SYMBIOTIC BRIGHT SOLITARY WAVE SOLUTIONS OF COUPLED NONLINEAR SCHRÖDINGER EQUATIONS

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ABSTRACT. Conventionally, bright solitary wave solutions can be obtained in self-focusing nonlinear Schrödinger equations with attractive self-interaction. However, when self-interaction becomes repulsive, it seems impossible to have bright solitary wave solution. Here we show that there exists symbiotic bright solitary wave solution of coupled nonlinear Schrödinger equations with repulsive self-interaction but strongly attractive interspecies interaction. For such coupled nonlinear Schrödinger equations in two and three dimensional domains, we prove the existence of least energy solutions and study the location and configuration of symbiotic bright solitons. We use Nehari's manifold to construct least energy solutions and derive their asymptotic behaviors by some techniques of singular perturbation problems.

1. INTRODUCTION

In this paper, we study symbiotic bright solitary wave solutions of two-component system of time-dependent nonlinear Schrödinger equations called Gross-Pitaevskii equations given by

$$\begin{cases} i\hbar\partial_t\psi_1 = -\frac{\hbar^2}{2m}\Delta\psi_1 + \tilde{V}_1(x)\psi_1 + U_{11}|\psi_1|^2\psi_1 + U_{12}|\psi_2|^2\psi_1, \\ i\hbar\partial_t\psi_2 = -\frac{\hbar^2}{2m}\Delta\psi_2 + \tilde{V}_2(x)\psi_2 + U_{22}|\psi_2|^2\psi_2 + U_{12}|\psi_1|^2\psi_2, \ x \in \Omega, \ t > 0. \end{cases} \tag{1.1}$$

which models a binary mixture of Bose-Einstein condensates with two different hyperfine states called a double condensate. Here $\Omega \subseteq \mathbb{R}^N (N \le 3)$ is the domain for condensate dwelling, ψ_j's are corresponding condensate wave functions, $\hbar$ is the Planck constant divided by 2π and m is atom mass. The constants $U_{jj} \sim a_{jj}$, $j = 1,2$, and $U_{12} \sim a_{12}$, where a_{jj} is the intraspecies scattering length of the j-th hyperfine state and a_{12} is the interspecies scattering length. Besides, $\tilde{V}_j$ is the trapping potential for the j-th hyperfine state. In physics, the usual trapping potential is given by

$$\tilde{V}_j(x) = \sum_{k=1}^{N} \tilde{a}_{j,k}(x_k - \tilde{z}_{j,k})^2 \quad \text{for } x = (x_1, \cdots, x_N) \in \Omega, j = 1, 2\,,$$

where $\tilde{a}_{j,k} \ge 0$ is the associated axial frequency, and $\tilde{z}_j = (\tilde{z}_{j,1}, \cdots, \tilde{z}_{j,N})$ is the center of the trapping potential $\tilde{V}_j$.

When the constant U_{jj} is negative and large enough, self-interaction of the j-th hyperfine state is strongly attractive and the associated condensate tends to increase its density at the centre of the trap potential in order to lower the interaction energy (cf. [32]).

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This may result in spikes and bright solitons which can be observed experimentally in three dimensional domain (cf. [8]). Conversely, when the constant U_{jj} becomes positive, self-interaction on the j-th hyperfine state turns into repulsion which cannot support the existence of bright solitons. To create bright solitons while each self-repulsive state cannot support a soliton by itself, the interspecies attraction may open a way to make two-component solitons called symbiotic bright solitons. Recently, symbiotic bright solitons in only one dimensional domain have been investigated as the interspecies scattering length a_{12} is negative and sufficiently large (cf. [28]). However, in two and three dimensional domains, the existence of symbiotic bright solitons has not yet been proved. In this paper, we want to show the existence of such solitons by studying the least energy solutions of two-component system of nonlinear Schrödinger equations.

To obtain symbiotic bright solitons in a double condensate, we may set $\psi_1(x,t) = u(x)\, e^{i\,\tilde\lambda_1 t}$, $\psi_2(x,t) = v(x)\, e^{i\,\tilde\lambda_2 t}$ and use Feshbach resonance to let U_{jj}'s, $\tilde\lambda_j$'s and $\tilde a_{j,k}$'s be very large quantities. By rescaling and some simple assumptions, the system (1.1) with very large U_{jj}'s, $\tilde\lambda_j$'s and $\tilde a_{j,k}$'s is equivalent to the following singularly perturbed problem:

$$\begin{cases} \varepsilon^2\Delta u - V_1(x)u + \mu_1 u^3 + \beta u v^2 = 0 \quad \text{in } \Omega, \\ \varepsilon^2\Delta v - V_2(x)v + \mu_2 v^3 + \beta u^2 v = 0 \quad \text{in } \Omega, \\ u, v > 0 \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where u and v are corresponding condensate amplitudes, $\varepsilon > 0$ is a small parameter, and $\beta \sim -a_{12} \neq 0$ is a coupling constant. Here we may use the zero Dirichlet boundary condition which may come from [13]. To study symbiotic bright solitons of double condensates, we consider two cases of the domain Ω. One is to set Ω as the entire space $\mathbf{R}^N (N \le 3)$. The other is to set Ω as a bounded smooth domain in $\mathbf{R}^N$. The constants $\mu_j \sim -U_{jj} \le 0\,, j = 1,2\,,$ give repulsive self-interaction, and $\beta \sim -a_{12} > 0$ means attractive interaction of solutions u and v. Moreover, $V_j > 0\,, j = 1,2$ are the associated trapping potentials.

Another motivation of studying the problem (1.2) may come from the formation of bright solitons in a mixture of a degenerate Fermi gas with a Bose-Einstein condensate in the presence of a sufficiently attractive boson-fermion interaction. Recently, there have been successful observations and associated experimental and theoretical studies of mixtures of a degenerate Fermi gas and a Bose-Einstein condensate (cf. [10], [24] and [25]). Recently, the corresponding model has been given by

$$\begin{cases} i\hbar\partial_t\varphi^B = -\frac{\hbar^2}{2m_B}\Delta\varphi^B + V_B(x)\varphi^B + g_B N_B|\varphi^B|^2\varphi^B + g_{BF}\displaystyle\sum_{j=1}^{N_F}|\varphi_j^F|^2\varphi^B\,, \\ i\hbar\partial_t\varphi_j^F = -\frac{\hbar^2}{2m_F}\Delta\varphi_j^F + V_F(x)\varphi_j^F + g_{BF}\,N_B\,|\varphi^B|^2\varphi_j^F\,,\ x\in\Omega,\ t>0\,, j = 1,\cdots,N_F\,, \end{cases} \tag{1.3}$$

where N_B and N_F are the numbers, m_B and m_F are the mass of bosons and fermions, V_B and V_F are trap potentials, φ^B and φ_j^F's are wave functions of Bose-Einstein condensate and individual fermions, respectively. When the constant g_B is positive i.e. repulsive self-interaction, and the constant g_{BF} is negative and large enough enough i.e. strongly attractive interspecies interaction, bright solitons may appear in such a system. Using the system (1.3) (cf. [17]), a novel scheme to realize bright solitons in one-dimensional atomic quantum gases (i.e. the domain Ω is one dimensional) can be found. Here we want to study bright solitons in two and three-dimensional atomic quantum gases i.e. the domain Ω is of two and three dimensional. As for the problem (1.2), we may set $\varphi^B = u(x)\, e^{i\,\tilde\lambda_1 t}/\sqrt{N_B}$, $\varphi_j^F = v_j(x)\, e^{i\,\tilde\lambda_2 t}$ and suitable scales on $m_B, m_F, V_B, V_F, g_B, g_{BF}$ and $\tilde\lambda_j$'s. Then the system (1.3) can be transformed into

$$\begin{cases} \varepsilon^2\Delta u - V_1(x)u + \mu_1 u^3 + \beta u \sum_{j=1}^{N_F} v_j^2 = 0 \quad \text{in } \Omega, \\ \varepsilon^2\Delta v_j - V_2(x)v_j + \beta u^2 v_j = 0 \quad \text{in } \Omega, \quad j = 1, \cdots, N_F\,, \\ u, v_j > 0 \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega\,, \end{cases} \tag{1.4}$$

which can be generalized as a singular perturbation problem given by

$$\begin{cases} \varepsilon^2\Delta u - V_1(x)u + \mu_1 u^3 + \beta u \sum_{j=1}^{m} v_j^2 = 0 \quad \text{in } \Omega, \\ \varepsilon^2\Delta v_j - V_2(x)v_j + \mu_2 v_j^3 + \beta u^2 v_j = 0 \quad \text{in } \Omega, \quad j = 1, \cdots, m\,, \\ u, v_j > 0 \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega\,, \end{cases} \tag{1.5}$$

where $\mu_j \leq 0, j = 1, 2$ are constants and $m = N_F \in \mathbb{N}$. In particular, the problem (1.5) becomes the problem (1.2) as $m = 1$.

In this paper, we study the asymptotic behavior of so-called least-energy solutions of the problem (1.2) which may give symbiotic bright solitons in two and three dimensional domains. By this, we mean

(1) $(u_\varepsilon, v_\varepsilon)$ is a solution of (1.2),

(2) $E_{\varepsilon,\Omega,V_1,V_2}[u_\varepsilon, v_\varepsilon] \leq E_{\varepsilon,\Omega,V_1,V_2}[u, v]$ for any nontrivial solution (u, v) of (1.2),

where $E_{\varepsilon,\Omega,V_1,V_2}[u, v]$ is the energy functional defined as follows:

$$\begin{aligned} E_{\varepsilon,\Omega,V_1,V_2}[u, v] \;:=\; & \frac{\varepsilon^2}{2}\int_\Omega |\bigtriangledown u|^2 + \frac{V_1}{2}\int_\Omega u^2 - \frac{\mu_1}{4}\int_\Omega u^4 \\ & + \frac{\varepsilon^2}{2}\int_\Omega |\bigtriangledown v|^2 + \frac{V_2}{2}\int_\Omega v^2 - \frac{\mu_2}{4}\int_\Omega v^4 \\ & - \frac{\beta}{2}\int_\Omega u^2 v^2\,, \end{aligned} \tag{1.6}$$

for $u, v \in H_0^1(\Omega)$. Actually, it is easy to generalize our results to the problem (1.5) for $m \in \mathbb{N}$. In the case of $\Omega = \mathbf{R}^N, N = 2, 3$, the least energy solution is also called ground state. In our previous papers [20], [21] and [22], we studied the existence and asymptotics of least energy solutions when μ_1 and μ_2 are positive constants. Hereafter, we study the case that both μ_1 and μ_2 are non-positive constants.

As $\beta \leq \sqrt{\mu_1\mu_2}$, it is obvious that

$$\int_\Omega [\varepsilon^2|\nabla u|^2 + V_1 u^2 + \varepsilon^2|\nabla v|^2 + V_2 v^2] = \int_\Omega [2\beta u^2 v^2 + \mu_1 u^4 + \mu_2 v^4] \leq 0 \tag{1.7}$$

for any (u, v) satisfying the problem (1.2) and hence $u, v \equiv 0$. To get nontrivial solutions of the problem (1.2), the assumption $\beta > \sqrt{\mu_1\mu_2}$ is necessary. So throughout the paper, we assume that

$$\mu_1 \le 0, \quad \mu_2 \le 0, \quad \beta > \sqrt{\mu_1\mu_2}\,. \tag{1.8}$$

To study least energy solutions, we define a Nehari manifold

$$N(\varepsilon, \Omega, V_1, V_2) = \left\{ (u, v) \in H_0^1(\Omega) \times H_0^1(\Omega) \middle| \begin{array}{l} \int_\Omega [\varepsilon^2 |\bigtriangledown u|^2 + V_1 u^2 + \varepsilon^2 |\bigtriangledown v|^2 + V_2 v^2] \\ = \int_\Omega [2\beta u^2 v^2 + \mu_1 u^4 + \mu_2 v^4] \end{array} \right\}. \tag{1.9}$$

Note that here, unlike [20]-[22], the Nehari manifold $N(\varepsilon, \Omega, V_1, V_2)$ has only one constraint. On such a manifold, we consider the minimization problem given by

$$c_{\varepsilon,\Omega,V_1,V_2} := \inf_{\substack{(u,v)\in N(\varepsilon,\Omega,V_1,V_2),\\ u,v\ge 0,\\ u,v\not\equiv 0}} E_{\varepsilon,\Omega,V_1,V_2}[u, v]\,. \tag{1.10}$$

When $\varepsilon = 1$, $V_j \equiv \lambda_j > 0, j = 1, 2$ i.e. constant trapping potentials and the domain $\Omega = \mathbf{R}^N$, the Euler-Lagrange equations of the problem (1.10) are

$$\begin{cases} \Delta u - \lambda_1 u + \mu_1 u^3 + \beta u v^2 = 0 \text{ in } \mathbf{R}^N, \\ \Delta v - \lambda_2 v + \mu_2 v^3 + \beta u^2 v = 0 \text{ in } \mathbf{R}^N, \\ u, v \to 0 \text{ as } |y| \to +\infty. \end{cases} \tag{1.11}$$

For such a problem, we have

Theorem 1.1. *Assume that (1.8) holds. Then $c_{1,\mathbf{R}^N,\lambda_1,\lambda_2}$ is attained and hence the problem (1.11) admits a ground state solution which is radially symmetric and strictly decreasing.*

Now we consider the existence of ground state solutions for nonconstant trapping potentials. Namely, we consider the problem of coupled nonlinear Schrödinger equations given by

$$\begin{cases} \varepsilon^2 \Delta u - V_1(x) u + \mu_1 u^3 + \beta u v^2 = 0 \text{ in } \mathbf{R}^N, \\ \varepsilon^2 \Delta v - V_2(x) v + \mu_2 v^3 + \beta u^2 v = 0 \text{ in } \mathbf{R}^N, \\ u, v \to 0 \text{ as } |y| \to +\infty, \end{cases} \tag{1.12}$$

where V_j's satisfy

$$0 < b_j^0 = \inf_{x\in\mathbf{R}^N} V_j(x) \le \lim_{|x|\to\infty} V_j(x) = b_j^\infty \le +\infty, \quad j = 1, 2\,. \tag{1.13}$$

Then we have the following theorem on the existence of ground state solutions of the problem (1.12).

Theorem 1.2. *If either $b_1^\infty + b_2^\infty = +\infty$ or*

$$c_{\varepsilon,\mathbf{R}^N,V_1,V_2} < c_{\varepsilon,\mathbf{R}^N,b_1^\infty,b_2^\infty} \tag{1.14}$$

Then $c_{\varepsilon,\mathbf{R}^N,V_1,V_2}$ is attained and hence the problem (1.12) admits a ground state solution.

Our next theorem is to show the asymptotic behavior of these ground state solutions as follows:

Theorem 1.3. *Assume (1.8) and*

$$\inf_{x\in\mathbb{R}^n} c_{1,\mathbf{R}^N,V_1(x),V_2(x)} < c_{1,\mathbf{R}^N,b_1^\infty,b_2^\infty} \tag{1.15}$$

hold. Then

(i) $c_{\varepsilon,\mathbf{R}^N,V_1,V_2}$ *is attained and the problem (1.12) admits a ground state solution* $(u_\varepsilon, v_\varepsilon)$.

(ii) *Let* P^ε *and* Q^ε *be the unique local maximum points of* u_ε *and* v_ε *respectively. Let* $u_\varepsilon(P^\varepsilon+\varepsilon y) := U_\varepsilon(y), v_\varepsilon(Q^\varepsilon+\varepsilon y) := V_\varepsilon(y)$. *Then as* $\varepsilon \to 0$, $(U_\varepsilon, V_\varepsilon) \to (U, V)$, *where* (U, V) *satisfies (1.11). Furthermore,*

$$\frac{|P^\varepsilon - Q^\varepsilon|}{\varepsilon} \to 0\,, \quad c_{1,\mathbf{R}^N,V_1(P^\varepsilon),V_2(Q^\varepsilon)} \to \inf_{x\in\mathbf{R}^N} c_{1,\mathbf{R}^N,V_1(x),V_2(x)}\,. \tag{1.16}$$

Remark 1. *In general, the condition (1.15) is difficult to check. However, if* $\inf_{x\in\mathbf{R}^N} V_j(x) < \lim_{|x|\to+\infty} V_j(x), j = 1, 2$, *then (1.15) is satisfied.*

Theorem 1.3 can be extended to general bounded domains. Firstly, we set Ω as a bounded smooth domain and trapping potentials V_j's as constants λ_j's. Namely, we consider the following system

$$\begin{cases} \varepsilon^2\Delta u - \lambda_1 u + \mu_1 u^3 + \beta u v^2 = 0 \quad \text{in } \Omega, \\ \varepsilon^2\Delta v - \lambda_2 v + \mu_2 v^3 + \beta u^2 v = 0 \quad \text{in } \Omega, \\ u, v > 0 \ \text{ in } \Omega, \\ u = v = 0 \ \text{ on } \partial\Omega\,. \end{cases} \tag{1.17}$$

The asymptotic behavior of corresponding least energy solutions can be characterized by

Theorem 1.4. *For any* $\beta > \sqrt{\mu_1\mu_2}$ *and* ε *sufficiently small, the problem (1.17) has a least energy solution* $(u_\varepsilon, v_\varepsilon)$. *Let* P_ε *and* Q_ε *be the local maximum points of* u_ε *and* v_ε, *respectively. Then* $|P_\varepsilon - Q_\varepsilon|/\varepsilon \to 0$,

$$d(P_\varepsilon, \partial\Omega) \to \max_{P\in\Omega} d(P, \partial\Omega),\ d(Q_\varepsilon, \partial\Omega) \to \max_{P\in\Omega} d(P, \partial\Omega)\,, \tag{1.18}$$

and $u_\varepsilon(x), v_\varepsilon(x) \to 0$ *in* $C^1_{loc}(\bar{\Omega}\backslash\{P_\varepsilon, Q_\varepsilon\})$. *Furthermore, as* $\varepsilon \to 0$, $(U_\varepsilon, V_\varepsilon) \to (U_0, V_0)$ *which is a least-energy solution of (1.11), where*

$$U_\varepsilon(y) := u_\varepsilon(P_\varepsilon + \varepsilon y), \quad V_\varepsilon(y) := v_\varepsilon(P_\varepsilon + \varepsilon y)\,.$$

By Theorem 1.4, we may generalize Theorem 1.3 to bounded smooth domains. The main idea may follow the proof of Corollary 2.7 in [22]. Moreover, by the same arguments of Theorems 1.1-1.4, one may get similar results for the problem (1.5).

As $\mu_1, \mu_2 > 0$, the assumption $\beta < \beta_0$ is essential in our previous works (cf. [20]-[22]) for the existence and the asymptotic behaviors of ground state (least energy) solutions, where $0 < \beta_0 < \sqrt{\mu_1\mu_2}$ is a small constant. For larger β's, results of ground and bound state solutions can be found in [1], [3], [33] and [34]. On the other hand, when the sign of μ_j's becomes negative i.e. $\mu_1, \mu_2 \le 0$, the assumption of β's can be changed as $\beta > \sqrt{\mu_1\mu_2}$ which is sufficient to prove the existence and the asymptotic behaviors of ground state solutions (see Theorem 1.1-1.4). These are new results of two and three dimensional bright solitary wave solutions for negative μ_j's.

Conventionally, there has been a vast literature on the study of concentration phenomena for single singularly perturbed nonlinear Schrödinger equations with attractive self-interaction. See [2], [4], [5], [6], [29], [30], [31], [9], [14], [15], [16], [18], [23], [37], [38], [36] and the references therein. In particular, a good survey can be found in [26] and [27]. However, until now, there are only few papers working on systems of coupled nonlinear Schrödinger equations, especially for two and three dimensional Bose-Einstein condensates. This paper seems to be the first in showing rigorously that strong interspecies attraction may produce symbiotic bright solitons in two and three dimensional Bose-Einstein condensates even though self-interactions are repulsive.

The organization of this paper is as follows:
In Section 2, we extend the classical Nehari's manifold approach to a system of semilinear elliptic equations in order to find a least energy solution to the problem (1.2). Hereafter, we need the condition $\beta > \sqrt{\mu_1 \mu_2}$ for strong interspecies attraction. Using approximation argument and energy upper bound, we may show Theorem 1.1, 1.2 and Theorem 1.3 in Section 3 and 4, respectively. In Section 5, we follow the same ideas of [20] to complete the proof of Theorem 1.4.

Throughout this paper, unless otherwise stated, the letter C will always denote various generic constants which are independent of ε, for ε sufficiently small. The constant $\sigma \in (0, \frac{1}{100})$ is a fixed small constant.

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2. Nehari's Manifold Approach : Existence of a Least-Energy Solution to (1.2)

In this section, we use Nehari's manifold approach to obtain a least energy solution to (1.2). Nehari's manifold approach has been used successfully in the study of single equations. Conti et al [7] have used Nehari's manifold to study solutions of competing species systems which are related to an optimal partition problem in N-dimensional domains. In our previous paper [20], we also used Nehari's manifold approach to find least energy solutions and symbiotic bright solitons.

We consider the following minimization problem

$$c_{\varepsilon,\Omega,V_1,V_2} := \inf_{\substack{(u,v)\in N(\varepsilon,\Omega,V_1,V_2),\\ u,v\geq 0,\\ u,v\not\equiv 0}} E_{\varepsilon,\Omega,V_1,V_2}[u,v] \tag{2.1}$$

where $N(\varepsilon,\Omega,V_1,V_2)$ and $E_{\varepsilon,\Omega,V_1,V_2}$ are defined in Section 1. Note that, for $N \leq 3$, by the compactness of Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$, $N(\varepsilon,\Omega,V_1,V_2)$ and $c_{\varepsilon,\Omega,V_1,V_2}$ are well-defined. Now we want to show that

Theorem 2.1. *Let Ω be a smooth and bounded domain in $\mathbf{R}^N, N \leq 3$. Suppose that $\beta > \sqrt{\mu_1\mu_2}$. Then for ε sufficiently small, $c_{\varepsilon,\Omega,V_1,V_2}$ can be attained by some $(u_\varepsilon, v_\varepsilon) \in N(\varepsilon,\Omega,V_1,V_2)$ satisfying*

$$C_1\varepsilon^N \leq \int_\Omega u_\varepsilon^4 \leq C_2\varepsilon^N,\ C_1\varepsilon^N \leq \int_\Omega v_\varepsilon^4 \leq C_2\varepsilon^N, \tag{2.2}$$

where C_1, C_2 are two positive constants independent of ε and Ω.

We first note that if $(u,v) \in N(\varepsilon,\Omega,V_1,V_2)$, then

$$\begin{aligned} E_{\varepsilon,\Omega,V_1,V_2}[u,v] &= \frac{1}{4}\left(\varepsilon^2\int_\Omega |\nabla u|^2 + \int_\Omega V_1u^2 + \varepsilon^2\int_\Omega |\nabla v|^2 + \int_\Omega V_2v^2\right) \\ &= \frac{1}{4}\left[\mu_1\int_\Omega u^4 + 2\beta\int_\Omega u^2v^2 + \mu_2\int_\Omega v^4\right]. \end{aligned} \tag{2.3}$$

Let (u_n,v_n) be a minimizing sequence. Then by Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ for $1 < q < \frac{2N}{N-2}$, we see that $u_n \to u_\varepsilon$, $v_n \to v_\varepsilon$(up to a subsequence) for some functions $u_\varepsilon \geq 0$, $v_\varepsilon \geq 0$ in $L^4(\Omega)$ and hence

$$E_{\varepsilon,\Omega,V_1,V_2}[u_n,v_n] \to \frac{1}{4}\left[\mu_1\int_\Omega u_\varepsilon^4 + 2\beta\int_\Omega u_\varepsilon^2v_\varepsilon^2 + \mu_2\int_\Omega v_\varepsilon^4\right] = c_{\varepsilon,\Omega,V_1,V_2}. \tag{2.4}$$

By (2.4) and the weak lower semicontinuity of the H^1 norm, we have

$$c_{\varepsilon,\Omega,V_1,V_2} \geq \frac{1}{4}\left(\varepsilon^2\int_\Omega |\nabla u_\varepsilon|^2 + \int_\Omega V_1u_\varepsilon^2 + \varepsilon^2\int_\Omega |\nabla v_\varepsilon|^2 + \int_\Omega V_2v_\varepsilon^2\right), \tag{2.5}$$

and

$$\varepsilon^2\int_\Omega |\nabla u_\varepsilon|^2 + \int_\Omega V_1u_\varepsilon^2 + \varepsilon^2\int_\Omega |\nabla v_\varepsilon|^2 + \int_\Omega V_2v_\varepsilon^2 \leq \mu_1\int_\Omega u_\varepsilon^4 + 2\beta\int_\Omega u_\varepsilon^2v_\varepsilon^2 + \mu_2\int_\Omega v_\varepsilon^4. \tag{2.6}$$

Next we consider for $t > 0$,

$$\beta_{(u,v)}(t) = E_{\varepsilon,\Omega,V_1,V_2}[\sqrt{t}u, \sqrt{t}v]. \tag{2.7}$$

Our first claim is

Claim 1. *If $2\beta\int_\Omega u^2v^2 + \mu_1\int_\Omega u^4 + \mu_2\int_\Omega v^4 > 0$, then $\beta_{(u,v)}(t)$ attains a unique maximum point t_0, where*

$$t_0 = \frac{\int_\Omega[\varepsilon^2|\nabla u|^2 + V_1u^2 + \varepsilon^2|\nabla v|^2 + V_2v^2]}{\int_\Omega[2\beta u^2v^2 + \mu_1u^4 + \mu_2v^4]}. \tag{2.8}$$

Furthermore, $(\sqrt{t_0}u, \sqrt{t_0}v) \in N(\varepsilon,\Omega,V_1,V_2)$.

Proof. Since

$$\begin{aligned} \beta_{(u,v)}(t) &= t\left[\frac{\varepsilon^2}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega V_1u^2 + \frac{\varepsilon^2}{2}\int_\Omega |\nabla v|^2 + \frac{1}{2}\int_\Omega V_2v^2\right] \\ &\quad -t^2\left[\frac{\mu_1}{4}\int_\Omega u^4 + \frac{\mu_2}{4}\int_\Omega v^4 + \frac{1}{2}\beta\int_\Omega u^2v^2\right], \end{aligned}$$

then the proof follows by simple calculations. We omit the details here. □

By Claim 1 and proper choice of (u,v), it is easy to check that the Nehari manifold $N(\varepsilon,\Omega,V_1,V_2)$ is nonempty. Our second claim is

Claim 2. *The inequalities of (2.2) hold if $\beta > \sqrt{\mu_1\mu_2}$.*

Proof. We first prove the upper bound of $c_{\varepsilon,\Omega,V_1,V_2}$. Since $\beta > \sqrt{\mu_1\mu_2}$, there exists $\alpha \neq 0$ such that $2\beta\alpha^2 + \mu_1\alpha + \mu_2 > 0$. In fact, we may set $\alpha = -\frac{\mu_2}{\mu_1}$ if $\mu_j < 0, j = 1, 2$. For ε sufficiently small, we choose a test function w such that support$(w) \subset B_\varepsilon(P)$ where $P \in \Omega$. Let $(u, v) = (\alpha w, w)$. Then $\int_\Omega [2\beta u^2v^2 + \mu_1 u^4 + \mu_2 v^4] > 0$. By Claim 1, there exists $t_0 > 0$ independent of ε such that $(\sqrt{t_0}u, \sqrt{t_0}v) \in N(\varepsilon, \Omega, V_1, V_2)$. Hence we obtain

$$c_{\varepsilon,\Omega,V_1,V_2} \leq C\varepsilon^N \,, \tag{2.9}$$

where C is a positive constant independent of ε and Ω. Combining (2.9) with (2.3), we obtain that

$$\int_\Omega [\varepsilon^2|\nabla u_\varepsilon|^2 + V_1 u_\varepsilon^2 + \varepsilon^2|\nabla v_\varepsilon^2| + V_2 v_\varepsilon^2] \leq C_2\varepsilon^N \,. \tag{2.10}$$

For (2.10), we may rescale spatial variables by ε and apply the standard Gagliardo-Nirenberg-Sobolev inequality in $\mathbf{R}^N$ (cf. [11]). Consequently,

$$\int_\Omega u_\varepsilon^4 \leq C_2\varepsilon^N, \quad \int_\Omega v_\varepsilon^4 \leq C_2\varepsilon^N \,, \tag{2.11}$$

where C_2 is a positive constant independent of ε and Ω.

For lower bound estimates, the definition of the manifold $N(\varepsilon, \Omega, V_1, V_2)$ may give

$$\int_\Omega [\varepsilon^2|\nabla u|^2 + V_1 u^2 + \varepsilon^2|\nabla v^2| + V_2 v^2] \leq 2\beta \int_\Omega u^2v^2 \,,$$

for any $(u, v) \in N(\varepsilon, \Omega, V_1, V_2)$. On the other hand, as for (2.11), we may rescale spatial variables by ε and apply the standard Gagliardo-Nirenberg-Sobolev inequality in $\mathbf{R}^N$ (cf. [11]) to derive

$$\int_\Omega [\varepsilon^2|\nabla u|^2 + V_1 u^2 + \varepsilon^2|\nabla v^2| + V_2 v^2] \geq C\varepsilon^{N/2}\left[(\int_\Omega u^4)^{1/2} + (\int_\Omega v^4)^{1/2}\right] \geq C\varepsilon^{N/2}(\int_\Omega u^2v^2)^{1/2}$$

for any $(u, v) \in N(\varepsilon, \Omega, V_1, V_2)$, and hence we obtain that for any $(u, v) \in N(\varepsilon, \Omega, V_1, V_2)$, $(u, v) \not\equiv (0, 0)$,

$$\int_\Omega u^2v^2 \geq C\varepsilon^N \,, \tag{2.12}$$

where C is a positive constant independent of ε and Ω. Due to $\int_\Omega u^2v^2 \leq \left(\int_\Omega u^4\right)^{1/2} \left(\int_\Omega v^4\right)^{1/2}$, (2.11) and (2.12) may yield lower bound estimates $\int_\Omega u_\varepsilon^4 \geq C_1\varepsilon^N$ and $\int_\Omega v_\varepsilon^4 \geq C_1\varepsilon^N$, where C_1 is a positive constant independent of ε and Ω.

□

Finally we claim that

Lemma 2.2. *$(u_\varepsilon, v_\varepsilon)$ is a least-energy solution of (1.2).*

Proof. By Claim 2 and (2.6), we have $2\beta \int_\Omega u_\varepsilon^2 v_\varepsilon^2 + \mu_1 \int_\Omega u_\varepsilon^4 + \mu_2 \int_\Omega v_\varepsilon^4 > 0$. Moreover, by Claim 1, there exists $t_0 > 0$ such that $(\sqrt{t_0}u_\varepsilon, \sqrt{t_0}v_\varepsilon) \in N(\varepsilon, \Omega, V_1, V_2)$ i.e.

$$\varepsilon^2 \int_\Omega |\nabla u_\varepsilon|^2 + \int_\Omega V_1 u_\varepsilon^2 + \varepsilon^2 \int_\Omega |\nabla v_\varepsilon|^2 + \int_\Omega V_2 v_\varepsilon^2 = t_0 \left[\mu_1 \int_\Omega u_\varepsilon^4 + 2\beta \int_\Omega u_\varepsilon^2 v_\varepsilon^2 + \mu_2 \int_\Omega v_\varepsilon^4\right] . \tag{2.13}$$

Consequently, (2.6) and (2.13) may give

$$t_0 \leq 1\,. \tag{2.14}$$

On the other hand,

$$E_{\varepsilon,\Omega,V_1,V_2}[\sqrt{t_0}u_\varepsilon, \sqrt{t_0}v_\varepsilon] \geq c_{\varepsilon,\Omega,V_1,V_2} = \frac{1}{4}\left[\mu_1 \int_\Omega u_\varepsilon^4 + 2\beta \int_\Omega u_\varepsilon^2 v_\varepsilon^2 + \mu_2 \int_\Omega v_\varepsilon^4\right], \tag{2.15}$$

$$E_{\varepsilon,\Omega,V_1,V_2}[\sqrt{t_0}u_\varepsilon, \sqrt{t_0}v_\varepsilon] = t_0^2\frac{1}{4}\left[\mu_1 \int_\Omega u_\varepsilon^4 + 2\beta \int_\Omega u_\varepsilon^2 v_\varepsilon^2 + \mu_2 \int_\Omega v_\varepsilon^4\right]. \tag{2.16}$$

Since $t_0 > 0$, (2.15) and (2.16) imply that $t_0 \geq 1$. Thus by (2.14), we obtain $t_0 = 1$ and $(u_\varepsilon, v_\varepsilon) \in N(\varepsilon, \Omega, V_1, V_2)$. Therefore, $(u_\varepsilon, v_\varepsilon)$ attains the minimum $c_{\varepsilon,\Omega,V_1,V_2}$.

Now we want to claim that $(u_\varepsilon, v_\varepsilon)$ is a nontrivial solution of (1.2). Since $(u_\varepsilon, v_\varepsilon)$ is an energy minimizer on the Nehari manifold $N(\varepsilon, \Omega, V_1, V_2)$, there exists a Lagrange multiplier α such that

$$\bigtriangledown E_{\varepsilon,\Omega,V_1,V_2}[u_\varepsilon, v_\varepsilon] + \alpha \bigtriangledown G[u_\varepsilon, v_\varepsilon] = 0\,, \tag{2.17}$$

where

$$G[u, v] = \int_\Omega [\varepsilon^2|\bigtriangledown u|^2 + V_1 u^2 + \varepsilon^2|\bigtriangledown v|^2 + V_2 v^2] - \int_\Omega [\mu_1 u^4 + 2\beta u^2 v^2 + \mu_2 v^4]\,. \tag{2.18}$$

Acting (2.17) with $(u_\varepsilon, v_\varepsilon)$, and making use of the fact that $(u_\varepsilon, v_\varepsilon) \in N(\varepsilon, \Omega, V_1, V_2)$, we see that

$$\alpha \int_\Omega 2[\varepsilon^2|\bigtriangledown u_\varepsilon|^2 + V_1 u_\varepsilon^2 + \varepsilon^2|\bigtriangledown v_\varepsilon|^2 + V_2 v_\varepsilon^2] - 8\alpha \int_\Omega [\mu_1 u_\varepsilon^4 + 2\beta u_\varepsilon^2 v_\varepsilon^2 + \mu_2 v_\varepsilon^4] = 0\,,$$

and

$$\alpha \int_\Omega [\mu_1 u_\varepsilon^4 + 2\beta u_\varepsilon^2 v_\varepsilon^2 + \mu_2 v_\varepsilon^4] = 0\,.$$

Since $(u_\varepsilon, v_\varepsilon) \not\equiv (0, 0)$ and

$$\int_\Omega [\mu_1 u_\varepsilon^4 + 2\beta u_\varepsilon^2 v_\varepsilon^2 + \mu_2 v_\varepsilon^4] = \int_\Omega [\varepsilon^2|\bigtriangledown u_\varepsilon|^2 + V_1 u_\varepsilon^2 + \varepsilon^2|\bigtriangledown v_\varepsilon|^2 + V_2 v_\varepsilon^2] > 0\,,$$

then $\alpha = 0$. This proves that

$$\bigtriangledown E_{\varepsilon,\Omega,V_1,V_2}[u_\varepsilon, v_\varepsilon] = 0$$

and hence $(u_\varepsilon, v_\varepsilon)$ is a critical point of $E_{\varepsilon,\Omega,V_1,V_2}[u, v]$ and satisfies (1.2). By Hopf boundary Lemma, it is easy to show that $u_\varepsilon > 0$ and $v_\varepsilon > 0$. Therefore, we may complete the proof of this Lemma and Theorem 2.1.

□

Another useful characterization of $c_{\varepsilon,\Omega,V_1,V_2}$ is given as follows:

Lemma 2.3. *If $\beta > \sqrt{\mu_1\mu_2}$, then we have*

$$c_{\varepsilon,\Omega,V_1,V_2} = \inf_{\substack{u,v\in H_0^1(\Omega),\ u\not\equiv 0, v\not\equiv 0,\\ \int_\Omega[2\beta u^2v^2+\mu_1u^4+\mu_2v^4]>0}} \sup_{t>0} E_{\varepsilon,\Omega,V_1,V_2}[\sqrt{t}u,\sqrt{t}v] \tag{2.19}$$

$$= \inf_{\substack{u,v\in H_0^1(\Omega),\ u\not\equiv 0, v\not\equiv 0,\\ \int_\Omega[2\beta u^2v^2+\mu_1u^4+\mu_2v^4]>0}} \frac{\int_\Omega[|\nabla u|^2+V_1u^2+|\nabla v|^2+V_2v^2]}{(\int_\Omega[2\beta u^2v^2+\mu_1u^4+\mu_2v^4])^{\frac{1}{2}}}.$$

Proof. The last identity in (2.19) follows from simple calculations. To prove (2.19), we denote the right hand side of (2.19) by m_ε. From Theorem 2.1, $c_{\varepsilon,\Omega,V_1,V_2}$ is attained at $(u_\varepsilon, v_\varepsilon) \in N(\varepsilon,\Omega,V_1,V_2)$. Moreover, by Claim 1 in Theorem 2.1, $E_{\varepsilon,\Omega,V_1,V_2}[\sqrt{t}u_\varepsilon,\sqrt{t}v_\varepsilon]$ attains its maximum at $t=1$. Hence

$$m_\varepsilon \le c_{\varepsilon,\Omega,V_1,V_2} = E_{\varepsilon,\Omega,V_1,V_2}[u_\varepsilon,v_\varepsilon] = \sup_{t>0} E_{\varepsilon,\Omega,V_1,V_2}[\sqrt{t}u_\varepsilon,\sqrt{t}v_\varepsilon]. \tag{2.20}$$

On the other hand, fix $u,v\in H_0^1(\Omega)$ such that $u,v\ge 0$ and $\int_\Omega[2\beta u^2v^2+\mu_1u^4+\mu_2v^4]>0$. Let t_0 be a critical point of $\beta_{(u,v)}(t)$. Then $(\sqrt{t_0}u,\sqrt{t_0}v)\in N(\varepsilon,\Omega,V_1,V_2)$,

$$c_{\varepsilon,\Omega,V_1,V_2} \le E_{\varepsilon,\Omega,V_1,V_2}(\sqrt{t_0}u,\sqrt{t_0}v) \le \sup_{t>0} E_{\varepsilon,\Omega,V_1,V_2}[\sqrt{t}u,\sqrt{t}v]$$

and hence $c_{\varepsilon,\Omega,V_1,V_2} \le m_\varepsilon$. Therefore, we may complete the proof of this Lemma. □

3. Proofs of Theorem 1.1 and Theorem 1.2

In this section, we prove Theorem 1.1 and Theorem 1.2 by approximation argument. Fix a ball $\Omega = B_k$, where k is a large parameter tending to infinity. By Theorem 2.1, each $c_{\varepsilon,B_k,V_1,V_2}$ is attained by (u_k,v_k) a least energy solution of the following problem:

$$\begin{cases} \varepsilon^2\triangle u(x) - V_1(x)u(x) + \mu_1u^3 + \beta uv^2 = 0 \text{ in } B_k, \\ \varepsilon^2\triangle v(x) - V_2(x)v(x) + \mu_2v^3 + \beta u^2v = 0 \text{ in } B_k, \\ u,v>0 \ \text{ in } B_k,\ u=v=0 \text{ on } \partial B_k. \end{cases} \tag{3.1}$$

By examining the argument in the proof of Theorem 2.1, we may obtain the following estimates:

$$C_1\varepsilon^N \le \int_{B_k} u_k^4 \le C_2\varepsilon^N, \quad C_1\varepsilon^N \le \int_{B_k} v_k^4 \le C_2\varepsilon^N\,, \tag{3.2}$$

where C_1 and C_2 are positive constants independent of $0<\varepsilon\le 1$ and $k\ge 1$. By the system (3.1) and (3.2), we may derive that

$$\int_{B_k}[\varepsilon^2|\nabla u_k|^2 + V_1u_k^2 + \varepsilon^2|\nabla v_k|^2 + V_2v_k^2] \le C_3\varepsilon^N\,, \tag{3.3}$$

where C_3 is a positive constant independent of $0<\varepsilon\le 1$ and $k\ge 1$. We may extend each u_k and v_k equal to 0 outside B_k, respectively. Then (3.3) may give

$$||u_k||_{H^1(\mathbf{R}^N)} + ||v_k||_{H^1(\mathbf{R}^N)} \le C_4\varepsilon^{N/2}\,, \tag{3.4}$$

where C_4 is a positive constant independent of $0<\varepsilon\le 1$ and $k\ge 1$.

Now we study the asymptotic behavior of u_k, v_k as $k \to \infty$. Due to (3.4), we obtain that as $k \to \infty$, $u_k \rightharpoonup \bar{u}, v_k \rightharpoonup \bar{v}$, where $\bar{u}, \bar{v} \geq 0$ and $\bar{u}, \bar{v} \in H^1(\mathbf{R}^N)$. Moreover, the standard elliptic regularity theorem may give that $(\bar{u}, \bar{v})$ is a solution of the system

$$\begin{cases} \varepsilon^2 \Delta \bar{u} - V_1 \bar{u} + \mu_1 \bar{u}^3 + \beta \bar{v}^2 \bar{u} &= 0 \quad \text{in } \mathbf{R}^N\,, \\ \varepsilon^2 \Delta \bar{v} - V_2 \bar{v} + \mu_2 \bar{v}^3 + \beta \bar{u}^2 \bar{v} &= 0 \quad \text{in } \mathbf{R}^N\,. \end{cases} \tag{3.5}$$

Then we have the following lemma, whose proof is exactly same as those of Theorem 3.3 in [22].

Lemma 3.1.

(a) *As $k \to \infty$, $c_{\varepsilon, B_k, V_1, V_2} \to c_{\varepsilon, \mathbf{R}^N, V_1, V_2}$,*

(b) *If $\bar{u} \not\equiv 0, \bar{v} \not\equiv 0$, then $(\bar{u}, \bar{v})$ is a solution of (1.12) and attains $c_{\varepsilon, \mathbf{R}^N, V_1, V_2}$, i.e. $(\bar{u}, \bar{v})$ is a ground state solution of (1.12).*

It remains to show that $\bar{u} \not\equiv 0, \bar{v} \not\equiv 0$. Note that if $\bar{u} \equiv 0$, then $\bar{v}$ satisfies

$$\varepsilon^2 \Delta \bar{v} - V_2 \bar{v} + \mu_2 \bar{v}^3 = 0\,. \tag{3.6}$$

Due to $\mu_2 \leq 0$, it is obvious that $\bar{v} \equiv 0$. Therefore, we only need to exclude the case that $\bar{u} \equiv \bar{v} \equiv 0$.

Suppose $V(x) \equiv \lambda_1$ and $V_2(x) \equiv \lambda_2$. Then by the Maximum Principle and Moving Plane Method, both u_k and v_k are radially symmetric, strictly decreasing and satisfy

$$\begin{cases} \varepsilon^2 \Delta u_k - \lambda_1 u_k + \mu_1 u_k^3 + \beta u_k v_k^2 = 0 \quad \text{in } \ B_k, \\ \varepsilon^2 \Delta v_k - \lambda_2 v_k + \mu_2 v_k^3 + \beta u_k^2 v_k = 0 \quad \text{in } \ B_k\,, \\ u_k = u_k(r), v_k = v_k(r) > 0 \ \text{ in } \ B_k, \\ u = v = 0 \ \text{ on } \ \partial B_k. \end{cases} \tag{3.7}$$

Here we have used the fact that $\lambda_j > 0$, $\mu_j \leq 0\,, j = 1, 2$ and $\beta > 0$. Moreover, since the origin 0 is the maximum point of u_k and v_k, then $\Delta u_k(0)\,, \Delta v_k(0) \leq 0$ and $u_k(0)\,, v_k(0) > 0$. Hence by (3.7), we have

$$\beta (v_k(0))^2 \geq -\mu_1 (u_k(0))^2 + \lambda_1, \quad \beta (u_k(0))^2 \geq -\mu_2 (v_k(0))^2 + \lambda_2\,.$$

Consequently, as $k \to +\infty$,

$$\begin{aligned} \beta (v_0(0))^2 &\geq -\mu_1 (u_0(0))^2 + \lambda_1 \geq \lambda_1\,, \\ \beta (u_0(0))^2 &\geq -\mu_2 (v_0(0))^2 + \lambda_2 \geq \lambda_2\,. \end{aligned} \tag{3.8}$$

Here we have used the fact that $\mu_j \leq 0$ and $(u_k, v_k) \to (u_0, v_0)$ in $C^2_{loc}(\mathbf{R}^N)$. Therefore, (3.8) may imply that $u_0 \not\equiv 0, v_0 \not\equiv 0$ and $(u_0, v_0) \in N(1, \mathbf{R}^N, \lambda_1, \lambda_2)$ is a minimizer of $c_{1, \mathbf{R}^N, \lambda_1, \lambda_2}$.

On the other hand, any minimizer of $c_{1, \mathbf{R}^N, \lambda_1, \lambda_2}$, called (U_0, V_0), must satisfy

$$\begin{cases} \Delta U_0 - \lambda_1 U_0 + \mu_1 U_0^3 + \beta U_0 V_0^2 = \ 0 \ \text{ in } \ \mathbf{R}^N, \\ \Delta V_0 - \lambda_2 V_0 + \mu_2 V_0^3 + \beta U_0^2 V_0 = \ 0 \ \text{ in } \ \mathbf{R}^N, \\ U_0, V_0 > 0, U_0, V_0 \in H^1(\mathbf{R}^N). \end{cases} \tag{3.9}$$

Due to $\beta > 0$, the problem (3.9) is of cooperative systems. By the moving plane method (cf. [35]), (U_0, V_0) must be radially symmetric and strictly decreasing. This may complete the proof of Theorem 1.1.

To finish the proof of Theorem 1.2, we divide the proof into two cases as follows:

Case 1: either $b_1^\infty = \infty$ or $b_2^\infty = \infty$.

Proof. In this case, we note that

$$\begin{aligned} c_{\varepsilon,B_k,V_1,V_2} &= \frac{1}{4}\int_{B_k}\left[\mu_1 u_k^4 + 2\beta u_k^2 v_k^2 + \mu_2 v_k^4\right] \\ &\le C_3\varepsilon^N\,, \end{aligned}$$

and

$$\begin{aligned} c_{\varepsilon,B_k,V_1,V_2} &= \frac{1}{4}\int_{B_k}\left[\varepsilon^2|\nabla u_k|^2 + V_1 u_k^2 + \varepsilon^2|\nabla v_k|^2 + V_2 v_k^2\right] \\ &\ge C_4\varepsilon^{N/2}\left(\sqrt{\int_{B_k} u_k^4} + \sqrt{\int_{B_k} v_k^4}\right). \end{aligned}$$

Consequently,

$$C_5\varepsilon^N \le c_{\varepsilon,B_k,V_1,V_2} \le C_6\varepsilon^N, \tag{3.10}$$

where C_5, C_6 are independent of $\varepsilon \le 1,\ k \ge 1$. This gives

$$\int_{B_k}[\varepsilon^2|\nabla u_k|^2 + V_1 u_k^2 + \varepsilon^2|\nabla v_k|^2 + V_2 v_k^2] \le C_7\varepsilon^N.$$

By Sobolev's embedding (since $N \le 3$),

$$\int_{B_k} u_k^6 \le C_8\varepsilon^N, \quad \int_{B_k\cap\{|x|\ge R\}} u_k^2 \le C_9\varepsilon^N \cdot \frac{1}{\min\limits_{|x|\ge R} V_1(x)}\,. \tag{3.11}$$

Hence

$$\begin{aligned} \int_{B_k\cap\{|x|\ge R\}} u_k^4 &\le \left(\int_{B_k\cap\{|x|\ge R\}} u_k^2\right)^{1/2}\left(\int_{B_k\cap\{|x|\ge R\}} u_k^6\right)^{1/2} \\ &\le C_{10}\varepsilon^N \cdot \left(\frac{1}{\min\limits_{|x|\ge R} V_1(x)}\right)^{1/2}. \end{aligned} \tag{3.12}$$

By (3.2) and (3.12), we have

$$\int_{B_k\cap\{|x|\le R\}} u_k^4 \ge \left(C_1 - \frac{C_{10}}{\sqrt{\min\limits_{|x|\ge R} V_1(x)}}\right)\varepsilon^N. \tag{3.13}$$

Thus if $u_k \rightharpoonup \overline{u}$, then $\overline{u} \ge 0$ and

$$\int_{B_R} \overline{u}^4 \ge \left(C_1 - \frac{C_{10}}{\sqrt{\min\limits_{|x|\ge R} V_1(x)}}\right)\varepsilon^N. \tag{3.14}$$

Due to $b_1^\infty = +\infty$, we may choose R large enough such that $C_1 - \frac{C_{10}}{\sqrt{\min_{|x|\ge R} V_1(x)}} \ge \frac{1}{2}C_1$. Consequently, $\int_{B_R}\overline{u}^4 \ge \frac{1}{2}C_1\varepsilon^N$ and hence $\overline{u} \not\equiv 0$.

□

Case 2: $b_j^\infty < +\infty,\ j = 1, 2$

Proof. Suppose $\overline{u} \equiv \overline{v} \equiv 0$. Then

$$u_k, v_k \to 0 \text{ in } \mathbb{C}^2_{loc}(\mathbf{R}^N). \tag{3.15}$$

Let M and R be such that

$$|V_j(x) - b_j^\infty| < \frac{1}{M} \quad \text{for } \ |x| \geq R\,. \tag{3.16}$$

Let $\chi_R(x)$ be a smooth cut-off function such that $\chi_R(x) = 1$ for $|x| \leq R$, $\chi_R(x) = 0$ for $|x| \geq 2R$. Now we set

$$\widetilde{u}_k = u_k(1 - \chi_R)\,, \quad \widetilde{v}_k = v_k(1 - \chi_R)\,. \tag{3.17}$$

Then we have

$$\int_{\mathbf{R}^N} |\nabla \widetilde{u}_k|^2 = \int_{\mathbf{R}^N} |\nabla u_k|^2 - 2\int_{\mathbf{R}^N} \nabla u_k \cdot \nabla(u_k\chi_R) + \int_{\mathbf{R}^N} |\nabla(u_k\chi_R)|^2,$$

and

$$\lim_{k\to+\infty} \left(\left| \int_{\mathbf{R}^N} \nabla u_k \cdot \nabla(u_k\chi_R) \right| + \int_{\mathbf{R}^N} |\nabla u_k\chi_R|^2 \right) = 0\,.$$

Now we denote $o(1)$ as the terms that approach zero as $k \to \infty$. Thus we can write

$$\int_{\mathbf{R}^N} |\nabla \widetilde{u}_k|^2 = \int_{\mathbf{R}^N} |\nabla u_k|^2 + o(1). \tag{3.18}$$

Similarly,

$$\int_{\mathbf{R}^N} |\nabla \widetilde{v}_k|^2 = \int_{\mathbf{R}^N} |\nabla v_k|^2 + o(1), \int_{\mathbf{R}^N} V_1\widetilde{u}_k^p = \int_{\mathbf{R}^N} V_1 u_k^p + o(1), \int_{\mathbf{R}^N} V_2\widetilde{v}_k^p = \int_{\mathbf{R}^N} V_2 v_k^p + o(1)$$

for all $2 \leq p \leq 6$. Hence $E_{\varepsilon,B_k,V_1,V_2}[u_k, v_k] = c_{\varepsilon,B_k,V_1,V_2} = E_{\varepsilon,B_k,V_1,V_2}[\widetilde{u}_k, \widetilde{v}_k] + o(1)$. Moreover,

$$\begin{aligned}
&\int_{\mathbf{R}^N} [\varepsilon^2|\nabla\widetilde{u}_k|^2 + b_1^\infty\widetilde{u}_k^2 + \varepsilon^2|\nabla\widetilde{v}_k|^2 + b_2^\infty\widetilde{v}_k^2] \qquad (3.19)\\
&- \int_{\mathbf{R}^N} [\mu_1\widetilde{u}_k^4 + 2\beta\widetilde{u}_k^2\widetilde{u}_k^2 + \mu_1\widetilde{v}_k^4]\\
&= \int_{\mathbf{R}^N} (b_1^\infty - V_1(x))\widetilde{u}_k^2 + \int_{\mathbf{R}^N} (b_2^\infty - V_2(x))\widetilde{v}_k^2 + o(1)\\
&= O\Big(\frac{1}{M}\int_{\mathbf{R}^N} (\widetilde{u}_k^2 + \widetilde{v}_k^2)\Big) + o(1)\\
&= O\Big(\frac{1}{M}\Big) + o(1),\ j = 1, 2\,.
\end{aligned}$$

Similarly, we have

$$\int_{\mathbf{R}^N} [2\beta\widetilde{u}_k^2\widetilde{v}_k^2 + \mu_1\widetilde{u}_k^4 + \mu_2\widetilde{v}_k^4] = \int_{\mathbf{R}^N} [2\beta u_k^2 v_k^2 + \mu_1 u_k^4 + \mu_2 v_k^4] + o(1)C\varepsilon^N. \tag{3.20}$$

Hence by (3.19), (3.20) and (2.8) of Claim 1 in Theorem 2.1, we see that the unique critical point $\widetilde{t}$ of the function $E_{\varepsilon,\mathbf{R}^N,b_1^\infty,b_2^\infty}[\sqrt{t}\widetilde{u}_k,\sqrt{t}\widetilde{v}_k]$ satisfies

$$|\widetilde{t}-1| = O\Big(\frac{1}{M}\Big) + o(1)\,, \tag{3.21}$$

which yields

$$\begin{aligned} E_{\varepsilon,\mathbf{R}^N,b_1^\infty,b_2^\infty}\left[\sqrt{\widetilde{t}}\widetilde{u}_k,\sqrt{\widetilde{t}}\widetilde{v}_k\right] =& E_{\varepsilon,\mathbf{R}^N,b_1^\infty,b_2^\infty}[\widetilde{u}_k,\widetilde{v}_k] + O\Big(\frac{1}{M}\Big) + o(1) \\ =& E_{\varepsilon,\mathbf{R}^N,V_1,V_2}[\widetilde{u}_k,\widetilde{v}_k] + O\Big(\frac{1}{M}\Big) + o(1) \\ =& E_{\varepsilon,\mathbf{R}^N,V_1,V_2}[u_k,v_k] + O\Big(\frac{1}{M}\Big) + o(1) \\ =& c_{\varepsilon,B_k,V_1,V_2} + O\Big(\frac{1}{M}\Big) + o(1). \end{aligned}$$

On the other hand,

$$\left(\sqrt{\widetilde{t}}\widetilde{u}_k,\sqrt{\widetilde{t}}\widetilde{v}_k\right) \in N(\varepsilon,\mathbf{R}^N,b_1^\infty,b_2^\infty) \tag{3.22}$$

and then

$$E_{\varepsilon,\mathbf{R}^n,b_1^\infty,b_2^\infty}\left[\sqrt{\widetilde{t}}\widetilde{u}_k,\sqrt{\widetilde{t}}\widetilde{v}_k\right] \geq c_{\varepsilon,\mathbf{R}^N,b_1^\infty,b_2^\infty} \tag{3.23}$$

Consequently, $c_{\varepsilon,\mathbf{R}^N,b_1^\infty,b_2^\infty} \leq c_{\varepsilon,B_k,V_1,V_2} + O\big(\frac{1}{M}\big) + o(1)$. Letting $M\to+\infty$ and $k\to+\infty$, we obtain $c_{\varepsilon,\mathbf{R}^N,b_1^\infty,b_2^\infty} \leq c_{\varepsilon,\mathbf{R}^N,V_1,V_2}$ which may contradict with (1.14). Therefore, we may complete the proof of Theorem 1.2. □

4. Proof of Theorem 1.3

In this section, we study the asymptotic behavior of $(u_\varepsilon,v_\varepsilon)$ as $\varepsilon\to0$. Firstly, the energy upper bound is stated as follows:

Lemma 4.1. *For $\beta>0$ and $0<\varepsilon<<1$,*

$$c_{\varepsilon,\mathbf{R}^N,V_1,V_2} \leq \varepsilon^N[\inf_{x\in\mathbf{R}^N} c_{1,\mathbf{R}^N,V_1(x),V_2(x)} + o(1)]. \tag{4.1}$$

PROOF. Fix a point $x_0\in\mathbf{R}^N$. Let (U_0,V_0) be a minimizer of $c_{1,\mathbf{R}^N,V_1(x_0),V_2(x_0)}$. We set $u(x)=U_0(\frac{x-x_0}{\varepsilon}), v(x)=V_0(\frac{x-x_0}{\varepsilon})$ and then use (2.19) to compute the upper bound of $c_{\varepsilon,\mathbf{R}^N,V_1,V_2}$. Due to $c_{\varepsilon,\mathbf{R}^N,\lambda_1,\lambda_2}=\varepsilon^N c_{1,\mathbf{R}^N,\lambda_1,\lambda_2}$, the rest of the proof is simple and thus omitted. □

Let $u_\varepsilon(P^\varepsilon)=\sup\limits_{x\in\mathbf{R}^N}u_\varepsilon(x)$ and $v_\varepsilon(Q^\varepsilon)=\sup\limits_{x\in\mathbf{R}^N}v_\varepsilon(x)$. We want to claim that $\sup\limits_{\varepsilon>0}(|P^\varepsilon|+|Q^\varepsilon|)<+\infty$. To this end, we need to show that both u_ε and v_ε are uniformly bounded. In fact, as for the proof of (3.11), we have

$$\int_{\mathbf{R}^N}(u_\varepsilon^q+v_\varepsilon^q)\leq c\varepsilon^N,\ 2\leq q\leq6. \tag{4.2}$$

The equation of u_ε gives

$$\begin{aligned}\varepsilon^2\triangle u_\varepsilon =& V_1 u_\varepsilon - \mu_1 u_\varepsilon^3 - \beta u_\varepsilon v_\varepsilon^2\\ \geq& -\beta v_\varepsilon^2 u_\varepsilon\\ =& -C(x)u_\varepsilon \quad \text{in } \mathbb{R}^N.\end{aligned}$$

Let $\widetilde{U}_\varepsilon(y) = u_\varepsilon(\varepsilon\, y)$, and $C_\varepsilon(y) = C(\varepsilon\, y)$. Then

$$\triangle\widetilde{U}_\varepsilon + C_\varepsilon(y)\widetilde{U}_\varepsilon \geq 0 \quad \text{in } \mathbb{R}^N, \quad \text{and } C_\varepsilon \in L^3(\mathbf{R}^N). \tag{4.3}$$

By the subsolution estimate (Theorem 8.17 of [12])

$$|\widetilde{U}_\varepsilon(y)| \leq C\left(\int_{B(y,1)} |\widetilde{U}_\varepsilon|^2\right)^{1/2}, \tag{4.4}$$

where $C > 0$ is independent of ε. Hence by (4.2) and (4.4), we see that $||\widetilde{U}_\varepsilon||_{L^\infty} \leq C$ and hence $0 < u_\varepsilon \leq C$. Similarly, we may obtain $0 < v_\varepsilon \leq C$.

Claim 3: *If $|P^\varepsilon| \to +\infty$, then $b_1^\infty < +\infty$.* Suppose $b_1^\infty = +\infty$. Since P^ε is a local maximum point of u_ε, then $\triangle u_\varepsilon(P^\varepsilon) \leq 0$. Hence by the equation of u_ε, we may obtain

$$V_1(P^\varepsilon)u_\varepsilon(P^\varepsilon) - \mu_1 u_\varepsilon^3(P^\varepsilon) - \beta u_\varepsilon(P^\varepsilon)v_\varepsilon^2(P^\varepsilon) = \varepsilon^2\triangle u_\varepsilon(P^\varepsilon) \leq 0,$$

which implies that

$$V_1(P^\varepsilon) \leq \beta v_\varepsilon^2(P^\varepsilon) \leq C, \tag{4.5}$$

and hence

$$|P^\varepsilon| \leq C_0. \tag{4.6}$$

Therefore, we may complete the proof of Claim 3. Moreover, we may also claim that $b_2^\infty < +\infty$. In fact, suppose $b_2^\infty = +\infty$. Set $U_\varepsilon(y) := u_\varepsilon(P^\varepsilon + \varepsilon y), V_\varepsilon(y) := v_\varepsilon(P^\varepsilon + \varepsilon y)$. Then $U_\varepsilon \to U_0$ in $C^2_{loc}(\mathbf{R}^N)$ and $V_\varepsilon \to V_0$ in $C^2_{loc}(\mathbf{R}^N)$, where (U_0, V_0) satisfies

$$\Delta U_0 - b_1^\infty U_0 + \mu_1 U_0^3 + \beta U_0 V_0^2 = 0 \text{ in } \mathbf{R}^N. \tag{4.7}$$

Hence by (4.5), we may obtain $V_0(0) > 0$, and then $V_0 \not\equiv 0$. This implies that

$$\begin{aligned} c_{\varepsilon,\mathbf{R}^N,V_1,V_2} &= \frac{1}{4}\int_{\mathbf{R}^N} [\varepsilon^2|\nabla u_\varepsilon|^2 + V_1 u_\varepsilon^2 + \varepsilon^2|\nabla v_\varepsilon|^2 + V_2 v_\varepsilon^2]\\ &\geq \frac{1}{4}\int_{|x|>R} [\varepsilon^2|\nabla u_\varepsilon|^2 + V_1 u_\varepsilon^2 + \varepsilon^2|\nabla v_\varepsilon|^2 + V_2 v_\varepsilon^2]\\ &\geq \frac{1}{4}\int_{|x|>R} V_2 v_\varepsilon^2\\ &\geq C\varepsilon^N\left[\inf_{|x|>R} V_2(x)\right]\end{aligned}$$

which contradicts with (4.1). Here we have used the hypothesis that $b_2^\infty = +\infty$. Thus we may assume that $b_1^\infty < +\infty$ and $b_2^\infty < \infty$. As before, $(U_\varepsilon, V_\varepsilon)$ converges to (U_0, V_0) satisfying

$$\Delta U_0 - b_1^\infty U_0 + \mu_1 U_0^3 + \beta U_0 V_0^2 = 0,\ \Delta V_0 - b_2^\infty V_0 + \mu_1 V_0^3 + \beta V_0 U_0^2 = 0 \text{ in } \mathbf{R}^N. \tag{4.8}$$

Then again $V_0 \not\equiv 0$ since otherwise, $(U_0, V_0) \equiv (0,0)$ which is impossible. Moreover,

$$\begin{aligned} c_{\varepsilon,\mathbf{R}^N,V_1,V_2} &= \frac{1}{4}\int_{\mathbf{R}^N} [\varepsilon^2|\nabla u_\varepsilon|^2 + V_1 u_\varepsilon^2 + \varepsilon^2|\nabla v_\varepsilon|^2 + V_2 v_\varepsilon^2] \\ &\geq \frac{1}{4}\int_{|x|>R} [\varepsilon^2|\nabla u_\varepsilon|^2 + V_1 u_\varepsilon^2 + \varepsilon^2|\nabla v_\varepsilon|^2 + V_2 v_\varepsilon^2] \\ &\geq \varepsilon^N \frac{1}{4}\int_{\mathbf{R}^N} [|\nabla U_0|^2 + b_1^\infty U_0^2 + |\nabla V_0|^2 + b_2^\infty V_0^2] + o(\varepsilon^N) \\ &\geq \varepsilon^N [c_{1,\mathbf{R}^N,b_1^\infty,b_2^\infty} + o(1)] \end{aligned}$$

which may contradict with (4.1). Therefore, we complete the proof of $\sup_{\varepsilon>0} |P^\varepsilon| + |Q^\varepsilon| < +\infty$.

Let $(P^\varepsilon, Q^\varepsilon) \to (P^0, Q^0)$. As before, $(U_\varepsilon, V_\varepsilon) = (u_\varepsilon(P^\varepsilon + \varepsilon y), v_\varepsilon(P^\varepsilon + \varepsilon y)) \to (U_0, V_0)$, where (U_0, V_0) satisfies

$$\begin{cases} \triangle U - V_1(P^0)U + \mu_1 U^3 + \beta U V^2 = 0 & \text{in } \mathbf{R}^N\,, \\ \triangle V - V_2(P^0)V + \mu_2 V^3 + \beta U^2 V = 0 & \text{in } \mathbf{R}^N\,. \end{cases}$$

Then by the strong Maximum Principle, $U_0, V_0 > 0$. Furthermore, we have

$$\lim_{\varepsilon\to 0} \varepsilon^{-N} c_{\varepsilon,\mathbf{R}^N,V_1,V_2} \geq c_{1,\mathbf{R}^N,V_1(P^0),V_2(P^0)}.$$

Hence by Lemma 4.1,

$$c_{1,\mathbf{R}^N,V_1(P^0),V_2(P^0)} \leq \inf_{x\in\mathbf{R}^N} c_{1,\mathbf{R}^N,V_1(x),V_2(x)}\,,$$

i.e. $c_{1,\mathbf{R}^N,V_1(P^0),V_2(P^0)} = \inf_{x\in\mathbf{R}^N} c_{1,\mathbf{R}^N,V_1(x),V_2(x)}$.

It remains to show that $\frac{|P^\varepsilon - Q^\varepsilon|}{\varepsilon} \to 0$. In fact, if $\frac{|P^\varepsilon - Q^\varepsilon|}{\varepsilon} \to +\infty$, then similar arguments may give

$$\lim_{\varepsilon\to 0} \varepsilon^{-N} c_{\varepsilon,\mathbf{R}^N,V_1,V_2} \geq c_{1,\mathbf{R}^N,V_1(P^0),V_2(P^0)} + c_{1,\mathbf{R}^N,V_1(Q^0),V_2(Q^0)} \geq 2 \inf_{x\in\mathbf{R}^N} c_{1,\mathbf{R}^N,V_1(x),V_2(x)}$$

which is impossible. On the other hand, if $\frac{|P^\varepsilon - Q^\varepsilon|}{\varepsilon} \to c \neq 0$, then U_0 and V_0 may have different maximum points. This may contradict with the fact that both U_0 and V_0 are radially symmetric and strictly decreasing. Thus $\frac{|P^\varepsilon - Q^\varepsilon|}{\varepsilon} \to 0$. The uniqueness of $P^\varepsilon, Q^\varepsilon$ may follow from Claim 8 of [20]. Therefore, we may complete the proof of Theorem 1.3.

5. Proof of Theorem 1.4

In this section, we follow the same ideas of [20] to prove Theorem 1.4. As for the proof of Lemma 4.2 in [20], the upper bound of $c_{\varepsilon,\Omega,\lambda_1,\lambda_2}$ is given by

Lemma 5.1. *For $\beta > \sqrt{\mu_1\mu_2}$,*

$$c_{\varepsilon,\Omega,\lambda_1,\lambda_2} \leq \varepsilon^N \left\{ c_{1,\mathbf{R}^N,\lambda_1,\lambda_2} + c_1\, e^{-2\sqrt{\lambda_1}(1-\sigma)R_\varepsilon} + c_2\, e^{-2\sqrt{\lambda_2}(1-\sigma)R_\varepsilon} \right\}, \tag{5.1}$$

where $R_\varepsilon = \frac{1}{\varepsilon}\max_{P\in\Omega} d(P,\partial\Omega)$ and c_j's are positive constants.

Furthermore, the asymptotic behavior of $(u_\varepsilon, v_\varepsilon)$'s can be summarized as follows:

Lemma 5.2. *For ε sufficiently small, u_ε has only one local maximum point P_ε and v_ε has only one local maximum point Q_ε such that*

$$\frac{d(P_\varepsilon, \partial\Omega)}{\varepsilon} \to +\infty, \quad \frac{d(Q_\varepsilon, \partial\Omega)}{\varepsilon} \to +\infty, \quad \frac{|P_\varepsilon - Q_\varepsilon|}{\varepsilon} \to 0. \tag{5.2}$$

Let $U_\varepsilon(y) := u_\varepsilon(P_\varepsilon + \varepsilon y)$, $V_\varepsilon(y) := (Q_\varepsilon + \varepsilon y)$. Then $(U_\varepsilon, V_\varepsilon) \to (U_0, V_0)$, where (U_0, V_0) is a least-energy solution of (1.11). Moreover,

$$\varepsilon\,|\bigtriangledown u_\varepsilon| + |u_\varepsilon| \le Ce^{-\sqrt{\lambda_1}(1-\sigma)\frac{|x-P_\varepsilon|}{\varepsilon}}, \quad \varepsilon\,|\bigtriangledown v_\varepsilon| + |v_\varepsilon| \le Ce^{-\sqrt{\lambda_2}(1-\sigma)\frac{|x-Q_\varepsilon|}{\varepsilon}}. \tag{5.3}$$

Now we want to complete the proof of Theorem 1.4. We may assume that, passing to a subsequence, that P_ε (or Q_ε) $\to x_0 \in \bar{\Omega}$. Thus

$$d_\varepsilon = d(P_\varepsilon, \partial\Omega) \to d_0 := d(x_0, \partial\Omega), \quad \text{as } \varepsilon \to 0.$$

Note that d_0 may be zero. Given $\sigma > 0$ a small constant, we may choose $d_0' > 0$ and $\sigma' > 0$ slightly smaller than σ such that

$$\mathrm{vol}(B(x_0, d_0')) = \mathrm{vol}(\Omega \cap B(x_0, d_0 + \sigma)) \quad \text{and} \quad d_0' < d_0 + \sigma'.$$

Besides, we may set η_ε as a C^∞ cut-off function such that

$$\begin{cases} \eta_\varepsilon(s) = 1 & \text{for } 0 \le s \le d_\varepsilon + \sigma', \\ \eta_\varepsilon(s) = 0 & \text{for } s > d_\varepsilon + \sigma, \\ 0 \le \eta_\varepsilon \le 1, & |\eta_\varepsilon'| \le C. \end{cases}$$

Let $\tilde{u}_\varepsilon(x) = u_\varepsilon \eta_\varepsilon(|P_\varepsilon - x|)$ and $\tilde{v}_\varepsilon(x) = v_\varepsilon \eta_\varepsilon(|Q_\varepsilon - x|)$. Then we have

$$\lim_{\varepsilon\to 0} \varepsilon^{-N} \int_\Omega [2\beta \tilde{u}_\varepsilon^2 \tilde{v}_\varepsilon^2 + \mu_1 \tilde{u}_\varepsilon^4 + \mu_2 \tilde{v}_\varepsilon^4] = \int_{\mathbf{R}^N} [2\beta U_0^2 V_0^2 + \mu_1 U_0^4 + \mu_2 V_0^4] > 0. \tag{5.4}$$

Hence

$$\int_\Omega [2\beta \tilde{u}_\varepsilon^2 \tilde{v}_\varepsilon^2 + \mu_1 \tilde{u}_\varepsilon^4 + \mu_2 \tilde{v}_\varepsilon^4] > 0,$$

as ε sufficiently small.

By the decay estimate (5.3) and Lemma 2.3, we obtain that

$$\begin{aligned} c_{\varepsilon,\Omega,\lambda_1,\lambda_2} &\ge E_{\varepsilon,\Omega,\lambda_1,\lambda_2}[tu_\varepsilon, tv_\varepsilon] \\ &\ge E_{\varepsilon,\tilde{\Omega},\lambda_1,\lambda_2}[t\tilde{u}_\varepsilon, t\tilde{v}_\varepsilon] - \varepsilon^N \exp\left[-\frac{2\sqrt{\lambda_1}}{\varepsilon}(d_\varepsilon + \sigma')\right] - \varepsilon^N \exp\left[-\frac{2\sqrt{\lambda_2}}{\varepsilon}(d_\varepsilon + \sigma')\right] \end{aligned} \tag{5.5}$$

for all $t \in [0, 2]$, where $\tilde{\Omega} = \Omega \cap B(x_\varepsilon, d_\varepsilon + \sigma)$ and x_ε can be P_ε or Q_ε. Let $R_\varepsilon = \frac{d_\varepsilon'}{\varepsilon}$, where d_ε' is chosen such that

$$\mathrm{vol}(B(0, d_\varepsilon')) = \mathrm{vol}(\Omega \cap B(x_\varepsilon, d_\varepsilon + \sigma)).$$

Using Schwartz's symmetrization, we have

$$\int_{B(0,d_\varepsilon')} (\tilde{u}_\varepsilon^*)^2 (\tilde{v}_\varepsilon^*)^2 \ge \int_{\tilde{\Omega}} \tilde{u}_\varepsilon^2 \tilde{v}_\varepsilon^2$$

and then

$$\int_{B(0,d'_\varepsilon)} [2\beta(\tilde{u}^*_\varepsilon)^2(\tilde{v}^*_\varepsilon)^2 + \mu_1(\tilde{u}^*_\varepsilon)^4 + \mu_2(\tilde{v}^*_\varepsilon)^4] \geq \int_{\tilde{\Omega}} [2\beta\tilde{u}^2_\varepsilon\tilde{v}^2_\varepsilon + \mu_1\tilde{u} - \varepsilon^4 + \mu_2\tilde{v}^4_\varepsilon] > 0\,. \quad (5.6)$$

Thus

$$E_{\varepsilon,B(0,d'_\varepsilon),\lambda_1,\lambda_2}[t\tilde{u}^*_\varepsilon, t\tilde{v}^*_\varepsilon] \leq E_{\varepsilon,\tilde{\Omega},\lambda_1,\lambda_2}[t\tilde{u}_\varepsilon, t\tilde{v}_\varepsilon]\,, \quad \forall\, t \in [0,2]\,. \quad (5.7)$$

Here we have used the fact that $\beta > 0$.

By (5.6) and Claim 1 of Theorem 2.1, there exists $t^* \in (0,2]$ such that

$$E_{\varepsilon,B(0,d'_\varepsilon),\lambda_1,\lambda_2}[t^*\tilde{u}^*_\varepsilon, t^*\tilde{v}^*_\varepsilon] \geq E_{\varepsilon,B(0,d'_\varepsilon),\lambda_1,\lambda_2}[t\tilde{u}^*_\varepsilon, t\tilde{v}^*_\varepsilon]\,, \quad \forall t \geq 0\,.$$

Then by (5.5) and (5.7),

$$\begin{aligned}
& E_{\varepsilon,B(0,d'_\varepsilon),\lambda_1,\lambda_2}[t^*\tilde{u}^*_\varepsilon, t^*\tilde{v}^*_\varepsilon] \\
\leq\ & E_{\varepsilon,\tilde{\Omega},\lambda_1,\lambda_2}[t^*\tilde{u}_\varepsilon, t^*\tilde{v}_\varepsilon] \\
\leq\ & c_{\varepsilon,\Omega,\lambda_1,\lambda_2} + \varepsilon^N \exp\left[-\frac{2\sqrt{\lambda_1}}{\varepsilon}(d_\varepsilon + \sigma')\right] + \varepsilon^N \exp\left[-\frac{2\sqrt{\lambda_2}}{\varepsilon}(d_\varepsilon + \sigma')\right], \\
& E_{\varepsilon,B(0,d'_\varepsilon),\lambda_1,\lambda_2}[t^*\tilde{u}^*_\varepsilon, t^*\tilde{v}^*_\varepsilon] \\
=\ & \sup_{t>0} E_{\varepsilon,B(0,d'_\varepsilon),\lambda_1,\lambda_2}[t\tilde{u}^*_\varepsilon, t\tilde{v}^*_\varepsilon] \\
\geq\ & \varepsilon^N \inf_{\substack{u,v\geq 0,\\ u\not\equiv 0, v\not\equiv 0,\\ (u,v)\in N(1,R_\varepsilon,\lambda_1,\lambda_2)}} E_{1,B_{R_\varepsilon},\lambda_1,\lambda_2}[u,v] \\
\geq\ & \varepsilon^N \left\{ c_{1,\mathbf{R}^N,\lambda_1,\lambda_2} + c_3 \exp\left[-\frac{2(1+\sigma)\sqrt{\lambda_1}}{\varepsilon}(d_\varepsilon + o(1))\right]\right\} \\
& +\varepsilon^N c_4 \exp\left[-\frac{2(1+\sigma)\sqrt{\lambda_2}}{\varepsilon}(d_\varepsilon + o(1))\right],
\end{aligned}$$

where c_j's are positive constants. Here the last inequality may follow from Lemma 5.1 and Theorem 4.1 of [20]. Thus

$$c_{\varepsilon,\Omega,\lambda_1,\lambda_2} \geq \varepsilon^N \left\{ \begin{array}{l} c_{1,\mathbf{R}^N,\lambda_1,\lambda_2} + c_3 \exp\left[-\frac{2(1+\sigma)\sqrt{\lambda_1}}{\varepsilon}(d_\varepsilon + o(1))\right] \\ +c_4 \exp\left[-\frac{2(1+\sigma)\sqrt{\lambda_2}}{\varepsilon}(d_\varepsilon + o(1))\right] \end{array} \right\}. \quad (5.8)$$

Combining the lower and upper bound of $c_{\varepsilon,\Omega,\lambda_1,\lambda_2}$, we obtain

$$\begin{aligned}
& c_3 \exp\left[-\frac{2(1+\sigma)\sqrt{\lambda_1}}{\varepsilon}(d_\varepsilon + o(1))\right] + c_4 \exp\left[-\frac{2(1+\sigma)\sqrt{\lambda_2}}{\varepsilon}(d_\varepsilon + o(1))\right] \\
\leq\ & c_1 \exp\left[-\frac{2(1-\sigma)\sqrt{\lambda_1}}{\varepsilon}(d_0 + o(1))\right] + c_2 \exp\left[-\frac{2(1-\sigma)\sqrt{\lambda_2}}{\varepsilon}(d_0 + o(1))\right].
\end{aligned}$$

This then shows that $d(P_\varepsilon, , \partial\Omega), d(Q_\varepsilon, \partial\Omega) \to \max_{P\in\Omega} d(P, \partial\Omega)$ since $|P_\varepsilon - Q_\varepsilon| \to 0$. □

References

[1] A. Ambrosetti and E. Colorado, Bound and ground states of coupled nonlinear Schrodinger equations, *Comptes Rendus Math. Acad. Sci. Paris* **342** (2006), 453-458.

[2] Ambrosetti, A.; Malchiodi, A.; Ni, W.-M. Singularly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres, Part I. *Comm. Math. Phys.* **235** (2003), 427-466.

[3] T. Bartsch, Z.Q. Wang and J. Wei, Bound states for a coupled Schrödinger system, preprint.

[4] P. Bates, E.N. Dancer and J. Shi, Multi-spike stationary solutions of the Cahn-Hilliard equation in higher-dimension and instability, *Adv. Diff. Eqns* 4 (1999), 1-69.

[5] Byeon, J.; Wang, Z.-Q. Standing waves with a critical frequency for nonlinear Schrödinger equations, *Arch. Rat. Mech. Anal.* **165** (2002), 295–316.

[6] Cingolani, S.; Lazzo M. Multiple positive solutions to nonlinear Schrödinger equations with competing potential functions. *J. Differential Equations* **160** (2000) 118–138.

[7] M. Conti, S. Terracini, G. Verzini, Nehari's Problem and competing species system. *Ann. I. H. Poincaré* 19(2002), no. 6, 871-888.

[8] S. L. Cornish, S. T. Thompson and C. E. Wieman, Formation of bright matter-wave solitons during the collapse of Bose-Einstein condensates, *Phys. Rev. Lett.* **96**, 170401(1-4) (2006).

[9] E.N. Dancer and S. Yan, Multipeak solutions for a singular perturbed Neumann problem, *Pacific J. Math.* 189(1999), 241-262.

[10] B. DeMarco and D. S. Jin, Onset of Fermi Degeneracy in a Trapped Atomic Gas, *Science* **285**, 1703 (1999).

[11] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[12] D. Gilbarg and N. S. Trudinger,*Elliptic Partial Differential Equations of Second order*, 2nd.ed, Springer-Verlag,1983.

[13] V. L. Ginzburg and L. P. Pitaevskii, *Zh. Eksperim. Theor. Fys.* **34** 1240(1958) [*Sov. Phys. JETP,* **7** 858(1958).]

[14] C. Gui and J. Wei, On multiple mixed interior and boundary peak solutions for some singularly perturbed Neumann problems, *Can. J. Math.* 52(2000), 522-538.

[15] Jeanjean, L.; Tanaka K., Singularly perturbed elliptic problems with superlinear or asymptotically linear nonlinearities. *Calc. Var. Partial Differential Equations* **21** (2004), no. 3, 287-318.

[16] Kang, X.; Wei, J. On interacting bumps of semi-classical states of nonlinear Schrödinger equations. *Adv. Diff. Eqn.* **5**(7-9) (2000), 899-928.

[17] T. Karpiuk, M. Brewczyk, S. Ospelkaus-Schwarzer, K. Bongs, M. Gajda, and K. Rzazewski, Soliton Trains in Bose-Fermi Mixtures, *Phys. Rev. Lett.* **93** (2004), 100401-(1-4).

[18] Y.-Y. Li, On a singularly perturbed equation with Neumann boundary condition, *Comm. PDE* 23(1998), 487- 545.

[19] E. Lieb and M. Loss, Analysis, American Mathematical Society, 1996.

[20] T. C. Lin and J. Wei, Spikes in two coupled nonlinear Schrödinger equations, *Ann. I. H. Poincaré Analyse. Non.* **22**, no.4(2005) 403-439.

[21] T. C. Lin and J. Wei, Ground state of N coupled nonlinear Schrödinger equations in $\mathbf{R}^N, n \leq 3$, *Comm. Math. Phys.* **255**, (2005) 629-653.

[22] T. C. Lin and J. Wei, Spikes in two-component systems of nonlinear Schrödinger equations with trapping potentials, *J. Diff. eqns.* **229**, (2006) 538-569.

[23] A. Malchiodi and M. Montenegro, Multidimensional boundary layers for a singularly perturbed Neumann problem. *Duke Math. J.* 124 (2004), no. 1, 105–143.

[24] G. Modugno, G. Roati, F. Riboli, F. Ferlaino, R. J. Brecha, and M. Inguscio, Collapse of a Degenerate Fermi Gas, *Science* **297**, 2240 (2002).

[25] K. Molmer, Bose Condensates and Fermi Gases at Zero Temperature, *Phys. Rev. Lett.* **80**, 1804-1807 (1998).

[26] W.-M. Ni, Diffusion, cross-diffusion, and their spike-layer steady states, *Notices of Amer. Math. Soc.* 45(1998), 9-18.

[27] W.-M. Ni, Qualitative properties of solutions to elliptic problems, in *Stationary partial differential equations.* Vol. I, 157-233, Handb. Differ. Equ., North-Holland, Amsterdam, 2004.

[28] Victor M. Perez-Garcia and Juan Belmonte Beitia, Symbiotic solitons in heteronuclear multicomponent Bose-Einstein condensates, *Phys. Rev. A* **72** (2005) 033620(1-5).

[29] M. del Pino and P. Felmer, Spike-layered solutions of singularly perturbed elliptic problems in a degenerate setting, *Indiana Univ. Math. J.* 48 (1999), no. 3, 883–898.

[30] del Pino, M.; Felmer, P. Semi-classcal states for nonlinear Schrödinger equations, *J. Funct. Anal.* **149** (1997), 245–265.

[31] del Pino, M.; Felmer, P. Semi-classical states of nonlinear Schrödinger equations: a variational reduction method. *Math. Ann.* **324** (2002), no. 1, 1-32.

[32] L. Pitaevskii, S. Stringari, Bose-Einstein Condensation, Oxford Univ. Press, Oxford, 2003.

[33] A. Pomponio, Coupled nonlinear Schrödinger systems with potentials, *Journal of Differential Equations* **227** (2006), 258-281.

[34] B. Sirakov, Least Energy Solitary Waves for a System of Nonlinear Schrödinger Equations in $\mathbb{R}^n$, to appear in *Comm. Math. Phys.*

[35] W. Troy, Symmetry properties in systems of semilinear elliptic equations, *J. Diff. Eqns.* 42 (1981), no. 3, 400-413.

[36] Z.-Q. Wang, Existence and symmetry of multi-bump solutions for nonlinear Schrödinger equations. *J. Differential Equations* **159** (1999), no. 1, 102–137.

[37] J. Wei, On the construction of single-peaked solutions to a singularly perturbed semilinear Dirichlet problem, *J. Diff. Eqns* 129 (1996), 315-333.

[38] J. Wei and M. Winter, Stationary solutions for the Cahn-Hilliard equation, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 15 (1998), 459-492.

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