Course 2- Homework Assignment 2 (Due Date: Next Monday by 3:30pm)

You only need to hand in solutions of the **three problems** from the following nine problems. Extra credits will be given if you hand in more problems.

1. Let $w = w(r)$ be a solution of

$$\Delta w - w + w^p = 0, w > 0 \text{ in } \mathcal{R}^N, w(+\infty) = 0$$

Discuss the eigenvalues of the following eigenvalue problem

$$\Delta \phi - \phi + \lambda w^{p-1} \phi = 0$$

Hint: Find out λ_1, λ_2 and their eigenfunctions.

2. Let ϕ satisfy

$$\Delta \phi - \phi + f = 0, \text{in } R^N, |f| \leq C_1 e^{-\frac{1}{2}|x|}$$

and ϕ be bounded. Show that

$$|\phi| \leq C_2 e^{-\frac{1}{2}|x|}$$

Hint: for r large consider the following comparison function

$$h = Ce^{-\frac{1}{2}|x|} + \epsilon e^{\frac{1}{2}|x|}$$

Then use Maximum Principle and let $\epsilon \to 0$.

3. (a) For $L > \pi$ show that there exists a solution to

$$w^{''} - w + w^2 = 0, 0 < x < L, w^{'}(x) < 0, w^{'}(0) = 0, w^{'}(L) = 0$$

Call this solution $w_L(x)$.

Hint: show that for $L \leq \pi$ only constant solution exists. Use the minimization

$$c = \min E[u] = \min \frac{\int_0^L (|u^{'}|^2 + u^2)}{(\int_0^L u^{p+1})^{\frac{2}{p+1}}}$$

(b) Show that the principal eigenvalue λ_1 corresponding eigenvalue problem

$$L_0(\phi) = \phi^{''} - \phi + 2w_L \phi = \lambda \phi, \phi^{'}(0) = \phi^{'}(L) = 0$$

is positive, and the second eigenvalue λ_2 is negative.

Hint: use the variational characterization. For the second part, note that $w_L^{'}$ satisfies the equation.

4. Continue from Problem 3. (a) Assume that

$$\int_0^L w_L L_0^{-1}(w_L) > 0$$

Show that the nonlocal eigenvalue problem

$$\phi^{''} - \phi + 2w_L\phi - \frac{2\int_0^L w_L\phi}{\int_0^L w_L^2} w_L^2 = \lambda\phi$$

is stable.

Hint: same as in class.

(b) Assume that

$$\int_0^L w_L L_0^{-1}(w_L) > 0$$

Show that the nonlocal eigenvalue problem

$$\phi^{''} - \phi + 2w_L\phi - \frac{2\int_0^L w_L\phi}{\int_0^L w_L^2} w_L^2 = \lambda\phi$$

has a unique Hopf bifurcation point.

Hint: same as in class.

5. State the **Contraction Mapping Principle** and discuss an application of this principle in partial differential equations

6. State the **Fredholm Alternatives** and discuss an application of this theorem in partial differential equations

7. Consider the following singularly perturbed problem

$$\epsilon^2 u^{''} - V(x)u + Q(x)u^p = 0$$

Find the necessary conditions for the locations x_0 at which a single spike solution may be constructed.

Hint: The solution to

$$U^{''} - \lambda U + \mu U^p = 0$$

is given by

$$U = (\frac{\lambda}{\mu})^{\frac{1}{p-1}} w(\sqrt{\lambda} y)$$

where w is the solution of

$$w^{''} - w + w^p = 0$$

8. Consider the following singularly perturbed problem

$$\epsilon^2 (u_{rr} + \frac{N-1}{r} u_r) - V(r)u + u^p = 0$$

Find the necessary condition for the radius r_0 at which a single ring solution may be constructed. Hence a ring solution is a spike type solution concentrating on a ring $r = r_0$.

9. Find out the Green's function for

$$G^{''} + \frac{N-1}{r} G - G + \delta_{r=r_0} = 0$$

and find Green's representation formula for solutions of

$$u^{''} + \frac{N-1}{r}u^{'} - u + f(r) = 0$$

Hint: see paper of Ni-Wei, On positive solutions concentrating on spheres for the Gierer-Meinhardt system J. Diff. Eqns. 221(2006), 158-189.