STABLE SPIKE CLUSTERS FOR THE ONE-DIMENSIONAL GIERER-MEINHARDT SYSTEM

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ABSTRACT. We consider the Gierer-Meinhardt system with precursor inhomogeneity and two small diffusivities in an interval

$$\begin{cases} A_t = \varepsilon^2 A'' - \mu(x) A + \frac{A^2}{H}, & x \in (-1,1),\ t > 0, \\ \tau H_t = D H'' - H + A^2, & x \in (-1,1),\ t > 0, \\ A'(-1) = A'(1) = H'(-1) = H'(1) = 0, \end{cases}$$

$$\text{where} \quad 0 < \varepsilon \ll \sqrt{D} \ll 1,$$

$$\tau \geq 0 \text{ and } \tau \text{ is independent of } \varepsilon.$$

A **spike cluster** is the combination of several spikes which all approach the same point in the singular limit. We rigorously prove the existence of a steady-state spike cluster consisting of N spikes near a non-degenerate local minimum point t^0 of the smooth inhomogeneity $\mu(x)$, i.e. we assume that $\mu'(t^0) = 0,\ \mu''(t^0) > 0$. Here N is an arbitrary positive integer. Further, we show that this solution is linearly stable. We explicitly compute all eigenvalues, both large (of order $O(1)$) and small (of order $o(1)$). The main features of studying the Gierer-Meinhardt system in this setting are as follows: (i) it is biologically relevant since it models a hierarchical process (pattern formation of small-scale structures induced by a pre-existing large-scale inhomogeneity); (ii) it contains three different spatial scales two of which are small: the $O(1)$ scale of the precursor inhomogeneity $\mu(x)$, the $O(\sqrt{D})$ scale of the inhibitor diffusivity and the $O(\varepsilon)$ scale of the activator diffusivity; (iii) the expressions can be made explicit and often have a particularly simple form.

1. INTRODUCTION

In his pioneering work [31] in 1952, Turing studied how pattern formation could start from an unpatterned state. He explained the onset of pattern formation by the presence of spatially varying instabilities combined with the absence of spatially homogeneous instabilities. This approach is now commonly called *Turing diffusion-driven instability*. Since then many reaction-diffusion systems in biological modeling have been proposed and the occurrence of pattern formation has been investigated based on the approach of Turing instability [31]. One of the most widely used class of biological pattern-formation models consists of the activator-inhibitor

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type models which are based on real-world interactions such as those encountered in experiments and observations on seashells, animal skin patterns, embryological development, cell signalling pathways and many more. Among these, one of the most popular models is the Gierer-Meinhardt system [9], [16], [19], which in one dimension with a precursor-inhomogeneity and two small diffusivities can be stated as follows:

$$\begin{cases} A_t = \varepsilon^2 \Delta A - \mu(x) A + \frac{A^2}{H}, & x \in (-1,1),\ t > 0, \\ \tau H_t = D\Delta H - H + A^2, & x \in (-1,1),\ t > 0, \\ A'(-1) = A'(1) = H'(-1) = H'(1) = 0, \end{cases} \qquad (1.1)$$

$$\text{where} \quad 0 < \varepsilon \ll \sqrt{D} \ll 1,$$

$\tau \geq 0$ and τ is independent of ε.

In the standard Gierer-Meinhardt system without precursor it is assumed that $\mu(x) \equiv 1$.

Precursor gradients in reaction-diffusion systems have been investigated in earlier work. The original Gierer-Meinhardt system [9], [16], [19] has been introduced with precursor gradients. These precursors were proposed to model the localization of the head structure in the coelenterate *Hydra*. Gradients have also been used in the Brusselator model to restrict pattern formation to some fraction of the spatial domain [13]. In that example, the gradient carries the system in and out of the pattern-forming part of the parameter range (across the Turing bifurcation), thus effectively confining the domain where peak formation can occur. A similar localization effect has been used to model segmentation patterns in the fruit fly *Drosophila melanogaster* in [15] and [12].

In this paper, we study the Gierer-Meinhardt system with precursor and prove the existence and stability of a cluster, which consists of N spikes approaching the same limiting point.

More precisely, we prove the existence of a steady-state spike cluster consisting of N spikes near a non-degenerate local minimum point t^0 of the inhomogeneity $\mu(x) \in C^3(\Omega)$, i.e. we assume that $\mu'(t^0) = 0$, $\mu''(t^0) > 0$. Further, we show that this solution is linearly stable.

We explicitly compute all eigenvalues, both large (of order $O(1)$) and small (of order $o(1)$). The main features of studying the Gierer-Meinhardt system in this setting are as follows: (i) it is biologically relevant since it models a hierarchical process (pattern formation of small-scale structures induced by a pre-existing inhomogeneity)

(ii) it is important to note that this system contains three different spatial scales two of which are small (i.e. $o(1)$):

(a) The $O(1)$ scale of the precursor $\mu(x)$,

(b) The $O(\sqrt{D})$ scale of the inhibitor diffusivity,

(c) The $O(\varepsilon)$ scale of the activator diffusivity.

Consequently there are the two small quantities $\sqrt{D}$ and $\frac{\varepsilon}{\sqrt{D}}$ which play an important role throughout the paper.

(iii) the expressions can be made explicit and often have a particularly simple form.

Let us now summarize the analytical approach employed in our paper. The existence proof is based on Liapunov-Schmidt reduction. The stability of the cluster is shown by first separating the eigenvalues into two cases: large eigenvalues which tend to a nonzero limit and small eigenvalues which tend to zero in the limit $D \to 0$ and $\frac{\varepsilon}{\sqrt{D}} \to 0$. Large eigenvalues are then explored by deriving suitable nonlocal eigenvalue problems and studying them using results of [36] and a compactness argument of Dancer [4]. Small eigenvalues are calculated explicitly by using asymptotic analysis with rigorous error estimates.

We shall establish the existence and stability of positive $N-$peaked steady-state spike clusters to (1.1). The steady-state problem for positive solutions of (1.1) is the following:

$$\begin{cases} \varepsilon^2 A'' - \mu(x)A + \frac{A^2}{H} = 0 & x \in (-1,1), \\ DH'' - H + A^2 = 0 & x \in (-1,1), \\ A(x) > 0, H(x) > 0, & x \in (-1,1), \\ A'(-1) = A'(1) = H'(-1) = H'(1) = 0. \end{cases} \tag{1.2}$$

To have a nontrivial spike cluster, we assume throughout the paper that

$$N \geq 2. \tag{1.3}$$

Before stating our main results, let us review some precious results on pattern formation for the Gierer-Meinhardt system (1.1) , in particular concerning spiky patterns.

1. I. Takagi [30] proved the existence of N-spike steady-state solutions of (1.1) in an interval for homogeneous coefficients (i.e. $\mu(x) = 1$) in the regime $\varepsilon \ll 1$ and $D \gg 1$, where N is an arbitrary positive integer. For these solutions, the spikes are identical copies of each other and their maxima are located at

$$t_j = -1 + \frac{2j-1}{N}, \quad j = 1, \ldots, N,$$

The proof in [30] is based on symmetry and the implicit function theorem.

2. In [14] (using matched asymptotic expansions) and [43] (based on rigorous proofs), the following stability result has been shown: for the N-spike steady-state solution derived in item 1 and $0 \leq \tau < \tau_0(N)$, where $\tau_0(N) > 0$ is independent of ε, there are numbers $D_1 > D_2 > \cdots > D_N > \cdots$ (which have been computed explicitly) such that the N-spike steady-state is stable for for $D < D_N$ and unstable for $D > D_N$.

3. In [14] (using matched asymptotic expansions) and [33] (based on rigorous analysis) the following existence and stability results have been shown: for a certain parameter range of D, the Gierer-Meinhardt system (1.1) with $\mu(x) = 1$ has **asymmetric** $N-$spike steady-state solutions, which consist of exact copies of precisely two different spikes with distinct amplitudes. They can be considered as bifurcating solutions from those in item 1 such that the amplitudes start to differ at the bifurcation point (saddle-node bifuraction). The stability of these asymmetric $N-$peaked solutions has been studied in [33].

4. In [45] the existence and stability of $N-$peaked steady states for the Gierer-Meinhardt system with precursor inhomogeneity has been shown. These spikes have different amplitudes. In particular, the results imply that precursor inhomogeneities can induce instabilities. Single-spike solutions for the Gierer-Meinhardt system with precursor including spike motion have been studied in [32].

5. In [42] the existence of symmetric and asymmetric multiple spike clusters in an interval has been shown.

Compared to each of the items listed above, the setting and results in our paper have marked differences. We now consider two small parameters, D and $\frac{\varepsilon}{\sqrt{D}}$ which results in new types of behavior. The leading-order asymptotic expression of the large and small eigenvalues depend on the index of the eigenvalue quadratically, whereas in items 1 and 2 this relation is oscillatory (involving trigonometric functions).

In our study, the spikes in leading order have equal amplitudes and uniform spacing, although there is precursor inhomogeneity in the system, in contrast to item 3. The amplitudes, positions and eigenvalues in our study can be characterized explicitly and have a simpler form than in item 4. We can also prove the stability of clusters not merely their existence as in item 5. In particular, we show here that the clusters may be stable, whereas in item 5 they are expected to be unstable.

In the shadow system case ($D = \infty$) the existence of single- or N-peaked solutions has been established in [10, 11, 21, 22] and other papers. It is interesting to remark that symmetric and asymmetric patterns can also be obtained for the Gierer-Meinhardt system on the real line, see [5, 6]. We refer to [23] for the SLEP approch for the existence and stability of multi-layered solutions for reaction-diffusion systems. For two-dimensional domains the existence and stability of multi-peaked steady states has been proved in [38, 39, 40] and results similar to items 1 and 2 have been derived. Hopf bifurcation has been established in [4, 34, 35, 40]. The repulsive dynamics of multiple spikes has been studied in [7].

Another study with three different spatial scales, two of which are small, considers a consumer chain model allowing for a novel type of spiky clustered pattern which is stable for certain parameters [46].

The model in our paper shows some similarity to variational models for material microstructure [1, 20, 48]. In both models the solutions have two small scales. However, in our case we have two parameters to control each of them independently, whereas in the microstructure case they are expressions of different orders depending on the same small parameter and so they are related to one another.

Results on the existence and stability of multi-spike steady states have been reviewed and put in a general context in [47].

This paper has the following structure: In Section 2 we state our main results on existence and stability and present four highlights of their proofs. In Section 3 we introduce some preliminaries. In Sections 4–5 and Appendices A-B we prove the existence of steady-state spike clusters: in Section 4 we introduce suitable approximate solutions, in Appendix A we compute their error, in Appendix B we use the Liapunov-Schmidt method to reduce the existence of solutions of (1.2) to a finite-dimensional problem, in Section 5 we solve this finite-dimensional reduced problem. In Sections 6–7 and Appendix C we prove the stability of these steady-state spike clusters: in Section 6 we study the large eigenvalues of the linearized operator and show that it has diagonal form. We give a complete description of their asymptotic behavior which is stated in Lemma 12. In Section 7 we characterize the small eigenvalues of the linearized operator and show that they all have negative real part. This includes deriving the eigenvalues of a matrix which is needed to compute the small eigenvalues explicitly. We give a complete description of their asymptotic behavior in leading order which can be found in Lemma 13. Our approach here is to interpret the main matrix as the finite-difference approximation of a suitable ordinary differential equation, compute the solution of this approximation explicitly and get the eigenvectors by taking the values of this solution at uniformly spaced points. In Appendix C we perform the technical analysis needed to derive the small eigenvalues. In Section 8 we conclude with a discussion of our results with respect to the bridging of length scales and the hierarchy of multi-stage biological processes.

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2. MAIN RESULTS ON EXISTENCE AND STABILITY

In this section, we state our main results on existence and stability of solutions and present four highlights of our approach.

We first need to introduce some essential notation. Let $L^2(-1,1)$ and $H^2(-1,1)$ denote Lebesgue and Sobolev space, respectively. Let the function w be the unique solution (ground state) of the problem

$$\begin{cases} w'' - w + w^2 = 0, & y \in \mathbb{R}, \\ w > 0, & w(0) = \max_{y\in\mathbb{R}} w(y), \\ w(y) \to 0 & \text{as } |y| \to \infty. \end{cases} \tag{2.1}$$

Then $w(y)$ can explicitly be written as

$$w(y) = \frac{3}{2}\cosh^{-2}\left(\frac{y}{2}\right). \tag{2.2}$$

Elementary calculations give

$$\int_{\mathbb{R}} w^2(z)\,dz = 6, \quad \int_{\mathbb{R}} w^3(z)\,dz = 7.2, \quad \int_{\mathbb{R}} (w')^2(z)\,dz = 1.2. \tag{2.3}$$

Let

$$\Omega = (-1,1).$$

For $z \in (-1,1)$, let $G_D(x,z)$ be the Green's function defined by

$$\begin{cases} DG_D''(x,z) - G_D(x,z) + \delta_z(x) = 0, & x \in (-1,1), \\ G_D'(-1,z) = G_D'(1,z) = 0, \end{cases} \tag{2.4}$$

where $G_D'(x,z) = \frac{\partial}{\partial x}G_D(x,z)$ (and the lefthand and righthand limits are considered for $x = z$). We calculate

$$G_D(x,z) = \begin{cases} \frac{\theta}{\sinh(2\theta)}\cosh[\theta(1+x)]\cosh[\theta(1-z)], & -1 < x < z, \\ \frac{\theta}{\sinh(2\theta)}\cosh[\theta(1-x)]\cosh[\theta(1+z)], & z < x < 1, \end{cases} \tag{2.5}$$

where

$$\theta = D^{-1/2}.$$

Let $t^0 \in (-1,1)$ and set

$$\mu^0 = \mu(t^0). \tag{2.6}$$

Let $\hat{\xi}^0$ be defined by

$$\hat{\xi}^0 = \frac{1}{2\sqrt{D}G_D(t^0,t^0)(\mu^0)^{3/2}}. \tag{2.7}$$

We set

$$\xi_\varepsilon := \frac{2\sqrt{D}}{\varepsilon\int_{\mathbb{R}} w^2(z)\,dz}. \tag{2.8}$$

Our first result is about the existence of an N-spike cluster solution near a non-degenerate minimum point of the precursor.

Theorem 1. *(Existence of an N-spike cluster.)*

Let N be a positive integer and $t^0 \in (-1, 1)$. We assume that $\mu \in C^3(-1, 1)$ and

$$\mu'(t^0) = 0,\ \mu''(t^0) > 0. \tag{2.9}$$

Then, for $\varepsilon \ll \sqrt{D} \ll 1$, problem (1.2) has an N-spike cluster solution which concentrates at t^0. In particular, it satisfies

$$A_\varepsilon(x) \sim \sum_{k=1}^{N} \xi_\varepsilon \hat{\xi}^0 \mu^0 w\left(\sqrt{\mu^0}\,\frac{x - t_k^\varepsilon}{\varepsilon}\right), \tag{2.10}$$

$$H_\varepsilon(t_k^\varepsilon) \sim \xi_\varepsilon \hat{\xi}^0, \quad k = 1, \ldots, N, \tag{2.11}$$

$$t_k^\varepsilon \to t^0, \quad k = 1, \ldots, N, \tag{2.12}$$

where μ^0 has been defined in (2.6), $\hat{\xi}^0$ has been introduced in (2.7) and ξ_ε has been defined in (2.8).

Next we state our second result which concerns the stability of the N-spike cluster steady states given in Theorem 1.

Theorem 2. *(Stability of an N-spike cluster.)*

For $\varepsilon \ll \sqrt{D} \ll 1$, let $(A_\varepsilon, H_\varepsilon)$ be an N-spike cluster steady state given in Theorem 1. Then there exists $\tau_0 > 0$ independent of ε and $\sqrt{D}$ such that the N-spike cluster steady state $(A_\varepsilon, H_\varepsilon)$ is linearly stable for all $0 \le \tau < \tau_0$.

Remark 3. *For the stability, we assume that $0 \le \tau < \tau_0$ for some $\tau_0 > 0$. Stability in the case where τ is large has been investigated in* [35] *for single spikes and those results on Hopf bifurcation are expected to carry over to the case of an N-spike cluster considered here. We remark that stability in the case of large τ for the shadow system has been studied in* [4, 34]. *It turns out that this Hopf bifurcation leads to oscillations of the amplitudes.*

Remark 4. *Previous studies of the precursor case can be found among others in* [2], [27], [28]. *We also refer to results for the dynamics of pulses in heterogeneous media* [24], [49].

The proofs of both Theorems 1 and 2 will follow the approach in [47], where we reviewed and discussed many results on the existence and stability of multi-spike steady states.

Next we present some highlights of the proofs of Theorems 1 and 2 in an informal manner. We will give reference to the full proofs which will follow in later sections.

Highlight 1: For the proof in Theorem 1 we use Liapunov-Schmidt reduction to derive a reduced problem which will determine the positions of the spikes. This reduced problem in leading order is given by

$$W_0(\mathbf{t}) \sim c_1 \sum_{k,|k-s|=1} e^{-|t_s-t_k|/\sqrt{D}} \left(-\frac{t_s-t_k}{|t_s-t_k|}\right) + c_2\sqrt{D}\mu''(t^0)(t_s-t^0), \quad s=1,\dots,N, \tag{2.13}$$

where $c_1,\ c_2>0$ are constants which are independent of the small parameters and $\mathbf{t}=(t_1,\dots,t_N)$ are the positions of the spikes (compare (5.4)). We need to solve $W_0(\mathbf{t})=0$, which implies

$$t_s - t_{s-1} \sim \sqrt{D}\log\frac{1}{D}, \quad s=2,\dots,N,$$

(compare (5.6)). The distance between neighboring spikes in the cluster is small (converging to zero) and in leading order it is the same between any pair of neighbors.

Highlight 2: The large eigenvalues with $\lambda_\varepsilon \to \lambda_0 \neq 0$ and their corresponding eigenfunctions

$$\phi_{\varepsilon,i}(y) \to \phi_i(y), \quad y=\frac{x-t_i}{\varepsilon}, \quad i=1,\dots,N,$$

where $\phi_{\varepsilon,i}(y)$ is the restriction of the rescaled eigenfunction of the activator A_ε near t_i, in the limit $\max\left(\frac{\varepsilon}{\sqrt{D}}, D\right) \to 0$ solve the nonlocal eigenvalue problem (NLEP)

$$\Delta_y\phi_i - \phi_i + 2w\phi_i - \frac{2\int_{\mathbb{R}} w\phi_i\,dy}{\int_{\mathbb{R}} w^2\,dy} w^2 = \lambda_0\phi_i, \quad i=1,\dots,N,$$

(see (6.6)). This nonlocal eigenvalue problem has diagonal form. Thus each spike only interacts with itself and not with the other spikes.

It follows that the spike cluster is stable with the respect to large eigenvalues.

Highlight 3: The small eigenvalues $\lambda_\varepsilon \to 0$ in leading order are given by the eigenvalues of the matrix

$$-\varepsilon^2 c_3 \mathcal{M}(\mathbf{t}^0),$$

where $c_3>0$ is independent of the small parameters, $\mathbf{t}^0=(t^0,\dots,t^0)$ and

$$\mathcal{M}(\mathbf{t}^0)_{i,j=1}^N \sim \mu''(t^0)$$

$$\times\left[\log\frac{1}{D}\left[-(i-1)(N+1-i)\delta_{i,j-1} - i(N-i)\delta_{i,j+1} + [(i-1)(N+1-i)+i(N-i)]\delta_{i,j}\right] + 4\delta_{i,j}\right]$$

with $\delta_{N,0}=\delta_{1,N+1}=0$ (compare 7.13)).

Highlight 4: We determine all the eigenvalues of the matrix $\mathcal{M}(\mathbf{t}^0)$ (see Highlight 3) explicitly by a method based on exactly finding a finite-difference approximation to a suitable ordinary differential equation.

These eigenvalues are given by

$$\lambda_{n,\varepsilon} \sim -\varepsilon^2 \log \frac{1}{D} c_3 \mu''(t^0) n(n+1), \quad n = 1, \ldots, N-1.$$

Further, there is an eigenvalue of smaller size given by

$$\lambda_{0,\varepsilon} \sim -\varepsilon^2 4 c_3 \mu''(t^0)$$

(compare Lemma 13).

This implies that the spike cluster is stable with respect to small eigenvalues.

3. Preliminaries: scaling property, Green's function and eigenvalue problems

In this section we will provide some preliminaries which will be needed later for the existence and stability proofs.

Let w be the ground state solution given in (2.1). By a simple scaling argument, the function

$$w_a(y) = a w(\sqrt{a}\, y) \tag{3.1}$$

is the unique solution of the problem

$$\begin{cases} w_a'' - a w_a + w_a^2 = 0 \ \ y \in \mathbb{R}, \\ w_a > 0, \quad w_a(0) = \max_{y \in \mathbb{R}} w_a(y), \quad w_a(y) \to 0 \ \text{ as } |y| \to \infty. \end{cases} \tag{3.2}$$

We compute

$$\int_{\mathbb{R}} w_a^2(y)\, dy = a^{3/2} \int_{\mathbb{R}} w^2(z)\, dz, \ \int_{\mathbb{R}} w_a^3(y)\, dy = a^{5/2} \int_{\mathbb{R}} w^3(z) dz,$$
$$\int_{\mathbb{R}} (w_a')^2(y)\, dy = a^{5/2} \int_{\mathbb{R}} (w')^2(z)\, dz. \tag{3.3}$$

We set

$$K_D(|x-z|) = \frac{1}{2\sqrt{D}} e^{-\frac{1}{\sqrt{D}}|x-z|} \tag{3.4}$$

to be the singular part of $G_D(x,z)$. Let the regular part H_D of G_D be defined by $H_D = K_D - G_D$. Note that $H_D(x,z)$ belongs to the space $C^\infty((-1,1)\times(-1,1))$.

By (2.5) we have

$$G_D(t^0,t^0) = K_D(0)\left(1 + O\left(e^{-2(d_0-\eta_0)/\sqrt{D}}\right)\right), \tag{3.5}$$

where $d_0 = \min(1-t^0, t^0+1)$ and $\eta_0 > 0$ is an arbitrary but fixed constant.

For $\hat{\xi}^0$, we estimate

$$\hat{\xi}^0 = \frac{1}{2\sqrt{D} G_D(t^0,t^0)(\mu^0)^{3/2}} = \frac{1}{(\mu^0)^{3/2}} + O\left(e^{-2(d_0-\eta_0)/\sqrt{D}}\right) \tag{3.6}$$

by (3.4), (3.5).

Let us denote $\frac{\partial}{\partial t_i}$ as ∇_{t_i}. When $i \neq j$, we can define $\nabla_{t_i} G(t_i, t_j)$ in the classical way because the function is smooth. When $i = j$, then $K_D(|t_i - t_j|) = K_D(0) = \frac{1}{2\sqrt{D}}$ is a constant independent of t_i and we define

$$\nabla_{t_i} G_D(t_i, t_i) := - \frac{\partial}{\partial x}\Big|_{x=t_i} H(x, t_i).$$

Similarly, we define

$$\nabla_{t_i}\nabla_{t_j} G_D(t_i, t_j) = \begin{cases} -\frac{\partial}{\partial x}|_{x=t_i}\frac{\partial}{\partial y}|_{y=t_i} H_D(x,y) \ \text{ if } \ i=j, \\ \nabla_{t_i}\nabla_{t_j} G_D(t_i,t_j) \ \text{ if } \ i \neq j. \end{cases} \tag{3.7}$$

For convenience and clarity, we introduce a re-scaled version of the Green's function which has a finite limit as $D \to 0$. Thus we set

$$\hat{G}_D(x,z) = 2\sqrt{D} G_D(x,z), \tag{3.8}$$

$$\hat{K}_D(x,z) = 2\sqrt{D} K_D(x,z), \tag{3.9}$$

$$\hat{H}_D(x,z) = 2\sqrt{D} H_D(x,z). \tag{3.10}$$

Next we consider the stability of a system of nonlocal eigenvalue problems (NLEPs). We first recall the following result:

Theorem 5. *Consider the nonlocal eigenvalue problem*

$$\phi'' - \phi + 2w\phi - \gamma \frac{\int_{\mathbb{R}} w\phi \, dy}{\int_{\mathbb{R}} w^2 \, dy} w^2 = \alpha\phi. \tag{3.11}$$

(1) (Appendix E of [14]*.) If $\gamma < 1$, then there is a positive eigenvalue to (3.11).*
(2) (Theorem 1.4 of [36]*.) If $\gamma > 1$, then for any nonzero eigenvalue α of (3.11) we have*

$$Re(\alpha) \leq -c < 0.$$

(3) If $\gamma \neq 1$ and $\alpha = 0$, then

$$\phi = c_0 w'$$

for some constant c_0.

Next we consider the following system of nonlocal eigenvalue problems:

$$L\Phi := \Phi'' - \Phi + 2w\Phi - 2\frac{\int_{\mathbb{R}} w\Phi \, dy}{\int_{\mathbb{R}} w^2 \, dy} w^2, \tag{3.12}$$

where

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix} \in (H^2(\mathbb{R}))^N.$$

Set

$$L_0 u := u'' - u + 2wu, \tag{3.13}$$

where $u \in H^2(\mathbb{R})$.

Then the conjugate operator of L under the scalar product in $L^2(\mathbb{R})$ is given by

$$L^*\Psi = \Psi'' - \Psi + 2w\Psi - 2\frac{\int_{\mathbb{R}} w^2 \Psi \, dy}{\int_{\mathbb{R}} w^2 \, dy} w, \tag{3.14}$$

where

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix} \in (H^2(\mathbb{R}))^N.$$

Then we have the following result.

Lemma 6. [43] *We have*

$$\textit{Ker}(L) = X_0 \oplus X_0 \oplus \cdots \oplus X_0, \tag{3.15}$$

where

$$X_0 = \textit{span}\,\{w'(y)\}$$

and

$$\textit{Ker}(L^*) = X_0 \oplus X_0 \oplus \cdots \oplus X_0. \tag{3.16}$$

Proof. The system (3.12) is in diagonal form. Suppose

$$L\Phi = 0.$$

For $l = 1, 2, \ldots, N$ the l-th equation of system (3.12) is given by

$$\Phi_l'' - \Phi_l + 2w\Phi_l - 2\frac{\int_{\mathbb{R}} w\Phi_l \, dy}{\int_{\mathbb{R}} w^2 \, dy} w^2 = 0. \tag{3.17}$$

By Theorem 5 (3) with $\gamma = 2$, we have

$$\Phi_l \in X_0, \quad l = 1, \ldots, N \tag{3.18}$$

and (3.15) follows.

To prove (3.16), we proceed in a similar way for L^*. The l-th equation of (3.14) is given as follows:

$$\Psi_l'' - \Psi_l + 2w\Psi_l - 2\frac{\int_{\mathbb{R}} w^2\Psi_l \, dy}{\int_{\mathbb{R}} w^2 \, dy} w = 0. \tag{3.19}$$

Multiplying (3.19) by w and integrating, we obtain

$$\int_{\mathbb{R}} w^2 \Psi_l \, dy = 0.$$

Thus all the nonlocal terms vanish and we have

$$L_0\Psi_l = 0, \quad l = 1, \dots, N. \tag{3.20}$$

By Theorem 5 (3) with $\gamma = 0$, this implies

$$\Psi_l \in X_0, \quad l = 1, \dots, N.$$

□

As a consequence of Lemma 6, we have

Lemma 7. [43] *The operator*

$$L : (H^2(\mathbb{R}))^N \to (L^2(\mathbb{R}))^N,\ L\Phi = \Phi'' - \Phi + 2w\Phi - 2\frac{\int_{\mathbb{R}} w\Phi\, dy}{\int_{\mathbb{R}} w^2\, dy} w^2,$$

is invertible if it is restricted as follows

$$L : (X_0 \oplus \cdots \oplus X_0)^{\perp} \cap (H^2(\mathbb{R}))^N \to (X_0 \oplus \cdots \oplus X_0)^{\perp} \cap (L^2(\mathbb{R}))^N.$$

Moreover, L^{-1} is bounded.

Proof. This result follows from the Fredholm Alternative and Lemma 6.

□

Finally, we study the eigenvalue problem for L:

$$L\Phi = \alpha\Phi. \tag{3.21}$$

We have

Lemma 8. *For any nonzero eigenvalue α of (3.21) we have Re$(\alpha) \le -c < 0$.*

Proof. Let (Φ, α) satisfy the system (3.21). Suppose Re$(\alpha) \ge 0$ and $\alpha \ne 0$. Then the l-th equation of (3.21) becomes

$$\Phi_l'' - \Phi_l + 2w\Phi_l - 2\frac{\int_{\mathbb{R}} w\Phi_l}{\int_{\mathbb{R}} w^2} w^2 = \alpha\Phi_l.$$

By Theorem 5 (2) we conclude that

$$\text{Re}(\alpha) \le -c < 0.$$

□

Throughout the paper, let $C,\ c$ denote generic constants which may change from line to line.

4. EXISTENCE PROOF I: APPROXIMATE SOLUTIONS

Let $t^0 \in (-1,1)$ be a non-degenerate minimum point of the precursor inhomogeneity, i.e. we assume that (2.9) is satisfied. In this section, we construct an approximation to a spike cluster solution to (1.2) which concentrates at t^0.

The approximate cluster consists of the spikes $\mu_i w\left(\sqrt{\mu_i}\frac{x-t_i}{\varepsilon}\right)$ which are centered at the points t_i and have the scaling factors $\mu_i = \mu(t_i)$, where $i = 1, \dots, N$.

Let Ω_η denote the set of all $\mathbf{t} = (t_1, t_2, \dots, t_N) \in \Omega^N$ with $-1 < t_1 < t_2 \cdots < t_N < 1$ satisfying (4.1) and (4.2), where

$$\left| \frac{t_s - t_{s-1}}{\sqrt{D}} - \log\frac{1}{D} + \log\log\frac{1}{D} + \log\left(\frac{5\mu''(t^0)}{16\mu^0}\right) + \log[(s-1)(N+1-s)]\right| \le \eta \tag{4.1}$$

for $s = 2, \dots, N$,

$$\left| \frac{1}{N}\sum_{k=1}^{N} t_k - t^0 \right| \le \eta \log\frac{1}{D} \tag{4.2}$$

and $\eta > 0$ is a constant which is small enough and will be chosen in Section 5. The reason for assuming (4.1) and (4.2) will become clear in Section 5 when we solve the reduced problem.

We further denote

$$\mathbf{t}^0 = (t^0, t^0, \dots, t^0) \tag{4.3}$$

and set

$$\Omega_0 = \{\mathbf{t}^0\}. \tag{4.4}$$

To simplify our notation, for $\mathbf{t} \in \Omega_\eta$ and $k = 1, \dots, N$, we set

$$w_k(x) = \mu_k w\left(\sqrt{\mu_k}\,\frac{x - t_i}{\varepsilon}\right) \tag{4.5}$$

and

$$\tilde{w}_k(x) = \mu_k w\left(\sqrt{\mu_k}\,\frac{x - t_k}{\varepsilon}\right) \cdot \chi\left(\left|\frac{x - t_k}{\delta_\varepsilon}\right|\right), \tag{4.6}$$

where χ is a smooth cut-off function which satisfies the conditions

$$\chi(x) = 1 \text{ for } |x| < \frac{1}{2}, \quad \chi(x) = 0 \text{ for } |x| > \frac{3}{4}, \quad \chi \in C_0^\infty(\mathbb{R}) \tag{4.7}$$

and

$$\varepsilon \ll \delta_\varepsilon \ll \frac{10}{\sqrt{\mu^0}}\varepsilon \log\frac{1}{\varepsilon}. \tag{4.8}$$

Using (4.1), we have $|t_i - t^0| = O\left(\sqrt{D}\log\frac{1}{D}\right)$ for $i = 1, \dots, N$. This implies

$$|\mu(t_i) - \mu(t^0)| = O\left(D\left(\log\frac{1}{D}\right)^2\right), \tag{4.9}$$

$$\mu'(t_i) = \mu''(t^0)(t_i - t^0) + O\left(D\left(\log\frac{1}{D}\right)^2\right) = O\left(\sqrt{D}\log\frac{1}{D}\right), \tag{4.10}$$

$$\mu''(t_i) = \mu''(t^0) + O\left(\sqrt{D}\log\frac{1}{D}\right) = O(1),\ \mu''(t^0) = O(1). \tag{4.11}$$

$$\mu'''(t_i) = O(1),\ \mu'''(t^0) = O(1). \tag{4.12}$$

To simplify notation, we set

$$\mu_i = \mu(t_i), \quad i = 1, \dots, N. \tag{4.13}$$

Further, we compute, using (2.5),

$$\hat{G}_D(t_i, t_i) = \hat{K}_D(0)\left(1 + O(e^{-2(d_0-\eta_0)/\sqrt{D}})\right) = 1 + O(e^{-2(d_0-\eta_0)/\sqrt{D}}), \tag{4.14}$$

where $d_0 = \min(1 - t^0, t^0 + 1)$, $\eta_0 > 0$ is an arbitrary but fixed constant (compare (3.5)). We have

$$\hat{G}_D(t_i, t_s) = O\left(D\log\frac{1}{D}\right),\ \hat{K}_D(t_i, t_s) = O\left(D\log\frac{1}{D}\right) \text{ for } |i - s| = 1, \tag{4.15}$$

$$\hat{G}_D(t_i, t_s) = O\left(\left(D\log\frac{1}{D}\right)^2\right),\ \hat{K}_D(t_i, t_s) = O\left(\left(D\log\frac{1}{D}\right)^2\right) \text{ for } |i - s| = 2. \tag{4.16}$$

Generally, we have

$$\hat{G}_D(t_i, t_s) = O\left(\left(D\log\frac{1}{D}\right)^{|i-s|}\right),\ \hat{K}_D(t_i, t_s) = O\left(\left(D\log\frac{1}{D}\right)^{|i-s|}\right) \text{ for } |i - s| \geq 1. \tag{4.17}$$

For the derivatives, we estimate

$$\frac{\partial^k}{\partial t_i^k}\hat{G}_D(t_i, t_s) = O\left(\left(D\log\frac{1}{D}\right)^{|i-s|} D^{-k/2}\right) \text{ for } |i - s| \geq 1, \quad k = 1, 2, \dots \tag{4.18}$$

$$\frac{\partial^k}{\partial t_i^k}\hat{K}_D(t_i, t_s) = O\left(\left(D\log\frac{1}{D}\right)^{|i-s|} D^{-k/2}\right) \text{ for } |i - s| \geq 1, \quad k = 1, 2, \dots \tag{4.19}$$

and analogous results hold for the mixed derivatives.

By rescaling $\hat{A} = \xi_\varepsilon A,\ \hat{H} = \xi_\varepsilon H$ with ξ_ε defined in (2.8), it follows that (1.2) is equivalent to the following system for the rescaled functions $\hat{A}, \hat{H}$:

$$\begin{cases} \varepsilon^2 \hat{A}'' - \mu(x)\hat{A} + \frac{\hat{A}^2}{\hat{H}} = 0, \quad x \in (-1, 1), \\ D\hat{H}'' - \hat{H} + \xi_\varepsilon \hat{A}^2 = 0, \quad x \in (-1, 1), \\ \hat{A}(x) > 0, \hat{H}(x) > 0 \quad \text{in } (-1, 1), \\ \hat{A}'(-1) = \hat{A}'(1) = \hat{H}'(-1) = \hat{H}'(1) = 0. \end{cases} \tag{4.20}$$

From now on, we shall work with (4.20) and drop the hats. Next we rewrite (4.20) as a single equation with a nonlocal term.

For a function $A \in H^2(-1,1)$, we define $T[A]$ to be the solution of

$$\begin{cases} D(T[A])'' - T[A] + \xi_\varepsilon A^2 = 0, \quad -1 < x < 1, \\ (T[A])'(-1) = (T[A])'(1) = 0. \end{cases} \tag{4.21}$$

It is easy to see that the solution $T[A]$ is unique and positive. Then (4.20) becomes

$$\mathcal{S}_\varepsilon[A] := \varepsilon^2 A'' - A + \frac{A^2}{T[A]} = 0, \quad A > 0, \quad A'(-1) = A'(1) = 0. \tag{4.22}$$

For $\mathbf{t} \in \Omega_\eta$, we define an approximate solution to (4.22) as follows:

$$A(x) = w_{\varepsilon,\mathbf{t}}(x) = \sum_{k=1}^{N} \hat{\xi}_k \tilde{w}_k(x), \quad x \in \Omega, \tag{4.23}$$

where $\mathbf{t} \in \Omega_\eta$ and $\tilde{w}_k$ has been defined in (4.6).

Next we are now going to determine the amplitudes $\hat{\xi}_k$ to leading order. Let us first compute

$$\tau_s := T[w_{\varepsilon,\mathbf{t}}](t_s). \tag{4.24}$$

From (4.21), we have

$$\begin{aligned} \tau_s &= \xi_\varepsilon \int_{-1}^{1} G_D(t_s, z) w_{\varepsilon,\mathbf{t}}^2(z)\, dz \\ &= \xi_\varepsilon \int_{-1}^{1} G_D(t_s, z) \left[\sum_{k=1}^{N} \hat{\xi}_k^2 \tilde{w}_k^2(z) + \sum_{k \neq l} \hat{\xi}_k \hat{\xi}_l \tilde{w}_k(z) \tilde{w}_l(z) \right] dz = I_1, \end{aligned} \tag{4.25}$$

where I_1 is defined by the last equality.

We have

$$\xi_\varepsilon \int_\Omega G_D(t_s, x) \tilde{w}_k^2(x)\, dx = \xi_\varepsilon \int_\Omega G_D(t_s, x) \left(\mu_k w \left(\sqrt{\mu_k}\, \frac{x - t_k}{\varepsilon} \right) \right)^2 dx\, (1 + O(\varepsilon^{10})).$$

For $k \neq s$, we compute

$$\begin{aligned} \xi_\varepsilon \int_\Omega G_D(t_s, x) \tilde{w}_k^2(x)\, dx &= \xi_\varepsilon \varepsilon (\mu_k)^{3/2} G_D(t_s, t_k) \left[\int_{\mathbb{R}} w^2(y)\, dy + O\left(\frac{\varepsilon}{\sqrt{D}} \right) \right] \\ &= (\mu_k)^{3/2} \hat{G}_D(t_s, t_k) \left[1 + O\left(\frac{\varepsilon}{\sqrt{D}} \right) \right] \\ &= (\mu_0)^{3/2} \hat{G}_D(t_s, t_k) \left[1 + O\left(\frac{\varepsilon}{\sqrt{D}} + D \left(\log \frac{1}{D} \right)^2 \right) \right] \\ &= (\mu^0)^{3/2} O\left(D \log \frac{1}{D} \right) \left[1 + O\left(\frac{\varepsilon}{\sqrt{D}} + D \left(\log \frac{1}{D} \right)^2 \right) \right] \\ &= O\left(D \log \frac{1}{D} \right), \end{aligned} \tag{4.26}$$

using (2.8), (3.8), (4.9) and (4.17). For $k = s$, we have

$$\xi_\varepsilon \int_\Omega G_D(t_s, x) \tilde{w}_s^2(x)\, dx$$

$$= \xi_\varepsilon \int_\Omega \left[\frac{1}{2\sqrt{D}} e^{-|t_s - x|/\sqrt{D}} - H_D(t_s, x) \right] \tilde{w}_s^2(x)\, dx$$

$$= \xi_\varepsilon \varepsilon (\mu_s)^{3/2} G_D(t_s, t_s) \left[\int_{\mathbb{R}} w^2(y)\, dy + O\left(\frac{\varepsilon}{\sqrt{D}} \right) \right]$$

$$= (\mu_s)^{3/2} \hat{G}_D(t_s, t_s) + O\left(\frac{\varepsilon}{\sqrt{D}} \right)$$

$$= (\mu_0)^{3/2} \hat{G}_D(t_s, t_s) + O\left(\frac{\varepsilon}{\sqrt{D}} + D\left(\log \frac{1}{D} \right)^2 \right), \tag{4.27}$$

using (2.8), (3.8) and (4.9). Next, for $k \neq l$, we have

$$\xi_\varepsilon \int_\Omega \hat{G}_D(t_s, z) \tilde{w}_k(z) \tilde{w}_l(z)\, dz = 0 \tag{4.28}$$

by (4.6). Combining (4.26), (4.27) and (4.28), we have

$$I_1 = \sum_{k=1}^N \hat{\xi}_k^2 (\mu_k)^{3/2} \hat{G}_D(t_s, t_k) \left[1 + O\left(\frac{\varepsilon}{\sqrt{D}} \right) \right]$$

$$= \hat{\xi}_s^2 (\mu_s)^{3/2} \hat{G}_D(t_s, t_s) \left[1 + O\left(\frac{\varepsilon}{\sqrt{D}} + D \log \frac{1}{D} \right) \right]. \tag{4.29}$$

Substituting (4.29) into (4.25), we conclude that

$$T[w_{\varepsilon,\mathbf{t}}](t_s) = \tau_s = \hat{\xi}_s^2 (\mu_s)^{3/2} \hat{G}_D(t_s, t_s) \left[1 + O\left(\frac{\varepsilon}{\sqrt{D}} + D \log \frac{1}{D} \right) \right]$$

$$= \hat{\xi}_s^2 (\mu^0)^{3/2} \left[1 + O\left(\frac{\varepsilon}{\sqrt{D}} + D\left(\log \frac{1}{D} \right)^2 \right) \right]. \tag{4.30}$$

We now choose $\hat{\xi}_s$ such that $\tau_s = \hat{\xi}_s$. Then (4.30) has a unique solution which satisfies

$$\hat{\xi}_s = \frac{1}{(\mu^0)^{3/2}} \left[1 + O\left(\frac{\varepsilon}{\sqrt{D}} + D\left(\log \frac{1}{D} \right)^2 \right) \right]. \tag{4.31}$$

This concludes the construction of the approximate solution.

The next two parts of the existence proof have been moved to the appendix since they are quite technical and follow the approaches in previous papers:

Appendix A: Existence Proof II – error terms

Appendix B: Existence Proof III – Liapunov Schmidt reduction.

In the next section, we continue with the discussion of the reduced problem, which concludes the Proof of Theorem 1.

5. EXISTENCE PROOF IV: REDUCED PROBLEM

In this section, we solve the reduced problem. This completes the proof of our main existence result given by Theorem 1.

By Lemma 19, for every $\mathbf{t} \in \Omega_\eta$, there exists a unique solution $\phi_{\epsilon,\mathbf{t}} \in \mathcal{K}_{\epsilon,\mathbf{t}}^\perp$ such that

$$\mathcal{S}_\varepsilon[w_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}] = v_{\varepsilon,\mathbf{t}} \in \mathcal{C}_{\epsilon,\mathbf{t}}. \tag{5.1}$$

We need to determine $\mathbf{t}^\varepsilon = (t_1^\varepsilon, t_2^\varepsilon, \dots, t_N^\varepsilon) \in \Omega_\eta$ such that

$$\mathcal{S}_\varepsilon[w_{\varepsilon,\mathbf{t}^\varepsilon} + \phi_{\varepsilon,\mathbf{t}^\varepsilon}] \perp \mathcal{C}_{\epsilon,\mathbf{t}^\varepsilon}, \tag{5.2}$$

which implies $\mathcal{S}_\varepsilon[w_{\varepsilon,\mathbf{t}^\varepsilon} + \phi_{\varepsilon,\mathbf{t}^\varepsilon}] = 0$.

To this end, let

$$W_{\epsilon,s}(\mathbf{t}) := \frac{1}{\varepsilon\sqrt{D}\log\frac{1}{D}} \int_\Omega \mathcal{S}_\varepsilon[w_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}] \frac{d\tilde{w}_s}{dx}\,dx,$$
$$W_\epsilon(\mathbf{t}) := (W_{\epsilon,1}(\mathbf{t}), \dots, W_{\epsilon,N}(\mathbf{t})) : \Omega_\eta \to \mathbb{R}^N.$$

Then the map $W_\epsilon(\mathbf{t})$ is continuous in $\mathbf{t} \in \Omega_\eta$ and it remains to find a zero of the vector field $W_\varepsilon(\mathbf{t})$.

We compute

$$\frac{1}{\varepsilon\sqrt{D}\log\frac{1}{D}} \int_\Omega \mathcal{S}_\varepsilon[w_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}] \frac{d\tilde{w}_s}{dx}\,dx$$
$$= \frac{1}{\varepsilon\sqrt{D}\log\frac{1}{D}} \int_\Omega \left[\mathcal{S}_\varepsilon[w_{\varepsilon,\mathbf{t}}] + \mathcal{S}_\varepsilon'[w_{\varepsilon,\mathbf{t}}](\phi_{\varepsilon,\mathbf{t}}) + O(\|\phi_{\varepsilon,\mathbf{t}}\|^2_{H^2(I_\varepsilon)})\right] \frac{d\tilde{w}_s}{dx}\,dx.$$

We first compute the main term given by

$$\frac{1}{\varepsilon\sqrt{D}\log\frac{1}{D}} \int_\Omega \mathcal{S}_\varepsilon[w_{\varepsilon,\mathbf{t}}] \frac{d\tilde{w}_s}{dx}\,dx = c_s \tag{5.3}$$

Let $x = t_s + \varepsilon y$. By (9.10), we have

$$\frac{1}{\varepsilon\sqrt{D}\log\frac{1}{D}} \int_\Omega \mathcal{S}_\varepsilon[w_{\varepsilon,\mathbf{t}}] \frac{d\tilde{w}_s}{dx}\,dx = c_{s,1} + c_{s,2},$$

where

$$c_{s,1} = -\frac{1}{D\log\frac{1}{D}} \int_\Omega \frac{x - t_s}{\varepsilon} \tilde{w}_s^2 \frac{dw_s}{dx}\,dx \sum_{k\neq s} (\mu_k)^{3/2} \hat{\xi}_k^2 e^{-|t_s - t_k|/\sqrt{D}} \left(-\frac{t_s - t_k}{|t_s - t_k|}\right)$$
$$+O\left(\frac{\varepsilon}{\sqrt{D}} + e^{-2(d_0 - \eta_0)/\sqrt{D}}\right)$$
$$= \frac{1}{D\log\frac{1}{D}} \frac{1}{3} \int_{\mathbb{R}} w_{\mu_s}^3\,dy \sum_{k\neq s} (\mu_k)^{3/2} \hat{\xi}_k^2 e^{-|t_s - t_k|/\sqrt{D}} \left(-\frac{t_s - t_k}{|t_s - t_k|}\right)$$
$$+O\left(\frac{\varepsilon}{\sqrt{D}} + e^{-2(d_0 - \eta_0)/\sqrt{D}}\right)$$

$$= \frac{1}{D\log\frac{1}{D}}\frac{1}{3}(\mu_s)^{5/2}\int_{\mathbb{R}} w^3\,dy \sum_{k\neq s}(\mu_k)^{3/2}\hat{\xi}_k^2 e^{-|t_s-t_k|/\sqrt{D}}\left(-\frac{t_s-t_k}{|t_s-t_k|}\right)$$

$$+O\left(\frac{\varepsilon}{\sqrt{D}}+e^{-2(d_0-\eta_0)/\sqrt{D}}\right)$$

$$= \frac{2.4}{D\log\frac{1}{D}}(\mu_s)^{5/2}\sum_{k\neq s}(\mu_k)^{3/2}\hat{\xi}_k^2 e^{-|t_s-t_k|/\sqrt{D}}\left(-\frac{t_s-t_k}{|t_s-t_k|}\right)$$

$$+O\left(\frac{\varepsilon}{\sqrt{D}}+e^{-2(d_0-\eta_0)/\sqrt{D}}\right)$$

$$= \frac{2.4}{D\log\frac{1}{D}}(\mu_0)^4(\hat{\xi}^0)^2\sum_{k\neq s} e^{-|t_s-t_k|/\sqrt{D}}\left(-\frac{t_s-t_k}{|t_s-t_k|}\right)+O\left(\frac{\varepsilon}{\sqrt{D}}+\sqrt{D}\log\frac{1}{D}\right)$$

and

$$c_{s,2} = -\frac{1}{\varepsilon\sqrt{D}\log\frac{1}{D}}\mu''(t^0)(t_s-t^0)\hat{\xi}_s\varepsilon\int_\Omega \frac{x-t_s}{\varepsilon}\tilde{w}_s\frac{d\tilde{w}_s}{dx}\,dx+O\left(\frac{\varepsilon}{\sqrt{D}}+\sqrt{D}\log\frac{1}{D}\right)$$

$$= -\frac{1}{\sqrt{D}\log\frac{1}{D}}\mu''(t^0)(t_s-t^0)\hat{\xi}_s\int_{\mathbb{R}} y w_{\mu_s}\frac{dw_{\mu_s}}{dy}\,dy+O\left(\frac{\varepsilon}{\sqrt{D}}+\sqrt{D}\log\frac{1}{D}\right)$$

$$= \frac{1}{\sqrt{D}\log\frac{1}{D}}(\mu_s)^{3/2}\mu''(t^0)(t_s-t^0)\frac{1}{2}\hat{\xi}_s\int_{\mathbb{R}} w^2\,dy+O\left(\frac{\varepsilon}{\sqrt{D}}+\sqrt{D}\log\frac{1}{D}\right)$$

$$= \frac{2}{\sqrt{D}\log\frac{1}{D}}(\mu_s)^{3/2}\hat{\xi}_s\mu''(t^0)(t_s-t^0)+O\left(\frac{\varepsilon}{\sqrt{D}}+\sqrt{D}\log\frac{1}{D}\right)$$

$$= \frac{3}{\sqrt{D}\log\frac{1}{D}}(\mu^0)^{3/2}\hat{\xi}^0\mu''(t^0)(t_s-t^0)+O\left(\frac{\varepsilon}{\sqrt{D}}+\sqrt{D}\log\frac{1}{D}\right).$$

In summary, we have

$$c_s = \frac{2.4}{D\log\frac{1}{D}}(\mu_0)^4\sum_{k\neq s}(\hat{\xi}^0)^2 e^{-|t_s-t_k|/\sqrt{D}}\left(-\frac{t_s-t_k}{|t_s-t_k|}\right)$$

$$+3\sqrt{D}(\mu^0)^{3/2}\hat{\xi}^0\mu''(t^0)(t_s-t^0)+O\left(\frac{\varepsilon}{\sqrt{D}}+\sqrt{D}\log\frac{1}{D}\right),\quad s=1,\ldots,N. \tag{5.4}$$

Next we estimate

$$\frac{1}{\varepsilon\sqrt{D}\log\frac{1}{D}}\int_\Omega \mathcal{S}_\varepsilon'[w_{\varepsilon,\mathbf{t}}](\phi_{\varepsilon,\mathbf{t}})\frac{d\tilde{w}_s}{dx}\,dx$$

$$= \frac{1}{\varepsilon\sqrt{D}\log\frac{1}{D}}\int_\Omega\left[\varepsilon^2\phi_{\varepsilon,\mathbf{t}}''-\mu(x)\phi_{\varepsilon,\mathbf{t}}+\frac{2w_{\varepsilon,\mathbf{t}}}{T[w_{\varepsilon,\mathbf{t}}]}\phi_{\varepsilon,\mathbf{t}}-\frac{w_{\varepsilon,\mathbf{t}}^2}{(T[w_{\varepsilon,\mathbf{t}}])^2}(T'[w_{\varepsilon,\mathbf{t}}]\phi_{\varepsilon,\mathbf{t}})\right]\frac{d\tilde{w}_s}{dx}\,dx$$

$$= \frac{1}{\varepsilon\sqrt{D}\log\frac{1}{D}}\int_\Omega\left[\varepsilon^2\phi_{\varepsilon,\mathbf{t}}''-\mu_s\phi_{\varepsilon,\mathbf{t}}+\frac{2w_{\varepsilon,\mathbf{t}}}{T[w_{\varepsilon,\mathbf{t}}]}\phi_{\varepsilon,\mathbf{t}}-\frac{w_{\varepsilon,\mathbf{t}}^2}{(T[w_{\varepsilon,\mathbf{t}}])^2}(T'[w_{\varepsilon,\mathbf{t}}]\phi_{\varepsilon,\mathbf{t}})\right]\frac{d\tilde{w}_s}{dx}\,dx$$

$$+\frac{1}{\varepsilon\sqrt{D}\log\frac{1}{D}}\int_\Omega -(\mu(x)-\mu(t_s))\phi_{\varepsilon,\mathbf{t}}\frac{d\tilde{w}_s}{dx}\,dx$$

$$= \frac{1}{\varepsilon\sqrt{D}\log\frac{1}{D}}\int_\Omega\left[\left[\frac{1}{T[w_{\varepsilon,\mathbf{t}}]}-\frac{1}{\hat{\xi}_s}\right]2\hat{\xi}_s\tilde{w}_s\phi_{\varepsilon,\mathbf{t}}-\frac{(\hat{\xi}_s\tilde{w}_s)^2}{(T[w_{\varepsilon,\mathbf{t}}])^2}(T'[w_{\varepsilon,\mathbf{t}}]\phi_{\varepsilon,\mathbf{t}})\right]\frac{d\tilde{w}_s}{dx}\,dx$$
$$+\frac{1}{\varepsilon\sqrt{D}\log\frac{1}{D}}\int_\Omega -(\mu(x)-\mu(t_s))\phi_{\varepsilon,\mathbf{t}}\frac{d\tilde{w}_s}{dx}\,dx$$
$$= O\left(\frac{\varepsilon}{\sqrt{D}}\right)$$

which follows from (10.9) and (10.10). This implies

$$W_{\varepsilon,s}(\mathbf{t}) = \frac{2.4}{D\log\frac{1}{D}}(\mu_0)^4\sum_{k\neq s}(\hat{\xi}^0)^2 e^{-|t_s-t_k|/\sqrt{D}}\left(-\frac{t_s-t_k}{|t_s-t_k|}\right)$$
$$+3\sqrt{D}(\mu^0)^{3/2}\hat{\xi}^0\mu''(t^0)(t_s-t^0)+O\left(\frac{\varepsilon}{\sqrt{D}}+\sqrt{D}\log\frac{1}{D}\right),\quad s=1,\ldots,N. \tag{5.5}$$

Now, for given small $\varepsilon > 0$, we have to determine $\mathbf{t}^\varepsilon \in \Omega_\eta$ such that $W_{\varepsilon,s}(\mathbf{t}^\varepsilon) = 0$ for $s = 1,\ldots,N$.

We first consider the limiting case which only takes into account the leading terms and set

$$W_0(\mathbf{t}) = 2.4\frac{1}{D\log\frac{1}{D}}(\mu^0)^4(\hat{\xi}^0)^2\sum_{k,|k-s|=1} e^{-|t_s-t_k|/\sqrt{D}}\left(-\frac{t_s-t_k}{|t_s-t_k|}\right)$$
$$+3\sqrt{D}(\mu^0)^{3/2}\hat{\xi}^0\mu''(t^0)(t_s-t^0).$$

We compute $W_0(\mathbf{t}^*) = 0$, where $\mathbf{t}^*$ satisfies

$$\frac{t_s^*-t_{s-1}^*}{\sqrt{D}} = \log\frac{1}{D}-\log\log\frac{1}{D}$$
$$-\log\left(\frac{5\mu''(t^0)}{16\mu^0}\right)-\log[(s-1)(N+1-s)]+O\left(\frac{\log\log\frac{1}{D}}{\log\frac{1}{D}}\right), \tag{5.6}$$
$$\frac{1}{N}\sum_{k=1}^N t_k^* = t^0. \tag{5.7}$$

By (5.6) and (5.7), we have $\mathbf{t}^* \in \Omega_\eta$ if D is small enough.

We need to find $\mathbf{t}^\varepsilon \in \Omega_\eta$ such that $W_\varepsilon(\mathbf{t}^\varepsilon) = 0$.

Setting $\mathbf{e} = (1,1\ldots,1)^T$, we have

$$\frac{c}{\sqrt{D}\log\frac{1}{D}} \le \|DW_0(\mathbf{t}^*)\mathbf{e}\| \le \frac{C}{\sqrt{D}\log\frac{1}{D}}$$

and

$$\frac{c}{\sqrt{D}} \le \|DW_0(\mathbf{t}^*)\mathbf{v}\| \le \frac{C}{\sqrt{D}}\|\mathbf{v}\| \quad \text{if } \mathbf{v}\cdot\mathbf{e}=0.$$

For $\mathbf{t} \in \Omega_\varepsilon$, we expand

$$W_\varepsilon(\mathbf{t}) = W_\varepsilon(\mathbf{t}) - W_0(\mathbf{t}) + W_0(\mathbf{t}) - W_0(\mathbf{t}^*) + W_0(\mathbf{t}^*)$$
$$= O\left(\frac{\varepsilon}{\sqrt{D}}\right) \quad \text{by (5.4)}$$

$$+DW_0(\mathbf{t}^*)\cdot(\mathbf{t}-\mathbf{t}^*)+R_0(\mathbf{t}-\mathbf{t}^*)$$
$$+W_0(\mathbf{t}^*),$$

where $R_0(\tau)=D^2W_0(\mathbf{t}^*)(\tau,\tau)$. Decomposing $\tau=\mathbf{v}+\alpha\mathbf{e}$, where $\mathbf{v}\cdot\mathbf{e}=0$, we estimate

$$|R_0(\tau)|\leq\frac{c_4}{D}|\mathbf{v}|^2+\frac{c_5}{\sqrt{D}\log\frac{1}{D}}\alpha|\mathbf{v}|+\frac{c_6}{\sqrt{D}\log\frac{1}{D}}\alpha^2.$$

Noting that for $\mathbf{t}^*+\tau\in\Omega_\eta$, we have $|\mathbf{v}|\leq\eta\sqrt{D}$ and $\alpha\leq\eta\sqrt{D}\log\frac{1}{D}$, we get

$$|R_0(\tau)|\leq\eta^2\left(c_4+c_5\sqrt{D}+c_6\sqrt{D}\log\frac{1}{D}\right).$$

This implies

$$|(DW_0(\mathbf{t}^*))^{-1}R_0(\tau)\mathbf{v}|\leq c_7\eta^2\sqrt{D}$$

and

$$|(DW_0(\mathbf{t}^*))^{-1}R_0(\tau)\alpha\mathbf{e}|\leq c_7\eta^2\sqrt{D}\log\frac{1}{D}.$$

Setting $\tau=\mathbf{t}-\mathbf{t}^*$, we have to determine τ such that

$$-(DW_0(\mathbf{t}^*))^{-1}[W_\varepsilon(\mathbf{t}^*+\tau)-W_0(\mathbf{t}^*+\tau)+R_0(\tau)]=\tau$$

and so τ must be a fixed point of the mapping

$$\tau\to M_{\varepsilon,D}(\tau):=-(DW_0(\mathbf{t}^*))^{-1}[W_\varepsilon(\mathbf{t}^*+\tau)-W_0(\mathbf{t}^*+\tau)+R_0(\tau)],\quad B_1\to B_1,$$

where $B_1=\Omega_\eta-\mathbf{t}^*$ (pointwise). We estimate

$$\|M_{\varepsilon,D}(\tau)\|=\|-(DW_0(\mathbf{t}^*))^{-1}[W_\varepsilon(\mathbf{t}^*+\tau)-W_0(\mathbf{t}^*+\tau)+R_0(\tau)]\|$$
$$\leq C\left(\frac{\varepsilon}{\sqrt{D}}\cdot\sqrt{D}\log\frac{1}{D}+\eta^2\sqrt{D}\log\frac{1}{D}\right)\sqrt{D}\log\frac{1}{D}.$$

Using projections, we have

$$\|M_{\varepsilon,D}(\tau)\cdot\mathbf{v}\|\leq\ C(\frac{\varepsilon}{\sqrt{D}}+\eta^2\Big)\sqrt{D}\quad\text{ if }\mathbf{v}\cdot\mathbf{e}=0$$

and

$$\|M_{\varepsilon,D}(\tau)\cdot(\alpha\mathbf{e})\|\leq C\left(\frac{\varepsilon}{\sqrt{D}}+\eta^2\right)\sqrt{D}\log\frac{1}{D}.$$

We now determine when the mapping $M_{\varepsilon,D}$ maps from B_1 into B_1 for $\max\left(\frac{\varepsilon}{\sqrt{D}},D\right)$ small enough. We need to have

$$C\left(\frac{\varepsilon}{\sqrt{D}}+\eta^2\right)\leq\eta. \tag{5.8}$$

Now (5.8) is satisfied if we choose

$$\eta=2C\frac{\varepsilon}{\sqrt{D}} \tag{5.9}$$

and we assume

$$C\eta^2=4C^3\frac{\varepsilon^2}{D}\leq C\frac{\varepsilon}{\sqrt{D}}. \tag{5.10}$$

Note that (5.10) is satisfied if

$$\frac{\varepsilon}{\sqrt{D}} \leq \frac{1}{4C^2}$$

which holds if $\frac{\varepsilon}{\sqrt{D}}$ is small enough since $\frac{1}{4C^2}$ is a constant which is independent of ε and D.

By Brouwer's fixed point theorem, the mapping $M_{\varepsilon,D}$ possesses a fixed point $\tau^\varepsilon \in B_1$. Then $\mathbf{t}^\varepsilon = \mathbf{t}^* + \tau^\varepsilon \in \Omega_\eta$ is the desired solution which satisfies $W_\varepsilon(\mathbf{t}^\varepsilon) = 0$.

Thus we have proved the following proposition.

Proposition 9. *For* $\max\left(\frac{\epsilon}{\sqrt{D}}, D\right)$ *small enough, there exist points* $\mathbf{t}^\epsilon \in \Omega_\eta$ *with* $\mathbf{t}^\epsilon \to \mathbf{t}^0$ *such that* $W_\epsilon(\mathbf{t}^\varepsilon) = 0$.

Finally, we complete the proof of Theorem 1.

Proof. By Proposition 9, there exists $\mathbf{t}^\varepsilon \to \mathbf{t}^0$ such that $W_\varepsilon(\mathbf{t}^\varepsilon) = 0$. Written differently, we have $\mathcal{S}_\varepsilon[w_{\varepsilon,\mathbf{t}^\varepsilon} + \phi_{\varepsilon,\mathbf{t}^\varepsilon}] = 0$. Let $A_\varepsilon = \xi_\varepsilon(w_{\varepsilon,\mathbf{t}^\varepsilon} + \phi_{\varepsilon,\mathbf{t}^\varepsilon}), H_\varepsilon = \xi_\varepsilon T[w_{\varepsilon,\mathbf{t}^\varepsilon} + \phi_{\varepsilon,\mathbf{t}^\varepsilon}]$. By the Maximum Principle, $A_\varepsilon > 0, H_\varepsilon > 0$. Moreover $(A_\varepsilon, H_\varepsilon)$ satisfies all the properties of Theorem 1.

□

6. STABILITY PROOF I: LARGE EIGENVALUES

In this section, we study the large eigenvalues which satisfy $\lambda_\varepsilon \to \lambda_0 \neq 0$ as $\max\left(\frac{\varepsilon}{\sqrt{D}}, D\right) \to 0$.

First we consider the special case $\tau = 0$. Then we need to analyze the eigenvalue problem

$$\tilde{L}_{\varepsilon,\mathbf{t}^\varepsilon}\phi_\varepsilon = \varepsilon^2\Delta\phi_\varepsilon - \mu(x)\phi_\varepsilon + \frac{2A_\varepsilon\phi_\varepsilon}{T[A_\varepsilon]} - \frac{A_\varepsilon^2}{(T[A_\varepsilon])^2}(T'[A_\varepsilon]\phi_\varepsilon) = \lambda_\varepsilon\phi_\epsilon, \tag{6.1}$$

where λ_ε is some complex number, $A_\varepsilon = w_{\varepsilon,\mathbf{t}^\varepsilon} + \phi_{\varepsilon,\mathbf{t}^\varepsilon}$ with $\mathbf{t}^\varepsilon \in \Omega_\eta$ determined in the previous section,

$$\phi_\varepsilon \in H_N^2(\Omega). \tag{6.2}$$

and for $\phi \in L^2(\Omega)$ the function $T'[A]\phi$ is defined as the unique solution of

$$\begin{cases} D\Delta(T'[A]\phi) - (T'[A]\phi) + 2\xi_\varepsilon A\phi = 0, & -1 < x < 1, \\ (T'[A]\phi)'(-1) = (T'[A]\phi)'(1) = 0. \end{cases} \tag{6.3}$$

Because we study the large eigenvalues, there exists some small $c > 0$ such that $|\lambda_\varepsilon| \geq c > 0$ for $\max\left(\frac{\epsilon}{\sqrt{D}}, D\right)$ small enough. We are looking for a condition under which $\text{Re}(\lambda_\varepsilon) \leq c < 0$ for all eigenvalues λ_ε of (6.1), (6.2) if $\max\left(\frac{\varepsilon}{\sqrt{D}}, D\right)$ is small enough, where c is independent of ε and D. If $\text{Re}(\lambda_\varepsilon) \leq -c$, then λ_ε is a stable large eigenvalue. Therefore, for the rest of this section, we assume that $\text{Re}(\lambda_\varepsilon) \geq -c$ and study the stability properties of such eigenvalues.

We first derive the limiting problem of (6.1), (6.2) as $\max\left(\frac{\varepsilon}{\sqrt{D}}, D\right) \to 0$ which will be given by a system of NLEPs. Let us assume that

$$\|\phi_\varepsilon\|_{H^2(\Omega_\varepsilon)} = 1.$$

We cut off ϕ_ε as follows: Introduce

$$\phi_{\varepsilon,j}(y) = \phi_\varepsilon(y)\chi\left(\left|\frac{\varepsilon y}{\delta_\varepsilon}\right|\right), \tag{6.4}$$

where $y = (x - t_j)/\varepsilon$ for $x \in \Omega$, the cut-off function χ was introduced in (4.7) and δ_ε satisfies (4.8) .

From (6.1), (6.2), using Re$(\lambda_\varepsilon) \geq -c$ and $\|\phi_{\varepsilon,\mathbf{t}^\varepsilon}\|_{H^2(\Omega_\varepsilon)} = O\left(\frac{\varepsilon}{\sqrt{D}}\right)$, it follows that

$$\phi_\varepsilon = \sum_{j=1}^N \phi_{\varepsilon,j} + O\left(\frac{\varepsilon}{\sqrt{D}}\right) \quad \text{in } H^2(\Omega_\varepsilon). \tag{6.5}$$

Then, by a standard procedure, we extend $\phi_{\varepsilon,j}$ to a function defined on $\mathbb{R}$ such that

$$\|\phi_{\varepsilon,j}\|_{H^2(\mathbb{R})} \leq C\|\phi_{\varepsilon,j}\|_{H^2(\Omega_\varepsilon)}, \qquad j = 1, \ldots, N.$$

Since $\|\phi_\varepsilon\|_{H^2(\Omega_\varepsilon)} = 1$, $\|\phi_{\varepsilon,j}\|_{H^2(\Omega_\varepsilon)} \leq C$. By taking a subsequence, we may also assume that $\phi_{\varepsilon,j} \to \phi_j$ as $\max\left(\frac{\varepsilon}{\sqrt{D}}, D\right) \to 0$ in $H^1(\mathbb{R})$ for $j = 1, \ldots, N$.

Taking the limit $\max\left(\frac{\varepsilon}{\sqrt{D}}, D\right) \to 0$ with $\lambda_\varepsilon \to \lambda_0$ in (6.1) we get

$$\Delta_y\phi_i - \mu_i\phi_i + 2\tilde{w}_i\phi_i$$

$$-2\lim_{D\to 0}\left(\sum_{k=1}^N \hat{G}_D(t_i^0, t_k^0)\int_{\mathbb{R}} \hat{\xi}_k^0\tilde{w}_k\phi_k\,dy\right)\left(\sum_{k=1}^N \hat{G}_D(t_i^0, t_k^0)\int_{\mathbb{R}}\left(\hat{\xi}_k^0\tilde{w}_k\right)^2\,dy\right)^{-1}\tilde{w}_i^2 = \lambda_0\phi_i.$$

Using the transformation $\tilde{y} = \sqrt{\mu}\,y$ and the relations

$$\hat{G}_D(t_i^0, t_j^0) = \delta_{ik} + O\left(D\log\frac{1}{D}\right),$$

$$\hat{\xi}_k^0 = \frac{1}{(\mu^0)^{3/2}}\left[1 + O\left(D\log\frac{1}{D}\right)\right]$$

this implies that

$$\Delta_y\phi_i - \phi_i + 2w\phi_i - \frac{2\int_{\mathbb{R}} w\phi_i\,dy}{\int_{\mathbb{R}} w^2\,dy}w^2 = \lambda_0\phi_i, \quad i = 1, \ldots, N, \tag{6.6}$$

where $\phi_i \in H^2(\mathbb{R}^N)$.

Then we have

Theorem 10. *Let λ_ε be an eigenvalue of (6.1) and (6.2) such that Re$(\lambda_\varepsilon) > -c$ for some $c > 0$.*

(1) Suppose that (for suitable sequences $\max\left(\frac{\varepsilon_n}{\sqrt{D_n}}, D_n\right) \to 0$) we have $\lambda_{\varepsilon_n} \to \lambda_0 \neq 0$. Then λ_0 is an eigenvalue of the problem (NLEP) given in (6.6).

(2) Let $\lambda_0 \neq 0$ *with Re*$(\lambda_0) > 0$ *be an eigenvalue of the problem (NLEP) given in (6.6). Then for* $\max\left(\frac{\varepsilon}{\sqrt{D}}, D\right)$ *small enough, there is an eigenvalue* λ_ε *of (6.1) and (6.2) with* $\lambda_\varepsilon \to \lambda_0$ *as* $\max\left(\frac{\varepsilon}{\sqrt{D}}, D\right) \to 0$.

Proof. (1) of Theorem 10 follows by asymptotic analysis similar to Appendix B.

To prove (2) of Theorem 10, we follow a compactness argument of Dancer [4]. The main idea of his approach is as follows: Let $\lambda_0 \neq 0$ be an eigenvalue of problem (6.6) with Re$(\lambda_0) > 0$.

Then can we rewrite (6.1) as follows:

$$\phi_\varepsilon = -R_\varepsilon(\lambda_\varepsilon)\left[\frac{2A\phi_\varepsilon}{T[A]} - \frac{A^2}{T[A]}T'[A]\phi_\varepsilon\right], \tag{6.7}$$

where $R_\varepsilon(\lambda_\varepsilon)$ is the inverse of $-\Delta+(\mu(x)+\lambda_\varepsilon)$ in $H^2(\mathbb{R})$ (which exists if Re$(\lambda_\varepsilon) > -\min_{x\in\mathbb{R}} \mu(x)$ or Im$(\lambda_\varepsilon) \neq 0$) and the nonlocal operators have been defined in (4.21) and (6.3), respectively.

The main property is that $R_\varepsilon(\lambda_\varepsilon)$ is a compact operator if $\max\left(\frac{\varepsilon}{\sqrt{D}}, D\right)$ is small enough. The rest of the argument follows in the same way as in [4].

□

We now study the stability of (6.1), (6.2) for large eigenvalues explicitly and prove Theorem 2.

By Lemma 8, for any nonzero eigenvalue λ_0 in (6.6) we have

$$\text{Re}(\lambda_0) \leq c_0 < 0 \qquad \text{for some } c_0 > 0.$$

Thus by Theorem 10 (1), for $\max\left(\frac{\varepsilon}{\sqrt{D}}, D\right)$ small enough, all nonzero large eigenvalues of (6.1), (6.2) have strictly negative real parts. More precisely, all eigenvalues λ_ε of (6.1), (6.2), for which $\lambda_\varepsilon \to \lambda_0 \neq 0$ holds, satisfy Re$(\lambda_\varepsilon) \leq -c < 0$.

When studying the case $\tau > 0$, we have to deal with nonlocal eigenvalue problems as in (3.11), for which the coefficient γ of the nonlocal term is a function of $\tau\alpha$. Let $\gamma = \gamma(\tau\alpha)$ be a complex function of $\tau\alpha$. Let us suppose that

$$\gamma(0) \in \mathbb{R}, \quad |\gamma(\tau\alpha)| \leq C \ \text{ for Re}(\alpha) = \alpha_R \geq 0,\ \tau \geq 0, \tag{6.8}$$

where C is a generic constant which is independent of τ and α. In our case the following simple example of a function $\gamma(\tau\alpha)$ satisfying (6.8) is relevant:

$$\gamma(\alpha) = \frac{2}{\sqrt{1+\tau\alpha}},$$

where $\sqrt{1+\tau\alpha}$ denotes the principal branch of the square root function, compare [35].

Now we have

Lemma 11. [43] *Consider the nonlocal eigenvalue problem*

$$\phi'' - \phi + 2w\phi - \gamma(\tau\alpha)\frac{\int_{\mathbb{R}} w\phi\, dy}{\int_{\mathbb{R}} w^2\, dy} w^2 = \alpha\phi, \tag{6.9}$$

where $\gamma(\tau\alpha)$ satisfies (6.8). Then there is a small number $\tau_0 > 0$ such that for $\tau < \tau_0$,
(1) if $\gamma(0) < 1$, then there is a positive eigenvalue to (3.11);
(2) if $\gamma(0) > 1$, then for any nonzero eigenvalue α of (6.9), we have

$$Re(\alpha) \le -c_0 < 0.$$

Proof. Lemma 11 follows from Theorem 5 by a regular perturbation argument. To make sure that the perturbation argument works, we have to show that if $\alpha_R \ge 0$ and $0 \le \tau < 1$, then $|\alpha| \le C$, where C is a generic constant which is independent of τ. In fact, multiplying (6.9) by the conjugate $\bar{\phi}$ of ϕ and integration by parts, we obtain that

$$\int_{\mathbb{R}} (|\phi'|^2 + |\phi|^2 - 2w|\phi|^2)\, dy = -\alpha \int_{\mathbb{R}} |\phi|^2\, dy - \gamma(\tau\alpha)\frac{\int_{\mathbb{R}} w\phi\, dy}{\int_{\mathbb{R}} w^2\, dy} \int_{\mathbb{R}} w^2\bar{\phi}\, dy. \tag{6.10}$$

From the imaginary part of (6.10), we obtain that

$$|\alpha_I| \le C_1 |\gamma(\tau\alpha)|,$$

where $\alpha = \alpha_R + \sqrt{-1}\alpha_I$ and C_1 is a positive constant (independent of τ). By assumption (6.8), $|\gamma(\tau\alpha)| \le C$ and so $|\alpha_I| \le C$. Taking the real part of (6.10) and noting that

$$\text{l.h.s. of (6.10)} \ge C \int_{\mathbb{R}} |\phi|^2 \quad \text{for some } C \in \mathbb{R},$$

we obtain that $\alpha_R \le C_2$, where C_2 is a positive constant (independent of $\tau > 0$). Therefore, $|\alpha|$ is uniformly bounded and hence a perturbation argument gives the desired conclusion.

□

Now Theorem 10 can be extended to the case $\tau > 0$ for eigenvalues such that $\mathrm{Re}(\tau\lambda_\varepsilon) \ge -\frac{1}{2}$. Then by Lemma 11 it follows that for $0 \le \tau < \tau_0$ all eigenvalues λ_ε of (6.1), (6.2), for which $\lambda_\varepsilon \to \lambda_0 \ne 0$ holds, satisfy $\mathrm{Re}(\lambda_\varepsilon) \le -c < 0$.

For $\tau \ge 0$, the large eigenvalues in the limit are determined explicitly by the following result from [47]:

Lemma 12. *Let $\lambda = \sqrt{-1}\lambda_I$ be an eigenvalue of the problem*

$$\Delta\phi - \phi + 2w\phi - \frac{2}{\sqrt{1+\tau\lambda}} \frac{\int_{\mathbb{R}} w\phi}{\int_{\mathbb{R}} w^2} w^2 = \lambda\phi, \phi \in H^1(\mathbb{R}), \tag{6.11}$$

where

$$\tau \ge 0,\ \lambda \in \mathbb{C},\ \lambda = \lambda_R + i\lambda_I, \lambda_R \ge 0$$

and we take the principal branch of $\sqrt{1+\tau\lambda}$. *Then* λ *is a solution of the algebraic equation*

$$\frac{\sqrt{1+\tau\lambda}}{2}-1 = -{}_4F_3\left\{\begin{matrix} 1, & 3, & -\frac{1}{2}, & 2 & ; \\ & & & & 1 \\ & 2+\gamma, & 2-\gamma, & \frac{5}{2} & ; \end{matrix}\right\}$$
$$+\frac{2\lambda}{3}b_1\frac{\Gamma(1+\gamma)\Gamma(\frac{5}{2})}{\Gamma(\gamma+\frac{3}{2})}{}_3F_2\left\{\begin{matrix} 2+\gamma, & -\frac{3}{2}+\gamma, & 1+\gamma & ; \\ & & & 1 \\ & 1+2\gamma, & \frac{3}{2}+\gamma & ; \end{matrix}\right\}, \quad (6.12)$$

where $\gamma=\sqrt{1+\lambda}$ *and* b_1 *is given by*

$$b_1=\frac{9}{24}\frac{(\gamma-1)^3\gamma}{(\gamma-3/2)(\gamma-1/2)2^{2\gamma}}\frac{\pi}{\sin(\pi(\gamma-1))}. \quad (6.13)$$

Here for two sequences $a_1,a_2,\dots,a_A$ and $b_1,b_2,\dots,b_B$ we let the series

$$1+\frac{a_1a_2\cdots a_A}{b_1b_2\cdots b_B}\frac{z}{1!}+\frac{(a_1+1)(a_2+1)\cdots(a_A+1)}{(b_1+1)(b_2+1)\cdots(b_B+1)}\frac{z^2}{2!}+\cdots \quad (6.14)$$
$$=:{}_AF_B\left\{\begin{matrix} a_1, & a_2, & \dots, & a_A & ; \\ & & & & z \\ b_1, & b_2, & \dots, & b_B & ; \end{matrix}\right\}$$

be the generalized Gauss function or generalized hypergeometric function.

In conclusion, we have finished the study of the large eigenvalues (of order $O(1)$) and derived results on their stability properties.

It remains to study the small eigenvalues (of order $o(1)$) which will be done in the next section.

7. Stability proof II: characterization of small eigenvalues

Now we study the eigenvalue problem (6.1), (6.2) with respect to small eigenvalues. Namely, we assume that $\lambda_\varepsilon\to 0$ as $\max\left(\frac{\varepsilon}{\sqrt{D}},D\right)\to 0$.

Let

$$\bar{w}_\varepsilon=w_{\varepsilon,\mathbf{t}^\varepsilon}+\phi_{\varepsilon,\mathbf{t}^\varepsilon},\quad \bar{H}_\varepsilon=T[w_{\varepsilon,\mathbf{t}^\varepsilon}+\phi_{\varepsilon,\mathbf{t}^\varepsilon}], \quad (7.1)$$

where $\mathbf{t}^\varepsilon=(t_1^\varepsilon,\dots,t_N^\varepsilon)\in\Omega_\eta$.

After re-scaling, the eigenvalue problem (6.1), (6.2) becomes

$$\begin{cases} \varepsilon^2\Delta\phi_\varepsilon-\mu(x)\phi_\varepsilon+2\frac{\bar{w}_\varepsilon}{\bar{H}_\varepsilon}\phi_\epsilon-\frac{\bar{w}_\varepsilon^2}{\bar{H}_\varepsilon^2}\psi_\varepsilon=\lambda_\varepsilon\phi_\epsilon, \\ D\Delta\psi_\varepsilon-\psi_\varepsilon+2\xi_\varepsilon\bar{w}_\varepsilon\phi_\epsilon=\lambda_\varepsilon\tau\psi_\varepsilon, \\ \phi_\varepsilon'(-1)=\phi_\varepsilon'(1)=\psi_\varepsilon'(-1)=\psi_\varepsilon'(1)=0. \end{cases} \quad (7.2)$$

Throughout this section, we denote

$$\mu_j=\mu(t_j^\varepsilon),\quad \mu_j'=\mu'(t_j^\varepsilon),\quad \mu_j''=\mu''(t_j^\varepsilon).$$

By the implicit function theorem, there exists a (locally) unique solution $\hat{\xi}(\mathbf{t}) = (\hat{\xi}_1(\mathbf{t}), \ldots, \hat{\xi}_N(\mathbf{t}))$ of the equation

$$\sum_{j=1}^{N} \hat{G}_D(t_i, t_j)\hat{\xi}_j^2 \mu_j^{3/2} = \hat{\xi}_i, \quad i = 1, \ldots, N. \tag{7.3}$$

Moreover, $\hat{\xi}(\mathbf{t})$ is C^1 for $\mathbf{t} \in \Omega_\eta$.

We have the estimates

$$\hat{\xi}(\mathbf{t}^\varepsilon) = O(1), \quad \hat{\xi}_i(\mathbf{t}^\varepsilon) - \hat{\xi}_j(\mathbf{t}^\varepsilon) = O\left(D\left(\log\frac{1}{D}\right)^2\right).$$

As a preparation, we first compute the derivatives of $\hat{\xi}(\mathbf{t})$.

Now from (7.3) we calculate

$$\nabla_{t_j}\hat{\xi}_i = 2\sum_{l=1}^{N} \hat{G}_D(t_i, t_l)\hat{\xi}_l \mu_l^{3/2} \nabla_{t_j}\hat{\xi}_l + \frac{\partial}{\partial t_j}(\hat{G}_D(t_i, t_j))\hat{\xi}_j^2 \mu_j^{3/2} + \frac{3}{2}\hat{G}_D(t_i, t_j)\hat{\xi}_j^2 \mu_j^{1/2}\mu_j' \quad \text{for } i \neq j,$$

$$\begin{aligned}
\nabla_{t_i}\hat{\xi}_i &= 2\sum_{l=1}^{N} \hat{G}_D(t_i, t_l)\hat{\xi}_l \mu_l^{3/2} \nabla_{t_i}\hat{\xi}_l + \sum_{l=1}^{N} \frac{\partial}{\partial t_i}(\hat{G}_D(t_i, t_l))\hat{\xi}_l^2 \mu_l^{3/2} \\
&\quad + \frac{3}{2}\hat{G}_D(t_i, t_i)\hat{\xi}_i^2 \mu_i^{1/2}\mu_i' \\
&= 2\sum_{l=1}^{N} \hat{G}_D(t_i, t_l)\hat{\xi}_l \mu_l^{3/2} \nabla_{t_i}\hat{\xi}_l + \nabla_{t_i}\hat{G}_D(t_i, t_i)\hat{\xi}_i^2 \mu_i^{3/2} - \frac{5}{4}\hat{\xi}_i \frac{\mu'(t_i)}{\mu(t_i)} \\
&\quad + \frac{3}{2}\hat{G}_D(t_i, t_i)\hat{\xi}_i^2 \mu_i^{1/2}\mu_i' + F_i(\mathbf{t}) \qquad (7.4) \\
&= 2\hat{G}_D(t_i, t_i)\hat{\xi}_i \mu_i^{3/2} \nabla_{t_i}\hat{\xi}_i - \frac{5}{4}\hat{\xi}_i \frac{\mu'(t_i)}{\mu(t_i)} + F_i(\mathbf{t}) + O\left(\sqrt{D}\log\frac{1}{D}\right) \quad i = 1, \ldots, N.
\end{aligned}$$

Here $F(\mathbf{t})$ is the vector field

$$F(\mathbf{t}) = (F_1(\mathbf{t}), \ldots, F_N(\mathbf{t})),$$

where

$$F_i(\mathbf{t}) = \frac{5}{4}\hat{\xi}_i \frac{\mu'(t_i)}{\mu_i} + \sum_{l=1}^{N} \nabla_{t_i}\hat{G}_D(t_i, t_l)\hat{\xi}_l^2 \mu_l^{3/2}, \quad i = 1, \ldots, N. \tag{7.5}$$

We compute

$$F_i(\mathbf{t}) = \frac{5}{4}\hat{\xi}_i \frac{\mu'(t_i)}{\mu_i} + \sum_{l, |l-i|=1} \nabla_{t_i}\hat{K}_D(t_i, t_l)\hat{\xi}_l^2 \mu_l^{3/2} + O\left(D^{3/2}\log\frac{1}{D}\right), \quad i = 1, \ldots, N, \tag{7.6}$$

by (3.5), (4.17).

Thus (7.4) implies that

$$\nabla_{\mathbf{t}}\hat{\xi}(\mathbf{t}) = O\left(\sqrt{D}\log\frac{1}{D}\right). \tag{7.7}$$

Set

$$(\mathcal{M}(\mathbf{t}))_{i,j} = \left(\frac{\partial F_i(\mathbf{t})}{\partial t_j}\right). \tag{7.8}$$

By the reduced problem (see Section 5), we have $F(\mathbf{t}^\varepsilon) = 0$ at $\mathbf{t}^\varepsilon = (t_1^\varepsilon, \ldots, t_N^\varepsilon)$. In addition, if $\mathcal{M}(\mathbf{t}^\varepsilon))$ is positive definite, then we will show that all small eigenvalues have negative real part when $0 \le \tau < \tau_0$ for some $\tau_0 > 0$.

Next we compute $\mathcal{M}(\mathbf{t})$ using (7.4).

For $i = j$, we have

$$\begin{aligned}
&\sum_{l=1}^N \nabla_{t_i}^2 \hat{G}_D(t_i, t_l)\hat{\xi}_l^2 \mu_l^{3/2} \\
&= \sum_{l=1}^N \nabla_{t_i}^2 \hat{K}_D(t_i, t_l)\hat{\xi}_l^2 \mu_l^{3/2} \left(1 + O(e^{-2(d_0-\eta_0)\sqrt{D}})\right) \\
&= \left[\nabla_{t_i}^2 \hat{K}_D(t_i, t_{i-1})\hat{\xi}_{i-1}^2 \mu_{i-1}^{3/2} + \nabla_{t_i}^2 \hat{K}_D(t_i, t_{i+1})\hat{\xi}_{i+1}^2 \mu_{i+1}^{3/2}\right] \left(1 + O\left(D \log \frac{1}{D}\right)\right) \\
&= \left[\nabla_{t_i}^2 \hat{K}_D(t_i, t_{i-1})(\hat{\xi}^0)^2(\mu^0)^{3/2} + \nabla_{t_i}^2 \hat{K}_D(t_i, t_{i+1})(\hat{\xi}^0)^2(\mu^0)^{3/2}\right] \left[1 + O\left(D\left(\log \frac{1}{D}\right)^2\right)\right].
\end{aligned} \tag{7.9}$$

For $|i-j| = 1$, we compute in case $j = i-1$

$$\begin{aligned}
&\sum_{l=1}^N \nabla_{t_{i-1}}(\nabla_{t_i} \hat{G}_D(t_i, t_l))\hat{\xi}_l^2 \mu_l^{3/2} \\
&= \sum_{l=1}^N \nabla_{t_{i-1}}(\nabla_{t_i} \hat{K}_D(t_i, t_l))\hat{\xi}_l^2 \mu_l^{3/2} \left[1 + O(e^{-2(d_0-\eta_0)\sqrt{D}})\right] \\
&= \nabla_{t_{i-1}}(\nabla_{t_i} \hat{K}_D(t_i, t_{i-1}))\hat{\xi}_{i-1}^2 \mu_{i-1}^{3/2} \left[1 + O\left(D \log \frac{1}{D}\right)\right] \\
&= \nabla_{t_{i-1}}(\nabla_{t_i} \hat{K}_D(t_i, t_{i-1}))(\hat{\xi}^0)^2(\mu^0)^{3/2} \left[1 + O\left(D\left(\log \frac{1}{D}\right)^2\right)\right] \\
&= -\nabla_{t_i}^2 \hat{K}_D(t_i, t_{i-1})(\hat{\xi}^0)^2(\mu^0)^{3/2} \left[1 + O\left(D\left(\log \frac{1}{D}\right)^2\right)\right]
\end{aligned} \tag{7.10}$$

and a similar result holds for $j = i+1$. For $|i-j| \ge 2$, we have

$$\sum_{l=1}^N \nabla_{t_j}\left(\nabla_{t_i} \hat{G}_D(t_i, t_l)\right)\hat{\xi}_l^2 \mu_l^{3/2} = O\left(D\left(\log \frac{1}{D}\right)^2\right). \tag{7.11}$$

This implies

$$\mathcal{M}(\mathbf{t}^\varepsilon) = (m_{ij}(\mathbf{t}^\varepsilon))_{i,j=1}^N = (m_{ij}(\mathbf{t}^0))_{i,j=1}^N \left[1 + O\left(D\left(\log \frac{1}{D}\right)^2\right)\right], \tag{7.12}$$

where

$$m_{ij}(\mathbf{t}) = (\hat{\xi}^0)^2(\mu^0)^{3/2}\Bigg[\nabla^2_{t_i}[\hat{K}_D(t_i,t_{i-1})+\hat{K}_D(t_i,t_{i+1})]\delta_{i,j}$$

$$-\nabla^2_{t_i}\hat{K}_D(t_i,t_{i-1})\delta_{i,j+1}-\nabla^2_{t_i}\hat{K}_D(t_i,t_{i+1})\delta_{i,j-1}\Bigg]+\frac{5}{4}\hat{\xi}_i\frac{\mu_i''}{\mu_i}\delta_{i,j}$$

$$+2\sum_{j=1}^N\nabla_{t_i}\hat{K}_D(t_i,t_{i-1})\hat{\xi}_{i-1}\nabla_{t_j}\hat{\xi}_{i-1}\mu_{i-1}^{3/2}+2\sum_{j=1}^N\nabla_{t_i}\hat{K}_D(t_i,t_{i+1})\hat{\xi}_{i+1}\nabla_{t_j}\hat{\xi}_{i+1}\mu_{i+1}^{3/2}$$

$$+\frac{3}{2}\nabla_{t_i}\hat{K}_D(t_i,t_{i-1})\hat{\xi}_{i-1}^2\mu_{i-1}^{1/2}\mu_{i-1}'\delta_{i,j+1}+\frac{3}{2}\nabla_{t_i}\hat{K}_D(t_i,t_{i+1})\hat{\xi}_{i+1}^2\mu_{i+1}^{1/2}\mu_{i+1}'\delta_{i,j-1}$$

$$+\frac{5}{4}\left[\nabla_{t_i}\hat{\xi}_i\frac{\mu_i'}{\mu_i}-\hat{\xi}_i\frac{(\mu_i')^2}{\mu_i^2}\right]\delta_{i,j}.$$

Therefore, using (5.6), (5.7) and the estimate (7.7) we have

$$m_{ij}(\mathbf{t}^\varepsilon) = \frac{5}{16}(\hat{\xi}^0)^2(\mu_0)^{1/2}\mu''(t_i^\varepsilon)\log\frac{1}{D}$$

$$\times\Bigg[-(i-1)(N+1-i)\delta_{j,i-1}-i(N-i)\delta_{j,i+1}+[(i-1)(N+1-i)+i(N-i)]\delta_{i,j}\Bigg]$$

$$+\frac{5}{4}\hat{\xi}^0(\mu^0)^{-1}\mu''(t_i^\varepsilon)\delta_{i,j}+O\left(D\left(\log\frac{1}{D}\right)^2\right)$$

$$=\frac{5}{16}(\hat{\xi}^0)^2(\mu_0)^{1/2}\mu''(t^0)$$

$$\times\left[\log\frac{1}{D}\left[-(i-1)(N+1-i)\delta_{i,j-1}-i(N-i)\delta_{i,j+1}+[(i-1)(N+1-i)+i(N-i)]\delta_{i,j}\right]+4\delta_{i,j}\right]$$

$$+O\left(\sqrt{D}\log\frac{1}{D}\right). \tag{7.13}$$

The matrix $\mathcal{M}(\mathbf{t}^\varepsilon)$ will be the leading-order contribution to the small eigenvalues (compare Lemma 20 and the comments following it). Thus we study the spectrum of the symmetric $N\times N$-matrix $\mathcal{A}$ defined by

$$a_{s,s}=(s-1)(N-s+1)+s(N-s),\quad s=1,\dots,N, \tag{7.14}$$

$$a_{s,s+1}=a_{s+1,s}=-s(N-s),\quad s=1,\dots,N-1,$$

$$a_{s,t}=0,\quad |s-t|>1.$$

We will show

Lemma 13. *The eigenvalues of the matrix $\mathcal{A}$ are given by*

$$\lambda_n=n(n+1),\quad n=0,1,\dots,N-1. \tag{7.15}$$

The corresponding eigenvectors are computed recursively from (7.17).

The matrix $\mathcal{A}$ has eigenvalue $\lambda_1 = 0$ with eigenvector $v_1 = \mathbf{e}$. To compute the other eigenvalues and eigenvectors of $\mathcal{A}$, we remark that this problem is equivalent to finding a suitable finite-difference approximation $\tilde{u}$ of the differential equation

$$h^2 x(1-x)u'' + \lambda u = 0, \quad u'(0) = u'(1) = 0 \tag{7.16}$$

in the interval $(0,1)$ for uniform stepsize $h = \frac{1}{N}$.

More precisely, we identify

$$v_{ik} = \tilde{u}(x_{k-1/2}) \text{ with } x_k = \frac{k}{N} \text{ and } x_{k-1/2} = \frac{k-1/2}{N} \text{ for } k = 1, \ldots, N,$$

where in (7.16) we replace $x(1-x)u''(x)$ by

$$\frac{1}{h^2}\Bigg[x_{k-1}(1-x_{k-1})\tilde{u}(x_{k-3/2}) + x_k(1-x_k)\tilde{u}(x_{k+1/2})$$
$$-[x_k(1-x_k) + x_{k-1}(1-x_{k-1})]\tilde{u}(x_{k-1/2})\Bigg]$$
$$= (k-1)(N-k+1)\tilde{u}(t_{k-3/2}) + k(N-k)\tilde{u}(t_{k+1/2})$$
$$-[(k-1)(N-k+1) + k(N-k)]\tilde{u}(t_{k-1/2}).$$

To determine the eigenvectors v_i, we have to solve this finite-difference problem exactly. We assume that the solutions are given by polynomials of degree n (which will be shown later and n will be specified). Using Taylor expansion around $x = x_{k-1/2}$ and the identities

$$x_{k-1}(1-x_{k-1}) - x_k(1-x_k) = -h(1-2x_{k-1/2})$$

and

$$x_{k-1}(1-x_{k-1}) + x_k(1-x_k) = 2x_{k-1/2}(1-x_{k-1/2}) - \frac{h^2}{2},$$

the finite-difference problem is equivalent to

$$\left(2x(1-x) - \frac{h^2}{4}\right)\sum_{l=1}^{[n/2]} \frac{h^{2l-2}}{(2l)!}\tilde{u}^{(2l)}(x)$$
$$+(1-2x)\sum_{l=1}^{[n/2]} \frac{h^{2l-2}}{(2l-1)!}\tilde{u}^{(2l-1)}(x) + \lambda_n \tilde{u}(x) = 0, \quad n = 0, \ldots, N-1.$$

Substituting the ansatz

$$\tilde{u}(x) = \sum_{k=0}^{n} a_k x^k$$

into this equation, considering the coefficient of the power x^k, $k = 0, \ldots, n$, implies that

$$(\lambda_n - k(k+1))a_k + (k+1)^2 a_{k+1}$$
$$+ \sum_{l=2}^{[n/2]+1} 2\frac{h^{2l-2}}{(2l)!}\frac{(k+2l-1)!}{(k-1)!}\left[\frac{k+l}{k}a_{k+2l-1} - a_{k+2l-2}\right]$$

$$-\sum_{l=1}^{[n/2]} \frac{h^{2l}}{2(2l)!}\frac{(k+2l)!}{k!}a_{k+2l} = 0,$$

(7.17)

where for $k=0$ we put $(0-1)!=1$ in the second line of (7.17).

For $k=n,\ n=0,1,\dots,N-1$, this gives

$$(\lambda_n - n(n+1))a_n + (n+1)^2 a_{n+1} = 0.$$

Thus, if $\lambda_n = n(n+1)$, we have $a_{n+1}=0$ and the solution $\tilde{u}(x)$ is indeed a polynomial with degree n. After choosing the leading coefficient $a_n \neq 0$ arbitrarily, from (7.17) we compute a_{n-1}, $a_{n-2},\ \dots, a_0$ recursively in a unique way. Then we set $v_n = (\tilde{u}(t_{1-1/2}), \tilde{u}(t_{2-1/2}), \dots, \tilde{u}(t_{N-1/2}))$.

There are two cases. **Case 1.** $n<N$: Then $v_n \neq 0$ since otherwise we would have $\tilde{u}\equiv 0$, in contradiction to the fact that we have chosen $\tilde{u}$ to be a nontrivial eigenfunction with $a_n \neq 0$. Thus (λ_n, v_n) is an eigenpair for $\mathcal{A}$. The eigenvectors $v_n,\ n=1,\dots,N$ are linearly independent. From Case 1, we get N eigenpairs with eigenvalues $\lambda_n = n(n+1)$ for $n=0,\dots,N-1$.

Case 2. $n \geq N$: Then $v_n = 0$ although $\tilde{u} \not\equiv 0$. The resulting eigenfunctions for $\mathcal{A}$ are trivial and so in this case there are no new eigenpairs.

Thus we have found N eigenpairs with linearly independent eigenvectors.

Remark 14. *The eigenvector v_0 with eigenvalue $\lambda_0 = 0$ corresponds to a rigid translation of all N spikes.*

The leading eigenpair for mutual movement of spikes is (λ_1, v_1).

The eigenvector for $\lambda_1 = 2$ can be computed as follows:

$$\tilde{u}(x) = 1-2x,\ 0<x<1$$

$$v_{1,k} = \tilde{u}(t_{k-1/2}),\ k=1,\dots,N,$$

$$v_{1,k} = 1 - \frac{2(k-1/2)}{N} = \frac{N-2k+1}{N}.$$

The components of $v_{1,k}$ are linearly increasing and have odd symmetry around the center of the spike cluster which corresponds to $k=\frac{N+1}{2}$ or $x=\frac{1}{2}$.

Remark 15. *The stability of the small eigenvalues follows from the results in* [29] *but the eigenvalues have not been determined explicitly.*

The technical analysis for the small eigenvalues has been postponed to Appendix C.

8. Discussion

We conclude this paper with a discussion of our results. We have considered a particular biological reaction-diffusion system with two small diffusivities, the Gierer-Meinhardt system with precursor. We have proved the existence and stability of cluster solutions which have three different length scales: a scale of order O(1) coming from the precursor inhomogeneity and two small scales which are of the same size as the square roots of the small diffusivities. In particular, the cluster solution can be stable for a suitable choice of parameter values.

Such systems and their solutions play an important role in biological modeling to account for the bridging of length scales, e.g. between genetic, nuclear, intra-cellular, cellular and tissue levels. Our solutions incorporate and combine multiple scales in a robust and stable manner. A particular example of biological multi-scale patterns concerns the pattern formation of hypostome, tentacles, and foot in *hydra*. Meinhardt's model [18] correctly describes the following experimental observation: with tentacle-specific antibodies, Bode et al. [3] have shown that after head removal tentacle activation first reappears at the very tip of the gastric column. Then this activation becomes shifted away from the tip to a new location, where the tentacles eventually appear. There are different lengthscales involved for this tentacle pattern: diameter of the gastric column, distance between tentacles, and diameter of tentacles.

Systems of the type considered in this paper are also a key to understanding the hierarchy of multi-stage biological processes such as in signalling pathways, where typically first large-scale structures appear which induce patterns on successively smaller scales. In our example, the multi-spike cluster is a typical small-scale pattern which is established near a pre-existing large-scale precursor inhomogeneity. The precursor can represent previous information from an earlier stage of development leading to the formation of fine structure at the present time. An example of hierarchical pattern formation is seen in the determination of cell states for segmentation in *Drosophila* wings. In this case three different hierarchy levels are involved in the process: maternal positional information, gap genes, and pair rule genes [17].

9. Appendix A: Existence proof II – error terms

In this section, we compute the error terms caused by the approximate solutions in Section 4. We begin by considering the spatial dependence of the inhibitor near the spikes which is given by the difference $T[w_{\varepsilon,\mathbf{t}}](x_s) - T[w_{\varepsilon,\mathbf{t}}](t_s)$ for $\mathbf{x} = (x_1, \ldots, x_N) \in \Omega_\eta$ and $\mathbf{t} \in \Omega_\eta$, where the nonlocal operator $\mathcal{T}[A]$ has been defined in (4.21) and the approximate solution has been introduced in (4.23).

To simplify our notation, we let

$$H_{\varepsilon,\mathbf{t}}(x_s) = T[w_{\varepsilon,\mathbf{t}}](x_s). \tag{9.1}$$

Let $x_s = t_s + \varepsilon y$. We calculate

$$\begin{aligned} & H_{\varepsilon,\mathbf{t}}(t_s+\varepsilon y) - H_{\varepsilon,\mathbf{t}}(t_s) \\ &= \xi_\varepsilon \int_\Omega [G_D(t_s+\varepsilon y, x) - G_D(t_s, x)] \left(\sum_{k=1}^N \hat{\xi}_k^2 \tilde{w}_k^2(x) + \sum_{k\neq l} \hat{\xi}_k \hat{\xi}_l \tilde{w}_k(x)\tilde{w}_l(x) \right) dx \\ &= J_1, \end{aligned} \tag{9.2}$$

where ξ_ε has been introduced in (2.8) and J_1 is defined by the last equality. For J_1, we have

$$\begin{aligned} J_1 &= \xi_\varepsilon \int_\Omega [G_D(t_s+\varepsilon y, x) - G_D(t_s, x)] \left(\sum_{k=1}^N \hat{\xi}_k^2 \tilde{w}_k^2(x) + \sum_{k\neq l} \hat{\xi}_k \hat{\xi}_l \tilde{w}_k(x)\tilde{w}_l(x) \right) dx \\ &= \xi_\varepsilon \sum_{k=1}^N \hat{\xi}_k^2 \int_\Omega [G_D(t_s+\varepsilon y, x) - G_D(t_s, x)]\, \tilde{w}_k^2\, dx. \end{aligned} \tag{9.3}$$

by (4.6). We further compute

$$\begin{aligned} & \xi_\varepsilon \int_\Omega [G_D(t_s+\varepsilon y, x) - G_D(t_s, x)]\, \tilde{w}_k^2\, dx \\ &= \xi_\varepsilon \int_\Omega \left[\frac{1}{2\sqrt{D}} \left(e^{-|t_s+\varepsilon y - x|/\sqrt{D}} - e^{-|t_s - x|/\sqrt{D}} \right) - (H_D(t_s+\varepsilon y, x) - H_D(t_s, x)) \right] \tilde{w}_k^2\, dx \\ &= \xi_\varepsilon \frac{1}{2\sqrt{D}} \int_\Omega \left(e^{-|t_s+\varepsilon y - x|/\sqrt{D}} - e^{-|t_s - x|/\sqrt{D}} \right) \tilde{w}_k^2\, dx \left(1 + O\left(e^{-2(d_0-\eta_0)/\sqrt{D}} \right) \right). \end{aligned} \tag{9.4}$$

using (4.14). Let $x = t_k + \varepsilon \tilde{z}$. For $k = s$, we have

$$\begin{aligned} & \xi_\varepsilon \frac{1}{2\sqrt{D}} \int_\Omega \left(e^{-|t_s+\varepsilon y - x|/\sqrt{D}} - e^{-|t_s - x|/\sqrt{D}} \right) \tilde{w}_s^2(x)\, dx \\ &= \xi_\varepsilon \frac{\varepsilon}{2\sqrt{D}} \int_{\mathbb{R}} \left(e^{-\varepsilon|y-\tilde{z}|/\sqrt{D}} - e^{-\varepsilon|\tilde{z}|/\sqrt{D}} \right) w_{\mu_s}^2(\tilde{z})\, d\tilde{z} \left(1 + O(\varepsilon^{10}) \right) \\ &= \xi_\varepsilon \frac{\varepsilon}{2\sqrt{D}} \left[\frac{\varepsilon}{\sqrt{D}} \int_{\mathbb{R}} (|\tilde{z}| - |y-\tilde{z}|)\, w_{\mu_s}^2(\tilde{z})\ d\tilde{z} + O\left(\frac{\varepsilon^2}{D} y^2 \right) \right] \left(1 + O(\varepsilon^{10}) \right) \\ &= \xi_\varepsilon \frac{\varepsilon}{2\sqrt{D}} \left[\frac{\varepsilon}{\sqrt{D}} \int_{\mathbb{R}} (|\tilde{z}| - |y-\tilde{z}|)\, w_{\mu^0}^2(\tilde{z})\ d\tilde{z} + O\left(\frac{\varepsilon^2}{D} y^2 \right) \right] \left(1 + O\left(D \left(\log \frac{1}{D} \right)^2 + \varepsilon^{10} \right) \right) \\ &= \frac{1}{\int_{\mathbb{R}} w^2(y)\, dy} \left[\frac{\varepsilon}{\sqrt{D}} T_0(y) + O\left(\frac{\varepsilon^2}{D} y^2 \right) \right] \left(1 + O\left(D \left(\log \frac{1}{D} \right)^2 + \varepsilon^{10} \right) \right), \end{aligned} \tag{9.5}$$

where w_{μ_s} has been defined in (3.1) and

$$T_0(y) = \int_{\mathbb{R}} (|\tilde{z}| - |y-\tilde{z}|)\, w_{\mu^0}^2\, d\tilde{z} \tag{9.6}$$

is an even function, using (2.8) and (4.9). For $k \neq s$, we have

$$\begin{aligned} & \xi_\varepsilon \frac{1}{2\sqrt{D}} \int_\Omega \left(e^{-|t_s+\varepsilon y - x|/\sqrt{D}} - e^{-|t_s - x|/\sqrt{D}} \right) \tilde{w}_k^2(x)\, dx \\ &= \xi_\varepsilon \frac{\varepsilon}{2\sqrt{D}} \int_{\mathbb{R}} \left(e^{-|t_s - t_k + \varepsilon(y-\tilde{z})|/\sqrt{D}} - e^{-|t_s - t_k - \varepsilon\tilde{z}|/\sqrt{D}} \right) w_{\mu_k}^2(\tilde{z})\, d\tilde{z} \left(1 + O(\varepsilon^{10}) \right) \end{aligned}$$

$$= \xi_\varepsilon \frac{\varepsilon}{2\sqrt{D}} (\mu_k)^{3/2} \left[e^{-|t_s-t_k|/\sqrt{D}} \left(-\frac{t_s-t_k}{|t_s-t_k|} \right) \frac{\varepsilon y}{\sqrt{D}} + O\left(D \log \frac{1}{D} \frac{\varepsilon^2}{D} y^2 \right) \right]$$

$$\times \int_{\mathbb{R}} w^2(\tilde{z})\, d\tilde{z} \left(1 + O(\varepsilon^{10})\right)$$

$$= (\mu^0)^{3/2} \left[e^{-|t_s-t_k|/\sqrt{D}} \left(-\frac{t_s-t_k}{|t_s-t_k|} \right) \frac{\varepsilon y}{\sqrt{D}} + O\left(\varepsilon^2 \log \frac{1}{D} y^2 \right) \right]$$

$$\times \left(1 + O\left(D \left(\log \frac{1}{D} \right)^2 + \varepsilon^{10} \right) \right), \tag{9.7}$$

using (4.9) and (2.8). Combining (9.5) and (9.7), we have

$$H_{\varepsilon,\mathbf{t}}(t_s + \varepsilon y) - H_{\varepsilon,\mathbf{t}}(t_s)$$

$$= \left(\hat{\xi}_s^2 \frac{1}{\int_{\mathbb{R}} w^2(y)\, dy} \frac{\varepsilon}{\sqrt{D}} \int_{\mathbb{R}} (|\tilde{z}| - |y - \tilde{z}|)\, w_{\mu_s}^2(\tilde{z})\ d\tilde{z} + O\left(\frac{\varepsilon^2}{D} y^2 \right) \right.$$

$$\left. + \sum_{k \neq s} \hat{\xi}_k^2 (\mu_k)^{3/2} e^{-|t_s-t_k|/\sqrt{D}} \left(-\frac{t_s-t_k}{|t_s-t_k|} \right) \frac{\varepsilon y}{\sqrt{D}} + O\left(\varepsilon^2 \log \frac{1}{D} y^2 \right) \right)$$

$$\times \left(1 + O\left(D \left(\log \frac{1}{D} \right)^2 + \varepsilon^{10} \right) \right). \tag{9.8}$$

Remark 16. *(i) The second line in (9.8) is an even function in the inner variable y which will drop out in many subsequent computations due to symmetry.*

(ii) The third line in (9.8) is an odd function in the inner variable y. For $\mathbf{t} \in \Omega_\eta$, we have

$$e^{-|t_s-t_k|/\sqrt{D}} = O\left(D \log \frac{1}{D} \right), \quad |k-s| = 1,$$

$$e^{-|t_s-t_k|/\sqrt{D}} = O\left(\left(D \log \frac{1}{D} \right)^2 \right), \quad |k-s| \geq 2.$$

Thus the third line in (9.8) is of exact order $O\left(\varepsilon \sqrt{D} \log \frac{1}{D} y \right)$.

Next we compute and estimate the error terms of the Gierer-Meinhardt system (4.20) for the approximate solution $w_{\varepsilon,\mathbf{t}}$. We recall that a steady state for (4.20) is given by $\mathcal{S}_\varepsilon[A] = 0$, where

$$\mathcal{S}_\varepsilon[A] := \varepsilon^2 A'' - A + \frac{A^2}{T[A]} \tag{9.9}$$

and $T[A]$ is defined by (4.21), combined with Neumann boundary conditions $A'(-1) = A'(1) = 0$. We now compute the error term

$$\mathcal{S}_\varepsilon[w_{\varepsilon,\mathbf{t}}] = \mathcal{S}_\varepsilon \left[\sum_{s=1}^N \hat{\xi}_s \tilde{w}_s \right]$$

$$= \varepsilon^2 \Delta \left(\sum_{s=1}^N \hat{\xi}_s \tilde{w}_s \right) - \mu(x) \sum_{s=1}^N \hat{\xi}_s \tilde{w}_s + \frac{\left(\sum_{s=1}^N \hat{\xi}_s \tilde{w}_s \right)^2}{H_{\varepsilon,\mathbf{t}}}$$

$$= \Bigg[\sum_{s=1}^N \left(\varepsilon^2 \Delta (\hat{\xi}_s \tilde{w}_s) - \mu_s \hat{\xi}_s \tilde{w}_s + \frac{\left(\hat{\xi}_s \tilde{w}_s\right)^2}{H_{\varepsilon,\mathbf{t}}(t_s)} \right) - \sum_{s=1}^N (\mu(x) - \mu(t_s)) \hat{\xi}_s \tilde{w}_s$$

$$- \sum_{s=1}^N \frac{\left(\hat{\xi}_s \tilde{w}_s\right)^2}{(H_{\varepsilon,\mathbf{t}}(t_s))^2} [H_{\varepsilon,\mathbf{t}}(x) - H_{\varepsilon,\mathbf{t}}(t_s)] \left(1 + O\left(\frac{\varepsilon}{\sqrt{D}} |y| \right) \right) \Bigg] \left(1 + O\left(e^{-2(d_0 - \eta_0)/\sqrt{D}} \right) \right)$$

$$\Bigg[= - \sum_{s=1}^N \left(\mu'(t_s) \varepsilon y \right) \hat{\xi}_s w_{\mu^s}$$

$$- \sum_{s=1}^N \hat{\xi}_s^2 w_{\mu^s}^2 \frac{1}{\int_{\mathbb{R}} w^2(y)\, dy} \frac{\varepsilon}{\sqrt{D}} \int_{\mathbb{R}} \left(|\tilde{z}| - |y - \tilde{z}| \right) w_{\mu_s}^2 (\tilde{z})\ d\tilde{z}$$

$$+ \sum_{s=1}^N w_{\mu^s}^2 \sum_{k \neq s} \hat{\xi}_k^2 (\mu_k)^{3/2} e^{-|t_s - t_k|/\sqrt{D}} \left(- \frac{t_s - t_k}{|t_s - t_k|} \right) \frac{\varepsilon y}{\sqrt{D}} \Bigg]$$

$$\times \left(1 + O\left(D \left(\log \frac{1}{D} \right)^2 + \varepsilon^{10} \right) \right)$$

$$= \Bigg[- \sum_{s=1}^N \left(\mu''(t^0)(t_s - t^0) \varepsilon y \right) \hat{\xi}_s w_{\mu_s}$$

$$- \sum_{s=1}^N \hat{\xi}_s^2 w_{\mu^s}^2 \frac{1}{\int_{\mathbb{R}} w^2(y)\, dy} \frac{\varepsilon}{\sqrt{D}} \int_{\mathbb{R}} \left(|\tilde{z}| - |y - \tilde{z}| \right) w_{\mu_s}^2 (\tilde{z})\ d\tilde{z} + O\left(\frac{\varepsilon^2}{D} y^2 \right)$$

$$+ \sum_{s=1}^N w_{\mu^s}^2 \sum_{k \neq s} \hat{\xi}_k^2 (\mu_k)^{3/2} e^{-|t_s - t_k|/\sqrt{D}} \left(- \frac{t_s - t_k}{|t_s - t_k|} \right) \frac{\varepsilon y}{\sqrt{D}} + O\left(\varepsilon^2 \log \frac{1}{D} y^2 \right) \Bigg]$$

$$\times \left(1 + O\left(D \left(\log \frac{1}{D} \right)^2 + \varepsilon^{10} \right) \right). \tag{9.10}$$

Now we readily have the estimate

$$\| \mathcal{S}_\varepsilon [w_{\varepsilon,\mathbf{t}}] \|_{L^2\left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right)} = O\left(\frac{\varepsilon}{\sqrt{D}} \right). \tag{9.11}$$

Remark 17. *The estimates derived in this section will be needed to conclude the existence proof using Liapunov-Schmidt reduction in Appendix B. In particular, they will imply an explicit formula for the positions of the spikes in Section 5.*

10. Appendix B: Existence proof III – Liapunov-Schmidt reduction

In this section, we study the linear operator defined by

$$\tilde{L}_{\varepsilon,\mathbf{t}} := \mathcal{S}'_\epsilon[A]\phi = \varepsilon^2\Delta\phi - \mu(x)\phi + \frac{2A\phi}{T[A]} - \frac{A^2}{(T[A])^2}(T'[A]\phi),$$

$$\tilde{L}_{\varepsilon,\mathbf{t}} : H^2(\Omega) \to L^2(\Omega),$$

where $A = w_{\varepsilon,\mathbf{t}}$ and $T'[A]$ has been defined in (6.3).

We will prove results on its invertibility after suitable projections. This will have important implications on the existence of solutions of the nonlinear problem including bounds in suitable norms. The proof uses the method of Liapunov-Schmidt reduction which was also considered in [10], [11], [8], [25], [26] and [37] and other works.

We define the approximate kernel and co-kernel of the operator $\tilde{L}_{\varepsilon,\mathbf{t}}$, respectively, as follows:

$$\mathcal{K}_{\varepsilon,\mathbf{t}} := \text{span}\left\{\frac{d\tilde{w}_i}{dx}\middle| i = 1,\dots,N\right\} \subset H^2(\Omega),$$

$$\mathcal{C}_{\varepsilon,\mathbf{t}} := \text{span}\left\{\frac{d\tilde{w}_i}{dx}\middle| i = 1,\dots,N\right\} \subset L^2(\Omega).$$

Recall that the vectorial linear operator L has been introduced in (3.12) as follows:

$$L\Phi := \Delta\Phi - \Phi + 2w\Phi - 2\frac{\int_{\mathbb{R}} w\Phi}{\int_{\mathbb{R}} w^2}w^2, \tag{10.1}$$

where

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix} \in (H^2(\mathbb{R}))^N.$$

By Lemma 7, we know that

$$L : (X_0 \oplus \cdots \oplus X_0)^\perp \cap (H^2(\mathbb{R}))^N \to (X_0 \oplus \cdots \oplus X_0)^\perp \cap (L^2(\mathbb{R}))^N$$

with $X_0 = \text{span}\left\{\frac{dw}{dy}\right\}$ is invertible and possesses a bounded inverse.

We also introduce the orthogonal projection $\pi^\perp_{\varepsilon,\mathbf{t}} : L^2(\Omega) \to \mathcal{C}^\perp_{\varepsilon,\mathbf{t}}$ and study the operator $L_{\varepsilon,\mathbf{t}} := \pi^\perp_{\varepsilon,\mathbf{t}} \circ \tilde{L}_{\varepsilon,\mathbf{t}}$. We will show that $L_{\varepsilon,\mathbf{t}} : \mathcal{K}^\perp_{\varepsilon,\mathbf{t}} \to \mathcal{C}^\perp_{\varepsilon,\mathbf{t}}$ is invertible with a bounded inverse provided $\max\left(\frac{\varepsilon}{\sqrt{D}}, D\right)$ is small enough. In proving this, we will use the fact that this system is the limit of the operator $L_{\varepsilon,\mathbf{t}}$ as $\max\left(\frac{\varepsilon}{\sqrt{D}}, D\right) \to 0$. This statement is contained in the following proposition.

Proposition 18. *There exist positive constants $\bar{\delta}, \lambda$ such that for $\max\left(\frac{\varepsilon}{\sqrt{D}}, D\right) \in (0, \bar{\delta})$ and all $\mathbf{t} \in \Omega_\eta$ we have*

$$\|L_{\varepsilon,\mathbf{t}}\phi_\varepsilon\|_{L^2(\Omega_\varepsilon)} \geq \lambda\|\phi_\varepsilon\|_{H^2(\Omega_\varepsilon)}. \tag{10.2}$$

Further, the map

$$L_{\varepsilon,\mathbf{t}} = \pi_{\varepsilon,\mathbf{t}} \circ \tilde{L}_{\varepsilon,\mathbf{t}} : \mathcal{K}^{\perp}_{\varepsilon,\mathbf{t}} \to \mathcal{C}^{\perp}_{\varepsilon,\mathbf{t}}$$

is surjective.

Proof. Suppose (10.2) is false. Then there exist sequences $\{\varepsilon_k\}$, $\{D_k\}$, $\{\mathbf{t}^k\}$, $\{\phi^k\}$ such that $\max\left(\frac{\varepsilon_k}{\sqrt{D_k}}, D_k\right) \to 0$, $\mathbf{t}^k \in \Omega_\eta$, $\phi^k = \phi_{\varepsilon_k} \in \mathcal{K}^{\perp}_{\varepsilon_k,\mathbf{t}^k}$, $k = 1, 2, \ldots$ and

$$\|L_{\epsilon_k,\mathbf{t}^k}\phi^k\|_{L^2(\Omega_{\varepsilon_k})} \to 0 \qquad \text{as } k \to \infty, \tag{10.3}$$

$$\|\phi^k\|_{H^2(\Omega_{\varepsilon_k})} = 1, \qquad k = 1,\, 2,\, \ldots\,. \tag{10.4}$$

We define $\phi_{\varepsilon,i}$, $i = 1, 2, \ldots, N$ and $\phi_{\varepsilon,N+1}$ as follows:

$$\phi_{\varepsilon,i}(x) = \phi_\varepsilon(x)\chi\left(\frac{x - t_i}{\delta_\varepsilon}\right), \quad x \in \Omega, \tag{10.5}$$

$$\phi_{\varepsilon,N+1}(x) = \phi_\varepsilon(x) - \sum_{i=1}^{N} \phi_{\varepsilon,i}(x), \quad x \in \Omega.$$

At first (after rescaling), the functions $\phi_{\varepsilon,i}$ are only defined on Ω_ε. However, by a standard result they can be extended to $\mathbb{R}$ such that their norm in $H^2(\mathbb{R})$ is bounded by a constant independent of $\varepsilon,\ D$ and $\mathbf{t}$ for $\max\left(\frac{\varepsilon}{\sqrt{D}}, D\right)$ small enough. In the following we will study this extension. For simplicity of notation we keep the same notation for the extension. Since for $i = 1, 2, \ldots, N$ each sequence $\{\phi_i^k\} := \{\phi_{\varepsilon_k,i}\}$ ($k = 1, 2, \ldots$) is bounded in $H^2_{loc}(\mathbb{R})$ it has a weak limit in $H^2_{loc}(\mathbb{R})$, and therefore also a strong limit in $L^2_{loc}(\mathbb{R})$ and $L^\infty_{loc}(\mathbb{R})$. Call these limits ϕ_i. Then $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix}$ solves the system $L\phi = 0$. By Lemma 6, $\phi \in \text{Ker}(L) = X_0 \oplus \cdots \oplus X_0$. Since $\phi_{\varepsilon_k} \in K^{\perp}_{\varepsilon_k,t_k}$ by taking $k \to \infty$ we get $\phi \in \text{Ker}(L)^{\perp}$. Therefore, $\phi = 0$.

By elliptic estimates we have $\|\phi_{\varepsilon_k,i}\|_{H^2(\mathbb{R})} \to 0$ as $k \to \infty$ for $i = 1, 2, \ldots, N$.

Furthermore, $\phi_{\varepsilon_k,N+1} \to \phi_{N+1}$ in $H^2(\mathbb{R})$, where Φ_{N+1} satisfies

$$\Delta\phi_{N+1} - \phi_{N+1} = 0 \quad \text{in } \mathbb{R}.$$

Therefore we conclude $\phi_{N+1} = 0$ and $\|\phi^k_{N+1}\|_{H^2(\mathbb{R})} \to 0$ as $k \to \infty$. This contradicts $\|\phi^k\|_{H^2(\Omega_{\varepsilon_k})} = 1$.

To complete the proof of Proposition 18 we just need to show that the conjugate operator to $L_{\varepsilon,\mathbf{t}}$ (denoted by $L^*_{\varepsilon,\mathbf{t}}$) is injective from $\mathcal{K}^{\perp}_{\varepsilon,\mathbf{t}}$ to $\mathcal{C}^{\perp}_{\varepsilon,\mathbf{t}}$.

The proof for $L^*_{\varepsilon,\mathbf{t}}$ follows along the same lines as for $L_{\varepsilon,\mathbf{t}}$ and is omitted.

□

Now we are in the position to solve the equation

$$\pi^{\perp}_{\epsilon,\mathbf{t}} \circ \mathcal{S}_\epsilon[w_{\varepsilon,\mathbf{t}} + \phi] = 0. \tag{10.6}$$

Since $L_{\epsilon,\mathbf{t}}|_{K^{\perp}_{\epsilon,\mathbf{t}}}$ is invertible (call the inverse $L^{-1}_{\epsilon,\mathbf{t}}$) we can rewrite this as

$$\phi = -(L^{-1}_{\epsilon,\mathbf{t}} \circ \pi^{\perp}_{\epsilon,\mathbf{t}} \circ \mathcal{S}_\epsilon[w_{\varepsilon,\mathbf{t}})] - (L^{-1}_{\epsilon,\mathbf{t}} \circ \pi^{\perp}_{\epsilon,\mathbf{t}} \circ \mathcal{N}_{\epsilon,\mathbf{t}}[\phi]) \equiv \mathcal{M}_{\epsilon,\mathbf{t}}[\phi], \tag{10.7}$$

where

$$\mathcal{N}_{\varepsilon,\mathbf{t}}[\phi] = \mathcal{S}_\varepsilon[w_{\varepsilon,\mathbf{t}} + \phi] - \mathcal{S}_\varepsilon[w_{\varepsilon,\mathbf{t}}] - \mathcal{S}'_\varepsilon[w_{\varepsilon,\mathbf{t}}]\phi \tag{10.8}$$

and the operator $\mathcal{M}_{\epsilon,\mathbf{t}}$ is defined by (10.7) for $\phi \in H^2(\Omega_\epsilon)$. We are going to show that the operator $\mathcal{M}_{\epsilon,\mathbf{t}}$ is a contraction on

$$B_{\epsilon,\delta} \equiv \{\phi \in H^2(\Omega_\epsilon) \,:\, \|\phi\|_{H^2(\Omega_\epsilon)} < \delta\}$$

for suitably chosen δ if $\max\left(\frac{\varepsilon}{\sqrt{D}}, D\right)$ is small enough. By (9.11) and Proposition 18 we have

$$\|\mathcal{M}_{\epsilon,\mathbf{t}}[\phi]\|_{H^2(\Omega_\epsilon)} \le \lambda^{-1}\left(\|\pi^{\perp}_{\epsilon,\mathbf{t}} \circ \mathcal{N}_{\epsilon,\mathbf{t}}[\phi]\|_{L^2(\Omega_\varepsilon)} + \left\|\pi^{\perp}_{\epsilon,\mathbf{t}} \circ \mathcal{S}_\varepsilon[w_{\varepsilon,\mathbf{t}}]\right\|_{L^2(\Omega_\varepsilon)}\right)$$

$$\le \lambda^{-1} C\left(c(\delta)\delta + \frac{\varepsilon}{\sqrt{D}}\right)\|\phi\|_{H^2(\Omega_\epsilon)},$$

where $\lambda > 0$ is independent of $\delta > 0$, $\varepsilon > 0$, $D > 0$ and $c(\delta) \to 0$ as $\delta \to 0$. Similarly, we show that

$$\|\mathcal{M}_{\epsilon,\mathbf{t}}[\phi] - \mathcal{M}_{\epsilon,\mathbf{t}}[\phi']\|_{H^2(\Omega_\varepsilon)} \le \lambda^{-1} C(c(\delta)\delta)\|\phi - \phi'\|_{H^2(\Omega_\varepsilon)},$$

where $c(\delta) \to 0$ as $\delta \to 0$. If we choose

$$\delta = 2\lambda^{-1}\left\|\pi^{\perp}_{\epsilon,\mathbf{t}} \circ \mathcal{S}_\varepsilon[w_{\varepsilon,\mathbf{t}}]\right\|_{L^2(\Omega_\epsilon)}$$

then, for $\max\left(\frac{\epsilon}{\sqrt{D}}, D\right)$ small enough, the operator $\mathcal{M}_{\epsilon,\mathbf{t}}$ is a contraction on $B_{\varepsilon,\delta}$. The existence of a fixed point $\phi_{\epsilon,\mathbf{t}}$ now follows from the standard contraction mapping principle and $\phi_{\epsilon,\mathbf{t}}$ is a solution of (10.7).

We have thus proved

Lemma 19. *There exists $\overline{\delta} > 0$ such that for every pair of $\epsilon, \mathbf{t}$ with $0 < \epsilon < \overline{\delta}$ and $\mathbf{t} \in \Omega_\eta$, there exists a unique $\phi_{\epsilon,\mathbf{t}} \in K^{\perp}_{\epsilon,\mathbf{t}}$ satisfying $\mathcal{S}_\epsilon[w_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}] \in \mathcal{C}_{\varepsilon,\mathbf{t}}$. Furthermore, we have the estimate*

$$\|\phi_{\epsilon,\mathbf{t}}\|_{H^2(\Omega_\epsilon)} \le C\left(\frac{\epsilon}{\sqrt{D}}\right). \tag{10.9}$$

Using the symmetry discussed in Remark 16, we can decompose

$$\phi_{\varepsilon,\mathbf{t}} = \phi_{\varepsilon,\mathbf{t},1} + \phi_{\varepsilon,\mathbf{t},2}, \tag{10.10}$$

where $\phi_{\varepsilon,\mathbf{t},1}$ is an even function in the inner variable y which can be estimated as

$$\|\phi_{\varepsilon,\mathbf{t},1}\|_{H^2\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right)} = O\left(\frac{\varepsilon}{\sqrt{D}}\right)$$

and $\phi_{\varepsilon,\mathbf{t},2}$ is an odd function in the inner variable y which can be estimated as

$$\|\phi_{\varepsilon,\mathbf{t},2}\|_{H^2\left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right)} = O\left(\varepsilon\sqrt{D}\log\frac{1}{D}\right).$$

11. Appendix C: Stability Proof III – technical analysis of small eigenvalues

In this section we perform a technical analysis of the small eigenvalues and conclude the proof of Theorem 2.

First let us define

$$\tilde{w}_{\varepsilon,j}(x) = \chi\left(\frac{x-t_j^\varepsilon}{\delta_\varepsilon}\right)\bar{w}_\varepsilon(x), \quad j=1,\ldots,N, \tag{11.11}$$

where $\chi(x)$ is given in (4.7) and δ_ε satisfies (4.8). We define similar to Section 5

$$\mathcal{K}^{new}_{\varepsilon,\mathbf{t}^\varepsilon} := \text{span}\left\{\varepsilon\tilde{w}'_{\varepsilon,j} \,:\, j=1,\ldots,N\right\} \subset H^2(\Omega_\varepsilon),$$

$$\mathcal{C}^{new}_{\varepsilon,\mathbf{t}^\varepsilon} := \text{span}\left\{\varepsilon\tilde{w}'_{\varepsilon,j} \,:\, j=1,\ldots,N\right\} \subset L^2(\Omega_\varepsilon).$$

Then it is easy to see that

$$\bar{w}_\varepsilon(x) = \sum_{j=1}^N \tilde{w}_{\varepsilon,j}(x) + O(\varepsilon^{10}). \tag{11.12}$$

Note that $\tilde{w}_{\varepsilon,j}$ satisfies

$$\varepsilon^2\Delta\tilde{w}_{\varepsilon,j} - \mu(x)\tilde{w}_{\varepsilon,j} + \frac{(\tilde{w}_{\varepsilon,j})^2}{\bar{H}_\varepsilon} + O(\varepsilon^{10}) = 0.$$

Further, we have $\tilde{w}_{\varepsilon,j}(x) = \hat{\xi}_j w_j\left(\frac{x-t_j^\varepsilon}{\varepsilon}\right) + O\left(\frac{\varepsilon}{\sqrt{D}} + D\left(\log\frac{1}{D}\right)^2\right)$ in $H^2(\Omega_\varepsilon)$, where w_j has been defined in(4.5).

Thus $\tilde{w}'_{\varepsilon,j} := \frac{d\tilde{w}_{\varepsilon,j}}{dx}$ satisfies

$$\varepsilon^2\Delta\tilde{w}'_{\varepsilon,j} - \mu(x)\tilde{w}'_{\varepsilon,j} + \frac{2\tilde{w}_{\varepsilon,j}}{\bar{H}_\varepsilon}\tilde{w}'_{\varepsilon,j} - \frac{\tilde{w}^2_{\varepsilon,j}}{(\bar{H}_\varepsilon)^2}\bar{H}'_\varepsilon - \mu'(x)\tilde{w}_{\varepsilon,j} + O(\varepsilon^9) = 0. \tag{11.13}$$

Let us now decompose

$$\phi_\varepsilon = \varepsilon\sum_{j=1}^N a_j^\varepsilon\tilde{w}'_{\varepsilon,j} + \phi_\varepsilon^\perp, \tag{11.14}$$

where a_j^ε are complex numbers and $\phi_\varepsilon^\perp \perp \mathcal{K}^{new}_{\varepsilon,\mathbf{t}^\varepsilon}$. Note that the scaling factor ε has been introduced to ensure that $\phi_\varepsilon = O(1)$ in $H^2(\Omega_\varepsilon)$.

Suppose that $\|\phi_\varepsilon\|_{H^2(\Omega_\varepsilon)} = 1$. Then $|a_j^\varepsilon| \leq C$.

The decomposition of ϕ_ε given in (11.14) implies

$$\psi_\varepsilon = \varepsilon \sum_{j=1}^N a_j^\varepsilon \psi_{\varepsilon,j} + \psi_\varepsilon^\perp, \tag{11.15}$$

where $\psi_{\varepsilon,j}$ satisfies

$$D\Delta\psi_{\varepsilon,j} - \psi_{\varepsilon,j} + 2\xi_\varepsilon \bar{w}_\varepsilon \tilde{w}'_{\varepsilon,j} = 0, \quad \psi'_{\varepsilon,j}(-1) = \psi'_{\varepsilon,j}(1) = 0 \tag{11.16}$$

and $\psi_\varepsilon^\perp$ is given by

$$D\Delta\psi_\varepsilon^\perp - \psi_\varepsilon^\perp + 2\xi_\varepsilon \bar{w}_\varepsilon \phi_\varepsilon^\perp = 0, \quad (\psi_\varepsilon^\perp)'(-1) = (\psi_\varepsilon^\perp)'(1) = 0. \tag{11.17}$$

Substituting the decompositions of ϕ_ε and ψ_ε into (7.2) we have, using (11.13),

$$\begin{aligned} &\varepsilon \sum_{j=1}^N a_j^\varepsilon \left(\frac{(\tilde{w}_{\varepsilon,j})^2}{\bar{H}_\varepsilon^2} \bar{H}'_\varepsilon - \frac{(\bar{w}_\varepsilon)^2}{\bar{H}_\varepsilon^2} \psi_{\varepsilon,j} \right) + \varepsilon \sum_{j=1}^N a_j^\varepsilon \mu'(x) \tilde{w}_{\varepsilon,j} \\ &+\varepsilon^2 \Delta\phi_\varepsilon^\perp - \mu(x)\phi_\varepsilon^\perp + 2\frac{\bar{w}_\varepsilon}{\bar{H}_\varepsilon}\phi_\varepsilon^\perp - \frac{\bar{w}_\varepsilon^2}{\bar{H}_\varepsilon^2}\psi_\varepsilon^\perp - \lambda_\varepsilon \phi_\varepsilon^\perp + O(\varepsilon^9) \\ &= \lambda_\varepsilon \left(\varepsilon \sum_{j=1}^N a_j^\varepsilon \tilde{w}'_{\varepsilon,j} \right). \end{aligned} \tag{11.18}$$

We first compute

$$\begin{aligned} I_2 \;&:=\; \varepsilon \sum_{j=1}^N a_j^\varepsilon \left(\frac{(\tilde{w}_{\varepsilon,j})^2}{\bar{H}_\varepsilon^2} \bar{H}'_\varepsilon - \frac{(\bar{w}_\varepsilon)^2}{\bar{H}_\varepsilon^2} \psi_{\varepsilon,j} \right) \\ &=\; \varepsilon \sum_{j=1}^N a_j^\varepsilon \frac{(\tilde{w}_{\varepsilon,j})^2}{\bar{H}_\varepsilon^2} [-\psi_{\varepsilon,j} + \bar{H}'_\varepsilon] - \varepsilon \sum_{j=1}^N a_j^\varepsilon \psi_{\varepsilon,j} \sum_{k\neq j} \frac{(\tilde{w}_{\varepsilon,k})^2}{\bar{H}_\varepsilon^2} + O(\varepsilon^9) \\ &=\; \varepsilon \sum_{j=1}^N a_j^\varepsilon \frac{(\tilde{w}_{\varepsilon,j})^2}{\bar{H}_\varepsilon^2} \left[-\psi_{\varepsilon,j} + \bar{H}'_\varepsilon\right] - \varepsilon \sum_{j=1}^N \sum_{k\neq j} a_k^\varepsilon \psi_{\varepsilon,k} \frac{(\tilde{w}_{\varepsilon,j})^2}{\bar{H}_\varepsilon^2} + O(\varepsilon^9). \end{aligned}$$

We estimate I_2 as follows

$$\begin{aligned} I_2 &= -\varepsilon \sum_{j=1}^N \sum_{k=1}^N a_k^\varepsilon \frac{(\tilde{w}_{\varepsilon,j})^2}{\bar{H}_\varepsilon^2} \left[\psi_{\varepsilon,k} - \bar{H}'_\varepsilon \delta_{jk}\right] + O(\varepsilon^9) \\ &= -\varepsilon \sum_{j=1}^N \sum_{|k-j|=1} a_k^\varepsilon \frac{(\tilde{w}_{\varepsilon,j})^2}{\bar{H}_\varepsilon^2} \psi_{\varepsilon,k} + O(\varepsilon^9) + O\left(\varepsilon D^{3/2} \left(\log \frac{1}{D} \right)^2 \right). \end{aligned} \tag{11.19}$$

Let us also put

$$\tilde{L}_\varepsilon \phi_\varepsilon^\perp := \varepsilon^2 \Delta\phi_\varepsilon^\perp - \mu(x)\phi_\varepsilon^\perp + \frac{2\bar{w}_\varepsilon}{\bar{H}_\varepsilon}\phi_\varepsilon^\perp - \frac{\bar{w}_\varepsilon^2}{\bar{H}_\varepsilon^2}\psi_\varepsilon^\perp \tag{11.20}$$

and

$$\mathbf{a}^\varepsilon := (a_1^\varepsilon, \ldots, a_N^\varepsilon)^T. \tag{11.21}$$

Multiplying both sides of (11.18) by $\tilde{w}'_{\varepsilon,l}$ and integrating over $(-1,1)$, we obtain, using (3.3),

$$\begin{aligned}
\text{r.h.s.} &= \varepsilon\lambda_\varepsilon \sum_{j=1}^N a_j^\varepsilon \int_{-1}^1 \tilde{w}'_{\varepsilon,j}\tilde{w}'_{\varepsilon,l}\,dx \\
&= \lambda_\varepsilon a_l^\varepsilon \hat{\xi}_l^2 \int_{\mathbb{R}} (w_l'(y))^2\,dy \, \left(1+O\left(\frac{\varepsilon}{\sqrt{D}}+D\left(\log\frac{1}{D}\right)^2\right)\right) \qquad (11.22)\\
&= \lambda_\varepsilon a_l^\varepsilon \hat{\xi}_l^2 \mu_l^{5/2} \int_{\mathbb{R}} (w'(z))^2\,dz \, \left(1+O\left(\frac{\varepsilon}{\sqrt{D}}+D\left(\log\frac{1}{D}\right)^2\right)\right) \qquad (11.23)
\end{aligned}$$

and, using (11.19),

$$\begin{aligned}
\text{l.h.s.} &= -\varepsilon \sum_{k=1}^N a_k^\varepsilon \int_{-1}^1 \frac{\tilde{w}_{\varepsilon,l}^2}{\bar{H}_\varepsilon^2}\left[\psi_{\varepsilon,k}-\bar{H}'_\varepsilon\delta_{lk}\right]\tilde{w}'_{\varepsilon,l}\,dx \\
&\quad +\varepsilon \sum_{j=1}^N a_j^\varepsilon \int_{-1}^1 \mu'\tilde{w}_{\varepsilon,j}\tilde{w}'_{\varepsilon,l}\,dx \\
&\quad +\int_{-1}^1 \frac{\tilde{w}_{\varepsilon,l}^2}{\bar{H}_\varepsilon^2}(\bar{H}'_\varepsilon\phi_\varepsilon^\perp)\,dx \\
&\quad -\int_{-1}^1 \frac{\tilde{w}_{\varepsilon,l}^2}{\bar{H}_\varepsilon^2}(\psi_\varepsilon^\perp w'_{\varepsilon,l})\,dx+\int_{-1}^1 \mu'\phi_\varepsilon^\perp w_{\varepsilon,l}\,dx \\
&= (J_{1,l}+J_{2,l}+J_{3,l}+J_{4,l}+J_{5,l}),
\end{aligned}$$

where $J_{i,l},\ i=1,2,3,4,5$ are defined by the last equality.

The following is the key lemma.

Lemma 20. *We have*

$$\begin{aligned}
J_{1,l} &= -\varepsilon^2\left(\frac{1}{3}\int_{\mathbb{R}} w^3\,dy\right)\hat{\xi}_l\mu_l^{5/2}\Big[-\nabla^2_{t_l^\varepsilon}\hat{K}_D(t_l^\varepsilon,t_{l-1}^\varepsilon)\hat{\xi}_{l-1}^2\mu_{l-1}^{3/2}a_{l-1}^\varepsilon \qquad (11.24)\\
&\quad -\nabla^2_{t_l^\varepsilon}\hat{K}_D(t_l^\varepsilon,t_{l+1}^\varepsilon)\hat{\xi}_{l+1}^2\mu_{l+1}^{3/2}a_{l+1}^\varepsilon \\
&\quad +\nabla^2_{t_l^\varepsilon}\hat{K}_D(t_l^\varepsilon,t_{l+1}^\varepsilon)\hat{\xi}_{l+1}^2\mu_{l+1}^{3/2}+\nabla^2_{t_l^\varepsilon}\hat{K}_D(t_l^\varepsilon,t_{l-1}^\varepsilon)\hat{\xi}_{l-1}^2\mu_{l-1}^{3/2}]a_l^\varepsilon\Big] \\
&\quad +O\left(\varepsilon^2\left(\frac{\varepsilon}{\sqrt{D}}+D\left(\log\frac{1}{D}\right)^2\right)\right),
\end{aligned}$$

$$J_{2,l} = -\varepsilon^2 \left(\frac{5}{12} \int_{\mathbb{R}} w^3 \, dy \right) \hat{\xi}_l^2 \mu_l^{3/2} \mu_l'' a_l^\varepsilon + O\left(\varepsilon^2 \left(\frac{\varepsilon}{\sqrt{D}} + D \left(\log \frac{1}{D} \right)^2 \right) \right), \quad (11.25)$$

$$J_{3,l} = O\left(\varepsilon^2 \left(\frac{\varepsilon}{\sqrt{D}} + D \left(\log \frac{1}{D} \right)^2 \right) \right), \quad (11.26)$$

$$J_{4,l} = O\left(\varepsilon^2 \left(\frac{\varepsilon}{\sqrt{D}} + D \left(\log \frac{1}{D} \right)^2 \right) \right), \quad (11.27)$$

$$J_{5,l} = O\left(\varepsilon^2 \left(\frac{\varepsilon}{\sqrt{D}} + D \left(\log \frac{1}{D} \right)^2 \right) \right), \quad (11.28)$$

where a_l^ε has been defined in (11.21) and

$$a_l^0 = \lim_{\varepsilon \to 0} a_l^\varepsilon, \quad \mathbf{a}^0 = (a_1^0, \ldots, a_N^0). \quad (11.29)$$

Proof. We prove Theorem 1 by using Lemma 20. We compute

$$\begin{aligned} \text{l.h.s.} &= J_{1,l} + J_{2,l} + O\left(\varepsilon^2 \left(\frac{\varepsilon}{\sqrt{D}} + D \left(\log \frac{1}{D} \right)^2 \right) \right) \\ &= -\varepsilon^2 \left(\frac{1}{3} \int_{\mathbb{R}} w^3 \, dy \right) \hat{\xi}_l \mu_l^{5/2} \Big[- \nabla^2_{t_l^\varepsilon} \hat{K}_D(t_l^\varepsilon, t_{l-1}^\varepsilon) \hat{\xi}_{l-1}^2 \mu_{l-1}^{3/2} a_{l-1}^\varepsilon \\ & \quad - \nabla^2_{t_l^\varepsilon} \hat{K}_D(t_l^\varepsilon, t_{l+1}^\varepsilon) \hat{\xi}_{l+1}^2 \mu_{l+1}^{3/2} a_{l+1}^\varepsilon \\ & \quad + \nabla^2_{t_l^\varepsilon} \hat{K}_D(t_l^\varepsilon, t_{l+1}^\varepsilon) \hat{\xi}_{l+1}^2 \mu_{l+1}^{3/2} + \nabla^2_{t_l^\varepsilon} \hat{K}_D(t_l^\varepsilon, t_{l-1}^\varepsilon) \hat{\xi}_{l-1}^2 \mu_{l-1}^{3/2}] a_l^\varepsilon \Big] \\ & \quad -\varepsilon^2 \left(\frac{5}{12} \int_{\mathbb{R}} w^3 \, dy \right) \hat{\xi}_l^2 \mu_l^{3/2} \mu_l'' a_l^\varepsilon + O\left(\varepsilon^2 \left(\frac{\varepsilon}{\sqrt{D}} + D \left(\log \frac{1}{D} \right)^2 \right) \right). \end{aligned}$$

Comparing with r.h.s. and recalling the computation of $\mathcal{M}(t^0)$ at (7.12), we obtain

$$\begin{aligned} & -2.4 \varepsilon^2 \hat{\xi}^0 (\mu^0)^{5/2} \mathcal{M}(\mathbf{t}^0) \mathbf{a}_\varepsilon \left(1 + O\left(\frac{\varepsilon}{\sqrt{D}} + \sqrt{D} \log \frac{1}{D} \right) \right) \\ &= \lambda_\varepsilon (\mu^0)^{5/2} (\hat{\xi}^0)^2 \mathbf{a}_\varepsilon \int_{\mathbb{R}} (w'(y))^2 \, dy \left(1 + O\left(\frac{\varepsilon}{\sqrt{D}} + \sqrt{D} \log \frac{1}{D} \right) \right), \end{aligned} \quad (11.30)$$

using (2.3). Equation (11.30) shows that the small eigenvalues λ_ε of (7.2) are given by

$$\lambda_\varepsilon \sim -2\varepsilon^2 \hat{\xi}^0 \sigma \left(\mathcal{M}(\mathbf{t}^0) \right),$$

using (2.3).

Arguing as in Theorem 10, this shows that if all the eigenvalues of $\mathcal{M}(t^0)$ have positive real part, then the small eigenvalues are stable. On the other hand, if $\mathcal{M}(t^0)$ has an eigenvalue with negative real part, then there are eigenfunctions and eigenvalues to make the system unstable.

This proves Theorem 2.

□

Next we prove Lemma 20.

Proof. We first study the asymptotic behavior of $\psi_{\varepsilon,j}$.

Lemma 21. *We have*

$$(\psi_{\varepsilon,k}-\bar{H}'_\varepsilon\delta_{kl})(t_l^\varepsilon)=-\nabla_{t_k^\varepsilon}\hat{K}_D(t_l^\varepsilon,t_k^\varepsilon)\hat{\xi}_k^2\mu_k^{3/2}[\delta_{k,l-1}+\delta_{k,l+1}]-\delta_{kl}\sum_{m,|m-l|=1}\nabla_{t_l^\varepsilon}\hat{K}_D(t_l^\varepsilon,t_m^\varepsilon)\hat{\xi}_m^2\mu_m^{3/2}$$
$$+O\left(\sqrt{D}\log\frac{1}{D}\left(\frac{\varepsilon}{\sqrt{D}}+\left(D\log\frac{1}{D}\right)^2\right)\right). \tag{11.31}$$

Proof. Note that for $l\neq k$, we have

$$\begin{aligned}\psi_{\varepsilon,k}(t_l^\varepsilon) &= 2\xi_\varepsilon\int_{-1}^1 G_D(t_l^\varepsilon,z)\bar{w}_\varepsilon\tilde{w}'_{\varepsilon,k}\,dz\\ &= -\nabla_{t_k^\varepsilon}\hat{G}_D(t_k^\varepsilon,t_l^\varepsilon)\hat{\xi}_k^2\mu_k^{3/2}+O\left(\sqrt{D}\log\frac{1}{D}\left(\frac{\varepsilon}{\sqrt{D}}+\left(D\log\frac{1}{D}\right)^2\right)\right) \quad (11.32)\\ &= -\nabla_{t_k^\varepsilon}\hat{K}_D(t_k^\varepsilon,t_l^\varepsilon)\hat{\xi}_k^2\mu_k^{3/2}+O\left(\sqrt{D}\log\frac{1}{D}\left(\frac{\varepsilon}{\sqrt{D}}+D\left(\log\frac{1}{D}\right)^2\right)\right).\end{aligned}$$

Next we compute $\psi_{\varepsilon,l}-\bar{H}'_\varepsilon$ near t_l^ε:

$$\begin{aligned}\bar{H}_\varepsilon(x) &= \xi_\varepsilon\int_{-1}^1 G_D(x,z)\bar{w}_\varepsilon^2\,dz\\ &= \xi_\varepsilon\int_{-\infty}^{+\infty}K_D(|z|)\tilde{w}_{\varepsilon,l}^2(x+z)dz-\xi_\varepsilon\int_{-1}^1 H_D(x,z)\tilde{w}_{\varepsilon,l}^2\,dz\\ &\quad+\xi_\varepsilon\sum_{k\neq l}\int_{-1}^1 G_D(x,z)\tilde{w}_{\varepsilon,k}^2\,dz+O\left(\frac{\varepsilon}{\sqrt{D}}+D\left(\log\frac{1}{D}\right)^2\right).\end{aligned}$$

So

$$\begin{aligned}\bar{H}'_\varepsilon(x) &= \xi_\varepsilon\int_{-\infty}^{+\infty}K_D(|z|)(2\tilde{w}_{\varepsilon,l}(x+z)\tilde{w}'_{\varepsilon,l}(x+z))\,dz-\xi_\varepsilon\int_{-1}^1\nabla_x H_D(x,z)\tilde{w}_{\varepsilon,l}^2\,dz\\ &\quad+\xi_\varepsilon\sum_{k\neq l}\int_{-1}^1\nabla_x G_D(x,z)\tilde{w}_{\varepsilon,k}^2\,dz+O\left(\sqrt{D}\log\frac{1}{D}\left(\frac{\varepsilon}{\sqrt{D}}+D\left(\log\frac{1}{D}\right)^2\right)\right)\\ &= \sum_{k,|k-l|=1}\nabla_x\hat{K}_D(x,t_k^\varepsilon)\hat{\xi}_k^2\mu_k^{3/2}\\ &\quad+O\left(\sqrt{D}\log\frac{1}{D}\left(\frac{\varepsilon}{\sqrt{D}}+D\left(\log\frac{1}{D}\right)^2\right)\right). \qquad (11.33)\end{aligned}$$

Thus

$$\begin{aligned}\bar{H}_\varepsilon'(x) - \psi_{\varepsilon,l}(x) &= -\xi_\varepsilon \int_{-1}^{1} \nabla_x H_D(x,z)\tilde{w}_{\varepsilon,l}^2\,dz + \xi_\varepsilon \sum_{k\neq l} \int_{-1}^{1} \nabla_x G_D(x,z)\tilde{w}_{\varepsilon,k}^2\,dz \\ &\quad +2\xi_\varepsilon \int_{-1}^{1} H_D(x,z)\tilde{w}_{\varepsilon,l}\tilde{w}_{\varepsilon,l}'\,dz \\ &\quad +O\left(\sqrt{D}\log\frac{1}{D}\left(\frac{\varepsilon}{\sqrt{D}} + D\left(\log\frac{1}{D}\right)^2\right)\right) \\ &= \sum_{k,|k-l|=1} \nabla_x \hat{K}_D(x,t_k^\varepsilon)\hat{\xi}_k^2\mu_k^{3/2} \\ &\quad +O\left(\sqrt{D}\log\frac{1}{D}\left(\frac{\varepsilon}{\sqrt{D}} + D\left(\log\frac{1}{D}\right)^2\right)\right). \qquad (11.34)\end{aligned}$$

Therefore, we have,

$$\begin{aligned}\bar{H}_\varepsilon'(t_l^\varepsilon) - \psi_{\varepsilon,l}(t_l^\varepsilon) &= -\xi_\varepsilon \int_{-1}^{1} \nabla_{t_l^\varepsilon} H_D(t_l^\varepsilon,z)\tilde{w}_{\varepsilon,l}^2\,dz + \xi_\varepsilon \sum_{k\neq l} \int_{-1}^{1} \nabla_{t_l^\varepsilon} G_D(t_l^\varepsilon,z)\tilde{w}_{\varepsilon,k}^2\,dz \\ &\quad -\nabla_{t_l^\varepsilon} H_D(t_l^\varepsilon,t_l^\varepsilon)\hat{\xi}_l^2\mu_l^{3/2} + O\left(\sqrt{D}\log\frac{1}{D}\left(\frac{\varepsilon}{\sqrt{D}} + D\left(\log\frac{1}{D}\right)^2\right)\right) \\ &= \sum_{k=1}^{N} \nabla_{t_l^\varepsilon} \hat{G}_D(t_l^\varepsilon,t_k^\varepsilon)\hat{\xi}_k^2\mu_k^{3/2} - \nabla_{t_l^\varepsilon} H_D(t_l^\varepsilon,t_l^\varepsilon)\hat{\xi}_l^2\mu_l^{3/2} \\ &\quad +O\left(\sqrt{D}\log\frac{1}{D}\left(\frac{\varepsilon}{\sqrt{D}} + D\left(\log\frac{1}{D}\right)^2\right)\right) \\ &= \sum_{k,|k-l|=1} \nabla_{t_l^\varepsilon} \hat{K}_D(t_l^\varepsilon,t_k^\varepsilon)\hat{\xi}_k^2\mu_k^{3/2} \\ &\quad +O\left(\sqrt{D}\log\frac{1}{D}\left(\frac{\varepsilon}{\sqrt{D}} + D\left(\log\frac{1}{D}\right)^2\right)\right). \qquad (11.35)\end{aligned}$$

Combining (11.33) and (11.35), we have shown (11.31).

□

Similar to the proof of Lemma 21, the following result is derived.

Lemma 22. *We have*

$$\begin{aligned}&\psi_{\varepsilon,k}(t_l^\varepsilon + \varepsilon y) - \psi_{\varepsilon,k}(t_l^\varepsilon) \qquad (11.36)\\ &= -\varepsilon y \nabla_{t_l^\varepsilon}\nabla_{t_k^\varepsilon} \hat{G}_D(t_l^\varepsilon,t_k^\varepsilon)\hat{\xi}_k^2\mu_k^{3/2}\left[1 + O\left(\frac{\varepsilon}{\sqrt{D}}y\right) + O\left(D\left(\log\frac{1}{D}\right)^2\right)\right] \\ &= -\varepsilon y \nabla_{t_l^\varepsilon}\nabla_{t_k^\varepsilon} \hat{K}_D(t_l^\varepsilon,t_k^\varepsilon)\hat{\xi}_k^2\mu_k^{3/2}[\delta_{l,k-1} + \delta_{l,k+1}]\left[1 + O\left(\frac{\varepsilon}{\sqrt{D}}y\right) + O\left(D\left(\log\frac{1}{D}\right)^2\right)\right]\end{aligned}$$

for $l \neq k$ and

$$(\psi_{\varepsilon,l} - \bar{H}'_\varepsilon)(t_l^\varepsilon + \varepsilon y) - (\psi_{\varepsilon,l} - \bar{H}'_\varepsilon)(t_l^\varepsilon) \tag{11.37}$$

$$= -\varepsilon y \sum_{m=1}^N \nabla^2_{t_l^\varepsilon} \hat{G}_D(t_l^\varepsilon, t_m^\varepsilon)\hat{\xi}_m^2 \mu_m^{3/2} \left[1 + O\left(\frac{\varepsilon}{\sqrt{D}} y\right) + O\left(D\left(\log\frac{1}{D}\right)^2\right)\right]$$

$$= -\varepsilon y \sum_{m,|m-l|=1} \nabla^2_{t_l^\varepsilon} \hat{K}_D(t_l^\varepsilon, t_m^\varepsilon)\hat{\xi}_m^2 \mu_m^{3/2} \left[1 + O\left(\frac{\varepsilon}{\sqrt{D}} y\right) + O\left(D\left(\log\frac{1}{D}\right)^2\right)\right].$$

For $J_{1,l}$, we compute

$$\begin{aligned}
J_{1,l} &= -\varepsilon \sum_{k=1}^N a_k^\varepsilon \int_{-1}^1 \frac{\tilde{w}_{\varepsilon,l}^2}{\bar{H}_\varepsilon^2} \left[\psi_{\varepsilon,k} - \bar{H}'_\varepsilon \delta_{lk}\right] \tilde{w}'_{\varepsilon,l}\, dx \\
&= -\varepsilon \sum_{k=1}^N a_k^\varepsilon \int_{-1}^1 \frac{\tilde{w}_{\varepsilon,l}^2}{\bar{H}_\varepsilon^2} \left[\psi_{\varepsilon,k}(t_l^\varepsilon) - \bar{H}'_\varepsilon(t_l^\varepsilon)\delta_{lk}\right] \tilde{w}'_{\varepsilon,l}\, dx \\
&\quad -\varepsilon \sum_{k=1}^N a_k^\varepsilon \int_{-1}^1 \frac{\tilde{w}_{\varepsilon,l}^2}{\bar{H}_\varepsilon^2} \left(\left[\psi_{\varepsilon,k}(x) - \bar{H}'_\varepsilon(x)\delta_{lk}\right] - \left[\psi_{\varepsilon,k}(t_l^\varepsilon) - \bar{H}'_\varepsilon(t_l^\varepsilon)\delta_{lk}\right]\right) \tilde{w}'_{\varepsilon,l}\, dx \\
&= J_{6,l} + J_{7,l}.
\end{aligned}$$

For $J_{6,l}$, we use (11.33) and Lemma 21 to obtain

$$\begin{aligned}
J_{6,l} &= -\frac{2}{3}\varepsilon \sum_{k=1}^N a_k^\varepsilon \int_{-1}^1 \frac{\tilde{w}_{\varepsilon,l}^3}{\bar{H}_\varepsilon^3} \bar{H}'_\varepsilon \left[\psi_{\varepsilon,k}(t_l^\varepsilon) - \bar{H}'_\varepsilon(t_l^\varepsilon)\delta_{lk}\right]\, dx \\
&= -\frac{2}{3}\varepsilon^2 \sum_{k=1}^N a_k^\varepsilon \left(\int_{\mathbb{R}} w_l^3\, dy\right) \bar{H}'_\varepsilon(t_l^\varepsilon) \left[\psi_{\varepsilon,k}(t_l^\varepsilon) - \bar{H}'_\varepsilon(t_l^\varepsilon)\delta_{lk}\right] \\
&\quad \times \left[1 + O\left(\frac{\varepsilon}{\sqrt{D}} + D\left(\log\frac{1}{D}\right)^2\right)\right] \\
&= \varepsilon^2 \sum_{k=1}^N a_k^\varepsilon \left(\frac{2}{3}\int_{\mathbb{R}} w^3\, dy\right) \mu_l^{5/2} \\
&\quad \left[\nabla_{t_k^\varepsilon} \hat{G}_D(t_l^\varepsilon, t_k^\varepsilon)\hat{\xi}_k^2 \mu_k^{3/2} - \sum_{k=1}^N \nabla_{t_l^\varepsilon} \hat{G}_D(t_l^\varepsilon, t_k^\varepsilon)\hat{\xi}_k^2 \mu_k^{3/2}\right] \\
&\quad \times \left[\sum_{j=1}^N \nabla_{t_l^\varepsilon} \hat{G}_D(t_l^\varepsilon, t_j^\varepsilon)\hat{\xi}_j^2 \mu_j^{3/2}\right] \left[1 + O\left(\frac{\varepsilon}{\sqrt{D}} + D\left(\log\frac{1}{D}\right)^2\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \varepsilon^2 \sum_{k=1}^N a_k^\varepsilon \left(\frac{2}{3}\int_{\mathbb{R}} w^3\,dy\right)\mu_l^{5/2} \\
&\quad \left[\nabla_{t_k^\varepsilon}\hat{K}_D(t_l^\varepsilon,t_k^\varepsilon)\hat{\xi}_k^2\mu_k^{3/2}[\delta_{l,k-1}+\delta_{l,k+1}] - \sum_{k,|k-l|=1}\nabla_{t_l^\varepsilon}\hat{K}_D(t_l^\varepsilon,t_k^\varepsilon)\hat{\xi}_k^2\mu_k^{3/2}\right] \\
&\quad \times\left[\sum_{j,|j-l|=1}\nabla_{t_l^\varepsilon}\hat{K}_D(t_l^\varepsilon,t_j^\varepsilon)\hat{\xi}_j^2\mu_j^{3/2}\right] + O\left(\varepsilon^2 D\left(\log\frac{1}{D}\right)^2\left(\frac{\varepsilon}{\sqrt{D}}+D\left(\log\frac{1}{D}\right)^2\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \varepsilon^2 \sum_{k=1}^N a_k^\varepsilon \left(\frac{2}{3}\int_{\mathbb{R}} w^3\,dy\right)\mu_l^{5/2} \\
&\quad \left[\nabla_{t_k^\varepsilon}\hat{K}_D(t_l^\varepsilon,t_k^\varepsilon)\hat{\xi}_k^2\mu_k^{3/2}[\delta_{l,k-1}+\delta_{l,k+1}] - \sum_{k,|k-l|=1}\nabla_{t_l^\varepsilon}\hat{K}_D(t_l^\varepsilon,t_k^\varepsilon)\hat{\xi}_k^2\mu_k^{3/2}\right] \\
&\quad \times\left[\sum_{j,|j-l|=1}\nabla_{t_l^\varepsilon}\hat{K}_D(t_l^\varepsilon,t_j^\varepsilon)\hat{\xi}_j^2\mu_j^{3/2}\right] + O\left(\varepsilon^2 D\left(\log\frac{1}{D}\right)^2\left(\frac{\varepsilon}{\sqrt{D}}+D\left(\log\frac{1}{D}\right)^2\right)\right) \\
&= O\left(\varepsilon^2 D\left(\log\frac{1}{D}\right)^2\right) + O\left(\varepsilon^2 D\left(\log\frac{1}{D}\right)^2\left(\frac{\varepsilon}{\sqrt{D}}+D\left(\log\frac{1}{D}\right)^2\right)\right) \\
&= O\left(\varepsilon^2 D\left(\log\frac{1}{D}\right)^2\right). \qquad (11.38)
\end{aligned}
$$

Similarly, we compute, using Lemma 22, (7.10) and (7.11),

$$
\begin{aligned}
J_{7,l} &= \varepsilon^2\hat{\xi}_l\int_{\mathbb{R}}\left(yw_l^2w_l'(y)\right)\,dy\sum_{k=1}^N\left(\nabla_{t_l^\varepsilon}\nabla_{t_k^\varepsilon}\hat{G}_D(t_l^\varepsilon,t_k^\varepsilon)\hat{\xi}_k^2\mu_k^{3/2}+\sum_{m=1}^N\nabla^2_{t_l^\varepsilon}\hat{G}_D(t_l^\varepsilon,t_m^\varepsilon)\hat{\xi}_m^2\mu_m^{3/2}\delta_{k,l}\right)a_k^\varepsilon \\
&\quad +O\left(\varepsilon^2\sqrt{D}\log\frac{1}{D}\left(\frac{\varepsilon}{\sqrt{D}}+D\left(\log\frac{1}{D}\right)^2\right)\right) \\
&= -\varepsilon^2\left(\frac{1}{3}\int_{\mathbb{R}}w^3\,dy\right)\left[\mu_l^{5/2}\sum_{k,|k-l|=1}\nabla_{t_l^\varepsilon}\nabla_{t_k^\varepsilon}\hat{K}_D(t_l^\varepsilon,t_k^\varepsilon)\hat{\xi}_k^2\mu_k^{3/2}a_k^\varepsilon\right. \\
&\quad \left.+\sum_{m,|m-l|=1}^N\nabla^2_{t_l^\varepsilon}\hat{K}_D(t_l^\varepsilon,t_m^\varepsilon)\hat{\xi}_m^2\mu_m^{3/2}a_l^\varepsilon\right]+O\left(\varepsilon^2\sqrt{D}\log\frac{1}{D}\left(\frac{\varepsilon}{\sqrt{D}}+D\left(\log\frac{1}{D}\right)^2\right)\right) \\
&= -\varepsilon^2\left(\frac{1}{3}\int_{\mathbb{R}}w^3\,dy\right)\mu_l^{5/2}\left[-\sum_{k,|k-l|=1}\nabla^2_{t_l^\varepsilon}\hat{K}_D(t_l^\varepsilon,t_k^\varepsilon)\hat{\xi}_k^2\mu_k^{3/2}a_k^\varepsilon\right. \\
&\quad \left.+\sum_{k,|k-l|=1}^N\nabla^2_{t_l^\varepsilon}\hat{K}_D(t_l^\varepsilon,t_k^\varepsilon)\hat{\xi}_k^2\mu_k^{3/2}a_l^\varepsilon\right]+O\left(\varepsilon^2\sqrt{D}\log\frac{1}{D}\left(\frac{\varepsilon}{\sqrt{D}}+D\left(\log\frac{1}{D}\right)^2\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= -\varepsilon^2 \left(\frac{1}{3}\int_{\mathbb{R}} w^3\,dy\right)\mu_l^{5/2} \sum_{k,|k-l|=1} \nabla^2_{t_l^\varepsilon} \hat{K}_D(t_l^\varepsilon, t_k^\varepsilon)\hat{\xi}_k^2 \mu_k^{3/2}(a_l^\varepsilon - a_k^\varepsilon) \\
&\quad +O\left(\varepsilon^2\sqrt{D}\log\frac{1}{D}\left(\frac{\varepsilon}{\sqrt{D}} + D\left(\log\frac{1}{D}\right)^2\right)\right) \\
&= -\varepsilon^2 \left(\frac{1}{3}\int_{\mathbb{R}} w^3\,dy\right)\mu_l^{5/2}\Big[-\nabla^2_{t_{l-1}^\varepsilon} \hat{K}_D(t_l^\varepsilon, t_{l-1}^\varepsilon)\hat{\xi}_{l-1}^2 \mu_{l-1}^{3/2} a_{l-1}^\varepsilon \\
&\quad -\nabla^2_{t_{l+1}^\varepsilon} \hat{K}_D(t_l^\varepsilon, t_{l+1}^\varepsilon)\hat{\xi}_{l+1}^2 \mu_{l+1}^{3/2} a_{l+1}^\varepsilon \\
&\quad +\nabla^2_{t_l^\varepsilon} \hat{K}_D(t_l^\varepsilon, t_{l+1}^\varepsilon)\hat{\xi}_l^2 \mu_{l+1}^{3/2} + \nabla^2_{t_l^\varepsilon} \hat{K}_D(t_l^\varepsilon, t_{l-1}^\varepsilon)\hat{\xi}_l^2 \mu_{l-1}^{3/2}] a_l^\varepsilon \Big] \\
&\quad +O\left(\varepsilon^2\sqrt{D}\log\frac{1}{D}\left(\frac{\varepsilon}{\sqrt{D}} + D\left(\log\frac{1}{D}\right)^2\right)\right). \qquad (11.39)
\end{aligned}
$$

Combining (11.38) and (11.39), we obtain (11.24).

For $J_{2,l}$, integration by parts gives

$$
\begin{aligned}
J_{2,l} &= \varepsilon \sum_{j=1}^N a_j^\varepsilon \int_{-1}^1 \mu' \tilde{w}_{\varepsilon,j} \tilde{w}'_{\varepsilon,l}\,dx \\
&= -\frac{\varepsilon a_l^\varepsilon}{2}\int_{-1}^1 \mu'' \tilde{w}_{\varepsilon,l}^2\,dx \left[1 + O\left(\frac{\varepsilon}{\sqrt{D}} + D\left(\log\frac{1}{D}\right)^2\right)\right] \\
&= -\frac{\varepsilon^2 a_l^\varepsilon}{2}\hat{\xi}_l^2 \mu_l^{3/2}\mu_l'' \int_{\mathbb{R}} w^2\,dy \left[1 + O\left(\frac{\varepsilon}{\sqrt{D}} + D\left(\log\frac{1}{D}\right)^2\right)\right]
\end{aligned}
$$

and (11.25) follows.

These are the main terms. The remaining terms are small and we will show that they are of the order $O\left(\varepsilon^2\left(\frac{\varepsilon}{\sqrt{D}} + D\left(\log\frac{1}{D}\right)^2\right)\right)$.

Similar to the proof of Proposition 18, it can be shown that $\tilde{L}_\varepsilon$ is invertible from $(\mathcal{K}_\varepsilon^{new})^\perp$ to $(\mathcal{C}_\varepsilon^{new})^\perp$ with uniformly bounded inverse for max $\left(\frac{\varepsilon}{\sqrt{D}}, D\right)$ small enough. By (11.18), (11.19), Lemma 21 and the fact that $\tilde{L}_\varepsilon$ is uniformly invertible, we deduce that

$$
\|\phi_\varepsilon^\perp\|_{H^2(\Omega_\varepsilon)} = O\left(\varepsilon\sqrt{D}\log\frac{1}{D}\right). \qquad (11.40)
$$

Then we have by the equation for $\psi_\varepsilon^\perp$

$$
\psi_\varepsilon^\perp(t_j^\varepsilon) = 2\xi_\varepsilon \int_{-1}^1 G_D(t_j^\varepsilon, z)\bar{w}_\varepsilon \phi_\varepsilon^\perp\,dz = O\left(\varepsilon\sqrt{D}\log\frac{1}{D}\right).
$$

Further, we estimate

$$
\psi_\varepsilon^\perp(t_j^\varepsilon + \varepsilon y) - \psi_\varepsilon^\perp(t_j^\varepsilon) = 2\xi_\varepsilon \int_{-1}^1 [G_D(t_j^\varepsilon + \varepsilon y, z) - G_D(t_j^\varepsilon + \varepsilon y, z)]\bar{w}_\varepsilon \phi_\varepsilon^\perp\,dz
$$

$$= 2\varepsilon y \xi_\varepsilon \int_{-1}^{1} \nabla_{t_j^\varepsilon} G_D(t_j^\varepsilon y, z) \bar{w}_\varepsilon \phi_\varepsilon^\perp \, dz \left[1 + O\left(\frac{\varepsilon}{\sqrt{D}} + D\left(\log\frac{1}{D}\right)^2 \right) \right]$$
$$= O\left(\varepsilon\sqrt{D}\log\frac{1}{D}\, \varepsilon\sqrt{D}\log\frac{1}{D}\, y \right) = O\left(\varepsilon^2 D \left(\log\frac{1}{D}\right)^2 y \right). \qquad (11.41)$$

These estimates of $\psi_\varepsilon^\perp$ and $\phi_\varepsilon^\perp$ are important for the rest of the proof.

For $J_{3,l}$, we have by (11.33), (11.40)

$$J_{3,l} = \bar{H}_\varepsilon'(t_l^\varepsilon) \int_{-1}^{1} w_l^2 \phi_\varepsilon^\perp \, dx \left[1 + O\left(\frac{\varepsilon}{\sqrt{D}} + D\left(\log\frac{1}{D}\right)^2 \right) \right]$$
$$= O\left(\sqrt{D}\log\frac{1}{D} \varepsilon \|\phi_\varepsilon^\perp\|_{H^2(\Omega_\varepsilon)} \right) = O\left(\varepsilon^2 D \left(\log\frac{1}{D}\right)^2 \right)$$

which proves (11.26).

For $J_{4,l}$, we decompose

$$J_{4,l} = J_{8,l} + J_{9,l},$$

where

$$J_{8,l} = -\int_{-1}^{1} \frac{\tilde{w}_{\varepsilon,l}^2}{\bar{H}_\varepsilon^2} (\psi_\varepsilon^\perp(t_l^\varepsilon) \tilde{w}_{\varepsilon,l}') \, dx, \qquad (11.42)$$

$$J_{9,l} = -\int_{-1}^{1} \frac{\tilde{w}_{\varepsilon,l}^2}{\bar{H}_\varepsilon^2} (\psi_\varepsilon^\perp(x) - \psi_\varepsilon^\perp(t_l^\varepsilon)) \tilde{w}_{\varepsilon,l}' \, dx. \qquad (11.43)$$

For $J_{8,l}$, we have using (11.33), (11.41)

$$\begin{aligned} J_{8,l} &= -\psi_\varepsilon^\perp(t_l^\varepsilon) \int_{-1}^{1} \frac{\tilde{w}_{\varepsilon,l}^2}{\bar{H}_\varepsilon^2} \tilde{w}_{\varepsilon,l}' \, dx \\ &= \frac{2}{3}\psi_\varepsilon^\perp(t_l^\varepsilon) \int_{-1}^{1} \frac{\tilde{w}_{\varepsilon,l}^3}{\bar{H}_\varepsilon^3} \bar{H}_\varepsilon' \, dx \left[1 + O\left(\frac{\varepsilon}{\sqrt{D}} + D\left(\log\frac{1}{D}\right)^2 \right) \right] \\ &= -\varepsilon\frac{2}{3} \bar{H}_\varepsilon'(t_l^\varepsilon) \psi_\varepsilon^\perp(t_l^\varepsilon) \mu_l^{5/2} \left(\int_{\mathbb{R}} w^3 \, dy \right) \left[1 + O\left(\frac{\varepsilon}{\sqrt{D}} + D\left(\log\frac{1}{D}\right)^2 \right) \right] \\ &= O\left(\varepsilon\sqrt{D}\log\frac{1}{D}\, \varepsilon\sqrt{D}\log\frac{1}{D} \right) = O\left(\varepsilon^2 D \left(\log\frac{1}{D}\right)^2 \right). \end{aligned} \qquad (11.44)$$

For $J_{9,l}$, we have using (11.41)

$$J_{9,l} = -\int_{-1}^{1} \frac{\tilde{w}_{\varepsilon,l}^2}{\bar{H}_\varepsilon^2} (\psi_\varepsilon^\perp(x) - \psi_\varepsilon^\perp(t_l^\varepsilon)) \tilde{w}_{\varepsilon,l}' \, dx = O\left(\varepsilon^2 D \left(\log\frac{1}{D}\right)^2 \right). \qquad (11.45)$$

Now (11.27) follows from (11.44), (11.45).

Finally, we estimate using (11.40) and $\mu'(t_i) = O\left(\sqrt{D}\log\frac{1}{D}\right)$ that

$$J_{5,l} = \int_{-1}^{1} \mu' \phi_\varepsilon^\perp w_{\varepsilon,l} \, dx = O\left(\varepsilon\, \varepsilon\sqrt{D}\log\frac{1}{D} \sqrt{D}\log\frac{1}{D} \right) = O\left(\varepsilon^2 D \left(\log\frac{1}{D}\right)^2 \right) \qquad (11.46)$$

and (11.28) follows.

□

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