72) must be false and so $\mathcal{L}_1$ cannot have any positive eigenvalues.

Since

$$Q[\phi, \phi] = -\int_0^L (\mathcal{L}_1 \phi)\phi \, dy,$$

we get $Q[\phi, \phi] \geq 0$ for all $\phi$ with equality if and only if $\phi = cw_L$ for some constant $c$.

This finished the proof.

For the uniqueness and transversality of the Hopf bifurcation for some positive $\tau = \tau_0$ we refer to [?].

## 1.3 Extensions to Higher Dimensions

In the previous subsections, we have studied the one-dimensional case. In the proofs we have used two key ingredients:

(H1c) The operator $\mathcal{L}$ is invertible.

(H2c) The integral $\int_0^L w_L \mathcal{L}^{-1} w_L \, dy$ is positive.

We now consider the case of general domains in $\mathcal{R}^n, N \geq 2$, namely the problem

$$\begin{cases} A_t = \Delta A - A + \frac{A^2}{\xi}, \; x \in \Omega_L, \; t > 0, \\ \tau \xi_t = -\xi + \frac{1}{|\Omega_L|} \int_{\Omega_L} A^2 \, dx, \\ A > 0, \, \frac{\partial A}{\partial \nu} = 0 \;\; \text{on } \partial \Omega_L, \end{cases} \tag{1.73}$$

$\Omega_L = \frac{1}{\epsilon}\Omega \subset \mathcal{R}^n$ with $L = \frac{1}{\epsilon}$ denotes the rescaled domain and we assume it is a smooth and bounded. Letting the dimension satisfy $N \leq 5$, then the exponent 2 is subcritical. A steady state (1.73) is given by

$$A = \xi u, \quad \xi^{-1} = \frac{1}{|\Omega_L|} \int_{\Omega_L} u^2 \, dx, \tag{1.74}$$

where $u$ solves

$$\begin{cases} \Delta u - u + u^2 = 0, \quad u > 0 \;\; \text{in } \; \Omega_L, \\ \frac{\partial u}{\partial \nu} = 0 \;\; \text{on } \partial \Omega_L. \end{cases} \tag{1.75}$$

The energy minimising solution $w_L(x)$ of (1.75) is defined by

$$E[w_L] = \inf_{u \in H^1(\Omega_L), u \not\equiv 0} E[u], \tag{1.76}$$

where

$$E[u] = \frac{\int_{\Omega_L} (|\nabla u|^2 + u^2)\, dy}{(\int_{\Omega_L} u^3\, dy)^{2/3}}.$$

Then

$$A_L = \xi_L w_L, \quad \xi_L^{-1} = \frac{1}{|\Omega_L|} \int_{\Omega_L} w_L^2\, dx \tag{1.77}$$

is a steady-state to the shadow system (1.73). Letting

$$\mathcal{L}[\phi] = \Delta\phi - \phi + 2w_L\phi,$$

then we have

**Lemma 1.9** *Consider the following eigenvalue problem:*

$$\begin{cases} \mathcal{L}\phi = \lambda\phi, \ \ in\ \Omega_L, \\ \frac{\partial\phi}{\partial\nu} = 0 \ \ on\ \partial\Omega_L. \end{cases} \tag{1.78}$$

*Then $\lambda_1 > 0$ and $\lambda_2 \le 0$.*

The proof of this lemma follows that of Lemma 1.1.

Next we make two key assumptions:

$$(H1c) \qquad\qquad \mathcal{L}^{-1} \ \text{ exists.}$$

$$(H2c) \qquad\qquad \int_{\Omega_L} w_L(\mathcal{L}^{-1}w_L)\, dy > 0.$$

Then we have the following result:

**Theorem 4** *Assume that (H1c) and (H2c) are valid. Then for, $\tau$ small enough, the steady state $(A_L, \xi_L)$ is linearly stable. There is a unique $\tau = \tau_c$ such that $(A_L, \xi_L)$ is stable for $\tau < \tau_c$, unstable for $\tau > \tau_c$, and there is a Hopf bifurcation at $\tau = \tau_c$. This Hopf bifurcation is transversal.*

The proof of Theorem 4 goes along the same lines as for one dimension.

If $L$ is large, by [**?**] and[**?**] we know that that (H1) is valid and (H2) holds for $N \le 3$. This recovers the results of [**?**].

For general $\epsilon$, it is hard to verify (H1c) and (H2c). We expect that (H1c) is valid for generic domains.

# 2 The Gierer-Meinhardt system with saturation

We investigate the shadow Gierer-Meinhardt system with saturation:

$$\begin{cases} A_t = \epsilon^2 \Delta A - A + \frac{A^2}{\xi(1+kA^2)}, \quad A > 0 \quad \text{in } \Omega \times (0,\infty), \\ \tau \xi_t = -\xi + \frac{1}{|\Omega|} \int_\Omega A^2(x)\,dx, \; \xi > 0 \quad \text{in } (0,\infty), \\ \frac{\partial A}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0,\infty). \end{cases} \tag{2.1}$$

Concerning the existence of steady states, we can no longer rescale with respect to the amplitude as we did for the system in case $k = 0$ without saturation. Thus it is impossible to reduce the existence problem for steady states to that of a single partial differential equation alone. Instead, we consider a system of a partial differential equation coupled to an algebraic equation:

$$\begin{cases} \epsilon^2 \Delta A - A + \frac{A^2}{\xi(1+kA^2)} = 0, \quad A > 0 \quad \text{in } \Omega, \\ \xi = \frac{1}{|\Omega|} \int_\Omega A^2(x)\,dx, \quad \xi > 0, \\ \frac{\partial A}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Firstly, we solve the parametrised ground state equation

$$\begin{cases} \Delta w_\delta - w_\delta + \frac{w_\delta^2}{1+\delta w_\delta^2} = 0, \quad w_\delta > 0 \quad \text{in } \mathcal{R}^n, \\ w_\delta(0) = \max_{y \in \mathcal{R}^n} w_\delta(y), \quad w_\delta(y) \to 0 \text{ as } |y| \to \infty. \end{cases} \tag{2.3}$$

Secondly, we consider the algebraic equation

$$\delta \left( \int_{\mathcal{R}^n} w_\delta^2(y)\,dy \right)^2 = k_0, \tag{2.4}$$

where

$$k_0 = \lim_{\epsilon \to 0} 4k\epsilon^{-2n} |\Omega|^2. \tag{2.5}$$

We remark that by introducing saturation the type of nonlinearity changes from convex in (**??**) to bistable in (2.3).

For the stability part, , we study NLEP

$$\begin{cases} \Delta \phi - \phi + \left( \frac{2w_\delta}{1+\delta w_\delta^2} - \frac{2\delta w_\delta^3}{(1+\delta w_\delta^2)^2} \right) \phi - 2 \frac{\int_{\mathcal{R}^n} w_\delta \phi}{\int_{\mathcal{R}^n} w^2} \frac{w_\delta^2}{1+\delta w_\delta^2} = \lambda \phi \quad \text{in } \mathcal{R}^n, \\ \phi \in H^1(\mathcal{R}^n), \quad \lambda \in \mathcal{C}. \end{cases} \tag{2.6}$$

In the one-dimensional case, we will give a complete study. In higher dimensions, we will derive sufficient conditions on $k$ to ensure the existence and stability of solutions.

We state our result in the one-dimensional case. Setting $\Omega = (0, 1)$, we have

**Theorem 5** *Assume that*

$$\lim_{\epsilon \to 0} 4k\epsilon^{-2}|\Omega|^2 = k_0 \in [0, +\infty). \tag{2.7}$$

*Then for each $k_0 \geq 0$ and for $\epsilon > 0$ small enough, (2.2) has a steady state $(u_\epsilon, \xi_\epsilon)$ such that*

*(a) $A_\epsilon(x) = (1 + o(1))\xi_\epsilon w_{\delta_\epsilon}(\frac{x}{\epsilon})$, where $\delta_\epsilon \to \delta$, $\delta$ is the unique solution to (2.4) and $w_{\delta_\epsilon}$ is the unique solution to (2.3),*

*(b) $\xi_\epsilon = (2 + o(1))(\epsilon \int_{\mathcal{R}} w_{\delta_\epsilon}^2)^{-1}$.*

*If $\tau$ is small enough, the steady state $(A_\epsilon, \xi_\epsilon)$ is linearly stable for (2.1).*

In case of higher dimensions, the statement is more involved. Let $Q \in \partial\Omega$. Denoting the mean curvature function at $Q$ by $H(Q)$, we we call $Q$ a nondegenerate critical point of $H(Q)$, if we have

$$\partial_i H(Q) = 0, i = 1, \ldots, n-1, \quad \det(\partial_i \partial_j H(Q)) \neq 0,$$

where $\partial_i$ denotes the $i-$th tangential derivative. Then we have

**Theorem 6** *Consider dimensions $n = 2, 3, \ldots$. Assume that*

$$\lim_{\epsilon \to 0} 4k\epsilon^{-2n}|\Omega|^2 = k_0 \in [0, +\infty) \tag{2.8}$$

*and that $Q_0 \in \partial\Omega$ is a nondegenerate critical point of $H(Q)$.*

*Then for each $k_0 \geq 0$ and for $\epsilon$ small enough, (2.2) admits a steady-state solution $(A_\epsilon, \xi_\epsilon)$ such that*

*(a) $A_\epsilon(x) = (1 + o(1))\xi_\epsilon w_{\delta_\epsilon}(\frac{x - Q_\epsilon}{\epsilon})$, where $\delta_\epsilon \to \delta$, $\delta$ is a solution to (2.4) and $w_{\delta_\epsilon}$ is the unique solution to (2.3),*

*(b) $Q_\epsilon \to Q_0$,*

*(c) $\xi_\epsilon = (2 + o(1))(\epsilon^n \int_{\mathcal{R}^n} w_{\delta_\epsilon}^2)^{-1}$.*

*If $Q_0$ is a nondegenerate local maximum point of $H(Q)$, then there is $\hat{k}_0 > 0$ such that in case $n \leq 3$ and $\tau$ small enough, for all $k_0 \in (0, \hat{k}_0)$ the steady state $(A_\epsilon, \xi_\epsilon)$ is linearly stable for (2.1).*

## 2.1 The parametrised ground state

In this subsection, we consider (2.3) and (2.4).

First we note that for $\delta = 0$ (2.3) becomes (**??**). For $\delta$, we use the scaling

$$w_\delta(y) = \frac{1}{\sqrt{\delta}} v\left(\frac{y}{\delta^{\frac{1}{4}}}\right) \tag{2.9}$$

and change (2.3) equivalent problem

$$\begin{cases} \Delta v + g(v) = 0, \quad v > 0 \quad \text{in } \mathcal{R}^n, \\ v(0) = \max_{y \in \mathcal{R}^n} v(y), \quad v(y) \to 0 \text{ as } |y| \to \infty. \end{cases} \tag{2.10}$$

where

$$g(v) = -\sqrt{\delta} v + \frac{v^2}{1+v^2}. \tag{2.11}$$

Now for each $\delta \in (0, \frac{1}{4})$, the equation $g(v) = 0$ has exactly two roots for $v > 0$ given by

$$t_1(\delta) = \frac{1 - \sqrt{1-4\delta}}{2\sqrt{\delta}}, \quad t_2(\delta) = \frac{1 + \sqrt{1-4\delta}}{2\sqrt{\delta}}. \tag{2.12}$$

Next we study

$$c(\delta) = \int_0^{t_2(\delta)} g(s)\, ds. \tag{2.13}$$

We calculate

$$c(\delta) = -\sqrt{\delta}\frac{(t_2(\delta))^2}{2} + t_2(\delta) - \arctan(t_2(\delta)).$$

To study $c(\delta)$, we consider the function

$$\rho(t) = \frac{t - \arctan\,(t)}{t^2}$$

which is well-defined for $t \in [0, +\infty)$. Further, $\rho(t)$ has a unique critical point $t_*$ which solves

$$\arctan t = \frac{2t + t^3}{2(1+t^2)}, \quad t > 0. \tag{2.14}$$

Numerically we get $t_* = 1.514\ldots < \frac{\pi}{2}$. Setting

$$\delta_* = (2\rho(t_*))^2, \tag{2.15}$$

it is easy to see that

$$c(\delta) \begin{cases} > 0 & \text{for } \delta < \delta_*, \\ = 0 & \text{for } \delta = \delta_*, \\ < 0 & \text{for } \delta > \delta_*. \end{cases} \tag{2.16}$$

Next we state a few properties of the function $g(v)$.

**Lemma 2.1** *For each $\delta \in (0, \delta_*)$, the function $g(v)$ has the following properties:*
*(g1) $g \in C^3(\mathcal{R}, \mathcal{R})$, $\quad g(0) = 0$, $\quad g'(0) < 0$.*
*(g2) There exist $b, c > 0$ such that $b < c$, $g(b) = g(c) = 0$, $g(v) > 0$ in $(-\infty, 0) \cup (b, c)$, and $g(v) < 0$ in $(0, b) \cup (c, +\infty)$.*
*(g3) $\int_0^c g(v)dv > 0$.*
*(g4) Let $\theta$ number such $\theta > b$ and $G(\theta) = 0$, where*

$$G(\theta) = \int_0^\theta g(s)ds.$$

*Further, let $\rho$ be the smallest number such that $\frac{g(u)}{u-\rho}$ is nonincreasing for $u \in (\rho, c)$. Then either*
*(i) $\theta \geq \rho$, or*
*(ii) $\theta < \rho$ and $K_g(u)$ is nonincreasing in $(\theta, \rho)$, where*

$$K_g(u) = \frac{ug'(u)}{g(u)}.$$

*Further, we have $K_g(u) \geq K_g(\theta)$ for $u \in (b, \theta)$ and $K_g(u) \leq K_g(\rho)$ for $u \in (0, b) \cup (\rho, c)$.*

**Proof.**
For the proof of Lemma 2.1 we refer to [**?**]. The proof is elementary and we note that $K_g(u) \to \pm\infty$ as $u \to \pm b$ if $g'(b) > 0$.
Next we state some important properties of $w_\delta$.

**Lemma 2.2** *For each $\delta \in [0, \delta_*)$, (2.3) possesses a unique solution, denoted by $w_\delta$, such that*
*(i) $w_\delta \in C^\infty(\mathcal{R}^n)$.*
*(ii) $w_\delta > 0$ is radially symmetric and $w_\delta'(r) < 0$ for $r \neq 0$.*
*(iii) $w_\delta$ and its first- and second-order derivatives decay exponentially at infinity, i.e., for every $\tilde{\delta} > 0$ there is $c_1 > 0$ such that*

$$|w_\delta(y)| \leq c_1 e^{-(1-\tilde{\delta})|y|},$$

$$\left|\frac{\partial w_\delta}{\partial y_i}(y)\right| \le c_1 e^{-(1-\tilde{\delta})|y|}, \quad i = 1, \dots, n,$$

$$\left|\frac{\partial^2 w_\delta}{\partial y_i y_j}(y)\right| \le c_1 e^{-(1-\tilde{\delta})|y|}, \quad i, j, = 1, \dots, n.$$

*(iv) The largest eigenvalue of the operator*

$$L_\delta = \Delta - 1 + \frac{2w_\delta}{1+\delta w_\delta^2} - \frac{2\delta w_\delta^3}{(1+\delta w_\delta^2)^2} \; : \; H^2(\mathcal{R}^n) \to L^2(\mathcal{R}^n), \tag{2.17}$$

*denoted by* $\lambda_1 = \lambda_1(L_\delta)$*, is positive and simple. Its eigenfunction* $\phi$ *is radially symmetric and it can be chosen to be positive.*

*(v) The second largest eigenvalue of* $L_\delta$ *is* 0*. Its kernel consists of the translation modes and has dimension* $n$*. Namely,* $\lambda_2(L_\delta) = 0$ *and*

$$\textit{Kernel}\left(\Delta - 1 + \frac{2w_\delta}{1+\delta w_\delta^2} - \frac{2\delta w_\delta^3}{(1+\delta w_\delta^2)^2}\right) = \textit{span}\left\{\frac{\partial w_\delta}{\partial y_1}, \dots, \frac{\partial w_\delta}{\partial y_n}\right\}. \tag{2.18}$$

**Proof:** By Lemma 2.1, $g(v) = -\sqrt{\delta} v + \frac{v^2}{1+v^2}$ satisfies conditions (g1)-(g4). By Proposition 1.3 of [**?**], Lemma 2.2 holds. To prove this lemma, we first show the statements of Lemma 2.2 for (2.10). Then they follow for the transformed function (2.3). We refer to [**?**, **?**, **?**] for related results.

Now we provide some information about the dependence of $w_\delta$ on $\delta$ state some relevant identities.

**Lemma 2.3** *(1)* $w_\delta(y)$ *is* $C^1$ *in* $\delta$ *for all* $\delta \in (0, \delta_*)$ *and* $y \in \mathcal{R}^n$*,*

*(2)* $w_\delta(y) \to t_2(\delta_*)/\sqrt{\delta_*}$ *in* $C^2_{\mathrm{loc}}(\mathcal{R}^n)$ *as* $\delta \to \delta_*$*.*

*(3) We have*

$$L_\delta w_\delta = \frac{w_\delta^2}{1+\delta w_\delta^2} - \frac{2\delta w_\delta^4}{(1+\delta w_\delta^2)^2}, \tag{2.19}$$

$$L_\delta \frac{dw_\delta}{d\delta} = \frac{w_\delta^4}{(1+\delta w_\delta^2)^2}, \tag{2.20}$$

$$L_\delta(y \cdot \nabla w_\delta) = 2\left(w_\delta - \frac{w_\delta^2}{1+\delta w_\delta^2}\right), \tag{2.21}$$

$$L_\delta\left(w_\delta + 2\delta \frac{dw_\delta}{d\delta} + \frac{1}{2} y \cdot \nabla w_\delta\right) = w_\delta, \tag{2.22}$$

$$L_\delta\left(w_\delta + 2\delta\frac{dw_\delta}{d\delta}\right) = \frac{w_\delta^2}{1+\delta w_\delta^2}. \tag{2.23}$$

**Proof:** (1) Lemma 2.2 gives the uniqueness of $w_\delta$ and the result follows. (2) Noting that $w_\delta \le t_2(\delta)/\sqrt{\delta}$ and taking the limit $\delta \to \delta_*$, we have that $w_\delta$ converges in $C^2_{\rm loc}(\mathcal{R}^n)$ to a solution of

$$\Delta u - u + \frac{u^2}{1+\delta_* u^2} = 0, \quad y \in \mathcal{R}^n, \quad u = u(|y|)$$

which admits only constant solutions. Further, this constant is $t_2(\delta_*)/\sqrt{\delta_*}$ since $w_\delta(0) \to t_2(\delta_*)/\sqrt{\delta_*}$. (2) follows.

(3) The identities (2.19) and (2.20) are computed directly. (2.21) are derived using Pohozaev's identity. Finally, (2.22) and (2.23) follow from (2.19) – (2.21).

Next we consider an algebraic equation.

**Lemma 2.4** *For each fixed $k_0 > 0$, there exists $\delta \in (0, \delta_*)$ such that*

$$k_0 = \delta\left(\int_{\mathcal{R}^n} w_\delta^2(y)\,dy\right)^2. \tag{2.24}$$

*holds.*

**Proof:** Let $\beta(\delta) = \delta\left(\int_{\mathcal{R}^n} w_\delta^2(y)\,dy\right)^2$. Then function $\beta(\delta)$ is continuous and $\beta(0) = 0$. Next we consider the asymptotic behaviour of $w_\delta$ as $\delta \to \delta_*$. By Lemma 2.3 (2), we have $w_\delta(|y|) \to t_2(\delta_*)/\sqrt{\delta_*}$ in $C^2_{\rm loc}(\mathcal{R}^n)$ as $\delta \to \delta_*$. Hence we get

$$\beta(0) = 0, \quad \beta(\delta) \to \infty \text{ as } \delta \to \delta_*. \tag{2.25}$$

Finally, using the mean-value theorem, for each $k_0 \in (0, +\infty)$, there exists $\delta \in (0, \delta_*)$ such that $\beta(\delta) = k_0$.

**Remark 2.1** *To show the uniqueness of the solution $\delta \in (0, \delta^*)$ to (2.24), we compute*

$$\frac{d\beta}{d\delta} = \left[\int_{\mathcal{R}^n} w_\delta^2(y)\,dy + 4\delta\int_{\mathcal{R}^n} w_\delta\frac{dw_\delta}{d\delta}\,dy\right]\int_{\mathcal{R}^n} w_\delta^2(y)\,dy. \tag{2.26}$$

Then we claim that

**Lemma 2.5**

$$\left.\int_{\mathcal{R}^n} w_\delta\frac{dw_\delta}{d\delta}\,dy\right|_{\delta=0} > 0. \tag{2.27}$$

**Proof:** From (2.20) and (2.22), we get

$$\int_{\mathcal{R}^n} w_\delta \frac{dw_\delta}{d\delta}\,dy\bigg|_{\delta=0} = \int_{\mathcal{R}^n} w_0 L_0^{-1}(w_0^4)\,dy$$

$$= \int_{\mathcal{R}^n} w_0^4 (L_0^{-1} w_0)\,dy = \left(1 - \frac{n}{10}\right) \int_{\mathcal{R}^n} w_0^5\,dy > 0.$$

Thus the solution to (2.24) is unique if $k$ is small enough. We expect that Lemma 2.5 holds for any $\delta \in (0, \delta_*)$ and show that this is true for the one-dimensional case:

**Lemma 2.6** *In case $n = 1$, for any $\delta \in (0, \delta_*)$ we have*

$$\frac{d}{d\delta}\left(\int_{\mathcal{R}} w_\delta^2\,dy\right) > 0. \tag{2.28}$$

**Proof.** The proof of Lemma 2.6 is technical and we refer to [**?**].

## 2.2 Stability of spikes for the shadow Gierer-Meinhardt system with saturation

Let $(A_\epsilon, \xi_\epsilon)$ be the steady state given in Theorems 5 and 6. Linearising around the steady state $(A_\epsilon, \xi_\epsilon)$, we have

$$\epsilon^2 \Delta\phi - \phi + \frac{2A_\epsilon \phi}{\xi_\epsilon (1 + kA_\epsilon^2)} - \frac{2kA_\epsilon^3 \phi}{\xi_\epsilon (1 + kA_\epsilon^2)^2} - \frac{A_\epsilon^2}{\xi_\epsilon^2 (1 + kA_\epsilon^2)}\,\eta = \lambda\phi, \tag{2.29}$$

$$-\eta + \frac{2}{|\Omega|}\int_\Omega A_\epsilon \phi\,dx = \tau\lambda\eta, \tag{2.30}$$

where $(\phi, \eta) \in H_N^2(\Omega) \times \mathcal{R}$.

In case $\tau = 0$, we have

$$\eta = \frac{2}{|\Omega|}\int_\Omega A_\epsilon \phi\,dx. \tag{2.31}$$

Inserting (2.31) into (2.29), rescaling and taking the limit as $\epsilon \to 0$, we obtain NLEP [**?**]

$$\Delta\phi - \phi + \frac{2w_\delta \phi}{1 + \delta w_\delta^2} - \frac{2\delta w_\delta^3 \phi}{(1 + \delta w_\delta^2)^2} - \frac{2\int_{\mathcal{R}^n} w_\delta \phi\,dy}{\int_{R^2} w_\delta^2\,dy}\,\frac{w_\delta^2}{1 + \delta w_\delta^2} = \lambda\phi. \tag{2.32}$$

To study (2.32), we will derive the following key result:

**Theorem 7** *Consider the case $n \leq 3$. Assume that $\delta \in [0, \delta_{**})$, where $\delta_{**} > 0$ is defined by*

$$\delta_{**} = \sup \left\{ \delta \in (0, \delta_*) \ : \int_{\mathcal{R}^n} w_s \frac{dw_s}{ds} > 0, \ \textit{for all} \ \ s \in (0, \delta) \right\}. \tag{2.33}$$

*Then for all nonzero eigenvalues $\lambda$ of (2.32), we have $Re(\lambda) \leq -c_0$ for some $c_0 > 0$.*

**Remark 2.2** *By Lemma 2.5, we have $\delta_{**} > 0$. By Lemma 2.6, for $n = 1$ we get $\delta_{**} = \delta_*$. Hence we have the following result.*

**Corollary 2.1** *Let $n = 1$. Then for all nonzero eigenvalues $\lambda$ of (2.32) and all $\delta \in [0, \delta_*)$, it holds that $Re(\lambda) \leq -c_0$ for some $c_0 > 0$.*

To prove Theorem 7, we use a continuation argument. In case $\delta = 0$, Theorem 7 has been proved in Chapter 3 and follows from the following key inequality:

**Lemma 2.7** *(Lemma 5.1 of [**?**]). Assume that $n \leq 3$. Then we have*

$$\int_{\mathcal{R}^n} (|\nabla \phi|^2 + |\phi|^2 - 2w_0^2 |\phi|^2)\, dy + \frac{2 \int_{\mathcal{R}^n} w_0 \phi_0 \, dy \int_{\mathcal{R}^n} w_0^2 \phi \, dy}{\int_{\mathcal{R}^n} w_0^2 \, dy}$$

$$- \frac{\left(\int_{\mathcal{R}^n} w_0 \phi \, dy\right)^2}{\left(\int_{\mathcal{R}^n} w_0^2 \, dy\right)^2} \int_{\mathcal{R}^n} w_0^3 \, dy \geq c_1 d_{L^2}(\phi, X_1), \tag{2.34}$$

*where*

$$X_1 = \left\{ w_0, \frac{\partial w_0}{\partial y_j}, \, j = 1, \ldots, n \right\}$$

*and $d_{L^2}$ is the $L^2$-distance.*

**Proof of Theorem 7:**

We use the continuation method and begin by restricting $\phi$ to the Sobolev space of radially symmetric functions given by

$$\phi \in H_r^2(\mathcal{R}^n) = \{\phi \in H^2(\mathcal{R}^n) \, : \, \phi = \phi(|y|)\}.$$

This is possible due to the argument in [**?**] and [**?**]. Then multiplication of (2.32) by the conjugate function $\overline{\phi}$ of $\phi$ and integration gives

$$Q_\delta[\phi_R, \phi_R] + Q_\delta[\phi_I, \phi_I] = -\lambda \int_{\mathcal{R}^n} |\phi|^2 \, dy, \tag{2.35}$$

where

$$Q_\delta[u,u]=\int_{\mathcal{R}^n}\left(|\nabla u|^2+u^2-\frac{2w_\delta^2u^2}{1+\delta w_\delta^2}+\frac{2\delta w_\delta^3u^2}{(1+\delta w_\delta^2)^2}\right)dy \tag{2.36}$$

$$+2\frac{\int_{\mathcal{R}^n}w_\delta u\,dy}{\int_{\mathcal{R}^n}w_\delta^2\,dy}\int_{\mathcal{R}^n}\frac{w_\delta^2u}{1+\delta w_\delta^2}\,dy$$

and $\phi_R=\mathrm{Re}(\phi)$, $\phi_I=\mathrm{Im}(\phi)$ are the real and the imaginary parts of $\phi$, respectively.

To prove Theorem 7, it remains to show that $Q_\delta$ is positive definite for $\delta\in[0,\delta_{**})$. We rewrite $Q_\delta$ as follows:

$$Q_\delta[u,u]=-(\mathcal{L}_\delta u,u),$$

where

$$\mathcal{L}_\delta u=\Delta u-u+\frac{2w_\delta}{1+\delta w_\delta^2}u-\frac{2\delta w_\delta^3}{(1+\delta w_\delta^2)^2}u-\frac{\int_{\mathcal{R}^n}w_\delta u\,dy}{\int_{\mathcal{R}^n}w_\delta^2\,dy}\,\frac{w_\delta^2}{1+\delta w_\delta^2}$$

$$-\frac{w_\delta}{\int_{\mathcal{R}^n}w_\delta^2\,dy}\int_{\mathcal{R}^n}\frac{w_\delta^2u}{1+\delta w_\delta^2}\,dy. \tag{2.37}$$

Then we have that

$$Q_\delta\quad\text{is positive definite}\qquad\Longleftrightarrow\ \mathcal{L}_\delta\quad\text{has negative spectrum only.} \tag{2.38}$$

By inequality (2.34), the principal eigenvalue of $\mathcal{L}_\delta$ is negative for $\delta=0$. Considering varying $\delta$, we assume that for some $\delta\in(0,\delta_*)$, the principal eigenvalue of $\mathcal{L}_\delta$ vanishes. Equivalently, for some function $\phi\in H_r^2(\mathcal{R}^n)$ we have

$$\mathcal{L}_\delta\phi=0. \tag{2.39}$$

Next we rewrite (2.39) as

$$L_\delta\phi=\frac{\int_{\mathcal{R}^n}w_\delta\phi\,dy}{\int_{\mathcal{R}^n}w_\delta^2\,dy}\,\frac{w_\delta^2}{1+\delta w_\delta^2}+\int_{\mathcal{R}^n}\frac{w_\delta^2\phi}{1+\delta w_\delta^2}\,dy\frac{w_\delta}{\int_{\mathcal{R}^n}w_\delta^2\,dy}.$$

Now by Lemma 2.2 the inverse operator $L_\delta^{-1}$ exists and we have

$$\phi=\frac{\int_{\mathcal{R}^n}w_\delta\phi\,dy}{\int_{\mathcal{R}^n}w_\delta^2\,dy}\left(L_\delta^{-1}\frac{w_\delta^2}{1+\delta w_\delta^2}\right)+\int_{\mathcal{R}^n}\frac{w_\delta^2\phi}{1+\delta w_\delta^2}\,dy\frac{L_\delta^{-1}w_\delta}{\int_{\mathcal{R}^n}w_\delta^2\,dy}. \tag{2.40}$$

To solve (2.40), we set $A = \int_{\mathcal{R}^n} w_\delta \phi \, dy$ and $B = \int_{\mathcal{R}^n} \frac{w_\delta^2 \phi}{1+\delta w_\delta^2} \, dy$. Then we get

$$\begin{cases} A = \dfrac{\int_{\mathcal{R}^n} w_\delta L_\delta^{-1} \frac{w_\delta^2}{1+\delta w_\delta^2} \, dy}{\int_{\mathcal{R}^n} w_\delta^2 \, dy} A + \dfrac{\int_{\mathcal{R}^n} w_\delta L_\delta^{-1} w_\delta \, dy}{\int_{\mathcal{R}^n} w_\delta^2 \, dy} B \\ B = \dfrac{\int_{\mathcal{R}^n} \frac{w_\delta^2}{1+\delta w_\delta^2} L_\delta^{-1} \frac{w_\delta^2}{1+\delta w_\delta^2} \, dy}{\int_{\mathcal{R}^n} w_\delta^2 \, dy} A + \dfrac{\int_{\mathcal{R}^n} \frac{w_\delta^2}{1+\delta w_\delta^2} L_\delta^{-1} w_\delta \, dy}{\int_{\mathcal{R}^n} w_\delta^2 \, dy} B. \end{cases} \tag{2.41}$$

Using Lemma 2.2 and noting that $\phi \in H_r^2(\mathcal{R}^n)$, we cannot have $L_\delta \phi = 0$ and $\phi \neq 0$. This implies $A^2 + B^2 \neq 0$.

Then (2.41) has nontrivial solutions if and only if

$$\begin{vmatrix} 1 - \dfrac{\int_{\mathcal{R}^n} w_\delta L_\delta^{-1} \frac{w_\delta^2}{1+\delta w_\delta^2} \, dy}{\int_{\mathcal{R}^n} w_\delta^2 \, dy} & -\dfrac{\int_{\mathcal{R}^n} w_\delta L_\delta^{-1} w_\delta \, dy}{\int_{\mathcal{R}^n} w_\delta^2 \, dy} \\ -\dfrac{\int_{\mathcal{R}^n} \frac{w_\delta^2}{1+\delta w_\delta^2} L_\delta^{-1} \frac{w_\delta^2}{1+\delta w_\delta^2} \, dy}{\int_{\mathcal{R}^n} w_\delta^2 \, dy} & 1 - \dfrac{\int_{\mathcal{R}^n} \frac{w_\delta^2}{1+\delta w_\delta^2} L_\delta^{-1} w_\delta \, dy}{\int_{\mathcal{R}^n} w_\delta^2 \, dy} \end{vmatrix} = 0, \tag{2.42}$$

which is equivalent to

$$\left( 1 - \frac{\int_{\mathcal{R}^n} \frac{w_\delta^2}{1+\delta w_\delta^2} L_\delta^{-1} w_\delta \, dy}{\int_{\mathcal{R}^n} w_\delta^2 \, dy} \right)^2 - \frac{1}{\left( \int_{\mathcal{R}^n} w_\delta^2 \, dy \right)^2} \left( \int_{\mathcal{R}^n} w_\delta L_\delta^{-1} w_\delta \, dy \right) \left( \int_{\mathcal{R}^n} \frac{w_\delta^2}{1+\delta w_\delta^2} L_\delta^{-1} \frac{w_\delta^2}{1+\delta w_\delta^2} \, dy \right) = 0. \tag{2.43}$$

Using the identities (2.19)–(2.23), we compute

$$\begin{aligned} \int_{\mathcal{R}^n} \frac{w_\delta^2}{1+\delta w_\delta^2} L_\delta^{-1} w_\delta dy &= \int_{\mathcal{R}^n} w_\delta L_\delta^{-1} \frac{w_\delta^2}{1+\delta w_\delta^2} dy = \int_{\mathcal{R}^n} w_\delta \left( w_\delta + 2\delta \frac{dw_\delta}{d\delta} \right) dy \\ &= \int_{\mathcal{R}^n} w_\delta^2 \, dy + 2\delta \int_{\mathcal{R}^n} w_\delta \frac{dw_\delta}{d\delta} \, dy, \end{aligned} \tag{2.44}$$

$$\begin{aligned} \int_{\mathcal{R}^n} w_\delta L_\delta^{-1} w_\delta \, dy &= \int_{\mathcal{R}^n} w_\delta \left( w_\delta + 2\delta \frac{dw_\delta}{d\delta} + \frac{1}{2} y \cdot \nabla w_\delta \right) dy \\ &= \left( 1 - \frac{n}{4} \right) \int_{\mathcal{R}^n} w_\delta^2 \, dy + 2\delta \int_{\mathcal{R}^n} w_\delta \frac{dw_\delta}{d\delta} \, dy, \end{aligned} \tag{2.45}$$

$$\int_{\mathcal{R}^n} \frac{w_\delta^2}{1+\delta w_\delta^2} L_\delta^{-1} \frac{w_\delta^2}{1+\delta w_\delta^2}\, dy = \int_{\mathcal{R}^n} \frac{w_\delta^2}{1+\delta w_\delta^2} \left( w_\delta + 2\delta \frac{dw_\delta}{d\delta} \right) dy$$

$$= \int_{\mathcal{R}^n} \frac{w_\delta^3}{1+\delta w_\delta^2}\, dy + 2\delta \int_{\mathcal{R}^n} \frac{w_\delta^2}{1+\delta w_\delta^2} \frac{dw_\delta}{d\delta}\, dy. \tag{2.46}$$

Multiplication of (2.21) by $\frac{1}{2}\frac{dw_\delta}{d\delta}$, use of (2.20) and integration gives

$$\int_{\mathcal{R}^n} \frac{w_\delta^2}{1+\delta w_\delta^2} \frac{dw_\delta}{d\delta}\, dy - \int_{\mathcal{R}^n} w_\delta \frac{dw_\delta}{d\delta}\, dy = \int_{\mathcal{R}^n} \frac{w_\delta^4}{(1+\delta w_\delta^2)^2} \left( -\frac{1}{2} y \cdot \nabla w_\delta \right) dy$$

and so

$$\int_{\mathcal{R}^n} \frac{w_\delta^2}{1+\delta w_\delta^2} \frac{dw_\delta}{d\delta}\, dy = \int_{\mathcal{R}^n} w_\delta \frac{dw_\delta}{d\delta}\, dy + \frac{n}{2} \int_{\mathcal{R}^n} \gamma_\delta(w_\delta)\, dy, \tag{2.47}$$

where

$$\gamma_\delta(w_\delta) = \int_0^{w_\delta} \frac{s^4}{(1+\delta s^2)^2}\, ds.$$

Finally, using

$$h(\delta) := \left( 2\delta \int_{\mathcal{R}^n} w_\delta \frac{dw_\delta}{d\delta}\, dy \right)^2 - \left( \left(1 - \frac{n}{4}\right) \int_{\mathcal{R}^n} w_\delta^2\, dy + 2\delta \int_{\mathcal{R}^n} w_\delta \frac{dw_\delta}{d\delta}\, dy \right)$$

$$\times \left( \int_{\mathcal{R}^n} \frac{w_\delta^3}{1+\delta w_\delta^2}\, dy + n\delta \int_{\mathcal{R}^n} \gamma_\delta(w_\delta)\, dy + 2\delta \int_{\mathcal{R}^n} w_\delta \frac{dw_\delta}{d\delta}\, dy \right)$$

$$= -2\delta \int_{\mathcal{R}^n} w_\delta \frac{dw_\delta}{d\delta}\, dy \left( \left(1 - \frac{n}{4}\right) \int_{\mathcal{R}^n} w_\delta^2\, dy + \int_{\mathcal{R}^n} \frac{w_\delta^3}{1+\delta w_\delta^2}\, dy + n\delta \int_{\mathcal{R}^n} \gamma_\delta(w_\delta)\, dy \right)$$

$$- \left(1 - \frac{n}{4}\right) \int_{\mathcal{R}^n} w_\delta^2\, dy \left( \int_{\mathcal{R}^n} \frac{w_\delta^3}{1+\delta w_\delta^2}\, dy + n\delta \int_{\mathcal{R}^n} \gamma_\delta(w_\delta)\, dy \right), \tag{2.48}$$

Hence (2.43) can be written as

$$h(\delta) = 0. \tag{2.49}$$

We remark that

$$1 - \frac{n}{4} > 0 \quad \text{since we consider the case } n \le 3.$$

Now, for $0 \le \delta \le \delta_{**}$, we have $h(\delta) < 0$ and so we must have $\delta > \delta_{**}$. Since we have assumed that $\delta \in [0, \delta_{**})$ we arrive at a contradiction.

Theorem 7 follows.

**Remark 2.3** *1) By the proof of Theorem 7, the number $\delta_{**}$ can be replaced by*

$$\delta_{***} = \sup\{\delta \in (0, \delta_0) \,:\, h(s) < 0, \quad s \in (0, \delta)\}. \tag{2.50}$$

*2) Another sufficient condition for stability can be stated as follows. Note that*

$$\int_{\mathcal{R}^n} \frac{w_\delta^3}{1+\delta w_\delta^2}\, dy = \int_{\mathcal{R}^n} w_\delta^2\, dy + \int_{\mathcal{R}^n} |\nabla w_\delta|^2\, dy > \int_{\mathcal{R}^n} w_\delta^2\, dy. \tag{2.51}$$

*Thus we have*

$$\frac{\left(1-\frac{n}{4}\right)\int_{\mathcal{R}^n} w_\delta^2\, dy \left(\int_{\mathcal{R}^n} \frac{w_\delta^3}{1+\delta w_\delta^2}\, dy + \int_{\mathcal{R}^n} n\delta\gamma_\delta(w_\delta)\, dy\right)}{\left(1-\frac{n}{4}\right)\int_{\mathcal{R}^n} w_\delta^2\, dy + \int_{\mathcal{R}^n} \frac{w_\delta^3}{1+\delta w_\delta^2}\, dy + \int_{\mathcal{R}^n} n\delta\gamma_\delta(w_\delta)\, dy}$$

$$> \frac{\left(1-\frac{n}{4}\right)\int_{\mathcal{R}^n} w_\delta^2\, dy}{\left(2-\frac{n}{4}\right)} = \frac{4-n}{8-n}\int_{\mathcal{R}^n} w_\delta^2\, dy.$$

*Now $h(\delta) < 0$ is guaranteed if*

$$\frac{4-n}{8-n}\int_{\mathcal{R}^n} w_\delta^2\, dy + 2\delta \int_{\mathcal{R}^n} w_\delta \frac{dw_\delta}{d\delta}\, dy > 0. \tag{2.52}$$

*Therefore, setting*

$$\delta_{****} = \sup\left\{\delta \in (0, \delta_*) \,:\, \frac{4-n}{8-n}\int_{\mathcal{R}^n} w_s^2\, dy + 2s\int_{\mathcal{R}^n} w_s \frac{dw_s}{ds}\, dy > 0, \quad \textit{for all } s \in (0, \delta)\right\}, \tag{2.53}$$

*Theorem 7 is valid for $\delta \in (0, \delta_{****})$.*

**Proof of Theorem 5 and Theorem 6:** Now we finish the proofs of our main theorems. Concerning the existence of solutions to (2.2), we use the scaling

$$A = \xi u, \quad \xi^{-1} = \frac{1}{|\Omega|}\int_\Omega u^2\, dx. \tag{2.54}$$

Then (2.2) is equivalent to

$$\begin{cases} \epsilon^2 \Delta u - u + \frac{u^2}{1+\delta u^2} = 0, u > 0, \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \end{cases} \tag{2.55}$$

coupled with the algebraic equation

$$\delta \left(2\epsilon^{-n} \int_\Omega u^2\, dx\right)^2 = k_\epsilon := 4k\epsilon^{-2n}|\Omega|^2. \tag{2.56}$$

By assumption (2.7), $\lim_{\epsilon\to 0} k_\epsilon = k_0 \in [0, +\infty)$. Lemma 2.4 implies that there exists $\delta_1 \in (0, \delta_*)$ such that

$$\delta_1 \left(\int_{\mathcal{R}^n} w_{\delta_1}^2\, dy\right)^2 = k_0. \tag{2.57}$$

Next we observe that $w_\delta$ is uniformly bounded in $H^1(\mathcal{R}^n)$ for $\delta \in (0, \delta_1)$, where the bound may depend on $\delta_1$.

By Lemma 2.2, for each fixed $\delta \in (0, \delta_1)$ we have that $w_\delta$ is nondegenerate. Then Theorem 1.1 of [?] and Theorem 1.1 of [?] (see also Theorem 4.5 of [?]) imply that for $\epsilon$ small enough problem (2.55) has a single boundary spike steady state $u_{\epsilon,\delta}$ which is unique, nondegenerate and possesses a unique local maximum point $Q_{\epsilon,\delta}$ which converges to $Q_0$ as $\epsilon \to 0$. Note that in the one-dimensional case, this follows from the implicit function theorem, whereas in higher dimensions we use Liapunov-Schmidt reduction.

Finally, we solve the algebraic equation

$$\beta_\epsilon(\delta) := \delta \left(2\epsilon^{-n} \int_\Omega u_{\epsilon,\delta}^2\, dx\right)^2 = k_\epsilon. \tag{2.58}$$

Using $\beta_\epsilon(0) = 0$ and

$$\lim_{\epsilon\to 0} \beta_\epsilon(\delta) \to \beta(\delta) = \delta \left(\int_{\mathcal{R}^n} w_\delta^2\, dy\right)^2,$$

where the convergence is uniform in $\delta \in (0, \delta_1)$, we derive that $\lim_{\epsilon\to 0} \beta_\epsilon(\delta_1) \to \delta_1(\int_{\mathcal{R}^n} w_{\delta_1}^2)^2 = k_0$. Since $u_{\epsilon,\delta}$ is unique and nondegenerate, $\beta_\epsilon$ is a continuous function of $\delta$. Using the mean-value theorem and considering $\epsilon$ small enough, for $k_\epsilon \in (0, k_0)$ there is $\delta_\epsilon \in (0, \delta_1)$ such that $\beta_\epsilon(\delta_\epsilon) = k_\epsilon$. Note that $\delta_\epsilon$ may not be unique. Since $k_0 \in [0, \infty)$ may be chosen arbitrarily, we get a solution for any $k_\epsilon \in [0, \infty)$.

Then $A_\epsilon = \xi_\epsilon u_{\epsilon,\delta_\epsilon}$, $\xi_\epsilon = \left(\frac{1}{|\Omega|}\int_\Omega u_{\epsilon,\delta_\epsilon}^2\, dx\right)^{-1}$ is a solution required in Theorems 5 and 6, respectively.

The existence part of the proof follows.

To investigate the stability of the solution $(A_\epsilon, \xi_\epsilon)$, we consider the eigenvalue problem

$$\begin{cases} \epsilon^2 \Delta\phi_\epsilon - \phi_\epsilon + \left(\dfrac{2A_\epsilon}{\xi_\epsilon(1 + kA_\epsilon^2)} - \dfrac{2kA_\epsilon^3}{\xi_\epsilon(1 + kA_\epsilon^2)^2}\right)\phi_\epsilon - \dfrac{A_\epsilon^2}{\xi_\epsilon^2(1 + kA_\epsilon^2)}\eta_\epsilon = \lambda_\epsilon\phi_\epsilon \text{ in } \Omega, \\ -\eta_\epsilon + \dfrac{2}{|\Omega|}\displaystyle\int_\Omega A_\epsilon\phi_\epsilon\, dx = \tau\lambda_\epsilon\eta_\epsilon. \end{cases} \tag{2.59}$$

Following the method in [?], we consider two cases separately. In Case 1, let $\lambda_\epsilon \to \lambda_0 \in \mathcal{C}$ with $\lambda_0 \neq 0$, the so-called large eigenvalues. Then, similarly to Chapter 4, we show that $\lambda_0$ satisfies

$$\Delta\phi_0 - \phi_0 + \left(\frac{2w_\delta}{1+\delta w_\delta^2} - \frac{2\delta w_\delta^3}{(1+\delta w_\delta^2)^2}\right)\phi_0 - \frac{2}{1+\tau\lambda_0}\frac{w_\delta^2}{1+\delta w_\delta^2}\frac{\int_{\mathcal{R}^n} w_\delta\phi_0\,dy}{\int_{\mathcal{R}^n} w_\delta^2\,dy} = \lambda_0\phi_0. \qquad (2.60)$$

By Theorem 7, for $n \leq 3$ and $\delta \in (0, \delta_{**})$, (2.60) is stable for $\tau$ small enough, i.e., for all eigenvalues of (2.60) with $\lambda_0 \neq 0$ we have $\mathrm{Re}(\lambda_0) \leq -c_0$ for some $c_0 > 0$. For $n = 1$, by Corollary 2.1, we may take $\delta_{**} = \delta_*$. This shows that the large eigenvalues are all stable.

Finally, we consider Case 2, for which $\lambda_\epsilon \to 0$, the small eigenvalues. In that in the one-dimensional case, $\lambda_\epsilon$ is bounded away from zero. Thus we only have to consider the higher-dimensional case. Then the proof follows closely Theorem 1.3 of [?].

The stability part of the proof is completed.