BOUNDARY INTERFACE FOR THE ALLEN-CAHN EQUATION

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ABSTRACT. We consider the Allen-Cahn equation

$$\varepsilon^2 \Delta u + u - u^3 = 0 \ \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \ \text{ on } \partial\Omega,$$

where Ω is a smooth and bounded domain in $\mathbb{R}^n$ such that the mean curvature is positive at each boundary point. We show that there exists a sequence $\varepsilon_j \to 0$ such that the Allen-Cahn equation has a solution u_{ε_j} with an interface which approaches the boundary as $j \to +\infty$.

1. Introduction

The aim of this paper is to construct a solution with an interface near the boundary to the Allen-Cahn equation

$$\left\{ \begin{array}{l} \varepsilon^2 \Delta u + u - u^3 = 0 \ \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \ \text{ on } \partial\Omega, \end{array} \right. \tag{1.1}$$

where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, Ω is a bounded smooth domain in $\mathbb{R}^n$, $\varepsilon > 0$ is a small parameter, and $\nu(x)$ denotes the unit outer normal at $x \in \partial\Omega$.

Problem (1.1) and its parabolic counterpart have been a subject of extensive research for many years. In order to describe some known results, we define the Allen-Cahn functional (see [2])

$$J_\varepsilon[u] = \int_\Omega [\frac{\varepsilon^2}{2} |\nabla u|^2 - F(u)], \ \text{where} \ \ F(u) = -\frac{1}{4}(1-u^2)^2.$$

The set $\{x \in \Omega \mid u(x) = 0\}$ is called the *interface* of u. Let $\mathrm{Per}_\Omega(A)$ be the relative perimeter of the set $A \subset \Omega$. Using $\Gamma-$convergence techniques (see [18]), Kohn and Sternberg in [11] showed a general result stating that in a neighborhood of an isolated local minimizer of Per_Ω there exists a local minimizer to the functional J_ε. They further used this idea to show the existence of a stable solution for (1.1) in two dimensional, non-convex domains, such as a dumb-bell. Since then, the existence of solutions with a single interface intersecting the boundary has been established and studied by many authors, see [1], [6], [8], [10], [20], [21], [22], [23], [24], [25] and the references therein.

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In [17], Ni and the authors showed that in the case of a ball, there are radially symmetric solutions with arbitrarily many interfaces near the boundary. For related results see also [4], [5], [3] and references therein. In this paper, we extend this result removing the symmetry, with a single boundary layer.

Our main result is the following.

Theorem 1.1. *Assume that the mean curvature k of $\partial\Omega$ is everywhere positive. Then there exists a sequence $\varepsilon_j \to 0$ such that problem (1.1) has a solution $u_{\varepsilon_j}(x)$ with the following property*

$$u_{\varepsilon_j}(x) = H(\frac{x_n}{\varepsilon_j} - \frac{1}{2\sqrt{2}} \log \frac{1}{\varepsilon_j} - R_{\varepsilon_j}(\overline{z})) + O(\sqrt{\varepsilon_j}).$$

Here, if x is near the boundary, we parameterize it with $\overline{z}$ and x_n, $\overline{z} \in \partial\Omega$ being the closest point to $\partial\Omega$ and $x_n = d(x, \partial\Omega)$, while $H(y)$ is the unique hetheroclinic solution of

$$H'' + H - H^3 = 0, \;\; H(0) = 0, \;\; H(\pm\infty) = \pm 1. \tag{1.2}$$

The function $R_\varepsilon(\overline{z})$ satisfies $R_\varepsilon(\overline{z}) = \gamma_1 \log \frac{1}{k(\overline{z})} + \gamma_2 + \gamma_3 \sqrt{\varepsilon} + O(\varepsilon)$, where the γ_i's are universal constants (with $\gamma_1 > 0$).

To explain the features of this result we first discuss its heuristic derivation, and then our rigorous approach to the problem.

As already remarked, the functional J_ε can be interpreted as an approximation (via Γ-convergence) of the area functional for the interface of a two-phase mixture. In [17] it was noticed that, when the domain is the unit ball of $\mathbb{R}^n$, there is an interplay between the perimeter of an interface near the boundary and its *interaction* with the boundary itself. Indeed (still in the radial case), the surface contribution to the energy of an interface H (scaled in ε) with radius r_0 is proportional to εr_0^{n-1}. To understand the interaction with $\partial\Omega$, one can reason as follows: since Neumann boundary conditions are required, to match them one can add to H a term $H'\left(\frac{r-r_0}{\varepsilon}\right) e^{-\frac{\sqrt{2}(1-r)}{\varepsilon}}$: the presence of the exponential function is justified by the asymptotics of H at infinity, see (2.2) below. Considering the functional J_ε applied to this new function, one verifies that its expansion in ε behaves qualitatively like $G_\varepsilon(r_0) := \varepsilon \left(r_0^{n-1} - e^{-\frac{-2\sqrt{2}(1-r_0)}{\varepsilon}} \right)$: we refer to Section 4 in [17] for a rigorous derivation of this formula. As one can see, there is a local maximum of G_ε for $1 - r_0 \simeq \frac{\varepsilon}{2\sqrt{2}} \log \frac{1}{\varepsilon}$, which suggests the existence of a stationary point of J_ε. In [17] this was indeed proved using a one-dimensional Lyapunov-Schmidt reduction.

Removing the symmetry, the above heuristic argument still applies when the mean curvature of $\partial\Omega$ is positive. However, the previous approach completely breaks down and one needs to use different arguments. The reason is due to some underlying resonance phenomena: these can be seen looking at the above function G_ε, which possesses an unstable critical point. Considering the linearization of (1.1) at an approximate solution, it turns out that this instability generates a small negative eigenvalue of order ε. In the radial case, see [17], this is the only small one, and can be taken care of with the Lyapunov-Schmidt reduction. In general more resonance occurs, due to the *vibration modes* of $\partial\Omega$: qualitatively, the small eigenvalue found before generates a sequence of eigenvalues of the form $-\varepsilon + \varepsilon^2 \overline{\lambda}_j \simeq -\varepsilon + \varepsilon^2 j^{\frac{2}{n-1}}$, where the $\overline{\lambda}_j$'s are the eigenvalues of the Laplace-Beltrami operator of $\partial\Omega$, whose asymptotics is given by the well known Weyl's formula. As one can see, we have an increasing number of negative eigenvalues, many of them accumulate to zero and sometimes, depending on the value of ε, we even have the presence of a kernel: this clearly causes difficulties if one wants to apply local inversion arguments.

To tackle this problem we take advantage of an approach used in [14], [15] (see also [12], [13]), where similar resonance phenomena were handled for another class of singularly perturbed equations. The main ingredient is to look at the eigenvalues (of the linearized problem) as functions of ε, and to estimate their derivative with respect to ε. This can be rigorously done using a linear perturbation theorem due to T.Kato, see Section 2, and by characterizing the resonant eigenfunctions. This result gives us indeed invertibility along a suitable sequence $\varepsilon_j \to 0$, and the norm of the inverse operator along this sequence has an upper bound of order $\varepsilon_j^{-\frac{n+1}{2}}$. This loss of uniform bounds as $j \to +\infty$ should be expected, since more and more eigenvalues are accumulating near zero. Anyway, we are able to deal with this further difficulty by choosing approximate solutions with a sufficiently high accuracy.

The plan of the paper is the following: in Section 2 we collect some preliminary results concerning the profile H, the expressions of the quantities under interest in local coordinates, and some spectral results. In Section 3 we turn to the construction of approximate solutions, which is done by a careful analysis of the linearized equation in the x_n component (following the notation of Theorem 1.1 x_n stands for the distance from the boundary of Ω). Finally in Section 4 we prove our main result applying Kato's theorem and a contraction mapping argument: the main ingredient here is the characterization of the eigenfunctions corresponding to small eigenvalues, which is done mainly via Fourier analysis. We collect some technical results in an appendix.

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2. Some Notation and Preliminary facts

In this section we introduce a list of useful facts, some of analytic nature involving the *standard profile* H of the Allen-Cahn equation, and other of more geometric content, concerning the use of local coordinates and the *Weyl's asymptotic formula* for the eigenvalues of elliptic operators. We finally recall a classical result due to T.Kato about the differentiability of eigenvalues of operators depending on a real parameter.

First of all, we adopt the convention that large constants depending only on Ω are denoted by C, and are allowed to vary, attaining larger and larger values. With the same convention, we write $O(t)$ to denote quantities which remains uniformly bounded by $C|t|$ as t tends to zero, and write $o(t)$ to denote those which tend to zero faster than $|t|$ in this limit. The symbols $\gamma_j, j = 1, ...$ will denote universal constants. Throughout the paper, we tacitely use the standard convention of summing upper and lower indices which are repeated. Elliptic operators will usually have a positive coefficient of second order derivatives, and we will count their eigenvalues in decreasing order.

Let H be the unique solution of (1.2): it is easy to see that

$$H(y) = \tanh(\frac{\sqrt{2}}{2} y), \tag{2.1}$$

and the following estimates hold

$$\begin{cases} H(y) - 1 = -A_0 e^{-\sqrt{2}|y|} + O(e^{-(2\sqrt{2})|y|}) & \text{for } y \to +\infty; \\ H(y) + 1 = A_0 e^{-\sqrt{2}|y|} + O(e^{-(2\sqrt{2})|y|}) & \text{for } y \to -\infty; \\ H'(y) = \sqrt{2} A_0 e^{-\sqrt{2}|y|} + O(e^{-(2\sqrt{2})|y|}) & \text{for } |y| \to +\infty, \end{cases} \tag{2.2}$$

where $A_0 > 0$ is a fixed constant. As a consequence of (2.2), we have

$$3 \int_{\mathbb{R}} (1 - H^2) H' e^{-\sqrt{2} y} = - \int_{\mathbb{R}} (H''' - 2H') e^{-\sqrt{2} y} = 4 A_0. \tag{2.3}$$

Let

$$Z = 3(1 - H^2) H',$$

$$f(u) = u - u^3, \qquad g(u) = 3u - u^3.$$

Then we have the following well-known result: for a proof, see Lemma 4.1 in [19].

Lemma 2.1. *Consider the following eigenvalue problem*

$$\phi'' + f'(H)\phi = \lambda\phi, \qquad \phi \in H^1(\mathbb{R}). \tag{2.4}$$

Then one has

$$\lambda_1 = 0, \quad \phi_1 = cH'; \qquad \lambda_2 < 0, \tag{2.5}$$

where the λ_i's denote the eigenvalues in decreasing order (counted with multiplicity), with corresponding eigenfunctions $(\phi_i)_i$. As a consequence (by Fredholm's alternative), given any function $g \in L^2(\mathbb{R})$ satisfying $\int_{\mathbb{R}} gZ = 0$, the following problem has a unique solution

$$\phi'' + f'(H)\phi = g \ \text{ in } \mathbb{R}, \qquad \int_{\mathbb{R}} Z\phi = 0. \tag{2.6}$$

Furthermore, there exists a positive constant C such that $\|\phi\|_{H^1(\mathbb{R})} \le C\|g\|_{L^2(\mathbb{R})}$.

Next, we scale the equation (1.1) by $\frac{1}{\varepsilon}$ to obtain

$$\begin{cases} \Delta u + u - u^3 = 0 & \text{in } \Omega_\varepsilon, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{2.7}$$

where $\Omega_\varepsilon = \Omega/\varepsilon$. To consider the scaled problem (2.7), it is convenient to introduce suitable coordinates in a tubular neighborhood of $\partial\Omega_\varepsilon$. For $z \in \partial\Omega_\varepsilon$, we can parameterize the points y in a neighborhood of the boundary (with distance from the boundary less or equal than $\frac{\delta}{\varepsilon}$, with δ small and fixed) as

$$y = z - x_n\nu(\varepsilon z), \tag{2.8}$$

where $\nu(\varepsilon z)$ stands for the unit outward normal to Ω at εz.

In the following, we let $\overline{g}$ denote the metric on $\partial\Omega$ (inherited from $\mathbb{R}^n$), $\overline{g}_\varepsilon$ the one on $\partial\Omega_\varepsilon$, and g_ε the flat metric of Ω_ε, which will be expressed in the above coordinates (z, x_n). If $z_1, \dots, z_{n-1}$ is a local set of coordinates on Ω_ε, and if $(\overline{g}_\varepsilon)_{ij}$ denote the corresponding components of the metric tensor, then the components of g_ε are given by

$$(g_\varepsilon)_{IJ} = \begin{pmatrix} (\overline{g}_\varepsilon)_{ij} - \varepsilon x_n(A_i^l g_{jl} + A_j^k g_{ik}) + \varepsilon^2 x_n^2 A_i^l A_j^k g_{lk} & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.9}$$

for $I, J = 1, \dots, n$, and $i, j = 1, \dots, n-1$. In the last formula, (A_j^i) are the components of the second fundamental form namely, setting $\overline{z} = \varepsilon z$ and $\overline{z}_j = \varepsilon z_j$, they are defined by $\frac{\partial \nu}{\partial \overline{z}_i} = A_i^j \frac{\partial \overline{z}}{\partial \overline{z}_j}$. To deduce (2.9), we notice that

$$\frac{\partial y}{\partial z_i} = \frac{\partial z}{\partial z_i} - \varepsilon x_n \frac{\partial \nu(\varepsilon z)}{\partial z_i}; \qquad \frac{\partial y}{\partial x_n} = -\nu(\varepsilon z).$$

Hence, since $(g_\varepsilon)_{ij} = \langle \frac{\partial y}{\partial z_i}, \frac{\partial y}{\partial z_j} \rangle$, and since $\nu(\varepsilon z)$ is perpendicular to $\frac{\partial z}{\partial z_i}$, then we get immediately (2.9).

The eigenvalues of the matrix (A_i^j) (with respect to the metric $\overline{g}$) are called *principal curvatures* of $\partial\Omega$, and will be denoted by $(k_i(\varepsilon z))_i$, $i = 1, \dots, n-1$. In the following, we let

$$k(\overline{z}) = \sum_{i=1}^{n-1} k_i(\overline{z}), \qquad \overline{z} \in \partial\Omega,$$

denote the *mean curvature* of $\partial\Omega$ (scaled by a factor $n-1$). This function enters in the expansion of the volume element in terms of ε, since from (2.9) (see also [15], p. 123, with an obvious change of notation) one finds

$$dV_{g_\varepsilon} = \sqrt{g_\varepsilon} dx_n dz = (1 - \varepsilon x_n k(\varepsilon z)) dV_{\overline{g}_\varepsilon} dx_n + O(\varepsilon^2 x_n^2) dV_{\overline{g}_\varepsilon} dx_n. \tag{2.10}$$

In a general system of coordinates with metric $g = (g_{IJ})_{IJ}$, the expression for the Laplace-Beltrami operator acting on a function u is the following

$$\Delta_g u = \frac{1}{\sqrt{\det g}} \partial_I \left(g^{IJ} \sqrt{\det g} \partial_J u \right), \tag{2.11}$$

where g^{IJ} are the entries of the inverse matrix of $(g_{IJ})_{IJ}$.

Then after some elementary computations, using the block form of the matrix in (2.9), one finds

$$\Delta_y u = u_{nn} - \sum_{i=1}^{n-1} \frac{k_i(\varepsilon z)}{1 - \varepsilon x_n k_i(\varepsilon z)} \varepsilon u_n + \varepsilon^2 \Delta_{\partial\Omega_{\varepsilon x_n}} u. \tag{2.12}$$

Here we are using the standard notation $u_n = \frac{\partial u}{\partial x_n}$, $u_{nn} = \frac{\partial^2}{\partial x_n^2}$, while $\Delta_{\partial\Omega_{\varepsilon x_n}} u$ stands for the operator in (2.11) freezing the coordinate x_n, namely summing over $i, j = 1, \dots, n-1$

$$\Delta_{\partial\Omega_{\varepsilon x_n}} u = \frac{1}{\sqrt{\det g}} \partial_i \left(g^{ij} \sqrt{\det g} \partial_j u \right) (z, x_n).$$

This operator is nothing but the Laplace-Beltrami operator for the metric $g_{\partial\Omega_{\varepsilon x_n}}$ on $\partial\Omega_\varepsilon$ with coefficients $((g_\varepsilon)_{ij}(\cdot, x_n))_{ij}$ in the coordinates $z_i, \dots, z_{n-1}$. With respect to this metric, one can introduce a corresponding gradient $\nabla_{\partial\Omega_{\varepsilon x_n}}$, defined by duality as

$$\langle \nabla_{\partial\Omega_{\varepsilon x_n}} u, v \rangle_{\partial\Omega_{\varepsilon x_n}} = (g_\varepsilon)^{ij}(\cdot, x_n) \frac{\partial u}{\partial z_i} v_j, \qquad \text{if} \quad v = v_j \frac{\partial}{\partial z_j} \in T_{\partial\Omega_\varepsilon}.$$

From the expression of g_{ij} in (2.9) then one finds the estimates

$$|\nabla_{\partial\Omega_{\varepsilon x_n}} u|^2 := (g_\varepsilon)^{ij}(\cdot, x_n) \frac{\partial u}{\partial z_i} \frac{\partial u}{\partial z_j} = (1 + O(\varepsilon x_n)) |\nabla_{\overline{g}_\varepsilon} u|^2; \tag{2.13}$$

$$\int_{\partial\Omega_\varepsilon} u \Delta_{\partial\Omega_{\varepsilon x_n}} v dV_{g_{\partial\Omega_{\varepsilon x_n}}} = \int_{\partial\Omega_\varepsilon} \langle \nabla_{\overline{g}_\varepsilon} u, \nabla_{\overline{g}_\varepsilon} u \rangle dV_{\overline{g}_\varepsilon} + O(\varepsilon x_n) \|\nabla_{\overline{g}_\varepsilon} u\|_{L^2(\partial\Omega_\varepsilon)} \|\nabla_{\overline{g}_\varepsilon} v\|_{L^2(\partial\Omega_\varepsilon)}, \tag{2.14}$$

for every $u, v \in H^1(\partial\Omega_\varepsilon)$.

In the following, for a fixed small constant $0 < \tau < 1$, we let

$$\Gamma_\tau = \left\{ y \in \Omega_\varepsilon \ : \ dist(y, \partial\Omega_\varepsilon) < \varepsilon^{-\tau} \right\} \simeq \partial\Omega_\varepsilon \times (0, \varepsilon^{-\tau}).$$

Considering then a function $u \in H^1(\Gamma_\tau)$, depending on the variables (z, x_n) through the parameterization in (2.8), we can freeze the variable x_n and define the gradient $\nabla_{\overline{g}_\varepsilon}$ on the variables z. With this convention, and using again formula (2.9) one obtains

$$\int_{\Gamma_\tau} |\nabla_{g_\varepsilon} u|^2 dV_{g_\varepsilon} = (1 + O(\varepsilon^{1-\tau})) \int_{\Gamma_\tau} |u_n|^2 dx_n dV_{\overline{g}_\varepsilon} + (1 + O(\varepsilon^{1-\tau})) \int_{\Gamma_\tau} |\nabla_{\overline{g}_\varepsilon} u|^2 dx_n dV_{\overline{g}_\varepsilon}. \tag{2.15}$$

For later purposes, it is convenient to consider a basis of eigenfunctions $(\varphi_j)_j$ for the following eigenvalue problem

$$-\Delta_{\partial\Omega} \varphi_j = \lambda_j k(\overline{z}) \varphi_j; \qquad \overline{z} \in \partial\Omega, \tag{2.16}$$

satisfying the normalization conditions $\int_{\partial\Omega} k(\overline{z}) \varphi_i \varphi_j dV_{\overline{g}} = \delta_{ij}$. Such eigenvalues can be obtained for example using the Rayleigh quotient, namely if M_j denote the family of j-dimensional subspaces of $H^1(\partial\Omega)$, then one has

$$\lambda_j = \inf_{M \in M_j} \sup_{\varphi \in M, \varphi \neq 0} \frac{\int_{\partial\Omega} |\nabla_{\partial\Omega} \varphi|^2}{\int_{\partial\Omega} k(\overline{z}) \varphi^2}.$$

It is standard to check the following Weyl's asymptotic formula ([7])

$$\lambda_j \simeq C_\Omega j^{\frac{2}{n-1}} \qquad \text{as } j \to +\infty,$$

for some constant C_Ω depending only on Ω.

We finally recall the following theorem due to T. Kato, ([9], page 444) which will be fundamental for us in order to obtain invertibility of the linearized equation.

Theorem 2.2. *Let $T(\chi)$ denote a differentiable family of operators from an Hilbert space X into itself, where χ belongs to an interval containing* 0. *Let $T(0)$ be a self-adjoint operator of the form* Identity - compact *and let $\sigma(0) = \sigma_0 \neq 1$ be an eigenvalue of $T(0)$. Then the eigenvalue $\sigma(\chi)$ is differentiable at* 0 *with respect to χ. The derivative of σ is given by*

$$\frac{\partial \sigma}{\partial \chi} = \left\{ \textit{eigenvalues of } P_{\sigma_0} \circ \frac{\partial T}{\partial \chi}(0) \circ P_{\sigma_0} \right\},$$

where $P_{\sigma_0} : X \to X_{\sigma_0}$ denotes the projection onto the σ_0-eigenspace X_{σ_0} of $T(0)$.

3. Approximate Solutions

Since our existence result is based on local inversion arguments, as explained in the introduction, we need first to find approximate solutions, and this is what this section is devoted to. We will take these functions identically equal to 1 outside some fixed neighborhood of $\partial\Omega$, and hence (after a scaling of the domain) it is sufficient to restrict our attention to the set

$\Gamma := \partial\Omega_\varepsilon \times (0, \frac{\delta}{\varepsilon})$. On Γ, we use the coordinates (z, x_n) introduced in the previous section. Then equation (1.1) becomes

$$\begin{cases} u_{nn} - \sum_{i=1}^{n-1} \frac{k_i(\varepsilon z)}{1-\varepsilon x_n k_i(\varepsilon z)} \varepsilon u_n - \varepsilon^2 \Delta_{\partial\Omega_{\varepsilon x_n}} u + u - u^3 = 0, (z, x_n) \in \Gamma, \\ u_n(\cdot, 0) = 0, \quad u(\cdot, \frac{\delta}{\varepsilon}) = 1. \end{cases} \tag{3.1}$$

Let

$$k(\varepsilon z) = \sum_{i=1}^{n-1} k_i(\varepsilon z), \qquad\qquad I_\varepsilon = [0, \frac{\delta}{\varepsilon}].$$

We define a norm

$$\|h\|_* = \sup_{\overline{z} \in \partial\Omega,\ x_n \in I_\varepsilon} |e^{\sigma(x_n - \frac{1}{2\sqrt{2}} \log \frac{1}{\varepsilon})_+} h(\overline{z}, x_n)| \tag{3.2}$$

where $\sigma > 0$ is a suitable small number (to be fixed later) and $x_+ = \max(x, 0)$. Similarly, for a positive integer l we set

$$\|h\|_{*,l} = \sup_{|\alpha| \leq l} \sup_{\overline{z} \in \partial\Omega,\ x_n \in I_\varepsilon} |e^{\sigma(x_n - \frac{1}{2\sqrt{2}} \log \frac{1}{\varepsilon})_+} D^\alpha_{\overline{z}} h(z, x_n)|, \tag{3.3}$$

where α stands for a multi-index. In this section we prove the following theorem.

Theorem 3.1. *If σ is sufficiently small, then for every $l \in \mathbb{N}$ and for any constant C the following property holds. Let $h(\overline{z}, x_n)$ be such that*

$$\|h\|_{*,l} < C. \tag{3.4}$$

Then there exists $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$, $z \in \partial\Omega$ and h satisfying (3.4), there exists a unique solution $u_\varepsilon(\overline{z}, x_n; h)$ to the problem

$$\begin{cases} u_{nn} - \sum_{i=1}^{n-1} \frac{k_i(\overline{z})}{1-\varepsilon x_n k_i(\overline{z})} \varepsilon u_n + u - u^3 = \varepsilon^2 h(\overline{z}, x_n), x_n \in I_\varepsilon, \\ u_n(0) = 0, \quad u(\frac{\delta}{\varepsilon}) = 1 \end{cases} \tag{3.5}$$

which satisfies

$$u_\varepsilon(\overline{z}, x_n; h) = \overline{u}_\varepsilon(\overline{z}, x_n) + O(\varepsilon^{\frac{3}{2}}) \tag{3.6}$$

in the $\|\cdot\|_*$ norm, where

$$\overline{u}_\varepsilon(\overline{z}, x_n) = H(x_n - \frac{1}{2\sqrt{2}} \log \frac{1}{\varepsilon} - R_\varepsilon(\overline{z})) + \frac{1}{\sqrt{2}} H'(-\frac{1}{2\sqrt{2}} \log \frac{1}{\varepsilon} - R_\varepsilon(\overline{z})) e^{-\sqrt{2} x_n}$$

$$+\varepsilon \left[\tilde{\phi}_0(x_n - \frac{1}{2\sqrt{2}} \log \frac{1}{\varepsilon} - R_\varepsilon(\overline{z})) + \gamma_0 H(x_n - \frac{1}{2\sqrt{2}} \log \frac{1}{\varepsilon} - R_\varepsilon(\overline{z})) e^{-2\sqrt{2} R_\varepsilon(\overline{z})} e^{-\sqrt{2} x_n} \right].$$

Here $R_\varepsilon(\overline{z})$ satisfies

$$R_\varepsilon(\overline{z}) = \gamma_1 \log \frac{1}{k(\overline{z})} + \gamma_2 + \gamma_3 \sqrt{\varepsilon} + O(\varepsilon), \tag{3.7}$$

and $\tilde{\phi}_0 \in H^1(\mathbb{R})$ *is the unique solution of*

$$\phi_{nn} + (1-3H^2)\phi - k(\overline{z})(H' + 3\gamma_4(H^2-1)e^{-\sqrt{2}y}) = 0, \qquad \int_{\mathbb{R}} \phi(1-H^2)H' = 0, \tag{3.8}$$

(here γ_4 *is the unique number such that (3.8) is solvable, see Lemma 2.1), and* $\gamma_i, i = 1,2,3$ *are universal constants.*

Moreover (for some constant C *depending on* Ω *and* l*), we have*

$$\|u_\varepsilon(\overline{z}, x_n; h)\|_{*,l} \le C \tag{3.9}$$

and if h_1, h_2 *satisfy* (3.4) *then*

$$\|u_\varepsilon(\overline{z}, x_n; h_1) - u_\varepsilon(\overline{z}, x_n; h_2)\|_{*,l} \le C\varepsilon \|h_1 - h_2\|_{*,l}. \tag{3.10}$$

Once we have Theorem 3.1, we can prove the main result of this section, concerning existence of approximate solutions to (2.7).

Theorem 3.2. *For each fixed integer* $K \ge 3$*, there exists an approximate solution* u_ε^K *satisfying (3.6) and*

$$\|u_{nn}^K - \sum_{i=1}^{n-1} \frac{k_i(\varepsilon z)}{1-\varepsilon x_n k_i(\varepsilon z)} \varepsilon u_n^K + \varepsilon^2 \Delta_{\partial\Omega_{\varepsilon x_n}} u^K + u^K - (u^K)^3\|_{*,2} \le C\varepsilon^K, \tag{3.11}$$

where $\Delta_{\partial\Omega_{\varepsilon x_n}}$ *is the operator defined in the previous section.*

Proof: Theorem 3.2 is proved by the following iteration:

$$u^0(\overline{z}, x_n) = u_\varepsilon(\overline{z}, x_n; 0), \ \ h_0 = 0;$$

$$u^k(\overline{z}, x_n) = u_\varepsilon(\overline{z}, x_n; h_{k-1}), \ \ h_{k-1} = -\Delta_{\partial\Omega_{\varepsilon x_n}} u^{k-1},$$

where $k = 1, ..., K-2$.

Let us consider $K = 3$ case first. Observe that u^0 satisfies

$$u_{nn}^0 - \sum_{i=1}^{n-1} \frac{k_i(\varepsilon z)}{1-\varepsilon x_n k_i(\varepsilon z)} \varepsilon u_n^0 + u^0 - (u^0)^3 = 0,$$

while u^1 solves

$$u_{nn}^1 - \sum_{i=1}^{n-1} \frac{k_i(\varepsilon z)}{1-\varepsilon x_n k_i(\varepsilon z)} \varepsilon u_n^1 + u^1 - (u^1)^3 + \varepsilon^2 \Delta_{\partial\Omega_{\varepsilon x_n}} u^0 = 0.$$

By (3.9), for any $l \in \mathbb{N}$ we have

$$\|u^0\|_{*,l} \le C,$$

and by (3.10)

$$\|(u^1 - u^0)\|_{*,l-2} \le C\varepsilon$$

which then implies that u^1 satisfies

$$\|u^1_{nn} - \sum_{i=1}^{n-1} \frac{k_i(\varepsilon z)}{1 - \varepsilon x_n k_i(\varepsilon z)} \varepsilon u^1_n + \varepsilon^2 \Delta_{\partial \Omega_{\varepsilon x_n}} u^1 + u^1 - (u^1)^3\|_{*, l-4} \leq C \varepsilon^3.$$

For $K > 3$ (choosing l in the initial step sufficiently large depending on K), we can prove (3.11) using an induction argument.

□

Remark 3.3. *The approximate solution u^K_ε constructed in Theorem 3.2 is actually unique (since the solution in Theorem 3.1 is unique), and smooth in ε.*

In the rest of this section we prove Theorem 3.1: we begin with a series of Lemmas, the first of which establishes the existence of u_ε.

Lemma 3.4. *For ε sufficiently small, there exists a solution $u_\varepsilon(\overline{z}, x_n; h)$ to (3.5) satisfying (3.6).*

Proof: the proof of the existence follows from a standard Lyapunov-Schmidt reduction method.

First, we choose an approximate solution. For a large C, let $R \in [\frac{1}{C}, C]$ be a fixed number and let $\chi(t)$ be a cut-off function such that $\chi(t) = 1$ for $t < \frac{\delta}{8}$, $\chi(t) = 0$ for $t > \frac{\delta}{2}$. We set

$$H_{\varepsilon, R}(x_n) := \Big(H(x_n - \frac{1}{2\sqrt{2}} \log \frac{1}{\varepsilon} - R) + \rho_{\varepsilon, R} e^{-\sqrt{2} x_n} \Big) \chi(\varepsilon x_n) + 1 - \chi(\varepsilon x_n), \tag{3.12}$$

where

$$\rho_{\varepsilon, R} = \frac{1}{\sqrt{2}} H^{'}(-\frac{1}{2\sqrt{2}} \log \frac{1}{\varepsilon} - R). \tag{3.13}$$

Observe that by (2.2)

$$\rho_{\varepsilon, R} = A_0 \sqrt{\varepsilon} e^{-\sqrt{2} R} + O(\varepsilon e^{-2\sqrt{2} R}). \tag{3.14}$$

Then we define an operator

$$\mathbb{S}[u] := u_{nn} - \sum_{i=1}^{n-1} \frac{k_i(\overline{z})}{1 - \varepsilon x_n k_i(\overline{z})} \varepsilon u_n + u - u^3 - \varepsilon^2 h(\overline{z}, x_n). \tag{3.15}$$

It is easy to show that

$$\mathbb{S}[H_{\varepsilon, R}] = -\varepsilon k(\overline{z}) H^{'}_{\varepsilon, R} + 3(1 - H^2) \rho_{\varepsilon, R} e^{-\sqrt{2} x_n} - \varepsilon^2 h(\overline{z}, x_n) \tag{3.16}$$

$$-\varepsilon^2 x_n \sum_{i=1}^{n-1} k_i^2(\overline{z}) H^{'}_{\varepsilon, R} + f^{''}(H) \frac{1}{2} (\rho_{\varepsilon, R})^2 e^{-2\sqrt{2} x_n} + o(\varepsilon^2) e^{-\sqrt{2}(x_n - 1/(2\sqrt{2}) \log 1/\varepsilon)_+}.$$

Since $(1-H^2)e^{-\sqrt{2}x_n} = O(\sqrt{\varepsilon}(1-H^2)e^{-\sqrt{2}y})$, where $y = x_n - \frac{1}{2\sqrt{2}}\log\frac{1}{\varepsilon} - R$, we see that if $\sigma < \sqrt{2}$ in the definition of $\|\cdot\|_*$ then

$$\|\mathbb{S}[H_{\varepsilon,R}]\|_* \leq C\varepsilon. \tag{3.17}$$

Let

$$h_{\varepsilon,R} = H'_{\varepsilon,R} + \frac{1}{\sqrt{2}}H''_{\varepsilon,R}(0)e^{-\sqrt{2}x_n}, \quad Z_{\varepsilon,R} = 3(1-H^2_{\varepsilon,R})H'_{\varepsilon,R}.$$

We divide now the proof of the lemma into two steps.

Step 1: *there exists a unique solution $\phi_{\varepsilon,R}$ of*

$$\mathbb{S}[H_{\varepsilon,R}+\phi_{\varepsilon,R}] = c_{\varepsilon,R}Z_{\varepsilon,R}, \qquad \int_{I_\varepsilon}\phi_{\varepsilon,R}Z_{\varepsilon,R} = 0 \tag{3.18}$$

for some constant $c_{\varepsilon,R}$ (and with the same boundary conditions as in (3.5)*). Moreover, $\phi_{\varepsilon,R}$ is unique, differentiable in z and satisfies*

$$e^{\sigma(x_n - \frac{1}{2\sqrt{2}}\log\frac{1}{\varepsilon})_+}|\phi_{\varepsilon,R}| \leq C\varepsilon. \tag{3.19}$$

The proof is based on the contraction mapping theorem. Since the arguments are quite standard, see e.g. [16], we postpone the details to the appendix.

Step 2: *we can choose R such that $c_{\varepsilon,R} = 0$.* Multiplying (3.18) by $h_{\varepsilon,R}$ and integrating over I_ε, from a Taylor expansion we obtain

$$c_{\varepsilon,R}\int_{I_\varepsilon} Z_{\varepsilon,R}h_{\varepsilon,R} = \int_{I_\varepsilon}\mathbb{S}[H_{\varepsilon,R}]h_{\varepsilon,R} + \int_{I_\varepsilon}\left[\phi_{\varepsilon,R,nn} + (1-3H^2_{\varepsilon,R})\phi_{\varepsilon,R}\right]h_{\varepsilon,R} + O(\varepsilon^2|\log\varepsilon|), \tag{3.20}$$

and we have

$$\int_{I_\varepsilon}\left[\phi_{\varepsilon,R,nn} + (1-3H^2_{\varepsilon,R})\phi_{\varepsilon,R}\right]h_{\varepsilon,R} = \int_{I_\varepsilon}\left[h_{\varepsilon,R,nn} + (1-3H^2_{\varepsilon,R})h_{\varepsilon,R}\right]\phi_{\varepsilon,R} + O(\varepsilon) = O(\varepsilon^2).$$

The left hand side of (3.20) can be estimated as

$$c_{\varepsilon,R}\int_{I_\varepsilon} Z_{\varepsilon,R}h_{\varepsilon,R} = c_{\varepsilon,R}(\int_{\mathbb{R}} 3(1-H^2)(H')^2 + O(\sqrt{\varepsilon})),$$

while for the first term in the right-hand side we can use (3.16) to obtain

$$\int_{I_\varepsilon}\mathbb{S}[H_{\varepsilon,R}]h_{\varepsilon,R} = \varepsilon k(\varepsilon z)\int_{I_\varepsilon} H'_{\varepsilon,R}h_{\varepsilon,R} + \rho_{\varepsilon,R}\int_{I_\varepsilon} 3(1-H^2)h_{\varepsilon,R}e^{-\sqrt{2}x_n} + O(\varepsilon^2|\log\varepsilon|).$$

Note that

$$\int_{I_\varepsilon} H'_{\varepsilon,R}h_{\varepsilon,R} = \int_{\mathbb{R}}(H')^2 + O(\varepsilon) = c_0 + O(\varepsilon),$$

$$\rho_{\varepsilon,R}\int_{I_\varepsilon} 3(1-H^2)h_{\varepsilon,R}e^{-\sqrt{2}x_n} = \sqrt{\varepsilon}\rho_{\varepsilon,R}e^{-\sqrt{2}R}\int_{\mathbb{R}} 3(1-H^2)H'e^{-\sqrt{2}y}dy + c_2\varepsilon^{3/2} + O(\varepsilon^2|\log\varepsilon|)$$

$$= c_1 \varepsilon e^{-2\sqrt{2}R} + c_2 \varepsilon^{3/2} + O(\varepsilon^2),$$

where by (2.3)

$$c_1 = 3A_0 \int_{\mathbb{R}} (1 - H^2) H' e^{-\sqrt{2}y} = 6\sqrt{2} A_0^2,$$

and where c_2 is a fixed constant (independent of h).

In conclusion we have

$$\int_{I_\varepsilon} \mathbb{S}[H_{\varepsilon,R}] H'_{\varepsilon,R} = \varepsilon c_0 k(\overline{z}) - \varepsilon c_1 e^{-2\sqrt{2}R} + c_2 \varepsilon^{3/2} + O(\varepsilon^2 |\log \varepsilon|), \tag{3.21}$$

and hence we derive that $c_{\varepsilon,R} = 0$ if and only if the following holds

$$c_1 e^{-2\sqrt{2}R} = c_0 k(\varepsilon z) + c_2 \sqrt{\varepsilon} + O(\varepsilon). \tag{3.22}$$

The latter equation has clearly a unique solution $R = R_\varepsilon(z; h)$ satisfying (3.7).

To show that u_ε has the expansion (3.6), we use the equation satisfied by $\phi_\varepsilon = \phi_{\varepsilon,R}$. Letting $\phi_\varepsilon = \varepsilon \hat{\phi}_\varepsilon$, then $\hat{\phi}_\varepsilon$ solves

$$\hat{\phi}_{\varepsilon,nn} + f'(H_{\varepsilon,R}) \hat{\phi}_\varepsilon - k(\varepsilon z) H'_{\varepsilon,R} + 3(1 - H^2_{\varepsilon,R}) \frac{\rho_{\varepsilon,R}}{\varepsilon} e^{-\sqrt{2}x_n} - 6 H_{\varepsilon,R} \frac{\rho^2_{\varepsilon,R}}{\varepsilon} e^{-2\sqrt{2}x_n} = F_{1,\varepsilon},$$

with

$$\|F_{1,\varepsilon}\|_* = O(\sqrt{\varepsilon}); \qquad \int_{I_\varepsilon} Z_{\varepsilon,R} \hat{\phi}_\varepsilon = 0.$$

We rewrite the above equation in the following way, recalling (3.14)

$$\hat{\phi}_{\varepsilon,nn} + f'(H_{\varepsilon,R}) \hat{\phi}_\varepsilon - k(\varepsilon z) H'_{\varepsilon,R} + 3(1 - H^2_{\varepsilon,R})(1 + o(1)) A_0 e^{-\sqrt{2}\left(x_n + R - \frac{1}{2\sqrt{2}\log\frac{1}{\varepsilon}}\right)}$$
$$- 6 H_{\varepsilon,R} A_0^2 (1 + o(1)) e^{-2\sqrt{2}R} e^{-2\sqrt{2}x_n} = F_{1,\varepsilon}.$$

The last term on the left-hand side needs to be taken care of since, as $\varepsilon \to 0$, it is not uniformly bounded in $L^2(I_\varepsilon)$. To treat it, we can add the expression $\gamma_3 \rho^2_{\varepsilon,R} H e^{-2\sqrt{2}x_n}$ to $\hat{\phi}^\varepsilon$ so that we get indeed a localized error term with uniformly bounded L^2 norm. Precisely, we can write

$$\hat{\phi}_\varepsilon = \gamma_3 \rho^2_{\varepsilon,R} H e^{-2\sqrt{2}x_n} - c_{3,\varepsilon} \rho^2_{\varepsilon,R} H'_{\varepsilon,R} + \sqrt{\varepsilon} c_{4,\varepsilon} e^{-\frac{1}{1-x_n}} + c_{5,\varepsilon} e^{-\frac{1}{1-(\delta/\varepsilon - x_n)}} + \hat{\phi}_{\varepsilon,1},$$

where $c_{3,\varepsilon}, c_{4,\varepsilon}$ are some real constants which converge to some $c_3, c_4 \in \mathbb{R}$ as $\varepsilon \to 0$, $c_{5,\varepsilon} \to 0$, and where $\hat{\phi}_{\varepsilon,1}$ satisfies the conditions

$$\int_{I_\varepsilon} Z_{\varepsilon,R_\varepsilon} \hat{\phi}_{\varepsilon,1} = 0; \qquad (\hat{\phi}_{\varepsilon,1})_n(0) = \hat{\phi}_{\varepsilon,1}\left(\frac{\delta}{\varepsilon}\right) = 0.$$

It is easy to see that $\hat{\phi}_{\varepsilon,1}$ solves

$$\hat{\phi}_{\varepsilon,1,nn} + f'(H_{\varepsilon,R}) \hat{\phi}_{\varepsilon,1} - k(\overline{z}) H'_{\varepsilon,R} + 3(1 - H^2_{\varepsilon,R}) \frac{\rho_{\varepsilon,R}}{\varepsilon} e^{-\sqrt{2}x_n} + 6(H^3_{\varepsilon,R} - H_{\varepsilon,R}) \frac{(\rho_{\varepsilon,R})^2}{\varepsilon} e^{-2\sqrt{2}x_n} = F_{2,\varepsilon},$$

with

$$\|F_{2,\varepsilon}\|_* = O(\sqrt{\varepsilon}); \qquad \int_{I_\varepsilon} Z_{\varepsilon,R}\hat{\phi}_{\varepsilon,1} = 0.$$

Letting $\varepsilon \to 0$ and using the expression of $H_{\varepsilon,R}$, we deduce that $\hat{\phi}_{\varepsilon,1} = \tilde{\phi}_0 + O(\sqrt{\varepsilon})$ in the $\|\cdot\|_*$ norm, where $\tilde{\phi}_0$ satisfies

$$\tilde{\phi}_{0,nn} + f'(H)\tilde{\phi}_0 - k(\overline{z})(H' + 3\gamma_4(1-H^2)e^{-\sqrt{2}y}) = 0, \qquad \int_{\mathbb{R}} \tilde{\phi}_0(1-H^2)H' = 0, \tag{3.23}$$

and where $\gamma_4 \in \mathbb{R}$ is such that $\int_{\mathbb{R}} Z(H' + 3\gamma_4(1-H^2)e^{-\sqrt{2}y}) = 0$. This concludes the proof. □

Let

$$R_0(\overline{z}) = \gamma_1 \log\frac{1}{k(\overline{z})} + \gamma_2 + \gamma_3\sqrt{\varepsilon}, \qquad H_0(x_n) = H(x_n - \frac{1}{2\sqrt{2}}\log\frac{1}{\varepsilon} - R_0(\overline{z})),$$

$$h_0(x_n) = H_0' + \frac{1}{\sqrt{2}}H_0''(0)e^{-\sqrt{2}x_n}. \tag{3.24}$$

Note that

$$R_\varepsilon(\overline{z}) = R_0(\overline{z}) + O(\varepsilon).$$

Our next lemma shows the uniqueness of u_ε.

Lemma 3.5. *The solution constructed in Lemma 3.4 is unique. Moreover, letting $\mathbb{L}$ denote the linearized operator at u_ε*

$$\mathbb{L}\phi = \phi_{nn} - \sum_{i=1}^{n-1}\frac{k_i(\overline{z})}{1-\varepsilon x_n k_i(\overline{z})}\varepsilon\phi_n + (1-3u_\varepsilon^2)\phi,$$

we have

$$\|\phi\|_* \le \frac{C}{\varepsilon}\|\mathbb{L}\phi\|_*. \tag{3.25}$$

More precisely, if

$$\mathbb{L}\phi = f, \tag{3.26}$$

with the boundary conditions $\phi_n(0) = 0$ and $\phi(\frac{\delta}{\varepsilon}) = 0$, then we have

$$\phi = ch_0 + \phi^\perp, \qquad \text{where } |c| = \frac{1}{\varepsilon}O\left(\left|\int_{I_\varepsilon} fh_0\right|\right) \text{ and } \|\phi^\perp\|_* = O(\|f\|_*). \tag{3.27}$$

Furthermore, the eigenvalues for the following problem

$$\mathbb{L}\phi_0 + \lambda_\varepsilon\phi_0 = 0 \tag{3.28}$$

satisfy

$$\lambda_1^\varepsilon = -\varepsilon\gamma_5 k(\overline{z}) + O(\varepsilon^{3/2}), \qquad \lambda_2^\varepsilon \ge \gamma_6 > 0 \tag{3.29}$$

for some positive constants $\gamma_5, \gamma_6 > 0$.

Proof: we first show (3.29). Let $(\phi_0, \lambda_\varepsilon)$ satisfy (3.28): by Lemma 2.1 it is easy to see that either $\lambda_\varepsilon \to 0$, or $\lambda_\varepsilon \geq \gamma_6 > 0$. We discuss the first case (the following argument will be used repeatedly) decomposing ϕ_0 as

$$\phi_0 = c_\varepsilon h_0 + \phi_0^\perp, \qquad \int_{I_\varepsilon} \phi^\perp h_0 = 0. \tag{3.30}$$

Then $\phi_0^\perp$ satisfies

$$\mathbb{L}\phi_0^\perp + \lambda_\varepsilon(\phi_0^\perp) = -c_\varepsilon \mathbb{L} h_0 - c_\varepsilon \lambda_\varepsilon h_0, \tag{3.31}$$

where

$$\mathbb{L} h_0 = 3(H^2 - u_\varepsilon^2)H_0' + \frac{3}{\sqrt{2}}(1 - u_\varepsilon^2)H_0''(0)e^{-\sqrt{2}x_n} - \varepsilon k(\overline{z})h_0' + O(\varepsilon^{3/2}) \tag{3.32}$$

(in the $\|\cdot\|_*$ norm). Since $\lambda_\varepsilon \to 0$ and $\int_{I_\varepsilon} \phi_0^\perp h_0 = 0$, from Lemma 2.1 we derive that

$$\|\phi_0^\perp\|_* \leq C|c_\varepsilon|(\varepsilon + |\lambda_\varepsilon|). \tag{3.33}$$

Now multiplying (3.28) by h_0 and integrating over I_ε, we obtain

$$\int_{I_\varepsilon} (\mathbb{L}\phi_0^\perp)h_0 = c_\varepsilon \left[\int_{I_\varepsilon} (-\mathbb{L}h_0)h_0 - \lambda_\varepsilon \int_{I_\varepsilon} h_0^2 \right]. \tag{3.34}$$

By (3.33) and some computation, the left hand side of (3.34) can be estimated as

$$\begin{aligned}\int_{I_\varepsilon} (\mathbb{L}\phi_0^\perp)h_0 &= \int_{I_\varepsilon} (h_0'' + (1 - 3u_\varepsilon^2)h_0)\phi_0^\perp + O(\sqrt{\varepsilon}c_\varepsilon(\varepsilon + |\lambda_\varepsilon|)) \\ &= \int_{I_\varepsilon} 3(H_0^2 - u_\varepsilon^2)\phi^\perp + O(\sqrt{\varepsilon}c_\varepsilon(\varepsilon + |\lambda_\varepsilon|)) = O(\sqrt{\varepsilon}c_\varepsilon(\varepsilon + |\lambda_\varepsilon|)),\end{aligned}$$

while for the right-hand side we have

$$c_\varepsilon \left[\int_{I_\varepsilon} (-\mathbb{L}h_0)h_0 - \lambda_\varepsilon \int_{I_\varepsilon} h_0^2 \right]$$

$$= c_\varepsilon \left[- \int_{I_\varepsilon} 3(H_0^2 - u_\varepsilon^2)(H_0')^2 - \frac{3}{\sqrt{2}} H_0''(0) \int_{I_\varepsilon} (1 - u_\varepsilon^2)H_0' e^{-\sqrt{2}x_n} + \lambda_\varepsilon \int_{I_\varepsilon} h_0^2 + O(\varepsilon^{3/2}) \right].$$

Notice that

$$\begin{aligned}\frac{3}{\sqrt{2}} H''(0) \int_{I_\varepsilon} (1 - u_\varepsilon^2)H_0' e^{-\sqrt{2}x_n} &= \sqrt{2}A_0 \varepsilon e^{-2\sqrt{2}R_0} \int_{\mathbb{R}} 3(1 - H^2)H' e^{-\sqrt{2}y} + O(\varepsilon^{3/2}) \\ &= 4\sqrt{2}A_0^2 \varepsilon e^{-2\sqrt{2}R_0} + O(\varepsilon^{3/2}),\end{aligned} \tag{3.35}$$

and that

$$\int_{I_\varepsilon} h_0^2 = \int_{\mathbb{R}} (H')^2 + O(\varepsilon).$$

Recall that

$$u_\varepsilon = H_{\varepsilon,R_\varepsilon} + \phi_\varepsilon = H_0 + H_{\varepsilon,R_\varepsilon} - H_0 + \phi_\varepsilon$$

$$= H_0 + H_0^{'}(R_0 - R_\varepsilon) + \frac{1}{\sqrt{2}}\rho_{\varepsilon,R_\varepsilon} e^{-\sqrt{2}x_n} + \phi_\varepsilon + O(\varepsilon^2|\log\varepsilon|) \tag{3.36}$$

in the $\|\cdot\|_*$ norm. Therefore it follows that

$$\int_{I_\varepsilon} 3(H_0^2 - u_\varepsilon^2)(H_0^{'})^2 = \int_{I_\varepsilon} (-6H_0)(H_0^{'})^2(H_{\varepsilon,R_\varepsilon} - H_0 + \phi_\varepsilon) + O(\varepsilon^{3/2})$$

$$= \int_{I_\varepsilon} (-6H_0)(H_0^{'})^2(H_{\varepsilon,R_\varepsilon} - H_0) - \int_{I_\varepsilon} (H_0^{''''} + f^{'}(H_0)H_0^{''})(\phi_\varepsilon) + O(\varepsilon^{3/2})$$

$$= \sqrt{\varepsilon} e^{-\sqrt{2}R_\varepsilon}\rho_{\varepsilon,R_\varepsilon} \int_{\mathbb{R}} (-6H)(H^{'})^2 e^{-\sqrt{2}y} - \int_{I_\varepsilon} (\mathbb{L}\phi_\varepsilon)H_0^{''} + O(\varepsilon^{3/2})$$

$$= \sqrt{\varepsilon} e^{-\sqrt{2}R_\varepsilon}\rho_{\varepsilon,R_\varepsilon} \int_{\mathbb{R}} (-6H)(H^{'})^2 e^{-\sqrt{2}y} + \int_{I_\varepsilon} S[H_{\varepsilon,R_\varepsilon}]H_0^{''} + O(\varepsilon^{3/2})$$

$$= \sqrt{\varepsilon} e^{-\sqrt{2}R_\varepsilon}\rho_{\varepsilon,R_\varepsilon} \int_{\mathbb{R}} ((-6H)(H^{'})^2 + 3(1-H^2)H^{''})e^{-\sqrt{2}y} + O(\varepsilon^{3/2}).$$

Using the fact that $H^{''''} - H^{''} + 3(1-H^2)H^{''} = 6H(H^{'})^2$ we have

$$\int_{I_\varepsilon} 3(H_0^2 - u_\varepsilon^2)(H_0^{'})^2 = 4\sqrt{2}A_0^2 \varepsilon e^{-\sqrt{2}R_0} + O(\varepsilon^{3/2}). \tag{3.37}$$

Combining (3.35) and (3.37), we obtain

$$\lambda_\varepsilon = -\frac{8\sqrt{2}A_0^2}{\int_{\mathbb{R}}(H^{'})^2}\varepsilon e^{-2\sqrt{2}R_0} + O(\varepsilon^{3/2}) = -\gamma_5 \varepsilon k(\overline{z}) + O(\varepsilon^{3/2}), \tag{3.38}$$

where $\gamma_5 > 0$, which proves (3.29). The proof of (3.27) follows from similar arguments. Finally, the uniqueness of u_ε can be deduced from (3.29), and this concludes the proof. □

As a consequence of Lemma 3.5, we deduce an improvement on the estimate of u_ε, concerning its regularity with respect to $\overline{z}$.

Lemma 3.6. *If $\|h\|_{*,l} \le C$ for some integer l, then*

$$\|u_\varepsilon(\overline{z}, x_n; h)\|_{*,l} \le C. \tag{3.39}$$

Proof: we consider the simplest case: $D_{\overline{z}}^\alpha = \frac{\partial}{\partial \overline{z}_1}$. Differentiating (3.5) with respect to $\overline{z}_1$, we obtain that $v = D_{\overline{z}_1}^\alpha u_\varepsilon(\overline{z}, x_n; h)$ satisfies

$$\mathbb{L}v + \varepsilon D_{\overline{z}_1} k(\overline{z})(u_\varepsilon)_n + O(\varepsilon^2) = 0$$

in the norm $\|\cdot\|_{*,l-1}$. By (3.29), (3.39) follows immediately. Similarly we have the estimates for the higher-order derivatives. □

We are interested next in the Lipschitz dependence of u_ε with respect to h.

Lemma 3.7. *If* $\|h_1\|_*, \|h_2\|_* \leq C$ *and if* $u_\varepsilon(\overline{z}, x_n; h_i)$ *are the corresponding solutions of* (3.5), *then the following estimate holds true*

$$\|u_\varepsilon(\overline{z}, x_n; h_1) - u_\varepsilon(\overline{z}, x_n; h_2)\|_* \leq C\varepsilon\|h_1 - h_2\|_*. \tag{3.40}$$

More precisely, following the notation in the proof of Lemma 3.5, we have the estimate

$$u_\varepsilon(\overline{z}, x_n; h_1) - u_\varepsilon(\overline{z}, x_n; h_2) = d_0 h_0 + \psi_0, \tag{3.41}$$

where

$$d_0 = d_0(z) = O(\varepsilon\|h_1 - h_2\|_*), \qquad \|\psi_0\|_* = O(\varepsilon^2\|h_1 - h_2\|_*). \tag{3.42}$$

Proof: let $\phi = u(\overline{z}, x_n; h_1) - u(\overline{z}, x_n; h_2)$. Then by the expansion in formula (3.6), since we control $R_\varepsilon(\overline{z})$ with a precision of order ε, we have $\|\phi\|_* = O(\varepsilon)$. It is easy to see that ϕ satisfies

$$\mathbb{L}^{(2)}\phi - 6u_\varepsilon(\overline{z}, x_n; h_2)\phi^2 + O(\|\phi\|_*^3) + \varepsilon^2(h_1 - h_2) = 0$$

in the $\|\cdot\|_*$ norm, where $\mathbb{L}^{(2)}\phi = \phi_{nn} - \varepsilon\sum_{i=1}^{n-1}\frac{k_i(\overline{z})}{1-\varepsilon x_n k_i(\overline{z})}\phi_n + (1 - 3u_\varepsilon(\overline{z}, x_n; h_2)^2)\phi$.

Decomposing ϕ as before

$$\phi = d_0 h_0 + \psi_0$$

and using (3.27), we see that

$$\|\psi_0\|_* = O(\varepsilon^2\|h_1 - h_2\|_*);$$
$$d_0 = \frac{1}{\varepsilon}O\left(\left|\int_{I_\varepsilon} u_\varepsilon(dh_0 + \psi)^2 h_0\right|\right) + O(\varepsilon\|h_1 - h_2\|_*).$$

Since $\int_{\mathbb{R}} H(H')^3 = 0$, the same argument as in the proof of Lemma 3.5 gives (3.42). $\square$

Lemma 3.8. *If* h_1, h_2 *and* $u_\varepsilon(\overline{z}, x_n; h_i)$ *are as in the previous lemma, then we have*

$$\|u(\overline{z}, x_n; h_1) - u(\overline{z}, x_n; h_2))\|_{*,l} \leq C\varepsilon\|h_1 - h_2\|_{*,l}. \tag{3.43}$$

Moreover, for any multi-index α *with* $|\alpha| \leq l$ *we have*

$$D^\alpha_{\overline{z}}(u_\varepsilon(\overline{z}, x_n; h_1) - u_\varepsilon(\overline{z}, x_n; h_2)) = d_\alpha h_0 + \psi_\alpha, \tag{3.44}$$

where

$$d_\alpha = d_\alpha(\overline{z}) = O(\varepsilon\|h_1 - h_2\|_{*,l}), \qquad \psi_\alpha = O(\varepsilon^2\|h_1 - h_2\|_{*,l}). \tag{3.45}$$

Proof: as before, let $\phi = u_\varepsilon(\overline{z}, x_n; h_1) - u_\varepsilon(\overline{z}, x_n; h_2)$. Then $D_{\overline{z}}\phi$ satisfies

$$\mathbb{L}^{(2)}D_{\overline{z}}\phi - (\varepsilon D_{\overline{z}}k + O(\varepsilon^2 x_n^2))\phi_n - 6\left(u_\varepsilon^{h_2}\phi D_{\overline{z}}\phi + 6u_\varepsilon^{h_2}\phi D_{\overline{z}}u_\varepsilon^{h_2} + \phi^2 D_{\overline{z}}u_\varepsilon^{h_2}\right)$$
$$+O(\|\phi\|_*^2)\phi + \varepsilon^2 D_{\overline{z}}(h_1 - h_2) = 0. \tag{3.46}$$

As before, we decompose $D_{\overline{z}}\phi$ as

$$D_{\overline{z}}\phi = d_1 h_0 + \psi_1.$$

Then using (3.41) (noting that $\int_{\mathbb{R}} H(H')^3 = 0$), the same technique as in Lemma 3.5 gives (3.42). By induction in the length of α, we obtain the desired estimate. □

Proof of Theorem 3.1. If follows immediately from Lemmas 3.4 to 3.8. □

Finally, we analyze the dependence of u_ε^K in ε: it is convenient first to scale the functions u_ε^K to Ω defining $\overline{u}_\varepsilon^K(\varepsilon x) = u_\varepsilon^K(x)$. Then, setting $v_\varepsilon^K(x) = \frac{\partial \overline{u}_\varepsilon^K}{\partial \varepsilon}(\varepsilon x)$, it is easy to see that for $K \geq 2$, v_ε^K satisfies

$$(v_\varepsilon^K)_{nn} - \sum_{i=1}^{n-1} \frac{k_i(\overline{z})}{1-\varepsilon x_n k_i(\overline{z})} \varepsilon (v_\varepsilon^K)_n + (1-3(u_\varepsilon^K)^2) v_\varepsilon^K + \frac{2}{\varepsilon}((u_\varepsilon^K)^3 - u_\varepsilon^K) = O(\varepsilon^2) \tag{3.47}$$

in the $\|\cdot\|_*$ norm. The same argument as in Lemma 3.5 gives the following asymptotic expansion in ε of v_ε^K.

Lemma 3.9. *For $K \geq 2$, and using the notation in Theorem 3.1, one has*

$$v_\varepsilon^K(\overline{z}, x_n) = \frac{\partial}{\partial \varepsilon} \overline{u}_\varepsilon(\overline{z}, x_n) + o(1) \qquad \text{as } \varepsilon \to 0$$

in the $\|\cdot\|_*$ norm.

Next we differentiate (3.47) with respect to x_n and let $\Phi_\varepsilon^K = \frac{\partial}{\partial x_n}(\frac{\partial u_\varepsilon^K}{\partial \varepsilon})$. Then Φ_ε^K satisfies

$$(\Phi_\varepsilon^K)_{nn} - \sum_{i=1}^{n-1} \frac{k_i(\varepsilon x')}{1-\varepsilon x_n k_i(\varepsilon x')} \varepsilon (\Phi_\varepsilon^K)_n + (1-3(u_\varepsilon^K)^2)\Phi_\varepsilon^K - 6 u_\varepsilon^K v_\varepsilon^K \frac{\partial u_\varepsilon^K}{\partial x_n} + \frac{2}{\varepsilon}(3(u_\varepsilon^K)^2 - 1)\frac{\partial u_\varepsilon^K}{\partial x_n} = o(\varepsilon) \tag{3.48}$$

in the $\|\cdot\|_*$ norm. This formula will be used for applying Kato's theorem in Subsection 4.3 below.

Remark 3.10. *The eigenvalue estimates in Lemma 3.5 also hold when we replace u_ε by u_ε^K. Furthermore, it is possible to prove that the eigenfunction ϕ_0 in* (3.28) *satisfies regularity estimates (in z) similar to those in* (3.9).

4. Proof of the main theorem

In this section we prove Theorem 1.1. As explained in the introduction, we are going to use the contraction mapping theorem: the main difficulty in the present case is that the problem is highly resonant, since many eigenvalues of the linearized equation are close to zero for ε tending to zero. To tackle this problem we apply Kato's theorem, showing the differentiability of the small eigenvalues with respect to ε and proving that for many epsilon's the linearized operator at approximate solutions is invertible, although the inverse operator will be rather big in norm. Anyhow, if we choose approximate solutions with a sufficient accuracy, even with a large operator we will get a contraction in a suitably small set.

We divide the arguments into three different subsections. In the first we characterize the eigenfunctions of the linearized equation corresponding to small eigenvalues. In the second we compute the derivative of small eigenvalues with respect to ε, and finally in the third we perform the contraction argument.

4.1. **Characterization of some eigenfunctions of the linearized operator.** In this section we study the eigenfunctions of the operator

$$L_\varepsilon \phi := \mathbb{L}\phi + \Delta_{\partial\Omega_{\varepsilon x_n}}$$

(recall the notation of Section 2) corresponding to suitably small eigenvalues. The reason is that, in order to apply Theorem 2.2, it is necessary to consider the projection onto the eigenspace of σ_0. Precisely, the eigenvalues of $P_{\sigma_0} \circ \frac{\partial T}{\partial \chi}(0) \circ P_{\sigma_0}$ can be found using the Rayleigh quotient

$$Q(u) = \frac{(P_{\sigma_0} \circ \frac{\partial T}{\partial \chi}(0) \circ P_{\sigma_0} u, u)_X}{(u,u)_X}, \qquad u \in X_{\sigma_0}, u \neq 0.$$

We also notice that, multiplying the eigenvalue equation (2.16) by φ_j and integrating by parts, given any fixed constant γ one finds

$$\varepsilon^2 \int_{\partial\Omega} |\nabla_{\partial\Omega}\varphi_j|^2 - \varepsilon\gamma \int_{\partial\Omega} k\varphi_j^2 = \varepsilon^2\lambda_j - \varepsilon\gamma := \lambda_{j,\varepsilon}. \tag{4.49}$$

Lemma 4.1. *Suppose the function ϕ satisfies (see the notation in Section 2)*

$$L_\varepsilon\phi + \lambda k(\varepsilon z)\phi = 0; \qquad \|\phi\|_{L^2(\Gamma_\tau)} = 1, \tag{4.50}$$

(with Neumann boundary conditions at $x_n = 0$ and Dirichlet boundary conditions at $x_n = \varepsilon^{-\tau}$) with $\lambda = o(\varepsilon)$ as $\varepsilon \to 0$. Let us write

$$\phi = \varphi(z)\phi_0(z, x_n) + \phi^\perp,$$

where $\phi_0(z)$ is the first eigenfunction (normalized in $L^2([0,\varepsilon^{-\tau}])$ with respect to the volume form of g_ε) of $\mathbb{L}(\varepsilon z)$ and where $\phi^\perp$ satisfies

$$\int_{[0,\varepsilon^{-\tau}]} \phi^\perp(z,x_n)\phi_0(z,x_n)dV_g(x_n) = 0 \qquad \text{for every } z \in \partial\Omega_\varepsilon.$$

Then, as $\varepsilon \to 0$ (if τ is sufficiently small), writing $\varphi(z) = \sum_j \alpha_j\varphi_j(\varepsilon z)$, one has the following bound

$$\|\phi^\perp\|^2_{H^1(\Gamma_\tau)} \leq \frac{C}{\varepsilon^{n-1}} \sum_j \alpha_j^2 \left(\varepsilon^4 + \varepsilon^4 j^{\frac{2}{n-1}}\right), \tag{4.51}$$

for some fixed constant C.

Proof. First of all we notice that, by standard elliptic theory, the function ϕ has the same regularity as u_ε, and we can assume it is of class C^2. Then, we can compute the value of $\varphi(z)$ simply using the orthogonality condition of $\phi^\perp$ by the formula

$$\varphi(z) = \frac{1}{\int_{[0,\varepsilon^{-\tau}]} \phi_0^2(z, x_n) dV_g(x_n)} \int_{[0,\varepsilon^{-\tau}]} \phi(\varepsilon z, x_n)\phi_0(\varepsilon z, x_n) dV_g(x_n).$$

Therefore also φ is a function of class C^2: from this remark in particular it follows that the series in (4.51) is convergent.

Next, in order to find estimates on $\phi^\perp$, we multiply the eigenvalue equation in (4.50) by $\phi^\perp$ and integrate on Γ_τ. Since $L_\varepsilon = \mathbb{L} + \Delta_{\partial\Omega_{\varepsilon x_n}}$, from the uniform invertibility of $\mathbb{L}$ on $\phi^\perp$, see Lemma 3.5 (we are actually substituting $[0, \delta/\varepsilon]$ with $[0, \varepsilon^{-\tau}]$, but this not affects the eigenvalue estimates), we find that $\int_{\Gamma_\tau} \phi^\perp \mathbb{L} \phi^\perp dV_{g_\varepsilon} \leq -C^{-1}(\|\phi_n^\perp\|^2_{L^2(\Gamma_\tau)} + \|\phi^\perp\|^2_{L^2(\Gamma_\tau)})$. Moreover we find from (2.14) that

$$\int_{\partial\Omega_\varepsilon} \phi^\perp \Delta_{\partial\Omega_{\varepsilon x_n}} \phi^\perp dV_{g_\varepsilon} = (1 + O(\varepsilon x_n)) \int_{\Gamma_\tau} |\nabla_{\overline{g}_\varepsilon} \phi^\perp|^2 dV_{\overline{g}_\varepsilon}.$$

From these observations and (2.15) we derive that $\int_{\Gamma_\tau} \phi^\perp L_\varepsilon \phi^\perp dV_{g_\varepsilon} \leq -C^{-1}\|\phi^\perp\|^2_{H^1(\Gamma_\tau)}$, and therefore

$$\begin{aligned} \frac{1}{C}\|\phi^\perp\|^2_{H^1(\Gamma_\tau)} \leq & \left| \int_{\Gamma_\tau} \mathbb{L}\phi_0 \varphi \phi^\perp dV_{g_\varepsilon} + \int_{\Gamma_\tau} \phi_0 \phi^\perp \Delta_{\partial\Omega_{\varepsilon x_n}} \varphi \, dV_{g_\varepsilon} \right| + \left| \int_{\Gamma_\tau} \varphi \phi^\perp \Delta_{\partial\Omega_{\varepsilon x_n}} \phi_0 dV_{g_\varepsilon} \right| \\ + & \left| 2 \int_{\Gamma_\tau} \langle \nabla_{\partial\Omega_{\varepsilon x_n}} \varphi, \nabla_{\partial\Omega_{\varepsilon x_n}} \phi_0 \rangle \phi^\perp dV_{g_\varepsilon} \right| + C|\lambda| \|\phi^\perp\|^2_{L^2(\Gamma_\tau)}. \end{aligned}$$

From the orthogonality condition on $\phi^\perp$ and from the fact that ϕ_0 is an eigenfunction for $\mathbb{L}$ (up to a small error), the first term on the right-hand side vanishes. Concerning the next two, from the slow dependence of ϕ_0 on z, from (2.10) and (2.13) we get the following estimate (using also the smallness of λ)

$$\|\phi^\perp\|_{H^1(\Gamma_\tau)} \leq C\varepsilon^2 \|\varphi\|_{L^2(\partial\Omega_\varepsilon)} + C\varepsilon \|\nabla_{\partial\Omega_{\varepsilon x_n}} \varphi\|_{L^2(\partial\Omega_\varepsilon)}.$$

Using the decomposition of φ into the eigenmodes of (2.16), the asymptotic formula for λ_j and a change of variables we find

$$\int_{\partial\Omega_\varepsilon} \varphi(z)^2 dV_{\overline{g}_\varepsilon} \leq C \int_{\partial\Omega_\varepsilon} k(\overline{z}) \varphi(z)^2 dV_{\overline{g}_\varepsilon} \leq \frac{1}{\varepsilon^{n-1}} \sum_j \alpha_j^2;$$

$$\int_{\partial\Omega_\varepsilon} |\nabla_{\partial\Omega_{\varepsilon x_n}} \varphi(z)|^2 dV_{\overline{g}_\varepsilon} \leq C \frac{1}{\varepsilon^{n-1}} \sum_j \varepsilon^2 j^{\frac{2}{n-1}} \alpha_j^2.$$

By the last three formulas, the proof is concluded. □

Lemma 4.2. *Suppose the same assumptions of Lemma 4.1 hold. Then, as* $\varepsilon \to 0$ *one has* $\|\phi^\perp\|_{H^1(\Gamma_\tau)} = o(\varepsilon)$.

Proof. The eigenvalue equation in (4.50) can be written as

$$\begin{aligned} L_\varepsilon \phi &= \phi_0 \Delta_{\partial\Omega_{\varepsilon x_n}} \varphi(z) + \varphi(z) \mathbb{L}\phi_0 + \varphi(z) \Delta_{\partial\Omega_{\varepsilon x_n}} \phi_0 \\ &+ 2\langle \nabla_{\partial\Omega_{\varepsilon x_n}} \varphi(z), \nabla_{\partial\Omega_{\varepsilon x_n}} \phi_0 \rangle + L_\varepsilon \phi^\perp = -\lambda k(\varepsilon z)\phi^\perp - \lambda k(\varepsilon z)\varphi(z)\phi_0. \end{aligned}$$

Using the fact that $\mathbb{L}\phi_0 = (\varepsilon\gamma k(\varepsilon z) + O(\varepsilon^{\frac{3}{2}}))\phi_0$ (we have set $\gamma = \gamma_5$, see Lemma 3.5 and the comments in the previous proof), then we have

$$\begin{aligned} L_\varepsilon \phi &= \phi_0 (\Delta_{\partial\Omega_{\varepsilon x_n}} + (\varepsilon\gamma k(\varepsilon z) + O(\varepsilon^{\frac{3}{2}}))\varphi(z) + \varphi(z) \Delta_{\partial\Omega_{\varepsilon x_n}} \phi_0 \\ &+ 2\langle \nabla_{\partial\Omega_{\varepsilon x_n}} \varphi(z), \nabla_{\partial\Omega_{\varepsilon x_n}} \phi_0 \rangle + L_\varepsilon \phi^\perp = -\lambda k(\varepsilon z)\phi^\perp - \lambda k(\varepsilon z)\varphi(z)\phi_0. \end{aligned} \tag{4.52}$$

Writing still $\varphi(z) = \sum_j \alpha_j \varphi_j(\varepsilon z)$, we let j_ε (depending on ε) be the first integer j such that $\varepsilon^2\lambda_j > \varepsilon^{\frac{1}{2}}$. We multiply then the last equation by $\sum_{j\geq j_\varepsilon} \alpha_j\varphi_j\phi_0$. Using the orthogonality of $\phi^\perp$ to ϕ_0, integrating by parts (to deal with the first term in the second line) (2.14) and the self-adjointness of L_ε we get

$$\begin{aligned} \frac{1}{\varepsilon^{n-1}} \sum_{j\geq j_\varepsilon} \varepsilon^2 \alpha_j^2 \lambda_j &\leq C(\varepsilon^2 + |\lambda|) \left(\frac{1}{\varepsilon^{n-1}} \sum_{j\geq j_\varepsilon} \alpha_j^2 \right)^{\frac{1}{2}} \left(\frac{1}{\varepsilon^{n-1}} \sum_{j} \alpha_j^2 \right)^{\frac{1}{2}} \\ &+ C\varepsilon \left(\frac{1}{\varepsilon^{n-1}} \sum_{j} \alpha_j^2 \right)^{\frac{1}{2}} \left(\frac{1}{\varepsilon^{n-1}} \sum_{j\geq j_\varepsilon} \varepsilon^2\lambda_j \alpha_j^2 \right)^{\frac{1}{2}} \\ &+ \left| \int_{\Gamma_\tau} \phi^\perp L_\varepsilon (\sum_{j\geq j_\varepsilon} \alpha_j\varphi_j\phi_0) dV_{g_\varepsilon} \right|. \end{aligned}$$

The last term can be evaluated as in the proof of the previous lemma as

$$\begin{aligned} \left| \int_{\Gamma_\tau} \phi^\perp L_\varepsilon (\sum_{j\geq j_\varepsilon} \alpha_j\varphi_j\phi_0) dV_{g_\varepsilon} \right| &\leq C\varepsilon \|\nabla_{\partial\Omega_{\varepsilon x_n}} \varphi\|_{L^2(\Gamma_\tau)} \|\phi^\perp\|_{L^2(\Gamma_\tau)} \\ &\leq C\varepsilon \|\phi^\perp\|_{L^2(\Gamma_\tau)} \left(\frac{1}{\varepsilon^{n-1}} \sum_{j\geq j_\varepsilon} \varepsilon^2\lambda_j\alpha_j^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Hence from the last two formulas and from the fact that $\lambda_j \gg 1$ for $j \geq j_\varepsilon$ we get

$$\left(\frac{1}{\varepsilon}^{n-1} \sum_{j\geq j_\varepsilon} \varepsilon^2\lambda_j\alpha_j^2 \right)^{\frac{1}{2}} \leq C\varepsilon \left(\left(\frac{1}{\varepsilon^{n-1}} \sum_j \alpha_j^2 \right)^{\frac{1}{2}} + \|\phi^\perp\|_{L^2(\Gamma_\tau)} \right). \tag{4.53}$$

We also notice that by the L^2 normalization of ϕ one has

$$\frac{1}{\varepsilon^{n-1}} \sum_j \alpha_j^2 + \|\phi^\perp\|^2_{L^2(\Gamma_\tau)} \leq C.$$

Then from Lemma 4.1 (dividing the j's into $\{j < j_\varepsilon\}$ and $\{j \geq j_\varepsilon\}$), recalling our definition of j_ε and (4.53) we find

$$\|\phi^\perp\|_{H^1(\Gamma_\tau)} \leq C\varepsilon^2 + C\varepsilon^{\frac{5}{4}} + C\varepsilon\left(\frac{1}{\varepsilon^{n-1}}\sum_{j\geq j_\varepsilon}\varepsilon^2\lambda_j\alpha_j^2\right)^{\frac{1}{2}} \leq C\varepsilon^{\frac{5}{4}} + C\varepsilon^2(1+\|\phi^\perp\|_{H^1(\Gamma_\tau)}).$$

Bringing the term $\varepsilon^2\|\phi^\perp\|_{H^1(\Gamma_\tau)}$ on the left-hand side we obtain the desired conclusion. □

4.2. **Differentiating the small eigenvalues with respect to ε.** In this subsection we differentiate some (suitably small) eigenvalues of L_ε with respect to the parameter ε. As an application we will obtain the invertibility of L_ε for a quite large family of (small) epsilon's. Then, as in [15], Proposition 4.3 (see also [14], Proposition 7.3), using Kato's theorem one can prove the following result, see the notation at the end of Section 3.

Proposition 4.3. *The eigenvalues λ of the problem*

$$L_\varepsilon u + \lambda k(\overline{z})u = 0, \qquad \text{in } \Gamma_\tau \tag{4.54}$$

(with Neumann boundary conditions at $x_n = 0$ and Dirichlet boundary conditions at $x_n = \varepsilon^{-\tau}$) are differentiable with respect to ε, and they satisfy the following estimates

$$T^1_{\lambda,\varepsilon} \leq \frac{\partial\lambda}{\partial\varepsilon} \leq T^2_{\lambda,\varepsilon}, \tag{4.55}$$

where

$$T^1_{\lambda,\varepsilon} = \inf_{u\in H_\lambda, u\neq 0}\frac{\int_{\Gamma_\tau}\left(\frac{2}{\varepsilon}|\nabla_{g_\varepsilon}u|^2 + 6u_\varepsilon^K v_\varepsilon^K u^2\right)dV_{g_\varepsilon}}{\int_{\Gamma_\tau}k(\overline{z})u^2 dV_{g_\varepsilon}};$$

$$T^2_{\lambda,\varepsilon} = \sup_{u\in H_\lambda, u\neq 0}\frac{\int_{\Gamma_\tau}\left(\frac{2}{\varepsilon}|\nabla_{g_\varepsilon}u|^2 + 6u_\varepsilon^K v_\varepsilon^K u^2\right)dV_{g_\varepsilon}}{\int_{\Gamma_\tau}k(\overline{z})u^2 dV_{g_\varepsilon}}.$$

Here H_λ stands for the eigenspace for (4.54) *corresponding to the eigenvalue λ, while v_ε^K is defined at the end of Section 3.*

Remark 4.4. *Differently from* [14] *and* [15], *we are considering the eigenvalue problem in $L^2(\Gamma_\tau)$ (with weight k) and not in $H^1(\Gamma_\tau)$. This gives rise to two (non substantial) differences in the above statement compared to its counterparts in* [14] *and* [15]. *First of all, in the latter ones it is assumed that the initial eigenvalue is different from* 1 *(as in Theorem 2.2), since using the H^1 norm the operator is of the form $Identity - compact$. This is necessary for the eigenvalue in order to have finite multiplicity and to be isolated: if instead we work with the $L^2(\Gamma_\tau)$ norm these properties are always satisfies.*

In addition since in the functional J_ε ε^2 appears explicitly as factor in the Dirichlet norm, the formulas in [14] *and* [15] *contain an extra term multiplying $\frac{2}{\varepsilon}$, which is not present here.*

We next give a further characterization of some eigenfunctions of L_ε, in addition to the ones in Lemmas 4.1 and 4.2, concerning in particular the function φ.

Lemma 4.5. *Suppose the assumptions of Lemma 4.1 hold true, except that we now use the normalization* $\|\phi\|_{H^1(\Gamma_\tau)} = 1$. *Then, in the above notation, if* $|\lambda| = O(\varepsilon^2)$ *we have*

$$\frac{1}{\varepsilon^{n-1}} \sum_{|\lambda_{j,\varepsilon}| \geq \varepsilon^{\frac{5}{4}}} \alpha_j^2 \varphi_j(\varepsilon z) = o(1); \qquad \frac{1}{\varepsilon^{n-1}} \sum_{|\lambda_{j,\varepsilon}| \geq \varepsilon^{\frac{5}{4}}} |\lambda_{j,\varepsilon}| \alpha_j^2 \varphi_j(\varepsilon z) = o(\varepsilon).$$

Proof. We define the sets

$$A_{1,\varepsilon} = \left\{ j \in \mathbb{N} \, : \, \lambda_{j,\varepsilon} < -\varepsilon^{\frac{5}{4}} \right\}; \qquad A_{2,\varepsilon} = \left\{ j \in \mathbb{N} \, : \, \lambda_{j,\varepsilon} > \varepsilon^{\frac{5}{4}} \right\},$$

and the functions

$$\overline{\varphi}_1(z) = \sum_{j \in A_{1,\varepsilon}} \alpha_j \varphi_j(\varepsilon z); \qquad \overline{\varphi}_2(z) = \sum_{j \in A_{2,\varepsilon}} \alpha_j \varphi_j(\varepsilon z);$$

$$\phi_1 = \overline{\varphi}_1(z) \phi_0; \qquad \phi_2 = \overline{\varphi}_2(z) \phi_0.$$

As one can easily see from the (weighted) orthogonality of $\overline{\varphi}_1$ and $\overline{\varphi}_2$, $\|\phi_1\|_{H_1(\Gamma_\tau)}$, $\|\phi_2\|_{H_1(\Gamma_\tau)}$ and $\|\varphi\phi_0\|_{L^2(\Gamma_\tau)}$ stay uniformly bounded as ε tends to zero. We multiply next the equation in (4.50) by ϕ_1 and integrate: by the regularity of ϕ_0 with respect to z, see Remark 3.10 and Lemma 4.2 we deduce

$$\begin{aligned} O(\varepsilon^2) &= \int_{\Gamma_\tau} \phi_1 L_\varepsilon \phi dV_{g_\varepsilon} = \int_{\Gamma_\tau} (\varphi\phi_0 + \phi^\perp) L_\varepsilon \phi_1 = o(\varepsilon) \|\phi_1\|_{H^1(\Gamma_\tau)} + \int_{\Gamma_\tau} \varphi \phi_0 L_\varepsilon \phi_1 dV_{g_\varepsilon} \\ &= O(\varepsilon^2) + \int_{\Gamma_\tau} \varphi \phi_0 L_\varepsilon \phi_1 dV_{g_\varepsilon}. \end{aligned}$$

From the expression of L_ε (see for example (4.52)) and (2.14) we get

$$\begin{aligned} O(\varepsilon^2) &= \int_{\Gamma_\tau} \varphi \phi_0 \left(\phi_0 \Delta_{\partial\Omega_{\varepsilon x_n}} \overline{\varphi}_1 - \varepsilon \gamma k \overline{\varphi}_1 + \overline{\varphi}_1 \Delta_{\partial\Omega_{\varepsilon x_n}} \phi_0 + 2 \langle \nabla_{\partial\Omega_{\varepsilon x_n}} \overline{\varphi}_1, \nabla_{\partial\Omega_{\varepsilon x_n}} \phi_0 \rangle \right) \\ &= -\frac{1 + O(\varepsilon^{1-\tau})}{\varepsilon^{n-1}} \sum_{j \in A_{1,\varepsilon}} \alpha_j^2 \lambda_{j,\varepsilon} + O(\varepsilon^2) \left(\frac{1}{\varepsilon^{n-1}} \sum_{j \in A_{1,\varepsilon}} \alpha_j^2 \right)^{\frac{1}{2}} \|\varphi\|_{L^2(\partial\Omega_\varepsilon)} \\ &+ O(\varepsilon) \left(\frac{1}{\varepsilon^{n-1}} \sum_{j \in A_{1,\varepsilon}} \alpha_j^2 \lambda_{j,\varepsilon} \right)^{\frac{1}{2}} \|\varphi\|_{L^2(\partial\Omega_\varepsilon)}. \end{aligned}$$

Then, using the fact that for $j \in A_{1,\varepsilon}$ it is $|\lambda_{j,\varepsilon}| \geq \varepsilon^{\frac{5}{4}} \gg \varepsilon^2$ and from the normalization of ϕ, one finds

$$\frac{1}{\varepsilon^{n-1}} \sum_{j \in A_{1,\varepsilon}} \alpha_j^2 \lambda_{j,\varepsilon} \leq C \varepsilon^2.$$

Still from the fact that $|\lambda_{j,\varepsilon}| > \varepsilon^{\frac{5}{4}}$ for $j \in A_{1,\varepsilon}$ one also derives

$$\frac{1}{\varepsilon^{n-1}} \sum_{j \in A_{1,\varepsilon}} \alpha_j^2 \leq C\varepsilon^{\frac{3}{4}}.$$

A similar argument, replacing $A_{1,\varepsilon}$ with $A_{2,\varepsilon}$ yields similar estimates, so we arrive to the conclusion. □

As an application of the above lemma, we obtain estimates of the derivatives of small eigenvalues of L_ε.

Lemma 4.6. *Suppose λ is as in Lemma 4.1, and assume that $|\lambda| = O(\varepsilon^2)$. Then, for ε sufficiently small the eigenvalue λ is differentiable with respect to ε, and there exists a positive constant c_Ω depending on Ω such that the derivative (which is possibly a multi-valued function) satisfies*

$$\left| \frac{\partial \lambda}{\partial \varepsilon} - c_\Omega \right| = o(1) \qquad \text{as } \varepsilon \to 0.$$

Proof. Suppose u is an eigenfunction of L_ε with eigenvalue λ. Using the eigenvalue equation and Proposition 4.3, we see that the numerator in Kato's formula can be substituted by the expression

$$\int_{\Gamma_\tau} \left(\frac{2}{\varepsilon}(1 - 3(u_\varepsilon^K)^2)u^2 + 6u_\varepsilon^K \frac{\partial \tilde{u}_\varepsilon^K(\varepsilon \cdot)}{\partial \varepsilon} u^2 \right) + O(\varepsilon)\|u\|^2.$$

By Lemmas 4.2 and 4.5 we can evaluate the latter integrand substituting to u the function

$$\phi = \phi_0 \overline{\varphi} := \phi_0 \sum_{|\lambda_{j,\varepsilon}| \leq \varepsilon^{\frac{5}{4}}} \alpha_j \varphi_j(\varepsilon z).$$

If λ we normalize φ so that

$$\int_{\Gamma_\tau} \phi_0^2 k(\varepsilon z) \overline{\varphi}^2(\varepsilon z) dV_{g_\varepsilon} = 1, \tag{4.56}$$

from the expansions of the metric coefficients in Section 2 we find that

$$\frac{\partial \lambda}{\partial \varepsilon} = \int_{\partial \Omega_\varepsilon} \int_0^{\varepsilon^{-\tau}} (1 - \varepsilon x_n k(\overline{z})) \left(\frac{2}{\varepsilon}(1 - 3(u_\varepsilon^K)^2)u^2 + 6u_\varepsilon^K \frac{\partial \tilde{u}_\varepsilon^K(\varepsilon \cdot)}{\partial \varepsilon} u^2 \right) \overline{\varphi}^2 \phi_0^2 dx_n dV_{\overline{g}_\varepsilon} + o(1).$$

In the appendix, we shall show that the latter integrand can be estimated as

$$\int_0^{\varepsilon^{-\tau}} (1 - \varepsilon x_n k(\overline{z})) \left(\frac{2}{\varepsilon}(1 - 3(u_\varepsilon^K)^2)u^2 + 6u_\varepsilon^K \frac{\partial \tilde{u}_\varepsilon^K(\varepsilon \cdot)}{\partial \varepsilon} u^2 \right) \phi_0^2 dx_n = \gamma_5 k(\overline{z}) + o(1) \tag{4.57}$$

as $\varepsilon \to 0$, where $\gamma_5 > 0$ is defined in (3.29). This, together with (4.56) and (4.57) concludes the proof of the lemma. □

Remark 4.7. *The above computations show that indeed $c_\Omega = \gamma_5$.*

4.3. **Invertibility of the linearized operator and proof of the main theorem.** In this subsection we prove our main theorem, showing that L_ε is invertible for a suitable sequence $\varepsilon_j \to 0$ (actually for a larger family of epsilon's) and applying the contraction mapping theorem.

Proposition 4.8. *For $K > 2$, let u_ε^K and L_ε be as above. Then for a suitable sequence $\varepsilon_j \to 0$, the operator $L_{\varepsilon_j} : H^2(\Gamma_\tau) \to L^2(\Gamma_\tau)$ is invertible and the inverse operator satisfies*

$$\left\| L_{\varepsilon_j}^{-1} \right\|_{L(L^2(\Gamma_\tau);H^2(\Gamma_\tau))} \leq \frac{C}{\varepsilon_j^{\frac{n+1}{2}}}, \qquad \text{for all } j \in \mathbb{N}.$$

PROOF. First of all we give an asymptotic estimate on the number N_ε of negative eigenvalues of L_ε (in our notation of (4.54)). We denote the eigenvalues of L_ε by $(\tilde{\lambda}_{j,\varepsilon})_j$, in non-decreasing order and counting them with multiplicity. From the Courant-Fisher characterization we can write $\tilde{\lambda}_{j,\varepsilon}$ in two different ways

$$-\tilde{\lambda}_{j,\varepsilon} = \sup_{M \in M_j} \inf_{u \in M, u \neq 0} \frac{\int_{\Gamma_\tau} u L_\varepsilon u dV_{g_\varepsilon}}{\int_{\Gamma_\tau} k u^2 dV_{g_\varepsilon}}; \qquad -\tilde{\lambda}_{j,\varepsilon} = \inf_{M \in M_{j-1}} \sup_{u \perp M, u \neq 0} \frac{\int_{\Gamma_\tau} u L_\varepsilon u dV_{g_\varepsilon}}{\int_{\Gamma_\tau} k u^2 dV_{g_\varepsilon}}. \tag{4.58}$$

Here M_j (resp. M_{j-1}) represents the family of j-dimensional (resp. $j-1$ dimensional) subspaces of $H^2(\Gamma_\tau)$, and the symbol $\perp$ denotes orthogonality with respect to the L^2 scalar product with weight k.

Using the first formula in (4.58) one can plug-in functions of the form $\phi = \varphi\phi_0$ so that (see (4.52))

$$L_\varepsilon \phi = (\phi_0 \Delta_{\partial\Omega_{\varepsilon x_n}} + (\varepsilon\gamma k(\varepsilon z) + O(\varepsilon^{\frac{3}{2}}))\varphi(z) + \varphi(z)\Delta_{\partial\Omega_{\varepsilon x_n}}\phi_0 + 2\langle \nabla_{\partial\Omega_{\varepsilon x_n}}\varphi(z), \nabla_{\partial\Omega_{\varepsilon x_n}}\phi_0 \rangle.$$

From the slow dependence of ϕ_0 in z and the Weyl's asymptotic formula one finds the lower bound

$$N_\varepsilon \geq (1 + o(1)) C_{1,\Omega} \varepsilon^{-\frac{n-1}{2}},$$

where $C_{1,\Omega}$ is a fixed constant depending on Ω.

To prove a similar upper bound, we write an arbitrary function $\phi \in H^2(\Gamma_\tau)$ as $\phi = \varphi\phi_0 + \phi^\perp$ (following the above notation). In the second expression for the eigenvalues in (4.58) we choose j to be the fist index such that $\lambda_{j-1,\varepsilon} > \varepsilon^{\frac{5}{4}}$ (from the Weyl's asymptotic formula we find that $j = (1+o(1))C_{1,\Omega}\varepsilon^{-\frac{n-1}{2}}$ as $\varepsilon \to 0$). We write $\varphi = \sum_l \alpha_l \varphi_l$, we set $\varepsilon_1 = \sum_{l \leq j-1} \alpha_l \varphi_l$, $\varphi_2 = \sum_{l \geq j} \alpha_l \varphi_l$ and we define $M_{j-1} = span\{\varphi_l \phi_0, l = 1, \dots, j-1\}$. From the orthogonality condition in Γ_τ and the expansions of the metric g_ε in Section 2 we deduce that $\|\varphi_1\phi_0\|_{L^2(\Gamma_\tau)} \leq C\varepsilon^{1-\tau}\|\phi\|_{L^2(\Gamma_\tau)}$. Therefore, with some simple calculations we can write that

$$\int_{\Gamma_\tau} (\varphi_1\phi_0) L_\varepsilon (\varphi_1\phi_0) dV_{g_\varepsilon} = O(\varepsilon^{3-\tau}) \|\phi\|_{L^2(\Gamma_\tau)}.$$

From our choice of j and the computations of the previous subsection we also deduce that (for τ sufficiently small)

$$\int_{\Gamma_\tau} (\varphi_2\phi_0) L_\varepsilon (\varphi_2\phi_0) dV_{g_\varepsilon} \geq C \varepsilon^{\frac{1}{2}} \|\varphi_2\phi_0\|^2_{L^2(\Gamma_\tau)}; \qquad \int_{\Gamma_\tau} \phi^\perp L_\varepsilon \phi^\perp dV_{g_\varepsilon} \geq C^{-1} \|\phi^\perp\|^2_{L^2(\Gamma_\tau)}$$

for some fixed constant C. Therefore, writing

$$\begin{aligned} \int_{\Gamma_\tau} \phi L_\varepsilon \phi dV_{g_\varepsilon} &= \int_{\Gamma_\tau} (\varphi_1\phi_0) L_\varepsilon (\varphi_1\phi_0) dV_{g_\varepsilon} + \int_{\Gamma_\tau} (\varphi_2\phi_0) L_\varepsilon (\varphi_2\phi_0) dV_{g_\varepsilon} + \int_{\Gamma_\tau} \phi^\perp L_\varepsilon \phi^\perp dV_{g_\varepsilon} \\ &+ 2 \int_{\Gamma_\tau} (\varphi_1\phi_0) L_\varepsilon (\varphi_2\phi_0) dV_{g_\varepsilon} + 2 \int_{\Gamma_\tau} (\varphi_1\phi_0) L_\varepsilon \phi^\perp dV_{g_\varepsilon} + 2 \int_{\Gamma_\tau} (\varphi_2\phi_0) L_\varepsilon \phi^\perp dV_{g_\varepsilon}, \end{aligned}$$

using the previous four estimates, the Hölder inequality, the fact that

$$\|\phi\|^2_{L^2(\Gamma_\tau)} (1 + o(1)) = \|\varphi_1\phi_0\|^2_{L^2(\Gamma_\tau)} + \|\varphi_2\phi_0\|^2_{L^2(\Gamma_\tau)} + \|\phi^\perp\|^2_{L^2(\Gamma_\tau)},$$

and the above asymptotics on j we deduce the lower bound

$$N_\varepsilon \leq (1 + o(1)) C_{1,\Omega} \varepsilon^{-\frac{n-1}{2}},$$

with the same constant as before. In conclusion we find that

$$N_\varepsilon \sim C_{1,\Omega} \varepsilon^{-\frac{n-1}{2}} \qquad \text{as } \varepsilon \to 0. \tag{4.59}$$

Next, for $l \in \mathbb{N}$, let $\varepsilon_l = 2^{-l}$. Then from (4.59) it follows that

$$N_{\varepsilon_{l+1}} - N_{\varepsilon_l} \sim C_{1,\Omega} \left(2^{(l+1)\frac{n-1}{2}} - 2^{l\frac{n-1}{2}} \right) = C_{1,\Omega} (2^{\frac{n-1}{2}} - 1) \varepsilon_l^{-\frac{n-1}{2}}. \tag{4.60}$$

By Lemma 4.6, the eigenvalues of L_ε bounded in absolute value by $o(\varepsilon)$ are decreasing in ε. Equivalently, by the last equation, the number of eigenvalues which become negative, when ε decreases from ε_l to ε_{l+1}, is of order $\varepsilon_l^{-\frac{n-1}{2}}$. Now we define

$$A_l = \{ \varepsilon \in (\varepsilon_{l+1}, \varepsilon_l) \; : \; ker L_\varepsilon \neq \emptyset \}; \qquad B_l = (\varepsilon_{l+1}, \varepsilon_l) \setminus A_l.$$

By (4.60) and the monotonicity (in ε) of the *small* eigenvalues, it follows that $\text{card}(A_l) < C \varepsilon_l^{-\frac{n-1}{2}}$, and hence there exists an interval (a_l, b_l) such that

$$(a_l, b_l) \subseteq B_l; \qquad |b_l - a_l| \geq C^{-1} \frac{\text{meas}(B_l)}{\text{card}(A_l)} \geq C^{-1} \varepsilon_l^{\frac{n+1}{2}}. \tag{4.61}$$

From Lemma 4.6 we deduce that

$$L_{\frac{a_l+b_l}{2}} \qquad \text{is invertible and} \qquad \left\| L^{-1}_{\frac{a_l+b_l}{2}} \right\|_{L(L^2(\Gamma_\tau); H^1(\Gamma_\tau))} \leq \frac{C}{\varepsilon_l^{\frac{n+1}{2}}}.$$

Now it is sufficient to set $\varepsilon_j = \frac{a_j+b_j}{2}$. This concludes the proof. ∎

We consider now the problem in the whole domain Ω_ε, and not only in the strip Γ_τ. Precisely, we first choose a cutoff function $\chi_\varepsilon(t)$ which is identically equal to 1 for $t \leq \frac{\varepsilon^{-\tau}}{2}$, and which is identically equal to 0 for $t \geq \frac{3}{4}\varepsilon^{-\tau}$. We then define the function $\hat{u}_\varepsilon^K$ by

$$\hat{u}_\varepsilon^K(z, x_n) = 1 - \chi(x_n) + \chi(x_n) u_\varepsilon^K(z, x_n).$$

It is easy to verify that, by the exponential convergence to 1 of u_ε^K in the interior of Ω_ε (and also by the decay of its derivative), that $\|S_\varepsilon(\hat{u}_\varepsilon^K)\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon^{K+1-\frac{n-1}{2}}$ and that $\|S_\varepsilon(\hat{u}_\varepsilon^K)\|_{L^\infty(\Omega_\varepsilon)} \leq C\varepsilon^{K+1}$.

We also extend the function $k : \Gamma_\tau \to \mathbb{R}$ to the whole Ω_ε in the following way, namely by setting

$$\hat{k}(y) = 1 - \chi(x_n/4) + \chi(x_n/4) k(\varepsilon z),$$

where we are using the same parameterization as in Theorem 1.1. We consider next the eigenvalue problem

$$\Delta u + F''(\hat{u}_\varepsilon^K) u + \lambda \hat{k} u = 0,$$

and we denote the eigenvalues by $(\hat{\lambda}_{j,\varepsilon})_j$, counted in decreasing order with their multiplicity.

As one can easily check, if λ is bounded from above (say, greater than -1), the corresponding eigenfunctions decay exponentially away from $\partial\Omega_\varepsilon$. Therefore, reasoning as for [14], Proposition 5.6 (see also [15], Proposition 5.1), one finds that there exists a constant C such that

$$|\hat{\lambda}_{j,\varepsilon} - \tilde{\lambda}_{j,\varepsilon}| \leq C e^{-\frac{C}{\varepsilon}} \qquad \text{provided } \hat{\lambda}_{j,\varepsilon} \leq 1 \text{ or } \tilde{\lambda}_{j,\varepsilon} \leq 1.$$

Therefore, by Proposition 4.8 and the last formula we obtain the following result.

Corollary 4.9. *For $k \in \mathbb{N}$, let $\hat{u}_\varepsilon^K$ be as above, and define the operator $\hat{L}_\varepsilon u = \Delta u - F''(u_\varepsilon^K)u$. Then for a suitable sequence $\varepsilon_j \to 0$, the operator $\hat{L}_\varepsilon : H^2(\Omega_\varepsilon) \to L^2(\Omega_\varepsilon)$ is invertible and the inverse operator satisfies $\left\| \hat{L}_\varepsilon^{-1} \right\|_{L(L^2(\Omega_\varepsilon);H^2(\Omega_\varepsilon))} \leq \frac{C}{\varepsilon_j^{\frac{n+1}{2}}}$, for all $j \in \mathbb{N}$.*

We are now in position to prove our main result.

Proof of Theorem 1.1 Let $(\varepsilon_j)_j$ be as in Corollary 4.9. Next, we simply apply the contraction mapping theorem, looking for a solution u_ε of the form

$$u_\varepsilon = \hat{u}_K^\varepsilon + w, \qquad w \in H^2(\Omega_\varepsilon).$$

Since $\hat{L}_\varepsilon$ is invertible for $\varepsilon = \varepsilon_j$, we can write

$$S_\varepsilon(\hat{u}_{k,\varepsilon} + w) = 0 \qquad \Leftrightarrow \qquad w = \hat{F}_\varepsilon(w) := -\hat{L}_\varepsilon^{-1}\left[S_\varepsilon(\hat{u}_K^\varepsilon) - 3\hat{u}_K^\varepsilon w^2 - w^3\right]. \tag{4.62}$$

For $r > 0$, it is convenient to introduce the set

$$\mathcal{B}_r = \left\{ w \in H^2(\Omega_\varepsilon) \cap L^\infty(\Omega_\varepsilon) \; : \; |||w||| \leq r \right\}.$$

where we have defined $|||w||| = \|w\|_{H^2(\Omega_\varepsilon)} + \|w\|_{L^\infty(\Omega_\varepsilon)}$.

By standard elliptic regularity results and by Corollary 4.9 one finds that there exists positive constants C (depending on Ω) and d (depending on the dimension n) such that

$$|||\hat{F}_\varepsilon(w)||| \leq C\varepsilon^{-d}\left(\varepsilon^{K+1-\frac{n-1}{2}} + |||w|||^2\right);$$

$$|||\hat{F}_\varepsilon(w_1) - \hat{F}_\varepsilon(w_2)||| \leq C\varepsilon^{-d}\left(|||w_1||| + |||w_2|||\right)\left(|||w_1 - w_2|||\right),$$

for $\varepsilon = \varepsilon_j$ and $w, w_1, w_2 \in H^2(\Omega_\varepsilon) \cap L^\infty(\Omega_\varepsilon)$. Then, letting $r = \varepsilon^l$, choosing first K sufficiently large and then l depending on k and d, and reasoning as in [15], Section 5, one can show that $\hat{F}_\varepsilon$ is a contraction for j sufficiently large. As a consequence, we find a solution of (4.62). This concludes the proof. ■

Remark 4.10. *Also, from the arguments of* [15], *Remark 5.3, one can show that the set of values ε for which $F''_\varepsilon(u^K_\varepsilon)$ is invertible (and for which our method produces solutions of* (1.1)*) has density converging to* 1 *in smaller and smaller right neighborhoods of the origin.*

5. Appendix

In this appendix we collect some useful estimates, which are technical in nature.

Proof of Step 1 in Lemma 3.4: the proof follows from a standard Lyapunov-Schmidt reduction technique. The key is an *a priori* estimate for the following linear problem: let (ϕ, h, c) satisfy

$$\phi_{nn} - \sum_{i=1}^{n-1} \frac{k_i(z)}{1 - \varepsilon x_n k_i(z)} \varepsilon \phi_n + (1 - 3H^2_{\varepsilon,R})\phi = h + cZ_{\varepsilon,R}, \qquad \int_{I-\varepsilon} \phi Z_{\varepsilon,R} = 0, \tag{5.63}$$

with the boundary conditions $\phi_n(0) = 0, \phi\left(\frac{\delta}{\varepsilon}\right) = 0$. Then for ε sufficiently small we have

$$\|\phi\|_* + |c| \leq C\|h\|_*. \tag{5.64}$$

This problem is nothing but the linearization of (3.5) at $H_{\varepsilon,R}$.

We prove the claim arguing by contradiction: suppose that there exists (ϕ, c, h) such that $\|h\|_* = o(1)$ and $\|\phi\|_* + |c| = 1$ as $\varepsilon \to 0$. Multiplying (5.63) by $H'_{\varepsilon,R}$ and integrating over $(0, \frac{\delta}{\varepsilon})$, using the equation satisfied by H' and integrating by parts we obtain that

$$|c| = o(1). \tag{5.65}$$

Now the right-hand side of (5.63) satisfies the estimate $\|h + cZ_{\varepsilon,R}\|_* = o(1)$. We first show that $\|\phi\|_{H^1(\mathbb{R})} = o(1)$: to show this we rewrite (5.63) as

$$\phi_{nn} - (1 - 3\overline{H}_{\varepsilon,R}(x_n)^2)\phi = F_{\varepsilon,R}(h, \phi), \tag{5.66}$$

where we have set

$$F_{\varepsilon,R}(h,\phi) = h + \sum_{i=1}^{n-1} \frac{k_i(z)}{1-\varepsilon x_n k_i(z)} \varepsilon \phi_n + 3(H_{\varepsilon,R}^2 - \overline{H}_{\varepsilon,R}^2)\phi + c\overline{Z}_{\varepsilon,R} + c(Z_{\varepsilon,R} - \overline{Z}_{\varepsilon,R});$$

$$\overline{H}_{\varepsilon,R} = H(x_n - \frac{1}{2\sqrt{2}} \log \frac{1}{\varepsilon} - R); \qquad \overline{Z}_{\varepsilon,R} = 3(1 - \overline{H}_{\varepsilon,R}^2)\overline{H}_{\varepsilon,R}'.$$

Now it is easy to show that $\|F_\varepsilon\|_{L^2(I_\varepsilon)} = o(1)(1+|c|) + o(1)\|\phi\|_{H^2(I_\varepsilon)}$ as $\varepsilon \to 0$. Therefore Lemma 2.1 and the contraction mapping theorem yield a solution (ϕ, c) of (5.63) for which $|c| + \|\phi\|_{H^1(I_\varepsilon)} = o(1)$. Then the estimate in the $\|\cdot\|_*$ norm (and hence (5.64)) follows from standard regularity results.

The next step consists in rewriting (3.18) as

$$\phi_{nn} - \sum_{i=1}^{n-1} \frac{k_i(z)}{1-\varepsilon x_n k_i(z)} \varepsilon \phi_n + (1 - 3H_{\varepsilon,R}^2)\phi = cZ_{\varepsilon,R} - \mathbb{S}_\varepsilon(H_{\varepsilon,R}) + (H_{\varepsilon,R} + \phi)^3 - H_{\varepsilon,R}^3 - 3H_{\varepsilon,R}^2\phi.$$

From a Taylor expansion of the later term one finds that

$$\|(H_{\varepsilon,R} + \phi)^3 - H_{\varepsilon,R}^3 - 3H_{\varepsilon,R}^2\phi\|_* = o(1)\|\phi\|_*^2 \qquad \text{as } \varepsilon \to 0$$

for a fixed constant C. Then the conclusion follows from the estimate (5.64) and another application of the contraction mapping theorem. ☐

Proof of (4.57): we consider the expression

$$\int_0^{\varepsilon^{-\tau}} (1 - \varepsilon x_n k(\overline{z})) \left(\frac{2}{\varepsilon}(1 - 3(u_\varepsilon^K)^2) + 6u_\varepsilon^K \frac{\partial \tilde{u}_\varepsilon^K}{\partial \varepsilon}(\varepsilon \cdot) \right) \phi_0^2 dx_n. \tag{5.67}$$

Note that the leading order term in (5.67) equals zero since

$$\int_{\mathbb{R}} (2(1 - 3H^2) - 6HH' t)(H')^2 = 0. \tag{5.68}$$

The latter equation follows because $H' t$ satisfies

$$(H' t)'' + f'(H)(H' t) + 2(H - H^3) = 0 \tag{5.69}$$

and

$$(H' t)''' + f'(H)(H' t)' + 2(1 - 3H^2)H' - 6H(H')^2 t = 0. \tag{5.70}$$

Thus we need to expand (5.67) to the next order in ε: to this end, we note that we can write

$$\phi_0 = \frac{\partial u_\varepsilon^K}{\partial x_n} + \varepsilon \phi_1 + o(\varepsilon) \tag{5.71}$$

in the $\|\cdot\|_*$ norm, where ϕ_1 satisfies (after a suitable translation in x_n)

$$\phi_1'' + (1 - 3H^2)\phi_1 = \gamma_5 k(\overline{z}) H' \qquad \text{in } [0, \varepsilon^{-\tau}], \tag{5.72}$$

which some boundary conditions at $x_n = 0$ which are described below.

To obtain (5.71), we let

$$\hat{h}_0 = \frac{\partial u_\varepsilon^K}{\partial x_n} + \frac{1}{\sqrt{2}}\tau_\varepsilon e^{-\sqrt{2}x_n}$$

where $\tau_\varepsilon = \frac{\partial^2 u_\varepsilon^K}{\partial x_n^2}(0)$. From equation (3.5), it is easy to see that $\hat{h}_0$ satisfies

$$\mathbb{L}\frac{\partial u_\varepsilon^K}{\partial x_n} = O(\varepsilon^2), \quad \frac{\partial u_\varepsilon^K}{\partial x_n}(0) = 0, \tag{5.73}$$

$$\mathbb{L}\hat{h}_0 = 3\tau_\varepsilon(1-u_\varepsilon^2)e^{-\sqrt{2}x_n} + O(\varepsilon^2), \quad \hat{h}_{0,n}(0) = 0. \tag{5.74}$$

We then decompose ϕ_0 as

$$\phi_0 = \hat{h}_0 + \varepsilon\hat{\phi}^\perp, \qquad \int_{I_\varepsilon} \hat{h}_0\hat{\phi}^\perp = 0$$

By simple computations, one can show

$$\mathbb{L}\hat{\phi}^\perp = \frac{3\tau_\varepsilon}{\sqrt{2}\varepsilon}(1-u_\varepsilon^2)e^{-\sqrt{2}x_n} - \frac{\lambda_\varepsilon}{\varepsilon}\phi_0 + O(\varepsilon). \tag{5.75}$$

Similarly to the proof of Lemma 3.5, we see that $\hat{\phi}^\perp \to \hat{\phi}_0^\perp$, which satisfies

$$(\hat{\phi}_0^\perp)'' + (1-3H^2)\hat{\phi}_0^\perp = \gamma_5 k(\overline{z})H' - 3k_7(\overline{z})(1-H^2)e^{-\sqrt{2}y}, \quad \int_{\mathbb{R}} \hat{\phi}_0^\perp H' = 0. \tag{5.76}$$

Here $\gamma_5 > 0$ is the constant defined by (3.29) and

$$k_7(\overline{z}) = -\lim_{\varepsilon\to 0}\left(\frac{\tau_\varepsilon}{\sqrt{2}\sqrt{\varepsilon}}\right)e^{-\sqrt{2}R(z)}. \tag{5.77}$$

We observe that in order to have solvability of this equation one needs

$$k_7(\overline{z}) = \gamma_5 k(\overline{z})\frac{\int_{\mathbb{R}}(H')^2}{\int_{\mathbb{R}}(3(1-H^2)e^{-\sqrt{2}y}H')dy} = \frac{\gamma_5 k(\overline{z})}{4A_0}\int_{\mathbb{R}}(H')^2 \tag{5.78}$$

where we have used the following identity, which can be proved using a n integration by parts

$$\int_{\mathbb{R}} 3(1-H^2)e^{-\sqrt{2}y}H' = 4A_0.$$

Finally, we let $\phi_1^\varepsilon = \hat{\phi}^\perp + \frac{\tau_\varepsilon}{\sqrt{2}\varepsilon}e^{-\sqrt{2}x_n}$. It is easy to see that $\phi_1^\varepsilon \to \phi_1$ and ϕ_1 satisfies (5.72), with boundary data such that

$$\phi_1^\varepsilon(0) \simeq \frac{\tau_\varepsilon}{\sqrt{2}\varepsilon}; \qquad (\phi_1^\varepsilon)'(0) \simeq -\frac{\tau_\varepsilon}{\varepsilon}. \tag{5.79}$$

Now we multiply (3.48) by $(1-\varepsilon x_n k(\overline{z}))\frac{\partial u_\varepsilon^K}{\partial x_n}$, use equation (5.74) and integrate by parts (noting that $\frac{\partial u_\varepsilon^K}{\partial x_n}(0) = 0, \Phi_\varepsilon^K(0) = 0$), to obtain

$$\int_0^{\varepsilon^{-\tau}} (1-\varepsilon x_n k(\overline{z}))\left(-6u_\varepsilon^K\frac{\partial \tilde{u}_\varepsilon^K}{\partial\varepsilon}\frac{\partial u_\varepsilon^K}{\partial x_n}\frac{\partial u_\varepsilon^K}{\partial x_n} + \frac{2}{\varepsilon}(3(u_\varepsilon^K)^2-1)\frac{\partial u_\varepsilon^K}{\partial x_n}\frac{\partial u_\varepsilon^K}{\partial x_n}\right) = o(\varepsilon). \tag{5.80}$$

This then implies

$$\begin{aligned}
&\int_0^{\varepsilon^{-\tau}} (1-\varepsilon x_n k(\overline{z}))\left(\frac{2}{\varepsilon}(1-3(u_\varepsilon^K)^2)+6u_\varepsilon^K\frac{\partial \tilde{u}_\varepsilon^K}{\partial\varepsilon}(\varepsilon\cdot)\right)\phi_0^2 dx_n \\
= &\int_0^{\varepsilon^{-\tau}} (1-\varepsilon x_n k(\overline{z}))\left(\frac{2}{\varepsilon}(1-3(u_\varepsilon^K)^2)(\frac{\partial u_\varepsilon^K}{\partial x_n})^2+6u_\varepsilon^K\frac{\partial \tilde{u}_\varepsilon^K}{\partial\varepsilon}(\frac{\partial u_\varepsilon^K}{\partial x_n})^2\right) \\
+ &\ 2\varepsilon\int_0^{\varepsilon^{-\tau}} (1-\varepsilon x_n k(\overline{z}))\left(\frac{2}{\varepsilon}(1-3(u_\varepsilon^K)^2)\frac{\partial u_\varepsilon^K}{\partial x_n}\phi_1+6u_\varepsilon^K\frac{\partial \tilde{u}_\varepsilon^K}{\partial\varepsilon}\frac{\partial u_\varepsilon^K}{\partial x_n}\phi_1\right)+o(\varepsilon) \\
= &\ o(\varepsilon)+2\varepsilon\int_0^{\varepsilon^{-\tau}} (1-\varepsilon x_n k(\overline{z}))\left(\frac{2}{\varepsilon}(1-3(u_\varepsilon^K)^2)\frac{\partial u_\varepsilon^K}{\partial x_n}\phi_1+6u_\varepsilon^K\frac{\partial \tilde{u}_\varepsilon^K}{\partial\varepsilon}\frac{\partial u_\varepsilon^K}{\partial x_n}\phi_1\right). \qquad (5.81)
\end{aligned}$$

Note that the main term in $\frac{\partial \tilde{u}_\varepsilon^K}{\partial\varepsilon}$ is

$$\frac{\partial \tilde{u}_\varepsilon^K}{\partial\varepsilon} \simeq (-\frac{x_n}{\varepsilon}+\frac{1}{2\sqrt{2}}\frac{1}{\varepsilon}+o(\frac{1}{\varepsilon}))H'.$$

Substituting the (more accurate) expansions for u_ε^K and $\frac{\partial u_\varepsilon^K}{\partial\varepsilon}$ to deal with the first term (see Theorem 3.2 and Lemma 3.9) and the equation (5.71) on ϕ_1 into (5.81), we arrive at

$$2\int_0^{\varepsilon^{-\tau}} (1-\varepsilon x_n k(\varepsilon z))\left(\frac{2}{\varepsilon}(1-3(u_\varepsilon^K)^2)+6u_\varepsilon^K\frac{\partial \tilde{u}_\varepsilon^K}{\partial\varepsilon}(\varepsilon\cdot)\right)\phi_0^2 dx_n \qquad (5.82)$$

$$= o(1)+2\int_0^{\varepsilon^{-\tau}} (2(1-3H^2)+6(-x_n+\frac{1}{2\sqrt{2}})HH')H'\phi_1$$

$$= o(1)+2\int_0^{\varepsilon^{-\tau}} [\mathbb{L}(-(H't)'+(\frac{1}{2\sqrt{2}}\log\frac{1}{\varepsilon}+R_\varepsilon+\frac{1}{2\sqrt{2}})H'')]\phi_1$$

$$= 2(2H''-\frac{1}{2\sqrt{2}}H''')(0)\phi_1(0)-2(H'-\frac{1}{2\sqrt{2}}H'')(0)\phi_1'(0)-2\int_0^{\varepsilon^{-\tau}}(\phi_1''+(1-3H^2)\phi_1)(H't)'$$

$$= 8A_0\frac{\tau_\varepsilon}{\sqrt{2}\sqrt{\varepsilon}}e^{-\sqrt{2}R}-2(\int_{\mathbb{R}}(H'(H't)')\gamma_5 k(\overline{z})+o(1)$$

$$= 8A_0 k_7(\overline{z})-(\int_{\mathbb{R}}(H')^2)\gamma_5 k(\overline{z})+o(1)$$

$$= \gamma_5\int_{\mathbb{R}}(H')^2 k(\overline{z})+o(1)>0.$$

Here we have (2.2), (5.72), (5.77), (5.78), (5.79), some integration by parts and (5.70). □

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