# AARMS SUMMER SCHOOL–LECTURE V: AN INTRODUCTION TO THE FINITE AND INFINITE DIMENSIONAL REDUCTION METHOD

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## 1. Introduction: What is finite dimensional Liapunov-Schmidt reduction method?

We briefly introduce the abstract set-up of the finite dimensional Lyapunov-Schmidt reduction (although it is always used in a framework that occurs often in bifurcation theory).

Let $X, Y$ be Banach spaces and $S(u)$ be a $C^1$ nonlinear map from $X$ to $Y$. To find a solution to the nonlinear equation

$$S(u) = 0, \tag{1.1}$$

a natural way is to find approximations first and then to look for genuine solutions as (small) perturbations of approximations. Assume that $U_\lambda$ are the approximations, where $\lambda \in \Lambda$ is the parameter (we think of $\Lambda$ as the configuration space). Writing $u = U_\lambda + \phi$, then solving $S(u) = 0$ amounts to solving

$$L[\phi] + E + N(\phi) = 0, \tag{1.2}$$

where

$$L[\phi] = S'(U_\lambda)[\phi], \; E = S(U_\lambda), \text{ and } N(\phi) = S(U_\lambda + \phi) - S(U_\lambda) - S'(U_\lambda)[\phi].$$

Here $S'(U_\lambda)$ is the Fréchet derivative of $S$ at $U_\lambda$, $E$ denotes the error of approximation, and $N(\phi)$ denotes the nonlinear term. In order to solve (1.2), we try to invert the linear operator $L$ so that we can rephrase the problem as a fixed point problem. That is, when $L$ has a uniformly bounded inverse in a suitable space, one can rewrite the equation (1.2) as

$$\phi = -L^{-1}[E + N(\phi)] = \mathcal{A}(\phi).$$

What is left is to use fixed point theorems such as contraction mapping theorem.

The finite dimensional Lyapunov-Schmidt reduction deals with the situation when the linear operator $L$ is Fredholm and its eigenfunction space associated to small eigenvalues has finite dimensional. Assuming that $\{\mathcal{Z}_1, \ldots, \mathcal{Z}_n\}$ is a basis of the eigenfunction space associated to small eigenvalues of $L$, we can divide the procedure of solving (1.2) into two steps:

[(i)] solving the projected problem for any $\lambda \in \Lambda$,

$$\begin{cases} L[\phi] + E + N(\phi) = \sum\limits_{j=1}^{n} c_j \mathcal{Z}_j, \\ \langle \phi, \mathcal{Z}_j \rangle = 0, \; \forall\, j = 1, \ldots, n, \end{cases}$$

where $c_j$ may be constant or function depending on the form of $\langle \phi, \mathcal{Z}_j \rangle$.

[(ii)] solving the reduced problem

$$c_j(\lambda) = 0, \ \forall j = 1, \ldots, n,$$

by adjusting $\lambda$ in the configuration space.

The original finite dimensional Liapunov-Schmidt reduction method was first introduced in a seminal paper by Floer and Weinstein [?] in their construction of single bump solutions to one dimensional nonlinear Schrodinger equations (Oh [?] generalized to high dimensional case)

$$\epsilon^2 \Delta u - V(x) u + u^p = 0, u > 0, \ u \in H^1(\mathbb{R}^N) \tag{1.3}$$

On the other hand, Bahri [?] and Bahri-Coron [?] developed the reduction method for critical exponent problems. In the last fifteen years, there are renewed efforts in refining the finite dimensional reduction method by many authors. When combined with variational methods, this reduction becomes "localized energy method". For subcritical exponent problems, we refer to Ambrosetti-Malchiodi [?], Gui-Wei [?], Malchiodi [?], Li-Nirenberg [?], Lin-Ni-Wei [?], Ao-Wei-Zeng [?], Wei-Yan [?] and the references therein. The localized energy method in degenerate setting is done by Byeon-Tanaka [?, ?]. For critical exponents, we refer to Bahri-Li-Rey [?], Del Pino-Felmer-Musso [?], Del Pino-Kowalczyk-Musso [?], Li-Wei-Xu [?], Rey-Wei [?, ?] and Wei-Yan [?] and the references therein. Many new features of the finite dimensional reduction are found in the references mentioned.

In the following we shall use the model problem (1.3) to give an introductory description of this method.

1.1. **Model Problem: Schrodinger equation in dimension N.** We start with the following model problem to illustrate the idea of finite dimensional reduction method:

$$\begin{cases} \varepsilon^2 \Delta u - V(x) u + u^p = 0 & \text{in } \mathbb{R}^N \\ 0 < u \ \text{ in } \mathbb{R}^N, & u(x) \to 0, \ \text{ as } |x| \to \infty \end{cases} \tag{1.4}$$

We consider $1 < p < \infty$ if $N \le 2$, and $1 < p < \frac{N+2}{N-2}$ if $N \ge 3$. Without loss of generality we assume that the function $V(x)$ is a positive function satisfying

$$0 < \alpha \le V(x) \le \beta < +\infty. \tag{1.5}$$

The basic building block that we consider is

$$\begin{cases} \Delta w - w + w^p = 0 & \text{in } \mathbb{R}^N \\ 0 < w \text{ in } \mathbb{R}^N, & w(x) \to 0, \ \text{ as } |x| \to \infty \end{cases} \tag{1.6}$$

We look for a solution $w = w(|x|)$, a radially symmetric solution. $w(r)$ satisfies the ordinary differential equation

$$\begin{cases} w'' + \frac{N-1}{r} w' - w + w^p = 0 & r \in (0, \infty) \\ w'(0) = 0, \, 0 < w \text{ in } (0, \infty) & w(|x|) \to 0, \ \text{ as } |x| \to \infty \end{cases} \tag{1.7}$$

We collect the following basic properties of $w$, whose proof can be found in the appendix of the book [?].

**Proposition 1.1.** *(a) There exist a solution $w(r)$ to (1.7);*

*(b) $w(r)$ satisfies the decay estimate $w(r) = A_0 r^{-\frac{N-1}{2}} e^r (1 + O(\frac{1}{r}))$;*

*(c) $w(r)$ is nondegenerate, i.e., the only bounded solution to*

$$L(\phi) = \Delta \phi + p w(x)^{p-1} \phi - \phi = 0, \quad \phi \in L^\infty(\mathbb{R}^N) \tag{1.8}$$

*is a linear combination of the functions* $\frac{\partial w}{\partial x_j}(x)$, $j = 1, \dots, N$.

We want to solve the problem

$$\begin{cases} \varepsilon^2 \Delta \tilde{u} - V(x)\tilde{u} + \tilde{u}^p = 0 & \text{in} \quad \mathbb{R}^N \\ \quad 0 < \tilde{u} \quad \text{in } \mathbb{R}^N & \tilde{u}(x) \to 0, \quad \text{as} \quad |x| \to \infty \end{cases} \tag{1.9}$$

We fix a point $\xi \in \mathbb{R}^N$. Observe that $U_{\varepsilon,\xi}(y) := V(\xi)^{\frac{1}{p-1}} w\left(\sqrt{V(\xi)}\frac{y-\xi}{\varepsilon}\right)$, is a solution of the rescaled equation

$$\varepsilon^2 \Delta u - V(\xi)u + u^p = 0.$$

We will look for a solution of (1.9) such $u_\varepsilon(x) \approx U_{\varepsilon,\xi}(y)$ for some $\xi \in \mathbb{R}^N$. We define $w_\lambda = \lambda^{\frac{1}{p-1}} w(\sqrt{\lambda}x)$.

Let us observe that if $\tilde{u}$ satisfies (1.9), then $u(x) = \tilde{u}(\varepsilon z)$ satisfies the problem

$$\begin{cases} \Delta u - V(\varepsilon z)u + u^p = 0 & \text{in} \quad \mathbb{R}^N \\ \quad 0 < u \quad \text{in } \mathbb{R}^N & u(x) \to 0, \text{ as } \ |x| \to \infty \end{cases} \tag{1.10}$$

Let $\xi' = \frac{\xi}{\varepsilon}$. We want a solution of (1.10) with the form $u(z) = w_\lambda(z - \xi') + \tilde{\phi}(z)$, with $\lambda = V(\xi)$ and $\tilde{\phi}$ being small compared with $w_\lambda(z - \xi')$.

1.2. **Equation in terms of $\phi$.** Let $\phi(x) = \tilde{\phi}(x - \xi')$. Then $\phi$ satisfies the equation

$$\Delta_x[w_\lambda(x) + \phi(x)] - V(\xi + \varepsilon x)[w_\lambda(x) + \phi(x)] + [w_\lambda(x) + \phi(x)]^p = 0.$$

We can write this equations as

$$\Delta\phi - V(\xi)\phi + pw_\lambda^{p-1}(x)\phi - E + B(\phi) + N(\phi) = 0 \tag{1.11}$$

where $E = (V(\xi + \varepsilon x) - V(\xi))w_\lambda(x)$, $B(\phi) = (V(\xi) - V(\xi + \varepsilon x))\phi$ and $N(\phi) = (w_\lambda + \phi)^p - w_\lambda^p - pw_\lambda^{p-1}\phi$.

We first consider the linear problem for $\lambda = V(\xi)$,

$$\begin{cases} L(\phi) = \Delta\phi - V(\xi + \varepsilon x)\phi + pw_\lambda(x)\phi = g - \sum_{i=1}^N c_i \frac{\partial w}{\partial x_i} \\ \qquad\qquad\qquad\qquad \int_{\mathbb{R}^N} \phi \frac{\partial w_\lambda}{\partial x_i} = 0, \quad i = 1, \dots, N \end{cases} \tag{1.12}$$

The $c_i's$ are defined such that

$$\int_{\mathbb{R}^N} (L(\phi) - g)\frac{\partial w_\lambda}{\partial x_i} dx = 0, i = 1, \dots, N \tag{1.13}$$

which is equivalent to

$$\int_{\mathbb{R}^N} (L(\frac{\partial w_\lambda}{\partial x_i})\phi - g\frac{\partial w_\lambda}{\partial x_i})dx = 0, i = 1, \dots, N \tag{1.14}$$

Denoting

$$L_0(\phi) = \Delta\phi - V(\xi)\phi + pw_\lambda(x)\phi$$

and using the fact that

$$L_0(\frac{\partial w_\lambda}{\partial x_i}) = 0$$

we see that (1.14) can be further simplified as follows

$$\int_{\mathbb{R}^N} ((V(\xi) - V(\xi + \epsilon x))\frac{\partial w_\lambda}{\partial x_i}\phi - g\frac{\partial w_\lambda}{\partial x_i})dx = 0, i = 1, \dots, N \tag{1.15}$$

Since

$$\int \frac{\partial w_\lambda}{\partial x_i}\frac{\partial w_\lambda}{\partial x_j} = \int_{\mathbb{R}^N} (\frac{\partial w}{\partial x_1})^2 \delta_{ij}$$

we find that

$$c_i = \frac{\int_{\mathbb{R}^N}((V(\xi) - V(\xi + \epsilon x))\frac{\partial w_\lambda}{\partial x_i}\phi - g\frac{\partial w_\lambda}{\partial x_i})dx}{\int_{\mathbb{R}^N}(\frac{\partial w_\lambda}{\partial x_1})^2}, i = 1, \ldots, N \tag{1.16}$$

In the following we shall solve the following:

Problem: Given $g \in L^\infty(\mathbb{R}^N)$ we want to find $\phi \in L^\infty(\mathbb{R}^N)$ solution to the problem (1.12)-(1.16).

1.3. **A linear problem.** Let us assume that $V \in C^1(\mathbb{R}^N)$, $\|V\|_{C^1} < \infty$. We assume in addition that $|\xi| \leq M_0$ and $0 < \alpha \leq V$. Then we have

**Proposition 1.2.** *There exists $\varepsilon_0$, $C_0 > 0$ such that $\forall 0 < \varepsilon \leq \varepsilon_0$, $\forall |\xi| \leq M_0$, $\forall g \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$, there exist a unique solution $\phi \in L^\infty(\mathbb{R}^N)$ to (1.12), $\phi = T[g]$ satisfies*

$$\|\phi\|_{C^1} \leq C_0 \|g\|_\infty$$

*Proof.* We divide the proof into two steps.

**Step 1-a priori estimates:** We first obtain *a priori estimates* of the problem (1.12) on bounded domains $B_R(0)$: There exist $R_0$, $\varepsilon_0$, $C_0$ such that $\forall \varepsilon < \varepsilon_0$, $R > R_0$, $|\xi| \leq M_0$ such that $\forall \phi, g \in L^\infty$ solving $L(\phi) = g - \sum_i c_i \frac{\partial w_\lambda}{\partial x_i}$ in $B_R$, $\int_{B_R} \phi \frac{\partial w_\lambda}{\partial x_i} = 0$ and $\phi = 0$ on $\partial B_R$, we have

$$\|\phi\|_{C^1(B_R)} \leq C_0 \|g\|_\infty$$

We prove first $\|\phi\|_\infty \leq C_0 \|g\|_\infty$. Assuming the opposite, then there exist sequences $\phi_n$, $g_n$, $\varepsilon \to 0$, $R_n \to \infty$, $|\xi_n| \leq M_0$ such that

$$L(\phi_n) = g_n - \sum_i c_i^n \frac{\partial w_\lambda}{\partial x_i}.$$

The first fact is that $c_i^n \to 0$ as $n \to \infty$. This fact follows just after multiplying the equation against $\frac{\partial w_\lambda}{\partial x_i}$ and integrating by parts, as we did in (1.16).

We observe that if $\Delta\phi = g$ in $B_2$ then there exist $C$ such that

$$\|\nabla\phi\|_{L^\infty(B_1)} \leq C[\|g\|_{L^\infty(B_2)} + \|\phi\|_{L^\infty(B_2)}]$$

where $B_1$ and $B_2$ are concentric balls. This implies that $\|\nabla\phi_n\|_{L^\infty(B)} \leq C$ a given bounded set $B$, $\forall n \geq n_0$. Hence passing to a subsequence we obtain $\phi_n \to \phi$ uniformly on compact sets, and $\phi \in L^\infty(\mathbb{R}^N)$. Observe that $\|\phi_n\|_\infty = 1$, and this implies that $\|\phi\|_\infty \leq 1$. We can also assume that up to a subsequence $\xi_n \to \xi_0$.

Since $\phi$ satisfies the equation $\Delta\phi - V(\xi_0)\phi + pw_{\lambda_0}^{p-1}(x)\phi = 0$, where $\lambda_0 = V(\xi_0)$, we have that $\phi \in \text{Span}\left\{\frac{\partial w_{\lambda_0}}{\partial x_1}, \ldots, \frac{\partial w_{\lambda_0}}{\partial x_N}\right\}$. Taking limits in the orthogonality condition (1.12) we obtain that $\int_{\mathbb{R}^N} \phi(w_{\lambda_0})_{\partial x_i} = 0$, $i = 1, \ldots, N$. This implies that $\phi = 0$ and hence $\|\phi_n\|_{L^\infty(B_M(0))} \to 0, \forall M < \infty$. Maximum principle yields that $\|\phi_n\|_{L^\infty(B_{R_n} \setminus B_{M_0}} \to 0$, since $|\phi_n| = o(1)$ on $\partial B_{R_n} \setminus B_{M_0}$ and $\|g_n\|_\infty \to 0$. Therefore we arrive at $\|\phi_n\|_\infty \to 0$, which is a contradiction. This implies that $\|\phi\|_{L^\infty(B_R)} \leq C_0 \|g\|_{L^\infty(B_R)}$ uniformly on large $R$. The $C^1$ estimate follows from elliptic local boundary estimates for elliptic operators.

**Step 2-Existence:** Recall that $g \in L^\infty$. We want to solve (1.12). We claim that solving (1.12) is equivalent to finding

$$\phi \in X = \{\psi \in H_0^1(B_R) : \int \psi \frac{\partial w_\lambda}{\partial x_i} = 0, \, i = 1, \ldots, N\}$$

such that

$$\int \nabla\phi\nabla\psi + \int V(\xi + \varepsilon x)\phi\psi - pw^{p-1}\phi\psi + \int g\psi = 0, \quad \forall \psi \in X.$$

Take general $\Psi \in H_0^1$. We can decompose into $\Psi = \psi - \sum_i \alpha_i \frac{\partial w_\lambda}{\partial x_i}$, with $\alpha_i = \frac{\int \Psi \frac{\partial w_\lambda}{\partial x_i}}{\int (\frac{\partial w_\lambda}{\partial x_i})^2}$. We have

$$-\int \Delta(\sum_i \alpha_i \frac{\partial w_\lambda}{\partial x_i})\nabla\phi + \int V(\xi)(\sum_i \alpha_i (\frac{\partial w_\lambda}{\partial x_i})\phi - pw^{p-1}(\sum_i \alpha_i \frac{\partial w_\lambda}{\partial x_i})\phi = 0$$

which implies that

$$\int \nabla\phi\nabla\Psi + \int V(\xi)\phi\Psi - pw^{p-1}\phi\Psi$$

$$-\int (V(\xi) - V(\xi + \varepsilon x))(\Psi - \sum_i \alpha_i \frac{\partial w_\lambda}{\partial x_i}) + \int g(\Psi - \sum_i \alpha_i \frac{\partial w_\lambda}{\partial x_i})$$

$$= \int [(V(\xi + \varepsilon x) - V(\xi))\phi + g](\Psi - \sum_i \alpha_i \frac{\partial w_\lambda}{\partial x_i})$$

Let $\Pi_X(\Psi) = \sum_i \alpha_i \frac{\partial w_\lambda}{\partial x_i}$. Then the above integral equals

$$\int \Pi_X([(V(\xi + \varepsilon x) - V(\xi))\phi + g]\phi)\Psi$$

This implies that

$$-\Delta\phi + V(\xi)\phi - pw^{p-1}\phi + \Pi_X([(V(\xi + \varepsilon x) - V(\xi))\phi + g]\phi) = 0.$$

The problem is formulated weakly as

$$\int \nabla\phi\nabla\psi + \int (V(\xi + \varepsilon x) - pw^{p-1})\phi\psi + \int g\psi = 0, \phi \in X, \forall \psi \in X$$

which can be written as $\phi = A[\phi] + \tilde{g}$, where $A$ is a compact operator. The a priori estimate implies that the only solution when $g = 0$ of this equation is $\phi = 0$. We conclude existence by Fredholm alternative. Finally we let $R \to +\infty$ and obtain the existence in the whole space, thanks to the a priori estimate in Step 1.

□

Next we consider the assembly of multiple spikes. We look for a solution of (1.10) which near $x_j = \xi_j' = \xi_j/\varepsilon$, $j = 1, \ldots, k$ looks like $v(x) \approx w_{\lambda_j}(x - \xi_j')$, $\lambda_j = V(\xi_j)$, where $w_\lambda = \lambda^{1/(p-1)} w(\sqrt{\lambda} y)$.

Let $\xi_1, \xi_2, \ldots \xi_k \in \mathbb{R}^N$ be such that $|\xi_j' - \xi_l'| \gg 1$, if $j \neq l$. We look for a solution $v(x) \approx \sum_{j=1}^k w_{\lambda_j}(x - \xi_j')$, $\lambda_j = V(\xi_j)$. We assume $V \in C^2(R^N)$ and $\|V\|_{C^2} < \infty$, $0 < \alpha \leq V$. We use the notation $W_j = w_{\lambda_j}(x - \xi_j')$, $\lambda_j = V(\xi_j)$ and $W = \sum_{j=1}^k W_j$.

Setting $v = W + \phi$, then $\phi$ solves the problem

$$\Delta\phi - V(\varepsilon x)\phi + pW^{p-1}\phi + E + N(\phi) = 0 \tag{1.17}$$

where

$$E = \Delta W - VW + W^p, \quad N(\phi) = (W+\phi)^p - W^p - pW^{p-1}\phi.$$

Observe that $\Delta W = \sum_j \Delta W_j = \sum_j \lambda_j W_j - W_j^p$. So we can write

$$E = \sum_j (\lambda_j - V(\varepsilon x))W_j + (\sum_j W_j)^p - \sum_j W_j^p.$$

Our next objective is to solve the approximate linearized projected problem.

1.4. **Linearized (projected) problem.** We use the following notation $Z_j^i = \frac{\partial W_j}{\partial x_i}$. The linearized projected problem is the following

$$\Delta\phi - V(\varepsilon x)\phi + pW^{p-1}\phi + g = \sum_{i,j} c_j^i Z_j^i, \tag{1.18}$$

with the orthogonality condition $\int \phi Z_j^i = 0$, $\forall i, j$. The $Z_j^i$'s are "nearly orthogonal" if the centers $\xi_j'$ are far away one to each other. The $c_j^i$'s are, by definition, the solution of the linear system

$$\int_{\mathbb{R}^N} (\Delta\phi - V(\varepsilon x)\phi + pW^{p-1}\phi + g)Z_{j_0}^{i_0} = \sum_{i,j} c_j^i \int_{\mathbb{R}^N} Z_j^i Z_{j_0}^{i_0},$$

for $i_0 = 1, \ldots, N$, $j_0 = 1, \ldots, k$. The $c_j^i$'s are indeed uniquely determined provided that $|\xi_l' - \xi_j'| > R_0 \gg 1$, because the matrix with coefficients $\alpha_{i,j,i_0,j_0} = \int Z_j^i Z_{j_0}^{i_0}$ is "nearly diagonal", which means

$$\alpha_{i,j,i_0,j_0} = \begin{cases} \frac{1}{N}\int |\nabla W_j|^2 & \text{if } (i,j) = (i_0, j_0), \\ o(1) & \text{if not} \end{cases}$$

Moreover by a similar argument leading to (1.15) we have

$$|c_{j_0}^{i_0}| \le C \sum_{i,j} \int |\phi| [|\lambda_j - V| + p|W^{p-1} - W_j^{p-1}|] |Z_j^i| + \int |g||Z_j^i| \le C(\|\phi\|_\infty + \|g\|_\infty)$$

with $C$ is uniform for large $R_0$. Furthermore if we rescale $x = \xi' + y$, we get

$$|(\lambda_j - V(\varepsilon x))Z_j^i| \le |(V(\xi_j) - V(\xi_j + \varepsilon y))| \left|\frac{\partial w_{\lambda_j}}{\partial y_i}\right| \le C\varepsilon e^{-\frac{\sqrt{\alpha}}{2}|y|},$$

because $|\frac{\partial w_{\lambda_j}}{\partial y_i}| \le Ce^{-|y|\sqrt{\lambda_j}}|y|^{-(N-1)/2}$. Observe also that

$$|(W^{p-1} - W_j^{p-1})Z_j^i| = |((1 - \sum_{l\neq j} \frac{W_l}{W_j})^{p-1} - 1)|W_j^{p-1} Z_j^i.$$

We estimate the interactions at each spike in two regions.

Observe that if $|x - \xi_j'| < \delta_0 \min_{j_1 \neq j_2} |\xi_{j_1}' - \xi_{j_2}'|$, then

$$\frac{W_l(x)}{W_j(x)} \approx \frac{e^{-\sqrt{\lambda_l}|x-\xi_l'|}}{e^{-\sqrt{\lambda_j}|x-\xi_j'|}} < \frac{e^{-\sqrt{\lambda_l}|x-\xi_l'|}}{e^{-\sqrt{\lambda_j}\delta_0 \min_{j_1\neq j_2}|\xi_{j_1}'-\xi_{j_2}'|}}$$

If $\delta_0 \ll 1$ but fixed, we conclude that $e^{-\sqrt{\lambda_l}|\xi_j'-\xi_l'| + \delta_0(\sqrt{\lambda_l} - \sqrt{\lambda_j})\min_{j_1\neq j_2}|\xi_{j_1}'-\xi_{j_2}'|} < e^{-\rho \min_{j_1\neq j_2}|\xi_{j_1}'-\xi_{j_2}'|} \ll 1$. Thus we conclude that if $|x - \xi_j'| < \delta_0 \min_{j_1\neq j_2}|\xi_{j_1}' - xi_{j_2}'|$ then

$$|(W^{p-1} - W_j^{p-1})Z_j^i| \le e^{-\rho \min_{j_1\neq j_2}|\xi_{j_1}'-\xi_{j_2}'|} e^{-\frac{\alpha}{2}|x-\xi_j'|}.$$

On the other hand if $|x-\xi_j'| > \delta_0 \min_{j_1\neq j_2}|\xi_{j_1}'-\xi_{j_2}'|$, then

$$|(W^{p-1}-W_j^{p-1})Z_j^i| \leq C|Z_j^i| \leq Ce^{-\rho\min_{j_1\neq j_2}|\xi_{j_1}'-\xi_{j_2}'|}e^{-\frac{\alpha}{2}|x-\xi_j'|}$$

As a conclusion we obtain the following estimate

$$|c_{j_0}^{i_0}| \leq C(\varepsilon + e^{-\rho\min_{j_1\neq j_2}|\xi_{j_1}'-\xi_{j_2}'|})\|\phi\|_\infty + \|g\|_\infty \tag{1.19}$$

**Lemma 1.1.** *Given $k \geq 1$, there exist $R_0, C_0, \varepsilon_0$ such that for all points $\xi_j'$ with $|\xi_{j_1}'-\xi_{j_2}'| > R_0$, $j=1,\dots,k$ and all $\varepsilon < \varepsilon_0$ then exist a unique solution $\phi$ to the linearized projected problem with*

$$\|\phi\|_\infty \leq C_0\|g\|_\infty.$$

*Proof.* As before we first prove the a priori estimate $\|\phi\|_\infty \leq C_0\|g\|_\infty$. If not there exist $\varepsilon_n \to 0$, $\|\phi_n\|_\infty = 1$, $\|g_n\| \to 0$, $\xi_j'^n$ with $\min_{j_1\neq j_2}|\xi_{j_1}'^n - \xi_{j_2}'^n| \to \infty$. We denote $W_n = \sum_j W_{j_n}$, and we have

$$\Delta\phi_n - V(\varepsilon_n x)\phi_n + pW_n^{p-1}\phi_n + g_n = \sum_{i,j}(c_j^i)_n(z_j^i)_n$$

Our first observation is that $(c_j^i)_n \to 0$ (which follows from the same estimate for $c_{j_0}^{i_0}$). Next we claim that $\forall R > 0$ $\|\phi_n\|_{L^\infty(B(\xi_j'^n,R))} \to 0$, $j=1,\dots,k$. If not, there exist $j_0$ $\|\phi_n\|_{L^\infty(B(\xi_{j_0}'^n,R))} \geq \gamma > 0$. We denote $\tilde{\phi_n}(y) := \phi_n(\xi_{j_0}'^n + y)$. We have $\|\tilde{\phi_n}\|_{L^\infty(B(0,R))} \geq \gamma > 0$. Since $|\Delta\tilde{\phi_n}| \leq C$, $\|\tilde{\phi}_n\|_\infty \leq 1$. This implies that $\|\nabla\tilde{\phi}_n\| \leq C$. Passing to a subsequence we may assume $\tilde{\phi}_n \to \tilde{\phi}$ uniformly on compacts sets. Observe that also $V(\varepsilon_n(\xi_{j_0}'^n + y)) = V(\varepsilon_n\xi_{j_0}'^n) + O(\varepsilon_n|y|) \to \lambda_{j_0}$ over compact sets and $W_n(\xi_{j_0}'^n + y) \to W_{\lambda_{j_0}}(y)$ uniformly on compact sets. This implies that $\tilde{\phi}$ is a solution of the problem

$$\Delta\tilde{\phi} - \lambda_{j_0}\tilde{\phi} + pw_{\lambda_0}^{p-1}p\,\tilde{-}\,1 = 0, \quad \int\tilde{\phi}\frac{\partial W_{\lambda_{j_0}}}{\partial y_i}dy = 0, i=1,\dots,N$$

Nondegeneracy of $w_{\lambda_{j_0}}$ implies that $\tilde{\phi} = \sum_i \alpha_i \frac{\partial w_{\lambda_{j_0}}}{\partial y_i}$. The orthogonality condition implies that $\alpha_i = 0$, $\forall i = 1,\dots,N$. This implies that $\tilde{\phi} = 0$ but $\|\tilde{\phi}\|_{L^\infty(B(0,R))} \geq \gamma > 0$, a contradiction.

Now we prove: $\|\phi_n\|_{L^\infty}(\mathbb{R}^N \setminus \cup_n B(\xi_j'^n, R)) \to 0$, provided that $R \gg 1$ and fixed so that $\phi_n \to 0$ in the sense of $\|\phi_n\|_\infty$ (again a contradiction). We will denote $\Omega_n = \mathbb{R}^N \setminus \cup_n B(\xi_j'^n, R)$. For $R \gg 1$ the equation for $\phi_n$ has the form

$$\Delta\phi_n - Q_n\phi_n + g_n = 0$$

where $Q_n = V(\varepsilon x) - pW_n^{p-1} \geq \frac{\alpha}{2} > 0$ for some $R$ sufficiently large (but fixed).

Let us take for $\sigma^2 < \alpha/2$

$$\bar{\phi} = \delta\sum_j e^{\sigma|x-\xi_j'^n|} + \mu_n.$$

We denote $\varphi(y) = e^{\sigma|y|}$, $r = |y|$. Observe that $\Delta\varphi - \alpha/2\varphi = e^{\sigma|y|}(\sigma^2 + \frac{N-1}{|y|} - \alpha/2) < 0$ if $|y| > R \gg 1$. Then

$$-\Delta\bar{\phi} + Q_n\bar{\phi} - g_n > -\Delta\bar{\phi} + \frac{\alpha}{2}\bar{\phi} - \|g_n\|_\infty > \frac{\alpha}{2}\mu_n - \|g_n\|_\infty > 0 \tag{1.20}$$

if we choose $\mu_n \geq \|g_n\|_\infty \frac{2}{\alpha}$. In addition we take $\mu_n = \sum_j \|\phi_n\|_{L^\infty(B(\xi_j^n,R))} + \|g_n\|_\infty \frac{2}{\alpha}$. Maximum principle implies that $\phi_n(x) \leq \bar{\phi}$ for all $x \in \Omega_n$. Taking $\delta \to 0$ this implies that $\phi_n(x) \leq \mu_n$, for all $x \in \Omega_n$. It is also true that $|\phi_n(x)| \leq \mu_n$ for all $x \in \Omega_n^c$, and this implies that $\|\phi_n\|_{L^\infty(\mathbb{R}^N)} \to 0$. □

**Remark:** If in addition we have the following decay for the error

$$\theta_n = \|g_n \left( \sum_j e^{-\rho|x-\xi_j'^n|} \right)^{-1} \|_\infty \to 0$$

with $\rho < \alpha/2$, then we can use as a barrier function

$$\bar{\phi} = \delta \sum_j e^{\sigma|x-\xi_j'^n|} + \mu_n \sum_j e^{-\rho|x-\xi_j'^n|}$$

with $\mu_n = e^{\rho R} \sum_j \|\phi_n\|_{L^\infty(B(\xi_j'^n,R))} + \theta_n$. It is easy to see that $\bar{\phi}$ is a super solution of the equation in $(\cup_j B(\xi_j, R))^c$ and we have $|\phi_n| \leq \bar{\phi}$. Letting $\delta \to 0$ we get $|\phi_n(x)| \leq \mu_n \sum_j e^{-\rho|x-\xi_j'^n|}$. As a conclusion we also get the a priori estimate

$$\|\phi \left( \sum_{j=1}^k e^{-\rho|x-\xi_j'|} \right)^{-1} \|_\infty \leq C \|g \left( \sum_{j=1}^k e^{-\rho|x-\xi_j'|} \right)^{-1} \|_\infty$$

provided that $0 \leq \rho < \alpha/2$, $|\xi_{j_1}' - \xi_{j_2}'| > R_0 \gg 1$, $\varepsilon < \varepsilon_0$.

We now give the proof of existence.

*Proof.* Let $g$ be compactly supported smooth functions. The weak formulation for

$$\Delta\phi - V(\varepsilon x)\phi + pW^{p-1}\phi + g = \sum_{i,j} c_j^i Z_j^i, \quad \int \phi Z_j^i = 0, \forall i, j \tag{1.21}$$

is to find $\phi \in X = \{\phi \in H^1(\mathbb{R}^N) : \int \phi Z_j^i = 0, \forall i, j\}$ such that

$$\int_{\mathbb{R}^N} \nabla\phi\nabla\psi + V\phi\psi - pw^{p-1}\phi\psi - g\psi = 0, \quad \forall \psi \in X. \tag{1.22}$$

Assume $\phi$ solves (1.21). For $g \in L^2$, we decompose $g = \tilde{g} + \Pi[g]$ where $\int \tilde{g} Z_j^i = 0$ for all $i, j$, and $\Pi$ is the orthogonal projection of $g$ onto the space spanned by the $Z_j^i$'s.

Let $\psi \in H^1(\mathbb{R}^N)$. We now use $\psi - \Pi[\psi]$ as a test function in (1.22). Then if $\varphi \in C_c^\infty(\mathbb{R}^N)$, then we have

$$\int_{\mathbb{R}^N} \nabla\varphi\nabla(\Pi[\psi]) = -\int_{\mathbb{R}^N} \Delta\varphi\Pi[\psi] = -\int_{\mathbb{R}^N} \Pi[\Delta\varphi]\psi. \tag{1.23}$$

On the other hand, we have $\Pi[\Delta\varphi] = \sum_{i,j} \alpha_{i,j} Z_j^i$, where

$$\sum \alpha_{i,j} \int Z_{i,j} Z_{i_0,j_0} = \int \Delta\varphi Z_{i_0}^{j_0} = \int \varphi \Delta Z_{i_0}^{j_0} \tag{1.24}$$

Then $\|\Pi[\Delta\varphi]\|_{L^2} \leq C\|\varphi\|_{H^1}$. By density argument it is also true for $\varphi \in H^1$ where $\Delta\varphi \in H^{-1}$. Therefore

$$\int \nabla\phi\nabla\psi + \int (V\phi - pW^{p-1}\phi - g)\psi = \int \Pi(V\phi - pW^{p-1}\phi + g)\psi \tag{1.25}$$

It follows that $\phi$ solves in weak sense

$$-\Delta\phi + V\phi - pW^{p-1}\phi - g = \Pi[-\Delta\phi + V\phi - pW^{p-1}\phi - g] \tag{1.26}$$

and $\Pi[-\Delta\phi + V\phi - pW^{p-1}\phi - g] = \sum_{i,j} c_i^j Z_i j$. Therefore by definition $\phi$ solves (1.22) implies that $\phi$ solves (1.26). Classical regularity gives that this weak solution is solution of (1.26) in strong sense, in particular $\phi \in L^\infty$ so that

$$\|\phi\|_\infty \le C\|g\|_\infty. \tag{1.27}$$

Now we give the proof of existence for (1.21). We take $g$ compactly supported. The equation (1.26) can be written in the following way (using Riesz theorem):

$$\langle \phi, \psi\rangle_{H^1} + \langle B[\phi], \psi\rangle_{H^1} = \langle \tilde{g}, \psi\rangle_{H^1} \tag{1.28}$$

or $\phi + B[\phi] = \tilde{g}$, $\phi \in X$. We claim that $B$ is a compact operator. Indeed if $\phi_n \rightharpoonup 0$ in $X$, then $\phi_n \to 0$ in $L^2$ over compacts and

$$|\langle B[\phi_n], \psi\rangle| \le |\int pW^{p-1}\phi_n\psi| \le (\int pw^{p-1}\phi_n^2)^{1/2}(\int pW^{p-1}\psi^2)^{1/2} \tag{1.29}$$

which yields

$$|\langle B[\phi_n], \psi\rangle| \le c(\int pW^{p-1}\phi_n^2)^{1/2}\|\psi\|_{H^1} \tag{1.30}$$

Take $\psi = B[\phi_n]$, which implies

$$\|B[\phi_n]\|_{H^1} \le c(\int pW^{p-1}\phi_n^2)^{1/2} \to 0. \tag{1.31}$$

This gives that $B$ is a compact operator.

Now we prove existence with the aid of Fredholm alternative. Problem (1.21) is solvable if for $\tilde{g} = 0$ the only solution to (1.22) is $\phi = 0$. But $\phi + B[\phi] = 0$ implies solve (1.21)(strongly) with $g = 0$. This implies $\phi \in L^\infty$, and the a priori estimate implies $\phi = 0$. Considering $g\Xi_{B_R(0)}$ we conclude that

$$\|\phi_R\|_\infty \le \|g\|_\infty \tag{1.32}$$

Taking $R \to \infty$ then along a subsequence $\phi_R \to \phi$ uniform over compacts we obtain a solution to (1.21).

□

Next we want to study the dependence and regularity of the solution with respect to the parameters. Let $g \in L^\infty$. We denote $\phi = T_{\xi'}[g]$, where $\xi' = (\xi'_1, \dots, \xi'_k)$. We want to analyze derivatives $\partial_{\xi'_{ji}} T_{\xi'}[g]$. We know that $\|T_{\xi'}[g]\| \le C_0\|g\|_\infty$. First we make a formal differentiation. We denote $\Phi = \frac{\partial\phi}{\partial\xi'_{i_0j_0}}$.

We have $\Delta\phi - V\phi + pW^{p-1}\phi + g = \sum_{i,j} c_j^i Z_j^i$ and $\int \phi Z_j^i = 0$, for all $i, j$. Formal differentiation yields

$$\Delta\Phi - V\Phi + pW^{p-1}\Phi + +\partial_{\xi_{i_0j_0}}(W^{p-1})\phi - \sum_{i,j} c_j^i \partial_{\xi_{i_0j_0}} Z_i^j = \sum_{i,j} \tilde{c}_j^i Z_j^i \tag{1.33}$$

where formally $\tilde{c}_i^j = \partial_{\xi_{i_0j_0}} c_i^j$. The orthogonality conditions is reduced to

$$\int_{\mathbb{R}^N} \Phi Z_j^i = \begin{cases} 0 & \text{if } j \ne j_0 \\ -\int \phi\partial_{\xi_{i_0j_0}Z_{j_0}^i} & \text{if } j = j_0 \end{cases} \tag{1.34}$$

Let us define $\tilde{\Phi} = \Phi - \sum_{i,j} \alpha_{i,j} Z_j^i$. We want $\int \tilde{\Phi} Z_j^i = 0$, for all $i, j$. We need

$$\sum_{i,j} \alpha_{i,j} \int Z_j^i Z_{\bar{j}}^{\bar{i}} = \begin{cases} 0 & \text{if } \bar{j} \neq j_0 \\ -\int \phi \partial_{\xi_{i_0 j_0}} Z_{j_0}^i & \text{if } \bar{j} = j_0 \end{cases} \tag{1.35}$$

The system has a unique solution and $|\alpha_{i,j}| \leq C\|\phi\|_\infty$ (since the system is almost diagonal). So we have the condition $\int \tilde{\Phi} Z_j^i = 0$, for all $i, j$. We add to the equation the term $\sum_{i,j} \alpha_{i,j}(\Delta - V + pW^{p-1})Z_j^i$, so $\tilde{\Phi}$ satisfies the equation$\Delta\phi - V\phi + pW^{p-1}\phi + g = \sum_{i,j} c_j^i Z_j^i$

$$\Delta\tilde{\Phi} - V\tilde{\Phi} + pW^{p-1}\tilde{\Phi} + \partial_{\xi_{i_0 j_0}}(W^{p-1})\phi - \sum_{i,j} c_j^i \partial_{\xi_{i_0 j_0}} Z_i^j = \sum_{i,j} \tilde{c}_j^i Z_j^i - \sum_{i,j} \alpha_{i,j}(\Delta - V + pW^{p-1})Z_j^i \tag{1.36}$$

This implies $\|\tilde{\Phi}\| \leq C(\|h\| + \|g\|) \leq C\|g\|_\infty$ and hence $\|\Phi\| \leq C\|g\|_\infty$.

The above formal procedure can be made rigorous by performing the analysis discretely, namely we consider solutions corresponding to $\xi$ and $\xi + h$ respectively. Then we consider the quotient and pass the limit in $h$. We omit the details. In conclusion the map $\xi \to \partial_\xi \phi$ is well defined and continuous (into $L^\infty$). Besides we also have $\|\partial_\xi \phi\|_\infty \leq C\|g\|_\infty$, and this implies

$$\|\partial_\xi T_\xi[\phi]\| \leq C\|g\| \tag{1.37}$$

1.5. **Nonlinear projected problem.** Consider now the nonlinear projected problem

$$\Delta\phi - V\phi + pw^{p-1}\phi + E + N(\phi) = \sum_{i,j} c_i^j Z_j^i, \quad \int \phi Z_i^j = 0, \forall i, j \tag{1.38}$$

We solve this by fixed point. We have $\phi = T(E + N(\phi)) =: M(\phi)$. We define $\Lambda = \{\phi \in C(\mathbb{R}^N) \cap L^\infty(R^N) : \|\phi\|_\infty \leq M\|E\|_\infty\}$. Remember that $E = \sum_i (\lambda_j - V(\varepsilon x))W_j + (\sum_j W_j)^p - \sum_j W_j^p$. Observe that

$$|E| \leq \varepsilon \sum_i e^{-\sigma|x - \xi'_j|} + ce^{-\delta_0 \min_{j_1 \neq j_2} |\xi'_{j_1} - \xi'_{j_2}|} \sum_j e^{-\sigma|x - \xi'_j|} \tag{1.39}$$

so, for existence we have $\|E\| \leq C[\varepsilon + e^{-\delta_0 \min_{j_1 \neq j_2} |\xi'_{j_1} - \xi'_{j_2}|}] =: \rho$ (see that $\rho$ is small). Contraction mapping implies there exists a unique solution $\phi = \Phi(\xi)$ and $\|\Phi(\xi)\| \leq M\rho$. The proof is standard and hence omitted.

1.6. **Differentiability in $\xi'$ of $\Phi(\xi')$.** As before the solutions obtained for the nonlinear projected problem has more regularity. In fact we can write the equation for $\Phi$ as

$$\Phi - T'_\xi(E'_\xi + N'_\xi(\phi)) = A(\Phi, \xi') = 0 \tag{1.40}$$

If $(D_\Phi A)(\Phi(\xi'), \xi')$ is invertible in $L^\infty$, then $\Phi(\xi')$ turns out to be of class $C^1$. This is a consequence of the fixed point characterization, i.e., $D_\Phi A(\Phi(\xi'), \xi') = I + o(1)$ (the order $o(1)$ is a direct consequence of fixed point characterization). Then it is invertible. Contraction mapping theorem yields the existence of $C^1$ derivative of $A(\Phi, \xi')$ in $(\phi, \xi')$. This implies $\Phi(\xi')$ is $C^1$. With a little bit of more work we can show that $\|D'_\xi \Phi(\xi')\| \leq C\rho$ (just using the derivative given by the implicit function theorem).

1.7. **Solving the reduced problem: direct method.** By (1.38), to solve (1.17), we need to find $\xi'$ such that the reduced problem

$$c_j^i = 0, \forall i, j \tag{1.41}$$

to get a solution to the original problem (1.10). There are two ways to solve the reduced problem (1.41): the first one is the direct method, and the second one is the variational reduction method. We describe the first method first by proving the following

**Theorem 1.** *(Oh [**?**]) Assume that $\xi_j^0, j = 1, ..., k$ are $k$ distinct non-degenerate critical points of $V$. Then there exist a solution $u_\varepsilon$ to the original problem with*

$$u_\varepsilon(x) \approx \sum_{j=1}^{k} w_{V(\xi_j^\varepsilon)}(x - \xi_j^\varepsilon/\varepsilon), \quad \xi_j^\varepsilon \to \xi_j^0$$

*Proof.* To solve the problem (1.41) we first obtain the asymptotic formula for $c_j^i$. To this end we multiply the equation (1.38) by $Z_{j_0}^{i_0}$ and integrate by parts. We obtain

$$\int_{\mathbb{R}^N} Z_j^i Z_{j_0}^{i_0} c_j^i = \int_{\mathbb{R}^N} (V(\xi_j + \epsilon x) - V(\xi_j)) w_{\xi_j} Z_{j_0}^{i_0} + O(\epsilon^2)$$

and thus

$$c_{j_0}^{i_0} \sim \partial_{i_0} V(\xi_j^0) + O(\epsilon)$$

The nondegeneracy of the critical point $\nabla V(\xi_j^0)$ and implicit function theorem yields the existence of $\xi_j = \xi_j^0 + O(\epsilon)$ such that (1.41) holds.

□

The direct method can be used to construct multiple spike solutions for problems *without variational structure*, such as Gierer-Meinhardt system. For this application we refer to [**?**].

1.8. **Solving the reduced problem: variational reduction.** If the problem concerned has a variational structure, it is more appropriate to use a variational reduction method to solve (1.41). This method gives much stronger results under very weak assumptions.

We now describe the procedure that we call Variational Reduction in which the problem of finding $\xi'$ with $c_j^i = 0$, for all $i, j$, is equivalent to finding a critical point of a reduced functional of $\xi'$.

Define an energy functional

$$J(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + V(\varepsilon x) v^2 - \frac{1}{p+1} \int_{\mathbb{R}^{N+1}} v_+^{p+1} \tag{1.42}$$

where $v \in H^1(\mathbb{R}^N)$ and $1 < p < \frac{N+2}{N-2}$. Since $p$ is subcritical, by standard elliptic regularity arguments and Maximum Principle $v$ is a solution of the problem

$$\Delta v - Vv + v^p = 0, v \to 0 \tag{1.43}$$

if and only if $v \in H^1(\mathbb{R}^N)$ and $J'(v) = 0$. Observe that $\langle J'(v), \varphi \rangle = \int \nabla v \nabla \varphi + V v \varphi - v_+^p \varphi$.

We will prove the following Variational Reduction Principle

**Theorem 2.** *$v = W_{\xi'_*} + \phi(\xi')$ is a solution of the original problem (for $\rho \ll 1$) if and only if*

$$\partial_{\xi'} J(W_{\xi'} + \phi(\xi'))|_{\xi'=\xi'_*} = 0. \tag{1.44}$$

*Proof.* Indeed, observe that $v(\xi') := W_{\xi'} + \phi(\xi')$ solves the problem $\Delta v(\xi') - V(\varepsilon x)v(\xi') + v(\xi')^p = \sum_{i,j} c_j^i Z_j^i$ and also that

$$\partial_{\xi'_{j_0 i_0}} J(v(\xi')) = \langle J'(v(\xi')), \partial_{\xi'_{j_0 i_0}} v(\xi')\rangle = -\sum_{j,i} c_j^i \int Z_j^i \partial_{\xi'_{j_0 i_0}} v = -\sum_{i,j} c_j^i \int Z_i^j (\partial_{\xi'_{j_0 i_0}} W_{\xi'} + \partial_{\xi'_{j_0 i_0}} \phi(\xi')). \tag{1.45}$$

Recall that $W_{\xi'} = \sum_{j=1}^k w_{\lambda_j}(x - \xi'_j)$,

$$\partial_{\xi'_{j_0 i_0}} W'_\xi = \partial_{\xi'_{j_0 i_0}} w_{\lambda_{j_0(\xi')}}(x-\xi'_j) = (\partial_\lambda w_\lambda (x-\xi'_{j_0}))|_{\lambda=\lambda_{j_0}} - \partial_{x_{i_0}} w_{\lambda_{j_0}}(x-\xi'_{j_0}) = O(e^{-\delta|x-\xi'_0|})o(\varepsilon) - Z_{j_0 {}^{i_0}(x)} \tag{1.46}$$

This is because $\partial_\lambda w_\lambda = O(e^{-\delta|x-\xi'_0|})$. On the other hand since $\int Z_i^j \phi(\xi') = 0$ we have

$$\int Z_i^j \partial_{\xi'_{j_0 i_0}} \phi(\xi') = -\int \phi(\xi') \partial_{\xi'_{j_0 i_0}} Z_i^j$$

which is small thanks to the fact that $|\phi| \le C\rho e^{-\delta|x-\xi'_{j_0}|}$. Finally, observe that

$$-\int Z_j^i (\partial_{\xi'_{j_0 i_0}} W'_\xi + \partial_{\xi'_{j_0 i_0}} \phi) = \int Z_j^i Z_{j_0}^{i_0} + O(\rho) \tag{1.47}$$

The matrix of these numbers is invertible provided $\rho \ll 1$.

□

We now discuss several applications of the reduction principle.

**Theorem 3.** *(del Pino and Felmer [?]) Assume that there exists an open, bounded set $\Lambda \subset \mathbb{R}^N$ such that*

$$\inf_{\partial\Lambda} V > \inf_{\Lambda} V, \tag{1.48}$$

*then there exist a solution to the original problem, $v_\varepsilon$ with $v_\varepsilon(x) = w_{V(\xi_\varepsilon)}((x - \xi_\varepsilon)/\varepsilon) + o(1)$ and $V(\xi_\varepsilon) \to \min_\Lambda V$, $\xi = \xi_\varepsilon$.*

**Theorem 4.** *(del Pino-Felmer [?]) Assume that $\Lambda_1, \ldots, \Lambda_k$ are disjoint bounded sets with*

$$\inf_{\Lambda_j} V < \inf_{\partial\Lambda_j} V, j = 1, \cdots, k.$$

*Then there exist a solution $u_\varepsilon$ to the original problem with*

$$u_\varepsilon(x) \approx \sum_{j=1}^k w_{V(\xi_j^\varepsilon)}(x - \xi_j^\varepsilon/\varepsilon), \quad \xi_j^\varepsilon \in \Lambda_j$$

*and $V(\xi_j^\varepsilon) \to \inf_{\Lambda_j} V$. The same result holds if the minimum is replaced by maximum.*

**Theorem 5.** *(Kang-Wei [?]) Let $\Gamma$ be a bounded open set such that*

$$\max_{\Gamma} V(x) > \max_{\partial\Gamma} V(x)$$

*Then for any positive integer $K$ there exists a solution $u_\epsilon$ such that*

$$u_\varepsilon(x) \approx \sum_{j=1}^k w_{V(\xi_j^\varepsilon)}(x - \xi_j^\varepsilon/\varepsilon), \quad \xi_j^\varepsilon \in \Lambda, V(\xi_j^\varepsilon) \to \max_{\Lambda} V(x)$$

*Proof.* Assume that $j = 1$ first so that $v(\xi') = W_{\xi'} + \phi(\xi')$. Then we can compute the reduced energy as follows:

$$J(v(\xi')) = J(W_{\xi'} + \phi(\xi')) + \langle J'(W'_\xi + \phi), -\phi\rangle + \frac{1}{2}J''(W'_\xi + (1-t)\phi)[\phi]^2 \quad (1.49)$$

(This follows from Taylor expansion of the function $\alpha(t) = J(W_{\xi'} + (1-t)\phi)$.) Observe that $\langle J'(W'_\xi + \phi), -\phi\rangle = \sum_{i,j} c^i_j \int Z^j_i \phi = 0$. Also observe that

$$J''(W'_\xi + (1-t)\phi)[\phi]^2 = \int |\nabla\phi|^2 + V(\varepsilon x)\phi^2 - p(W'_\xi + (1-t)\phi)\phi^2 = O(\varepsilon^2) \quad (1.50)$$

uniformly on $\xi'$ because $\nabla\phi, \phi = O(\varepsilon e^{-\delta|x-\xi'|})$. We call $\Phi(\xi) := J(v(\xi')) = J(W_{\xi'}) + O(\varepsilon^2)$, and

$$J(W_{\xi'}) = \frac{1}{2}\int |\nabla W_{\xi'}|^2 + V(\xi)W^2_{\xi'} - \frac{1}{p+1}\int W^{p+1}_{\xi'} + \int (V(\varepsilon x) - V(\xi'))W^2_{\xi'} \quad (1.51)$$

Taking $\lambda = V(\xi)$, we have that

$$\int |\nabla w_\lambda(x)|^2 = \lambda^{-N/2}\int |\nabla w(\lambda^1/2x)|^2\lambda^{1+2/(p-1)}\lambda^{N/2}dx = \lambda^{-N/2+p+1/p-1}|\nabla w(y)|^2 dy \quad (1.52)$$

and

$$\lambda\int w^2_\lambda(x) = \lambda^{-N/2p+1/p-1}\int w(y)^{p+1}dy \quad (1.53)$$

This implies that

$$\frac{1}{2}\int |\nabla W_{\xi'}|^2 + V(\xi^{'})W^2_{\xi'} - \frac{1}{p+1}\int W^{p+1}_{\xi'} = V(\xi')^{p+1/p-1-N/2}c_{p,N} \quad (1.54)$$

and we also have

$$\int (V(\varepsilon x) - V(\xi'))w_\lambda(x-\xi')^2 = O(\varepsilon) \quad (1.55)$$

uniformly in $\xi'$.

In summary we have the following asymptotic expansion of the reduced energy

$$\Phi(\xi) = J(v(\xi')) = V(\xi)^{p+1/p-1-N/2}c_{p,N} + O(\varepsilon) \quad (1.56)$$

To prove Theorem 3 we observe that $\frac{p+1}{p-1} - \frac{N}{2} > 0$. Then $\forall\varepsilon \ll 1$ we have

$$\inf_{\xi\in\Lambda}\Phi(\xi) < \inf_{\xi\in\partial\Lambda}\Phi(\xi) \quad (1.57)$$

and therefore $\Phi$ has a local minimum $\xi_\varepsilon \in \Lambda$ and $V(\xi_\varepsilon) \to \min_\Lambda V$. The same procedure also works for local maximums.

For several separated local minimums, the proof is similar. In fact when $|\xi_{j_1} - \xi_{j_2}| > \delta$, for all $j_1 \neq j_2$, we have $\rho = e^{-\delta_0\min_{j_1\neq j_2}|\xi'_{j_1}-\xi'_{j_2}|} + \varepsilon \leq e^{-\delta_0\delta/\varepsilon} + \varepsilon < 2\varepsilon$. So we obtain

$$|\nabla_x\phi(\xi')| + |\phi(\xi')| \leq C\varepsilon\sum_j e^{-\delta_0|x-\xi'_j|} \quad (1.58)$$

Now we get

$$J(v(\xi')) = \sum_j V(\varepsilon\xi^{'}_j)^{p+1/p-1-N/2}c_{p,N} + O(\varepsilon) \quad (1.59)$$

$\varepsilon\xi' = (\xi_1, \ldots, \xi_k)$ implies for several minimal points on the $\Lambda_j$ we have the result desired.

Finally we prove the existence of multiply interacting spikes. The computations are little bit involved since we have to measure precisely the interactions. The reduced energy functional takes the following form:

$$J(v(\xi')) = \sum_j V(\varepsilon\xi_j)^{p+1/p-1-N/2}(c_{p,N}+o(1))-(1+o(1))\sum_{i\neq j} e^{-\min_{i\neq j}(\sqrt{V(\xi_i),V(\xi_j)})|\xi^{'}-\xi_j^{'}|}. \tag{1.60}$$

We shall take the following configuration space

$$\Sigma = \{(\xi_1, ..., \xi_k) \mid \xi_i \in \Lambda, \min_{i\neq j} |\xi_i - \xi_j| > \rho\epsilon \log \frac{1}{\epsilon}\}$$

and prove that the following maximization problem attains a solution in the interior part of the set $\Sigma$:

$$\min_{(\xi_1,\ldots,\xi_k)\in\Sigma} J(v(\xi'))$$

□

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