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DEGREE COUNTING AND SHADOW SYSTEM FOR $SU(3)$ TODA SYSTEM: ONE BUBBLING

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Abstract. Here we initiate the program for computing the Leray-Schauder topological degree for $SU(3)$ Toda system. This program still contains a lot of challenging problems for analysts. The first step of our approach is to answer whether concentration phenomena holds or not. In this paper, we prove the concentration phenomena holds while ρ_1 crosses 4π, and $\rho_2 \notin 4\pi\mathbb{N}$. However, for $\rho_1 \geq 8\pi$, the question whether concentration holds or not still remains open up to now. The second step is to study the corresponding shadow system and its degree counting formula. The last step is to construct bubbling solution of $SU(3)$ Toda system via a non-degenerate solution of the shadow system. Using this construction, we succeed to calculate the degree for $\rho_1 \in (0,4\pi)\cup(4\pi,8\pi)$ and $\rho_2 \notin 4\pi\mathbb{N}$.

1. Introduction

Let (M,g) be a compact Riemann surface with volume 1, h_1^* and h_2^* be a C^1 positive function on M and $\rho_1,\rho_2 \in \mathbb{R}^+$. We consider the following $SU(3)$ Toda system on the compact surface M.

$$\begin{cases} \Delta u_1^* + 2\rho_1\left(\frac{h_1^* e^{u_1^*}}{\int_M h_1^* e^{u_1^*}} - 1\right) - \rho_2\left(\frac{h_2^* e^{u_2^*}}{\int_M h_2^* e^{u_2^*}} - 1\right) = 4\pi \sum_{q\in S_1} \alpha_q(\delta_q - 1), \\ \Delta u_2^* - \rho_1\left(\frac{h_1^* e^{u_1^*}}{\int_M h_1^* e^{u_1^*}} - 1\right) + 2\rho_2\left(\frac{h_2^* e^{u_2^*}}{\int_M h_2^* e^{u_2^*}} - 1\right) = 4\pi \sum_{q\in S_2} \beta_q(\delta_q - 1), \end{cases} \tag{1.1}$$

where Δ is the Beltrami-Laplace operator, $\alpha_q \geq 0$ for every $q \in S_1$, $\beta_q \geq 0$ for every $q \in S_2$ and δ_q is the Dirac measure at $q \in M$.

When the two equations in (1.1) are identical, i.e., $S_1 = S_2$, $\alpha_q = \beta_q$, $u_1^* = u_2^* = u^*$, $h_1^* = h_2^* = h^*$ and $\rho_1 = \rho_2 = \rho$, system (1.1) is reduced to the following mean field equation

$$\Delta u^* + \rho\left(\frac{h^* e^{u^*}}{\int_M h^* e^{u^*}} - 1\right) = 4\pi \sum_{q\in S_1} \alpha_q(\delta_q - 1). \tag{1.2}$$

Equation (1.1) and equation (1.2) arise in many physical and geometric problems. In physics, (1.2) or (1.1) are one of the limiting equations of the abelian gauge field theory or non-abelian Chern-Simons gauge field theory, one can see [16, 17, 34, 44, 45, 52] and references therein. In conformal geometry, a solution u^* of

$$\Delta_g u^* + e^{u^*} - 2K = 4\pi \sum_{q\in S_1} \alpha_q \delta_q \text{ in } M, \tag{1.3}$$

where $K(x)$ is the Gaussian curvature of the given metric g at $x \in M$, is equivalent to saying that the new metric $e^{2v} g$ (where $2v = u^* - \log 2$) has constant Gaussian curvature $\tilde{K} = 1$. By integrating (1.3), it is easy to see that (1.3) is a special case of (1.2) with $\rho = 4\pi \sum_{q\in S_1} \alpha_q$. Since u^* has a logarithmic singularity at each

$q \in S_1$, the new metric $e^{2v}g$ has a conic singularity at each q. Equation (1.3) has been extensively studied in the last three decades, see [9, 13, 29, 33, 37, 48] and references therein. However, when the number of the singularities is greater than three, there are very few existence results for equation (1.3). Studies on the case of four singularities are referred to [13] and [29]. For the recent development of the mean field equation (1.2), we refer the readers to [3, 4, 7, 10, 11, 12, 13, 15, 27, 39, 43, 44, 52].

For equation (1.2), we let the set Σ of the critical parameters be defined by

$$\begin{aligned}\Sigma :&= \big\{8N\pi + \Sigma_{q\in A} 8\pi(1+\alpha_q) \mid A \subseteq S_1,\ N \in \mathbb{N}\cup\{0\}\big\} \setminus \{0\}\\ &= \{8\pi\mathfrak{a}_k \mid k = 1,2,3,\cdots.\},\end{aligned}$$

where $\mathfrak{a}_k$ will be defined in (1.4). It was proved that if $\rho \notin \Sigma$, then the a-priori estimate for any solution of (1.2) holds in $C^2_{\mathrm{loc}}(M \setminus S_1)$. This a-priori bound was obtained by Li and Shafrir [26] for the case without singular sources, and by Bartolucci and Tarantello [3] for the general case with singular sources. After establishing the a-priori bound for a non-critical parameter ρ, it is natural to count the Leray-Schauder topological degree for the equation (1.2). It was proved by Li [25] that this degree counting should depend only on the topology of M for the case without singularities. In a series of papers [10]-[13], Chen and Lin has derived the topological degree counting formulas as described below.

We denote the topological degree of (1.2) for $\rho \notin \Sigma$ by d_ρ. By the homotopic invariant of the topological degree, d_ρ is a constant for $8\pi\mathfrak{a}_k < \rho < 8\pi\mathfrak{a}_{k+1},\ k = 0,1,2,\cdots$, where $\mathfrak{a}_0 = 0$. Set $d_m = d_\rho$ for $8\pi\mathfrak{a}_m < \rho < 8\pi\mathfrak{a}_{m+1}$. To state the result, we introduce the following generating function Ξ_0 :

$$\begin{aligned}\Xi_0(x) =&(1+x+x^2+x^3+\cdots)^{-\chi(M)+|S_1|}\Pi_{q\in S_1}(1-x^{1+\alpha_q})\\ =&1+\mathfrak{c}_1x^{\mathfrak{a}_1}+\mathfrak{c}_2x^{\mathfrak{a}_2}+\cdots+\mathfrak{c}_kx^{\mathfrak{a}_k}+\cdots.\end{aligned}$$

The degree d_m can be written in terms of $\mathfrak{c}_j$, as shown in the following theorem.

Theorem A. ([13]) *Let d_ρ be the Leray-Schauder degree for (1.2). Suppose $8\mathfrak{a}_k\pi < \rho < 8\mathfrak{a}_{k+1}\pi$. Then*

$$d_\rho = \sum_{j=0}^{k} \mathfrak{c}_j,$$

where $d_0 = 1$.

For the application, it often requires that $\alpha_q \in \mathbb{N}$ for all $q \in S_1$. In this case, $\Sigma = \{8m\pi \mid m \in \mathbb{N}\}$ and let $d_m = d_\rho$ for $\rho \in (8m\pi, 8(m+1)\pi)$. Then the generating function

$$\begin{aligned}\Xi_1(x) =&\sum_{k=0}^{\infty} d_kx^k = (1+x+x^2+\cdots)^{-\chi(M)+1+|S_1|}\Pi_{q\in S_1}(1-x^{\alpha_q+1})\\ =&(1+x+x^2+\cdots)^{-\chi(M)+1}\Pi_{q\in S_1}(1+x+x^2+\cdots+x^{\alpha_q}) \qquad (1.4)\end{aligned}$$

Clearly, we have $d_m \geq 1,\ \forall m$ provided $\chi(M) \leq 0$. Hence we can obtain the existence of the solution to (1.2) when the genus of M is nonzero. When M is a torus and $\sum_{q\in S_1} \alpha_q$ is an odd integer, by applying the Theorem A, we can get the degree formula for (1.3)

$$d = \frac{\Pi_{q\in S_1}(1+\alpha_q)}{2}.$$

Similarly, we could consider the following Toda system

$$\begin{cases} \Delta u_1^* + 2e^{u_1^*} - e^{u_2^*} - 2K = 4\pi \sum_{q\in S_1} \alpha_q \delta_q, \\ \Delta u_2^* + 2e^{u_2^*} - e^{u_1^*} - 2K = 4\pi \sum_{q\in S_2} \beta_q \delta_q, \end{cases} \tag{1.5}$$

on M, which is a natural generalization of (1.3), but is a special case of (1.1). In geometry, it is closely related to the classical Plücker formula for a holomorphic curve from M to $\mathbb{CP}^2$, the vortex points and α_q are exactly the branch points and its ramification index of this holomorphic curve. See [33] for more precise formulation and also [5, 6, 8, 14, 18, 23] for connection with different aspects of geometry. For the past decades, there are many studies for the $SU(3)$ Toda system, or more generally, system of equations with exponential nonlinearity. We refer the readers to [2, 19, 20, 24, 28, 30, 31, 32, 35, 36, 37, 38, 40, 41, 42, 45, 50, 51] and references therein.

In this paper, we want to initiate the program for computing the Leray-Schauder degree formula for the system (1.1). However, it seems still a very challenging problem even now. Hence in this article we shall consider the simplest (but nontrivial) case, described below. We assume

(i) $S_1, S_2 = \emptyset$,
(ii) $\rho_1 \in (0, 4\pi) \cup (4\pi, 8\pi)$ and $\rho_2 \notin \Sigma_1 = \{4N\pi \mid N \in \mathbb{N}\}$.

To eliminate the singularities on the right hand side of (1.1), we introduce the Green function $G(x,p)$:

$$-\Delta G(x,p) = \delta_p - 1 \text{ in } M, \quad \text{with } \int_M G(x,p) = 0.$$

and let

$$u_1(x) = u_1^*(x) - 4\pi \sum_{q\in S_1} \alpha_q G(x,q),\ u_2(x) = u_2^*(x) - 4\pi \sum_{q\in S_2} \beta_q G(x,q).$$

Then (1.1) is equivalent to the following system

$$\begin{cases} \Delta u_1 + 2\rho_1(\frac{h_1 e^{u_1}}{\int_M h_1 e^{u_1}} - 1) - \rho_2(\frac{h_2 e^{u_2}}{\int_M h_2 e^{u_2}} - 1) = 0, \\ \Delta u_2 - \rho_1(\frac{h_1 e^{u_1}}{\int_M h_1 e^{u_1}} - 1) + 2\rho_2(\frac{h_2 e^{u_2}}{\int_M h_2 e^{u_2}} - 1) = 0, \end{cases} \tag{1.6}$$

where $h_1, h_2 \geq 0$ in M and $h_1(x) = 0$ iff $x \in S_1$, $h_2 = 0$ iff $x \in S_2$. Near each $q \in S_1$, h_1 has the form in local coordinate:

$$h_1(x) = h_{1,q}(x)|x-q|^{2\alpha_q},\ \text{for } |x-q| \ll 1,\ \forall q \in S_1,$$

where $h_{1,q}(x) > 0$ for any $q \in S_1$. Near each $q \in S_2$, h_2 has the form in local coordinate:

$$h_2(x) = h_{2,q}(x)|x-q|^{2\beta_q},\ \text{for } |x-q| \ll 1,\ \forall q \in S_2,$$

where $h_{2,q}(x) > 0$ for any $q \in S_2$.

We notice that equation (1.6) is invariant by adding constant to the solutions. Hence we can always normalize u_1, u_2 to satisfy $\int_M u_1 = \int_M u_2 = 0$. Let $\mathring{H}^1$ be the space:

$$\mathring{H}^1 = \{u \in H^1(M) : \int_M u = 0\}.$$

From now on, we will restrict our discussion in $\mathring{H}^1 \times \mathring{H}^1$. In order to compute the Leray-Schauder degree of the system, we need to get well understand of the blow-up phenomena for (1.6). The first main issue for system is to determine the set of critical parameters, i.e., those $\rho = (\rho_1, \rho_2)$ such that the a-priori bounds for solutions of (1.1) fail. In [20], the authors claimed that if $\rho_i \notin 4\pi\mathbb{N},\ i = 1, 2$, then the a-priori bound for all solution exists. See Theorem 1.2 in [20]. However they did not give a proof of this fact, but just said that it follows immediately from Proposition 2.4 in [20], where the local masses of solutions (u_1, u_2) at a blow up point is calculated. In addition to Proposition 2.4 in [20], for their claim of Theorem 1.2, it requires that the concentration phenomena holds, i.e., $\frac{h_i e^{u_{ik}}}{\int_M h_i e^{u_{ik}}}$ tends to a sum of Dirac measures. However, as far as the authors know, the concentration has not been proved yet. Under the assumption (i) and (ii), we can show that concentration holds, i.e., if u_{1k} blows up at some points, then $\frac{h_1 e^{u_{1k}}}{\int_M h_1 e^{u_{1k}}}$ tends to a sum of Dirac measures.

Our first main theorem is the following a-priori estimate.

Theorem 1.1. *Suppose h_i are positive smooth functions and the assumption $(i)-(ii)$. Then there exists a positive constant c such that for any solution of equation (1.6), there holds:*

$$|u_1(x)|, |u_2(x)| \le c,\ \forall x \in M,\ i = 1, 2.$$

For the general case, we shall study the concentration phenomena for $\rho_1 \ge 8\pi$ in a future work. By Theorem 1.1, the Leray-Schauder degree $d^{(2)}_{\rho_1,\rho_2}$ for (1.1), or equivalently (1.6), is well-defined for $\rho_1 \in (0, 4\pi) \cup (4\pi, 8\pi)$ and $\rho_2 \notin 4\pi\mathbb{N}$. Clearly, $d^{(2)}_{\rho_1,\rho_2} = d^{(1)}_{\rho_2}$ if $0 < \rho_1 < 4\pi$, and $\rho_2 \notin 4\pi\mathbb{N}$. Hence, the main contribution of our paper is to compute the degree $d^{(2)}_{\rho_1,\rho_2}$ for $4\pi < \rho_1 < 8\pi$. By the homotopic invariant, for any fixed $\rho_2 \notin 4\pi\mathbb{N}$, $d^{(2)}_{\rho_1,\rho_2}$ is a constant for $\rho_1 \in (0, 4\pi)$, and the same holds true for $\rho_1 \in (4\pi, 8\pi)$. For the simplicity, we might let $d^{(2)}_-$ and $d^{(2)}_+$ denotes $d^{(2)}_{\rho_1,\rho_2}$ for $\rho_1 \in (0, 4\pi)$ and $\rho_1 \in (4\pi, 8\pi)$. Since $d^{(2)}_-$ is known by Theorem A, computing $d^{(2)}_+$ is equivalent to computing the difference of $d^{(2)}_+ - d^{(2)}_-$, which might be not zero due to the bubbling phenomena of (1.1) at $(4\pi, \rho_2)$.

To calculate $d^{(2)}_+ - d^{(2)}_-$, we need to compute the topological degree of the bubbling solution of (1.6) when ρ_1 crosses 4π, $\rho_2 \notin 4\pi\mathbb{N}$. For convenience, we rewrite (1.6) as

$$\begin{cases} \Delta v_1 + \rho_1\left(\frac{h_1 e^{2v_1 - v_2}}{\int_M h_1 e^{2v_1 - v_2}} - 1\right) = 0, \\ \Delta v_2 + \rho_2\left(\frac{h_2 e^{2v_2 - v_1}}{\int_M h_2 e^{2v_2 - v_1}} - 1\right) = 0, \end{cases} \tag{1.7}$$

where $v_1 = \frac{1}{3}(2u_1 + u_2),\ v_2 = \frac{1}{3}(u_1 + 2u_2)$. It is known that the Leray-Schauder degree for (1.6) and (1.7) are the same. So, our aim is to compute the degree contribution of the bubbling solution of (1.7) when ρ_1 crosses 4π, $\rho_2 \notin 4\pi\mathbb{N}$. We consider (v_{1k}, v_{2k}) to be a sequence of solutions of (1.7) with $(\rho_{1k}, \rho_{2k}) \to (4\pi, \rho_2)$, and assume $\max_M(v_{1k}, v_{2k}) \to \infty$. Then we have the following theorem

Theorem 1.2. *Let (v_{1k}, v_{2k}) be described as above. Then, the followings hold:*

(i)

$$\rho_{1k}\frac{h_1 e^{2v_{1k}-v_{2k}}}{\int_M h_1 e^{2v_{1k}-v_{2k}}} \to 4\pi\delta_p \text{ for some } p \in M, \tag{1.8}$$

(ii) $v_{2k} \to \frac{1}{2}w$ in $C^{2,\alpha}(M)$ *where (p, w) satisfies*

$$\nabla\big(\log(h_1 e^{-\frac{1}{2}w})(x) + 4\pi R(x,x)\big)\mid_{x=p}= 0, \tag{1.9}$$

and

$$\Delta w + 2\rho_2\Big(\frac{h_2 e^{w-4\pi G(x,p)}}{\int_M h_2 e^{w-4\pi G(x,p)}} - 1\Big) = 0. \tag{1.10}$$

Here $R(x,p)$ refers to the regular part of the Green function $G(x,p)$.

We write (1.9) and (1.10) as

$$\begin{cases} \Delta w + 2\rho_2\big(\frac{h_2 e^{w-4\pi G(x,p)}}{\int_M h_2 e^{w-4\pi G(x,p)}} - 1\big) = 0, \\ \nabla\big(\log(h_1 e^{-\frac{1}{2}w})(x) + 4\pi R(x,x)\big)\mid_{x=p}= 0. \end{cases} \tag{1.11}$$

The system of equation (1.11) is called the *shadow system* of (1.7). This kind of systems also appear while studying the self-dual system [22] for the Jackiw-Weinberg electroweak theory. After Theorem 1.2, it is natural to ask the following question: Given any pair of solution (p, w), can we find a bubbling solution (v_{1k}, v_{2k}) of (1.7) with $(\rho_{1k}, \rho_{2k}) \to (4\pi, \rho_2)$ such that (1.11) holds and v_{2k} converges to $\frac{1}{2}w$ in $C^{2,\alpha}(M)$. One of main results in this article is to give an answer of this question.

For the application in other problems, we want to consider a more general class of equation than (1.11). Let

$$Q = P_w \cup S = \{p_1^0, p_2^0, \cdots, p_m^0\} \cup S, \text{ where } P_w \cap S_1 = \emptyset,\ S \subseteq S_1,$$

and (P_w, w) be a solution of

$$\begin{cases} \Delta w + 2\rho_2\Big(\frac{h_2 e^{w-4\pi\sum_{j=1}^m G(x,p_j^0)-4\pi\sum_{q\in S}(1+\alpha_q)G(x,q)}}{\int_M h_2 e^{w-4\pi\sum_{j=1}^m G(x,p_j^0)-4\pi\sum_{q\in S}(1+\alpha_q)G(x,q)}} - 1\Big) = 0, \\ \nabla_{x_i} f_Q(p_1^0, p_2^0, \cdots, p_m^0) = 0, \end{cases} \tag{1.12}$$

where

$$\begin{aligned} f_Q(x_1, x_2, \cdots, x_m) = &\sum_{j=1}^m \big[\log(h_1 e^{-\frac{1}{2}w})(x_j) + 4\pi R(x_j, x_j)\big] + 4\pi\sum_{i,j=1, i\neq j}^m G(x_i, x_j) \\ &+ 8\pi\sum_{q\in S}\sum_{j=1}^m (1+\alpha_q)G(x_j, q). \end{aligned}$$

It is clear to see (1.12) is a shadow system of (1.11) corresponding some more complicate bubbling phenomena. Note that in (1.12), there might allow $(P_w \cup S) \cap S_2 \neq \emptyset$. We say (P_w, w) is called a *non-degenerate solution* of (1.12) if the linearized equation. i.e., for $(\phi,\overrightarrow{\nu})$, $\overrightarrow{\nu}=(\nu_1,\nu_2,\cdots,\nu_m)$, where $\nu_i\in\mathbb{R}^2$

$$\begin{cases}\Delta\phi+2\rho_2\frac{\overline{h}_2e^{w-4\pi\sum_{j=1}^mG(x,p_j^0)}}{\int_M\overline{h}_2e^{w-4\pi\sum_{j=1}^mG(x,p_j^0)}}\phi\\ \qquad -2\rho_2\frac{\overline{h}_2e^{w-4\pi\sum_{j=1}^mG(x,p_j^0)}}{\left(\int_M\overline{h}_2e^{w-4\pi\sum_{j=1}^mG(x,p_j^0)}\right)^2}\int_M\left(\overline{h}_2e^{w-4\pi\sum_{j=1}^mG(x,p_j^0)}\phi\right)\\ \qquad -8\pi\rho_2\frac{\overline{h}_2e^{w-4\pi\sum_{j=1}^mG(x,p_j^0)}\sum_{j=1}^m(\nabla G(x,p_j^0)\nu_j)}{\int_M\overline{h}_2e^{w-4\pi\sum_{j=1}^mG(x,p_j^0)}}\\ \qquad +8\pi\rho_2\frac{\overline{h}_2e^{w-4\pi\sum_{j=1}^mG(x,p_j^0)}\int_M\left(\overline{h}_2e^{w-4\pi\sum_{j=1}^mG(x,p_j^0)}\sum_{j=1}^m(\nabla G(x,p_j^0)\nu_j)\right)}{\left(\int_M\overline{h}_2e^{w-4\pi\sum_{j=1}^mG(x,p_j^0)}\right)^2}=0,\\ \nabla^2_{x_i}f_Q(p_1^0,p_2^0,\cdots,p_m^0)\nu_i+\mathcal{F}_i-\frac12\nabla\phi(p_i^0)=0,\ i=1,2,\cdots,m,\ \int_M\phi=0,\end{cases}\tag{1.13}$$

admits only trivial solution, i.e., $(\phi,\overrightarrow{\nu})=(0,0)$. Here

$$\overline{h}_2=h_2e^{-4\pi\sum_{q\in S}(1+\alpha_q)G(x,q)}\tag{1.14}$$

and

$$\mathcal{F}_i=8\pi\sum_{j=1,j\neq i}^m\nabla_x^2G(p_i^0,x)\mid_{x=p_j^0}\cdot\nu_j.\tag{1.15}$$

For the shadow system (1.12), the set of non-critical parameters Σ_2 is defined as

$$\Sigma_2=\{4N\pi+4\pi\sum_{q\in S_2\cup S}(1+\alpha_q),\ N\in\mathbb{N}\}.\tag{1.16}$$

Our third main result is the following.

Theorem 1.3. *Suppose $P_w=(p_1^0,p_2^0,\cdots,p_m^0)$ and (P_w,w) is a non-degenerate solution of (1.12) and the quantity $l(Q)\neq0$ [1]. Suppose $\alpha_q\notin\mathbb{N}$ for $q\in S$ and $\rho_2\notin\Sigma_2$. Then there exists a sequence of solutions (v_{1k},v_{2k}) of (1.7) with (ρ_{1k},ρ_2) such that $\lim_{k\to+\infty}\rho_{1k}=4m\pi+\sum_{q\in S}4\pi(1+\alpha_q)$. Furthermore we have:*

(i) $\rho_{1k}\frac{h_1e^{2v_{1k}-v_{2k}}}{\int_M h_1e^{2v_{1k}-v_{2k}}}\to4\pi\sum_{j=1}^m\delta_{p_j^0}+4\pi\sum_{q\in S}(1+\alpha_q)\delta_q$,
(ii) $v_{2k}\to\frac12w$ *in* $C^{1,\alpha}(M)$.

The proof of Theorem 1.3 will be given in section 4-5. The main difficulty for constructing such solutions would be the one for v_{1k} component. Here we follow the arguments in [11], [13] which procedures simultaneously have the advantage in computing the Morse index contributed by the bubbling solutions (v_{1k},v_{2k}). The calculation of the Morse index is the key step towards computing the Leray-Schauder degree for system (1.7), once the degree for the shadow system is known. For a given solution (P_w,w) of (1.12), we say μ is a eigenvalue of the linearized system of equation (1.12) if there exists a nontrivial pair $(\phi,\overrightarrow{\nu})=(\phi;\nu_1,\nu_2,\cdots,\nu_m)$

[1]The definition of $l(Q)$ is given in section 4.

such that

$$\begin{cases} \Delta\phi+2\rho_2\frac{\overline{h}_2e^{w-4\pi\sum_{j=1}^m G(x,p_j^0)}}{\int_M\overline{h}_2e^{w-4\pi\sum_{j=1}^m G(x,p_j^0)}}\phi \\ \quad -2\rho_2\frac{\overline{h}_2e^{w-4\pi\sum_{j=1}^m G(x,p_j^0)}}{\left(\int_M\overline{h}_2e^{w-4\pi\sum_{j=1}^m G(x,p_j^0)}\right)^2}\int_M\left(\overline{h}_2e^{w-4\pi\sum_{j=1}^m G(x,p_j^0)}\phi\right) \\ \quad -8\pi\rho_2\frac{\overline{h}_2e^{w-4\pi\sum_{j=1}^m G(x,p_j^0)}\sum_{j=1}^m(\nabla G(x,p_j^0)\nu_j)}{\int_M\overline{h}_2e^{w-4\pi\sum_{j=1}^m G(x,p_j^0)}} \\ \quad +8\pi\rho_2\frac{\overline{h}_2e^{w-4\pi\sum_{j=1}^m G(x,p_j^0)}\int_M\left(\overline{h}_2e^{w-4\pi\sum_{j=1}^m G(x,p_j^0)}\sum_{j=1}^m(\nabla G(x,p_j^0)\nu_j)\right)}{\left(\int_M\overline{h}_2e^{w-4\pi\sum_{j=1}^m G(x,p_j^0)}\right)^2}+\mu\phi=0, \\ \nabla^2_{x_i}f_Q(p_1^0,\cdots,p_m^0)\nu_i+\mathcal{F}_i-\frac{1}{2}\nabla\phi(p_i^0)+\mu\nu_i=0, i=1,2,\cdots,m,\quad \int_M\phi=0. \end{cases}$$

The Morse index of the solution (P_w,w) to (1.12) is the total number (counting multiplicity) of the negative eigenvalue of the linearized system, and the Leray-Schauder topological degree contributed by (P_w,w) is given by $(-1)^N$, where N is the Morse index.

From Theorem 1.3, it is known when $\rho_{1k}\to 4m\pi+4\pi\sum_{q\in S}(1+\alpha_q)$, there exists a sequence of solutions (v_{1k},v_{2k}) to (1.7) such that v_{1k} blow up at Q and $v_{2k}\to\frac{1}{2}w$. Our aim is to compute the topological degree of (1.7) contributed by those bubbling solutions satisfying the conclusion of Theorem 1.3, that is, all the bubbling solutions contained in $S_{\rho_1}(Q,w)\times S_{\rho_2}(Q,w)$. For the precise definition of $S_{\rho_i}(Q,w),\ i=1,2$, see section 4 in this paper. Let $d_T(Q,w)$ denotes the degree contributed by the solutions $(v_{1k},v_{2k})\in S_{\rho_1}(Q,w)\times S_{\rho_2}(Q,w)$ and $d_S(Q,w)$ denotes the degree of the shadow system (1.12) contributed by the Morse index of (P_w,w). We have the following theorem

Theorem 1.4. *Suppose $\alpha_q\notin\mathbb{N}$ for $q\in S$, (p_w,w) is a non-degenerate solution of (1.12) and $l(Q)\neq 0$. Let $d_T(Q,w)$ and $d_S(Q,w)$ are defined above. Then*

$$d_T(Q,w)=(-1)^n d_S(Q,w),$$

where $n=|Q|$.

We shall apply Theorem 3.3, a stronger version of Theorem 1.2, and Theorem 1.4 to calculate the degree of (1.6), or equivalently (1.7). Here we assume $S_1=S_2=\emptyset$, i.e., h_1,h_2 are $C^{2,\alpha}$ positive functions on M. It is still very difficult for us to compute the topological degree for $SU(3)$ Toda system while $S_1,S_2\neq\emptyset$. In general, the main difficulties are to prove the concentration and to get the topological degree for system (1.12). Until now, we are only able to over those difficulties under the assumption (i) and (ii). Our approach to obtain the degree of (1.11) is to use a homotopic deformation to decouple the system. However, this method can not work for (1.12) in general. The main difficulty is due to the collapse of the vortices.

In order to state our degree formulas for $SU(3)$ Toda system and the corresponding shadow system (1.11), we first introduce the following generating function

$$\Xi_1(x)=(1+x+x^2+x^3\cdots)^{-\chi(M)+1}=b_0+b_1x^1+b_2x^2+\cdots+b_kx^k+\cdots,$$

which is (1.4) provided $\alpha_q=0,\ \forall q\in S_1$. It is easy to see that

$$b_k=\begin{pmatrix} k-\chi(M) \\ k \end{pmatrix}, \tag{1.17}$$

where

$$\left(\begin{array}{c} k-\chi(M) \\ k \end{array}\right)=\begin{cases} \frac{(k-\chi(M))\cdots(1-\chi(M))}{k!}, & \text{if } k\geq 1\\ 1, & \text{if } k=0.\end{cases}$$

Theorem 1.5. *Assume $S_1=S_2=\emptyset$ and $\rho_2\notin 4N\pi, N\in\mathbb{N}$. The set of solutions (p,w) for (1.11) is pre-compact in the space $M\times \overset{\circ}{H}_1(M)$. Let d_S denotes the topological degree for* (1.11) *when $\rho_2\in(4k\pi,4(k+1)\pi)$. Then*

$$d_S=\chi(M)\cdot(b_k+b_{k-1}), \tag{1.18}$$

where $b_{-1}=0$.

Under the assumption $(i)-(ii)$ and the previous discussion, we can obtain the partial results on computing the Leray-Schauder degree for system (1.6), or equivalently (1.7).

Theorem 1.6. *Suppose $S_1=S_2=\emptyset$ and $d^{(2)}_{\rho_1,\rho_2}$ denotes the topological degree for (1.7) when $\rho_2\in(4k\pi,4(k+1)\pi)$, then*

$$d^{(2)}_{\rho_1,\rho_2}=\begin{cases} b_k, & \rho_1\in(0,4\pi),\\ b_k-\chi(M)(b_k+b_{k-1}), & \rho_1\in(4\pi,8\pi).\end{cases}$$

Set $d^{(2)}_k=d^{(2)}_{\rho_1,\rho_2}$ for $\rho_1\in(4\pi,8\pi)$ and $\rho_2\in(4k\pi,4(k+1)\pi)$. Then the generating function for $d^{(2)}_{\rho_1,k}$, $\rho_1\in(4\pi,8\pi)$ is

$$\Xi_2(x)=\sum_{k=0}^{\infty}d^{(2)}_k x^k=\big[1-\chi(M)(1+x)\big]\Xi_1(x).$$

As a consequence of Theorem 1.6, we have the following corollaries.

Corollary 1.7. *Suppose $S_1=S_2=\emptyset$, M is the sphere S^2, $\rho_1\in(4\pi,8\pi)$ and $\rho_2\in(4k\pi,4(k+1)\pi)$. Then*

$$d^{(2)}_{\rho_1,\rho_2}=\begin{cases} -1, & \text{if } k=0,\\ -1, & \text{if } k=1,\\ 2, & \text{if } k=2,\\ 0, & \text{if } k\geq 3.\end{cases}$$

Corollary 1.8. *Suppose $S_1=S_2=\emptyset$, $\rho_1,\rho_2\in(4\pi,8\pi)$, and $d^{(2)}_{2,2}$ denotes the topological degree for (1.6). We have*

$$d^{(2)}_{2,2}=(\chi(M))^2-3\chi(M)+1. \tag{1.19}$$

A consequence of the degree counting formula is the existence of (1.6). Suppose $S_2=\emptyset$ and $\chi(M)\leq 0$, then for any $(\rho_1,\rho_2)\in(4\pi,8\pi)\times(4k\pi,4(k+1)\pi)$, the system (1.6) has a solution.

This paper is organized as follows. In section 2, we prove Theorem 1.1, Theorem 1.2 and use the transversality theorem to show that there exists smooth function h_1^* and h_2^* such that any solution of shadow system (1.12) is non-degenerate. In section 3, we get the a-priori estimate for solutions of (1.7) when $\rho_1\to 4\pi$ and $\rho_2\notin 4\pi\mathbb{N}$. In section 4 and section 5, we use the solution of shadow system (1.12) to get a good approximation of some bubbling solutions of (1.7) and thereby prove Theorem 1.3 and Theorem 1.4 except some important estimates which are shown in section 8. In section 6, we prove Theorem 1.5 and derive the degree counting formula for the Shadow system (1.11). In section 7, we give a brief account for the Dirichlet problem on a bounded smooth domain of $\mathbb{R}^2$.

2. Shadow system

We shall prove Theorem 1.1 and Theorem 1.2. As mentioned in the Introduction, this result is not an immediate consequence of Proposition 2.4 in Jost-Lin-Wang [20], due to the fact of concentration has not yet been proved. Therefore, we want to provide a correct proof of this a-priori estimate. For the concentration phenomena in the general case, we shall discuss it in another paper.

For a sequence of bubbling solution (u_{1k}, u_{2k}) of (1.6). We set

$$\tilde{u}_{ik} = u_{ik} - \int_M h_i e^{u_{ik}},\ i = 1, 2.$$

Then $\tilde{u}_{ik}$ satisfy

$$\begin{cases} \Delta \tilde{u}_{1k} + 2\rho_1(h_1 e^{\tilde{u}_{1k}} - 1) - \rho_2(h_2 e^{\tilde{u}_{2k}} - 1) = 0, \\ \Delta \tilde{u}_{2k} - \rho_1(h_1 e^{\tilde{u}_{1k}} - 1) + 2\rho_2(h_2 e^{\tilde{u}_{2k}} - 1) = 0. \end{cases} \tag{2.1}$$

We define the blow up set for $\tilde{u}_{ik}$

$$\mathfrak{S}_i = \{p \in M \mid \exists \{x_k\},\ x_k \to p,\ \lim \tilde{u}_{ik}(x_k) \to +\infty\} \tag{2.2}$$

and define $\mathfrak{S} = \mathfrak{S}_1 \cup \mathfrak{S}_2$. We note that

$$u_{ik} = \tilde{u}_{ik} + \int_M h_i e^{u_{ik}} \geq \tilde{u}_{ik} + C e^{\int_M u_{ik}} \geq \tilde{u}_{ik} + C,$$

where we used the Jensen's inequality and h_i (here $h_i = h_i^*$) is a positive function in M. So, if p is a blow up point of $\tilde{u}_{ik}$, then p is also a blow up point of u_{ik}. For any $p \in \mathfrak{S}$, we define the local mass by

$$\sigma_{ip} = \lim_{\delta \to 0} \lim_{k \to +\infty} \frac{1}{2\pi} \int_{B_\delta(p)} \rho_i h_i e^{\tilde{u}_{ik}}. \tag{2.3}$$

Lemma 2.1. *If $\sigma_{1p}, \sigma_{2p} < \frac{2\pi}{3}$, we have $p \notin \mathfrak{S}$.*

Proof. The proof is a standard by using the argument in [7]. We provide a detail proof for the sake of completeness. Since $\sigma_{ip} \leq \frac{2\pi}{3}$, then we can choose small r_0, such that in $B_{r_0}(p)$, the following holds

$$\int_{B_{r_0}(p)} \rho_i h_i e^{\tilde{u}_{ik}} < \pi, \tag{2.4}$$

which implies $\int_{B_{r_0}(p)} \tilde{u}_{ik}^+ \leq C$, where C is some constant independent of k. In the following, C always denotes some generic constant independent of k, and may depend on the domain $B_{r_0}(p)$. For the first equation in (2.1), we decompose $\tilde{u}_{1k} = \sum_{j=1}^3 \tilde{u}_{1k,j}$, where $\tilde{u}_{1k,j}$ satisfy the following equation

$$\begin{cases} -\Delta \tilde{u}_{1k,1} = 2\rho_1 h_1 e^{\tilde{u}_{1k}} - \rho_2 h_2 e^{\tilde{u}_{2k}} & \text{in } B_{r_0}(p), & \tilde{u}_{1k,1} = 0 & \text{on } \partial B_{r_0}(p), \\ -\Delta \tilde{u}_{1k,2} = -2\rho_1 + \rho_2 & \text{in } B_{r_0}(p), & \tilde{u}_{1k,2} = 0 & \text{on } \partial B_{r_0}(p), \\ -\Delta \tilde{u}_{1k,3} = 0 & \text{in } B_{r_0}(p), & \tilde{u}_{1k,3} = \tilde{u}_{1k} & \text{on } \partial B_{r_0}(p). \end{cases} \tag{2.5}$$

For the first equation in (2.5), since

$$\int_{B_{r_0}(p)} \left|2\rho_1 h_1 e^{\tilde{u}_{1k}} - \rho_2 h_2 e^{\tilde{u}_{2k}}\right| < 3\pi,$$

By [7, Theorem 1], we have

$$\int_{B_{r_0}(p)} \exp((1+\delta)|\tilde{u}_{1k,1}|)dx \leq C, \tag{2.6}$$

where $\delta \in (0, \frac{1}{3})$. Therefore, we have

$$\int_{B_{r_0}(p)} |\tilde{u}_{1k,1}| \leq C. \tag{2.7}$$

For the second equation in (2.5), we can easily get

$$\int_{B_{r_0}(p)} |\tilde{u}_{1k,2}| \leq C, \text{ and } |\tilde{u}_{1k,2}| \leq C. \tag{2.8}$$

For the third equation in (2.5). By the mean value theorem for harmonic function we have

$$\begin{aligned}
\|\tilde{u}^+_{1k,3}\|_{L^\infty(B_{r_0/2}(p))} &\leq C\|\tilde{u}^+_{1k,3}\|_{L^1(B_{r_0}(p))} \\
&\leq C\Big[\|\tilde{u}^+_{1k}\|_{L^1(B_{r_0}(p))} + \|\tilde{u}_{1k,1}\|_{L^1(B_{r_0}(p))} + \|\tilde{u}_{1k,2}\|_{L^1(B_{r_0}(p))}\Big] \\
&\leq C.
\end{aligned} \tag{2.9}$$

From (2.8)-(2.9), we have

$$2\rho_1 h_1 e^{\tilde{u}_{1k,2}+\tilde{u}_{1k,3}} \leq C \text{ in } B_{r_0/2}(p). \tag{2.10}$$

By (2.6), (2.10) and Hölder inequality, we obtain

$$e^{\tilde{u}_{1k}} \in L^{1+\delta_1}(B_{r_0}(p))$$

with $\delta_1 > 0$ independent of k. Similarly, we have

$$e^{\tilde{u}_{2k}} \in L^{1+\delta_2}(B_{r_0}(p))$$

with $\delta_2 > 0$ independent of k. By using the standard elliptic estimate for the first equation in (2.5), we get $\|\tilde{u}_{1k,1}\|_{L^\infty}(B_{r_0/2}(p))$ is uniformly bounded. Combined with (2.8) and (2.9), we have $\tilde{u}_{1k}$ is uniformly bounded above in $B_{\frac{r_0}{2}}(p)$. Following a same process, we can also obtain $\tilde{u}_{2k}$ is uniformly bounded above in $B_{\frac{r_0}{2}}(p)$. Hence, we finish the proof of the lemma. □

From Lemma 2.1, we get if $p \in \mathfrak{S}$, either $\sigma_{1p} \geq \frac{2\pi}{3}$ or $\sigma_{2p} \geq \frac{2\pi}{3}$. Thus $|\mathfrak{S}| < \infty$. Therefore $\mathfrak{S}$ is discrete in M. In fact, in next lemma, we shall prove that if $p \in \mathfrak{S}_i$, σ_{ip} must be positive.

Lemma 2.2. *If $p \in \mathfrak{S}_i$, $\sigma_{ip} > 0$.*

Proof. We prove it by contradiction. Without loss of generality, we assume $\sigma_{2p} = 0$. First, we claim that there is a constant $C_K > 0$ that depends on the compact set K such that

$$|u_{ik}(x)| \leq C_K, \ \forall x \in K \subset\subset M \setminus \mathfrak{S}, \ i = 1, 2. \tag{2.11}$$

We only prove for $i=1$, the other one can be obtained similarly

$$\begin{aligned}u_{1k}(x) &= \int_M G(x,z)\Big(2\rho_1(h_1e^{\tilde u_{1k}}-1)-\rho_2(h_2e^{\tilde u_{2k}}-1)\Big)\\ &= \int_{M_1} G(x,z)\Big(2\rho_1(h_1e^{\tilde u_{1k}}-1)-\rho_2(h_2e^{\tilde u_{2k}}-1)\Big)\\ &\quad+\int_{M\setminus M_1} G(x,z)\Big(2\rho_1(h_1e^{\tilde u_{1k}}-1)-\rho_2(h_2e^{\tilde u_{2k}}-1)\Big),\end{aligned}$$

where $M_1=\cup_{p\in\mathfrak{S}}B_{r_0}(p)$ and r_0 is small enough to make $K\subset\subset M\setminus M_1$. It is easy to see that

$$\int_{M_1} G(x,z)\Big(2\rho_1(h_1e^{\tilde u_{1k}}-1)-\rho_2(h_2e^{\tilde u_{2k}}-1)\Big)=O(1),$$

because $G(x,z)$ is bounded due to the distance $d(x,z)\geq\delta_0>0$ for $z\in M_1$, and $x\in K$. In $M\setminus M_1$, we can see that $\tilde u_{ik}$ are bounded above by some constant depends on r_0, then it is not difficult to obtain that

$$\int_{M\setminus M_1} G(x,z)\Big(2\rho_1(h_1e^{\tilde u_{1k}}-1)-\rho_2(h_2e^{\tilde u_{2k}}-1)\Big)=O(1).$$

Therefore, we prove the claim. Since $\sigma_{2p}=0$, we can find some r_0, such that

$$\int_{B_{r_0}(p)}\rho_2h_2e^{\tilde u_{2k}}\leq\pi \tag{2.12}$$

for all k (passing to a subsequence if necessary) and $r_0\leq\frac12 d(p,\mathfrak{S}\setminus\{p\})$. On $\partial B_{r_0}(p)$, by (2.11)

$$|u_{1k}|,\ |u_{2k}|\leq C \text{ on } \partial B_{r_0}(p). \tag{2.13}$$

Let w_k satisfy the following equation

$$\begin{cases}\Delta w_k=\rho_1\big(\frac{h_1e^{u_{1k}}}{\int_M h_1e^{u_{1k}}}-1\big) & \text{in } B_{r_0}(p),\\ w_k=u_{1k} & \text{on } \partial B_{r_0}(p).\end{cases} \tag{2.14}$$

We set $w_k=w_{k1}+w_{k2}$ where w_{k1},w_{k2} satisfy

$$\begin{cases}\Delta w_{k1}=\rho_1\frac{h_1e^{u_{1k}}}{\int_M h_1e^{u_{1k}}} & \text{in } B_{r_0}(p), & w_{k1}=u_{1k} & \text{on } \partial B_{r_0}(p),\\ \Delta w_{k2}=-\rho_1 & \text{in } B_{r_0}(p), & w_{k1}=0 & \text{on } \partial B_{r_0}(p).\end{cases} \tag{2.15}$$

By maximum principle, we have $w_{k1}\leq\max_{\partial B_{r_0}(p)}u_{1k}\leq C$ by (2.13) for $x\in B_{r_0}(p)$. By elliptic estimate, we can easily get $|w_{k2}|\leq C$. Therefore,

$$w_k\leq C,\ \forall x\in B_{r_0}(p). \tag{2.16}$$

We set $u_{2k}=f_{k1}+f_{k2}+w_k$, where f_{k1} and f_{k2} satisfy

$$\begin{cases}\Delta f_{k1}=-2\rho_2\frac{h_2e^{u_{2k}}}{\int_M h_2e^{u_{2k}}} & \text{in } B_{r_0}(p), & f_{k1}=0 & \text{on } \partial B_{r_0}(p),\\ \Delta f_{k2}=2\rho_2 & \text{in } B_{r_0}(p), & f_{k2}=u_{2k}-w_k & \text{on } \partial B_{r_0}(p).\end{cases} \tag{2.17}$$

For the second equation in (2.17), we have

$$|f_{k2}|\leq|u_{2k}|+|w_k|=|u_{2k}|+|u_{1k}|\leq C \text{ on } \partial B_{r_0}(p),$$

Thus $|f_{k2}| \le C$ in $B_{r_0}(p)$. We denote $g_k = e^{f_{k2}+w_k}$, then the first equation in (2.16) can be written as

$$\Delta f_{k1} + 2\rho_2 \frac{h_2 e^{g_k}}{\int_M h_2 e^{u_{2k}}} e^{f_{k1}} = 0 \text{ in } B_{r_0}(p), \quad f_{k1} = 0 \text{ on } \partial B_{r_0}(p). \tag{2.18}$$

By using the Jensen's inequality, we have $\int_M h_2 e^{u_{2k}} \ge C e^{\int_M u_{2k}} \ge C > 0$. We set $V_k = 2\rho_2 \frac{h_2 e^{g_k}}{\int_M h_2 e^{u_{2k}}}$, and have $V_k \le C$, this C depends on r_0. Using (2.12), we get $\int_{B_{r_0}(p)} V_k e^{f_k} \le 2\pi$. By [7, Corollary 3], we have $|f_{k1}| \le C$ and

$$u_{2k} \le f_{k1} + f_{k2} + w_k \le C.$$

This leads to $\tilde u_{2k} = u_{2k} - \int_M h_2 e^{u_{2k}} \le C$, which contradicts to the assumption $\tilde u_{2k}$ blows up at p. Thus we finish the proof of this lemma. $\square$

By these two lemmas, we now begin to prove Theorem 1.1.

Proof of Theorem 1.1. We note that it is enough for us to prove $\tilde u_{ik}$ is uniformly bounded above. We shall prove it by contradiction.

First, we claim $\mathfrak{S}_1 \neq \emptyset$. If not, $\tilde u_{1k}$ is uniformly bounded above and $\tilde u_{2k}$ blows up. We decompose $u_{2k} = u_{2k,1} + u_{2k,2}$, where $u_{2k,1}$ and $u_{2k,2}$ satisfies the following

$$\begin{cases} \Delta u_{2k,1} - \rho_1(h_1 \tilde u_{1k} - 1) = 0, & \int_M u_{2k,1} = 0, \\ \Delta u_{2k,2} + 2\rho_2\left(\frac{\tilde h_{2k} e^{u_{2k,2}}}{\int_M \tilde h_{2k} e^{u_{2k,2}}} - 1\right) = 0, & \int_M u_{2k,2} = 0, \end{cases}$$

where $\tilde h_{2k} = h_2 e^{u_{2k,1}}$. By the L^p estimate, $u_{2k,1}$ is bounded in $W^{2,p}$ for any $p > 1$. Thus u_{1k} is bounded in $C^{1,\alpha}$ for any $\alpha \in (0,1)$, after passing to a subsequence if necessary, we gain $u_{2k,1}$ converges to u_0 in $C^{1,\alpha}$. As a consequence, $\tilde h_{2k} \to h_2 e^{u_0}$ in $C^{1,\alpha}$. Since $\tilde u_{2k}$ blows up, u_{2k} and $u_{2k,2}$ both blow up. Then applying the result of Li and Shafrir in [26], we have $\rho_2 \in 4\pi\mathbb{N}$, which contradicts to our assumption. Thus $\mathfrak{S}_1 \neq \emptyset$. Similarly, we can prove that $\mathfrak{S}_2 \neq \emptyset$.

We note that our argument above can be applied to the local case, which yields $\mathfrak{S}_1 \cap \mathfrak{S}_2 \neq \emptyset$. Suppose $\mathfrak{S}_1 \cap \mathfrak{S}_2 = \emptyset$. For any point $p \in \mathfrak{S}_2$, we consider the behavior of u_{1k} and u_{2k} in $B_{r_0}(p)$, where r_0 is small enough such that $B_{r_0}(p) \cap (\mathfrak{S} \setminus \{p\}) = \emptyset$. We decompose $\tilde u_{2k} = u_{2k,3} + u_{2k,4}$, where $u_{2k,3}$ and $u_{2k,4}$ satisfy

$$\begin{cases} \Delta u_{2k,3} - \rho_1(h_1 e^{\tilde u_{1k}} - 1) = 0 & \text{in } B_{r_0}(p), \quad u_{2k,3} = 0 \quad \text{on } \partial B_{r_0}(p), \\ \Delta u_{2k,4} + 2\rho_2(\tilde h_{2,k} e^{u_{2k,4}} - 1) = 0 & \text{in } B_{r_0}(p), \quad u_{2k,4} = \tilde u_{2k} \quad \text{on } \partial B_{r_0}(p), \end{cases} \tag{2.19}$$

where $\tilde h_{2,k} = h_2 e^{u_{2k,3}}$. By using $\tilde u_{1k}$ uniformly bounded from above in $B_{r_0}(p)$, we have $\tilde h_{2,k}$ converges in $C^{1,\alpha}(B_{r_0}(p))$. Since $u_{2k,4}$ blows up simply at p, we have

$$\left| u_{2k,4} - \log\left(\frac{e^{u_{2k,4}(p^{(k)})}}{\left(1 + \frac{\rho_2 \tilde h_{2,k}(p^{(k)}) e^{u_{2k,4}(p^{(k)})}}{4} |x - p^{(k)}|^2\right)^2} \right) \right| \le C, \tag{2.20}$$

where $u_{2k,4}(p^{(k)}) = \max_{B_{r_0}(p)} u_{2k,4}$. (2.20) is proved in [4] and [25]. From (2.20), we have

$$u_{2k,4} \to -\infty \text{ in } B_{r_0(p)} \setminus \{p\} \quad \text{and} \quad \rho_2 h_2 e^{\tilde u_{2k}} \to 4\pi\delta_p \text{ in } B_{r_0}(p), \tag{2.21}$$

which implies

$$\rho_2 = \lim_{k\to\infty} \int_M \rho_2 h_2 e^{\tilde u_{2k}} = 4\pi |\mathfrak{S}_2|, \tag{2.22}$$

a contradiction to our assumption $\rho_2 \notin 4\pi\mathbb{N}$, so $\mathfrak{S}_1 \cap \mathfrak{S}_2 \neq \emptyset$.

Let $p \in \mathfrak{S}_1 \cap \mathfrak{S}_2$, and $\sigma_{ip}, i = 1, 2$ be the local masses of them at p. Applying the result of Jost-Lin-Wang (Proposition 2.4 in [20]), we have $(\sigma_{1p}, \sigma_{2p})$ is one of $(2, 4)$, $(4, 2)$ and $(4, 4)$. By the assumption (ii), $\sigma_{1p} = 2$. Thus $\sigma_{2p} = 4$.

In the following, we claim $\tilde{u}_{2k}$ concentrate, i.e., $\tilde{u}_{2k} \to -\infty$ uniformly in any compact set of $M \setminus \mathfrak{S}_2$. Then,

$$\rho_2 h_2 e^{\tilde{u}_{2k}} \to 4\pi \sum_{q \in \mathfrak{S}_2 \setminus \{p\}} \delta_q + 8\pi\delta_p \text{ and } \rho_2 \in 4\pi\mathbb{N}, \tag{2.23}$$

which again yields a contradiction. This completes the proof of Theorem 1.1. The proof of this claim is given in Lemma 2.3 below. □

Lemma 2.3. *Suppose $\tilde{u}_{ik}, i = 1, 2$ both blow up at p, and let 2 and 4 be the local masses of $\tilde{u}_{1k}$ and $\tilde{u}_{2k}$ respectively. Then $\tilde{u}_{2k} \to -\infty$ in $B_{r_0}(p) \setminus \{p\}$.*

Proof. If the claim is not true, we have $\tilde{u}_{2k}$ is bounded by some constant C in $L^\infty(\partial B_{r_0}(p))$. Let $f_{1k} = -\rho_1(h_1 e^{\tilde{u}_{1k}} - 1) + 2\rho_2(h_2 e^{\tilde{u}_{2k}} - 1)$ and z_k be the solution of

$$\begin{cases} -\Delta z_k = f_{1k} & \text{in } B_{r_0}(p), \\ z_k = -C & \text{on } \partial B_{r_0}(P). \end{cases} \tag{2.24}$$

Note that $f_{1k} \to f_1$ uniformly in any compact set of $B_{r_0}(p) \setminus \{p\}$ and the integration of the RHS over $B_{r_0}(p)$ is $12\pi + o(1)$ as $r_0 \to 0$. By maximum principle, $\tilde{u}_{2k} \geq z_k$ in $B_{r_0}(p)$. In particular

$$\int_{B_{r_0}(p)} e^{z_k} \leq \int_{B_{r_0}(p)} e^{\tilde{u}_{2k}} < \infty.$$

On the other hand, using Green representation formula for z_k, we have

$$z_k(x) = -\int_{B_{r_0}(p)} \frac{1}{2\pi} \ln|x-y| \big(-\rho_1(h_1 e^{\tilde{u}_{1k}} - 1) + 2\rho_2(h_2 e^{\tilde{u}_{2k}} - 1) \big) + O(1), \tag{2.25}$$

where we used the regular part of the Green function is bounded. For any $x \in B_{r_0}(p) \setminus \{p\}$, we denote the distance between x and p by $2r$. From (2.25), we have

$$\begin{aligned} z_k(x) = &-\int_{B_{r_0}(p)} \frac{1}{2\pi} \ln|x-y| \big(-\rho_1(h_1 e^{\tilde{u}_{1k}} - 1) + 2\rho_2(h_2 e^{\tilde{u}_{2k}} - 1) \big) + O(1) \\ = &-\int_{B_{r_0}(p) \cap B_r(x)} \frac{1}{2\pi} \ln|x-y| \big(-\rho_1(h_1 e^{\tilde{u}_{1k}} - 1) + 2\rho_2(h_2 e^{\tilde{u}_{2k}} - 1) \big) \\ &-\int_{B_{r_0}(p) \setminus B_r(x)} \frac{1}{2\pi} \ln|x-y| \big(-\rho_1(h_1 e^{\tilde{u}_{1k}} - 1) + 2\rho_2(h_2 e^{\tilde{u}_{2k}} - 1) \big) + O(1). \end{aligned}$$

It is easy to see

$$\Big| \int_{B_{r_0}(p) \cap B_r(x)} \ln|x-y| \big(-\rho_1(h_1 e^{\tilde{u}_{1k}} - 1) + 2\rho_2(h_2 e^{\tilde{u}_{2k}} - 1) \big) \Big| \leq C,$$

due to $\tilde{u}_{ik}$ are uniformly bounded above in $B_r(x)$, $i = 1, 2$. Here C depends only on x. For $y \in B_{r_0}(p) \setminus B_r(x)$, we have $|x-y| \geq r$ and

$$\int_{B_{r_0}(p) \setminus B_r(x)} \ln|x-y| \big(-\rho_1(h_1 e^{\tilde{u}_{1k}} - 1) + 2\rho_2(h_2 e^{\tilde{u}_{2k}} - 1) \big) = (12\pi + o(1)) \ln|x-p| + O(1).$$

Therefore, we get $z_k(x)$ is uniformly bounded below by some constant that depends on x only. Thus, we have $z_k \to z$ in $C^2_{loc}(B_{r_0}(p) \setminus \{p\})$, where z satisfies

$$\begin{cases} -\Delta z = f_1 & \text{in } B_{r_0}(p) \setminus \{p\}, \\ z = -C & \text{on } \partial B_{r_0}(P). \end{cases}$$

For $\varphi \in C_0^\infty(B_{r_0}(p))$,

$$\begin{aligned} \lim_{k\to+\infty} \int_{B_{r_0}(p)} \varphi \Delta z_k &= \int_{B_{r_0}(p)} (\varphi(x) - \varphi(p))\Delta z_k + \varphi(p)(\int_{B_{r_0}(p)} f_1 + 12\pi) \\ &= \int_{B_{r_0}(p)} \varphi(x) f_1 + 12\pi\varphi(p). \end{aligned}$$

Thus $-\Delta z = f_1 + 12\pi\delta_p$. Therefore, we have $z(x) \geq 6 \log \frac{1}{|x-p|} + O(1)$ as $x \to p$, which implies $\int_{B_{r_0}(p)} e^z = \infty$, a contradiction. Hence

$$\tilde{u}_{2k} \to -\infty \text{ in } B_{r_0}(p) \setminus \{p\}. \tag{2.26}$$

□

Next, we prove Theorem 1.2 and derive the shadow system (1.11).

Proof of Theorem 1.2. As $\rho_{1k} \to 4\pi, \rho_{2k} \to \rho_2$ and $\rho_2 \notin 4\pi\mathbb{N}$, we consider a sequence of solutions (v_{1k}, v_{2k}) to (1.7) such that $\max_M(v_{1k}, v_{2k}) \to +\infty$. We claim $\max_M(\tilde{u}_{1k}, \tilde{u}_{2k}) \to +\infty$. Otherwise, $\tilde{u}_{1k}, \tilde{u}_{2k}$ are uniformly bounded above. From Green representation theorem and L^p estimate, we can get u_{1k}, u_{2k} are uniformly bounded. This implies v_{1k}, v_{2k} are uniformly bounded, which contradicts to our assumption. Let $\mathfrak{S}_i$ denotes the blow up point of $\tilde{u}_{ik},\ i = 1, 2$ as before.

We claim $\mathfrak{S}_2 = \emptyset$ and $\mathfrak{S}_1$ consists of one point only. Suppose first $\mathfrak{S}_2 \neq \emptyset$. From the proof of Theorem 1.1, if $\mathfrak{S}_1 \cap \mathfrak{S}_2 = \emptyset$, then $\tilde{u}_{2k}$ would concentrate, i.e., $\tilde{u}_{2k} \to -\infty$, $\forall x \in M \setminus \mathfrak{S}_2$, which implies $\rho_2 = \lim_{k\to+\infty} \int_M h_2 e^{\tilde{u}_{2k}} \in 4\pi\mathbb{N}$, a contradiction. Thus $\mathfrak{S}_1 \cap \mathfrak{S}_2 \neq \emptyset$. Suppose $q \in \mathfrak{S}_1 \cap \mathfrak{S}_2$, from Proposition 2.4 in [20] and the condition $\rho_{1k} < 8\pi$, we conclude $\sigma_{1q} = 2, \sigma_{2q} = 4$. By Lemma 2.3, we have $\tilde{u}_{2k}$ concentrate, which implies $\rho_2 \in 4\pi\mathbb{N}$, a contradiction again. Hence $\mathfrak{S}_2 = \emptyset$. By Lemma 2.2, $\tilde{u}_{2k}$ is uniformly bounded from above in M. Since $\max_M(\tilde{u}_{1k}, \tilde{u}_{2k}) \to +\infty$, we get $\mathfrak{S}_1 \neq \emptyset$. By the fact $\rho_{1k} \to 4\pi$, we have $\mathfrak{S}_1$ contains only one point.

We write the equation for $v_{ik},\ i = 1, 2$ as

$$\begin{cases} \Delta v_{1k} + \rho_{1k}(\frac{h_1 e^{2v_{1k} - v_{2k}}}{\int_M h_1 e^{2v_{1k} - v_{2k}}} - 1) = 0, \\ \Delta v_{2k} + \rho_{2k}(\frac{h_2 e^{2v_{2k} - v_{1k}}}{\int_M h_2 e^{2v_{2k} - v_{1k}}} - 1) = 0. \end{cases} \tag{2.27}$$

Since $\tilde{u}_{2k}$ is uniformly bounded above, the second equation of (2.27) implies that v_{2k} is uniformly bounded in M and converges to some function $\frac{1}{2}w$ in $C^{1,\alpha}(M)$. From the first equation of (2.27) and $\rho_{1k} \to 4\pi$, v_{1k} blows up at only one point, say $p \in M$.

We write the first equation in (2.27) as

$$\Delta v_{1k} + \rho_{1k}(\frac{\tilde{h}_k e^{2v_{1k}}}{\int_M \tilde{h}_k e^{2v_{1k}}} - 1) = 0, \tag{2.28}$$

where $\tilde{h}_k = h_1 e^{-v_{2k}}$. We define $\tilde{v}_{1k} = v_{1k} - \frac{1}{2} \log \int_M \tilde{h}_k e^{2v_{1k}}$. Due to the $C^{1,\alpha}$ convergence of $\tilde{h}_k$, $\tilde{v}_{1k}$ simply blows up at p by a result of Li [25] (one can also see

[4]), i.e., the following inequality holds:

$$\Big|2\tilde{v}_{1k} - \log \frac{e^{\lambda_k}}{\big(1+\frac{\rho_{1k}\tilde{h}_k(p^{(k)})e^{\lambda_k}}{4}|x-p^{(k)}|^2\big)^2}\Big| < c \text{ for } |x-p^{(k)}| < r_0, \tag{2.29}$$

where $\lambda_k = 2\tilde{v}_{1k}(p^{(k)}) = \max_{x\in B_{r_0}(p)} 2\tilde{v}_{1k}$. By using this sharp estimate, we get

$$\tilde{v}_{1k} \to -\infty \text{ in } M\setminus\{p\}, \quad \rho_{1k}\frac{h_1 e^{2v_{1k}-v_{2k}}}{\int_M h_1 e^{2v_{1k}-v_{2k}}} \to 4\pi\delta_p, \tag{2.30}$$

and

$$\nabla\big(\log(h_1 e^{-\frac{1}{2}w}) + 4\pi R(x,x)\big)\,|_{x=p}= 0, \tag{2.31}$$

which proves (1.8) and (1.9).

In the following, we claim $v_{2k} \to \frac{1}{2}w$ in $C^{2,\alpha}(M)$. From this claim and (2.28), it is easy to get

$$v_{1k} \to 8\pi G(x,p) \text{ in } C^{2,\alpha}(M\setminus\{p\}).$$

Combined with $v_{2k} \to \frac{1}{2}w$ in $C^{2,\alpha}(M)$, we have w satisfies the following equation

$$\Delta w + 2\rho_2\Big(\frac{h_2 e^{w-4\pi G(x,p)}}{\int_M h_2 e^{w-4\pi G(x,p)}} - 1\Big) = 0. \tag{2.32}$$

This proves (1.10). Therefore, we finish the proof of Theorem 1.2. The proof of the claim is given in the following Lemma 2.4. □

Lemma 2.4. *Let v_{1k}, v_{2k} be a sequence of blow up solutions of (2.27), which v_{1k} blows at p and $v_{2k} \to \frac{1}{2}w$ in $C^{1,\alpha}(M)$. Then $v_{2k} \to \frac{1}{2}w$ in $C^{2,\alpha}(M)$.*

Proof. By (2.29), we have

$$|\lambda_k - \log\int_M \tilde{h}_k e^{2v_{1k}}| < c. \tag{2.33}$$

To prove $v_{2k} \to \frac{1}{2}w$ in $C^{2,\alpha}$, we need the following estimate

$$\Big|2\nabla\tilde{v}_{1k} - \nabla\Big(\log\frac{e^{\lambda_k}}{\big(1+\frac{\rho_{1k}h(p)e^{\lambda_k}}{4}|x-p|^2\big)^2}\Big)\Big| < c \text{ for } |x-p| < r_0, \tag{2.34}$$

where (2.34) comes from the error estimate of [10, Lemma 4.1]. We write

$$h_2 e^{2v_{2k}-v_{1k}} = h_2 e^{-v_{1k}} e^{2v_{2k}}.$$

By (2.29) and (2.34), it is not difficult to show

$$\nabla\big(h_2 e^{-v_{1k}}\big) \in L^\infty(M).$$

Therefore, by classical elliptic regularity and Sobolev inequality, we can show that

$$v_{2k} \to \frac{1}{2}w \text{ in } C^{2,\alpha} \text{ for any } \alpha\in(0,1). \tag{2.35}$$

Then we finished the proof of this lemma. □

After deriving the shadow system (1.11), we show the non-degeneracy of (1.12) by applying the well-known transversality theorem.

For convenience, we write (1.12) as

$$\begin{cases} \Delta w + 2\rho_2\Big(\frac{h_2^* e^{w-4\pi\sum_{j=1}^m G(x,p_j^0)-F_0(x)}}{\int_M h_2^* e^{w-4\pi\sum_{j=1}^m G(x,p_j^0)-F_0(x)}} - 1\Big) = 0, \\ \nabla\big(\log(h_1^* e^{-\frac{1}{2}w}) + 4\pi R(x,x) + F_i(x)\big)\mid_{x=p_i^0} = 0,\ i = 1, 2, \cdots, m, \end{cases} \tag{2.36}$$

where

$$F_0(x) = 4\pi \sum_{q\in S_2} \beta_q G(x,q) + 4\pi \sum_{q\in S} (1+\alpha_q) G(x,q),$$

and

$$F_i(x) = 8\pi \sum_{j=1, j\neq i}^{m} G(x, p_j^0) + 8\pi \sum_{q\in S}(1+\alpha_q)G(x,q) - 4\pi \sum_{q\in S_1} \alpha_q G(x,q).$$

In order to show (2.36) has a non-degenerate solution, we need the following theorem, which can be found in [1], [46] and references therein. First, we recall that

Theorem 2.5. *Let $F : \mathcal{H} \times \mathcal{B} \to \mathcal{E}$ be a C^k map. $\mathcal{H}$, $\mathcal{B}$ and $\mathcal{E}$ Banach manifolds with $\mathcal{H}$ and $\mathcal{E}$ separable. If 0 is a regular value of F and $F_b = F(\cdot, b)$ is a Fredholm map of index $< k$, then the set $\{b \in \mathcal{B} : 0$ is a regular value of $F_b\}$ is residual in $\mathcal{B}$.*

We say $y \in \mathcal{E}$ is a regular value if every point $x \in F^{-1}(y)$ is a regular point, where $x \in \mathcal{H} \times \mathcal{B}$ is a regular point of F if $D_x F : T_x(\mathcal{H} \times \mathcal{B}) \to T_{F(x)}\mathcal{E}$ is onto. We say a set A is a residual set if A is a countable intersection of open dense sets, see [1], which implies A is dense in B (B is a Banach space), see [21].

Following the notations in Theorem 2.5, we denote

$$\mathcal{H} = \big(M_*^m \setminus \Gamma^m\big) \times \mathring{W}^{2,\mathfrak{p}}(M),\ \mathcal{B} = C^{2,\alpha}(M) \times C^{2,\alpha}(M),\ \mathcal{E} = (\mathbb{R}^2)^m \times \mathring{W}^{0,\mathfrak{p}}(M),$$

where

$$M_* = M \setminus S_1,\ \Gamma^m := \{(x_1, x_2, \cdots, x_m) \mid x_i \in M_*, x_i = x_j \text{ for some } i = j\},$$
$$\mathring{W}^{2,\mathfrak{p}}(M) := \{f \in W^{2,\mathfrak{p}} \mid \int_M f = 0\},\ \mathring{W}^{0,\mathfrak{p}}(M) := \{f \in L^{\mathfrak{p}} \mid \int_M f = 0\},$$

and

$$C^{2,\alpha}(M) = \{f \in C^{2,\alpha}(M)\}.$$

Remark 1. Clearly, Theorem 2.5 is local in nature. Even though $M_*^m \setminus \Gamma_m$ is not a complete manifold, we can follow the proof of the Transversality Theorem in [46] with minor modification to get Theorem 2.5, see [46, 47].

We consider the map

$$T(w, P_w, h_1^*, h_2^*) = \begin{bmatrix} \Delta w + 2\rho_2(\frac{h_2^* e^{w-4\pi\sum_{i=1}^m G(x,p_i^0)-F_0(x)}}{\int_M h_2^* e^{w-4\pi\sum_{i=1}^m G(x,p_i^0)-F_0(x)}} - 1) \\ \nabla\log\big(h_1^* e^{-\frac{1}{2}w} + 4\pi R(x,x)\big)(p_1^0) + \nabla F_1(p_1^0) \\ \vdots \\ \nabla\log\big(h_1^* e^{-\frac{1}{2}w} + 4\pi R(x,x)\big)(p_m^0) + \nabla F_m(p_m^0) \end{bmatrix}. \tag{2.37}$$

Clearly, T is C^1. Next, we claim

(i) $T(\cdot,\cdot,h_1^*,h_2^*)$ is a Fredholm map of index 0,
(ii) 0 is a regular value of T.

For the first claim, after computation, we get

$$T'_{w,P_w}(w,P_w,h_1^*,h_2^*)[\phi,\nu_1,\cdots,\nu_m]=\begin{bmatrix} T_0(w,P_w,h_1^*,h_2^*)[\phi,\nu_1,\cdots,\nu_m]\\ T_1(w,P_w,h_1^*,h_2^*)[\phi,\nu_1,\cdots,\nu_m]\\ \vdots\\ T_m(w,P_w,h_1^*,h_2^*)[\phi,\nu_1,\cdots,\nu_m]\end{bmatrix}, \tag{2.38}$$

where

$$\begin{aligned} T_0(w,P_w,h_1^*,h_2^*)[\phi,\nu_1,\cdots,\nu_m]=&\Delta\phi+2\rho_2\frac{\hat h_2e^w}{\int_M\hat h_2e^w}\phi-2\rho_2\frac{\hat h_2e^w}{(\int_M\hat h_2e^w)^2}\int_M\hat h_2e^w\phi\\ &-8\pi\rho_2\sum_{i=1}^m\frac{\hat h_2e^w}{\int_M\hat h_2e^w}\nabla G(x,p_i^0)\cdot\nu_i\\ &+8\pi\rho_2\sum_{i=1}^m\frac{\hat h_2e^w}{(\int_M\hat h_2e^w)^2}\int_M\hat h_2e^w\nabla G(x,p_i^0)\cdot\nu_i, \end{aligned}$$

$$\begin{aligned} T_i(w,P_w,h_1^*,h_2^*)[\phi,\nu_1,\cdots,\nu_m]=&\nabla_x^2(\log h_1^*e^{-\frac12w}+4\pi R(x,x)+F_i)\mid_{x=p_i^0}\cdot\nu_i\\ &+\mathcal{F}_i-\frac12\nabla\phi(p_i^0)\text{ for }i=1,2,\cdots,m, \end{aligned}$$

where

$$\hat h_2=h_2^*e^{-4\pi\sum_{i=1}^mG(x,p_i^0)-F_0(x)}$$

and

$$\mathcal{F}_i=8\pi\sum_{j=1,j\neq i}^m\nabla_x^2G(p_i^0,x)\mid_{x=p_j^0}\cdot\nu_j.$$

We decompose

$$T'_{w,P_w}[\phi,\nu_1,\cdots,\nu_m]=\begin{bmatrix}T_{01}\\T_{11}\\ \vdots\\ T_{m1}\end{bmatrix}[\phi,\nu_1,\cdots,\nu_m]+\begin{bmatrix}T_{02}\\T_{12}\\ \vdots\\ T_{m2}\end{bmatrix}[\phi,\nu_1,\cdots,\nu_m], \tag{2.39}$$

where

$$\begin{aligned} T_{01}(w,P_w,h_1^*,h_2^*)[\phi,\nu_1,\cdots,\nu_m]&=\Delta\phi+2\rho_2\frac{\hat h_2e^w}{\int_M\hat h_2e^w}\phi-2\rho_2\frac{\hat h_2e^w}{(\int_M\hat h_2e^w)^2}\int_M\hat h_2e^w\phi,\\ T_{02}(w,P_w,h_1^*,h_2^*)[\phi,\nu_1,\cdots,\nu_m]&=-8\pi\rho_2\sum_{i=1}^m\frac{\hat h_2e^w}{\int_M\hat h_2e^w}\nabla G(x,p_i^0)\nu_i\\ &\quad+8\pi\rho_2\sum_{i=1}^m\frac{\hat h_2e^w}{(\int_M\hat h_2e^w)^2}\int_M\hat h_2e^w\nabla G(x,p_i^0)\nu_i, \end{aligned}$$

$T_{i1}=0,\quad T_{i2}=T_i,$ for $i=1,2,\cdots,m.$

We define $\mathfrak{T}_1 = \begin{bmatrix} T_{01} \\ T_{11} \\ \vdots \\ T_{m1} \end{bmatrix}$ and $\mathfrak{T}_2 = \begin{bmatrix} T_{02} \\ T_{12} \\ \vdots \\ T_{m2} \end{bmatrix}$. We can easily see that $\mathfrak{T}_1$ is symmetric, it follows from the basic theory of elliptic operators that $\mathfrak{T}_1$ is a Fredholm operator of index 0. Combining the Sobolev inequality and $(\mathbb{R}^2)^m$ is a finite Euclidean space, we can show that $\mathfrak{T}_2$ is a compact operator. Therefore, by the standard linear operator theory [21], we get $\mathfrak{T}_1 + \mathfrak{T}_2$ is also a Fredholm linear operator with index 0. Hence, we prove the first claim that T is a Fredholm map with index 0.

It remains to show that 0 is a regular value. We derive the differentiation of the operator T with respect to h_1^* and h_2^*,

$$T'_{h_1^*}(w, P_w, h_1^*, h_2^*)[H_1] = \begin{bmatrix} 0 \\ \frac{\nabla H_1}{h_1^*}(p_1^0) - \frac{\nabla h_1^*}{(h_1^*)^2} H_1(p_1^0) \\ \vdots \\ \frac{\nabla H_1}{h_1^*}(p_m^0) - \frac{\nabla h_1^*}{(h_1^*)^2} H_1(p_m^0) \end{bmatrix},$$

and

$$T'_{h_2^*}(w, P_w, h_1^*, h_2^*)[H_2] = \begin{bmatrix} 2\rho_2 \frac{\hat{h}_2 e^w}{\int_M \hat{h}_2 e^w} \frac{H_2}{h_2^*} - 2\rho_2 \frac{\hat{h}_2 e^w}{(\int_M \hat{h}_2 e^w)^2} \int_M \hat{h}_2 e^w \frac{H_2}{h_2^*} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

By choosing $\nu_1 = \nu_2 = \cdots = \nu_m = 0$, and H_1 such that $\frac{\nabla H_1}{h_1^*} - \frac{\nabla h_1^*}{(h_1^*)^2} H_1 = \frac{1}{2}\nabla\phi$ at $p_i^0, i = 1, 2, \cdots, m$. We get

$$\begin{aligned} &T'_{w,P_w}(w, P_w, h_1^*, h_2^*)[\phi, \nu_1, \cdots, \nu_m] + T'_{h_1}(w, P_w, h_1^*, h_2^*)[H_1] \\ &= \begin{bmatrix} \Delta\phi + 2\rho_2 \frac{\hat{h}_2 e^w}{\int_M \hat{h}_2 e^w} \phi - 2\rho_2 \frac{\hat{h}_2 e^w}{(\int_M \hat{h}_2 e^w)^2} \int_M \hat{h}_2 e^w \phi \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \end{aligned}$$

Next, we claim that the vector space spanned by $T'_{w,P_w}(w, P_w, h_1^*, h_2^*)[\phi, \nu_1, \cdots, \nu_m]$, $T'_{h_1^*}(w, P_w, h_1^*, h_2^*)[H_1]$ and $T'_{h_2^*}(w, P_w, h_1^*, h_2^*)[H_2]$ contains $\begin{bmatrix} f \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ for all $f \in \mathring{W}^{0,p}$.

It is enough for us to prove that only $\phi = 0$ can satisfy

$$\phi \in \mathrm{Ker}\Big\{\Delta \cdot + 2\rho_2 \frac{\hat{h}_2 e^w}{\int_M \hat{h}_2 e^w} \cdot - 2\rho_2 \frac{\hat{h}_2 e^w}{(\int_M \hat{h}_2 e^w)^2} \int_M \hat{h}_2 e^w \cdot \Big\}$$

and

$$\Big\langle \phi,\ 2\rho_2 \frac{\hat{h}_2 e^w}{\int_M \hat{h}_2 e^w} \frac{H_2}{h_2^*} - 2\rho_2 \frac{\hat{h}_2 e^w}{(\int_M \hat{h}_2 e^w)^2} \int_M \hat{h}_2 e^w \frac{H_2}{h_2^*} \Big\rangle = 0,$$

for all $H_2 \in C^{2,\alpha}(M)$. We set

$$L = \Delta \cdot +2\rho_2 \frac{\hat{h}_2 e^w}{\int_M \hat{h}_2 e^w} \cdot -2\rho_2 \frac{\hat{h}_2 e^w}{(\int_M \hat{h}_2 e^w)^2} \int_M \hat{h}_2 e^w \cdot .$$

Using $\phi \in \mathrm{Ker}(L)$, we obtain that for any $\overline{H}_2 \in W^{0,p}(M)$,

$$\int_M \Big(\Delta\phi + 2\rho_2 \frac{\hat{h}_2 e^w}{\int_M \hat{h}_2 e^w} \phi - 2\rho_2 \frac{\hat{h}_2 e^w}{(\int_M \hat{h}_2 e^w)^2} \int_M \hat{h}_2 e^w \phi \Big) \cdot \overline{H}_2 = 0, \tag{2.40}$$

Since $C^{2,\alpha}(M)$ is dense in $W^{0,p}(M)$ and

$$\Big\langle \phi, 2\rho_2 \frac{\hat{h}_2 e^w}{\int_M \hat{h}_2 e^w} \overline{H}_2 - 2\rho_2 \frac{\hat{h}_2 e^w}{(\int_M \hat{h}_2 e^w)^2} \int_M \hat{h}_2 e^w \overline{H}_2 \Big\rangle = 0,$$

we deduce

$$\int_M \Delta\phi \cdot \overline{H}_2 = 0, \quad \forall\, \overline{H}_2 \in W^{0,p}(M). \tag{2.41}$$

Thus

$$\Delta\phi = 0 \text{ in } M, \quad \int_M \phi = 0. \tag{2.42}$$

So $\phi \equiv 0$. Therefore the claim is proved.

On the other hand, we choose two functions, $H_{1,1}$ and $H_{1,2}$ such that

$$H_{1,i}(p_j^0) = 0, \nabla H_{1,i}(p_j^0) = 0,\ 2 \le j \le m,\ i = 1, 2.$$

Based on this choice, we can further make such $H_{1,i}, i = 1, 2$ that

$$\frac{\nabla H_{1,1}}{h_1^*}(p_1^0) - \frac{\nabla h_1^*}{(h_1^*)^2} H_{1,1}(p_1^0) = (1, 0),$$

and

$$\frac{\nabla H_{1,2}}{h_1^*}(p_1^0) - \frac{\nabla h_1^*}{(h_1^*)^2} H_{1,2}(p_1^0) = (0, 1).$$

Then it is not difficult to see that (By setting $\phi = 0, \nu_1 = \nu_2 = \cdots = \nu_m = 0$)

$$\begin{bmatrix} 0 \\ c \\ 0 \\ \vdots \\ 0 \end{bmatrix} \subset DT(w, P_w, h_1^*, h_2^*)[\phi, \nu_1, \cdots, \nu_m, H_1, H_2]$$

for all $c \in \mathbb{R}^2$. Similarly, we can show

$$\begin{bmatrix} 0 \\ c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} \subset DT(w, P_w, h_1^*, h_2^*)[\phi, \nu_1, \cdots, \nu_m, H_1, H_2]$$

for all $c_i \in \mathbb{R}^2,\ i = 1, 2, \cdots, m$. Therefore, we proved that the differential map is onto. As a consequence, 0 is a regular point of T. By Theorem 2.5, we have

$$\Big\{ (h_1^*, h_2^*) \in \mathcal{B} : 0 \text{ is a regular value of } T(\cdot, \cdot, h_1^*, h_2^*) \Big\}$$

is residual in $\mathcal{B}$. Since $T(w, P_w, h_1^*, h_2^*)$ is a Fredholm map of index 0 for fixed h_1^*, h_2^*, we have

$$\Big\{(h_1^*, h_2^*) \in \mathcal{B} : \text{ the solution } (w, P_w) \text{ of } T(\cdot, \cdot, h_1^*, h_2^*) = 0 \text{ is nondegenerate}\Big\}$$

is residual in $\mathcal{B}$. Thus, we can choose $h_1^*, h_2^* > 0$ such that the solution of (1.12) is non-degenerate.

3. Apriori estimate

In this section, we shall prove that all the blow up solutions of (1.7) must be contained in the set $S_{\rho_1}(p, w) \times S_{\rho_2}(p, w)$ when $\rho_1 \to 4\pi$, $\rho_2 \notin 4\pi\mathbb{N}$, where the definition of $S_{\rho_i}(p, w), i = 1, 2$ is given in (3.14) and (3.15) of this section.

To simplify our description, we may assume M has a flat metric near a neighborhood of each blow up point. Of course we can modify our arguments without any difficulty for the general case, as in [11].

We start to define the set $S_{\rho_i}(p, w)$. For any given non-degenerate solution (p, w) of (1.11), we set

$$h = h_1 e^{-\frac{1}{2}w}. \tag{3.1}$$

Note that

$$\nabla_x(\log h + 4\pi R(x, x)) \mid_{x=p} = \nabla_x\big(\log h(x) + 8\pi R(x, p)\big) \mid_{x=p} = 0, \tag{3.2}$$

whenever (p, w) is a solution of shadow system (1.11). For q such that $|q - p| \ll 1$ and large $\lambda > 0$, we set

$$U(x) = \lambda - 2\log\big(1 + \frac{\rho_1 h(q)}{4} e^{\lambda}|x - q|^2\big), \tag{3.3}$$

and $U(x)$ satisfies the following equation

$$\Delta U(x) + 2\rho_1 h(q) e^U = 0 \text{ in } \mathbb{R}^2, \quad U(q) = \max_{\mathbb{R}^2} U(x) = \lambda. \tag{3.4}$$

Let

$$H(x) = \exp\Big\{\log\frac{h(x)}{h(q)} + 8\pi R(x, q) - 8\pi R(q, q)\Big\} - 1, \tag{3.5}$$

and

$$s = \lambda + 2\log\Big(\frac{\rho_1 h(q)}{4}\Big) + 8\pi R(q, q) + \frac{\Delta H(q)}{\rho_1 h(q)}\frac{\lambda^2}{e^{\lambda}}. \tag{3.6}$$

Let $\sigma_0(t)$ be a cut-off function:

$$\sigma_0(t) = \begin{cases} 1, & \text{if } |t| < r_0, \\ 0, & \text{if } |t| \geq 2r_0. \end{cases}$$

Set $\sigma(x) = \sigma_0(|x - q|)$ and

$$J(x) = \begin{cases} \big(H(x) - \nabla H(p) \cdot (x - p)\big)\sigma, & x \in B_{2r_0}(q), \\ 0, & x \notin B_{2r_0}(q). \end{cases}$$

Let $\eta(x)$ satisfy

$$\begin{cases} \Delta\eta + 2\rho_1 h(q)e^U(\eta + J(x)) = 0 \quad \text{on } \mathbb{R}^2, \\ \eta(q) = 0, \nabla\eta(q) = 0. \end{cases} \tag{3.7}$$

The existence of η was proved in [11]. Furthermore, we have the following lemma

Lemma 3.1. *Let $R = \sqrt{\frac{\rho_1 h(q)}{4}e^\lambda}$. For $h \in C^{2,\alpha}(M)$ and large λ. there exists a solution η satisfying (3.7) and the following*

(i) $\eta(x) = -\frac{4\Delta H(q)}{\rho_1 h(q)}e^{-\lambda}[\log(R|x-q|+2)]^2 + O(\lambda e^{-\lambda})$ on $B_{2r_0}(q)$,

(ii) $\eta, \nabla_x\eta, \partial_q\eta, \partial_\lambda\eta, \nabla_x\partial_q\eta, \nabla_x\partial_\lambda\eta = O(\lambda^2 e^{-\lambda})$ on $B_{2r_0}(q)$.

The proof of Lemma 3.1 was given in [11].

We set

$$\begin{cases} v_q(x) = \Big(U(x) + \eta(x) + 8\pi(R(x,q) - R(q,q)) + s\Big)\sigma(x) \\ \qquad\qquad +8\pi G(x,q)(1-\sigma(x)), \\ \overline{v}_q = \frac{1}{|M|}\int_M v_q, \\ v_{q,\lambda,a} = a(v_q - \overline{v}_q). \end{cases} \tag{3.8}$$

Note that $v_q(x)$ depends on q and λ. Next, we define $O^{(1)}_{q,\lambda}$ and $O^{(2)}_{q,\lambda}$:

$$O^{(1)}_{q,\lambda} = \Big\{\phi \in \mathring{H}^1(M) \;\Big|\; \int_M \nabla\phi\cdot\nabla v_q = \int_M \nabla\phi\cdot\nabla\partial_q v_q = \int_M \nabla\phi\cdot\nabla\partial_\lambda v_q = 0\Big\}, \tag{3.9}$$

and

$$O^{(2)}_{q,\lambda} = \Big\{\psi \in W^{2,\mathfrak{p}}(M) \;\Big|\; \int\psi = 0\Big\}, \quad \mathfrak{p} > 2. \tag{3.10}$$

For each (q,λ), we define

$$t = \lambda + 8\pi R(q,q) + 2\log\frac{\rho_1 h(q)}{4} + \frac{\Delta H(q)}{\rho_1 h(q)}\lambda^2 e^{-\lambda} - \overline{v}_q. \tag{3.11}$$

For $\rho_1 \neq 4\pi$, we define $\lambda(\rho_1)$ such that

$$\rho_1 - 4\pi = \frac{\Delta\log h(p) + 8\pi - 2K(p)}{h(p)}\lambda(\rho_1)e^{-\lambda(\rho_1)}, \tag{3.12}$$

where (p,w) is the non-degenerate solution of (1.11) and $K(p)$ denotes the Gaussian curvature of p. By using the equation (1.10), we have $e^{-4\pi G(x,p)}\ |_{x=p} = 0$ and $\Delta w(p) = 2\rho_2$. Thus

$$\Delta\log h(p) + 8\pi - 2K(p) = \Delta\log h_1(p) - \rho_2 + 8\pi - 2K(p). \tag{3.13}$$

Obviously, $\lambda(\rho_1)$ can be well-defined only if

$$\Delta\log h_1(p) - \rho_2 + 8\pi - 2K(p) \neq 0.$$

Let c be a positive constant, which will be chosen later. By using ρ_1, we set

$$\begin{aligned} S_{\rho_1}(p,w) = \Big\{v_1 = \frac{1}{2}v_{q,\lambda,a} + \phi \;\Big|\; & |q-p| \le c\lambda(\rho_1)e^{-\lambda(\rho_1)}, |\lambda - \lambda(\rho_1)| \le c\lambda(\rho_1)^{-1}, \\ & |a-1| \le c\lambda(\rho_1)^{-\frac{1}{2}}e^{-\lambda(\rho_1)}, \phi \in O^{(1)}_{q,\lambda} \text{ and } \|\phi\|_{H^1(M)} \le c\lambda(\rho_1)e^{-\lambda(\rho_1)}\Big\}, \end{aligned} \tag{3.14}$$

and

$$S_{\rho_2}(p,w)=\Big\{v_2=\frac{1}{2}w+\psi\ \Big|\ \psi\in O^{(2)}_{q,\lambda}\text{ and }\|\psi\|_*\le c\lambda(\rho_1)e^{-\lambda(\rho_1)}\Big\},\tag{3.15}$$

where $\|\psi\|_*=\|\psi\|_{W^{2,\mathfrak{p}}(M)}$.

Now suppose (v_{1k},v_{2k}) is a sequence of bubbling solutions of (1.7) such that v_{1k} blows up at p and weakly converges to $4\pi G(x,p)$, while $v_{2k}\to\frac{1}{2}w$ in $C^{2,\alpha}(M)$. Then we want to prove that

$$(v_{1k},v_{2k})\in S_{\rho_1}(p,w)\times S_{\rho_2}(p,w).$$

First of all, we prove the following lemma.

Lemma 3.2. *Let (v_{1k},v_{2k}) be a sequence of blow up solutions of (1.7), which v_{1k} blows up at p, weakly converges to $4\pi G(x,p)$ and $v_{2k}\to\frac{1}{2}w$ in $C^{2,\alpha}(M)$. Suppose (p,w) is a non-degenerate solution of (1.11) and*

$$\Delta\log h_1(p)-\rho_2+8\pi-2K(p)\neq 0.\tag{3.16}$$

Then there exist $c>0$, q_k, λ_k, a_k, ϕ_k, ψ_k such that

$$v_{1k}=\frac{1}{2}v_{q_k,\lambda_k,a_k}+\phi_k,\qquad v_{2k}=\frac{1}{2}w+\psi_k,\tag{3.17}$$

and $(v_{1k},v_{2k})\in S_{\rho_1}(p,w)\times S_{\rho_2}(p,w)$.

Remark 2. Because the proof of this lemma is very long, we describe the process briefly. First of all, we have to obtain a good approximation of v_{1k}. Since v_{2k} converges to $\frac{1}{2}w$ in $C^{2,\alpha}(M)$, this fine estimate can be obtained by the same proof in Lemma [10]. Next, we substitute v_{1k} into the second equation of v_{2k}. Then we use the non-degeneracy of (1.11) to get the sharp estimates of ψ_k and $|\tilde q_k-p|$, where $\psi_k=v_{2k}-\frac{1}{2}w$ and $\tilde q_k$ is the point where v_{1k} obtains its maximal value. After that, we get the lemma. In the following proof, we use the same notation as the proof of Theorem 1.2.

Proof. Let v_{1k} and v_{2k} be a sequence of blow up solutions of (1.7).

$$\begin{cases}\Delta v_{1k}+\rho_{1k}\big(\frac{h_1e^{2v_{1k}-v_{2k}}}{\int_M h_1e^{2v_{1k}-v_{2k}}}-1\big)=0,\\ \Delta v_{2k}+\rho_{2k}\big(\frac{h_2e^{2v_{2k}-v_{1k}}}{\int_M h_2e^{2v_{2k}-v_{1k}}}-1\big)=0.\end{cases}\tag{3.18}$$

For convenience, we write the first equation in (3.18) as,

$$\Delta v_{1k}+\rho_{1k}\Big(\frac{\tilde h_ke^{2v_{1k}}}{\int_\Omega\tilde h_ke^{2v_{1k}}}-1\Big)=0,\tag{3.19}$$

where

$$\tilde h_k=h_1e^{-v_{2k}}=he^{-\psi_k}\text{ and }\psi_k=v_{2k}-\frac{1}{2}w.\tag{3.20}$$

Since $\tilde h_k\to h$ in $C^{2,\alpha}(M)$, all the estimates in [10] can be applied to our case here, although in [10] the coefficient $\tilde h_k$ is independent of k. In the followings (up to (3.28) below), we sketch the estimates in [10, 11] which will be used here. We denote $\tilde q_k$ to be the maximal point of $\tilde v_{1k}$ near p, where $\tilde v_{1k}=v_{1k}-\frac{1}{2}\log\int_M\tilde h_ke^{2v_{1k}}$. Let

$$\lambda_k=2\tilde v_{1k}(\tilde q_k)-\log\int_M\tilde h_ke^{2v_{1k}}.$$

In the local coordinate near $\tilde{q}_k$, we set

$$\tilde{U}_k(x) = \log \frac{e^{\lambda_k}}{(1 + \frac{\rho_{1k}\tilde{h}_k(q_k)}{4} e^{\lambda_k}|x - q_k|^2)^2}.$$

where q_k is chosen such that

$$\nabla \tilde{U}_k(\tilde{q}_k) = \nabla \log h(\tilde{q}_k),$$

clearly $|q_k - \tilde{q}_k| = O(e^{-\lambda_k})$. Then the error term inside $B_{r_0}(q_k)$ is set by

$$\tilde{\eta}_k(x) = 2\tilde{v}_{1k} - \tilde{U}_k(y) - (8\pi R(x, q_k) - 8\pi R(q_k, q_k)), \tag{3.21}$$

and the error term outside $B_{r_0}(q_k)$ is set by

$$\xi_k(x) = 2v_{1k}(x) - 8\pi G(x, q_k). \tag{3.22}$$

By Green's representation for v_{1k}, it is not difficult to obtain

$$\xi_k(x) = O(\lambda_k e^{-\lambda_k}) \text{ for } x \in M \setminus B_{r_0}(q_k). \tag{3.23}$$

By a straightforward computation, the error term $\tilde{\eta}_k$ satisfies

$$\Delta\tilde{\eta}_k + 2\rho_{1k}\tilde{h}_k(q_k)e^{\tilde{U}_k}\tilde{H}_k(x, \tilde{\eta}_k) = 0, \tag{3.24}$$

where

$$\begin{aligned}\tilde{H}_k(x,t) =& \exp\{\log \frac{\tilde{h}_k(x)}{\tilde{h}_k(q_k)} + 8\pi\big(R(x, q_k) - R(q_k, q_k)\big) + t\} - 1\\ =& H_k(x) + t + O(|t|^2),\end{aligned}$$

and

$$H_k(x) = \exp\Big\{\log\frac{\tilde{h}_k(x)}{\tilde{h}_k(q_k)} + 8\pi R(x, q_k) - 8\pi R(q_k, q_k)\Big\} - 1.$$

We see that except for the higher-order term $O(|\tilde{\eta}_k|^2)$, equation (3.24) is exactly like (3.7). By Lemma 3.1, we can prove

$$\tilde{\eta}_k(x) = -\frac{4}{\rho_{1k}\tilde{h}_k(q_k)}\Delta \log H_k(q_k)e^{-\lambda_k}[\log(R_k|x - q_k| + 2)]^2 + O(\lambda_k e^{-\lambda_k}) \tag{3.25}$$

for $x \in B_{2r_0}(q_k)$, where $R_k = \sqrt{\frac{\rho_{1k}\tilde{h}_k(q_k)}{4}e^{\lambda_k}}$.

From [10, Theorem 1.1, Theorem 1.4 and Lemma 5.4], we have

$$\rho_{1k} - 4\pi = \frac{\Delta \log \tilde{h}_k(q_k) + 8\pi - 2K(q_k)}{\tilde{h}_k(q_k)}\lambda_k e^{-\lambda_k} + O(e^{-\lambda_k}), \tag{3.26}$$

$$2\overline{\tilde{v}}_{1k} + \lambda_k + 2\log\frac{\rho_{1k}\tilde{h}_k(q_k)}{4} + 8\pi R(q_k, q_k) + \frac{\Delta H_k(q_k)}{\rho_{1k}\tilde{h}_k(q_k)}\lambda_k^2 e^{-\lambda_k} = O(\lambda_k e^{-\lambda_k}), \tag{3.27}$$

and

$$|\nabla H_k(q_k)| = O(\lambda_k e^{-\lambda_k}). \tag{3.28}$$

Now we let η_k be defined as in (3.7), v_{q_k} and v_{q_k,λ_k,a_k} be defined as in (3.8) with $q = q_k$, $\lambda = \lambda_k$ and $a = a_k = 1$. By Lemma 3.1, (3.25) and (3.28), we have

$$\eta_k(x) = \tilde{\eta}_k + O(\lambda_k e^{-\lambda_k}) \text{ for } x \in B_{2r_0}(q_k). \tag{3.29}$$

Note that for $x \in B_{r_0}(q_k)$,

$$
\begin{aligned}
v_{q_k,\lambda_k,a_k} =&\tilde{U}_k(x) + \eta_k(x) + \big(8\pi R(x,q_k) - 8\pi R(q_k,q_k)\big) + \lambda_k + 2\log\frac{\rho_{1k}\tilde{h}_k(q_k)}{4}\\
&+ 8\pi R(q_k,q_k) + \frac{\Delta H_k(q_k)}{\rho_{1k}\tilde{h}_k(q_k)}\lambda_k^2 e^{-\lambda_k} - \overline{v}_{q_k},
\end{aligned}
$$

where $\overline{v}_{q_k}$ denotes the average of v_{q_k}. From [11, Lemma 2.2 and Lemma 2.3], we have

$$
v_{q_k} - 4\pi G(x,q_k) = O(\lambda_k e^{-\lambda_k}) \text{ in } M\setminus B_{2r_0}(q_k), \text{ and } \overline{v}_{q_k} = O(\lambda_k e^{-\lambda_k}). \tag{3.30}
$$

By (3.21), (3.27), (3.29) and (3.30), we have

$$
\begin{aligned}
2v_{1k} - v_{q_k,\lambda_k,a_k} =&2\tilde{v}_{1k} + \int_M \tilde{h}_k e^{2v_{1k}} - v_{q_k,\lambda_k,a_k}\\
=&2\tilde{v}_{1k} - \tilde{U}_k - \big(8\pi R(x,x) - 8\pi R(x,q_k)\big) - \eta_k(x) + O(\lambda_k e^{-\lambda_k})\\
=&\tilde{\eta}_k(x) - \eta_k(x) + O(\lambda_k e^{-\lambda_k}) = O(\lambda_k e^{-\lambda_k})
\end{aligned} \tag{3.31}
$$

for $x \in B_{r_0}(q_k)$. For $x \in M\setminus B_{2r_0}(q_k)$, by (3.22) and (3.30), we get

$$
2v_{1k} - v_{q_k,\lambda_k,a_k} = 2v_{1k} - 8\pi G(x,q_k) - (v_{q_k} - 8\pi G(x,q_k)) + \overline{v}_{q_k} = O(\lambda_k e^{-\lambda_k}).
$$

For the intermediate domain $B_{2r_0}(q_k)\setminus B_{r_0}(q_k)$, following a similar way, we can obtain that $2v_{1k} - v_{q_k,\lambda_k,a_k} = O(\lambda_k e^{-\lambda_k})$. Thus, we find a good approximation $\frac{1}{2}v_{q_k,\lambda_k,a_k}$ for v_{1k}. For convenience, we write

$$
v_{1k} = \frac{1}{2}v_{q_k,\lambda_k,a_k} + \phi_k, \text{ where } \|\phi_k\|_{L^\infty(M)} < \tilde{c}\lambda_k e^{-\lambda_k}, \tag{3.32}
$$

where $\tilde{c}$ is independent of ψ_k.

Next, we substitute (3.32) and $v_{2k} = \frac{1}{2}w + \psi_k$ into the second equation of (3.18), after computation, we obtain

$$
\mathcal{L}\psi_k = \mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3, \quad \int_M \psi_k = 0, \tag{3.33}
$$

where

$$
\begin{aligned}
L\psi_k =&\Delta\psi_k + 2\rho_2\frac{h_2 e^{w-4\pi G(x,p)}}{\int_M h_2 e^{w-4\pi G(x,p)}}\psi_k\\
&- 2\rho_2\frac{h_2 e^{w-4\pi G(x,p)}}{(\int_M h_2 e^{w-4\pi G(x,p)})^2}\int_M (h_2 e^{w-4\pi G(x,p)}\psi_k)\\
&- 4\pi\rho_2\frac{h_2 e^{w-4\pi G(x,p)}}{\int_M h_2 e^{w-4\pi G(x,p)}}\big(\nabla G(x,p)(q_k-p)\big)\\
&+ 4\pi\rho_2\frac{h_2 e^{w-4\pi G(x,p)}}{(\int_M h_2 e^{w-4\pi G(x,p)})^2}\int_M \big(h_2 e^{w-4\pi G(x,p)}(\nabla G(x,p)(q_k-p))\big),
\end{aligned}
$$

$$
\mathbb{I}_1 = -\rho_2\frac{h_2 e^{w+2\psi_k - v_{1k}}}{\int_M h_2 e^{w+2\psi_k - v_{1k}}} + \rho_2\frac{h_2 e^{w+2\psi_k - 4\pi G(x,q_k)}}{\int_M h_2 e^{w+2\psi_k-4\pi G(x,q_k)}},
$$

$$\begin{aligned}\mathbb{I}_2 =&\rho_2\frac{h_2e^{w-4\pi G(x,p)}}{\int_M h_2e^{w-4\pi G(x,p)}}-\rho_2\frac{h_2e^{w+2\psi_k-4\pi G(x,p)}}{\int_M h_2e^{w+2\psi_k-4\pi G(x,p)}}\\&+2\rho_2\frac{h_2e^{w-4\pi G(x,p)}}{\int_M h_2e^{w-4\pi G(x,p)}}\psi_k\\&-2\rho_2\frac{h_2e^{w-4\pi G(x,p)}}{(\int_M h_2e^{w-4\pi G(x,p)})^2}\int_M(h_2e^{w-4\pi G(x,p)}\psi_k),\end{aligned}$$

and

$$\begin{aligned}\mathbb{I}_3 =&-\rho_2\frac{h_2e^{w+2\psi_k-4\pi G(x,q_k)}}{\int_M h_2e^{w+2\psi_k-4\pi G(x,q_k)}}+\rho_2\frac{h_2e^{w+2\psi_k-4\pi G(x,p)}}{\int_M h_2e^{w+2\psi_k-4\pi G(x,p)}}\\&-4\pi\rho_2\frac{h_2e^{w-4\pi G(x,p)}}{\int_M h_2e^{w-4\pi G(x,p)}}(\nabla G(x,p)(q_k-p))\\&+4\pi\rho_2\frac{h_2e^{w-4\pi G(x,p)}}{(\int_M h_2e^{w-4\pi G(x,p)})^2}\int_M\big(h_2e^{w-4\pi G(x,p)}(\nabla G(x,p)(q_k-p))\big).\end{aligned}$$

We shall analyze the right hand side of (3.33) term by term in the following. For $\mathbb{I}_1$, we set

$$\mathfrak{E}_1=\exp\big(w+2\psi_k-4\pi G(x,q_k)\big)-\exp\big(w+2\psi_k-\frac{1}{2}v_{q_k,\lambda_k,a_k}-\phi_k\big).$$

For $x\in M\setminus B_{r_0}(q_k)$. We see that the difference between $4\pi G(x,q_k)$ and v_{q_k,λ_k,a_k} is of order $\lambda_ke^{-\lambda_k}$. As a consequence, $\mathfrak{E}_1=O(\lambda_ke^{-\lambda_k})$.

For $x\in B_{r_0}(q_k)$,

$$\begin{aligned}4\pi G(x,q_k)-\frac{1}{2}v_{q_k,\lambda_k,a_k}=&4\pi G(x,q_k)-4\pi R(x,q_k)-\log(\frac{\rho_1\tilde h_k(q_k)e^{\lambda_k}}{4})\\&+\log(1+\frac{\rho_1\tilde h_k(q_k)e^{\lambda_k}}{4}|x-q_k|^2)-\lambda_k\\&-\frac{1}{2}(\eta_k+\frac{\Delta H_k(q_k)}{\rho_1\tilde h_k(q_k)}\frac{\lambda_k^2}{e^{\lambda_k}})+O(\lambda_ke^{-\lambda_k})\\=&\log(\frac{4}{\rho_1\tilde h_k(q_k)e^{\hat\lambda_k}|x-q_k|^2}+1)\\&-\frac{1}{2}(\eta_k+\frac{\Delta H_k(q_k)}{\rho_1\tilde h_k(q_k)}\frac{\lambda_k^2}{e^{\lambda_k}})+O(\lambda_ke^{-\lambda_k}).\end{aligned}$$

Since $\phi_k=O(\lambda_ke^{-\lambda_k})$,

$$\exp\big(w+2\psi_k-\frac{1}{2}v_{q_k,\lambda_k,a_k}-\phi_k\big)=\exp\big(w+2\psi_k-\frac{1}{2}v_{q_k,\lambda_k,a_k}\big)+O(\lambda_ke^{-\lambda_k}).$$

Then, we have

$$
\begin{aligned}
&\exp\big(w+2\psi_k-4\pi G(x,q_k)\big)-\exp\big(w+2\psi_k-\frac{1}{2}v_{q_k,\lambda_k,a_k}-\phi_k\big)\\
&=\exp\big(w+2\psi_k-4\pi G(x,q_k)\big)-\exp\big(w+2\psi_k-\frac{1}{2}v_{q_k,\lambda_k,a_k}\big)+O(\lambda_k e^{-\lambda_k})\\
&=\exp\big(w+2\psi_k-4\pi G(x,q_k)\big)\big(1-\exp\big(4\pi G(x,q_k)-\frac{1}{2}v_{q_k,\lambda_k,a_k}\big)\big)+O(\lambda_k e^{-\lambda_k})\\
&=\exp\big(w+2\psi_k-4\pi G(x,q_k)\big)\Big(1-\exp\Big[\log(1+\frac{4}{\rho_1\tilde{h}_k e^{\lambda_k}|x-q_k|^2})\\
&\qquad+O(\eta_k+\frac{\Delta H_k(q_k)}{\rho_1\tilde{h}_k(q_k)}\frac{\lambda_k^2}{e^{\lambda_k}})+O(\lambda_k e^{-\lambda_k})\Big]\Big)
\end{aligned}
$$

When $|x-q_k|=O(e^{-\frac{\lambda_k}{2}})$, we have

$$\exp\big(w+2\psi_k-4\pi G(x,q_k)\big)=O(e^{-\lambda_k}),$$

and

$$\log(1+\frac{4}{\rho_1\tilde{h}_k e^{\lambda_k}|x-q_k|^2})+O(\eta_k+\frac{\Delta H_k(q_k)}{\rho_1\tilde{h}_k(q_k)}\frac{\lambda_k^2}{e^{\lambda_k}})=O\big(\log(e^{-\lambda_k}|x-q_k|^{-2})\big),$$

hence

$$\mathfrak{E}_1=O(\lambda_k e^{-\lambda_k})\text{ for }|x-q_k|=O(e^{-\frac{\lambda_k}{2}}).$$

When $|x-q_k|\gg e^{-\frac{\lambda_k}{2}}$, then

$$
\begin{aligned}
1-\exp\big(\log(1+\frac{4}{\rho_1\tilde{h}_k e^{\lambda_k}|x-q_k|^2})&+O(\eta_k+\frac{\Delta H_k(q_k)}{\rho_1\tilde{h}_k(q_k)}\frac{\lambda_k^2}{e^{\lambda_k}})\big)+O(\lambda_k e^{-\lambda_k})\\
&=O(\frac{4}{\rho_1\tilde{h}_k e^{\lambda_k}|x-q_k|^2}+\lambda_k e^{-\lambda_k}),
\end{aligned}
$$

as a result, we have

$$\mathfrak{E}_1=O(\lambda_k e^{-\lambda_k})\text{ for }r_0\geq|x-q_k|\gg e^{-\frac{\lambda_k}{2}}.$$

Thus, $\|\mathfrak{E}_1\|_{L^\infty(M)}=O(\lambda_k e^{-\lambda_k})$. This implies $\mathbb{I}_1=O(\lambda_k e^{-\lambda_k})$.

For the second term, it is easy to see that $\mathbb{I}_2=O(\|\psi_k\|_*)$. It remains to estimate $\mathbb{I}_3$. We divide it into three parts.

$$\mathbb{I}_3=\mathbb{I}_{31}+\mathbb{I}_{32}+\mathbb{I}_{33},$$

where

$$
\begin{aligned}
\mathbb{I}_{31}=&-\rho_2\frac{h_2e^{w+2\psi_k-4\pi G(x,q_k)}}{\int_M h_2e^{w+2\psi_k-4\pi G(x,q_k)}}+\rho_2\frac{h_2e^{w+2\psi_k-4\pi G(x,p)}}{\int_M h_2e^{w+2\psi_k-4\pi G(x,p)}}\\
&-4\pi\rho_2\frac{h_2e^{w+2\psi_k-4\pi G(x,p)}}{\int_M h_2e^{w+2\psi_k-4\pi G(x,p)}}\big(\nabla G(x,p)(q_k-p)\big),\\
&+4\pi\rho_2\frac{h_2e^{w+2\psi_k-4\pi G(x,p)}}{(\int_M h_2e^{w+2\psi_k-4\pi G(x,p)})^2}\int_M\Big(h_2e^{w+2\psi_k-4\pi G(x,p)}(\nabla G(x,p)(q_k-p))\Big),
\end{aligned}
$$

$$\begin{aligned}\mathbb{I}_{32} =&4\pi\rho_2\frac{h_2e^{w-4\pi G(x,p)}}{(\int_M h_2e^{w-4\pi G(x,p)})^2}\int_M\Big(h_2e^{w-4\pi G(x,p)}(\nabla G(x,p)(q_k-p))\Big)\\&-4\pi\rho_2\frac{h_2e^{w+2\psi_k-4\pi G(x,p)}}{(\int_M h_2e^{w+2\psi_k-4\pi G(x,p)})^2}\int_M\Big(h_2e^{w+2\psi_k-4\pi G(x,p)}(\nabla G(x,p)(q_k-p))\Big),\end{aligned}$$

and

$$\begin{aligned}\mathbb{I}_{33} =&4\pi\rho_2\frac{h_2e^{w+2\psi_k-4\pi G(x,p)}}{\int_M h_2e^{w+2\psi_k-4\pi G(x,p)}}\big(\nabla G(x,p)(q_k-p)\big)\\&-4\pi\rho_2\frac{h_2e^{w-4\pi G(x,p)}}{\int_M h_2e^{w-4\pi G(x,p)}}\big(\nabla G(x,p)(q_k-p)\big).\end{aligned}$$

It is not difficult to see

$$\mathbb{I}_{31}=O(|q_k-p|^2),\ \mathbb{I}_{32}=O(1)\|\psi\|_*|q_k-p|,\ \mathbb{I}_{33}=O(1)\|\psi\|_*|q_k-p|.$$

Then (3.33) can be written as

$$\mathcal{L}(\psi_k)=o(1)\|\psi_k\|_*+O(\|\psi_k\|_*^2+\lambda_ke^{-\lambda_k})+O(|p-q_k|^2). \tag{3.34}$$

By the definition of H_k and (3.28), we have

$$\nabla H_k(q_k)=\nabla\log h(q_k)-\nabla\psi(q_k)+8\pi\nabla R(q_k,q_k)=O(\lambda_ke^{-\lambda_k}). \tag{3.35}$$

By (3.2) and (3.35), we have

$$\begin{aligned}\nabla^2\big(\log h(p)+8\pi R(p,p)\big)(q_k-p)-\nabla\psi_k(p)=&\nabla\log h(q_k)-\nabla\psi(q_k)+8\pi\nabla R(q_k,q_k)\\&-\big(\nabla\log h(p)+8\pi\nabla R(p,p)\big)\\&+\nabla\psi(q_k)-\nabla\psi_k(p)\\&+O(|p-q|^2)\\=&\nabla H_k(q_k)-\nabla H(p)+O(|p-q|^\gamma\|\psi_k\|_*)\\&+O(|p-q|^2),\end{aligned} \tag{3.36}$$

where γ depends on $\mathfrak{p}$. We note that $\nabla H(p)=0$. From (3.34)-(3.36) and the non-degeneracy of p,w, we obtain

$$\|\psi_k\|_*+|p-q_k|\le C(\lambda_ke^{-\lambda_k}+o(1)\|\psi_k\|_*+\|\psi_k\|_*^2+|p-q_k|^2), \tag{3.37}$$

where C is a generic constant, independent of k and ψ_k. Therefore, we have

$$\psi_k=O(\lambda_ke^{-\lambda_k}),\ |p-q_k|=O(\lambda_ke^{-\lambda_k}). \tag{3.38}$$

As a conclusion of (3.12), (3.26) and (3.38), we have

$$\lambda_k-\lambda(\rho_1)=O(\lambda(\rho_1)^{-1}),\ \tilde h_k=h+O(\lambda(\rho_1)e^{-\lambda(\rho_1)}),\ |q_k-p|=O(\lambda(\rho_1)e^{-\lambda(\rho_1)}) \tag{3.39}$$

and

$$v_{2k}-\frac{1}{2}w=O(\lambda(\rho_1)e^{-\lambda(\rho_1)}). \tag{3.40}$$

We replace $\tilde h_k$ by h in the definition of v_q, we denote the new terms by v_q. By (3.38), we have

$$v_{q_k}-v_q=O(\lambda(\rho_1)e^{-\lambda(\rho_1)}).$$

We set

$$v_{q,\lambda,a}=v_q-\overline{v}_q. \tag{3.41}$$

By (3.32) and (3.41), we gain

$$v_{1k}-\frac{1}{2}v_{q,\lambda,a}=O(\lambda(\rho_1)e^{-\lambda(\rho_1)}). \tag{3.42}$$

By [11, Lemma 3.2], if we choose c in $S_{\rho_1}(p,w)$ big enough, there exists triplet (q^*,λ^*,a^*) and $\phi^*\in O^{(1)}_{q^*,\lambda^*}$ such that

$$v_{1k}=\frac{1}{2}v_{q^*,\lambda^*,a^*}+\phi^*_k, \tag{3.43}$$

where $q^*,\ \lambda^*,\ a^*$ satisfy the condition in $S_{\rho_1}(p,w)$. Therefore, we proved

$$(v_{1k},v_{2k})\in S_{\rho_1}(p,w)\times S_{\rho_2}(p,w).$$

$\square$

In conclusion, we have the following Theorem,

Theorem 3.3. *Suppose h_1,h_2 are two positive $C^{2,\alpha}$ function on M such that any solution (p,w) of (1.11) is non-degenerate and $\Delta\log h_1(p)-\rho_2+8\pi\neq 0$. Then there exists $\varepsilon_0>0$ and $C>0$ such that for any solution of (1.7) with $\rho_1\in(4\pi-\varepsilon_0,4\pi+\varepsilon_0),\rho_2\notin 4\pi\mathbb{N}$, either $|v_1|,|v_2|\leq C,\forall x\in M$ or $(v_1,v_2)\in S_{\rho_1}(p,w)\times S_{\rho_2}(p,w)$ for some solution (p,w) of (1.11).*

4. Approximate blow up solution

In the following two sections. We shall construct the blow up solutions of (1.7) in general case when

$$\rho_1\to\rho_*=4m\pi+4\pi\sum_{q\in S}\alpha_q \text{ and } \rho_2\notin\Sigma_2. \tag{4.1}$$

The construction of such bubbling solution is based on a non-degenerate solution of (1.12). For a given non-degenerate solution (P_w,w) of (1.12). We define the space $S_{\rho_1}(Q,w)$, $S_{\rho_2}(Q,w)$ for v_1 and v_2 respectively, where the definition of $S_{\rho_i}(Q,w)$ is given in (4.23)-(4.24). These two sets are generalization of the one defined in the previous section. Our aim is to compute the degree of the following nonlinear operator

$$\begin{pmatrix} v_1\\ v_2\end{pmatrix}=(-\Delta)^{-1}\begin{pmatrix} 2\rho_1\big(\frac{h_1e^{2v_1-v_2}}{\int_M h_1e^{2v_1-v_2}}-1\big)\\ 2\rho_2\big(\frac{h_2e^{2v_2-v_1}}{\int_M h_2e^{2v_2-v_1}}-1\big)\end{pmatrix}$$

in the space $S_{\rho_1}(Q,w)\times S_{\rho_2}(Q,w)$.

For a give non-degenerate solution (p_w,w) of (1.12). We define

$$Q=P_w\cup S=\{p_1^0,p_2^0,\cdots,p_m^0\}\cup S,\ P_w\cap S_1=\emptyset,\ S\subseteq S_1.$$

In order to simplify our notation, we relabel the index in Q and set

$$Q=\{p_1^0,p_2^0,\cdots,p_n^0\},\ n=m+|S|$$

and

$$\alpha_i=0 \text{ for } 1\leq i\leq m,\ \alpha_{m+i}=\alpha_{p^0_{m+i}},\ 1\leq i\leq n-m.$$

For each point p_j^0 in Q, we set

$$G_j^*(x) = 8\pi\Big((1+\alpha_j)R(x,p_j^0) + \sum_{1\le i\le n, i\ne j}(1+\alpha_i)G(x,p_i^0)\Big),\ j=1,2,\cdots,n. \quad (4.2)$$

We only consider the case $S \neq \emptyset$, because the construction for the case $S = \emptyset$ is similar. we may assume (relabeling the index if necessary)

$$\alpha_{m+1} = \cdots = \alpha_{m+l} > \alpha_{m+l+1} \geq \cdots \geq \alpha_{m+n},$$

We use G_j^* associate with Q to define $l(Q)$ as follows.

$$l(Q) = \sum_{j=m+1}^{m+l} \big(\frac{h_{p_j}(p_j^0)\rho_*}{4(1+\alpha_j)^2}\big)^{\frac{1}{1+\alpha_j}} e^{\frac{G_j^*(p_j^0)}{1+\alpha_j}} \big(\Delta \log(h_1^* e^{-\frac{1}{2}w})(p_j^0) + 2\rho_* - N^* - 2K(p_j^0)\big), \quad (4.3)$$

where $K(p)$ is the Gaussian curvature at p, $\rho_* = 4\pi\sum_{j=1}^n(1+\alpha_j)$, $N^* = 4\pi\sum_{q\in S_1}\alpha_q$, and

$$h_{p_j}(x) = \begin{cases} h(x), & \text{if } 1\le j\le m, \\ |x-p_j^0|^{-2\alpha_j}h(x), & \text{if } m+1\le j\le n. \end{cases}$$

Let $P = (p_1, p_2, \cdots, p_n)$ with $|p_i - p_i^0| \ll 1$ for $1 \le i \le m$ and $p_i = p_i^0$ for $m < i \le n$. For large $\lambda_j > 0,\ j = 1, 2, \cdots, n$, we set

$$U_j(x) = \lambda_j - 2\log\big(1 + \frac{\rho_1 h_{p_j}(p_j)}{4(1+\alpha_j)^2}e^{\lambda_j}|x-p_j|^{2(1+\alpha_j)}\big). \quad (4.4)$$

These $U_j(x)$ satisfy the following equation

$$\Delta U_j(x) + 2\rho_1 h_{p_j}(p_j)e^{U_j} = 0 \text{ in } \mathbb{R}^2, \quad U_j(p_j) = \max_{\mathbb{R}^2} U_j(x) = \lambda_j, \quad (4.5)$$

By using $h_{p_j}(x)$, we define $H_j(x,t)$,

$$H_j(x,t) = \exp\big\{\log\frac{h_{p_j}(x+p_j)}{h_{p_j}(p_j)} + (G_j^*(x+p_j) - G_j^*(p_j)) + t\big\} - 1, \quad (4.6)$$

For convenience, we set

$$J = \{1, 2, \cdots, n\}, \quad J_1 = \{1, 2, \cdots, m\}, \quad J_2 = \{m+1, m+2, \cdots, m+l\}.$$

Next, we construct the error terms near each point p_j for $j \in J_2$. With out loss of generality, we may assume $\nabla H_j(0,0) = (e_j, 0)$. Let $Q(x) = \frac{1}{2}\big(\nabla^2\log[H_j(0,0) + 1]x, x\big)$. Then the Taylor expansion gives

$$\log\frac{h_{p_j}(x+p_j)}{h_{p_j}(p_j)} + G_j^*(x+p_j) - G_j^*(p_j) = e_j x_1 + Q(x) + \text{higher order}, \quad (4.7)$$

and

$$H_j(x,t) = e_j x_1 + t + Q(x) + \frac{1}{2}(e_j x_1 + t)^2 + O(|x|^3 + t^3). \quad (4.8)$$

For $j \in J_2$, we let $\zeta_{1,j}(y)$ and $\zeta_{2,j}(y)$ be the solutions of

$$\begin{cases} \Delta\zeta_{1,j}(y) + 2\rho_1 h_{p_j}(p_j)|y|^{2\alpha_j}e^{U_j'(y)}(\zeta_{1,j}(y) + e_j y_1) = 0 \text{ in } \mathbb{R}^2, \\ \zeta_{1,j}(0) = 0, \quad |\zeta_{1,j}(y)| = O(|y|^{-2\alpha-1}) \text{ at } \infty, \end{cases} \quad (4.9)$$

and

$$\begin{cases} \Delta\zeta_{2,j}(y)+2\rho_1 h_{p_j}(p_j)|y|^{2\alpha_j}e^{U_j'(y)}\Big(\zeta_{2,j}(y)+Q(y)+\frac{1}{2}(e_jy_1+\zeta_{1,j}(y))^2\Big)=0 \text{ in } \mathbb{R}^2,\\ \zeta_{2,j}(0)=0, \quad |\zeta_{2,j}(y)|=O(\log|y|) \text{ at } \infty, \end{cases} \tag{4.10}$$

where

$$U_j'(y)=-2\log\Big(1+\frac{\rho_1 h_{p_j}(p_j)}{4(1+\alpha_j)^2}|y|^{2(1+\alpha_j)}\Big).$$

The existence of $\zeta_{1,j}$ and $\zeta_{2,j}$ has been proved in section 3 of [12]. Set $\epsilon_j=e^{-\frac{\lambda_j}{2(1+\alpha_j)}}$. For $j\in J\setminus J_1$, we define

$$\eta_j(x)=\epsilon_j\zeta_{1,j}(\epsilon_j^{-1}x)+\epsilon_j^2\zeta_{2,j}(\epsilon_j^{-1}x) \text{ for } |x|\le 2r_0. \tag{4.11}$$

By (4.8), (4.9) and (4.10), η_j satisfies,

$$\Delta\eta_j+2\rho_1 h_{p_j}(p_j)|x-p_j|^{2\alpha_j}e^{U_j(x)}\tilde{H}_j(x,\eta_j)=0, \tag{4.12}$$

where

$$\begin{aligned}\tilde{H}_j(x,\eta_j)=&e_jx_1+\eta_j+Q(x)+\frac{1}{2}(e_jx_1+\eta_j)^2-\frac{1}{2}(\epsilon_j^2\zeta_{2,j}(\epsilon_j^{-1}x))^2\\ &-\epsilon_j^2\zeta_{2,j}(\epsilon_j^{-1}x)\big(e_jx_1+\epsilon_j\zeta_{1,j}(\epsilon_j^{-1}x)\big).\end{aligned}$$

If $j\in J_1$, we set $\eta_j\equiv 0$.

For $j\in J$, we let

$$s_j=\lambda_j+2\log\Big(\frac{\rho_1 h_{p_j}(p_j)}{4(1+\alpha_j)^2}\Big)+8\pi(1+\alpha_j)R(p_j,p_j)+\frac{d_j}{2(1+\alpha_j)}\lambda_je^{-\frac{\lambda_j}{1+\alpha_j}}, \tag{4.13}$$

where

$$d_j=\frac{\pi}{(1+\alpha_j)\sin(\frac{\pi}{1+\alpha_j})}\Big(\frac{4(1+\alpha_j)^2}{\rho_1 h_{p_j}(p_j)}\Big)^{\frac{1}{1+\alpha_j}}\times\big(\Delta\log(h_1^*e^{-\frac{1}{2}w})(p_j)+2\rho_*-N^*-2K(p_j)\big),$$

if $j\in J\setminus J_1$, and $d_j=0$ for $j\in J_1$.

Let $\sigma(x)$ be a cut-off function:

$$\sigma(x)=\begin{cases}1, & \text{if } |x|<r_0,\\ 0, & \text{if } |x|\ge 2r_0,\end{cases}$$

and $\sigma_j(x)=\sigma(x-p_j)$. We set

$$\begin{aligned}v_{p_j}(x)=&\Big(U_j(x)+\eta_j(x)+8\pi(1+\alpha_j)(R(x,p_j)-R(p_j,p_j))+s_j\Big)\sigma_j(x)\\ &+8\pi(1+\alpha_j)G(x,p_j)(1-\sigma_j).\end{aligned} \tag{4.14}$$

We note that $\eta_j(x)+\frac{d_j}{2(1+\alpha_j)}\lambda_je^{-\frac{\lambda_j}{1+\alpha_j}}=O(e^{-\frac{\lambda_j}{1+\alpha_j}})$ if $m+1\le j\le n$ and $|x|\ge\delta>0$ for some $\delta>0$.

Next, we state two lemmas that shall be used later. One can see [12] for a proof

Lemma 4.1. *Let $\xi_j(x)=v_{p_j}(x)-8\pi(1+\alpha_j)G(x,p_j)$. Then for $r_0\le|x-p_j|\le 2r_0$, the followings hold,*

(1) *For $m+1\le j\le n$, $\xi_j(x),\partial_{\lambda_j}\xi_j(x),\nabla_x\xi_j(x),\Delta_x\xi_j(x)$ are $O(e^{-\frac{\lambda_j}{1+\alpha_j}})$.*
(2) *For $1\le j\le m$, $\xi_j(x),\partial_{p_j}\xi_j(x),\partial_{\lambda_j}\xi_j(x),\nabla_x\xi_j(x),\Delta_x\xi_j(x)$ are $O(\lambda_je^{-\lambda_j})$.*

and

Lemma 4.2. *For v_{p_j}, we have*

(1) *For $m+1 \le j \le n$, $\int_M v_{p_j},\ \int_M \partial_{\lambda_j} v_{p_j} = O(e^{-\frac{\lambda_j}{1+\alpha_j}})$,*
(2) *For $1 \le j \le m$, $\int_M v_{p_j},\ \int_M \partial_{\lambda_j} v_{p_j},\ \int_M \partial_{p_j} v_{p_j}(x) = O(\lambda_j e^{-\lambda_j})$.*

Now we are going to construct a good approximation of the blow up solution to (1.7). For each

$$P = (p_1, p_2, \cdots, p_n), \Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n, A = (a_1, a_2, \cdots, a_n) \in \mathbb{R}^n,$$

we set

$$v_{P,\Lambda,A} = \sum_{j=1}^{n} a_j \big(v_{p_j} - \overline{v}_{p_j}\big),$$

where v_{p_j} is constructed in (4.14) and $\overline{v}_{p_j}$ denotes the average of v_{p_j}. We define

$$t_j = \lambda_j + G_j^*(p_j) + 2\log\Big(\frac{\rho_* h_{p_j}(p_j)}{4(1+\alpha_j)^2}\Big) + \frac{d_j}{2(1+\alpha_j)}\lambda_j e^{-\frac{\lambda_j}{1+\alpha_j}} - \sum_{j=1}^{n} \overline{v}_{p_j} \text{ for } 1 \le j \le n. \tag{4.15}$$

We recall the term $l(Q)$:

$$l(Q) = \sum_{j\in J_2} \big(\frac{h_{p_j}(p_j^0)\rho_*}{4(1+\alpha_j)^2}\big)^{\frac{1}{1+\alpha_j}} e^{\frac{G_j^*(p_j^0)}{1+\alpha_j}} \big(\Delta \log(h_1^* e^{-\frac{1}{2}w})(p_j^0) + 2\rho_* - N^* - 2K(p_j^0)\big). \tag{4.16}$$

In the proof, we assume

$$l(Q) \neq 0, \tag{4.17}$$

and then define $\lambda(P)$ by

$$\rho_1 - \rho_* = \frac{\pi^2}{(1+\alpha_{m+1})\sin\frac{\pi}{1+\alpha_{m+1}}} \big(\frac{4(1+\alpha_{m+1})^2}{\rho_* h_{p_{m+1}}(p_{m+1}^0)}\big)^{\frac{2}{1+\alpha_{m+1}}} e^{-\frac{G_{m+1}^*(p_{m+1}^0)}{1+\alpha_{m+1}}} l(Q) e^{-\frac{\lambda(P)}{1+\alpha_{m+1}}}. \tag{4.18}$$

Remark 3. We note that the main components in $l(Q)$ are

$$\Delta \log(h_1^* e^{-\frac{1}{2}w})(p_j^0) + 2\rho_* - N^* - 2K(p_j^0) = \Delta \log h_1^* - \rho_2 + 2\rho_* - N^* - 2K(p_j^0). \tag{4.19}$$

For h_1^*, h_2^*, using Theorem 2.5, we can find a dense set in $C^{2,\alpha}(M) \times C^{2,\alpha}(M)$ to make all the solutions (P_w, w) of (1.12) non-degenerate. Based on this choice, we can choose such h_1^* that

$$\begin{aligned} &\|\Delta \log h_1^* - \rho_2 - 2K\|_{L^\infty(M)} < |2\rho_* - N^*|, && \text{if } 2\rho_* - N^* \neq 0,\\ &\|\Delta \log h_1^* - 2K\|_{L^\infty(M)} < \rho_2, && \text{if } 2\rho_* - N^* = 0, \end{aligned} \tag{4.20}$$

where ρ_* and N^* are given in (4.3). In conclusion, we can always choose h_1^*, h_2^* such that the solutions of (1.12) are non-degenerate and the term $l(Q) \neq 0$.

For each P,Λ, we define the following function space

$$
\begin{aligned}
O^{(1)}_{P,\Lambda} =\Big\{\phi\in \mathring{H}^1(M)\ \Big|\ &\int_M\nabla\phi\cdot\nabla v_{p_j}=\int_M\nabla\phi\cdot\nabla\partial_{p_j}v_{p_j}=\int_M\nabla\phi\cdot\nabla\partial_{\lambda_j}v_{p_j}=0\\
&\text{for } 1\le j\le m,\ \text{and}\ \int_M\nabla\phi\cdot\nabla v_{p_j}=\int_M\nabla\phi\cdot\nabla\partial_{\lambda_j}v_{p_j}=0\\
&\text{for } m+1\le j\le n\Big\}
\end{aligned}\tag{4.21}
$$

and

$$
O^{(2)}_{P,\Lambda}=\Big\{\psi\in W^{2,\mathfrak{p}}(M)\ \Big|\ \int_M\psi=0\Big\}.\tag{4.22}
$$

Let c_0 be a positive constant, which will be chosen later. By using (4.14), (4.15), (4.18), (4.21) and (4.22), we define

$$
\begin{aligned}
S_{\rho_1}(Q,w)=\Big\{v_1=\frac12 v_{P,\Lambda,A}+\phi\ \Big|\ &|p_j-p_j^0|\le c_0e^{-\frac{\lambda(P)}{1+\alpha_{m+1}}}\ \text{for } 1\le j\le m,\\
&|\lambda_{m+1}-\lambda(P)|\le c_0\lambda(P)^{-1},\ |t_j-t_1|\le c_0e^{-\frac{\lambda(P)}{1+\alpha_{m+1}}}\ \text{for } 2\le j\le n,\\
&|a_j-1|\le c_0\lambda(P)^{-\frac12}e^{-\frac{\lambda(P)}{1+\alpha_{m+1}}}\ \text{for } 1\le j\le n,\\
&\phi\in O^{(1)}_{P,\Lambda}\ \text{and}\ \|\phi\|_{H^1(M)}\le c_0e^{-\frac{\lambda(P)}{1+\alpha_{m+1}}}\Big\}
\end{aligned}\tag{4.23}
$$

and

$$
S_{\rho_2}(Q,w)=\Big\{v_2=\frac12 w+\psi\ \Big|\ \psi\in O^{(2)}_{P,\Lambda}\ \text{and}\ \|\psi\|_*\le c_0e^{-\frac{\lambda(P)}{1+\alpha_{m+1}}}\Big\},\tag{4.24}
$$

where $\|\psi\|_*=\|\psi\|_{W^{2,\mathfrak{p}}(M)},\ \mathfrak{p}>2$.

Next, we want to reduce the computation of the topological degree contributed from $\big(S_{\rho_1}(Q,w)\times S_{\rho_2}(Q,w)\big)$ to a easier problem.

Set

$$
T(v_1,v_2)=\begin{pmatrix}T_1(v_1,v_2)\\ T_2(v_1,v_2)\end{pmatrix}=\Delta^{-1}\begin{pmatrix}2\rho_1\big(\frac{h_1e^{2v_1-v_2}}{\int_M h_1e^{2v_1-v_2}}-1\big)\\ 2\rho_2\big(\frac{h_2e^{2v_2-v_1}}{\int_M h_2e^{2v_2-v_1}}-1\big)\end{pmatrix}.
$$

Since each solution in v_1 in $S_{\rho_1}(Q,w)$ can be represented by (P,Λ,A,ϕ), each solution in v_2 in $S_{\rho_2}(Q,w)$ can be represented by w and ψ. Therefore the nonlinear operator $2v_1+T_1(v_1,v_2)$ can be split according to this representation.

Let $v_1=\frac12 v_{P,\Lambda,A}+\phi\in S_{\rho_1}(Q,w)$. Recall

$$
t_j=s_j+8\pi\sum_{i\neq j}(1+\alpha_i)G(p_j,p_i)-\sum_{i=1}^n\overline{v}_{p_i}
$$

and for $x\in B_{r_0}(p_j)$

$$
\begin{aligned}
v_{P,\Lambda,A}(x)+\log\frac{h_{p_j}(x)}{h_{p_j}(p_j)}=&U_j+t_j+\log(H_j(x-p_j,\eta_j)+1)+(a_j-1)(U_j+s_j)\\
&+8\pi\sum_{i\neq j}(1+\alpha_i)(a_i-1)G(p_j,p_i)+O(|a_j-1|(|y|+|\eta_j|)),
\end{aligned}
$$

where $y=x-p_j$. The above together with (4.7) implies

$$
\begin{aligned}
\rho_1 h_1 e^{2v_1-v_2-2\phi+\psi} =&\rho_1 h e^{v_{P,\Lambda,A}} = \rho_1 h_{p_j}(p_j)|y|^{2\alpha_j} e^{U_j+t_j}\Big[1+(a_j-1)(U_j+s_j) \\
&+\sum_{i\neq j, i=1}^{n} 8\pi(1+\alpha_i)(a_i-1)G(p_j,p_i)+\eta_j+\nabla_y H_j(0,0)\cdot y \\
&+Q_j(y)+(a_j-1)(U_j+s_j)(\nabla_y H_j(0,0)\cdot y+\eta_j) \\
&+\frac{1}{2}(\eta_j+\nabla_y H_j\cdot y)^2+O(\tilde{\beta}_j)\Big],
\end{aligned}
\tag{4.25}
$$

where

$$
\tilde{\beta}_j=\lambda_{m+1}^2\sum_{i=1}^{n}\big(|a_i-1|^2+|a_i-1|(|\eta_j|+|y|)\big)+|\eta_j|^3+|y|^3.
$$

Therefore we have on $B_{r_0}(p_j)$

$$
\begin{aligned}
\rho_1 h_1 e^{2v_1-v_2} =&(1+\varphi)\rho_1 h e^{v_{P,\Lambda,A}}+(e^{\varphi}-1-\varphi)\rho_1 h e^{v_{P,\Lambda,A}} \\
=&\rho_1 h_{p_j}(p_j)|y|^{2\alpha_j}e^{U_j+t_j}\Big[1+(a_j-1)(U_j+s_j)+\eta_j+\nabla_y H_j\cdot y+Q_j(y) \\
&+(a_j-1)(U_j+s_j)(\nabla_y H_j(0,0)\cdot y+\eta_j)+\frac{1}{2}(\eta_j+\nabla_y H_j\cdot y)^2 \\
&+\sum_{i=1, i\neq j}^{n} 8\pi(a_i-1)(1+\alpha_i)G(p_j,p_i)+\varphi\Big]+\tilde{E}_j,
\end{aligned}
\tag{4.26}
$$

where

$$
\tilde{E}_j=(e^{\varphi}-1-\varphi)\rho_1 h e^{v_{P,\Lambda,A}}+\rho_1 h_{p_j}(p_j)|y|^{2\alpha_j}e^{U_j+t_j}O(\varphi^2+\tilde{\beta}_j),
\tag{4.27}
$$

and $\varphi=2\phi-\psi$.

Let $\epsilon_2>0$ be small. $\tilde{E}_j$ can be written into two parts

$$
\tilde{E}_j=\tilde{E}_j^{+}+\tilde{E}_j^{-},
$$

where

$$
\tilde{E}_j^{+}=\begin{cases}\tilde{E}_j & \text{if} \quad |\varphi|\geq\epsilon_2\\ 0 & \text{if} \quad |\varphi|<\epsilon_2,\end{cases}\qquad
\tilde{E}_j^{-}=\begin{cases}0 & \text{if} \quad |\varphi|\geq\epsilon_2\\ \tilde{E}_j & \text{if} \quad |\varphi|<\epsilon_2.\end{cases}
$$

Then

$$
\tilde{E}_j^{+}=O(e^{|\varphi|+2\lambda_j}) \text{ if } |\varphi|\geq\epsilon_2,
\tag{4.28}
$$

and

$$
\tilde{E}_j^{-}=\rho_1 h e^{U_j+\lambda_j}|y|^{2\alpha_j}O(\varphi^2+\tilde{\beta}_j).
\tag{4.29}
$$

Using the expression for $\rho_1 h_1 e^{2v_1-v_2}$ above, we obtain the following estimate for $\int_M \rho_1 h_1 e^{2v_1-v_2}$.

Lemma 4.3. *Let $\rho_* = \sum_{j=1}^n 4\pi(1+\alpha_j)$, $v_1 = \frac{1}{2}v_{P,\Lambda,A} + \phi \in S_{\rho_1}(Q,w)$ and $v_2 = \frac{1}{2}w + \psi \in S_{\rho_2}(Q,w)$. Then as $\rho_1 \to \rho_*, \rho_2 \notin \Sigma_2$, we have*

$$\begin{aligned}\int_M \rho_1 h_1 e^{2v_1 - v_2} =& \sum_{j=1}^n 4\pi(1+\alpha_j)e^{t_j}(1-\psi(p_j)) + \sum_{j\in J_2} \pi d_j e^{t_j} e^{-\frac{\lambda_j}{1+\alpha_j}} \\ &+ \sum_{j=1}^n 8\pi(1+\alpha_j)\lambda_j(a_j-1)e^{t_j} + O(\sum_{j=1}^n |a_j-1|e^{\lambda_1}) \\ &+ O(e^{\lambda_{m+1} - \frac{\lambda_{m+1}}{1+\alpha_{m+1}} - \varepsilon\lambda_{m+1}})\end{aligned} \tag{4.30}$$

for some $\varepsilon > 0$.

Proof. We note that $\lambda_{m+1} = \lambda_j + O(1)$ and $t_j = \lambda_j + O(1)$. By (4.26)

$$\begin{aligned}\int_M \rho_1 h_1 e^{2v_1-v_2} =& \sum_j \int_{B_{r_0}(p_j)} \{\rho_1 h_{p_j}(p_j)|y|^{2\alpha_j} e^{U_j + t_j}[1 + \cdots] + \tilde{E}_j\} \mathrm{d}y \\ &+ \int_{M \setminus \bigcup_j B_{r_0}(p_j)} \rho_1 h_1 e^{2v_1 - v_2}.\end{aligned}$$

By the explicit expression of U_j, we have

$$\int_{M \setminus \bigcup_j B_{r_0}(p_j)} \rho_1 h_1 e^{2v_1 - v_2} = O(1), \tag{4.31}$$

$$\int_{B_{r_0}(p_j)} \rho_1 h_{p_j}(p_j)|y|^{2\alpha_j} e^{U_j + t_j} \mathrm{d}y = 4\pi(1+\alpha_j)e^{t_j} + O(1), \tag{4.32}$$

$$\begin{aligned}\int_{B_{r_0}(p_j)} \rho_1 h_{p_j}(p_j)|y|^{2\alpha_j} e^{U_j+t_j}(a_j-1)(U_j+s_j)\mathrm{d}y =& 8\pi(1+\alpha_j)(a_j-1)\lambda_j e^{t_j} \\ &+ O(|a_j-1|e^{\lambda_j}),\end{aligned} \tag{4.33}$$

where $U_j + s_j = 2\lambda_j - 2\log\left(1 + \frac{\rho_1 h_{p_j}(p_j)}{4(1+\alpha_j)^2} e^{\lambda_j}|y|^{2(1+\alpha_j)}\right) + O(1)$ is used. By equation (4.12) of η_j for $j \in J \setminus J_1$, and the fact $\zeta_{2,j}(y)y_1$ and $\zeta_{2,j}(y)\zeta_{1,j}(y)$ are odd functions, we have

$$\begin{aligned}&\int_{B_{r_0}(p_j)} \rho_1 h_j(p_j)|y|^{2\alpha_j} e^{U_j+t_j}\Big(\eta_j + \nabla_y H_j \cdot y + Q_j + \frac{1}{2}(\eta_j + \nabla_y H_j \cdot y)^2\Big)dy \\ &= -\frac{1}{2}e^{t_j}\int_{B_{r_0}(p_j)} \Delta\eta_j \mathrm{d}y + O(e^{t_j}e^{-\frac{2\lambda_j}{1+\alpha_j}}) \\ &= -\frac{1}{2}e^{t_j}\int_{\partial B_{r_0}(p_j)} \frac{\partial \eta_j}{\partial\nu} + O(e^{t_j}e^{-\frac{2\lambda_j}{1+\alpha_j}}) \\ &= \pi d_j e^{t_j} e^{-\frac{\lambda_j}{1+\alpha_j}} + O(e^{t_j}e^{-\frac{2\lambda_j}{1+\alpha_j}}).\end{aligned} \tag{4.34}$$

It is not difficult to see

$$\int_{B_{r_0}(p_j)} |y|^{2\alpha_j} e^{U_j+t_j}(a_i-1)G(p_j,p_i)\mathrm{d}y = O(\sum_{i=1}^n |a_i-1|e^{\lambda_j}). \tag{4.35}$$

$$\int_{B_{r_0}(p_j)} |y|^{2\alpha_j} e^{U_j+t_j}(a_j-1)[(U_j+s_j)+1+\eta_j]\nabla_y H_j(0,0)\cdot y\mathrm{d}y$$
$$= O(|a_j-1|e^{\lambda_{m+1}-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}}), \tag{4.36}$$

where the cancelation occurs due to the oddness of $\nabla_y H_j(0,0)\cdot y$.

To estimate the terms involving ϕ and ψ, we use (4.14) to obtain

$$\Delta v_{p_j} = \Delta U_j + \Delta\eta_j + 8\pi(1+\alpha_j) \text{ for } x\in B_{r_0}(p_j),$$

and

$$\Delta v_{p_j} = \Delta(v_{p_j} - 8\pi(1+\alpha_j)G(x,p_j)) + 8\pi(1+\alpha_j) \text{ for } x\notin B_{r_0}(p_j).$$

This together with Lemma 4.1 implies

$$\begin{aligned}\int_{B_{r_0}(p_j)} 2\rho_1 h_{p_j}(p_j)|y|^{2\alpha_j}e^{U_j}\phi =& -\int_M \phi\Delta v_{p_j} + \int_{B_{r_0}(p_j)}\phi\Delta\eta_j + 8\pi(1+\alpha_j)\int_M\phi\\ &+\int_{M\backslash B_{r_0}(p_j)}\phi\Delta(v_{p_j}-8\pi(1+\alpha_j)G(x,p_j))\\ =&\int_M\nabla\phi\nabla v_{p_j} + 8\pi(1+\alpha_j)\int_M\phi + \int_{B_{r_0}(p_j)}\phi\Delta\eta_j\\ &+\int_{M\backslash B_{r_0}(p_j)}\phi\Delta(v_{p_j}-8\pi(1+\alpha_j)G(x,p_j)).\end{aligned} \tag{4.37}$$

To estimate $\int_{B_{r_0}(p_j)}\phi\Delta\eta_j$, we choose $r_0'\in(\frac{r_0}{2},r_0)$ such that

$$\int_{\partial B_{r_0'}(p_j)}|\phi|\mathrm{d}\sigma \le \frac{2}{r_0}\int_{B_{r_0}(p_j)}|\phi|\mathrm{d}x.$$

Hence

$$\Big|\int_{\partial B_{r_0'}(p_j)}\phi\frac{\partial\eta_j}{\partial\nu}\mathrm{d}\sigma\Big| \le C\max_{\partial B_{r_0'}(p_j)}\Big|\frac{\partial\eta_j}{\partial\nu}\Big|\|\phi\|_{H^1} = O(e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}})\|\phi\|_{H^1},$$

where by (4.12),

$$-\int_{\partial B_{r_0'}(p_j)}\frac{\partial\eta_j}{\partial\nu} = \int_{B_{r_0'}(p_j)}2\rho_1 h_{p_j}(p_j)|y|^{2\alpha_j}e^{U_j}\tilde{H}(x,\eta_j)\mathrm{d}x = O(e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}}).$$

Thus

$$\begin{aligned}\int_{B_{r_0}(p_j)}\phi\Delta\eta_j =& \int_{B_{r_0'}(p_j)}\phi\Delta\eta_j + O(e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}})\int_M|\phi|\\ =& -\int_{B_{r_0'}(p_j)}\nabla\phi\nabla\eta_j + O(e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}})\|\phi\|_{H^1}\\ \le& (\int_{B_{r_0'}(p_j)}|\nabla\phi|^2)^{\frac12}(\int_{B_{r_0'}(p_j)}|\nabla\eta_j|^2)^{\frac12} + O(e^{-\lambda_{m+1}})\int_M|\phi|\\ =& O(e^{-\frac{\lambda_{m+1}}{2(1+\alpha_{m+1})}})\|\phi\|_{H^1}.\end{aligned}$$

By (8.2), we have for some $\epsilon>0$

$$\Big|\int_{B_{r_0}(p_j)}\rho_1 h_{p_j}(p_j)|y|^{2\alpha_j}e^{U_j}\phi\Big|=O(e^{-\epsilon\lambda_{m+1}})\|\phi\|_{H^1}. \tag{4.38}$$

While, for the terms involving ψ, we have

$$\begin{aligned}\int_{B_{r_0}(p_j)}\rho_1 h_j(p_j)|y|^{2\alpha_j}e^{U_j}\psi=&\int_{B_{r_0}(p_j)}\rho_1 h_{p_j}(p_j)|y|^{2\alpha_j}e^{U_j}\psi(p_j)\\ &+\int_{B_{r_0}(p_j)}\rho_1 h_{p_j}(p_j)|y|^{2\alpha_j}e^{U_j}(\psi-\psi(p_j))\\ =&4\pi(1+\alpha_j)\psi(p_j)+O(e^{-\frac{3}{2(1+\alpha_{m+1})}\lambda_j}).\end{aligned} \tag{4.39}$$

For $\tilde{E}_j^+$, we have

$$\begin{aligned}\int_{B_{r_0}(p_j)}|\tilde{E}_j^+|=&O(1)\int_{B_{r_0}(p_j)\cap\{|\varphi-\overline{\varphi}|\geq\epsilon_2\}}e^{|\varphi|+2\lambda_j}\\ =&O(1)\int_{B_{r_0}(p_j)\cap\{|\varphi-\overline{\varphi}|\geq\epsilon_2\}}e^{|\varphi-\overline{\varphi}|+2\lambda_j},\end{aligned}$$

where $\overline{\varphi}=\frac{\int_{B_{r_0}(p_j)}\varphi}{\mathrm{vol}(B_{r_0}(p_j))}=O(\|\varphi\|_{H^1})=O(e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}})$ if λ_{m+1} is large. Write

$$e^{|\varphi-\overline{\varphi}|}=e^{|\varphi-\overline{\varphi}|(1-\frac{4\pi|\varphi-\overline{\varphi}|}{\|\varphi-\overline{\varphi}\|^2})}e^{\frac{4\pi|\varphi-\overline{\varphi}|^2}{\|\varphi-\overline{\varphi}\|^2}}.$$

Since $\|\varphi-\overline{\varphi}\|^{-2}=\|\varphi-\overline{\varphi}\|_{H^1}^{-2}\gg 2\lambda_j$, we have

$$e^{|\varphi-\overline{\varphi}|(1-\frac{4\pi|\varphi-\overline{\varphi}|}{\|\varphi-\overline{\varphi}\|^2})}\leq e^{\frac{\epsilon_2}{2}(1-\frac{2\pi\epsilon_2}{\|\varphi-\overline{\varphi}\|^2})}\ll e^{-2\lambda_j}\ \text{ for } \|\varphi-\overline{\varphi}\|\geq\frac{\epsilon_2}{2}.$$

Hence, by Moser-Trudinger inequality

$$\int_{B_{r_0}(p_j)\cap\{|\varphi-\overline{\varphi}|\geq\frac{\epsilon_2}{2}\}}e^{|\varphi-\overline{\varphi}|}\leq e^{-2\lambda_j}\int_{B_{r_0}(p_j)}\exp\big(\frac{4\pi|\varphi-\overline{\varphi}|^2}{\|\varphi-\overline{\varphi}\|^2}\big)\leq O(1)e^{-2\lambda_j}, \tag{4.40}$$

which implies $\int_{B_{r_0}(p_j)}|\tilde{E}_j^+|\leq O(1)$.

For $\tilde{E}_j^-$, (4.29) gives

$$\int_{B_{r_0}(p_j)}|\tilde{E}_j^-|\leq O(1)\int_{B_{r_0}(p_j)}(|\varphi|^2+\tilde{\beta}_j)\rho_1 h_{p_j}(p_j)|y|^{2\alpha_j}e^{U_j+t_j}. \tag{4.41}$$

By (8.3) in section 8, we can estimate the first term on the right hand side of (4.41) by

$$\begin{aligned}\int_{B_{r_0}(p_j)}\rho_1 h_j(p_j)|y|^{2\alpha_j}e^{U_j+t_j}|\varphi|^2\leq&O(1)e^{t_j}\Big(\int_M|\nabla\phi|^2+\|\psi\|_*^2\int_{B_{r_0}(p_j)}|y|^{2\alpha_j}e^{U_j}\Big)\\ =&O(1)e^{t_j}e^{-2\frac{\lambda_{m+1}}{1+\alpha_{m+1}}}=O(1)e^{\lambda_{m+1}-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}-\epsilon_1\lambda_{m+1}}.\end{aligned}$$

Since $v_1 \in S_{\rho_1}(Q,w)$, the term related to $\tilde{\beta}_j$ can be estimated as follows

$$\begin{aligned}
&\int_{B_{r_0}(p_j)} \rho_1 h_{p_j}(p_j)|y|^{2\alpha_j} e^{U_j+t_j}\tilde{\beta}_j \\
&= O(1)e^{t_j}\Big(\sum_{i=1}^n \lambda_{m+1}^2(|a_i-1|)^2 + \int_{B_{r_0}(p_j)} [|\eta_j|^3+|y|^3+|a_i-1|(|\eta_i|+|y|)]e^{U_j}\mathrm{d}y\Big) \\
&= O(e^{t_j}e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}-\epsilon\lambda_{m+1}})
\end{aligned}$$

for some $\epsilon>0$ and large λ_{m+1}. Therefore, we have

$$\int_{B_{r_0}(p_j)} |\tilde{E}_j^-| = O(1)e^{\lambda_{m+1}-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}-\epsilon\lambda_{m+1}}. \tag{4.42}$$

By (4.26) and (4.31)-(4.42), we obtain (4.30). Hence we finish the proof of Lemma 4.3. □

Now we want to express $2v_1+T_1(v_1,v_2)$ in a formula similar to (4.26). By Lemma 4.3, we expect that $\frac{e^{t_j}}{\int_M h_1 e^{2v_1-v_2}}-1$ is small. Indeed, by definition of $S_{\rho_1}(Q,w)$,

$$|t_j-t_1| = O(1)e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}}.$$

By Lemma 4.3 and the Taylor expansion of the exponential function,

$$\begin{aligned}
e^{-t_j}\int_M \rho_1 h_1 e^{2v_1-v_2} =& \rho_* + \sum_{i=1}^n 4\pi(1+\alpha_i)(t_i-t_j-\psi(p_i)) \\
&+ \sum_{i=1}^n \pi d_1 e^{-\frac{\lambda_i}{1+\alpha_i}} + \sum_{i=1}^n 8\pi(1+\alpha_i)\lambda_i(a_i-1) \\
&+ O(|a_i-1|) + O(e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}-\epsilon\lambda_{m+1}}).
\end{aligned}\tag{4.43}$$

Hence

$$\begin{aligned}
\frac{e^{t_j}}{\int_M h_1 e^{2v_1-v_2}}-1 =& \frac{1}{e^{-t_j}\int_M \rho_1 h_1 e^{2v_1-v_2}}\big(\rho_1 - \frac{\int_M \rho_1 h_1 e^{2v_1-v_2}}{e^{t_j}}\big) \\
=& \theta_j + O(|a_i-1|) + O(e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}-\epsilon\lambda_{m+1}}),
\end{aligned}\tag{4.44}$$

where θ_j is defined by

$$\begin{aligned}
\theta_j =& \frac{1}{\rho_*}\Big[(\rho_1-\rho_*) - \sum_{i=1}^n \pi d_i e^{-\frac{\lambda_i}{1+\alpha_i}} - \sum_{i=1}^n 4\pi(1+\alpha_i)(t_i-t_j-\psi(p_i)) \\
&- \sum_{i=1}^n 8\pi(1+\alpha_i)\lambda_i(a_i-1)\Big].
\end{aligned}\tag{4.45}$$

Let

$$\beta_j = \Big|\frac{e^{t_j}}{\int_M h_1 e^{2v_1-v_2}}-1\Big|^2 + \tilde{\beta}_j, \tag{4.46}$$

and

$$E_j = (e^{\varphi}-1-\varphi)\frac{2\rho_1 h_1 e^{2v_1-v_2}}{\int_M h_1 e^{2v_1-v_2}} + 2\rho_1 h_{p_j}(p_j)|y|^{2\alpha_j}e^{U_j}(O(\varphi^2)+O(\beta_j)). \tag{4.47}$$

Then in $B_{r_0}(p_j)$, we have by (4.26),

$$\begin{aligned}\frac{\rho_1 h_1 e^{2v_1-v_2}}{\int_M h_1 e^{2v_1-v_2}} =&(1+\varphi)\frac{\rho_1 h e^{v_{P,\Lambda,A}}}{\int_M h_1 e^{2v_1-v_2}} + (e^{\varphi}-1-\varphi)\frac{\rho_1 h e^{v_{P,\Lambda,A}}}{\int_M h_1 e^{2v_1-v_2}}\\ =&\rho_1 h_{p_j}(p_j)|y|^{2\alpha_j}e^{U_j}\Big[1+\big(\frac{e^{t_j}}{\int_M h_1 e^{2v_1-v_2}}-1\big)+(a_j-1)(U_j+s_j)\\ &+\sum_{i=1,i\neq j}^{n}8\pi(1+\alpha_i)(a_i-1)G(p_j,p_i)+\eta_j+\nabla_y H_j\cdot y\\ &+(a_j-1)(U_j+s_j)(\nabla_y H_j\cdot y+\eta_j)+\frac{1}{2}(\eta_j+\nabla_y H_j\cdot y)^2\\ &+Q_j(y)+\varphi\Big]+E_j. \end{aligned}\tag{4.48}$$

Thus, we have in $B_{r_0}(p_j)$,

$$\begin{aligned}\Delta(2v_1+T_1(v_1,v_2)) =&2\Delta v_1+\frac{2\rho_1 h_1 e^{2v_1-v_2}}{\int_M h_1 e^{2v_1-v_2}}-2\rho_1\\ =&a_j(\Delta U_j+\Delta\eta_j)+2\Delta\phi+\sum_{j=1}^{n}8\pi(1+\alpha_i)a_i+\frac{2\rho_1 h_1 e^{2v_1-v_2}}{\int_M h_1 e^{2v_1-v_2}}-2\rho_1\\ =&-2a_j\rho_1 h_{p_j}(p_j)|y|^{2\alpha_j}e^{U_j}\Big[1+\eta_j+\nabla_y H_j\cdot y+Q_j(y)\\ &-\epsilon_0^2\zeta_{2,j}(\epsilon_0^{-1}y)(\epsilon_0\zeta_{1,j}(\epsilon_0^{-1}y)+\nabla_y H_j\cdot y)-\frac{\epsilon_0^4}{2}\zeta_{2,j}^2(\epsilon_0^{-1}y)\\ &+\frac{1}{2}(\eta_j+\nabla_y H_j\cdot y)^2\Big]+2\Delta\phi+[\sum_{j=1}^{n}8\pi(1+\alpha_j)-2\rho_1]\\ &+\sum_{j=1}^{n}8\pi(1+\alpha_j)(a_j-1)+\frac{2\rho_1 h_1 e^{2v_1-v_2}}{\int_M h_1 e^{2v_1-v_2}}\\ =&2\Delta\phi+(2\rho_*-2\rho_1)+\sum_{j=1}^{n}8\pi(1+\alpha_j)(a_j-1)\\ &+2\rho_1 h_{p_j}(p_j)|y|^{2\alpha_j}e^{U_j}\Big[(U_j+s_j)(a_j-1)(\nabla_y H_j\cdot y+\eta_j)\\ &+(a_j-1)(U_j+s_j-1)+\sum_{i\neq j}8\pi(1+\alpha_i)(a_i-1)G(p_j,p_i)\\ &+(\frac{e^{t_j}}{\int_M h_1 e^{2v_1-v_2}}-1)+\varphi\Big]+\hat{E}_j, \end{aligned}\tag{4.49}$$

where $\epsilon_0=e^{-\frac{\lambda_{m+1}}{2(1+\alpha_{m+1})}}$ and

$$\hat{E}_j=E_j+2a_j\rho_1 h_{p_j}(p_j)|y|^{2\alpha_j}e^{U_j}\Big[\epsilon_0^2\zeta_{2,j}(\epsilon_0^{-1}y)(\eta_j+\nabla_y H_j\cdot y)+\frac{\epsilon_0^4}{2}\zeta_{2,j}^2(\epsilon_0^{-1}y)\Big].$$

On $B_{2r_0}(p_j)\setminus B_{r_0}(p_j)$, since $v_{p_j}-\overline{v}_{p_j}-8\pi(1+\alpha_j)G(x,p_j)$ is small, we write $\Delta(2v_1+T_1(v_1,v_2))$ as

$$\begin{aligned}&\Delta(2v_1+T_1(v_1,v_2))\\=&2\Delta\phi+a_j\Delta(v_{p_j}-8\pi(1+\alpha_j)G(x,p_j))\\&+2\rho_*-2\rho_1+\sum_{j=1}^{n}8\pi(1+\alpha_j)(a_j-1)\\&+\frac{2\rho_1 h}{\int_M h_1e^{2v_1-v_2}}e^{a_j(v_{p_j}-\overline{v}_{p_j}-8\pi(1+\alpha_j)G(x,p_j))+\sum_{i=1}^n 8\pi(1+\alpha_i)a_iG(x,p_i)+\varphi}.\end{aligned}\tag{4.50}$$

On $M\setminus\bigcup_j B_{2r_0}(p_j)$, we have

$$\begin{aligned}\Delta(2v_1+T_1(v_1,v_2))=&2\Delta\phi+2\rho_*-2\rho_1+\sum_{j=1}^{n}8\pi(1+\alpha_j)(a_j-1)\\&+\frac{2\rho_1 h}{\int_M h_1e^{2v_1-v_2}}e^{\sum_{i=1}^n 8\pi(1+\alpha_i)a_iG(x,p_i)+\varphi}.\end{aligned}\tag{4.51}$$

From (4.49)-(4.51), we have the following

Lemma 4.4. *Let* $v_1=\frac{1}{2}v_{P,\Lambda,A}+\phi\in S_{\rho_1}(Q,w)$, $v_2=\frac{1}{2}w+\phi$. *Then as* $\rho_1\to\rho_*=\sum_{j=1}^n 4\pi(1+\alpha_j)$,

(1)

$$\langle\nabla(2v_1+T_1(v_1,v_2)),\nabla\phi_1\rangle=2\mathfrak{B}(\phi,\phi_1)+O(e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}})\|\phi_1\|_{H_0^1(M)},\tag{4.52}$$

where

$$\mathfrak{B}(\phi,\phi_1):=\int_M\nabla\phi\cdot\nabla\phi_1-\sum_j\int_{B_{r_0}(p_j)}2\rho_1h_{p_j}(p_j)|y|^{2\alpha_j}e^{U_j}\phi\phi_1,$$

(2) *for* $1\le j\le m$,

$$\begin{aligned}&\langle\nabla(2v_1+T_1(v_1,v_2)),\nabla\partial_{p_j}v_{p_j}\rangle\\&=-8\pi\nabla_yH_j(0,0)+8\pi\nabla\psi(p_j)+O\Big(|\frac{e^{t_j}}{\int_M h_1e^{2v_1-v_2}}-1-\psi(p_j)|\\&\quad+|a_j-1|\lambda_j+e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}}\Big),\end{aligned}\tag{4.53}$$

(3) *for* $1\le j\le n$,

$$\begin{aligned}&\langle\nabla(2v_1+T_1(v_1,v_2)),\nabla\partial_{\lambda_j}v_{p_j}\rangle\\&=-16\pi(1+\alpha_j)(a_j-1)\lambda_j-8\pi(1+\alpha_j)(\theta_j-\psi(p_j))\\&\quad+O(\max_i|a_i-1|+e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}-\epsilon\lambda_{m+1}}),\end{aligned}\tag{4.54}$$

(4) *for* $1 \le j \le n$,

$$\begin{aligned}
&\langle \nabla(2v_1+T_1(v_1,v_2)), \nabla v_{p_j}\rangle \\
&\quad =\Big(2\lambda_j - 1 + 8\pi(1+\alpha_j)R(p_j,p_j) + 2\log\frac{\rho_1 h_{p_j}(p_j)}{4(1+\alpha_j)^2}\Big) \\
&\qquad \times \langle \nabla(2v_1+T_1(v_1,v_2)), \nabla\partial_{\lambda_j} v_{p_j}\rangle + 16\pi(1+\alpha_j)(a_j-1)\lambda_j \\
&\qquad + 8\pi(1+\alpha_j)\sum_{i\neq j} G(p_i,p_j)\langle \nabla(2v_1+T_1(v_1,v_2)), \nabla\partial_{\lambda_i} v_{p_i}\rangle \\
&\qquad + O(1)\|\phi\|_{H^1(M)} + O(e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}}),
\end{aligned} \tag{4.55}$$

We will prove Lemma 4.4 in the section 8 because the proof contains a lot of computations.

In order to know the Morse index for the solutions in $S_{\rho_1}(Q,w)$, we have to compute the Morse index of the bilinear form $\mathfrak{B}(\phi,\phi_1)$ in Lemma 4.4. For such bilinear form $\mathfrak{B}(\phi,\phi_1)$, we have the following lemma, due to Chen-Lin [13, Lemma 3.3].

Lemma 4.5. *Assume that all* $\alpha_j \ge 0$, *are not inters for* $j \in J \setminus J_1$. *Let* $P = (p_1,p_2,\cdots,p_n)$ *and* $\Lambda = (\lambda_1,\lambda_2,\cdots,\lambda_n)$, *where* $\mathrm{dist}(p_i,p_j) > 2r_0$ *for* $i \neq j$. *If* $\lambda(P)$ *is large, then the symmetric bilinear form*

$$\mathfrak{B}(\phi,\phi_1) := \int_M \nabla\phi\cdot\nabla\phi_1 - \sum_{j=1}^n \int_{B_{r_0}(p_j)} 2\rho_1 h_{p_j}(p_j)|y|^{2\alpha_j} e^{U_j}\phi\phi_1$$

is non-degenerate and has Morse index $\sum_{j=m+1}^n 2(1+[\alpha_j])$ *in* $O^{(1)}_{P,\Lambda}$, *where* $[\alpha_j]$ *denotes the greatest integer less than or equal to* α_j.

5. Deformation and Degree counting formula

In this section, we want to deform $2v_i + T_i(v_1,v_2)$ into a simple form which can be solvable. Obviously, $v_1 = \frac{1}{2}v_{P,\Lambda,A} + \phi,\ v_2 = \frac{1}{2}w + \psi$ is a solution of $2v_1 + T_1(v_1,v_2) = 0$, iff the left hand sides of (4.52)-(4.55) vanish. To solve the system (4.52)-(4.55) and $2v_2 + T_2(v_1,v_2) = 0$, we recall

$$\mathring{H}^1 = O^{(1)}_{P,\Lambda} \bigoplus \text{ the linear subspace spanned by } v_{p_j}, \partial_{\lambda_j} v_{p_j} \text{ and } \partial_{p_j} v_{p_j}$$

and deform $2v_i + T_i(v_1,v_2)$ to a simpler operator $2v_i + T_i^0(v_1,v_2)$ by defining the operator $2I + T_i^t,\ 0 \le t \le 1,\ i = 1,2$ through the following relations.

$$\begin{aligned}
\langle \nabla(2v_1+T_1^t(v_1,v_2)), \nabla\phi_1\rangle =& t\langle \nabla(2v_1+T_1(v_1,v_2)), \nabla\phi_1\rangle \\
&+ 2(1-t)\mathfrak{B}(\phi,\phi_1) \text{ for } \phi_1 \in O^{(1)}_{P,\Lambda};
\end{aligned} \tag{5.1}$$

$$\begin{aligned}
\langle \nabla(2v_1+T_1^t(v_1,v_2)), \nabla\partial_{p_j} v_{p_j}\rangle =& t\langle \nabla(2v_1+T_1(v_1,v_2)), \nabla\partial_{p_j} v_{p_j}\rangle \\
&+ (1-t)\big(-8\pi\nabla_y H_j(0,0) + 8\pi\nabla\psi(p_j)\big) \text{ for } 1 \le j \le m;
\end{aligned} \tag{5.2}$$

$$\langle \nabla(2v_1+T_1^t(v_1,v_2)),\nabla\partial_{\lambda_j}v_{p_j}\rangle = t\langle \nabla(2v_1+T_1(v_1,v_2)),\nabla\partial_{\lambda_j}v_{p_j}\rangle \\ -(1-t)8\pi(1+\alpha_j)\Big[2(a_j-1)\lambda_j+(\theta_j-\psi(p_j))\Big],\ \text{for } 1\le j\le n; \quad (5.3)$$

$$\langle \nabla(2v_1+T_1^t(v_1,v_2)),\nabla v_{p_j}\rangle = t\Big[\big(2\lambda_j+O(1)\big)\langle \nabla(2v_1+T_1^t(v_1,v_2)),\nabla\partial_{\lambda_j}v_{p_j}\rangle \\ +\sum_{i\neq j}O(1)\langle \nabla(2v_1+T_1^t(v_1,v_2)),\nabla\partial_{\lambda_i}v_{p_i}\rangle+O(1)\|\phi\|_{H^1} \\ +O(e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}})\Big]+16\pi(1+\alpha_j)(a_j-1)\lambda_j,\ \text{for } 1\le j\le n; \quad (5.4)$$

$$2v_2+T_2^t(v_1,v_2)=t(2v_2+T_2(v_1,v_2)) \\ +(1-t)\Big(w+2\psi-2\rho_2(-\Delta)^{-1}\big(\frac{h_2e^{w+2\psi-\sum_{j=1}^n 4\pi(1+\alpha_j)G(x,p_j)}}{\int_M h_2e^{w+2\psi-\sum_{j=1}^n 4\pi(1+\alpha_j)G(x,p_j)}}-1\big)\Big), \quad (5.5)$$

where those coefficients $O(1)$ are those terms appeared in (5.4) so that $T_1^1(v_1,v_2)=T_1(v_1,v_2)$. From the construction above, we have

$$2v_i+T_i(v_1,v_2)=2v_i+T_i^1(v_1,v_2),\ i=1,2.$$

When $t=0$, the operator T_i^0 is simpler than $T_i,\ i=1,2$. During the deformation from T_i^1 to $T_i^0,\ i=1,2$ we have

Lemma 5.1. *Let $\rho_*=\sum_{j=1}^n 4(1+\alpha_j)\pi$. Assume $(\rho_1-\rho_*)\neq 0$, and $\rho_2\notin\Sigma_2$. Then there is $\varepsilon_1>0$ such that $(2v_1+T_1^t(v_1,v_2),2v_2+T_2^t(v_1,v_2))\neq 0$ for $(v_1,v_2)\in\partial\big(S_{\rho_1}(Q,w)\times S_{\rho_2}(Q,w)\big)$ and $0\le t\le 1$ if $|\rho_1-\rho_*|<\varepsilon_1$ and ρ_2 is fixed.*

Proof. Assume $(v_1,v_2)\in\overline{S}_{\rho_1}(Q,w)\times\overline{S}_{\rho_2}(Q,w)$, where $\overline{S}_{\rho_i}(Q,w)$ denotes the closure of $S_{\rho_i}(Q,w)$, and $2v_i+T_i^t(v_1,v_2)=0,\ i=1,2$ for some $0\le t\le 1$. We will show that $(v_1,v_2)\notin\partial\big(S_{\rho_1}(Q,w)\times S_{\rho_2}(Q,w)\big)$.

From $\langle \nabla(2v_1+T_1^t(v_1,v_2)),\nabla\phi\rangle=0$, we have by Lemma 4.4

$$\|\phi\|_{H^1}^2\le O(e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}})\|\phi\|_{H^1}.$$

This implies

$$\|\phi\|_{H^1}=O(e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}})\le c_1e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}}, \quad (5.6)$$

for some constant c_1 [2] independent of c_0.

Using $\langle \nabla(2v_1+T_1^t(v_1,v_2)),\nabla\partial_{\lambda_j}v_{p_j}\rangle=0$ and $\langle \nabla(2v_1+T_1^t(v_1,v_2)),\nabla v_{p_j}\rangle=0$, (5.4) and (5.6) implies

$$16\pi\lambda_j(1+\alpha_j)(a_j-1)=O(e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}})\ \text{for } j=1,\cdots,n, \quad (5.7)$$

that is, when ρ_1 is close to ρ_*,

$$|a_j-1|=O(\lambda_{m+1}^{-1}e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}})<c_0\lambda_{m+1}^{-\frac{1}{2}}(P)e^{-\frac{\lambda(P)}{1+\alpha_{m+1}}}\ \text{for } 1\le j\le n. \quad (5.8)$$

By $\langle \nabla(2v_1+T_1^t(v_1,v_2)),\nabla_{\lambda_j}v_{p_j}\rangle=0$, we conclude from (4.54) and (5.8) that

$$\theta_j-\psi(p_j)+2\lambda_j(a_j-1)=O(\max|a_i-1|+e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}-\epsilon\lambda_{m+1}})=O(\lambda_{m+1}^{-1}e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}}). \quad (5.9)$$

---

[2]Here c_1 is independent of ψ, it can be shown in the proof of Lemma 4.4.

$$e^{t_j}\Big(\int_M h_1e^{2v_1-v_2}\Big)^{-1}-1-\theta_j=O(\max|a_i-1|+e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}-\epsilon\lambda_{m+1}})=O(e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}}). \tag{5.10}$$

Together with $\langle\nabla(2v_1+T_1^t(v_1,v_2)),\nabla\partial_{p_j}v_{p_j}\rangle=0$ for $j\le m$ and (5.8), (5.10) and part (2) of Lemma 4.4, we have

$$\begin{aligned}|\nabla_yH_j(0,0)-\nabla\psi(p_j)|&=O\Big(\lambda_j|a_j-1|+|\frac{e^{t_j}}{\int_M h_1e^{2v_1-v_2}}-1-\psi(p_j)|+e^{-\frac{\lambda_j}{1+\alpha_{m+1}}}\Big)\\&\le O(1)e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}},\end{aligned}$$

which implies

$$\begin{aligned}&\Big|\nabla_y^2H_j\cdot(p_j-p_j^0)+8\pi\sum_{i=1,i\neq j}^m\nabla_x^2G(x,p_j^0)\mid_{x=p_i^0}\cdot(p_i-p_i^0)-\nabla\psi(p_j^0)\Big|\\&\le O(1)e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}}+O(1)\|\psi\|_*|p_j-p_j^0|^\gamma+O(1)\sum_{i=1}^m|p_i-p_i^0|^2,\end{aligned} \tag{5.11}$$

where we used $\nabla_yH_j(p_j^0-p_j,0)=0$ and $\mathfrak{p}>2$.

For the second component, by (5.5), we have

$$\begin{aligned}0=&(1-t)\Big(\Delta w+2\Delta\psi+2\rho_2\Big(\frac{h_2e^{w+2\psi-\sum_{j=1}^n4\pi(1+\alpha_j)G(x,p_j)}}{\int_M h_2e^{w+2\psi-\sum_{j=1}^n4\pi(1+\alpha_j)G(x,p_j)}}-1\Big)\Big)\\&+t\Big(\Delta w+2\Delta\psi+2\rho_2\Big(\frac{h_2e^{w+2\psi-\frac12v_{P,\Lambda,A}-\phi}}{\int_M h_2e^{w+2\psi-\frac12v_{P,\Lambda,A}-\phi}}-1\Big)\Big).\end{aligned} \tag{5.12}$$

We set

$$\Theta=2\rho_2\frac{h_2e^{w+2\psi-\frac12v_{P,\Lambda,A}-\phi}}{\int_M h_2e^{w+2\psi-\frac12v_{P,\Lambda,A}-\phi}}-2\rho_2\frac{h_2e^{w+2\psi-\sum_{j=1}^n4\pi(1+\alpha_j)G(x,p_j)}}{\int_M h_2e^{w+2\psi-\sum_{j=1}^n4\pi(1+\alpha_j)G(x,p_j)}},$$

and claim

$$\|\Theta\|_{L^{\mathfrak{p}}(M)}\le c_2e^{-\frac{\lambda(P)}{1+\alpha_{m+1}}}, \tag{5.13}$$

where c_2 is a constant that independent of c_0 and $\mathfrak{p}$ is defined in $O_{P,\Lambda}^{(2)}$. By (5.6), it is not difficult to get

$$\exp\big(w+2\psi-\frac12v_{P,\Lambda,A}-\phi\big)=\exp(w+2\psi-\frac12v_{P,\Lambda,A})+\Theta_1,$$

where $\|\Theta_1\|_{\mathfrak{p}}\le c_3e^{-\frac{\lambda(P)}{1+\alpha_{m+1}}}$. By noting (5.8), it is enough for us to prove the following one.

$$\Big\|\exp(w+2\psi-\frac12v_{P,\Lambda,A})-\exp\big(w+2\psi-\sum_{j=1}^n4\pi(1+\alpha_j)a_jG(x,p_j)\big)\Big\|_{L^\infty(M)}\le c_4e^{-\frac{\lambda(P)}{1+\alpha_{m+1}}}. \tag{5.14}$$

Since the proof is long, we leave it in section 8. By (5.13), (5.12) can be written as

$$\Delta w+2\Delta\psi+2\rho_2\Big(\frac{h_2e^{w+2\psi-\sum_{j=1}^n4\pi(1+\alpha_j)G(x,p_j)}}{\int_M h_2e^{w+2\psi-\sum_{j=1}^n4\pi(1+\alpha_j)G(x,p_j)}}-1\Big)+t\Theta=0. \tag{5.15}$$

We expand the above equation,

$$\begin{aligned}\mathfrak{R} =&\Delta\psi + 2\rho_2\frac{\overline{h}_2 e^{w-4\pi\sum_{j=1}^m G(x,p_j^0)}}{\int_M \overline{h}_2 e^{w-4\pi\sum_{j=1}^m G(x,p_j^0)}}\psi\\ &- 2\rho_2\frac{\overline{h}_2 e^{w-4\pi\sum_{j=1}^m G(x,p_j^0)}}{\Big(\int_M \overline{h}_2 e^{w-4\pi\sum_{j=1}^m G(x,p_j^0)}\Big)^2}\int_M \big(\overline{h}_2 e^{w-4\pi\sum_{j=1}^m G(x,p_j^0)}\psi\big)\\ &- 4\pi\rho_2\frac{\overline{h}_2 e^{w-4\pi\sum_{j=1}^m G(x,p_j^0)}}{\int_M \overline{h}_2 e^{w-4\pi\sum_{j=1}^m G(x,p_j^0)}}\big(\sum_{j=1}^m \nabla G(x,p_j^0)(p_j-p_j^0)\big)\\ &+ 4\pi\rho_2\frac{\overline{h}_2 e^{w-4\pi\sum_{j=1}^m G(x,p_j^0)}}{\Big(\int_M \overline{h}_2 e^{w-4\pi\sum_{j=1}^m G(x,p_j^0)}\Big)^2}\int_M \Big(\overline{h}_2 e^{w-4\pi\sum_{j=1}^m G(x,p_j^0)}\big(\sum_{j=1}^m \nabla G(x,p_j^0)(p_j-p_j^0)\big)\Big),\end{aligned} \tag{5.16}$$

where $\mathfrak{R} = t\Theta + o(1)\|\psi\|_* + |p_j - p_j^0|^2$. By the non-degeneracy of (P_w, w) to (1.12), (5.11) and (5.16), we can get

$$\|\psi\|_* \le c_5 e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}} \text{ and } |p_j - p_j^0| \le c_6 e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}}. \tag{5.17}$$

Recall that

$$\begin{aligned}\theta_j =&\frac{1}{\rho_*}\Big((\rho_1-\rho_*) - \sum_{j=1}^n \pi d_j e^{-\frac{\lambda_j}{1+\alpha_j}} - \sum_{i=1}^n 4\pi(1+\alpha_i)(t_i - t_j - \psi(p_i))\\ &- \sum_{i=1}^n 8\pi(1+\alpha_i)\lambda_i(a_i-1)\Big).\end{aligned}$$

From (5.9), we obtain

$$\begin{aligned}O(1)\lambda_{m+1}^{-1}e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}} =&\sum_j 8\pi(1+\alpha_j)[\theta_j - \psi(p_j) + 2\lambda_j(a_j-1)]\\ =&2\Big(\rho_1 - \rho_* - \sum_{j=1}^n \pi d_j e^{-\frac{\lambda_j}{1+\alpha_j}}\Big),\end{aligned} \tag{5.18}$$

where all t_j cancel out with each other. Here $\rho_* = 4\pi\sum_j(1+\alpha_j)$ is used. By the definition of d_i in section 4, and the assumption

$$|t_j - t_1| \le c_0 e^{-\frac{\lambda(P)}{1+\alpha_{m+1}}},$$

we have

$$\begin{aligned}\sum_{j=1}^n \pi d_j e^{-\frac{\lambda_j}{1+\alpha_j}} =&\frac{\pi^2}{(1+\alpha_{m+1})\sin\frac{\pi}{1+\alpha_{m+1}}}\big(\frac{4(1+\alpha_{m+1})^2}{\rho_* h^*_{p_{m+1}}(p_{m+1})}\big)^{\frac{2}{1+\alpha_{m+1}}} e^{-\frac{G^*_{m+1}(p_{m+1})}{1+\alpha_{m+1}}}\\ &\times l(Q)e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}} + O(e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}-\epsilon\lambda_{m+1}}).\end{aligned} \tag{5.19}$$

Also, we have

$$\rho_1 - \rho_* = \frac{\pi^2}{(1+\alpha_{m+1})\sin\frac{\pi}{1+\alpha_{m+1}}}\big(\frac{4(1+\alpha_{m+1})^2}{\rho_* h^*_{p_{m+1}}(p^0_{m+1})}\big)^{\frac{2}{1+\alpha_{m+1}}} e^{-\frac{G^*_{m+1}(p^0_{m+1})}{1+\alpha_{m+1}}} l(Q)e^{-\frac{\lambda(P)}{1+\alpha_{m+1}}}.$$

Therefore, (5.18) and (5.19) imply

$$O(1)\lambda_{m+1}^{-1}e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}}=c(e^{-\frac{\lambda(P)}{1+\alpha_{m+1}}}-e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}})$$

for some $c\neq 0$. This in turn gives

$$|\lambda_{m+1}-\lambda(P)|\leq c_7\lambda(P)^{-1}. \tag{5.20}$$

for some c_7 independent of c_0.

Using (5.17) and (5.20), we have

$$\|\psi\|_*\leq c_8e^{-\frac{\lambda(P)}{1+\alpha_{m+1}}} \text{ and } |p_j-p_j^0|\leq c_9e^{-\frac{\lambda(P)}{1+\alpha_{m+1}}}. \tag{5.21}$$

To obtain estimates for $t_j-t_1, j\geq 2$, we note $\theta_j=O(e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}})$ by (5.9). Combined with (4.45), we have

$$\Big|t_j-\frac{1}{\rho_*}\sum_j 4\pi(1+\alpha_j)t_j\Big|=O(e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}})$$

and

$$\begin{aligned}|t_j-t_1|\leq&\Big|t_j-\frac{1}{\rho_*}\sum_j 4\pi(1+\alpha_j)t_j\Big|+\Big|\frac{1}{\rho_*}\sum_j 4\pi(1+\alpha_j)t_j-t_1\Big|\\ =&O(e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}})\leq c_{10}e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}}\end{aligned} \tag{5.22}$$

for $j\geq 2$, where c_{10} depends on $c_8,\ c_9$ and is independent of c_0. By choosing

$$c_0>c_1,c_7,c_8,c_9,c_{10},$$

we can get $v_2\notin\partial S_{\rho_2}(Q,w)$. From (5.6), (5.8), (5.20), (5.21), and (5.22), we obtain $v_1\notin\partial S_{\rho_1}(Q,w)$. Therefore,

$$(v_1,v_2)\notin\partial\Big(S_{\rho_1}(Q,w),S_{\rho_2}(Q,w)\Big).$$

The proof is completed. □

Then, we want to apply Lemma 4.5 and Lemma 5.1 to get the degree of the linear operator in $S_{\rho_1}(Q,w)\times S_{\rho_2}(Q,w)$ when ρ_1 crosses ρ_*.

To compute the term

$$\deg\Big(\big(2v_1+T_1(v_1,v_2),2v_2+T_2(v_1,v_2)\big);S_{\rho_1}(Q,w)\times S_{\rho_2}(Q,w),0\Big).$$

We set

$$S_1^*(Q,w)=\Big\{(P,\Lambda,A):\frac{1}{2}v_{P,\Lambda,A}+\phi\in S_{\rho_1}(Q,w),\phi\in O_{P,\Lambda}^1\Big\}$$

and define the map

$$\Phi_Q=(\Phi_{Q,1},\Phi_{Q,2},\Phi_{Q,3},\Phi_{Q,4}):$$

$$\begin{aligned}\Phi_{Q,1}^j&=\langle\nabla(2v_1+T_1^0(v_1,v_2)),\nabla\partial_{p_j}v_{p_j}\rangle+\langle\nabla 2v_2+T_2^0(v_1,v_2),0\rangle,\ \text{for } 1\leq j\leq m,\\ \Phi_{Q,2}^j&=\langle\nabla(2v_1+T_1^0(v_1,v_2)),\nabla\partial_{\lambda_j}v_{p_j}\rangle+\langle\nabla(2v_2+T_2^0(v_1,v_2)),0\rangle,\ \text{for } 1\leq j\leq n,\\ \Phi_{Q,3}^j&=\langle\nabla(2v_1+T_1^0(v_1,v_2)),\nabla v_{p_j}\rangle+\langle\nabla(2v_2+T_2^0(v_1,v_2)),0\rangle,\ \text{for } 1\leq j\leq n,\\ \Phi_{Q,4}&=\langle\nabla(2v_1+T_1^0(v_1,v_2)),0\rangle+(2v_2+T_2^0(v_1,v_2)).\end{aligned}$$

Clearly, by Lemma 4.5 and Lemma 5.1, we have

$$\begin{aligned}&\deg\Big(\big(2v_1+T_1(v_1,v_2),2v_2+T_2(v_1,v_2)\big);S_{\rho_1}(Q,w)\times S_{\rho_2}(Q,w),0\Big)\\&\quad=\deg\Big(\Phi_Q;S_1^*(Q,w)\times S_{\rho_2}(Q,w),0\Big).\end{aligned}\tag{5.23}$$

Next, we study the right hand side of (5.23) and prove Theorem 1.4.

Proof of Theorem 1.4. To compute the degree, we can simplify the problem by replacing Φ_Q by a new map $\hat{\Phi}_Q$ defined as follows: $\hat{\Phi}_{Q,1}=\Phi_{Q,1}$, $\hat{\Phi}_{Q,3}=\Phi_{Q,3}$, $\hat{\Phi}_{Q,4}=\Phi_{Q,4}$,

$$\begin{aligned}\hat{\Phi}^j_{Q,2}=&\Phi^j_{Q,2}-\frac{8\pi(1+\alpha_j)}{2\rho_*}\sum_{i=1}^n\Phi^i_{Q,3}+\Phi^j_{Q,3}\\=&-\frac{8\pi(1+\alpha_j)}{\rho_*}\Big[\rho-\rho_*-4\pi\sum_i[(1+\alpha_i)(t_i-t_j)]-\pi\sum_i d_ie^{-\frac{\lambda_i}{1+\alpha_i}}\Big].\end{aligned}\tag{5.24}$$

Clearly, we have

$$\frac{\partial\hat{\Phi}_{Q,1}}{\partial\Lambda}=\frac{\partial\hat{\Phi}_{Q,1}}{\partial A}=\frac{\partial\hat{\Phi}_{Q,2}}{\partial A}=\frac{\partial\hat{\Phi}_{Q,2}}{\partial\psi}=\frac{\partial\hat{\Phi}_{Q,3}}{\partial\psi}=\frac{\partial\hat{\Phi}_{Q,4}}{\partial A}=\frac{\partial\hat{\Phi}_{Q,4}}{\partial\Lambda}=0,\tag{5.25}$$

$$\Phi_Q(P,\Lambda,A,\psi)=0\text{ if and only in }\hat{\Phi}_Q(P,\Lambda,A,\psi)=0,\tag{5.26}$$

and

$$\deg\Big(\Phi_Q;S_1^*(Q,w)\times S_{\rho_2}(Q,w),0\Big)=\deg\Big(\hat{\Phi}_Q;S_1^*(Q,w)\times S_{\rho_2}(Q,w),0\Big).\tag{5.27}$$

Moreover if $\hat{\Phi}_{Q,1}=0$, $\hat{\Phi}_{Q,3}=0$ and $\hat{\Phi}_{Q,4}=0$ if and only if

$$(p_1,p_2,\cdots,p_m)=(p_1^0,p_2^0,\cdots,p_m^0),\ A=(1,1,\cdots,1),\ \psi=0,\tag{5.28}$$

and $\hat{\Phi}_{Q,2}=0$ if and only if

$$\begin{cases}t_1=t_2=\cdots=t_n,\\ \rho_1-\rho_*=\pi\sum_j d_je^{-\frac{\lambda_j}{1+\alpha_j}}.\end{cases}\tag{5.29}$$

It is not difficult to see that if $|\rho_1-\rho_*|$ is sufficiently small, equation (5.29) possesses a unique solution

$$\Lambda(P)=(\lambda_1,\lambda_2,\cdots,\lambda_n)$$

up to permutation. Hence $(P,\Lambda(P),A,0)$ is the solution of $\hat{\Phi}_Q$, where $A=(1,1,\cdots,1)$. By (5.25), the degree of $\hat{\Phi}_Q$ at $(P,\Lambda(P),A,0)$ depends on the number of negative eigenvalue for the following matrix

$$\mathcal{M}=\begin{bmatrix}\frac{\partial\hat{\Phi}_{Q,1}}{\partial P}, & \frac{\partial\hat{\Phi}_{Q,1}}{\partial\Lambda}, & \frac{\partial\hat{\Phi}_{Q,1}}{\partial A}, & \frac{\partial\hat{\Phi}_{Q,1}}{\partial\psi}\\ \frac{\partial\hat{\Phi}_{Q,2}}{\partial P}, & \frac{\partial\hat{\Phi}_{Q,2}}{\partial\Lambda}, & \frac{\partial\hat{\Phi}_{Q,2}}{\partial A}, & \frac{\partial\hat{\Phi}_{Q,2}}{\partial\psi}\\ \frac{\partial\hat{\Phi}_{Q,3}}{\partial P}, & \frac{\partial\hat{\Phi}_{Q,3}}{\partial\Lambda}, & \frac{\partial\hat{\Phi}_{Q,3}}{\partial A}, & \frac{\partial\hat{\Phi}_{Q,3}}{\partial\psi}\\ \frac{\partial\hat{\Phi}_{Q,4}}{\partial P}, & \frac{\partial\hat{\Phi}_{Q,4}}{\partial\Lambda}, & \frac{\partial\hat{\Phi}_{Q,4}}{\partial A}, & \frac{\partial\hat{\Phi}_{Q,4}}{\partial\psi}\end{bmatrix}$$

Here we say $\mu_{\mathcal{M}}$ is an eigenvalue of $\mathcal{M}$, if there exists $(\nu_1,\nu_2,\cdots,\nu_m)\in(\mathbb{R}^2)^m$, $(\lambda_1,\lambda_2,\cdots,\lambda_n)\in\mathbb{R}^n$, $(a_1,a_2,\cdots,a_n)\in\mathbb{R}^n$, and Ψ such that

$$\mathcal{M}\begin{bmatrix}\nu_1\\ \vdots\\ \nu_m\\ a_1\\ \vdots\\ a_n\\ \lambda_1\\ \vdots\\ \lambda_n\\ \Psi\end{bmatrix}=\mu_{\mathcal{M}}\begin{bmatrix}\nu_1\\ \vdots\\ \nu_m\\ a_1\\ \vdots\\ a_n\\ \lambda_1\\ \vdots\\ \lambda_n\\ (-\Delta)^{-1}\Psi\end{bmatrix},$$

where $\frac{\partial\hat{\Phi}^j_{Q,1}}{\partial\psi}[\Psi]=8\pi\nabla\Psi(p_j^0)$, and

$$\begin{aligned}\frac{\partial\hat{\Phi}_{Q,4}}{\partial\psi}[\Psi]=&\Psi-(-\Delta)^{-1}\Big(2\rho_2\frac{\overline{h}_2e^{w-4\pi\sum_{j=1}^m G(x,p_j^0)}}{\int_M\overline{h}_2e^{w-4\pi\sum_{j=1}^m G(x,p_j^0)}}\Psi\\ &-2\rho_2\frac{\overline{h}_2e^{w-4\pi\sum_{j=1}^m G(x,p_j^0)}}{\big(\int_M\overline{h}_2e^{w-4\pi\sum_{j=1}^m G(x,p_j^0)}\big)^2}\int_M\big(\overline{h}_2e^{w-4\pi\sum_{j=1}^m G(x,p_j^0)}\Psi\big)\Big).\end{aligned}$$

We set $N(T)$ as the number of the negative eigenvalue of matrix T,

$$\mathcal{M}_1=\begin{bmatrix}\frac{\partial\hat{\Phi}_{Q,1}}{\partial P}, & \frac{\partial\hat{\Phi}_{Q,1}}{\partial\psi}\\ \frac{\partial\hat{\Phi}_{Q,4}}{\partial P}, & \frac{\partial\hat{\Phi}_{Q,4}}{\partial\psi}\end{bmatrix}\text{ and }\mathcal{M}_2=\begin{bmatrix}\frac{\partial\hat{\Phi}_{Q,2}}{\partial\Lambda}, & \frac{\partial\hat{\Phi}_{Q,2}}{\partial A}\\ \frac{\partial\hat{\Phi}_{Q,3}}{\partial\Lambda}, & \frac{\partial\hat{\Phi}_{Q,3}}{\partial A}\end{bmatrix}.$$

By using (5.25),

$$N(\mathcal{M})=N(\mathcal{M}_1)+N(\mathcal{M}_2)=N(\mathcal{M}_1)+N\Big[\frac{\partial\hat{\Phi}_{Q,2}}{\partial\Lambda}\Big]+N\Big[\frac{\partial\hat{\Phi}_{Q,3}}{\partial A}\Big],$$

Therefore,

$$\begin{aligned}\deg\Big(\Phi_Q;S_1^*(Q,w)\times S_{\rho_2}(Q,w),0\Big)&=(-1)^{N(\mathcal{M})}=(-1)^{N(\mathcal{M}_1)}\times(-1)^{N(\mathcal{M}_2)}\\ &=(-1)^{N(\mathcal{M})}\times\mathrm{sgn}\Big(\det(\frac{\partial\hat{\Phi}_{Q,2}}{\partial\Lambda})\Big)\times\mathrm{sgn}\Big(\det(\frac{\partial\hat{\Phi}_{Q,3}}{\partial A})\Big).\end{aligned}$$

We first consider the last two terms on the right hand side of above equality. For $\det(\frac{\partial\hat{\Phi}_{Q,3}}{\partial A})$, it is easy to see that the sign of this value is positive, since it is a diagonal matrix with every term positive on diagonal. Therefore

$$\mathrm{sgn}\ \det(\frac{\partial\hat{\Phi}_{Q,3}}{\partial A})=1.$$

To compute $\det(\frac{\partial\hat{\Phi}_{Q,2}}{\partial\Lambda})$, we recall that

$$t_j=\lambda_j+\frac{d_j}{2(1+\alpha_j)^2}\lambda_je^{-\frac{\lambda_j}{1+\alpha_j}}-\sum_{j=1}^n\overline{v}_{p_j}+\text{constant}.$$

Thus

$$\frac{\partial t_j}{\partial\lambda_i}=\Big[1+(\frac{d_j}{2(1+\alpha_j)}-\frac{d_j}{2(1+\alpha_j)}\lambda_j)e^{-\frac{\lambda_j}{1+\alpha_j}}\Big]\delta_{ij}-\frac{\partial\overline{v}_{p_i}}{\partial\lambda_i}.$$

By (5.24), we have

$$\frac{\partial \hat{\Phi}^j_{Q,2}}{\partial \lambda_j} = -\sum_{i\neq j}(1+\alpha_i)\frac{\partial t_j}{\partial \lambda_j} + O(e^{-\frac{\lambda_j}{1+\alpha_j}}) = -\sum_{i\neq j}(1+\alpha_i)[1-\frac{d_j}{2(1+\alpha_j)^2}\lambda_j e^{-\frac{\lambda_j}{1+\alpha_j}}] + O(e^{-\frac{\lambda_j}{1+\alpha_j}})$$

and

$$\frac{\partial \hat{\Phi}^j_{Q,2}}{\partial \lambda_i} = (1+\alpha_i)\frac{\partial t_i}{\partial \lambda_i} - \frac{d_i}{4(1+\alpha_i)}e^{-\frac{\lambda_i}{1+\alpha_i}} = (1+\alpha_i)[1-\frac{d_i}{2(1+\alpha_i)^2}\lambda_i e^{-\frac{\lambda_i}{1+\alpha_i}}] + O(e^{-\frac{\lambda_i}{1+\alpha_i}})$$

for $i \neq j$, here we replace $\hat{\Phi}_{Q,2}$ by $\frac{\rho^*}{32\pi^2(1+\alpha_j)}\hat{\Phi}_{Q,2}$ (still denoted by $\hat{\Phi}_{Q,2}$). Denote

$$B = \sum_i (1+\alpha_i),\ E_i = 1-\frac{d_i}{2(1+\alpha_i)^2}\lambda_i e^{-\frac{\lambda_i}{1+\alpha_i}},\ \delta_j = \sum_i \frac{\partial \hat{\Phi}^j_{Q,2}}{\partial \lambda_i}. \tag{5.30}$$

Thus, we have

$$\begin{aligned}
&\det\Big[(\frac{\partial \hat{\Phi}^j_{Q,2}}{\partial \Lambda})\Big] \\
=&\det\begin{bmatrix}
(1+\alpha_1-B)E_1+(*) & (1+\alpha_2)E_2+(*) & \cdots & (1+\alpha_n)E_n+(*) \\
(1+\alpha_1)E_1+(*) & (1+\alpha_2-B)E_2+(*) & \cdots & (1+\alpha_n)E_n+(*) \\
\vdots & \vdots & \vdots & \vdots \\
(1+\alpha_1)E_1+(*) & (1+\alpha_2)E_2+(*) & \cdots & (1+\alpha_n-B)E_n+(*)
\end{bmatrix} \\
=&\det\begin{bmatrix}
\delta_1 & (1+\alpha_2)E_2+(*) & \cdots & (1+\alpha_n)E_n+(*) \\
\delta_2 & (1+\alpha_2-B)E_2+(*) & \cdots & (1+\alpha_n)E_n+(*) \\
\vdots & \vdots & \vdots & \vdots \\
\delta_n & (1+\alpha_2)E_2+(*) & \cdots & (1+\alpha_n-B)E_n+(*)
\end{bmatrix} \\
=&\det\begin{bmatrix}
\delta_1 & (1+\alpha_2)E_2+(*) & (1+\alpha_3)E_3+(*) & \cdots & (1+\alpha_n)E_n+(*) \\
\delta_2-\delta_1 & -BE_2+(*) & (*) & \cdots & (*) \\
\delta_3-\delta_1 & (*) & -BE_3+(*) & \cdots & (*) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\delta_n-\delta_1 & (*) & (*) & \cdots & -BE_n+(*)
\end{bmatrix} \\
=&\det\begin{bmatrix}
\sum_j \frac{1+\alpha_j}{B}\delta_j & (*) & (*) & \cdots & (*) \\
\delta_2-\delta_1 & -BE_2+(*) & (*) & \cdots & (*) \\
\delta_3-\delta_1 & (*) & -BE_3+(*) & \cdots & (*) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\delta_n-\delta_1 & (*) & (*) & \cdots & -BE_n+(*)
\end{bmatrix},
\end{aligned}$$

where all the terms $(*)$ is bounded by $O(e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}})$. Next, we consider $\sum_j(1+\alpha_j)\delta_j$,

$$\begin{aligned}
&\sum_j(1+\alpha_j)\delta_j\\
=&\sum_j(1+\alpha_j)\Big[-\sum_{i\neq j}(1+\alpha_i)\Big(1-(\frac{d_j}{2(1+\alpha_j)^2}\lambda_j-\frac{d_j}{2(1+\alpha_j)})e^{-\frac{\lambda_j}{1+\alpha_j}}-\frac{\partial\overline{v}_{p_j}}{\partial\lambda_j}\Big)\\
&-\frac{d_j}{4(1+\alpha_j)}e^{-\frac{\lambda_j}{1+\alpha_j}}+\Big(\sum_{i\neq j}(1+\alpha_i)\Big(1-(\frac{d_i}{2(1+\alpha_i)^2}\lambda_i-\frac{d_i}{2(1+\alpha_i)})e^{-\frac{\lambda_i}{1+\alpha_i}}-\frac{\partial\overline{v}_{p_i}}{\partial\lambda_i}\Big)\\
&-\frac{d_i}{4(1+\alpha_i)}e^{-\frac{\lambda_i}{1+\alpha_i}}\Big)\Big]\\
=&-\sum_j(1+\alpha_j)\sum_j\frac{d_j}{4(1+\alpha_j)}e^{-\frac{\lambda_j}{1+\alpha_j}}+O(e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}-\epsilon\lambda_{m+1}})\\
=&-\frac{B}{4(1+\alpha_{m+1})}\sum_j d_je^{-\frac{\lambda_j}{1+\alpha_j}}+O(e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}-\epsilon\lambda_{m+1}})\\
=&-\frac{B}{4\pi(1+\alpha_{m+1})}(\rho-\rho_*)+O(e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}-\epsilon\lambda_{m+1}})
\end{aligned}$$

for some $\epsilon>0$. Thus

$$\det\Big[(\frac{\partial\hat{\Phi}^j_{Q,2}}{\partial\Lambda})\Big]=(-1)^n\frac{B^{n-1}}{4\pi(1+\alpha_{m+1})}(\rho-\rho_*)+O(e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}-\epsilon\lambda_{m+1}}). \tag{5.31}$$

It remains to compute $N(\mathcal{M}_1)$. According to the definition, we have

$$\Big[\frac{\partial(\hat{\Phi}_{Q,1},\hat{\Phi}_{Q,4})}{\partial(P,\psi)}\Big]\begin{pmatrix}\nu_1\\ \nu_2\\ \vdots\\ \nu_m\\ \Psi\end{pmatrix}=\begin{pmatrix}-\mathcal{I}_1+\nabla\Psi(p_1^0)\\ -\mathcal{I}_2+\nabla\Psi(p_2^0)\\ \vdots\\ -\mathcal{I}_m+\nabla\Psi(p_m^0)\\ -\mathcal{I}_0\end{pmatrix}, \tag{5.32}$$

where

$$\mathcal{I}_i=\nabla_x^2H_i(0,0)\cdot\nu_i+8\pi\sum_{j=1,j\neq i}^m\nabla_x^2G(x,p_j^0)\mid_{x=p_i^0}\cdot\nu_j,\ i=1,2,\cdots,m,$$

and

$$\begin{aligned}
\mathcal{I}_0=&-\Psi+(-\Delta)^{-1}\Big(2\rho_2\frac{\overline{h}_2e^{w-4\pi\sum_{j=1}^mG(x,p_j^0)}}{\int_M\overline{h}_2e^{w-4\pi\sum_{j=1}^mG(x,p_j^0)}}\Psi\\
&-2\rho_2\frac{\overline{h}_2e^{w-4\pi\sum_{j=1}^mG(x,p_j^0)}}{\big(\int_M\overline{h}_2e^{w-4\pi\sum_{j=1}^mG(x,p_j^0)}\big)^2}\int_M\big(\overline{h}_2e^{w-4\pi\sum_{j=1}^mG(x,p_j^0)}\Psi\big)\\
&-4\pi\rho_2\frac{\overline{h}_2e^{w-4\pi\sum_{j=1}^mG(x,p_j^0)}}{\int_M\overline{h}_2e^{w-4\pi\sum_{j=1}^mG(x,p_j^0)}}\big(\sum_{j=1}^m\nabla G(x,p_j^0)\cdot\nu_j\big)\\
&+4\pi\rho_2\frac{\overline{h}_2e^{w-4\pi\sum_{j=1}^mG(x,p_j^0)}}{\big(\int_M\overline{h}_2e^{w-4\pi\sum_{j=1}^mG(x,p_j^0)}\big)^2}\int_M\Big[\overline{h}_2e^{w-4\pi\sum_{j=1}^mG(x,p_j^0)}\big(\sum_{j=1}^m\nabla G(x,p_j^0)\cdot\nu_j\big)\Big]\Big).
\end{aligned}$$

According to the definition of the topological degree for the solution to the shadow system (1.12), we can get $(-1)^{N(\mathcal{M}_1)}$ is exactly the Leray-Schauder topological degree contributed by (P_w, w). Therefore, we proved Theorem 1.4. $\square$

Proof of Theorem 1.3. Theorem 1.3 is a consequence of Theorem 1.4. $\square$

6. Proof of Theorem 1.5 and Theorem 1.6

This section is devoted to prove Theorem 1.5. We first introduce a deformation to decouple the system (1.11).

$$(S_t)\begin{cases} \Delta w + 2\rho_2(\frac{h_2 e^{w-4\pi G(x,p)}}{\int_M h_2 e^{w-4\pi G(x,p)}} - 1) = 0, \\ \nabla\big(\log(h_1 e^{-\frac{1}{2}w\cdot(1-t)}) + 4\pi R(x,x)\big)\mid_{x=p} = 0. \end{cases} \tag{6.1}$$

We can easily see that the system (6.1) is exactly (1.11) when $t=0$, and will be a decoupled system when $t=1$. During the deformation from (S_1) to (S_0), we have

Lemma 6.1. *Let $\rho_2 \notin 4\pi\mathbb{N}$. Then there is uniform constant C_{ρ_2} such that for all solutions to (6.1), we have $|w|_{L^\infty(M)} < C_{\rho_2}$.*

Proof. Since $\rho_2 \notin 4\pi\mathbb{N}$, then we can see any solution for the following equation

$$\Delta w + 2\rho_2(\frac{h_2 e^{w-4\pi G(x,p)}}{\int_M h_2 e^{w-4\pi G(x,p)}} - 1) = 0 \tag{6.2}$$

is uniformly bounded above. By using the classical elliptic estimate, we have $|w|_{C^1(M)} < C$. This constant C depends on ρ_2. $\square$

Proof of Theorem 1.5. It is known that the topological degree is independent of h_1 and h_2 as long as they are positive C^1 functions. By Remark 3 in section 4, we always can choose h_1 and h_2 such that the hypothesis of Theorem 3.3 holds.

Let d_S denote the Leray-Schauder degree for (1.1). By Lemma 6.1, computing the topological degree for (1.11) is reduced to compute the topological degree for system (6.1) when $t=1$,

$$\begin{cases} \Delta w + 2\rho_2(\frac{h_2 e^{w-4\pi G(x,p)}}{\int_M h_2 e^{w-4\pi G(x,p)}} - 1) = 0, \\ \nabla[\log h_1 + 4\pi R(x,x)]\mid_{x=p} = 0. \end{cases} \tag{6.3}$$

Since this is a decoupled system, the topological degree of (6.3) equals the product of the degree of first equation and degree contributed by the second equation. By Poincare-Hopf Theorem, the degree of the second equation is $\chi(M)$. On the other hand, by Theorem A, the topological degree for the first equation is $b_k + b_{k-1}$, where b_k is given (1.17). Therefore,

$$d_S = \chi(M)\cdot(b_k + b_{k-1}). \tag{6.4}$$

Combined with Lemma 6.1, we get Theorem 1.5. $\square$

Proof of Theorem 1.6. Theorem 1.6 is a consequence of Theorem 1.4, Theorem 1.5 and Theorem A. $\square$

7. The Dirichlet Problem

In this section we consider the Dirichlet problem of $SU(3)$ Toda system. Let Ω be a bounded smooth domain in $\mathbb{R}^2$ and $h_1,\ h_2$ are two positive $C^{2,\alpha}$ function in Ω. We consider

$$\begin{cases} \Delta u_1 + 2\rho_1 \frac{h_1 e^{u_1}}{\int_M h_1 e^{u_1}} - \rho_2 \frac{h_2 e^{u_2}}{\int_M h_2 e^{u_2}} = 0,\ u_1 = 0 \text{ on } \partial\Omega, \\ \Delta u_2 - \rho_1 \frac{h_1 e^{u_1}}{\int_M h_1 e^{u_1}} + 2\rho_2 \frac{h_2 e^{u_2}}{\int_M h_2 e^{u_2}} = 0,\ u_2 = 0 \text{ on } \partial\Omega. \end{cases} \tag{7.1}$$

By [31, Theorem 1.1], we know that the blow up never occurs on the boundary. Therefore, we can use all the arguments for (1.6) with minor modification to get the corresponding result for Dirichlet boundary problem (7.1).

Theorem 7.1. *Suppose h_1, h_2 are two positive $C^{2,\alpha}$ function in Ω and the assumption $(i), (ii)$. Then there exists a positive constant c such that for any solution of equation (7.1), there holds:*

$$|u_1(x)|, |u_2(x)| \leq c,\ \forall x \in M,\ i = 1, 2.$$

In order to state our degree formula for (7.1), we introduce the following generating function

$$\Xi_\Omega(x) = (1 + x + x^2 + x^3 \cdots)^{-\chi(\Omega)+1} = \mathfrak{b}_0 + \mathfrak{b}_1 x^1 + \mathfrak{b}_2 x^2 + \cdots + \mathfrak{b}_k x^k + \cdots,$$

where $\chi(\Omega)$ denotes the Euler characteristic number for Ω, then we have the following theorem

Theorem 7.2. *Suppose $d^{(2)}_{\rho_1,\rho_2}$ denotes the topological degree for (7.1) when $\rho_2 \in (4k\pi, 4(k+1)\pi)$, then*

$$d^{(2)}_{\rho_1,\rho_2} = \begin{cases} \mathfrak{b}_k, & \rho_1 \in (0, 4\pi), \\ \mathfrak{b}_k - \chi(\Omega)(\mathfrak{b}_k + \mathfrak{b}_{k-1}), & \rho_1 \in (4\pi, 8\pi), \end{cases}$$

where $\mathfrak{b}_{-1} = 0$.

Corollary 7.3. *If Ω is not simply connected, then (7.1) has a solution for $\rho_1 \in (0, 4\pi) \cup (4\pi, 8\pi)$ and $\rho_2 \notin 4\pi\mathbb{N}$.*

8. Proof of Lemma 4.4 and (5.14)

This section is devoted to prove Lemma 4.4 and (5.14). Let

$$\overline{v}_\alpha := \frac{\int_{B_{r_0}(0)} \frac{|y|^{2\alpha} e^\lambda}{(1+e^\lambda |y|^{2+2\alpha})^2} v(y) \mathrm{d}y}{\int_{\mathbb{R}^2} \frac{|y|^{2\alpha} e^\lambda}{(1+e^\lambda |y|^{2+2\alpha})^2} \mathrm{d}y} = \frac{1+\alpha}{\pi} \int_{B_{r_0}(0)} \frac{|y|^{2\alpha} e^\lambda}{(1+e^\lambda |y|^{2+2\alpha})^2} v(y) \mathrm{d}y.$$

Then we have the following Poincare-type inequality:

$$\int_{B_{r_0}(0)} \frac{|y|^{2\alpha} e^\lambda}{(1+e^\lambda |y|^{2+2\alpha})^2} \phi^2(y) \mathrm{d}y \leq c(\|\phi\|^2_{H^1(B_{r_0}(0))} + \overline{\phi}^2_\alpha) \tag{8.1}$$

for some constant c independent of λ (see [12, Lemma 6.2] for example). Using (8.1) we can prove the following result.

Lemma 8.1. *Let $P = (p_1, p_2, \cdots, p_n)$ and $\Lambda = (\lambda_1, \cdots, \lambda_n)$. Assume $\phi \in O^{(1)}_{P,\Lambda}$. Then there is a constant c and $\epsilon > 0$ such that for large λ_j*

$$\int_{B_{r_0}(p_j)} |y|^{2\alpha_j} e^{U_j} \phi \mathrm{d}y \le c e^{-\epsilon\lambda_j} \|\phi\|_{H^1}, \tag{8.2}$$

and

$$\int_{B_{r_0}(p_j)} |y|^{2\alpha_j} e^{U_j} \phi^2 \mathrm{d}y = O(1)\|\phi\|^2_{H^1}. \tag{8.3}$$

For a proof, see [12].

Proof of Lemma 4.4. We start with part (1). Let $\phi \in O^{(1)}_{P,\Lambda}$ and $\psi \in O^{(2)}_{P,\Lambda}$. Recall $2v_1 = v_{P,\Lambda,A} + 2\phi$, $\phi \in O^{(1)}_{P,\Lambda}$ and $v_2 = \frac{1}{2}w + \psi$, $\psi \in O^{(2)}_{P,\Lambda}$. We compute

$$\langle \nabla(2v_1 + T_1(v_1, v_2)), \nabla\phi_1\rangle = -\langle \Delta(2v_1 + T_1(v_1, v_2)), \phi_1\rangle.$$

Here we will use the decomposition of $\Delta(2v_1 + T_1(v_1, v_2))$ in (4.49)-(4.51).

$$\begin{aligned}\langle \nabla(2v_1 + T_1(v_1, v_2)), \nabla\phi_1\rangle =& \int 2\nabla\phi\cdot\nabla\phi_1 - \sum_j \int_{B_{r_0}(p_j)} 4\rho_1 h_{p_j}(p_j) e^{I_j}\phi\phi_1 \\ &+ \text{remainders} \\ :=& 2\mathfrak{B}(\phi,\phi_1) + \text{remainder terms.}\end{aligned}$$

Clearly, $\mathfrak{B}$ is a symmetric bilinear form in $O^{(1)}_{P,\Lambda}$. For the remainder terms, by (8.2) and $\phi_1 \in O^{(1)}_{P,\Lambda}$, we have for large λ_{m+1},

$$\Big|\Big(\frac{e^{t_j}}{\int_M h_1 e^{2v_1 - v_2}} - 1\Big)\int_{B_{r_0}(0)} \phi_1\rho_1 h_{p_j}(p_j)|y|^{2\alpha_j}e^{U_j}\mathrm{d}y\Big| = O(e^{-(\frac{1}{1+\alpha_{m+1}}+\epsilon)\lambda_{m+1}})\|\phi_1\|_{H^1}, \tag{8.4}$$

$$\begin{aligned}&\lambda_j(a_j - 1)\int_{B_{r_0}(p_j)} \nabla H_j\cdot y\rho_1 h_{p_j}(p_j)|y|^{2\alpha_j}e^{U_j}\phi_1\mathrm{d}y \\ \le& O(1)\lambda_j|a_j - 1|e^{-\frac{\lambda_{m+1}}{2(1+\alpha_{m+1})}}\Big[\int_{B_{r_0}(p_j)} |y|^{2+2\alpha_j}e^{U_j}\mathrm{d}y\Big]^{\frac{1}{2}}\|\phi_1\|_{H^1} \\ \le& O(1)\lambda_j|a_j - 1|e^{-\frac{\lambda_{m+1}}{2(1+\alpha_{m+1})}}\|\phi_1\|_{H^1} = o(1)e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}}\|\phi_1\|_{H^1}.\end{aligned} \tag{8.5}$$

Similarly, we have

$$\lambda_j(a_j - 1)\int_{B_{r_0}(p_j)} |y|^{2\alpha_j}e^{U_j}\eta_j\phi_1\mathrm{d}y = O(1)e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}}\|\phi_1\|_{H^1}. \tag{8.6}$$

By Lemma 8.1, we have for large λ_{m+1}

$$\begin{aligned}
&\int_{B_{r_0}(p_j)} \rho_1 h_{p_j}(p_j)|y|^{2\alpha_j} e^{U_j}(a_j-1)(U_j+s_j-1)\phi_1 \mathrm{d}y \\
&=2\lambda_j \int_{B_{r_0}(p_j)} \rho_1 h_{p_j}(p_j)|y|^{2\alpha_j} e^{U_j}(a_j-1)\phi_1 \mathrm{d}y \\
&\quad+\int_{B_{r_0}(p_j)} \rho_1 h_{p_j}(p_j)|y|^{2\alpha_j} e^{U_j}(a_j-1)(U_j-\lambda_j+O(1))\phi_1 \mathrm{d}y \\
&=2\lambda_j(a_j-1)O(e^{-\epsilon\lambda_j})\|\phi_1\|_{H^1} \\
&\quad+O(a_j-1)\Big(\int_{B_{r_0}(p_j)} |y|^{2\alpha_j} e^{U_j}(U_j-\lambda_j+O(1))^2 \mathrm{d}y\Big)^{\frac12}\Big(\int_{B_{r_0}(p_j)} |y|^{2\alpha_j} e^{U_j}\phi_1^2\Big)^{\frac12} \\
&=O(1)|a_j-1|\|\phi_1\|_{H^1}=O(1)e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}}\|\phi_1\|_{H^1}.
\end{aligned} \tag{8.7}$$

As for $\hat{E}_j$, we define E^+ and E^- as before:

$$E^+=\begin{cases}\hat{E}_j & \text{if } |\varphi|\geq\varepsilon_2\\ 0 & \text{if } |\varphi|<\varepsilon_2,\end{cases} \qquad E^-=\begin{cases}\hat{E}_j & \text{if } |\varphi|<\varepsilon_2\\ 0 & \text{if } |\varphi|\geq\varepsilon_2,\end{cases}$$

where ε_2 is a small number. Then we use (4.40) and similar argument there to obtain

$$\begin{aligned}
\int_{B_{r_0}(p_j)} |E^+\phi_1|\mathrm{d}y \leq&\Big(\int_{B_{r_0}(p_j)} |E^+|^2\mathrm{d}y\Big)^{\frac12}\Big(\int_{B_{r_0}(p_j)} \phi_1^2\Big)^{\frac12} \\
=&O(e^{-b\lambda_j})\|\phi_1\|_{H^1}
\end{aligned} \tag{8.8}$$

for any fixed $b>0$. For E^-, we use (4.29) and Lemma 8.1 to obtain

$$\begin{aligned}
\int_{B_{r_0}(p_j)} |E^-\phi_1|\mathrm{d}y \leq& O(1)\int_{B_{r_0}(p_j)} h_{p_j}(p_j)|y|^{2\alpha_j}e^{U_j}(O(\phi^2)+O(\beta_j))\phi_1\mathrm{d}y \\
=&O(\varepsilon_2)\Big(\int_{B_{r_0}(0)} |y|^{2\alpha_j}e^{U_j}\phi^2\Big)^{\frac12}\Big(\int_{B_{r_0}(0)} |y|^{2\alpha_j}e^{U_j}\phi_1^2\mathrm{d}y\Big)^{\frac12} \\
&+O(e^{-\frac{2\lambda_{m+1}}{1+\alpha_{m+1}}})\Big(\int_{B_{r_0}(0)} |y|^{2\alpha_j}e^{U_j}\phi_1^2\mathrm{d}y\Big)^{\frac12} \\
&+\int_{B_{r_0}(0)} |y|^{2\alpha_j}e^{U_j}|y|^3|\phi_1| \\
=&O(\varepsilon_2)\|\phi\|_{H^1}\|\phi_1\|_{H^1}+O(e^{-\frac{2\lambda_{m+1}}{1+\alpha_{m+1}}})\|\phi_1\|_{H^1} \\
&+\Big(\int_{B_{r_0}(0)} |y|^{2b\alpha_j}e^{bU_j}|y|^6\Big)^{\frac12}\Big(\int_{B_{r_0}(0)} e^{(2-b)U_j}|y|^{2(2-b)\alpha_j}\phi_1^2\mathrm{d}y\Big)^{\frac12} \\
=&O(\varepsilon_2 e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}})\|\phi_1\|_{H^1}+O(e^{-(2-\frac{b}{2})\frac{\lambda_{m+1}}{1+\alpha_{m+1}}})\|\phi_1\|_{H^1}
\end{aligned}$$

for $2>b>\frac{4}{2+\alpha_j}$.

$$\int_{B_{r_0}(p_j)} |E^-\phi_1|\mathrm{d}y=O(e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}})\|\phi_1\|_{H^1}, \tag{8.9}$$

provided that $\varepsilon_2 c_1 < 1$. By Lemma 4.1,

$$\int_{B_{2r_0}(p_j)\setminus B_{r_0}(p_j)} \Delta(v_{p_j} - 4\pi(1+\alpha_j)G(x,p_j))\phi_1 = O(e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}})\|\phi_1\|_{H^1}. \tag{8.10}$$

For the nonlinear term in $\Delta T_1(v_1,v_2)$ on $M\setminus\bigcup_{j=1}^n B_{r_0}(p_j)$, we first note that

$$\begin{aligned}\int_{M\setminus\cup_{j=1}^n B_{r_0}(p_j)} |e^{\varphi}\phi_1| =& O(1)\Big(\int_{|\varphi|\ge\varepsilon_2}|e^{\varphi}\phi_1| + \int_{|\varphi|<\varepsilon_2}|e^{\varepsilon_2}\phi_1|\Big)\\ =& O(1)\|\phi_1\|_{H^1}\end{aligned}$$

by (4.40). Using $\int_M h_1 e^{2v_1-v_2} = O(e^{\lambda_{m+1}})$, we have

$$\begin{aligned}\int_{M\setminus\cup_{j=1}^n B_{r_0}(p_j)} \frac{\rho_1 h_1 e^{2v_1-v_2}}{\int_M h_1 e^{2v_1-v_2}}|\phi_1| &= O(e^{-\lambda_{m+1}})\int_{M\setminus\cup_{j=1}^n B_{r_0}(p_j)} e^{\varphi}|\phi_1|\\ &= O(e^{-\lambda_{m+1}})\|\phi_1\|_{H^1}.\end{aligned} \tag{8.11}$$

In the end, we need to consider the term $\int_{B_{r_0}(p_j)} 2\rho_1 h_{p_j}(p_j)e^{U_j}\psi\phi_1$. We have

$$\begin{aligned}\int_{B_{r_0(p_j)}} 2\rho_1 h_{p_j}(p_j)e^{U_j}\psi\phi_1 =& \int_{B_{r_0(p_j)}} 2\rho_1 h_{p_j}(p_j)e^{U_j}\psi(p_j)\phi_1\\ &+ \int_{B_{r_0(p_j)}} 2\rho_1 h_{p_j}(p_j)e^{U_j}(\psi-\psi(p_j))\phi_1\\ =& o(e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}})\|\phi_1\|_{H^1},\end{aligned} \tag{8.12}$$

where we used (8.2). Combining (8.4)-(8.12), we obtain

$$\langle\nabla(2v_1+T_1(v_1,v_2)),\nabla\phi_1\rangle = \mathfrak{B}(\phi,\phi_1) + O(e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}})\|\phi_1\|_{H^1}.$$

This proves part (1).

Next, we prove part (3). On $B_{2r_0}(p_j)$, we have

$$\begin{aligned}\partial_{\lambda_j} v_{p_j} =& \Big(2 - \frac{\frac{\rho_1 h_{p_j}(p_j)}{2(1+\alpha_j)^2}e^{\lambda_j}|x-p_j|^{2(1+\alpha_j)}}{1+\frac{\rho_1 h_{p_j}(p_j)}{4(1+\alpha_j)^2}e^{\lambda_j}|x-p_j|^{2(1+\alpha_j)}} + O(\lambda_j e^{-\frac{\lambda_j}{1+\alpha_j}})\Big)\sigma_j\\ =& \Big[(1+\partial_{\lambda_j}U_j) + O(\lambda_j e^{-\frac{\lambda_j}{1+\alpha_j}})\Big]\sigma_j\end{aligned} \tag{8.13}$$

by the setting of v_{p_j}. On $M\setminus\bigcup_j B_{2r_0}(p_j)$, $\partial_{\lambda_j} v_{p_j} = 0$. We compute $\langle\nabla(2v_1 + T_1(v_1,v_2)),\nabla\partial_{\lambda_j}v_{p_j}\rangle = -\langle\Delta(2v_1+T_1(v_1,v_2)),\partial_{\lambda_j}v_{p_j}\rangle$ by using (4.49)-(4.51).

Since $\phi\in O^{(1)}_{P,\Lambda}$, we have

$$\int_M \nabla\phi\cdot\nabla\partial_{\lambda_j}v_{p_j} = 0.$$

Direct computation yields,

$$\int_{B_{r_0}(p_j)} \partial_{\lambda_j}v_{p_j} = \int_{B_{r_0}(p_j)} (1+\partial_{\lambda_j}U_j) + O(\lambda_j e^{-\frac{\lambda_j}{1+\alpha_j}}) = O(\lambda_j e^{-\frac{\lambda_j}{1+\alpha_j}}). \tag{8.14}$$

Hence,

$$(|\rho_* - \rho_1| + |a_i - 1|)\int_{B_{r_0}(p_j)} \partial_{\lambda_j}v_{p_j}\mathrm{d}y = O(e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}} - \epsilon\lambda_{m+1}}) \tag{8.15}$$

for some $\epsilon>0$. By (8.13),

$$
\begin{aligned}
&\int_{B_{r_0}(p_j)}\rho_1 h_{p_j}(p_j)|x-p_j|^{2\alpha_j}e^{U_j}\partial_{\lambda_j}v_{p_j}\mathrm{d}y\\
&\quad=\int_{B_{r_0}(p_j)}\rho_1 h_{p_j}(p_j)|x-p_j|^{2\alpha_j}e^{U_j}\Big(1+\partial_{\lambda_j}U_j+O(\lambda_j e^{-\frac{\lambda_j}{1+\alpha_j}})\Big)\mathrm{d}y\\
&\quad=\int_{\mathbb{R}^2}\rho_1 h_{p_j}(p_j)|x-p_j|^{2\alpha_j}e^{U_j}(1+\partial_{\lambda_j}U_j)\mathrm{d}z+O(\lambda_j e^{-\frac{\lambda_j}{1+\alpha_j}})\\
&\quad=4\pi(1+\alpha_j)+O(\lambda_j e^{-\frac{\lambda_j}{1+\alpha_j}}),
\end{aligned}\tag{8.16}
$$

and

$$
\begin{aligned}
&\int_{B_{r_0}(p_j)}2\rho_1 h_{p_j}(p_j)|x-p_j|^{2\alpha_j}e^{U_j}\Big[-2\log\Big(1+\frac{\rho_1 h_{p_j}(p_j)e^{\lambda_j}}{4(1+\alpha_j)^2}|x-p_j|^{2(1+\alpha_j)}\Big)\Big]\partial_{\lambda_j}v_{p_j}\\
&=\int_{\mathbb{R}^2}\frac{8(1+\alpha_j)^2r^{2\alpha_j}}{(1+r^{2(1+\alpha_j)})^2}[-2\log(1+r^{2(1+\alpha_j)})]\Big(\frac{2}{1+r^{2(1+\alpha_j)}}+O(\lambda_j e^{-\frac{\lambda_j}{1+\alpha_j}})\Big)\mathrm{d}z\\
&\quad+O(e^{-\lambda_j})\\
&=-8\pi(1+\alpha_j)+O(e^{-\frac{\lambda_j}{1+\alpha_j}}).
\end{aligned}\tag{8.17}
$$

(8.16) and (8.17) together give

$$
\begin{aligned}
&\int_{B_{r_0}(p_j)}\Big(2\rho_1 h_{p_j}(p_j)|x-p_j|^{2\alpha_j}e^{U_j}\big[(a_j-1)(U_j+s_j-1)\\
&\quad+\sum 8\pi(1+\alpha_i)(a_i-1)G(p_j,p_i)+(\frac{e^{t_j}}{\int_M h_1e^{2v_1-v_2}}-1)\big]\Big)\mathrm{d}y\\
&\quad=8\pi(1+\alpha_j)(2\lambda_j(a_j-1)+\frac{e^{t_j}}{\int_M h_1e^{2v_1-v_2}}-1)+O(1)\big(\max_i|a_1-1|\big).
\end{aligned}\tag{8.18}
$$

To estimate the term with $\phi\partial_{\lambda_j}v_{p_j}$ and $\psi\partial_{\lambda_j}v_{p_j}$, note that $\phi\in O^{(1)}_{P,\Lambda}$ implies

$$
\begin{aligned}
0&=\int_M\nabla\phi\nabla\partial_{\lambda_j}v_{p_j}=-\int_M\phi\Delta(\partial_{\lambda_j}v_{p_j})\\
&=-\int_{B_{r_0}(p_j)}\phi\Delta(\partial_{\lambda_j}U_j)\mathrm{d}y+O(\lambda_j e^{-\frac{\lambda_j}{1+\alpha_j}}\|\phi\|_{H^1})\\
&=\int_{B_{r_0}(p_j)}2\rho_1 h_{p_j}(p_j)|x-p_j|^{2\alpha_j}e^{U_j}\phi\partial_{\lambda_j}U_j\mathrm{d}y+O(\lambda_j e^{-\frac{\lambda_j}{1+\alpha_j}}\|\phi\|_{H^1})
\end{aligned}
$$

Together with Lemma 8.1, we conclude from the above

$$
\begin{aligned}
&\int_{B_{r_0}(p_j)}2\rho_1 h_{p_j}(p_j)|x-p_j|^{2\alpha_j}e^{U_j}\phi\partial_{\lambda_j}v_{p_j}\\
&\quad=\int_{B(r_0)(p_j)}2\rho_1 h_{p_j}(p_j)|x-p_j|^{2\alpha_j}e^{U_j}\phi(1+\partial_{\lambda_j}U_j+O(\lambda_j e^{-\frac{\lambda_j}{1+\alpha_j}}))\mathrm{d}y\\
&\quad=O(e^{-\frac{\epsilon\lambda_{m+1}}{1+\alpha_{m+1}}})\|\phi\|_{H^1}=O(e^{-\epsilon\lambda_{m+1}-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}}).
\end{aligned}\tag{8.19}
$$

While for $\psi\partial_{\lambda_j}v_{p_j}$, we have

$$\begin{aligned}
&\int_{B_{r_0(p_j)}}\rho_1 h_{p_j}(p_j)|x-p_j|^{2\alpha_j}e^{U_j}\psi\partial_{\lambda_j}v_{p_j}\\
=&\int_{B_{r_0(p_j)}}\rho_1 h_{p_j}(p_j)|x-p_j|^{2\alpha_j}e^{U_j}\psi(p_j)\partial_{\lambda_j}v_{p_j}\\
&+\int_{B_{r_0(p_j)}}\rho_1 h_{p_j}(p_j)|x-p_j|^{2\alpha_j}e^{U_j}(\psi-\psi(p_j))\partial_{\lambda_j}v_{p_j}\\
=&4\pi(1+\alpha_j)\psi(p_j)+O(e^{-\epsilon\lambda_j-\frac{\lambda_j}{1+\alpha_j}}),
\end{aligned}\tag{8.20}$$

The other integrals of $\partial_{\lambda_j}v_{p_j}$ with other terms in (4.49) would be smaller than $O(\max|a_i-1|+e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}-\epsilon\lambda_{m+1}})$. Since the computations are straightforward, we omit the details here.

Now we go to the integral over $M\setminus B_{r_0}(p_j)$. Since $\partial_{\lambda_j}v_{p_j}=0$ on $M\setminus B_{2r_0}(p_j)$, we only need to consider the integrals on $B_{2r_0}(p_j)$. On $B_{2r_0}(p_j)\setminus B_{r_0}(p_j)$, $e^{2v_1-v_2}=O(e^{\varphi})$, $\frac{\rho_1 h_1 e^{2v_1-v_2}}{\int_M h_1 e^{2v_1-v_2}}=O(e^{-\lambda_{m+1}})e^{\varphi}$ and $\partial_{\lambda_j}v_{p_j}=O(e^{-\frac{\lambda_{m+1}}{(1+\alpha_{m+1})}})$. By using Moser-Trudinger's inequality as in the proof in (4.40),

$$\int_{B_{2r_0}(p_j)\cap\{\varphi\ge\epsilon_2\}}e^{\varphi}\le e^{-2\lambda_j}\int_{B_{r_0}(p_j)}\exp(\frac{4\pi|\varphi-\overline{\varphi}|^2}{\|\varphi-\overline{\varphi}\|^2})\mathrm{d}y\le c_2e^{-2\lambda_j},\tag{8.21}$$

which implies

$$\int_{B_{2r_0}(p_j)}e^{|\varphi|}=O(1)\tag{8.22}$$

and

$$\int_{B_{2r_0}(p_j)\setminus B_{r_0}(p_j)}\frac{\rho_1 h_1 e^{2v_1-v_2}}{\int_M h_1e^{2v_1-v_2}}\partial_{\lambda_j}v_{p_j}=O(e^{-\epsilon\lambda_{m+1}-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}}).\tag{8.23}$$

By Lemma 4.2, we have

$$\int_{B_{2r_0}(p_j)\setminus B_{r_0}(p_j)}\Delta(v_{p_j}-\overline{v}_{p_j}-8\pi(1+\alpha_j)G(x,p_j))\cdot\partial_{\lambda_j}v_{p_j}=O(e^{-\frac{2\lambda_{m+1}}{1+\alpha_{m+1}}}).\tag{8.24}$$

By (4.44), (8.15), (8.18)-(8.24), we obtain,

$$\begin{aligned}
\langle\nabla(2v_1+T_1(v_1,v_2)),\nabla\partial_{\lambda_j}v_{p_j}\rangle=&-16\pi(1+\alpha_j)(a_j-1)\lambda_j-8\pi(1+\alpha_j)(\theta_j-\psi(p_j))\\
&+O(\max|a_i-1|+e^{-\epsilon\lambda_{m+1}-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}}).
\end{aligned}$$

This proves part (3).

For the proof of part (4), we write

$$\begin{aligned}
\langle\nabla(2v_1+T_1(v_1,v_2)),\nabla v_{p_j}\rangle&=\langle\nabla(2v_1+T_1(v_1,v_2),\nabla(v_{p_j}-\overline{v}_{p_j})\rangle\\
&=-\langle\Delta(2v_1+T_1(v_1,v_2)),v_{p_j}-\overline{v}_{p_j}\rangle.
\end{aligned}$$

First, we have $\langle 1, v_{p_j} - \overline{v}_{p_j}\rangle = 0$ and $\langle \Delta\phi, v_{p_j} - \overline{v}_{p_j}\rangle = 0$ on M because $\phi \in O^{(1)}_{P,\Lambda}$. To estimate the other terms, we note that

$$\begin{aligned} v_{p_j} - \overline{v}_{p_j} =& 2\lambda_j - 2\log\Big(1 + \frac{\rho_1 h_{p_j}(p_j)}{4(1+\alpha_j)^2} e^{\lambda_j}|x-p_j|^{2(1+\alpha_j)}\Big) \\ &+ 8\pi(1+\alpha_j)R(p_j,p_j) + 2\log\frac{\rho_1 h_{p_j}(p_j)}{4(1+\alpha_j)^2} + O(|x-p_j|) + O(\lambda_j e^{-\frac{\lambda_j}{1+\alpha_j}}). \end{aligned} \tag{8.25}$$

By scaling, we have

$$\begin{aligned} &(a_j - 1)\lambda_j \int_{\mathbb{R}^2} \rho_1 h_{p_j}(p_j)|x-p_j|^{2\alpha_j} e^{U_j} \log\Big(1 + \frac{\rho_1 h_{p_j}(p_j)}{4(1+\alpha_j)^2} e^{\lambda_j}|x-p_j|^{2(1+\alpha_j)}\Big)\mathrm{d}y \\ &\quad = 2\pi(1+\alpha_j)(a_j-1)\lambda_j. \end{aligned} \tag{8.26}$$

Let ϑ represent any constant term in

$$(a_j-1)(U_j+s_j-1), \quad \sum_{i\neq j}(a_i-1)(1+\alpha_i)G(p_j,p_i), \quad \frac{e^{t_j}}{\int_M h_1 e^{2v_1-v_2}} - 1.$$

For simplicity of notations, we set $W_j(x) = 2\rho_1 h_{p_j}(p_j)|x-p_j|^{2\alpha_j}e^{U_j}$. By comparing (8.13) and (8.25), we have

$$\begin{aligned} \int_{B_{r_0}(p_j)} W_j\vartheta(v_{p_j} - \overline{v}_{p_j})\mathrm{d}y =& \Big(2\lambda_j - 1 + 8\pi(1+\alpha_j)R(p_j,p_j) \\ &+ 2\log\frac{\rho_1 h_{p_j}(p_j)}{4(1+\alpha_j)^2}\Big)\int_{B_{r_0}(p_j)} W_j\vartheta\partial_{\lambda_j}v_{p_j} + O(e^{-\frac{\lambda_j}{1+\alpha_j}}), \end{aligned} \tag{8.27}$$

where (8.18) is used. It is also easy to see the integral of $v_{p_j} - \overline{v}_{p_j}$ with all remainder terms except ψ in (4.49) are smaller than $O(e^{-\frac{\lambda_j}{1+\alpha_j}} + \|\phi\|_{H^1})$. For example, by Lemma 8.1,

$$\begin{aligned} &\int_{B_{r_0}(p_j)} 2\rho_1 h_{p_j}(p_j)|x-p_j|^{2\alpha_j}e^{U_j}\phi(v_{p_j} - \overline{v}_{p_j}) \\ &\quad = \int_{B_{r_0}(p_j)} 2\rho_1 h_{p_j}(p_j)|x-p_j|^{2\alpha_j}e^{U_j}\phi \\ &\qquad \times \Big[\lambda_j + s_j - 2\log(1 + \frac{\rho_1 h_{p_j}(p_j)e^{\lambda_j}|x-p_j|^{2(1+\alpha_j)}}{4(1+\alpha_j)^2}) + O(|x-p_j|)\Big]\mathrm{d}y \\ &\quad = O(\lambda_j e^{-\epsilon\lambda_j})\|\phi\|_{H^1} + \Big(\int_{B_{r_0}(p_j)} 2\rho_1 h_{p_j}(p_j)e^{U_j}|x-p_j|^{2\alpha_j}\phi^2\Big)^{\frac{1}{2}} \\ &\qquad \times \Big(\int_{B_{r_0}(p_j)} 2\rho_1 h_{p_j}(p_j)e^{U_j}|x-p_j|^{2\alpha_j}\Big[\log(1 + \frac{\rho_1 h_{p_j}(p_j)e^{\lambda_j}|x-p_j|^{2(1+\alpha_j)}}{4(1+\alpha_j)^2}) \\ &\qquad\quad + O(|x-p_j|)\Big]^2\mathrm{d}y\Big)^{\frac{1}{2}} \\ &\quad = O(1)\|\phi\|_{H^1}. \end{aligned}$$

For ψ, we have

$$\begin{aligned}
&\int_{B_{r_0}(p_j)} 2\rho_1 h_{p_j}(p_j)|x-p_j|^{2\alpha_j} e^{U_j}\psi(v_{p_j}-\overline{v}_{p_j})\\
&=\int_{B_{r_0}(p_j)} 2\rho_1 h_{p_j}(p_j)|x-p_j|^{2\alpha_j} e^{U_j}\psi(p_j)(v_{p_j}-\overline{v}_{p_j})\\
&\quad+\int_{B_{r_0}(p_j)} 2\rho_1 h_{p_j}(p_j)|x-p_j|^{2\alpha_j} e^{U_j}(\psi-\psi(p_j))(v_{p_j}-\overline{v}_{p_j})\\
&=\int_{B_{r_0}(p_j)} 2\rho_1 h_{p_j}(p_j)|x-p_j|^{2\alpha_j} e^{U_j}\psi(p_j)\times\Big[\lambda_j+s_j\\
&\quad-2\log(1+\frac{\rho_1 h_{p_j}(p_j)e^{\lambda_j}|x-p_j|^{2(1+\alpha_j)}}{4(1+\alpha_j)^2})\Big]+O(e^{-\epsilon\lambda_{m+1}-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}})\\
&=8\pi\Big(2\lambda_j-1+8\pi(1+\alpha_j)R(p_j,p_j)+2\log\frac{\rho_1 h_{p_j}(p_j)}{4(1+\alpha_j)^2}\Big)(1+\alpha_j)\psi(p_j)\\
&\quad+O(e^{-\epsilon\lambda_{m+1}-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}}).
\end{aligned}\tag{8.28}$$

Since $v_{p_j}-\overline{v}_{p_j}=O(1)$ on $M\setminus B_{r_0}(p_j)$, by Lemma 4.1,

$$\int_{B_{2r_0}(p_j)\setminus B_{r_0}(p_j)}\Delta(v_{p_j}-8\pi(1+\alpha_j)G(x,p_j))(v_{p_j}-\overline{v}_{p_j})=O(e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}}).\tag{8.29}$$

For the integration on $B_{r_0}(p_i), i\neq j$, the dominant term can be estimated by

$$\begin{aligned}
&\int_{B_{r_0}(p_i)} 2\rho_1 h_{p_i}(p_i)|x-p_i|^{2\alpha_i}e^{U_i}\\
&\quad\times\Big[(a_i-1)(U_i-s_i-1)+\big(\frac{e^{t_i}}{\int_M h_1 e^{2v_1-v_2}}-1-\psi\big)\Big](v_{p_j}-\overline{v}_{p_j})\mathrm{d}y\\
&=(1+\alpha_j)(1+\alpha_i)\Big[128\pi^2 G(p_i,p_j)(a_i-1)\lambda_i\\
&\quad+64\pi^2 G(p_i,p_j)(\frac{e^{t_i}}{\int_M h_1 e^{2v_1-v_2}}-1-\psi(p_i))\Big]\\
&\quad+O(|a_i-1|)+O(e^{-\epsilon\lambda_{m+1}-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}})\\
&=64\pi^2(1+\alpha_i)(1+\alpha_j)G(p_i,p_j)\Big[(\frac{e^{t_i}}{\int_M h_1 e^{2v_1-v_2}}-1-\psi(p_i))+2(a_i-1)\lambda_i\Big]\\
&\quad+O(|a_i-1|+e^{-\epsilon\lambda_{m+1}-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}})\\
&=-8\pi(1+\alpha_j)G(p_i,p_j)\langle\nabla(2v_1+T_1(v_1,v_2)),\nabla\partial_{\lambda_i}v_{p_i}\rangle\\
&\quad+O(|a_i-1|+e^{-\epsilon\lambda_{m+1}-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}}),
\end{aligned}\tag{8.30}$$

where we use the proof part (3) of Lemma 4.4 in the above, it is easy to see that the other terms on $B_{r_0}(p_i)$ are bounded by $e^{-\epsilon\lambda_{m+1}-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}}$.

For the integration outside $\bigcup_j B_{r_0}(p_j)$, we have $(\int_M h_1 e^{2v_1-v_2})^{-1} = O(e^{-\lambda_j})$ and

$$\int_{B_{2r_0}(p_j)\setminus B_{r_0}(p_j)} \frac{\rho_1 h_1 e^{2v_1-v_2}}{\int_M h_1 e^{2v_1-v_2}}(v_{p_j}-\overline{v}_{p_j}) = \int_{B_{2r_0}(p_j)\setminus B_{r_0}(p_j)} O(e^{-\lambda_j})e^{\varphi} = O(e^{-\lambda_j}). \tag{8.31}$$

Similarly

$$\int_{M\setminus\bigcup_j B_{r_0}(p_j)} \frac{\rho_1 h_1 e^{2v_1-v_2}}{\int_M h_1 e^{2v_1-v_2}}(v_{p_j}-\overline{v}_{p_j}) = O(e^{-\lambda_j}). \tag{8.32}$$

Therefore, by (8.26)-(8.32), we have

$$\begin{aligned}
&\langle\nabla(2v_1+T_1(v_1,v_2)),\nabla(v_{p_j}-\overline{v}_{p_j})\rangle\\
=&\Big(2\lambda_j-1+8\pi(1+\alpha_j)R(p_j,p_j)+2\log\frac{\rho_1 h_{p_j}(p_j)}{4(1+\alpha_j)^2}\Big)\langle\nabla(2v_1+T_1(v_1,v_2)),\nabla_{\lambda_j}v_{p_j}\rangle\\
&+8\pi(1+\alpha_j)\sum_{i\neq j}G(p_j,p_i)\langle\nabla(2v_1+T_1(v_1,v_2)),\nabla_{\lambda_i}v_{p_i}\rangle+16\pi(a_j-1)\lambda_j(1+\alpha_j)\\
&+O(1)\|\phi\|_{H^1}+O(e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}}).
\end{aligned}\tag{8.33}$$

The proof of part (4) is complete.

Finally, we prove part (2), we note that

$$\langle\nabla(2v_1+T_1(v_1,v_2)),\nabla\partial_{p_j}v_{p_j}\rangle = \langle\nabla(2v_1+T_1(v_1,v_2)),\nabla\partial_{p_j}(v_{p_j}-\overline{v}_{p_j})\rangle.$$

From $\phi\in O^{(1)}_{P,\Lambda}$, we have

$$\int_M \nabla\phi\nabla\partial_{p_j}(v_{p_j}-\overline{v}_{p_j}) = 0. \tag{8.34}$$

Since $\int_M(v_{p_j}-\overline{v}_{p_j})=0$, we have $\int_M \partial_{p_j}(v_{p_j}-\overline{v}_{p_j})=0$ and

$$\int_M\Big[\rho_*-\rho_1+\sum_{j=1}^n 4\pi(1+\alpha_j)(a_j-1)\Big]\partial_{p_j}(v_{p_j}-\overline{v}_{p_j}) = 0. \tag{8.35}$$

On $B_{r_0}(p_j)$, by Lemma 4.1

$$\partial_{p_j}v_{p_j} = -\nabla_y U_j+\frac{\partial_{p_j}h(p_j)}{h(p_j)}(\partial_{\lambda_j}U_j-1)+2\partial_{p_j}\log h(p_j)+8\pi\partial_{p_j}R(x,p_j)+O(|x-p_j|). \tag{8.36}$$

Since $\nabla_y U_j$ is an odd function, we have

$$\int_{B_{r_0}(p_j)} \rho_1 h_{p_j}(p_j)|x-p_j|^{2\alpha_j}e^{U_j}(U_j+s_j-1)\nabla_y U_j \mathrm{d}y = O(1). \tag{8.37}$$

Hence, by Lemma 4.2 and the fact that $\partial_{\lambda_j}U_j$ is bounded,

$$\int_{B_{r_0}(p_j)} \rho_1 h_{p_j}(p_j)e^{U_j}(U_j+s_j-1)\partial_{p_j}(v_{p_j}-\overline{v}_{p_j}) = O(\lambda_j),$$

where (4.33) was used. For the other terms in (4.49), we have the following estimates.

$$\int_{B_{r_0}(p_j)} \rho_1 h_{p_j}(p_j) e^{U_j} \partial_{p_j}(v_{p_j} - \overline{v}_{p_j}) \mathrm{d}y = O(1), \tag{8.38}$$

$$\int_{B_{r_0}(p_j)} \rho_1 h_{p_j}(p_j) e^{U_j} O(|x-p_j|) \partial_{p_j}(v_{p_j} - \overline{v}_{p_j}) \mathrm{d}y = O(1), \tag{8.39}$$

$$\int_{B_{r_0}(p_j)} 2\rho_1 h_{p_j}(p_j) e^{U_j} \nabla H_j(p_j) \cdot (x-p_j) \nabla_y U_j \mathrm{d}y = (8\pi + O(e^{-\lambda_j})) \nabla H_j(p_j), \tag{8.40}$$

and

$$\begin{aligned} &\int_{B_{r_0}(p_j)} 2\rho_1 h_{p_j}(p_j) e^{U_j} \nabla H_j(p_j) \cdot (x-p_j) \partial_{p_j}(v_{p_j} - \overline{v}_{p_j}) \\ &= 8\pi \nabla H_j(p_j) + O(\lambda_j e^{-\frac{3}{2}\lambda_j}), \end{aligned} \tag{8.41}$$

where we used $\nabla H_j(p_j) = O(\lambda_j e^{-\lambda_j})$ for $v_1 \in S_{\rho_1}(Q, w)$.

By Lemma 4.2 and (8.36),

$$\int_{B_{r_0}(p_j)} \rho_1 h_{p_j}(p_j) e^{U_j} \phi \partial_{p_j}(v_{p_j} - \overline{v}_{p_j}) \mathrm{d}y = O(e^{-\epsilon\lambda_{m+1}}) \|\phi\|_{H^1(M)}. \tag{8.42}$$

While for the term $\frac{e^{t_j}}{\int_M h_1 e^{2v_1 - v_2}} - 1$ and ψ, we have

$$\begin{aligned} &\int_{B_{r_0}(p_j)} 2\rho_1 h_{p_j}(p_j) e^{U_j} \Big(\frac{e^{t_j}}{\int_M h_1 e^{2v_1 - v_2}} - 1 - \psi \Big) \partial_{p_j}(v_{p_j} - \overline{v}_{p_j}) \\ &= \int_{B_{r_0}(p_j)} 2\rho_1 h_{p_j}(p_j) e^{U_j} \Big(\frac{e^{t_j}}{\int_M h_1 e^{2v_1 - v_2}} - 1 - \psi(p_j) \Big) \partial_{p_j}(v_{p_j} - \overline{v}_{p_j}) \\ &\quad - \int_{B_{r_0}(p_j)} 2\rho_1 h_{p_j}(p_j) e^{U_j} (\psi - \psi(p_j)) \partial_{p_j}(v_{p_j} - \overline{v}_{p_j}) \\ &= -8\pi \nabla \psi(p_j) + O(|\frac{e^{t_j}}{\int_M h_1 e^{2v_1 - v_2}} - 1 - \psi(p_j)|) \end{aligned} \tag{8.43}$$

where we used

$$\int_{B_{r_0}(p_j)} 2\rho_1 h_{p_j}(p_j) e^{U_j} \nabla \psi(p_j) \cdot (x-p_j) \nabla_y U_j \mathrm{d}y = (8\pi + O(e^{-\lambda_j})) \nabla \psi(p_j),$$

and (8.38). Since $\partial_{p_j}(v_{p_j} - \overline{v}_{p_j}) = O(e^{\frac{1}{2}\lambda_j})$, as in the proof of part (3), we have

$$\int_{B_{r_0}(p_j)} E \partial_{p_j}(v_{p_j} - \overline{v}_{p_j}) \mathrm{d}y = O(e^{-\epsilon\lambda_{m+1} - \lambda_j}). \tag{8.44}$$

On $M \setminus \bigcup_j B_{r_0}(p_j)$, $\partial_{p_j}(v_{p_j} - \overline{v}_{p_j}) = O(1)$. Hence by Lemma 4.1,

$$\int_{B_{2r_0}(p_j) \setminus B_{r_0}(p_j)} \Delta(v_{p_j} - 8\pi(1+\alpha_j) G(x, p_j)) \cdot \partial_{p_j}(v_{p_j} - \overline{v}_{p_j}) = O(\lambda_j e^{-\lambda_j}).$$

Since $(\int_M h_1 e^{2v_1-v_2})^{-1} = O(e^{-\lambda_j})$, the integral of the products of $\partial_{p_j}(v_{p_j} - \overline{v}_{p_j})$ and the nonlinear term in (4.49)-(4.50) are of order

$$O(e^{-\lambda_j})\int_M e^{\varphi} = O(e^{-\lambda_j}).$$

The estimates above imply

$$\begin{aligned}\langle \nabla(2v_1+T_1(v_1,v_2)), \nabla\partial_{p_j}(v_{p_j}-\overline{v}_{p_j})\rangle =& -8\pi\nabla H_j(p_j) + 8\pi\nabla\psi(p_j)\\ &+ O\big(\Big|\frac{e^{t_j}}{\int_M h_1 e^{2v_1-v_2}} - 1 - \psi(p_j)\Big|\\ &+ |a_j-1|\lambda_j + e^{-\frac{\lambda_{m+1}}{1+\alpha_{m+1}}}\big). \qquad (8.45)\end{aligned}$$

This proves part (2) and hence the proof of Lemma 4.4 is complete. □

Next, we give a proof of (5.14).

Proof of (5.14): For convenience, we denote

$$\mathfrak{E}_2 = \exp(w + 2\psi - \frac{1}{2}v_{P,\Lambda,A}) - \exp\big(w + 2\psi - \sum_{j=1}^{n} 4\pi(1+\alpha_j)a_j G(x,p_j)\big).$$

For $x \in M \setminus \bigcup_{j=1}^{n} B_{r_0}(p_j)$. By Lemma 4.1, we have

$$\Big|\frac{1}{2}v_{P,\Lambda,A} - 4\pi\sum_{j=1}^{n}(1+\alpha_j)a_j G(x,p_j)\Big| \le \tilde{c}e^{-\frac{\lambda(p)}{1+\alpha_{m+1}}}$$

for some $\tilde{c}$ independent of c_0. Thus $|\mathfrak{E}_2| \le c_4 e^{-\frac{\lambda(p)}{1+\alpha_{m+1}}}$ in $M \setminus \bigcup_{j=1}^{n} B_{r_0}(p_j)$.

For $x \in B_{r_0}(p_j), j \in J_1$, we note

$$\begin{aligned}4\pi\sum_{j=1}^{n} a_j G(x,p_j) - \frac{1}{2}v_{P,\Lambda,A} =& 4\pi a_j G(x,p_j) - 4\pi a_j R(x,p_j) - a_j\log(\frac{\rho_1 h_{p_j}(p_j)e^{\lambda_j}}{4})\\ &+ a_j\log\Big(1 + \frac{\rho_1 h_{p_j}(p_j)e^{\lambda_j}}{4}|x-p_j|^2\Big) + O(e^{-\frac{\lambda(P)}{1+\alpha_{m+1}}})\\ =& a_j\log\Big(\frac{4}{\rho_1 h_{p_j}(p_j)e^{\lambda_j}|x-p_j|^2} + 1\Big) + O(e^{-\frac{\lambda(P)}{1+\alpha_{m+1}}}),\end{aligned}$$

where we have used $\overline{v}_{p_j} = O(e^{-\frac{\lambda(P)}{1+\alpha_{m+1}}})$. Then, we have

$$\begin{aligned}&\exp\big(w + 2\psi - 4\pi\sum_{j=1}^{n} a_j(1+\alpha_j)G(x,p_j)\big) - \exp\big(w + 2\psi - \frac{1}{2}v_{P,\Lambda,A}\big)\\ &= |x-p_j|^{2a_j}\Big(1 - \exp\big(4\pi\sum_{j=1}^{n} a_j G(x,p_j) - \frac{1}{2}v_{P,\Lambda,A}\big)\Big)\\ &= |x-p_j|^{2a_j}\Big(1 - \exp\big(a_j\log(1 + \frac{4}{\rho_1 h_{p_j}(p_j)e^{\lambda_j}|x-p_j|^2}) + O(e^{-\frac{\lambda(P)}{1+\alpha_{m+1}}})\big)\Big),\end{aligned} \qquad (8.46)$$

where we used

$$\exp\big(w+2\psi_k-4\pi\sum_{j=1}^{n}(1+\alpha_j)a_jG(x,p_j)\big)\sim|x-p_j|^{2a_j}\ \text{for}\ x\in B_{r_0}(p_j),\ j\in J_1.$$

When $|x-p_j|=O(e^{-\frac{\lambda_j}{2a_j}})$, we have

$$\exp\big(w+2\psi-4\pi\sum_{j=1}^{n}a_jG(x,p_j)\big)=O(|x-p_j|^{2a_j}),$$

and

$$1-\exp\big(a_j\log(1+\frac{4}{\rho_1 h_{p_j}e^{\lambda_j}|x-p_j|^2})+O(e^{-\frac{\lambda(P)}{1+\alpha_{m+1}}})\big)=O(e^{-a_j\lambda_j}|x-p_j|^{-2a_j}),$$

which implies

$$\mathfrak{E}_2=O(e^{-\frac{\lambda(P)}{1+\alpha_{m+1}}})\ \text{for}\ |x-p_j|=O(e^{-\frac{\lambda_j}{2a_j}}),\ j\in J_1.$$

When $|x-q_j|\gg e^{-\frac{\lambda_j}{2a_j}}$, then

$$1-\exp\Big(a_j\log(1+\frac{4}{\rho_1 h_{p_j}(p_j)e^{\lambda_j}|x-p_j|^2})\Big)=O(\frac{1}{e^{a_j\lambda_j}|x-p_j|^{2a_j}}).$$

As a result, we have the right hand side of (8.46) are of order $O(e^{-\frac{\lambda(P)}{1+\alpha_{m+1}}})$. Therefore

$$\mathfrak{E}_2=O(e^{-\frac{\lambda(P)}{1+\alpha_{m+1}}})\ \text{for}\ x\in B_{r_0}(p_j),\ j\in J_1.$$

For $x\in B_{r_0}(p_j), j\in J\setminus J_1$, we have

$$\begin{aligned}
&4\pi\sum_{j=1}^{n}(1+\alpha_j)a_jG(x,p_j)-\frac{1}{2}v_{P,\Lambda,A}\\
=&4\pi(1+\alpha_j)a_jG(x,p_j)-4\pi(1+\alpha_j)a_jR(x,p_j)-\frac{1}{2}a_j\lambda_j-a_j\log\Big(\frac{\rho_1h_{p_j}(p_j)}{4(1+\alpha_j)^2}\Big)\\
&-a_j\Big(\frac{1}{2}v_{p_j}(x)+\frac{1}{2}\eta_j+\frac{d_j}{4(1+\alpha_j)}\lambda_je^{-\frac{\lambda_j}{1+\alpha_j}}\Big)\\
=&a_j\log\Big(\frac{4(1+\alpha_j)^2}{\rho_1h_{p_j}(p_j)e^{\lambda_j}|x-p_j|^{2(1+\alpha_j)}}+1\Big)-\frac{1}{2}a_j\big(\eta_j+\frac{d_j}{2(1+\alpha_j)}\lambda_je^{-\frac{\lambda_j}{1+\alpha_j}}\big)\\
&+O(e^{-\frac{\lambda(P)}{1+\alpha_{m+1}}}).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\exp\big(w+2\psi-4\pi\sum_{j=1}^{n}(1+\alpha_j)a_jG(x,p_j)\big)-\exp\big(w+2\psi-\frac{1}{2}v_{P,\Lambda,A}\big)\\
=&O(1)|x-p_j|^{2a_j(1+\alpha_j)}\Big(1-\exp\big(4\pi\sum_{j=1}^{n}(1+\alpha_j)a_jG(x,p_j)-\frac{1}{2}v_{P,\Lambda,A}\big)\Big)\\
=&O(1)|x-p_j|^{2a_j(1+\alpha_j)}\Big(1-\exp\Big[a_j\log\big(\frac{4(1+\alpha_j)^2}{\rho_1h_{p_j}(p_j)e^{\lambda_j}|x-p_j|^{2(1+\alpha_j)}}+1\big)\\
&+\frac{1}{2}a_j\big(\eta_j+\frac{d_j}{2(1+\alpha_j)}\lambda_je^{-\frac{\lambda_j}{1+\alpha_j}}\big)+O\big(e^{-\frac{\lambda(P)}{1+\alpha_{m+1}}}\big)\Big]\Big),
\end{aligned}$$

where we used

$$\exp\big(w+2\psi_k-4\pi\sum_{j=1}^{n}(1+\alpha_j)a_jG(x,p_j)\big)\sim|x-p_j|^{2a_j(1+\alpha_j)}\text{ for }x\in B_{r_0}(p_j),\ j\in J\backslash J_1.$$

If $|x-p_j|=O(e^{-\frac{\lambda_j}{2a_j(1+\alpha_j)}})$, we have

$$\frac{1}{2}\big(\eta_j+\frac{d_j}{2(1+\alpha_j)}\lambda_je^{-\frac{\lambda_j}{1+\alpha_j}}\big)=O(1),$$

and

$$\begin{aligned}&|x-p_j|^{2a_j(1+\alpha_j)}\exp\Big(a_j\log\big(\frac{4(1+\alpha_j)^2}{\rho_1h_{p_j}(p_j)e^{\lambda_j}|x-p_j|^{2(1+\alpha_j)}}+1\big)+O(1)\Big)\\&=O(e^{-\frac{\lambda(P)}{1+\alpha_{m+1}}}).\end{aligned}$$

If $|x-p_j|\gg e^{-\frac{\lambda(P)}{2a_j(1+\alpha_j)}}$, we have

$$\frac{1}{2}\big(\eta_j+\frac{d_j}{2(1+\alpha_j)}\lambda_je^{-\frac{\lambda_j}{1+\alpha_j}}\big)=O(e^{-\frac{\lambda(P)}{1+\alpha_{m+1}}}|x-p_j|^{-2(1+\alpha_j)}),$$

and

$$\begin{aligned}&\exp\Big(a_j\log\big(\frac{4(1+\alpha_j)^2}{\rho_1h_{p_j}(p_j)e^{\lambda_j}|x-p_j|^{2(1+\alpha_j)}}+1\big)+O(e^{-\frac{\lambda(P)}{1+\alpha_{m+1}}}|x-p_j|^{-2(1+\alpha_j)})\Big)\\&=1+O\Big(\frac{1}{e^{a_j\lambda_j}|x-p_j|^{2a_j(1+\alpha_j)}}+e^{-a_j\frac{\lambda(P)}{1+\alpha_{m+1}}}|x-p_j|^{-2a_j(1+\alpha_j)}\Big).\end{aligned}$$

Then

$$\begin{aligned}\mathfrak{E}_2=&|x-p_j|^{2a_j(1+\alpha_j)}\Big(\frac{1}{e^{a_j\lambda_j}|x-p_j|^{2a_j(1+\alpha_j)}}+e^{-\frac{a_j\lambda(P)}{1+\alpha_{m+1}}}|x-p_j|^{-2a_j(1+\alpha_j)}\Big)\\=&O(e^{-\frac{\lambda(P)}{1+\alpha_{m+1}}}),\end{aligned}$$

where we used (5.8). As a conclusion, we have

$$\mathfrak{E}_2=O(e^{-\frac{\lambda(P)}{1+\alpha_{m+1}}})\text{ provided }x\in B_{r_0}(p_j),\ j\in J\setminus J_1.$$

Therefore, we get (5.14).

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