Course 2 - Homework Assignment 3 Solution

1. (a)

$$\begin{aligned} u''(x) &= f(x) \\ u'(x) &= -\int_x^\infty f(t)\,dt \\ u(x) &= \int_x^\infty \int_t^\infty f(s)\,dsdt \end{aligned}$$

As $x \to \infty$, the dummy variables $t, s \to \infty$ so

$$|u(x)| \le \int_x^\infty \int_t^\infty Ce^{-s}\,dsdt \le Ce^{-x}.$$

(b) Similarly,

$$u(x) = \int_{-\infty}^x \int_\infty^t f(s)\,dsdt.$$

(c) The condition is

$$\int_{-\infty}^\infty f(x)\,dx = 0.$$

Indeed, using the formula in (b),

$$\left|\int_{-\infty}^t f(s)\,ds\right| = \left|-\int_t^\infty f(s)\,ds\right| \le \int_t^\infty Ce^{-s}\,ds = Ce^{-t}$$

for large t, hence for all t. Then

$$|u(x)| \le \int_{-\infty}^x \left|\int_{-\infty}^t f(s)\,ds\right|\,dt \le C\int_{-\infty}^\infty e^{-|t|}\,dt \le C.$$

(d) The condition is

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^\infty xf(x)\,dx = 0,$$

that is, f is orthogonal to the kernels of $\dfrac{d^2}{dx^2}$. Note also that the second also reads

$$\int_{-\infty}^\infty \int_{-\infty}^t f(s)\,dsdt = 0.$$

These allow us to write, for $x \to \infty$,

$$u(x) = \int_{-\infty}^x \int_\infty^t f(s)\,dsdt = -\int_x^\infty \int_\infty^t f(s)\,dsdt = \int_x^\infty \int_t^\infty f(s)\,dsdt \le Ce^{-x}.$$

2. (a)

$$\begin{aligned} u_0(x) &= -2\log\cosh(x) + \log 2 \\ u_0'(x) &= -\frac{2\sinh(x)}{\cosh(x)} \\ u_0''(x) &= -\frac{2}{\cosh^2(x)} = -e^{u_0(x)}. \end{aligned}$$

(b) The translation invariance is clear, since the equation involves no x. For the scaling invariance, we see that

$$(u_0(\lambda x) + 2\log\lambda)'' + e^{u_0(\lambda x)+2\log\lambda} = \lambda^2 u_0''(\lambda x) + \lambda^2 e^{u_0(\lambda x)} = 0.$$

(c) Applying $-\dfrac{\partial}{\partial a}$ to $(u_0(x-a))'' + e^{u_0(x-a)} = 0$, we have

$$(u_0'(x-a))'' + e^{u_0(x-a)}u_0'(x-a) = 0,$$

that is,

$$(u_0')'' + e^{u_0}u_0' = 0.$$

The first kernel is $\phi_1 = u_0'(x) = -\dfrac{2\sinh(x)}{\cosh(x)} = -2\tanh(x)$.

The second one is given by

$$\phi_2(x) = \frac{\partial}{\partial\lambda}\bigg|_{\lambda=1} (u_0'(\lambda x) + 2\log\lambda) = xu_0'(x) + 2 = -2x\tanh(x) + 2.$$

(d) First we compute the Wronskian

$$W = \phi_1\phi_2' - \phi_1'\phi_2 = (u_0')^2 - 2u_0'' = \frac{4\sinh^2(x) + 4}{\cosh^2(x)} = 4.$$

By the variation of parameters formula, we have for any given a, b,

$$\begin{aligned} \phi(x) &= -\phi_1(x)\int_a^x \frac{\phi_2(t)f(t)}{W(t)}\,dt + \phi_2(x)\int_b^x \frac{\phi_1(t)f(t)}{W(t)}\,dt \\ &= -\tanh(x)\int_a^x (t\tanh(t)+1)f(t)\,dt + (x\tanh(x)+1)\int_b^x \tanh(t)f(t)\,dt \end{aligned}$$

For decay at the positive end, we take $a = b = \infty$ to obtain

$$\phi(x) = \tanh(x)\int_x^\infty (t\tanh(t)+1)f(t)\,dt - (x\tanh(x)+1)\int_x^\infty \tanh(t)f(t)\,dt.$$

(e) We require that f be orthogonal to the kernels ϕ_1 and ϕ_2, that is,

$$\int_{-\infty}^\infty \tanh(x)f(x)\,dx = \int_{-\infty}^\infty (x\tanh(x)+1)f(x)\,dx = 0.$$

In fact since $\phi_2(x)$ grows like x, the decay is a bit weaker:

$$|\phi(x)| \le C(1+|x|)e^{-|x|} \text{ for all } x.$$

3. If we define $\tilde{a}(r) = r^{N-1}a(r)$, then this is what has been discussed in the lectures. The necessary condition is $(\log \tilde{a})'(r_0) = 0$ and a sufficient condition is the non-degeneracy $(\log \tilde{a})''(r_0) \neq 0$. By direct computations,

$$\begin{aligned}
(\log \tilde{a})' &= \frac{\tilde{a}'}{\tilde{a}} \\
(\log \tilde{a})'' &= \frac{\tilde{a}''}{\tilde{a}} - \left(\frac{\tilde{a}'}{\tilde{a}}\right)^2 \\
\frac{\tilde{a}'}{\tilde{a}} &= \frac{a'}{a} + \frac{N-1}{r} \\
\frac{\tilde{a}''}{\tilde{a}} &= \frac{a''}{a} + \frac{2(N-1)}{r}\frac{a'}{a} + \frac{(N-1)(N-2)}{r^2}.
\end{aligned}$$

Therefore, the necessary condition is

$$(\log a)'(r_0) = -\frac{N-1}{r_0}$$

and a sufficient condition is

$$(\log a)''(r_0) \neq \frac{N-1}{r_0^2}.$$

4. (a)

$$\begin{aligned}
u_0(r) &= \log \frac{8}{(1+r^2)^2} = \log 8 - 2\log(1+r^2) \\
u_0'(r) &= -\frac{4r}{1+r^2} \\
u_0''(r) &= \frac{4(r^2-1)}{(1+r^2)^2} \\
u_0'' + \frac{1}{r}u_0' + e_0^u &= \frac{1}{(1+r^2)^2}(4r^2 - 4 - 4 - 4r^2 + 8) = 0.
\end{aligned}$$

(b) First note that the equation holds at λr:

$$u_0''(\lambda r) + \frac{1}{\lambda r}u_0'(\lambda r) + e^{u(\lambda r)} = 0.$$

Then

$$(u_0(\lambda r)+2\log\lambda)'' + \frac{1}{r}(u_0(\lambda r)+2\log\lambda)' + e^{u_0(\lambda r)+2\log\lambda} = \lambda^2\left(u_0''(\lambda r) + \frac{1}{\lambda r}u_0'(\lambda r) + e^{u(\lambda r)}\right) = 0.$$

(c) A kernel is

$$\left.\frac{\partial}{\partial\lambda}\right|_{\lambda=1}(u_0(\lambda r) + 2\log\lambda) = ru_0'(r) + 2 = \frac{2(1-r^2)}{1+r^2}.$$

Let us take

$$Z_0(r) = \frac{1-r^2}{1+r^2}.$$

(d) Let Z_1 be the other kernel. By Abel's formula, the Wronskian satisfies, up to a constant,

$$W(r) = Z_0(r)Z_1'(r) - Z_0'(r)Z_1(r) = e^{-\int \frac{1}{r}} = \frac{1}{r}.$$

Solving for $Z_1(r)$ (reduction of order), we have

$$\begin{aligned} Z_1(r) &= Z_0(r)\int^r \frac{W(s)}{Z_0^2(s)}\,ds \\ &= \frac{1-r^2}{1+r^2}\int^r \frac{(1+s^2)^2}{s(1-s^2)^2}\,ds \\ &= \frac{1-r^2}{1+r^2}\left(\log r - \frac{2}{r^2-1}\right) \\ &= \frac{1-r^2}{1+r^2}\log r + \frac{2}{1+r^2} \end{aligned}$$

By the variation of parameters formula, the solution satisfying

$$\phi'(0) = 0$$

is given by

$$\begin{aligned} \phi(r) &= AZ_0(r) - Z_0(r)\int_0^r \frac{Z_1(s)f(s)}{W(s)}\,ds + Z_1(r)\int_0^r \frac{Z_0(s)f(s)}{W(s)}\,ds \\ &= -\frac{1-r^2}{1+r^2}\int_0^r \frac{s((1-s^2)\log s + 2)}{1+s^2} f(s)\,ds + \frac{(1-r^2)\log r + 2}{1+r^2}\int_0^r \frac{s(1-s^2)}{1+s^2} f(s)\,ds. \end{aligned}$$

where A is an arbitrary constant.

5. Note that as $r \to \infty$,

$$Z_0(r) = -1 + O(\frac{1}{r^2})$$

,

$$Z_1(r) = -\log r + O(\frac{\log r}{r^2})$$

and

$$\begin{aligned} \int_0^r \frac{s((1-s^2)\log s + 2)}{1+s^2} f(s) &= \int_0^\infty \frac{s((1-s^2)\log s + 2)}{1+s^2} f(s) + O(\frac{\log r}{r^2}) \\ \int_0^r \frac{s(1-s^2)}{1+s^2} f(s) &= \int_0^\infty Z_0(s)f(s)sds + O(\frac{1}{r^2}) \end{aligned}$$

Hence we have as $r \to +\infty$

$$\phi(r) = A - \int_0^\infty Z_1(s)f(s)sds - \log r\int_0^\infty Z_0(s)f(s)sds + O(\frac{\log r}{r^2})$$

Thus a necessary condition for ϕ to be bounded is that

$$\int_0^\infty Z_0(s)f(s)sds = 0$$

Under the above condition we can choose

$$A = \int_0^\infty Z_1(s)f(s)sds$$

so that ϕ decays at $+\infty$:

$$\phi = O(\frac{\log r}{r^2})$$

Alternatively, we can define

$$\begin{aligned}\phi(r) &= Z_0(r)\int_r^\infty \frac{Z_1(s)f(s)}{W(s)}\,ds - Z_1(r)\int_r^\infty \frac{Z_0(s)f(s)}{W(s)}\,ds \\ &= \frac{1-r^2}{1+r^2}\int_r^\infty \frac{s((1-s^2)\log s+2)}{1+s^2}f(s)\,ds - \frac{(1-r^2)\log r+2}{1+r^2}\int_r^\infty \frac{s(1-s^2)}{1+s^2}f(s)\,ds,\end{aligned} \tag{1}$$

so for r large

$$\begin{aligned}|\phi(r)| &\leq \int_r^\infty s\log s\,|f(s)|\,ds + \log r\int_r^\infty s\,|f(s)|\,ds \\ &\leq 2\int_r^\infty s\log s\,|f(s)|\,ds \\ &\leq C\int_r^\infty \frac{s\log s}{1+s^4}\,ds \\ &\leq \frac{C}{r}\int_r^\infty \frac{\log s}{s^2}\,ds \\ &= O\left(\frac{\log r}{r^2}\right)\end{aligned}$$

However this definition (1) does not satisfy the initial condition

$$\phi^{'}(0) = 0$$

unless $\int_0^\infty Z_0(s)f(s)sds = 0$. In fact we calculate

$$\phi^{'}(0) = \lim_{r\to 0}\frac{1}{r}\int_0^\infty Z_0(s)f(s)sds$$