On The Number of Interior Peak Solutions for A Singularly Perturbed Neumann Problem

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Abstract

We consider the following singularly perturbed Neumann problem:

$$\epsilon^2 \Delta u - u + f(u) = 0 \text{ in } \Omega, \quad u > 0 \text{ in } \Omega \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega$$

where $\Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, $\epsilon > 0$ is a constant, Ω is a bounded smooth domain in $\mathbb{R}^N$ with its unit outward normal ν, and f is superlinear and subcritical. A typical f is $f(u) = u^p$ where $1 < p < +\infty$ when $N = 2$ and $1 < p < \frac{N+2}{N-2}$ when $N \geq 3$.

We show that there exists an $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$ and for each integer K bounded by

$$1 \leq K \leq \frac{\alpha_{N,\Omega,f}}{\epsilon^N (|\ln \epsilon|)^N}$$

where $\alpha_{N,\Omega,f}$ is a constant depending on N, Ω and f only, there exists a solution with K interior peaks. (An explicit formula for $\alpha_{N,\Omega,f}$ is also given.) As a consequence, we obtain that for ϵ sufficiently small, there exists at least $[\frac{\alpha_{N,\Omega f}}{\epsilon^N (|\ln \epsilon|)^N}]$ number of solutions. Moreover, for each $m \in (0, N)$ there exist solutions with energies in the order of ϵ^{N-m}.

1 Introduction

The main theme of this paper is the concentration phenomena of the following singularly perturbed elliptic problem

$$\text{(1.1)} \qquad \begin{cases} \epsilon^2 \Delta u - u + u^p = 0 \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \end{cases}$$

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where $\Delta = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, $\epsilon > 0$ is a constant, the exponent p satisfies $1 < p < \frac{N+2}{N-2}$ for $N \geq 3$ and $1 < p < \infty$ for $N = 2$, and Ω is a bounded smooth domain in $\mathbb{R}^N$ with its unit outward normal ν.

For the last fifteen years, problem (1.1) has received considerable attention as it has been shown that its solutions have rich and interesting structures. In particular, the various concentration phenomena exhibited by the solutions of (1.1) seem both mathematically intriguing and scientifically useful.

Although problem (1.1) takes a classical form of singular perturbations, the traditional techniques in that area did not seem helpful as the error terms appeared in the inner and outer expansions are exponentially small in $\epsilon > 0$.

In the papers [22, 28, 29], the authors studied the following "energy" functional in $H^1(\Omega)$ associated with (1.1) via a variational approach

$$E_\epsilon[u] = \frac{1}{2}\int_\Omega (\epsilon^2|\nabla u|^2 + u^2) - \frac{1}{p+1}\int_\Omega u_+^{p+1}, \tag{1.2}$$

where $u_+ = \max\{u, 0\}$.

It is easily seen that E_ϵ is not bounded above nor bounded below. However, Ni and Takagi observed in [28] that, among all possible solutions of (1.1), there is a "least-energy" solution u_ϵ; that is, a solution u_ϵ with minimal energy. Furthermore, they showed in [28, 29] that, *for each $\epsilon > 0$ sufficiently small, u_ϵ has exactly one (local) maximum point P_ϵ in $\bar{\Omega}$, and P_ϵ must be located on $\partial\Omega$ and near the "most curved" part of the $\partial\Omega$.* More precisely, *u_ϵ must tend to 0, as $\epsilon \to 0$, everywhere on $\bar{\Omega}$ except at P_ϵ, and $H(P_\epsilon) \to \max_{P \in \partial\Omega} H(P)$ where H denotes the mean curvature of the boundary $\partial\Omega$.* (Such points P_ϵ will be referred to as *peaks* or *spikes*.) The crucial idea in [28, 29] is to establish the following energy estimate for ϵ small

$$E_\epsilon(u_\epsilon) = \epsilon^N[C_1 - C_2 H(P_\epsilon)\epsilon + o(\epsilon)] \tag{1.3}$$

where C_1, C_2 are two positive constants. From (1.3), the above conclusion on the location of the peak P_ϵ is derived.

Since the publication of [29], problem (1.1) has received a great deal of attention and significant progress has been made. More specifically, solutions with multiple boundary peaks as well as multiple interior peaks have been established. It turns out that *a general guideline is that while multiple boundary spikes tend to cluster around the local minimum points of the boundary mean curvature $H(P)$, the location of the interior spikes are governed by the distance between the peaks as well as the boundary of $\partial\Omega$.* (See [3, 4, 6-10, 12-15, 19, 20, 29-36] and the references therein.) In particular, it was established in Gui and Wei [14] that *for any two given integers $k \geq 0, l \geq 0$ and $k + l > 0$, problem (1.1) has a solution with exactly k interior spikes and l boundary spikes* for every ϵ sufficiently small. This solution has its "energy" still at the level ϵ^N and again has the property that it tends to 0 everywhere in $\bar{\Omega}$ except at those $k + l$ points. Thus, in general we call such spiky solutions as *solutions with $0-$dimensional concentration sets.*

It seems natural to ask if problem (1.1) has solutions which "concentrate" on higher dimensional sets, e.g. curves, or surfaces. In this regards, we mention that it has been *conjectured* for a long time that *problem (1.1) actually possesses solutions which have* $m-$*dimensional concentration sets for every* $0 \leq m \leq n-1$. (See e.g. [27].) Furthermore, it is intuitively clear that the "energy" levels of many solutions with $m-$dimensional concentration sets would be at the order ϵ^{N-m}. Progress in this direction, although still limited, has also been made in [2, 23, 24, 25, 26].

In this paper we shall explore the question of *the maximal number of spikes*, in terms of the small parameter $\epsilon > 0$, *a solution of (1.1) could possibly have*. Our main result, Theorem 1.1 below, asserts that *for every positive integer* $K \leq C(\epsilon|\ln\epsilon|)^{-N}$, *where* C *is a suitable constant depending only on* p *and* Ω, *problem (1.1) has a solution with exactly* K *peaks.* As a corollary, one derives immediately that *problem (1.1) already possesses spiky solutions* u_ϵ (i.e. solutions with only $0-$dimensional concentration sets), *with "energy" levels*

$$E_\epsilon[u_\epsilon] \sim \epsilon^{N-m} \tag{1.4}$$

where m *ranges from* 0 *to* $N-1$. (In this paper we use "$A_\epsilon \sim B_\epsilon''$ to denote that there exist positive constants C' and C'' such that $C' \leq A_\epsilon/B_\epsilon \leq C''$ for ϵ small.) Thus it seems difficult to use different energy levels to characterize solutions with concentration sets of different dimensions. This further illustrates how complicated the simple problem (1.1) could be, and its richness seems somewhat surprising.

Our proof uses a "localized energy" method as in [13], but we now need to obtain accurate estimates of the "interactions" between two peaks as well as the "interactions" between a peak and the boundary of the domain Ω. This is not an easy task – it requires a thorough understanding of various approximations of the solutions of (1.1) we are looking for.

To state our main result, we shall include a slightly more general equation than (1.1), namely,

$$\begin{cases} \epsilon^2\Delta u - u + f(u) = 0 \ \text{ in } \Omega, \\ u > 0 \ \text{ in } \ \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \ \text{ on } \partial\Omega. \end{cases} \tag{1.5}$$

We will always assume that $f : \mathbb{R} \to \mathbb{R}$ is of class $C^{1+\sigma}$ for some $0 < \sigma \leq 1$ and satisfies the following conditions (f1)-(f2):

(f1) $f(u) \equiv 0$ for $u \leq 0$, $f(0) = f'(0) = 0$.
(f2) The following equation

$$\begin{cases} \triangle w - w + f(w) = 0, \ w > 0 \quad \text{ in } \mathbb{R}^N, \\ w(0) = \max_{y \in \mathbb{R}^N} \ w(y), \ w \to 0 \quad \text{at } \infty, \end{cases} \tag{1.6}$$

has a unique solution $w(y)$ and w is nondegenerate, i.e.

$$\text{Kernel}(\Delta - 1 + f'(w)) = \text{span}\left\{ \frac{\partial w}{\partial y_1}, ..., \frac{\partial w}{\partial y_N} \right\}. \tag{1.7}$$

Occasionally, we assume further that f satisfies (f3):

(f3) The principal eigenvalue of $\Delta - 1 + f^{'}(w)$ is positive. That is, there exists an eigenvalue $\lambda_1 > 0$ and the corresponding eigenfunction Φ_0 (which can be made positive and radially symmetric) satisfying

$$\Delta \Phi_0 - \Phi_0 + f^{'}(w)\Phi_0 = \lambda_1 \Phi_0, \Phi_0 \in H^1(\mathbb{R}^N). \tag{1.8}$$

One typical example of f is: $f(u) = u^p - a u^q$, where $a \geq 0, 1 < q < p < (\frac{N+2}{N-2})_+ (= \infty$ if $N = 2; = \frac{N+2}{N-2}$ if $N > 2)$. For the uniqueness of w, see [5], [17] and [18].

The energy functional associated with (1.5) is

$$E_\epsilon[u] = \frac{1}{2} \int_\Omega (\epsilon^2 |\nabla u|^2 + u^2) - \int_\Omega F(u), \ u \in H^1(\Omega) \tag{1.9}$$

where $F(u) = \int_0^u f(s) ds$.

We now state our main result in this paper.

Theorem 1.1. *Let f satisfy assumptions (f1)-(f2). Then there exists an $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$ and any positive integer K satisfying*

$$1 \leq K \leq \frac{\alpha_{N,\Omega,f}}{\epsilon^N (|\ln \epsilon|)^N}, \tag{1.10}$$

where $\alpha_{N,\Omega,f}$ is a constant depending on N, Ω and f only, problem (1.5) has a solution u_ε which possesses exactly K local maximum points $Q_1^\varepsilon, ..., Q_K^\epsilon$ such that

$$u_\epsilon(x) = \sum_{j=1}^K w(\frac{x - Q_j^\epsilon}{\epsilon}) + o(1) \tag{1.11}$$

and we have the following energy estimate

$$E_\epsilon[u_\epsilon] = \epsilon^N K I[w](1 + o(1)) \tag{1.12}$$

where $I[w]$ is the energy of w:

$$I[w] = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla w|^2 + w^2) - \int_{\mathbb{R}^N} F(w). \tag{1.13}$$

As a consequence, for each integer $m \in [0, N)$, there exists a solution u_ϵ to (1.5) with the following energy bound

$$E_\epsilon[u_\epsilon] \sim \epsilon^{N-m}. \tag{1.14}$$

When $m = N$, we have a solution u_ϵ with the following energy estimate

$$E_\epsilon[u_\epsilon] \sim (|\ln \epsilon|)^{-N}. \tag{1.15}$$

If we further assume that f satisfies (f3), then the Morse index of u_ϵ is at least K.

A simple corollary is the following result which gives a lower bound on the number of positive solutions to (1.5).

Theorem 1.2. *For ϵ sufficiently small, problem (1.5) has at least* $\left[\frac{\alpha_{N,\Omega,f}}{\epsilon^N|\ln\epsilon|^N}\right]$ *number of positive solutions.*

Remarks: 1. It seems that the upper bound for K is "almost" best possible. Note that when $N=1$, the upper bound is $\frac{C}{\epsilon}$. When $N\geq 2$, because of the boundary mean curvature, it seems that the best upper bound could be $\frac{C}{\epsilon^N(|\ln\epsilon|)^N}$, where the best constant C should depend on N, f, and the domain geometry.

2. The constant $\alpha_{N,\Omega,f}$ can be made more precise. Let $r\leq \max_{Q\in\Omega} d(Q,\partial\Omega)$ be a small positive number. We denote by $K_\Omega(r)$, the maximum number of non-overlapping balls with equal radius r packed in Ω. By checking the computations of the proof of Theorem 1.1 (see (5.4)), we can take

$$1\leq K\leq K_\Omega\left((\frac{6+4\sigma}{2\sigma}N+\frac{1}{200})\epsilon|\ln\epsilon|\right), \tag{1.16}$$

where

$$\sigma \text{ is the Hölder exponent of } f'.$$

Let

$$C_\Omega=\liminf_{r\to 0}\ r^N K_\Omega(r). \tag{1.17}$$

Then we may take

$$\alpha_{N,\Omega,f}=C_\Omega\left(\frac{2\sigma}{(6+4\sigma)N+\frac{\sigma}{100}}\right)^N. \tag{1.18}$$

3. Unlike [13], where the limiting location of the spikes $(Q_1^\epsilon,...,Q_K^\epsilon)$ can be identified as sphere-packing positions, we can not say much more about the locations of the spikes. This remains an interesting question.

To conclude the Introduction, we include a brief description of the background of (1.1). Problem (1.1) arises in many models concerning biological pattern formations. For instance, it gives rise to steady states in the Keller-Segel model of the chemotactic aggregation of the cellular slime molds ([19, 22]) and it also plays an important role in the Gierer-Meinhardt model describing the regeneration phenomena of *hydra*.

By a straightforward scaling argument, we can easily construct positive steady state solutions of the following system from solutions of (1.1)

$$\begin{cases} \mathcal{U}_t=d_1\Delta\mathcal{U}-\mathcal{U}+\frac{\mathcal{U}^p}{\xi^q} \ \text{ in } \Omega\times(0,+\infty),\\ \tau\xi_t=-\xi+\frac{1}{|\Omega|}\xi^{-s}\int_\Omega \mathcal{U}^r,\\ \frac{\partial\mathcal{U}}{\partial\nu}=0 \ \text{ on } \partial\Omega\times(0,+\infty),\end{cases} \tag{1.19}$$

where $d_1=\epsilon^2$, $|\Omega|$ is the measure of Ω, and p,q,r,s,τ are nonnegative and satisfy

$$0<\frac{p-1}{q}<\frac{r}{s+1}.$$

Problem (1.19) is, in turn, the shadow-system of the well-known activator-inhibitor system proposed by Gierer and Meinhardt following Turing's idea of *diffusion-driven instability*, in modelling the regeneration phenomenon of *hydra* in morphogenesis ([11, 32]), see also the survey [27]. More precisely, (1.19) is obtained formally by letting $d_2 \to +\infty$ in the following system

$$\text{(1.20)} \qquad \begin{cases} \mathcal{U}_t = d_1 \Delta \mathcal{U} - \mathcal{U} + \frac{\mathcal{U}^p}{\mathcal{V}^q} \ \text{ in } \Omega \times (0, +\infty), \\ \tau \mathcal{V}_t = d_2 \Delta \mathcal{V} - \mathcal{V} + \frac{\mathcal{U}^r}{\mathcal{V}^s} \ \text{ in } \Omega \times (0, +\infty), \\ \frac{\partial \mathcal{U}}{\partial \nu} = \frac{\partial \mathcal{V}}{\partial \nu} = 0 \ \text{ on } \partial\Omega \times (0, +\infty). \end{cases}$$

Indeed, the system (1.20) was motivated by biological experiments on *hydra* in morphogenesis. *Hydra*, an animal of a few millimeters in length, is made up of approximately 100, 000 cells of about 15 different types. It consists of a "head" region located at one end along its length. Typical experiments on *hydra* involve removing part of its "head" region and transplanting it to other parts of the body column. Then, a new "head" will form if and only of the transplanted area is sufficiently far from the (old) "head". These observation have led to the assumption of the existence of two chemical substances – a slowly diffusing (short-range) activator and a rapidly diffusing (long-range) inhibitor. In 1952, A. Turing argued, although diffusion is a smoothing and trivializing process in a single chemical, for systems of two or more chemicals, different diffusion rates could force the uniform steady states to become unstable and lead to nonhomogeneous distributions for such reactants. This is now known as the "diffusion-driven instability". Exploring this idea further, in 1972, Gierer and Meinhardt proposed the system (1.20) to model the above regeneration phenomenon of *hydra*.

The paper is organized as follows. Notation, preliminaries and some useful estimates are explained in Section 2. Section 3 contains the study of a linear problem which is the first step in the Liapunov-Schmidt reduction process. In Section 4 we solve a nonlinear problem which sets up a maximization problem in Section 5. Finally in Section 6 we show that the solution to the maximization problem is indeed a solution of (2.2) and satisfies all the properties of Theorem 1.1.

Throughout this paper, unless otherwise stated, the letter C will always denote various generic constants which are independent of ϵ and K, for ϵ sufficiently small.

2 Notation and Some Preliminary Analysis

In this section we introduce some notation and present some preliminary analysis on approximate solutions.

Without loss of generality, we may assume that $0 \in \Omega$. By the following rescaling:

$$\text{(2.1)} \qquad x = \epsilon z, \ z \in \Omega_\epsilon := \{\epsilon z \in \Omega\},$$

equation (1.5) becomes

$$\begin{cases} \Delta u - u + f(u) = 0 \ \text{ in } \Omega_\epsilon, \\ u > 0 \ \text{ in } \Omega_\epsilon, \quad \text{and } \frac{\partial u}{\partial \nu} = 0 \ \text{ in } \partial\Omega_\epsilon. \end{cases} \tag{2.2}$$

For $u \in H^2(\Omega_\epsilon)$, we put

$$S_\epsilon[u] = \Delta u - u + f(u). \tag{2.3}$$

Then (2.2) is equivalent to

$$S_\epsilon[u] = 0, u \in H^2(\Omega_\epsilon), u > 0 \text{ in } \Omega_\epsilon, \ \frac{\partial u}{\partial \nu} = 0 \ \text{ on } \partial\Omega_\epsilon. \tag{2.4}$$

Associated with problem (2.2) is the following energy functional

$$J_\epsilon[u] = \frac{1}{2}\int_{\Omega_\epsilon} (|\nabla u|^2 + u^2) - \int_{\Omega_\epsilon} F(u), \ \ u \in H^1(\Omega_\epsilon). \tag{2.5}$$

Note that $J_\epsilon = \epsilon^{-N} E_\epsilon$.

We define two inner products:

$$< u, v >_\epsilon = \int_{\Omega_\epsilon} uv, \ \text{for } u, v \in L^2(\Omega_\epsilon); \tag{2.6}$$

$$(u, v)_\epsilon = \int_{\Omega_\epsilon} (\nabla u \nabla v + uv), \ \text{for } u, v \in H^1(\Omega_\epsilon). \tag{2.7}$$

Let

$$M > \frac{6 + 2\sigma}{\sigma} N \tag{2.8}$$

be a fixed positive constant. Now we define a configuration space:

$$\Lambda := \left\{ (Q_1, ..., Q_K) \in \Omega^K \middle| \varphi(Q_1, ..., Q_K) \geq M\epsilon |\ln \epsilon| \right\} \tag{2.9}$$

where

$$\varphi(Q_1, ..., Q_K) = \min_{i,j,k=1,...,K, i\neq j} (|Q_i - Q_j|, 2d(Q_k, \partial\Omega)). \tag{2.10}$$

Let w be the unique solution of (1.6). By the well-known result of Gidas, Ni and Nirenberg [12], w is radially symmetric: $w(y) = w(|y|)$ and strictly decreasing: $w^{'}(r) < 0$ for $r > 0, r = |y|$. Moreover, we have the following asymptotic behavior of w :

(2.11)

$$w(r) = A_N r^{-\frac{N-1}{2}} e^{-r}(1 + O(\frac{1}{r})), \quad w^{'}(r) = -A_N r^{-\frac{N-1}{2}} e^{-r}(1 + O(\frac{1}{r})),$$

for r large, where $A_N > 0$ is a constant. Let $K(r)$ be the fundamental solution of $-\Delta + 1$ centered at 0. Then we have

$$w(r) = (A_0 + O(\frac{1}{r}))K(r), \ w^{'}(r) = (-A_0 + O(\frac{1}{r}))K(r), \ \text{for } r \geq 1 \tag{2.12}$$

where A_0 is a positive constant.

For $Q \in \Omega$, we define $w_{\epsilon,Q}$ to be the unique solution of

$$\Delta v - v + f(w(\cdot - \frac{Q}{\epsilon})) = 0 \text{ in } \Omega_\epsilon, \quad \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega_\epsilon. \tag{2.13}$$

We first analyze $w_{\epsilon,Q}$. To this end, set

$$\varphi_{\epsilon,Q}(x) = w(\frac{|x-Q|}{\epsilon}) - w_{\epsilon,Q}(\frac{x}{\epsilon}).$$

We state the following useful lemma on the properties of $\varphi_{\epsilon,Q}$.

Lemma 2.1. *Assume that* $\frac{M}{2}\epsilon|\ln\epsilon| \le d(Q,\partial\Omega) \le \delta$ *where* δ *is sufficiently small. We have*

$$\varphi_{\epsilon,Q} = -(A_0 + o(1))K(\frac{|x-Q^*|}{\epsilon}) + O(\epsilon^{\sqrt{2}M+N+1}) \tag{2.14}$$

where $Q^* = Q + 2d(Q,\partial\Omega)\nu_{\bar{Q}}$, $\nu_{\bar{Q}}$ *denotes the unit outer normal at* $\bar{Q} \in \partial\Omega$ *and* $\bar{Q}$ *is the unique point on* $\partial\Omega$ *such that* $d(\bar{Q},Q) = d(Q,\partial\Omega)$.

Proof. Let $\Psi_\epsilon(x)$ be the unique solution of

$$\epsilon^2\Delta\Psi_\epsilon - \Psi_\epsilon = 0 \text{ in } \Omega, \quad \frac{\partial\Psi_\epsilon}{\partial\nu} = 1 \text{ on } \partial\Omega. \tag{2.15}$$

It is easy to see that

$$0 < \Psi_\epsilon(x) \le \Psi_1(x) \le C, \text{ for } \epsilon < 1. \tag{2.16}$$

On the other hand, $\varphi_{\epsilon,Q}(x)$ satisfies

$$\epsilon^2\triangle v - v = 0 \text{ in } \Omega, \quad \frac{\partial v}{\partial\nu} = \frac{\partial}{\partial\nu}w(\frac{|x-Q|}{\epsilon}) \text{ on } \partial\Omega.$$

Using (2.11), we see that on $\partial\Omega$,

$$\frac{\partial}{\partial\nu}w(\frac{|x-Q|}{\epsilon}) = \frac{1}{\epsilon}w'(\frac{|x-Q|}{\epsilon})\frac{< x-Q,\nu >}{|x-Q|}$$
$$= -(A_N + o(1))\varepsilon^{\frac{N-3}{2}}e^{-\frac{|x-Q|}{\varepsilon}}\frac{< x-Q,\nu >}{|x-Q|^{\frac{N+1}{2}}},$$

which implies, by a comparison principle, that

$$|\varphi_{\epsilon,Q}(x)| \le C\epsilon^{\sqrt{2}M+N+1}\Psi_\epsilon(x), \text{ if } d(Q,\partial\Omega) \ge (\sqrt{2}M+N+2)\epsilon|\ln\epsilon|. \tag{2.17}$$

Therefore it remains to consider the case when $\frac{M}{2}\epsilon|\ln\epsilon| \le d(Q,\partial\Omega) \le (\sqrt{2}M+N+2)\epsilon|\ln\epsilon|$. In this case, we use the following comparison function

$$\varphi_1(x) = -(A_0 - \epsilon^{\frac{1}{4}})K(\frac{|x-Q^*|}{\epsilon}) + \epsilon^{\sqrt{2}M+N+1}\Psi_\epsilon.$$

For $x \in \partial\Omega, |x-Q| \le \epsilon^{3/4}$, we have

$$\frac{< x-Q,\nu >}{|x-Q|} = (1 + O(\epsilon^{\frac{1}{2}}|\ln\epsilon|))\frac{< x-Q^*,\nu >}{|x-Q^*|},$$

$$\frac{|x-Q|}{\epsilon} = (1+O(\epsilon^{\frac{1}{2}}|\ln\epsilon|))\frac{|x-Q^*|}{\epsilon},$$

and hence

$$\frac{\partial \varphi_{\epsilon,Q}}{\partial \nu}(x) \leq \frac{\partial}{\partial \nu}\left((-A_0+\epsilon^{\frac{1}{4}})K(\frac{|x-Q^*|}{\epsilon})\right) \leq \frac{\partial \varphi_1}{\partial \nu}.$$

For $x \in \partial\Omega, |x-Q| \geq \epsilon^{3/4}$, we have

$$\frac{\partial \varphi_{\epsilon,Q}}{\partial \nu} \leq Ce^{-\epsilon^{-1/4}} \leq 2\epsilon^{\sqrt{2}M+N+1} \leq \frac{\partial \varphi_1}{\partial \nu}.$$

Summarizing we have for $x \in \partial\Omega$,

$$\frac{\partial \varphi_{\epsilon,Q}}{\partial \nu} \leq \frac{\partial \varphi_1}{\partial \nu}.$$

By a comparison principle, we have

$$\varphi_{\epsilon,Q}(x) \leq \varphi_1(x), \text{ for } x \in \Omega.$$

Similarly, we obtain

$$\varphi_{\epsilon,Q}(x) \geq -(A_0+\epsilon^{\frac{1}{4}})K(\frac{|x-Q^*|}{\epsilon}) - \epsilon^{\sqrt{2}M+N+1}\Psi_\epsilon \text{ for } x \in \Omega.$$

□

For $\mathbf{Q} = (Q_1, ..., Q_K) \in \Lambda$, we define

(2.18) $$w_i(z) = w(|z-\frac{Q_i}{\epsilon}|), \ \ w_{\epsilon,\mathbf{Q}}(z) = \sum_{j=1}^{K} w_{\epsilon,Q_j}.$$

The next lemma analyzes $w_{\epsilon,\mathbf{Q}}$ in Ω_ϵ. To this end, we divide Ω_ϵ into $K+1$-parts:

(2.19) $$\Omega_{\epsilon,j} = \{|z-\frac{Q_j}{\epsilon}| \leq \frac{1}{2\epsilon}\varphi(\mathbf{Q})\}, j=1,...,K, \quad \Omega_{\epsilon,K+1} = \Omega_\epsilon \setminus \cup_{j=1}^{K} \Omega_{\epsilon,j}$$

where $\varphi(\mathbf{Q})$ is defined at (2.10).

Lemma 2.2. *For $z \in \Omega_{\epsilon,j}, j=1,...,K$, we have*

(2.20) $$w_{\epsilon,\mathbf{Q}} = w_{\epsilon,Q_j} + O(K\epsilon^{\frac{M}{2}}) = w(|z-\frac{Q_j}{\epsilon}|) + O(K\epsilon^{\frac{M}{2}}).$$

For $z \in \Omega_{\epsilon,K+1}$, we have

(2.21) $$w_{\epsilon,\mathbf{Q}} = O(K\epsilon^{\frac{M}{2}}).$$

Proof. For $k \neq j$ and $z \in \Omega_{\epsilon,j}$, we have

$$w_{\epsilon,Q_k}(z) = w(|z - \frac{Q_k}{\epsilon}|) - \varphi_{\epsilon,Q_k}(\epsilon z)$$

$$= O(e^{-|z-\frac{Q_k}{\epsilon}|} + e^{-|z-\frac{Q_k^*}{\epsilon}|} + \epsilon^{M+N+1}) = O(\epsilon^{\frac{M}{2}})$$

and so

$$\sum_{k\neq j} w_{\epsilon,Q_k} = O(K\epsilon^{\frac{M}{2}})$$

which proves (2.20). The proof of (2.21) is similar. □

Next we state a useful lemma about the interactions of two w's.

Lemma 2.3. *For $\frac{|Q_1-Q_2|}{\epsilon}$ large, it holds*

$$\int_{\mathbb{R}^N} f(w(|z - \frac{Q_1}{\epsilon}|))w(|z - \frac{Q_2}{\epsilon}|) = (\gamma + o(1))w(\frac{|Q_1 - Q_2|}{\epsilon}) \tag{2.22}$$

where

$$\gamma = \int_{\mathbb{R}^N} f(w(|y|))e^{-y_1}dy. \tag{2.23}$$

Remark: Note that $\gamma > 0$. See Lemma 4.7 of [31].

Proof. By (2.11), we have for $|\epsilon y| << |Q_1 - Q_2|$,

$$w(|y + \frac{Q_1 - Q_2}{\epsilon}|) = (A_N + o(1))(\frac{\epsilon}{|\epsilon y + Q_1 - Q_2|})^{\frac{N-1}{2}} e^{-|y+\frac{Q_1-Q_2}{\epsilon}|}$$

$$= w(\frac{|Q_1 - Q_2|}{\epsilon})e^{-<y,\frac{Q_1-Q_2}{|Q_1-Q_2|}>+o(|y|)}.$$

Thus by Lebesgue's Dominated Convergence Theorem

$$\int_{\mathbb{R}^N} f(w(|z - \frac{Q_1}{\epsilon}|))w(|z - \frac{Q_2}{\epsilon}|) = \int_{\mathbb{R}^N} f(w(|y|))w(|y + \frac{Q_1 - Q_2}{\epsilon}|)$$

$$= (1+o(1))w(\frac{|Q_1 - Q_2|}{\epsilon}) \int_{\mathbb{R}^N} f(w(|y|))e^{-<y,\frac{Q_1-Q_2}{|Q_1-Q_2|}>}dy = (\gamma+o(1))w(\frac{|Q_1 - Q_2|}{\epsilon}).$$

□

Let us define several quantities for later use:

$$B_\epsilon(Q_j) = -\int_{\Omega_\epsilon} f(w_j)\varphi_{\epsilon,Q_j},\ B_\epsilon(Q_i, Q_j) = \int_{\Omega_\epsilon} f(w_i)w_j. \tag{2.24}$$

Then we have

Lemma 2.4. *For $\mathbf{Q} = (Q_1, ..., Q_K) \in \Lambda$, it holds*

$$B_\epsilon(Q_j) = (\gamma + o(1))w(\frac{2d(Q_j, \partial\Omega)}{\epsilon}) + o(w(M|\ln\epsilon|)), \tag{2.25}$$

$$B_\epsilon(Q_i, Q_j) = (\gamma + o(1))w(\frac{|Q_i - Q_j|}{\epsilon}) + o(w(M|\ln\epsilon|)). \tag{2.26}$$

Proof. Note that $A_0 K(\frac{|x-Q^*|}{\epsilon}) = (1+o(1))w(\frac{|x-Q^*|}{\epsilon})$ and by Lemma 2.1

$$B_\epsilon(Q_j) = (1+o(1))\int_{\Omega_\epsilon} f(w_j)w(|z - \frac{Q_j^* - Q_j}{\epsilon}|) + O(\epsilon^{\sqrt{2}M+N+1})$$

$$= (\gamma+o(1))w(\frac{|Q_j - Q_j^*|}{\epsilon})+o(w(M|\ln\epsilon|)) = (\gamma+o(1))w(\frac{2d(Q_j,\partial\Omega)}{\epsilon})+o(w(M|\ln\epsilon|)).$$

(2.25) follows from Lemma 2.1. To prove (2.26), we note that

$$B_\epsilon(Q_i,Q_j) = \int_{\mathbb{R}^N} f(w(|y|))w(|y-\frac{Q_i-Q_j}{\epsilon}|) - \int_{\mathbb{R}^N\setminus\Omega_{\epsilon,Q_i}} f(w(|y|))w(|y-\frac{Q_i-Q_j}{\epsilon}|)$$

$$= (\gamma+o(1))w(\frac{|Q_i-Q_j|}{\epsilon}) + O(e^{-(1+\frac{\sigma}{2})\frac{d(Q_i,\partial\Omega)}{\epsilon}}e^{-\frac{d(Q_j,\partial\Omega)}{\epsilon}})$$

$$= (\gamma+o(1))w(\frac{|Q_i-Q_j|}{\epsilon}) + o(w(M|\ln\epsilon|)).$$

□

Finally we state the following which provides the key estimates on the energy expansion and error estimates. The proof of it is delayed to the appendix.

Lemma 2.5. *For any* $\mathbf{Q} = (Q_1,...,Q_K) \in \Lambda$ *and* ϵ *sufficiently small we have*

(2.27)

$$J_\epsilon[\sum_{i=1}^K w_{\epsilon,Q_j}] = KI[w] - \frac{1}{2}\sum_{i=1}^K B_\epsilon(Q_i) - \frac{1}{2}\sum_{i,j=1,\ldots,K,i\neq j} B_\epsilon(Q_i,Q_j) + o\Big(w(M|\ln\epsilon|)\Big),$$

and

$$\|S_\epsilon[\sum_{j=1}^K w_{\epsilon,Q_j}]\|_{L^q(\Omega_\epsilon)} \le CK^{\frac{q+1}{q}+\sigma}\epsilon^{\frac{M(1+\sigma)}{2}} \tag{2.28}$$

for any $q > \frac{N}{2}$.

3 An Auxiliary Linear Problem

In this section we study a linear theory which allows us to perform the finite-dimensional reduction procedure. The key to our argument is to show that the constants are independent of (ϵ, K).

Fix $\mathbf{Q} \in \Lambda$. We define the following functions

(3.1)

$$Z_{i,j} = (\Delta - 1)\left[\frac{\partial w_i}{\partial z_j}\chi_i(z)\right], \text{ where } \chi_i(z) = \chi\left(\frac{2|\epsilon z - Q_i|}{(M-1)\epsilon|\ln\epsilon|}\right) \qquad i = 1,...,K,\ j = 1,...,N,$$

where $\chi(t)$ is a smooth cut-off function such that $\chi(t) = 1$ for $|t| < 1$ and $\chi(t) = 0$ for $|t| > \frac{M^2}{M^2-1}$. Note that the support of $Z_{i,j}$ belongs to $B_{\frac{M^2-1}{2M}|\ln\epsilon|}(\frac{Q_i}{\epsilon})$.

In this section, we consider the following linear problem: Given $h \in L^2(\Omega_\epsilon)$, find a function ϕ satisfying

$$\begin{cases} L_\epsilon[\phi] := \Delta\phi - \phi + f'(w_{\epsilon,\mathbf{Q}})\phi = h + \sum_{k,l} c_{k,l} Z_{k,l}; \\ < \phi, Z_{i,j} >_\epsilon = 0, i = 1, ..., K, \ j = 1, ..., N, \quad \text{and } \frac{\partial\phi}{\partial\nu} = 0 \ \text{ on } \partial\Omega_\epsilon, \end{cases} \tag{3.2}$$

for some constants $c_{k,l}, k = 1, ..., K, l = 1, ..., N$. To this purpose, we define two norms

$$\|\phi\|_* = \|\phi\|_{W^{2,q}(\Omega_\epsilon)}, \quad \|f\|_{**} = \|f\|_{L^q(\Omega_\epsilon)}, \tag{3.3}$$

where $q > \frac{N}{2}$ is a fixed number.

We have the following result:

Proposition 3.1. *Let ϕ satisfy (3.2). Then for ϵ sufficiently small and $\mathbf{Q} \in \Lambda$, we have*

$$\|\phi\|_* \le C\|h\|_{**} \tag{3.4}$$

where C is a positive constant independent of ϵ, K and $\mathbf{Q} \in \Lambda$.

Proof. Arguing by contradiction, assume that

$$\|\phi\|_* = 1; \qquad \|h\|_{**} = o(1). \tag{3.5}$$

We multiply (3.2) by $\frac{\partial w_i}{\partial z_j}\chi_i(z)$ and integrate over Ω_ϵ to obtain

$$\sum_{k,l} c_{k,l} < Z_{k,l}, \frac{\partial w_i}{\partial z_j}\chi_i(z) >_\epsilon = - < h, \frac{\partial w_i}{\partial z_j}\chi_i(z) >_\epsilon$$
$$+ < \Delta\phi - \phi + f'(w_{\epsilon,\mathbf{Q}})\phi, \frac{\partial w_i}{\partial z_j}\chi_i(z) >_\epsilon . \tag{3.6}$$

From the exponential decay of w one finds

$$< h, \frac{\partial w_i}{\partial z_j}\chi_i(z) >_\epsilon = o(1).$$

Observe that $\frac{\partial w_i}{\partial z_j}\chi_i(z)$ satisfies

$$\Delta(\frac{\partial w_i}{\partial z_j}\chi_i(z)) - (\frac{\partial w_i}{\partial z_j}\chi_i(z)) + f^{'}(w_i)(\frac{\partial w_i}{\partial z_j}\chi_i(z)) = 2\nabla_z\frac{\partial w_i}{\partial z_j}\nabla_z\chi_i + (\Delta\chi_i)\frac{\partial w_i}{\partial z_j}. \tag{3.7}$$

Integrating by parts and using Lemma 2.2, we deduce

$$< \Delta\phi - \phi + f'(w_{\epsilon,\mathbf{Q}})\phi, \frac{\partial w_i}{\partial z_j}\chi_i(z) >_\epsilon = < (f'(w_{\epsilon,\mathbf{Q}}) - f^{'}(w_i))\frac{\partial w_i}{\partial z_j}\chi_i(z), \phi >_\epsilon + O(\epsilon^{\frac{M-1}{2}}\|\phi\|_*)$$
$$= O(K^\sigma \epsilon^{\frac{M\sigma}{2}}\|\phi\|_*) = o(\|\phi\|_*) = o(1)$$

where we have used the fact that $M > \frac{6+2\sigma}{\sigma} N$ and that

$$\|(f'(w_{\epsilon,\mathbf{Q}}) - f'(w_i))\frac{\partial w_i}{\partial z_j}\chi_i\|_{**} \leq C\||w_{\epsilon,\mathbf{Q}} - w_i|^\sigma|\frac{\partial w_i}{\partial z_j}\chi_i\|_* \leq K^\sigma \epsilon^{\frac{M\sigma}{2}}.$$

It is easy to see that

$$< Z_{i,j}, \frac{\partial w_i}{\partial z_j}\chi_i(z) >_\epsilon = -\int_{\mathbb{R}^N} f'(w)(\frac{\partial w}{\partial y_j})^2 dy + o(1). \tag{3.8}$$

On the other hand, for $k \neq i$ we have

$$< Z_{k,l}, \frac{\partial w_i}{\partial z_j}\chi_i(z) >_\epsilon = 0 \tag{3.9}$$

and for $k = i$ and $l \neq j$, we have

$$< Z_{i,l}, \frac{\partial w_i}{\partial z_j}\chi_i(z) >_\epsilon = O(\epsilon^M). \tag{3.10}$$

The left hand side of (3.6) becomes

$$c_{i,j} + \sum_{l \neq j} O(\epsilon^M c_{i,l}) = o(1)$$

and hence

$$c_{i,j} = o(1), \qquad i = 1, ..., K, j = 1, ..., N. \tag{3.11}$$

To obtain a contradiction, we define the following cut-off functions:

$$\phi_i = \phi\chi_i', \text{ where } \chi_i' = \chi\left(\frac{2|\epsilon z - Q_i|}{(M - M^{-1})\epsilon|\ln\epsilon|}\right), i = 1, ..., K. \tag{3.12}$$

Note that $\chi_i' = 1$ for $z \in B_{\frac{M^2-1}{2M}|\ln\epsilon|}(\frac{Q_i}{\epsilon})$ and the support of ϕ belongs to $B_{\frac{M}{2}|\ln\epsilon|}(\frac{Q_i}{\epsilon})$.

Then the conditions $< \phi, Z_{i,j} >_\epsilon = 0$ is equivalent to

$$< \phi_i, Z_{i,j} >_\epsilon = 0. \tag{3.13}$$

The equation for ϕ_i becomes

$$\Delta\phi_i - \phi_i + f'(w_{\epsilon,\mathbf{Q}})\phi_i = \sum_j c_{i,j} Z_{i,j} + h\chi_i' + 2\nabla\phi\nabla\chi_i' + (\Delta\chi_i')\phi \tag{3.14}$$

Lemma 2.2 yields

$$f'(w_{\epsilon,\mathbf{Q}})\phi_i = (f(w_i) + o(\epsilon^{M/2-N}))\phi_i. \tag{3.15}$$

Using (3.13) and (3.15), a contradiction argument similar to that of Proposition 3.2 of [13] gives

$$\|\phi_i\|^q_{W^{2,q}(\Omega_\epsilon)} \leq C\|h\chi_i'\|^q_{L^q(\Omega_\epsilon)} + C\|2\nabla\phi\nabla\chi_i' + (\Delta\chi_i')\phi\|^q_{L^q(\Omega_\epsilon)}. \tag{3.16}$$

Next, we decompose

$$\phi = \sum_{i=1}^{K} \phi_i + \Phi \tag{3.17}$$

where $\Phi = \phi(1 - \sum_{i=1}^{K} \chi_i^{'})$. Then the equation for Φ becomes

$$\Delta\Phi - \Phi + f^{'}(w_{\epsilon,\mathbf{Q}})\Phi = h(1 - \sum_{i=1}^{K} \chi_i^{'}) - 2\sum_{i=1}^{K} \nabla\phi\nabla\chi_i^{'} - \sum_{i=1}^{K}(\Delta\chi_i^{'})\phi. \tag{3.18}$$

By Lemma 2.2, $f^{'}(w_{\epsilon,\mathbf{Q}})\Phi = o(1)\Phi$. Standard regularity theory gives

$$\|\Phi\|^q_{W^{2,q}(\Omega_\epsilon)} \leq C\|h(1 - \sum_{i=1}^{K} \chi_i^{'})\|^q_{L^q(\Omega_\epsilon)} + C\|2\sum_{i=1}^{K} \nabla\phi\nabla\chi_i^{'} + \sum_{i=1}^{K}(\Delta\chi_i^{'})\phi\|^q_{L^q(\Omega_\epsilon)}. \tag{3.19}$$

(Observe that the constant C in the L^q-estimates of (3.19) is independent of $\epsilon < 1$. Inequality (3.19) in the case of Dirichlet boundary condition has been proved in Lemma 6.4 of [31]. Inequality (3.19) in the case of Neumann boundary condition can be proved similarly.)

Combining (3.17), (3.16) and (3.19), we obtain

$$\|\phi\|^q_{W^{2,q}(\Omega_\epsilon)} \leq C\|\sum_{i=1}^{K} \phi_i\|^q_{W^{2,q}(\Omega_\epsilon)} + C\|\Phi\|^q_{W^{2,q}(\Omega_\epsilon)} \leq C\sum_{i=1}^{K} \|\phi_i\|^q_{W^{2,q}(\Omega_\epsilon)} + C\|\Phi\|^q_{W^{2,q}(\Omega_\epsilon)}$$

$$\leq C\left(\sum_{i=1}^{K} \|h\chi_i^{'}\|^q_{L^q(\Omega_\epsilon)} + \|h(1-\sum_{i=1}^{K} \chi_i^{'})\|^q_{L^q(\Omega_\epsilon)}\right) + C\sum_{i=1}^{K} \|2\nabla\phi\nabla\chi_i^{'} + (\Delta\chi_i^{'})\phi\|^q_{L^q(\Omega_\epsilon)}$$

$$\leq C\|h\|^q_{L^q(\Omega_\epsilon)} + O(|\ln\epsilon|^{-1})\|\phi\|^q_{W^{2,q}(\Omega_\epsilon)}$$

since

$$\sum_{i=1}^{K}(\chi_i^{'})^q + (1 - \sum_{i=1}^{K} \chi_i^{'})^q \leq 2, \ |\nabla\chi^{'}| + |\Delta\chi^{'}| \leq C(|\ln\epsilon|)^{-1}. \tag{3.20}$$

This gives

$$\|\phi\|_{W^{2,q}(\Omega_\epsilon)} = o(1). \tag{3.21}$$

A contradiction to (3.5).

□

Proposition 3.2. *There exists* $\epsilon_0 > 0$ *such that for any* $0 < \epsilon < \epsilon_0$ *the following property holds true. Given* $h \in W^{2,q}(\Omega_\epsilon)$*, there exists a unique pair* $(\phi, \mathbf{c}) = (\phi, \{c_{i,j}\}_{i=1,...,K, j=1,...,N})$ *such that*

$$L_\epsilon[\phi] = h + \sum_{i,j} c_{i,j} Z_{i,j}, \tag{3.22}$$

$$< \phi, Z_{i,j} >_\epsilon = 0, i = 1, ..., K, \ j = 1, ..., N, \ \frac{\partial\phi}{\partial\nu} = 0 \ \text{ on } \partial\Omega_\epsilon. \tag{3.23}$$

Moreover, we have

$$\|\phi\|_* \le C\|h\|_{**} \tag{3.24}$$

for some positive constant C.

Proof. The bound in (3.24) follows from Proposition 3.1 and (3.11). Let us now prove the existence part. Set

$$\mathcal{H} = \left\{ u \in H^1(\Omega_\epsilon) | \left(u, (\Delta - 1)^{-1} Z_{i,j} \right)_\epsilon = 0 \right\}$$

where we define the inner product on $H^1(\Omega_\epsilon)$ as

$$(u, v)_\epsilon = \int_{\Omega_\epsilon} (\nabla u \nabla v + uv).$$

Note that, integrating by parts, one has for $\psi \in H^1(\Omega_\epsilon)$

$$\psi \in \mathcal{H} \qquad \text{if and only if} \qquad < \psi, Z_{i,j} >_\epsilon = 0, \quad i = 1, ..., K, j = 1, ..., N.$$

Observe that ϕ solves (3.22) and (3.23) if and only if $\phi \in \mathcal{H}$ satisfies

$$\int_{\Omega_\epsilon} (\nabla\phi\nabla\psi + \phi\psi) - < f'(w_{\epsilon,\mathbf{Q}})\phi, \psi >_\epsilon = < h, \psi >_\epsilon, \ \forall \psi \in \mathcal{H}.$$

This equation can be rewritten as

$$\phi + \mathcal{S}(\phi) = \bar{h} \qquad \text{in } \mathcal{H}, \tag{3.25}$$

where $\overline{h}$ is defined by duality and $\mathcal{S} : \mathcal{H} \to \mathcal{H}$ is a linear compact operator.

Using Fredholm's alternative, showing that equation (3.25) has a unique solution for each $\bar{h}$, is equivalent to showing that the equation has a unique solution for $\bar{h} = 0$, which in turn follows from Proposition 3.1 and our proof is complete. □

In the following, if ϕ is the unique solution given in Proposition 3.2, we set

$$\phi = \mathcal{A}_\epsilon(h). \tag{3.26}$$

Note that (3.24) implies

$$\|\mathcal{A}_\epsilon(h)\|_* \le C\|h\|_{**}. \tag{3.27}$$

4 Liapunov-Schmidt Reduction: A Nonlinear Problem

In this section we reduce problem (2.4) to a finite-dimensional one.

For ϵ small and for $\mathbf{Q} \in \Lambda$, we are going to find a function $\phi_{\epsilon,\mathbf{Q}}$ such that for some constants $c_{i,j}, j = 1, ..., N$, the following equation holds true

$$\begin{cases} \Delta(w_{\epsilon,\mathbf{Q}} + \phi) - (w_{\epsilon,\mathbf{Q}} + \phi) + f(w_{\epsilon,\mathbf{Q}} + \phi) = \sum_{k,l} c_{k,l} Z_{k,l} \text{ in } \Omega_\epsilon, \\ \\ < \phi, Z_{i,j} >_\epsilon = 0, j = 1, ..., N, \ \ \frac{\partial\phi}{\partial\nu} = 0 \text{ on } \partial\Omega_\epsilon. \end{cases} \tag{4.1}$$

The first equation in (4.1) can be written as

$$\Delta\phi - \phi + f'(w_{\epsilon,\mathbf{Q}})\phi = (-S_\epsilon[w_{\epsilon,\mathbf{Q}}]) + N_\epsilon[\phi] + \sum_{i,j} c_{i,j} Z_{i,j},$$

where

$$N_\epsilon[\phi] = -\left[f(w_{\epsilon,\mathbf{Q}} + \phi) - f(w_{\epsilon,\mathbf{Q}}) - f'(w_{\epsilon,\mathbf{Q}})\phi \right]. \tag{4.2}$$

Lemma 4.1. *For* $\mathbf{Q} \in \Lambda$ *and* ϵ *sufficiently small, we have for* $\|\phi\|_* + \|\phi_1\|_* + \|\phi_2\|_* \leq 1$,

$$\|N_\epsilon[\phi]\|_{**} \leq C\|\phi\|_*^{1+\sigma}; \tag{4.3}$$

$$\|N_\epsilon[\phi_1] - N_\epsilon[\phi_2]\|_{**} \leq C(\|\phi_1\|_*^\sigma + \|\phi_2\|_*^\sigma)\|\phi_1 - \phi_2\|_*. \tag{4.4}$$

Proof. Inequality (4.3) follows from the mean-value theorem. In fact, for all $z \in \Omega_\epsilon$ there holds

$$f(w_{\epsilon,\mathbf{Q}} + \phi) - f(w_{\epsilon,\mathbf{Q}}) = f'(w_{\epsilon,\mathbf{Q}} + \theta\phi)\phi.$$

Since f' is Hölder continuous with exponent σ, we deduce

$$|f(w_{\epsilon,\mathbf{Q}} + \phi) - f(w_{\epsilon,\mathbf{Q}}) - f'(w_{\epsilon,\mathbf{Q}})\phi| \leq C|\phi|^{1+\sigma},$$

which implies (4.3). The proof of (4.4) goes along the same way. □

Proposition 4.2. *For* $\mathbf{Q} \in \Lambda$ *and* ϵ *sufficiently small, there exists a unique* $\phi = \phi_{\epsilon,\mathbf{Q}}$ *such that (4.1) holds. Moreover,* $\mathbf{Q} \mapsto \phi_{\epsilon,\mathbf{Q}}$ *is of class* C^1 *as a map into* $W^{2,q}(\Omega_\epsilon) \cap \mathcal{H}$*, and we have*

$$\|\phi_{\epsilon,\mathbf{Q}}\|_* \leq rK^{\frac{q+1}{q}+\sigma}\epsilon^{\frac{M(1+\sigma)}{2}} \tag{4.5}$$

for some constant $r > 0$.

Proof. Let $\mathcal{A}_\epsilon$ be as defined in (3.26). Then (4.1) can be written as

$$\phi = \mathcal{A}_\epsilon\left[(-S_\epsilon[w_{\epsilon,\mathbf{Q}}]) + N_\epsilon[\phi] \right]. \tag{4.6}$$

Let r be a positive (large) number, and set

$$\mathcal{F}_r = \left\{ \phi \in \mathcal{H} \cap W^{2,q}(\Omega_\epsilon) : \|\phi\|_* < rK^{\frac{q+1}{q}+\sigma}\epsilon^{\frac{M(1+\sigma)}{2}} \right\}.$$

Define now the map $\mathcal{G}_\epsilon : \mathcal{F}_r \to \mathcal{H} \cap W^{2,q}(\Omega_\epsilon)$ as

$$\mathcal{G}_\epsilon[\phi] = \mathcal{A}_\epsilon\left[(-S_\epsilon[w_{\epsilon,\mathbf{Q}}]) + N_\epsilon[\phi] \right].$$

Solving (4.1) is equivalent to finding a fixed point for $\mathcal{G}_\epsilon$. By Lemma 2.5 and Lemma 4.1, for ϵ sufficiently small and r large we have

$$\|\mathcal{G}_\epsilon[\phi]\|_* \leq C\|S_\epsilon[w_{\epsilon,\mathbf{Q}}]\|_{**} + C\|N_\epsilon[\phi]\|_{**} < rK^{\frac{q+1}{q}+\sigma}\epsilon^{\frac{M(1+\sigma)}{2}},$$

$$\|\mathcal{G}_\epsilon[\phi_1] - \mathcal{B}_\epsilon[\phi_2]\|_* \le C\|N_\epsilon[\phi_1] - N_\epsilon[\phi_2]\|_* < \frac{1}{2}\|\phi_1 - \phi_2\|_*,$$

which shows that $\mathcal{G}_\epsilon$ is a contraction mapping on $\mathcal{F}_r$. Hence there exists a unique $\phi = \phi_{\epsilon,\mathbf{Q}} \in \mathcal{F}_r$ such that (4.1) holds.

Now we come to the differentiability of $\phi_{\epsilon,\mathbf{Q}}$. Consider the following map $H_\epsilon : \Lambda \times \mathcal{H} \cap W^{2,q}(\Omega_\epsilon) \times R^{NK} \to \mathcal{H} \cap W^{2,q}(\Omega_\epsilon) \times R^{NK}$ of class C^1

(4.7)

$$H_\epsilon(\mathbf{Q}, \phi, \mathbf{c}) = \begin{pmatrix} (\Delta - 1)^{-1}(S_\epsilon[w_{\epsilon,\mathbf{Q}} + \phi]) - \sum_{i,j} c_{i,j}(\Delta - 1)^{-1}Z_{i,j} \\ (\phi, (\Delta - 1)^{-1}Z_{1,1})_\epsilon \\ \vdots \\ (\phi, (\Delta - 1)^{-1}Z_{K,N})_\epsilon \end{pmatrix},$$

Equation (4.1) is equivalent to $H_\epsilon(\mathbf{Q}, \phi, \mathbf{c}) = 0$. We know that, given $\mathbf{Q} \in \Lambda$, there is a unique local solution $\phi_{\epsilon,\mathbf{Q}}, c_{\epsilon,\mathbf{Q}}$ obtained with the above procedure. We prove that the linear operator

$$\left.\frac{\partial H_\epsilon(\mathbf{Q}, \phi, \mathbf{c})}{\partial(\phi, \mathbf{c})}\right|_{(\mathbf{Q}, \phi_{\epsilon,\mathbf{Q}}, \mathbf{c}_{\epsilon,\mathbf{Q}})} : \mathcal{H} \cap W^{2,q}(\Omega_\epsilon) \times R^{NK} \to \mathcal{H} \cap W^{2,q}(\Omega_\epsilon) \times R^{NK}$$

is invertible for ϵ small. Then the C^1-regularity of $\mathbf{Q} \mapsto (\phi_{\epsilon,\mathbf{Q}}, c_{\epsilon,\mathbf{Q}})$ follows from the Implicit Function Theorem. Indeed we have

$$\left.\frac{\partial H_\epsilon(\mathbf{Q}, \phi, \mathbf{c})}{\partial(\phi, \mathbf{c})}\right|_{(\mathbf{Q}, \phi_{\epsilon,\mathbf{Q}}, \mathbf{c}_{\epsilon,\mathbf{Q}})} [\psi, \mathbf{d}] = \begin{pmatrix} (\Delta - 1)^{-1}(S'[w_{\epsilon,\mathbf{Q}} + \phi_{\epsilon,\mathbf{Q}}](\psi)) - \sum_{i,j} d_{ij}(\Delta - 1)^{-1}Z_{i,j} \\ (\psi, (\Delta - 1)^{-1}Z_{1,1})_\epsilon \\ \vdots \\ (\psi, (\Delta - 1)^{-1}Z_{K,N})_\epsilon \end{pmatrix}.$$

Since $\|\phi_{\epsilon,\mathbf{Q}}\|_*$ is small, the same proof as in that of Proposition 3.1 shows that $\frac{\partial H_\epsilon(\mathbf{Q},\phi,\mathbf{c})}{\partial(\phi,\mathbf{c})}|_{(\mathbf{Q},\phi_{\epsilon,\mathbf{Q}},\mathbf{c}_{\epsilon,\mathbf{Q}})}$ is invertible for ϵ small.

This concludes the proof of Proposition 4.2.

□

5 The reduced problem: A Maximization Procedure

In this section, we study a maximization problem.

Fix $\mathbf{Q} \in \Lambda$. Let $\phi_{\epsilon,\mathbf{Q}}$ be the solution given by Proposition 4.2. We define a new functional

(5.1) $$\mathcal{M}_\epsilon(\mathbf{Q}) = J_\epsilon[w_{\epsilon,\mathbf{Q}} + \phi_{\epsilon,\mathbf{Q}}] : \Lambda \to R.$$

We shall prove

Proposition 5.1. *For ϵ small, the following maximization problem*

(5.2) $$\max\{\mathcal{M}_\epsilon(\mathbf{Q}) : \mathbf{Q} \in \Lambda\}$$

has a solution $\mathbf{Q}^\epsilon \in \Lambda^\circ$-the interior of Λ.

Proof. Since $J_\epsilon[w_{\epsilon,\mathbf{Q}}+\phi_{\epsilon,\mathbf{Q}}]$ is continuous in $\mathbf{Q}$, the maximization problem has a solution. Let $\mathcal{M}_\epsilon(\mathbf{Q}^\epsilon)$ be the maximum where $\mathbf{Q}^\epsilon \in \Lambda$.

We claim that $\mathbf{Q}^\epsilon$ must stay in the interior of Λ.

We first obtain an asymptotic formula for $\mathcal{M}_\epsilon(\mathbf{Q})$. In fact for any $\mathbf{Q}\in\Lambda$, we have

$$\mathcal{M}_\epsilon(\mathbf{Q}) = J_\epsilon[w_{\epsilon,\mathbf{Q}}]$$

$$+\int_{\Omega_\epsilon}(\nabla w_{\epsilon,\mathbf{Q}}\nabla\phi_{\epsilon,\mathbf{Q}}+w_{\epsilon,\mathbf{Q}}\phi_{\epsilon,\mathbf{Q}})-\int_{\Omega_\epsilon} f(w_{\epsilon,\mathbf{Q}})\phi_{\epsilon,\mathbf{Q}}+O(\|\phi_{\epsilon,\mathbf{Q}}\|_*^2)$$

$$= J_\epsilon[w_{\epsilon,\mathbf{Q}}]+\int_{\Omega_\epsilon}(-S_\epsilon[w_{\epsilon,\mathbf{Q}}])\phi_{\epsilon,\mathbf{Q}}+O(\|\phi_{\epsilon,\mathbf{Q}}\|_*^2)$$

$$= J_\epsilon[w_{\epsilon,\mathbf{Q}}]+O(\|S_\epsilon[w_{\epsilon,\mathbf{Q}}]\|_{**}\|\phi_{\epsilon,\mathbf{Q}}\|_*)+O(\|\phi_{\epsilon,\mathbf{Q}}\|_*^2)$$

$$= J_\epsilon[w_{\epsilon,\mathbf{Q}}]+O(K^{2+\frac{2}{q}+2\sigma}\epsilon^{M(1+\sigma)}) = J_\epsilon[w_{\epsilon,\mathbf{Q}}]+o(w(M|\ln\epsilon|))$$

by Lemma 2.5, Proposition 4.2 and the choice of M at (2.8).

By Lemmas 2.4 and 2.5, we obtain

(5.3)
$$\mathcal{M}_\epsilon(\mathbf{Q}) = KI[w]-\frac{1}{2}(\gamma+o(1))\sum_{i=1}^K w(\frac{2d(Q_i,\partial\Omega)}{\epsilon})-\frac{1}{2}(\gamma+o(1))\sum_{i\neq j} w(\frac{|Q_i-Q_j|}{\epsilon})+o(w(M|\ln\epsilon|)).$$

Next, we obtain a lower bound for $\mathcal{M}_\epsilon$: Recall that $K_\Omega(r)$ is the maximum number of non-overlapping balls with equal radius r packed in Ω. Now we choose K such that

$$1\le K\le K_\Omega(\frac{M+2N}{2}\epsilon|\ln\epsilon|). \tag{5.4}$$

Let $\mathbf{Q}^0=(Q_1^0,...,Q_K^0)$ be the centers of arbitrary K balls among those $K_\Omega(\frac{M+2N}{2}\epsilon|\ln\epsilon|)$ balls. Certainly $\mathbf{Q}^0\in\Lambda$. Then we have

$$w(\frac{2d(Q_i^0,\partial\Omega)}{\epsilon})\le e^{-\frac{2d(Q_i^0,\partial\Omega)}{\epsilon}}\le\epsilon^{M+2N},\ w(\frac{|Q_i^0-Q_j^0|}{\epsilon})\le\epsilon^{M+2N}$$

and hence

(5.5)
$$\mathcal{M}_\epsilon(\mathbf{Q}^\epsilon)\ge\mathcal{M}_\epsilon(\mathbf{Q}^0)\ge KI[w]-\frac{K}{2}(\gamma+o(1))\epsilon^{M+2N}-\frac{K^2}{2}(\gamma+o(1))\epsilon^{M+2N}+o(w(M|\ln\epsilon|))$$
$$\ge KI[w]-K^2(\gamma+o(1))\epsilon^{M+2N}+o(w(M|\ln\epsilon|)).$$

On the other hand, if $\mathbf{Q}^\epsilon\in\partial\Lambda$, then either there exists (i,j) such that $|Q_i^\epsilon-Q_j^\epsilon|=M\epsilon|\ln\epsilon|$, or there exists a k such that $d(Q_k^\epsilon,\partial\Omega)=\frac{M}{2}\epsilon|\ln\epsilon|$. In both cases we have

$$\mathcal{M}_\epsilon(\mathbf{Q}^\epsilon)\le KI[w]-\frac{1}{2}(\gamma+o(1))w(M|\ln\epsilon|)+o(w(M|\ln\epsilon|)). \tag{5.6}$$

Combining (5.6) and (5.5), we obtain

$$w(M|\ln\epsilon|)\le 2K^2\epsilon^{M+2N}\le C\epsilon^M(|\ln\epsilon|)^{-2N} \tag{5.7}$$

which is impossible.

We conclude that $\mathbf{Q}^\epsilon \in \Lambda$. This completes the proof of Proposition 5.1.

□

Remark: Since $M > \frac{6+2\sigma}{\sigma}N$, we may choose $M = \frac{6+2\sigma}{\sigma}N + \frac{1}{400}$, such that (5.4) is exactly (1.16).

6 Proof of Theorem 1.1

In this section, we apply results in Section 4 and Section 5 to prove Theorem 1.1.

Proof of Theorem 1.1. By Proposition 4.1, there exists ϵ_0 such that for $0 < \epsilon < \epsilon_0$ we have a C^1 map which, to any $\mathbf{Q} \in \Lambda$, associates $\phi_{\epsilon,\mathbf{Q}}$ such that

(6.1) $$S_\epsilon[w_{\epsilon,\mathbf{Q}} + \phi_{\epsilon,\mathbf{Q}}] = \sum_{k=1,\ldots,K;l=1,\ldots,N} c_{kl} Z_{k,l},$$

$$< \phi_{\epsilon,\mathbf{Q}}, Z_{i,j} >_\epsilon = 0$$

for some constants $c_{kl} \in R^{KN}$.

By Proposition 5.1, we have $\mathbf{Q}^\epsilon \in \Lambda$, achieving the maximum of the maximization problem in Proposition 5.1. Let $u_\epsilon = w_{\epsilon,\mathbf{Q}^\epsilon} + \phi_{\epsilon,\mathbf{Q}^\epsilon}$. Then we have

$$D_{Q_{i,j}}|_{Q_i=Q_i^\epsilon} \mathcal{M}_\epsilon(\mathbf{Q}^\epsilon) = 0, i = 1, ..., K, j = 1, ..., N.$$

Hence we have

$$\int_{\Omega_\epsilon} \left[\nabla u_\epsilon \nabla \frac{\partial(w_{\epsilon,\mathbf{Q}} + \phi_{\epsilon,\mathbf{Q}})}{\partial Q_{i,j}}\bigg|_{Q_i=Q_i^\epsilon} + u_\epsilon \frac{\partial(w_{\epsilon,\mathbf{Q}} + \phi_{\epsilon,\mathbf{Q}})}{\partial Q_{i,j}}\bigg|_{Q_i=Q_i^\epsilon} \right.$$
$$\left. - f(u_\epsilon) \frac{\partial(w_{\epsilon,\mathbf{Q}} + \phi_{\epsilon,\mathbf{Q}})}{\partial Q_{i,j}}\bigg|_{Q_i=Q_i^\epsilon} \right] = 0,$$

which gives

(6.2) $$\sum_{k=1,\ldots,K;l=1,\ldots,N} c_{kl} \int_{\Omega_\epsilon} Z_{k,l} \frac{\partial(w_{\epsilon,\mathbf{Q}} + \phi_{\epsilon,\mathbf{Q}})}{\partial Q_{i,j}}\bigg|_{Q_i=Q_i^\epsilon} = 0.$$

We claim that (6.2) is a diagonally dominant system. In fact, since $< \phi_{\epsilon,\mathbf{Q}}, Z_{i,j} >_\epsilon = 0$, we have that

$$\int_{\Omega_\epsilon} Z_{k,l} \frac{\partial \phi_{\epsilon,\mathbf{Q}^\epsilon}}{\partial Q_{i,j}^\epsilon} = -\int_{\Omega_\epsilon} \phi_{\epsilon,\mathbf{Q}^\epsilon} \frac{\partial Z_{k,l}}{\partial Q_{i,j}^\epsilon} = 0 \text{ if } k \neq i.$$

If $k = i$, we have

$$\int_{\Omega_\epsilon} Z_{k,l} \frac{\partial \phi_{\epsilon,\mathbf{Q}^\epsilon}}{\partial Q_{k,j}^\epsilon} = -\int_{\Omega_\epsilon} \frac{\partial Z_{k,l}}{\partial Q_{k,j}^\epsilon} \phi_{\epsilon,\mathbf{Q}^\epsilon}$$

$$= \|\frac{\partial Z_{k,l}}{\partial Q_{k,j}^\epsilon}\|_{**}\|\phi_{\epsilon,\mathbf{Q}^\epsilon}\|_{**} = O(K^{\frac{q+1}{q}+\sigma}\epsilon^{\frac{M(1+\sigma)}{2}-1}) = O(\epsilon^{\frac{M(1+\sigma)}{2}-(\frac{q+1}{q}+\sigma)N-1}) = O(\epsilon^{\frac{M}{2}}).$$

For $k \neq i$, we have

$$\int_{\Omega_\epsilon} Z_{k,l}\frac{\partial w_{\epsilon,Q_i^\epsilon}}{\partial Q_{i,j}^\epsilon} = \int_{\Omega_\epsilon \cap B_{\frac{M}{2}|\ln\epsilon|}(\frac{Q_k^\epsilon}{\epsilon})} Z_{k,l}\frac{\partial w_{\epsilon,Q_i^\epsilon}}{\partial Q_{i,j}^\epsilon} = O(\epsilon^M).$$

For $k = i$, we have

$$\int_{\Omega_\epsilon} Z_{k,l}\frac{\partial w_{\epsilon,Q_k^\epsilon}}{\partial Q_{k,j}^\epsilon} = \int_{\Omega_\epsilon \cap B_{\frac{M}{2}|\ln\epsilon|}(\frac{Q_k^\epsilon}{\epsilon})} Z_{k,l}\frac{\partial w_{\epsilon,Q_k^\epsilon}}{\partial Q_{k,j}^\epsilon}$$

$$= -\epsilon^{-1}\delta_{lj}\int_{\mathbb{R}^N} f'(w)(\frac{\partial w}{\partial y_j})^2 + O(1).$$

For each (k, l), the off-diagonal term gives

$$O(\epsilon^{\frac{M}{2}}) + \sum_{k\neq i}\epsilon^M + \sum_{k=i,l\neq j} O(\epsilon) = O(\epsilon^{\frac{M}{2}} + K\epsilon^M + \epsilon) = o(1)$$

by our choice of $M > \frac{6+2\sigma}{\sigma}N$.

Thus equation (6.2) becomes a system of homogeneous equations for c_{kl} and the matrix of the system is nonsingular. So $c_{kl} \equiv 0, k = 1, ..., K, l = 1, ...N$.

Hence $u_\epsilon = \sum_{i=1}^K w_{\epsilon,Q_i^\epsilon} + \phi_{\epsilon,Q_1^\epsilon,...,Q_K^\epsilon}$ is a solution of (2.2).

By our construction and Maximum Principle, it is easy to see that $u_\epsilon > 0$ in Ω. Moreover by (5.3) and Lemma 2.2

$$J_\epsilon[u_\epsilon] = KI[w] + O(K^2\epsilon^M) + o(w(M|\ln\epsilon|)) = KI[w](1 + o(1)), \tag{6.3}$$

and hence $E_\epsilon[u_\epsilon] = KI[w](1 + o(1))$.

Furthermore, by Lemma 2.2, for $|z - \frac{Q_j^\epsilon}{\epsilon}| < \frac{M}{2}\epsilon|\ln\epsilon|$, we also have

$$u_\epsilon = w_{\epsilon,\mathbf{Q}^\epsilon} + \phi_{\epsilon,\mathbf{Q}^\epsilon} = w(z - \frac{Q_j^\epsilon}{\epsilon}) + O(K\epsilon^{\frac{M}{2}} + K^{\frac{q+1}{q}+\sigma}\epsilon^{\frac{M(1+\sigma)}{2}}) = w(z - \frac{Q_j^\epsilon}{\epsilon}) + O(K\epsilon^{\frac{M}{2}}),$$

since $\|\phi_{\epsilon,\mathbf{Q}^\epsilon}\|_{L^\infty(\Omega_\epsilon)} = O(\|\phi_{\epsilon,\mathbf{Q}^\epsilon}\|_{W^{2,q}(\Omega_\epsilon)})$. For $z \in \Omega_\epsilon \backslash (\cup_{j=1}^K B_{\frac{M}{2}|\ln\epsilon|}(\frac{Q_j^\epsilon}{\epsilon}))$, $u_\epsilon = O(K\epsilon^{\frac{M}{2}})$.

Note that at a local maximum point of u_ϵ, the value of u_ϵ must be great than some fixed positive value u_0. So u_ϵ can only have maximum points in $\cup_{j=1}^K B_R(\frac{Q_j^\epsilon}{\epsilon})$ for some $R \geq 1$. For each $j = 1, ..., K$, u_ϵ satisfies

$$\Delta u_\epsilon - u_\epsilon + f(u_\epsilon) = 0, z \in B_R(\frac{Q_j^\epsilon}{\epsilon}).$$

Let $\frac{\tilde{Q_j}^\epsilon}{\epsilon}$ be a local maximum point of u_ϵ in $B_R(\frac{Q_j^\epsilon}{\epsilon})$. Since $u_\epsilon(z) = w(z - \frac{Q_j^\epsilon}{\epsilon}) + o(1)$ for $z \in B_R(\frac{Q_j^\epsilon}{\epsilon})$, it is easy to see that $|Q_j^\epsilon - \tilde{Q}_j^\epsilon| = o(\epsilon)$. We look at the difference $\phi_\epsilon(z) = u_\epsilon(z) - w(z - \frac{\tilde{Q_j}^\epsilon}{\epsilon})$. Then similar to the proof of Step 2 of

Theorem 1.1 in [28], one can show that $\phi_\epsilon \to 0$ in $C^{2,\alpha}(B_{\frac{R}{2}}(\frac{Q_j^\epsilon}{\epsilon}))$ for $0 < \alpha < 1$ and hence u_ϵ can have only one local maximum point in $B_R(\frac{Q_j^\epsilon}{\epsilon})$. This shows that u_ϵ has exactly K local maximum points $\tilde{Q}_1^\epsilon, ..., \tilde{Q}_K^\epsilon$ such that $\tilde{Q}_j^\epsilon - Q_j^\epsilon = o(\epsilon), j = 1, ..., K$.

(6.3) also shows that if we take $K = [\epsilon^{-m}]$, then $J_\epsilon[u_\epsilon] \sim \epsilon^{-m}$, where $m < N$. (Hence $E_\epsilon \sim \epsilon^{N-m}$.) If we take $K = [\frac{\alpha_{N,\Omega,f}}{\epsilon^N |\ln \epsilon|^N}]$, then we have a solution u_ϵ with energy $J_\epsilon[u_\epsilon] \sim \frac{1}{\epsilon^N |\ln \epsilon|^N}$. (Hence $E_\epsilon \sim (|\ln \epsilon|)^{-N}$.)

Finally, if f also satisfies (f3), we show that the Morse index of u_ϵ is at least K. In fact, let Φ_0 be given by (1.8). Set

$$\Phi_{0,j}(z) = \Phi_0(z - \frac{Q_j^\epsilon}{\epsilon})\chi_j(z), j = 1, ..., K \tag{6.4}$$

where $\chi_j(z)$ is defined at (3.1). It is easy to see that for ϵ sufficiently small

$$\int_{\Omega_\epsilon} \left[(|\nabla \Phi_{0,j}|^2 + \Phi_{0,j}^2) - f'(u_\epsilon)\Phi_{0,j}^2 \right] = \int_{\Omega_\epsilon} \left[(|\nabla \Phi_{0,j}|^2 + \Phi_{0,j}^2) - f'(w_j)\Phi_{0,j}^2 \right] + \int_{\Omega_\epsilon} (f(w_j) - f'(u_\epsilon))\Phi_{0,j}^2$$

$$\leq -\frac{\lambda_1}{2} \int_{\Omega_\epsilon} \Phi_{0,j}^2 < 0, j = 1, ..., K. \tag{6.5}$$

This, together with the fact that the supports of $\Phi_{0,1}, ...\Phi_{0,K}$ are mutually disjoint, implies that the Morse index of u_ϵ is at least K.

□

Appendix: Proof of Lemma 2.5

We prove the energy expansion formula and error estimates given in Lemma 2.5. The main concern is that we have to make precise the dependence of the error terms on K.

We decompose

$$J_\epsilon[\sum_{j=1}^K w_{\epsilon,Q_j}] = \sum_{j=1}^K J_\epsilon[w_{\epsilon,Q_j}] + \frac{1}{2} \sum_{i \neq j} \int_{\Omega_\epsilon} (\nabla w_{\epsilon,Q_i} \nabla w_{\epsilon,Q_j} + w_{\epsilon,Q_i} w_{\epsilon,Q_j})$$

$$- \int_{\Omega_\epsilon} \left[F(\sum_{j=1}^K w_{\epsilon,Q_j}) - \sum_{j=1}^K F(w_{\epsilon,Q_j}) \right]$$

(1.1)
$$= \sum_{j=1}^{K} J_\epsilon[w_{\epsilon,Q_j}] + \frac{1}{2}\sum_{i\neq j}\int_{\Omega_\epsilon} f(w_i)w_{\epsilon,Q_j} - \int_{\Omega_\epsilon}\left[F(\sum_{j=1}^{K} w_{\epsilon,Q_j}) - \sum_{j=1}^{K} F(w_{\epsilon,Q_j})\right].$$

Observe that

$$J_\epsilon[w_{\epsilon,Q_i}] = \frac{1}{2}\int_{\Omega_\epsilon} f(w_i)w_{\epsilon,Q_i} - \int_{\Omega_\epsilon} F(w_{\epsilon,Q_i})$$
$$= \int_{\Omega_\epsilon}[\frac{1}{2}f(w_i)w_i - F(w_i)] + \frac{1}{2}B_\epsilon(Q_i) - \int_{\Omega_\epsilon}[F(w_{\epsilon,Q_i}) - F(w_i)]$$
$$= \int_{\Omega_\epsilon}[\frac{1}{2}f(w_i)w_i - F(w_i)] - \frac{1}{2}B_\epsilon(Q_i) - \int_{\Omega_\epsilon}[F(w_{\epsilon,Q_i}) - F(w_i) + f(w_i)\varphi_{\epsilon,Q_i}].$$

Note that

$$\int_{\Omega_\epsilon}[\frac{1}{2}f(w_i)w_i - F(w_i)] = \int_{\mathbb{R}^N}[\frac{1}{2}f(w_i)w_i - F(w_i)] - \int_{\mathbb{R}^N\setminus\Omega_\epsilon}[\frac{1}{2}f(w_i)w_i - F(w_i)]$$
$$= I[w] + O(\epsilon^{M(1+\frac{\sigma}{2})}),$$
$$\left|\int_{\Omega_\epsilon}[F(w_{\epsilon,Q_i}) - F(w_i) - f(w_i)\varphi_{\epsilon,Q_i}]\right| \leq C\int_{\Omega_\epsilon}(|w_i|^\sigma|\varphi_{\epsilon,Q_i}|^2 + |\varphi_{\epsilon,Q_i}|^{2+\sigma})$$
$$\leq C\epsilon^{M(1+\frac{\sigma}{2})}.$$

So

$$\sum_{i=1}^{K} J_\epsilon[w_{\epsilon,Q_i}] = KI[w] - \frac{1}{2}\sum_{i=1}^{K} B_\epsilon(Q_i) + O(K\epsilon^{M(1+\frac{\sigma}{2})}). \tag{1.2}$$

Next we have

$$\sum_{i\neq j}\int_{\Omega_\epsilon} f(w_i)w_{\epsilon,Q_j} = \sum_{i\neq j}\int_{\Omega_\epsilon} f(w_i)w_j - \sum_{i\neq j}\int_{\Omega_\epsilon} f(w_i)\varphi_{\epsilon,Q_j}$$

$$= \sum_{i\neq j} B_\epsilon(Q_i,Q_j) + \sum_{i\neq j} O(e^{-\frac{|Q_i-Q_j^*|}{\epsilon}}) = \sum_{i\neq j} B_\epsilon(Q_i,Q_j) + O(K^2\epsilon^{\sqrt{2}M}). \tag{1.3}$$

It remains to compute the last term in (1.1). To this end, we divide the domain into $(K+1)-$parts as in (2.19). On $\Omega_{\epsilon,K+1}$, we have

$$\int_{\Omega_{\epsilon,K+1}}\left[F(w_{\epsilon,\mathbf{Q}}) - \sum_{j=1}^{K} F(w_{\epsilon,Q_j})\right] = O\left(\int_{\Omega_{\epsilon,K+1}}(|\sum_{j=1}^{K} w_{\epsilon,Q_j}|^{2+\sigma} + \sum_{j=1}^{K} w_{\epsilon,Q_j}^{2+\sigma})\right)$$
$$= O(K^{2+\sigma}\epsilon^{M(1+\frac{\sigma}{2})}).$$

On $\Omega_{\epsilon,j}, j = 1, ..., K$, we have

$$\int_{\Omega_{\epsilon,j}}\left[F(w_{\epsilon,\mathbf{Q}}) - \sum_{j=1}^{K} F(w_{\epsilon,Q_j})\right] = \int_{\Omega_{\epsilon,j}}\left[F(w_{\epsilon,Q_j} + \sum_{l\neq j} w_{\epsilon,Q_l}) - F(w_{\epsilon,Q_j}) - \sum_{l\neq j} F(w_{\epsilon,Q_l})\right]$$

where the last term can be estimated as

$$\sum_{l\neq j}\int_{\Omega_{\epsilon,j}}|F(w_{\epsilon,Q_l})|\leq C\sum_{l\neq j}\int_{\Omega_{\epsilon,j}}|w_{\epsilon,Q_l}|^{2+\sigma}$$
$$\leq CKe^{-(1+\frac{\sigma}{2})\frac{\varphi(\mathbf{Q})}{\epsilon}}\leq CK\epsilon^{M(1+\frac{\sigma}{2})}.$$

So

$$\int_{\Omega_{\epsilon,j}}\left[F(w_{\epsilon,Q_j}+\sum_{l\neq j}w_{\epsilon,Q_l})-F(w_{\epsilon,Q_j})-\sum_{l\neq j}F(w_{\epsilon,Q_l})\right]$$
$$=\int_{\Omega_{\epsilon,j}}\left[F(w_{\epsilon,Q_j}+\sum_{l\neq j}w_{\epsilon,Q_l})-F(w_{\epsilon,Q_j})\right]+O(K\epsilon^{M(1+\frac{\sigma}{2})})$$
$$=\sum_{l\neq j}\int_{\Omega_{\epsilon,j}}f(w_{\epsilon,Q_j})w_{\epsilon,Q_l}+O(K\epsilon^{M(1+\frac{\sigma}{2})})+\int_{\Omega_{\epsilon,j}}\left(|w_{\epsilon,Q_j}|^{\sigma}(\sum_{l\neq j}w_{\epsilon,Q_l})^2+(\sum_{l\neq j}w_{\epsilon,Q_l})^{2+\sigma}\right)$$

where the last term can be estimated by

$$\int_{\Omega_{\epsilon,j}}\left(|w_{\epsilon,Q_j}|^{\sigma}(\sum_{l\neq j}w_{\epsilon,Q_l})^2+(\sum_{l\neq j}w_{\epsilon,Q_l})^{2+\sigma}\right)$$
$$\leq CK^2\epsilon^{M(1+\frac{\sigma}{2})}+CK^{2+\sigma}\epsilon^{M(1+\frac{\sigma}{2})}\leq CK^{2+\sigma}\epsilon^{M(1+\frac{\sigma}{2})}.$$

Note that

$$\int_{\Omega_{\epsilon,j}}f(w_{\epsilon,Q_j})w_{\epsilon,Q_l}=\int_{\Omega_{\epsilon,j}}f(w_j)w_{\epsilon,Q_l}+O(\int_{\Omega_{\epsilon,j}}|w_j|^{\sigma}|\varphi_{\epsilon,Q_j}|w_{\epsilon,Q_l})$$
$$=B_\epsilon(Q_j,Q_l)+O(\epsilon^{\sqrt{2}M}).$$

Thus

$$\int_{\Omega_\epsilon}\left[F(w_{\epsilon,Q_j}+\sum_{l\neq j}w_{\epsilon,Q_l})-F(w_{\epsilon,Q_j})-\sum_{l\neq j}F(w_{\epsilon,Q_l})\right]$$
$$=\sum_{l\neq j}B_\epsilon(Q_l,Q_j)+O(K^{3+\sigma}\epsilon^{M(1+\frac{\sigma}{2})}+K^2\epsilon^{\sqrt{2}M}).\tag{1.4}$$

Combining the estimates (1.2), (1.3) and (1.4) together, we arrive

$$J_\epsilon[w_{\epsilon,\mathbf{Q}}]=KI[w]-\frac{1}{2}\sum_{i=1}^{K}B_\epsilon(Q_i)-\frac{1}{2}\sum_{i\neq j}B_\epsilon(Q_i,Q_j)+O(K^{3+\sigma}\epsilon^{M(1+\frac{\sigma}{2})}+K^2\epsilon^{\sqrt{2}M})$$

Since $O(K^{3+\sigma}\epsilon^{M(1+\frac{\sigma}{2})}+K^2\epsilon^{\sqrt{2}M})=O(\epsilon^M\epsilon^{M\frac{\sigma}{2}-(3+\sigma)N}+\epsilon^{\sqrt{2}M-2N})=o(w(M|\ln\epsilon|))$ by our choice of M at (2.8), we obtain (2.27) of Lemma 2.5.

Finally we prove (2.28). Observe that

$$S_\epsilon[w_{\epsilon,\mathbf{Q}}]=f(\sum_{j=1}^{K}w_{\epsilon,Q_j})-\sum_{j=1}^{K}f(w_j)$$

In $\Omega_{\epsilon,K+1}$, we have

$$|S_\epsilon[\sum_{j=1}^K w_{\epsilon,Q_j}]| \leq C(|\sum_j w_{\epsilon,Q_j}|^{1+\sigma} + \sum_j w_j^{1+\sigma})$$

$$\leq CK^\sigma \sum_{j=1}^K w_{\epsilon,Q_j}^{1+\sigma} + \sum_j w_j^{1+\sigma}$$

$$\leq O(K^{1+\sigma} e^{-\frac{1+\sigma}{2\epsilon}\varphi(\mathbf{Q})}) \leq O(K^{1+\sigma} \epsilon^{\frac{M(1+\sigma)}{2}}).$$

Hence

$$\|S_\epsilon[\sum_{j=1}^K w_{\epsilon,Q_j}]\|_{L^q(\Omega_{\epsilon,K+1})} \leq O(K^{1+\sigma}\epsilon^{M(\frac{1+\sigma}{2})}). \tag{1.5}$$

In $\Omega_{\epsilon,j}$, $j = 1, ..., K$, we have

$$|S_\epsilon[\sum_{i=1}^K w_{\epsilon,Q_j}]| = |f(w_{\epsilon,Q_j}) - f(w_j)| + O\left(\sum_{l\neq j} |w_{\epsilon,Q_j}|^\sigma w_{\epsilon,Q_l} + (\sum_{l\neq j} w_{\epsilon,Q_l})^{1+\sigma}\right) + O(\sum_{l\neq j} w_l^{1+\sigma})$$

$$= |f(w_{\epsilon,Q_j}) - f(w_j)| + O\left(K\epsilon^{\frac{M(1+\sigma)}{2}} + K^{1+\sigma}e^{\frac{M(1+\sigma)}{2}}\right) + O(K\epsilon^{\frac{M(1+\sigma)}{2}})$$

$$= |f(w_{\epsilon,Q_j}) - f(w_j)| + O(K^{1+\sigma}e^{\frac{M(1+\sigma)}{2}}). \tag{1.6}$$

Therefore, we obtain

$$\int_{\Omega_{\epsilon,j}} (S_\epsilon[\sum_{j=1}^K w_{\epsilon,Q_j}])^q \leq C\int_{\Omega_{\epsilon,j}} |f(w_{\epsilon,Q_j}) - f(w_j)|^q + O(K^{q+q\sigma}e^{\frac{qM(1+\sigma)}{2}})$$

$$\leq \int_{\Omega_{\epsilon,j}} w_j^{q\sigma}|\varphi_{\epsilon,Q_j}|^q + O(K^{q+q\sigma}e^{\frac{qM(1+\sigma)}{2}})$$

$$\leq O(\epsilon^{\frac{qM(1+\sigma)}{2}} + K^{q+q\sigma}e^{\frac{qM(1+\sigma)}{2}})$$

and

$$\sum_{j=1}^K \int_{\Omega_{\epsilon,j}} (S_\epsilon[\sum_{j=1}^K w_{\epsilon,Q_j}])^q = O(K\epsilon^{\frac{qM(1+\sigma)}{2}} + K^{q+q\sigma+1}e^{\frac{qM(1+\sigma)}{2}}) = O(K^{q+q\sigma+1}\epsilon^{\frac{qM(1+\sigma)}{2}}). \tag{1.7}$$

Combining (1.5), (1.6) and (1.7), we obtain

$$\|S_\epsilon[\sum_{j=1}^K w_{\epsilon,Q_j}]\|_{L^q(\Omega_\epsilon)} \leq O(K^{1+\sigma}\epsilon^{\frac{M(1+\sigma)}{2}} + K^{1+\frac{1}{q}+\sigma}e^{\frac{M(1+\sigma)}{2}}) \leq CK^{1+\frac{1}{q}+\sigma}e^{\frac{M(1+\sigma)}{2}})$$

which proves (2.28) of Lemma 2.5.

□

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