On Non-topological Solutions of the $\mathbf{A}_2$ and $\mathbf{B}_2$ Chern-Simons System*

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Abstract

For any rank 2 of simple Lie algebra, the relativistic Chern-Simons system has the following form:

$$\begin{cases} \Delta u_1 + (\sum_{i=1}^2 K_{1i}e^{u_i} - \sum_{i=1}^2 \sum_{j=1}^2 e^{u_i}K_{1i}e^{u_j}K_{ij}) = 4\pi \sum_{j=1}^{N_1} \delta_{p_j} \\ \Delta u_2 + (\sum_{i=1}^2 K_{2i}e^{u_i} - \sum_{i=1}^2 \sum_{j=1}^2 e^{u_i}K_{2i}e^{u_j}K_{ij}) = 4\pi \sum_{j=1}^{N_2} \delta_{q_j} \end{cases} \text{in } \mathbb{R}^2, \tag{0.1}$$

where K is the Cartan matrix of rank 2. There are three Cartan matrix of rank 2: $\mathbf{A}_2$, $\mathbf{B}_2$ and $\mathbf{G}_2$. A long-standing open problem for (0.1) is the question of the existence of non-topological solutions. In this paper, we consider the $\mathbf{A}_2$ and $\mathbf{B}_2$ case. We prove the existence of non-topological solutions under the condition that either $\sum_{j=1}^{N_1} p_j = \sum_{j=1}^{N_2} q_j$ or $\sum_{j=1}^{N_1} p_j \neq \sum_{j=1}^{N_2} q_j$ and $N_1, N_2 > 1, |N_1 - N_2| \neq 1$. We solve this problem by a perturbation from the corresponding $\mathbf{A}_2$ and $\mathbf{B}_2$ Toda system with one singular source.

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1 Introduction

1.1 Background

There are four types of simple non-exceptional Lie Algebra:$\mathbf{A}_m$, $\mathbf{B}_m$, $\mathbf{C}_m$, and $\mathbf{D}_m$ which Cartan subalgebra are $sl(m+1)$, $so(2m+1)$, $sp(m)$, and $so(2m)$ respectively. To each of them, a Toda system is associated. In geometry, solutions of Toda system is closely related to holomorphic curves in projective spaces. For example, the Toda system of type $\mathbf{A}_m$ can be derived from the classical Plücker formulas, and any holomorphic curve gives rise to a solution u of the Toda system, whose branch points correspond to the singularities of u. Conversely, we could integrate the Toda system, and any solution u gives rise to a holomorphic curve in $\mathbb{CP}^n$ at least locally. See [16] and reference therein. It is very interesting to note that the reverse process holds globally if the domain for the equation is $\mathbb{S}^2$ or $\mathbb{C}$. Any solution u of type $\mathbf{A}_m$ Toda system on $\mathbb{S}^2$ or $\mathbb{C}$ could produce a global holomorphic curves into $\mathbb{CP}^n$. This holds even when the solution u has singularities. We refer the readers to [16] for more precise statements of these results.

In physics, the Toda system also plays an important role in non-Abelian gauge field theory. One example is the relativistic Chern-Simons model proposed by Dunne [7, 8, 9] in order to explain the physics of high critical temperature superconductivity. See also [13], [14] and [15].

The model is defined in the (2+1) Minkowski space $\mathbb{R}^{1,2}$, the gauge group is a compact Lie group with a semi-simple Lie algebra $\mathcal{G}$. The Chern-Simons Lagrangian density $\mathcal{L}$ is defined by:

$$\mathcal{L} = -k\epsilon^{\mu\nu\rho} tr(\partial_\mu A_\nu A_\rho) + \frac{2}{3} A_\mu A_\nu A_\rho - tr((D_\mu \phi)^\dagger D^\mu \phi) - V(\phi, \phi^\dagger)$$

for a Higgs field ϕ in the adjoint representation of the compact gauge group G, where the associated semi-simple Lie algebra is denoted by $\mathcal{G}$ and the $\mathcal{G}-$valued gauge field A_α on $2+1$ dimensional Minkowski space $\mathbb{R}^{1,2}$ with metric diag{-1,1,1}. Here $k > 0$ is the Chern-Simons coupling parameter, tr is the trace in the matrix representation of $\mathcal{G}$ and V is the potential energy density of the Higgs field $V(\phi, \phi^\dagger)$ given by

$$V(\phi, \phi^\dagger) = \frac{1}{4k^2} tr(([[\phi, \phi^\dagger], \phi] - v^2\phi)^\dagger([[\phi, \phi^\dagger], \phi] - v^2\phi)),$$

where $v > 0$ is a constant which measures either the scale of the broken symmetry or the subcritical temperature of the system.

In general, the Euler-Lagrangian equation corresponding $\mathcal{L}$ is very difficult to study. So we restrict to consider solutions to be energy minimizers of the Lagrangian functional, and a self-dual system of first order derivatives could be derived from minimizing the energy functional:

$$\begin{array}{l} D_{-}\phi = 0 \\ F_{+-} = \frac{1}{k^2}\left[\phi - [[\phi, \phi^\dagger], \phi], \phi^\dagger\right], \end{array} \tag{1.1}$$

where $D_{-} = D_1 - iD_2$, and $F_{+-} = \partial_+ A_- - \partial_- A_+ + [A_+, A_-]$ with $A_\pm = A_1 + iA_2$, $\partial_\pm = \partial_1 \pm i\partial_2$. Here ∂_i and D_i are respectively the partial derivative and the gauge-covariant derivative w.r.t z_i, $i = 1, 2$.

In order to find non-trivial solutions which are not algebraic solutions of $[[\phi, \phi^\dagger], \phi] = \phi$, Dunne [8] has considered a simplified form of the self-dual system (1.1) in which both the gauge potential A and the Higgs field ϕ are algebraically restricted, for example, ϕ has the following form:

$$\phi = \sum_{a=1}^{r} \phi^a E_a,$$

where r is the rank of the Lie algebra $\mathcal{G}$, $\{E_{\pm a}\}$ is the family of the simple root step operators (with $E_{-a} = E^+$), and ϕ^a are complex-valued functions.

In this paper, we consider the case of rank 2. Let

$$u_a = \ln |\phi^a|^2.$$

Then equation (1.1) can be reduced to

$$\begin{cases} \Delta u_1 + \frac{1}{k^2}(\sum_{i=1}^2 K_{1i}e^{u_i} - \sum_{i=1}^2 \sum_{j=1}^2 e^{u_i} K_{1i} e^{u_j} K_{ij}) = 4\pi \sum\limits_{j=1}^{N_1} \delta_{p_j} \\ \Delta u_2 + \frac{1}{k^2}(\sum_{i=1}^2 K_{2i}e^{u_i} - \sum_{i=1}^2 \sum_{j=1}^2 e^{u_i} K_{2i} e^{u_j} K_{ij}) = 4\pi \sum\limits_{j=1}^{N_2} \delta_{q_j} \end{cases} \text{in } \mathbb{R}^2, \tag{1.2}$$

where $K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$ is the Cartan matrix of rank 2 of the Lie algebra $\mathcal{G}$, $\{p_1, \ldots, p_{N_1}\}$ and $\{q_1, \ldots, q_{N_2}\}$ are given vortex points. For the details of the process to derive (1.2), we refer to [8],[22],[31] and [32]. In this paper, without loss of generality, we assume $k = 1$.

It is known that there are three types of Cartan matrix of rank 2, given by

$$\mathbf{A}_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad \mathbf{B}_2 (= \mathbf{C}_2) = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}, \quad \mathbf{G}_2 = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}. \tag{1.3}$$

In this paper, we construct non-topological solutions in the case of $\mathbf{A}_2$ and $\mathbf{B}_2$. For $K = \mathbf{A}_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ and $K = \mathbf{B}_2 = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$, equation (1.2) becomes

$$\begin{cases} \Delta u_1 + 2e^{u_1} - e^{u_2} - (4e^{2u_1} - 2e^{2u_2} - e^{u_1+u_2}) = 4\pi \sum_{j=1}^{N_1} \delta_{p_j} \\ \Delta u_2 + 2e^{u_2} - e^{u_1} - (4e^{2u_2} - 2e^{2u_1} - e^{u_1+u_2}) = 4\pi \sum_{j=1}^{N_2} \delta_{q_j} \end{cases} \text{in } \mathbb{R}^2, \tag{1.4}$$

and

$$\begin{cases} \Delta u_1 + 2e^{u_1} - e^{u_2} - (4e^{2u_1} - 2e^{2u_2}) = 4\pi \sum_{j=1}^{N_1} \delta_{p_j} \\ \Delta u_2 + 2e^{u_2} - 2e^{u_1} - (4e^{2u_2} - 4e^{2u_1} - 2e^{u_1+u_2}) = 4\pi \sum_{j=1}^{N_2} \delta_{q_j} \end{cases} \text{in } \mathbb{R}^2, \tag{1.5}$$

respectively.

1.2 Previous Results

In the literature, a solution $\mathbf{u} = (u_1, u_2)$ to system (1.2) is called a *topological* solution if $\mathbf{u}$ satisfies

$$u_a(z) \to \ln \sum_{j=1}^{2} (K^{-1})_{aj} \quad \text{as } |z| \to +\infty, \ a = 1, 2,$$

and is called a *non-topological* solution if $\mathbf{u}$ satisfies

$$u_a(z) \to -\infty \quad \text{as } |z| \to +\infty,\ a = 1, 2. \tag{1.6}$$

The existence of topological solutions with arbitrary multiple vortex points was proved by Yang [32] more than fifteen years ago, not only for (1.4) and (1.5), but also for general Cartan matrix including $SU(N+1)$ case, $N \geq 1$. However, the existence of non-topological solutions is more difficult to prove. The first result was due to Chae and Imanuvilov [2] for the $SU(2)$ Abelian Chern-Simons equation which is obtained by letting $u_1(z) = u_2(z) = u(z)$ in the system (1.4) where u satisfies

$$\Delta u + e^u(1 - e^u) = 4\pi \sum_{j=1}^{N} \delta_{p_j} \ \text{in} \ \mathbb{R}^2. \tag{1.7}$$

Equation (1.7) is the $SU(2)$ Chern-Simons equation for the Abelian case. This relativistic Chern-Simons model was proposed by Jackiw-Weinberg [12] and Hong-Kim-Pac [11]. For the past more than twenty years, the existence and multiplicity of solutions to (1.7) with different nature (e.g. topological, non-topological, periodically constrained etc.) have been studied, see [1], [2], [3], [4], [5], [17], [18], [19], [20], [24], [25], [26], [27], [28] and references therein.

In [2], Chae and Imanuvilov proved the existence of non-topological solutions for (1.7) for any vortex points $(p_1, ..., p_N)$. For the question of existence of non-topological solutions for the system (1.4), an "answer" was given by Wang and Zhang [30] but their proof contains serious gaps. In fact they used a special solution of the Toda system as the approximate solution, but they did not have the full non-degeneracy of the linearised equation of the Toda system and their analysis for the linearised equation is incorrect. Thus, the existence of non-topological solutions has remained a long-standing open problem. Even for radially symmetric solutions (the case when all the vortices coincide), the ODE system of (1.4) or (1.5) is much subtle than equation (1.7). The classification of radial solution is an important issue for future study as long as bubbling solutions are concerned.

1.3 Main Results

In this paper, we give an affirmative answer to the existence of non-topological solutions for the system with Cartan matrix $\mathbf{A}_2$ and $\mathbf{B}_2$. Our main theorem can be stated as follows.

Theorem 1.1. *Let* $\{p_j\}_{j=1}^{N_1}$, $\{q_j\}_{j=1}^{N_2} \subset \mathbb{R}^2$. *If either*

(a) $\sum_{j=1}^{N_1} p_j = \sum_{j=1}^{N_2} q_j$;

or

(b) $\sum_{j=1}^{N_1} p_j \neq \sum_{j=1}^{N_2} q_j$ *and* $N_1, N_2 > 1$, $|N_1 - N_2| \neq 1$, *then there exists a non-topological solution* (u_1, u_2) *of problem (1.4) and (1.5) respectively.*

Non-topological solutions play very important role in the bubbling analysis of solutions to (1.2). Therefore, our result is only the first step towards understanding the solution structure of non-topological solution of (1.2). For further study on non-topological solutions for the Abelian case, we refer to [3] and [5].

We will prove Theorem 1.1 in three cases which we describe below:

Assumption (i):
$$\sum_{j=1}^{N_1} p_j = \sum_{j=1}^{N_2} q_j \ , \ N_1 = N_2; \tag{1.8}$$

Assumption (ii):
$$\sum_{j=1}^{N_1} p_j = \sum_{j=1}^{N_2} q_j \ , \ N_1 \neq N_2 \ , \ N_1, N_2 > 1 \tag{1.9}$$
$$\text{or} \ \sum_{j=1}^{N_1} p_j \neq \sum_{j=1}^{N_2} q_j \ , \ |N_1 - N_2| \neq 1 \ , \ N_1, N_2 > 1;$$

Assumption (iii):
$$\sum_{j=1}^{N_1} p_j = \sum_{j=1}^{N_2} q_j, \ N_1 \neq N_2, \ N_1 = 1 \text{ or } N_2 = 1. \tag{1.10}$$

If we can prove the existence of non-topological solutions under the above three assumptions separately, then it is easy to see that Theorem 1.1 is proved. So in the following, we will prove the theorem under the three assumptions respectively.

1.4 Sketch of the Proof for the case A_2

In the following, we will outline the sketch of our proof for the $\mathbf{A}_2$ case, the proof for $\mathbf{B}_2$ is similar.

First, let us recall the proof of the existence of non-topological solutions of Chae and Imanuvilov [2] for single $SU(2)$ equation (1.7). Getting rid of the Dirac measure, (1.7) is equivalent to

$$\Delta u + \Pi_{j=1}^N |z-p_j|^2 e^u = \Pi_{j=1}^N |z-p_j|^4 e^{2u} \text{ in } \mathbb{R}^2. \tag{1.11}$$

After a suitable scaling transformation, the equation (1.11) becomes

$$\Delta \tilde{U} + \Pi_{j=1}^N |z-\varepsilon p_j|^2 e^{\tilde{U}} = \varepsilon^2 \Pi_{j=1}^N |z-\varepsilon p_j|^4 e^{2\tilde{U}} \text{ in } \mathbb{R}^2 \tag{1.12}$$

where ε is small. Let $\varepsilon = 0$. The zeroth order equation of (1.12) is

$$\Delta U_0 + |z|^{2N} e^{U_0} = 0 \text{ in } \mathbb{R}^2, \int_{\mathbb{R}^2} |z|^{2N} e^{U_0} < +\infty. \tag{1.13}$$

Prajapat and Tarantello [23] gave a complete classification of solutions to (1.13). In fact all solutions to (1.13) are given by

$$U_{\lambda,a} = \log \frac{8(N+1)^2\lambda^2}{(\lambda^2 + |z^{N+1} - a|^2)^2}. \tag{1.14}$$

(Furthermore, in a recent paper [10], del Pino, Esposito and Musso proved that the solution $U_{\lambda,a}$ is actually non-degenerate (see also [16] for another proof).)

In [2], the authors have chosen the following radially symmetric initial approximation

$$U_0 = \log \frac{8(N+1)^2}{(1+|z|^{2(N+1)})^2}. \tag{1.15}$$

This radial solution is non-degenerate, i.e., the dimension of the kernel of the linearized operator is exactly three, and is spanned by $\frac{\partial U}{\partial a_1}, \frac{\partial U}{\partial a_2}, \frac{\partial U}{\partial \lambda}$. To obtain the next order term in ε, in the reduced problem, one then needs to solve the following equation

$$\Delta \phi + |z|^{2N} e^{U_0} \phi = \frac{|z|^{4N+2}}{(1+|z|^{2N+2})^4}. \tag{1.16}$$

Because of the special form of the right hand side, the solution to (1.16) can be found explicitly ([2]). Then by adding appropriate "small" non-radial functions, they used the standard implicit function theorem to prove their result. In this procedure, the number of solvability conditions (and so the number of free parameters) is exactly 2, since the codimension of the linearised problem as in (1.16) is 2, and the scaling parameter has been fixed.

Now we comment on the main ideas and difficulties for systems. As in [2], we will view equation (1.4) as a small perturbation of the $SU(3)$ Toda system with singular source. (Recently, the complete classification and nondegeneracy of solutions to the Toda system with singular source was obtained by Ye and the second and third authors in [16].)

Let us first consider the system (1.4). Similar to the procedure of leading to (1.12), after a suitable scaling transformation, the system (1.4) is transformed to

$$\begin{aligned} &\Delta \tilde{U}_1 + \Pi_{j=1}^{N_1} |z - \varepsilon p_j|^2 e^{2\tilde{U}_1 - \tilde{U}_2} \\ &= 2\varepsilon^2 \Pi_{j=1}^{N_1} |z - \varepsilon p_j|^4 e^{4\tilde{U}_1 - 2\tilde{U}_2} - \varepsilon^2 \Pi_{j=1}^{N_1} |z - \varepsilon p_j|^2 \Pi_{j=1}^{N_2} |z - \varepsilon q_j|^2 e^{\tilde{U}_1 + \tilde{U}_2}, \\ &\Delta \tilde{U}_2 + \Pi_{j=1}^{N_2} |z - \varepsilon q_j|^2 e^{2\tilde{U}_2 - \tilde{U}_1} \\ &= 2\varepsilon^2 \Pi_{j=1}^{N_2} |z - \varepsilon q_j|^4 e^{4\tilde{U}_2 - 2\tilde{U}_1} - \varepsilon^2 \Pi_{j=1}^{N_2} |z - \varepsilon q_j|^2 \Pi_{j=1}^{N_1} |z - \varepsilon p_j|^2 e^{\tilde{U}_1 + \tilde{U}_2}. \end{aligned} \tag{1.17}$$

When $\varepsilon = 0$, we obtain the following limiting system

$$\begin{cases} \Delta \tilde{U}_1 + |z|^{2N_1} e^{2\tilde{U}_1 - \tilde{U}_2} = 0 \ \text{ in } \mathbb{R}^2 \\ \Delta \tilde{U}_2 + |z|^{2N_2} e^{2\tilde{U}_2 - \tilde{U}_1} = 0 \ \text{ in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} |z|^{2N_1} e^{2\tilde{U}_1 - \tilde{U}_2} < +\infty \ , \ \int_{\mathbb{R}^2} |z|^{2N_2} e^{2\tilde{U}_2 - \tilde{U}_1} < +\infty \end{cases} \tag{1.18}$$

which is the $SU(3)$ Toda system with single source at the origin.

In a recent paper of Lin, Wei and Ye [16], the authors have completely classified all the solutions of (1.18). In fact all solutions are of the form (see Lemma 2.1 below)

$$\begin{cases} e^{-\tilde{U}_1} = c_1 + c_2 |z^{N_1+1} - a|^2 + c_3 |z^{N_1+N_2+2} - b z^{N_1+1} - d|^2 \\ e^{-\tilde{U}_2} = c_1' + c_2' |z^{N_2+1} - a'|^2 + c_3' |z^{N_1+N_2+2} - b' z^{N_2+1} - d'|^2. \end{cases} \tag{1.19}$$

The solutions written above depend on eight parameters $(\mathbf{a}, \mu) = (a_1, a_2, b_1, b_2, d_1, d_2, \mu_1, \mu_2)$. In particular, all c_i and c_i' depend on the variables μ_1, μ_2. See Lemma 2.1. Furthermore, they showed that the dimension of the kernel of the linearized operator is *eight*.

The main difficulty in this paper is the large dimension of kernels. Unlike the single equation (1.7) case in which the coefficients for determining the free parameters can be computed explicitly, there are no explicit formula for the coefficients, except in the case $\sum_{j=1}^{N_1} p_j = \sum_{j=1}^{N_2} q_j$, $N_1 = N_2$ which can be considered as the reminiscent of the $SU(2)$ scalar equation. To get over this difficulty, we make use of the two scaling parameters for solutions of the Toda system and introduce two more free parameters . Instead of solving the coefficient matrices for fixed scaling parameters, we only need to compute the two matrices in front of the two free scaling parameters we introduce.

Now let us be more specific. The term of order $O(\varepsilon)$ will satisfy (2.32) and (2.34) in Section 2. The $O(\varepsilon^2)$ term will satisfy (2.36). In this $O(\varepsilon^2)$ term ψ, we introduce **two free parameters** ξ_1, ξ_2 which play an important role in our proof. See Section 2.8 and 2.9. At last the solution we find will be of this form

$$\tilde{U} = \tilde{U}_{\mathbf{a}} + \varepsilon\Psi + \varepsilon^2\psi + \varepsilon^2 v, \tag{1.20}$$

where $\tilde{U}_{\mathbf{a}} = (\tilde{U}_{1,\mathbf{a}}, \tilde{U}_{2,\mathbf{a}})$ is given by (1.19), and $\mathbf{a}$ denote the parameters $(\mathbf{a}, \mu)$ for simplicity of notations. In order to solve in v, we need to solve a linearized problem:

$$\begin{cases} \Delta\phi_1 + |z|^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}}(2\phi_1 - \phi_2) = f_1 \\ \Delta\phi_2 + |z|^{2N_1} e^{2\tilde{U}_{2,0} - \tilde{U}_{1,0}}(2\phi_2 - \phi_1) = f_2 \end{cases} \tag{1.21}$$

where f_1 and f_2 are explicitly given. We use the Liapunov-Schmidt reduction method to solve it. It turns out that we can choose the perturbation $\mathbf{a}$ and μ such that we can get the solution.

Now we comment on the technical conditions. In the proof, we will choose (μ_1, μ_2) first, depending on the assumptions. In general, the reduced problem for $\mathbf{a}$ has the following form

$$\frac{1}{\varepsilon}\mathbf{B}\mathbf{a} + \frac{1}{\varepsilon}\mathbf{A}\mathbf{a}\cdot\mathbf{a} + \mathbf{Q}\mathbf{a} + O(|\mathbf{a}|^2) + \mathbf{a}_0 = O(\varepsilon), \tag{1.22}$$

where $\mathbf{A}, \mathbf{B}, \mathbf{Q}$ are matrices of size 6×6, and $\mathbf{a}_0 \in \mathbb{R}^6$. Furthermore, the matrix $\mathbf{Q}$ can be decomposed into

$$\mathbf{Q} = \xi_1\mathbf{Q}_1 + \xi_2\mathbf{Q}_2 + \mathcal{T} \tag{1.23}$$

where ξ_1 and ξ_2 are two free parameters. As we said before, we shall not attempt to compute the matrices $\mathbf{A}, \mathbf{B}$ and $\mathcal{T}$. Instead we focus on the two matrices $\mathbf{Q}_1$ and $\mathbf{Q}_2$. All we need to show is that at least of one of these two matrices is non-degenerate.

In case (a) of Theorem 1.1, i.e. $\sum_{j=1}^{N_1} p_j = \sum_{j=1}^{N_2} q_j$, by a shift of origin, we may assume that $\sum_{j=1}^{N_1} p_j = \sum_{j=1}^{N_2} q_j = 0$. In this case, the $\varepsilon-$term $\varepsilon\Psi$ vanishes and both $\mathbf{A}$ and $\mathbf{B}$ vanish. If $N_1 \neq N_2, N_1, N_2 > 1$ or $N_1 = N_2$, then $\mathbf{a}_0$ vanishes and we obtain a reduced problems (in terms of $\mathbf{a}$) as follows:

$$\mathbf{Qa} + O(|\mathbf{a}|^2) = O(\varepsilon). \tag{1.24}$$

If $N_1 \neq N_2, N_1 = 1$, or $N_2 = 1$, we use a different $O(\varepsilon^2)$ approximation ψ in (1.20) and we obtain the reduced problem as follows:

$$\mathbf{Qa} + O(|\mathbf{a}|^2) + \mathbf{a}_0 = O(\varepsilon). \tag{1.25}$$

See Section 2.9. In both cases, we can show that the matrix $\mathbf{Q}$ is non-degenerate and (1.22) can be solved by contraction mapping.

The case (b) is considerably more difficult. Since $\sum_{j=1}^{N_1} p_j \neq \sum_{j=1}^{N_2} q_j$, the $\varepsilon-$ term exists and presents great difficulty in solving the reduced problem (1.22) in $\mathbf{a}$. To show that $\mathbf{B} = 0$, we need $|N_1 - N_2| \neq 1$. To show that the $\mathbf{a}_0$ term vanishes, we need $N_1, N_2 > 1$. In this case the reduced problem now takes the form

$$\frac{1}{\varepsilon}\mathbf{Aa} \cdot \mathbf{a} + \mathbf{Qa} + O(|\mathbf{a}|^2) = O(\varepsilon). \tag{1.26}$$

where $\mathbf{Q}$ has the form of (1.23). By choosing large ξ_1 and $\xi_2 = 0$, we can solve (1.26) such that $|\mathbf{a}| \leq O(\varepsilon)$.

In summary, the technical condition we have imposed is to make sure that $\mathbf{B} = 0$ and that the quadratic term $\frac{1}{\varepsilon}\mathbf{Aa} \cdot \mathbf{a}$ and the $O(1)$ term can not coexist.

We remark that in the $SU(2)$ case, we may assume that $\sum_{j=1}^{N} p_j = 0$ and so there is no $\varepsilon-$ term. The reduced problem for $\mathbf{a}$ is considerably simpler since the kernel is three-dimensional only and $\mathbf{a}$ is basically one-dimensional.

1.5 Sketch of the Proof for B_2 Case

Since the idea of the proof for $\mathbf{B}_2$ case is the same as $\mathbf{A}_2$ case, we only mention the main difference.

For the $\mathbf{B}_2$ case, after suitable transformation, when $\varepsilon = 0$, the limiting equation (1.18) becomes:

$$\begin{cases} \Delta \tilde{U}_1 + |z|^{2N_1} e^{2\tilde{U}_1 - \tilde{U}_2} = 0 \\ \Delta \tilde{U}_2 + |z|^{2N_2} e^{2\tilde{U}_2 - 2\tilde{U}_1} = 0 \\ \int_{\mathbb{R}^2} |z|^{2N_1} e^{2\tilde{U}_1 - \tilde{U}_2} < +\infty, \quad \int_{\mathbb{R}^2} |z|^{2N_2} e^{2\tilde{U}_2 - 2\tilde{U}_1} < +\infty. \end{cases} \tag{1.27}$$

An immediate problem is the classification and non-degeneracy of the above system.

In [16], Lin, Wei and Ye obtained the classification and non-degeneracy results of the $SU(N+1)$ Toda system with singular sources. In the appendix, we use the results of [16] to obtain a complete classification and non-degeneracy of the $\mathbf{B}_2$ Toda system (1.27). See Theorem 4.1 and Corollary 4.1. This is new. In fact, the Toda system with $\mathbf{B}_2$ can be embedded into the $\mathbf{A}_3$Toda system under the group action $\tilde{U}_1 = \tilde{U}_3$. The dimension of the linearized operator is **ten** and all the solutions of (1.27) can be expressed as

$$e^{-\tilde{U}_1} = (\lambda_0 + \sum_{i=1}^{3} \lambda_i |P_i(z)|^2), \tag{1.28}$$

where

$$P_i(z) = z^{\mu_1' + \cdots + \mu_i'} + \sum_{j=0}^{i-1} c_{ij} z^{\mu_1' + \cdots + \mu_j'}, \tag{1.29}$$

$\mu_1' = \mu_3' = N_1 + 1$, $\mu_2' = N_2 + 1$ and c_{ij} are complex numbers and λ_i are positive numbers. The solutions of (1.27) depend on ten parameters $(\tilde{\mathbf{a}}, \lambda) = (c_{21}, c_{30}, c_{31}, c_{32}, \lambda_0, \lambda_1)$. See Theorem 4.1.

Once we get the classification and non-degeneracy results, the proof can go through as the $\mathbf{A}_2$ case. Of course the computations get harder as we now have to compute two matrices with 64 coefficients.

1.6 Organization of the Paper

The organization of the paper is the following. In Section 2, we prove Theorem 1.1 for the $\mathbf{A}_2$ case. In Section 2.1, we present several important preliminaries of analysis. We first formulate our problem in terms of the functional equations (Section 2.1). Then we apply the classification and non-degeneracy results of Lin-Wei-Ye (Section 2.2). In Section 2.3, we establish the invertibility properties of the linearized operator. Finally we obtain the next two orders $O(\varepsilon)$ and $O(\varepsilon^2)$ in Section 2.4 and Section 2.5 respectively. In Section 2.6, we solve a projected nonlinear problem based on the preliminary results. In Section 2.7, we prove the theorem under the **Assumption (i)**. Finally we prove our main theorem under the **Assumption (ii)** and **Assumption (iii)** in Section 2.8 and 2.9 respectively. In Section 3, we prove our main theorem for $\mathbf{B}_2$ case, the proof is similar as in Section 2. In the first part of the appendix, we show the classification and non-degeneracy results of the $\mathbf{B}_2$ Toda system with singular sources. We postpone all the technical calculations in the second and third part of the appendix.

Remark 1.1. *Since the proof for the $\mathbf{A}_2$ and $\mathbf{B}_2$ case are similar, we may use the same notations in Section 2 and Section 3.*

Remark 1.2. *We believe that the idea to deal with $\mathbf{B}_2$ case can be used to deal with the $\mathbf{G}_2$ case. But it will be more complicated.*

2 Proof of Theorem 1.1 in the A_2 Case

In this section, we consider the following system in $\mathbb{R}^2$:

$$\begin{cases} \Delta u_1 + 2e^{u_1} - e^{u_2} = 4e^{2u_1} - 2e^{2u_2} - e^{u_1+u_2} + 4\pi \sum_{j=1}^{N_1} \delta_{p_j} \\ \Delta u_2 + 2e^{u_2} - e^{u_1} = 4e^{2u_2} - 2e^{2u_1} - e^{u_1+u_2} + 4\pi \sum_{j=1}^{N_2} \delta_{q_j}. \end{cases} \tag{2.1}$$

2.1 Functional Formulation of the Problem

The aim of this section is to transform system (2.1) to a more convenient form. Defining

$$u_1 = \sum_{j=1}^{N_1} \ln|z-p_j|^2 + \tilde{u}_1, \quad u_2 = \sum_{j=1}^{N_2} \ln|z-q_j|^2 + \tilde{u}_2,$$

we obtain from (2.1) that $(\tilde{u}_1, \tilde{u}_2)$ satisfy

$$\begin{aligned}
&\Delta\tilde{u}_1 + 2\Pi_{j=1}^{N_1}|z-p_j|^2 e^{\tilde{u}_1} - \Pi_{j=1}^{N_2}|z-q_j|^2 e^{\tilde{u}_2} \\
&= 4\Pi_{j=1}^{N_1}|z-p_j|^4 e^{2\tilde{u}_1} - 2\Pi_{j=1}^{N_2}|z-q_j|^4 e^{2\tilde{u}_2} - \Pi_{j=1}^{N_1}|z-p_j|^2\Pi_{j=1}^{N_2}|z-q_j|^2 e^{\tilde{u}_1+\tilde{u}_2}, \\
&\Delta\tilde{u}_2 + 2\Pi_{j=1}^{N_2}|z-q_j|^2 e^{\tilde{u}_2} - \Pi_{j=1}^{N_1}|z-p_j|^2 e^{\tilde{u}_1} \\
&= 4\Pi_{j=1}^{N_2}|z-q_j|^4 e^{2\tilde{u}_2} - 2\Pi_{j=1}^{N_1}|z-p_j|^4 e^{2\tilde{u}_1} - \Pi_{j=1}^{N_2}|z-q_j|^2\Pi_{j=1}^{N_1}|z-p_j|^2 e^{\tilde{u}_1+\tilde{u}_2}.
\end{aligned}$$

Then, making a change of variables $z = \frac{\tilde{z}}{\varepsilon}$, and defining

$$\tilde{u}_1(z) = U_1(\tilde{z}) + (2N_1+2)\ln\varepsilon, \quad \tilde{u}_2(z) = U_2(\tilde{z}) + (2N_2+2)\ln\varepsilon,$$

we get the equations satisfied by U_1, U_2 are

$$\begin{cases}
\Delta U_1 + 2\Pi_{j=1}^{N_1}|\tilde{z}-\varepsilon p_j|^2 e^{U_1} - \Pi_{j=1}^{N_2}|\tilde{z}-\varepsilon q_j|^2 e^{U_2} = 4\varepsilon^2\Pi_{j=1}^{N_1}|\tilde{z}-\varepsilon p_j|^4 e^{2U_1} \\
\quad -2\varepsilon^2\Pi_{j=1}^{N_2}|\tilde{z}-\varepsilon q_j|^4 e^{2U_2} - \varepsilon^2\Pi_{j=1}^{N_1}|\tilde{z}-\varepsilon p_j|^2\Pi_{j=1}^{N_2}|\tilde{z}-\varepsilon q_j|^2 e^{U_1+U_2} \\
\\
\Delta U_2 + 2\Pi_{j=1}^{N_2}|\tilde{z}-\varepsilon q_j|^2 e^{U_2} - \Pi_{j=1}^{N_1}|\tilde{z}-\varepsilon p_j|^2 e^{U_1} = 4\varepsilon^2\Pi_{j=1}^{N_2}|\tilde{z}-\varepsilon q_j|^4 e^{2U_2} \\
\quad -2\varepsilon^2\Pi_{j=1}^{N_1}|\tilde{z}-\varepsilon p_j|^4 e^{2U_1} - \varepsilon^2\Pi_{j=1}^{N_2}|\tilde{z}-\varepsilon q_j|^2\Pi_{j=1}^{N_1}|\tilde{z}-\varepsilon p_j|^2 e^{U_1+U_2}.
\end{cases} \tag{2.2}$$

Let $\begin{pmatrix} \tilde{U}_1 \\ \tilde{U}_2 \end{pmatrix} = \mathbf{A}_2^{-1}\begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$, where $\mathbf{A}_2$ is the Cartan matrix $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. We obtain from (2.2) that $(\tilde{U}_1, \tilde{U}_2)$ satisfy

$$\begin{cases}
\Delta\tilde{U}_1 + \Pi_{j=1}^{N_1}|\tilde{z}-\varepsilon p_j|^2 e^{2\tilde{U}_1-\tilde{U}_2} \\
= 2\varepsilon^2\Pi_{j=1}^{N_1}|\tilde{z}-\varepsilon p_j|^4 e^{4\tilde{U}_1-2\tilde{U}_2} - \varepsilon^2\Pi_{j=1}^{N_1}|\tilde{z}-\varepsilon p_j|^2\Pi_{j=1}^{N_2}|\tilde{z}-\varepsilon q_j|^2 e^{\tilde{U}_1+\tilde{U}_2} \\
\\
\Delta\tilde{U}_2 + \Pi_{j=1}^{N_2}|\tilde{z}-\varepsilon q_j|^2 e^{2\tilde{U}_2-\tilde{U}_1} \\
= 2\varepsilon^2\Pi_{j=1}^{N_2}|\tilde{z}-\varepsilon q_j|^4 e^{4\tilde{U}_2-2\tilde{U}_1} - \varepsilon^2\Pi_{j=1}^{N_2}|\tilde{z}-\varepsilon q_j|^2\Pi_{j=1}^{N_1}|\tilde{z}-\varepsilon p_j|^2 e^{\tilde{U}_1+\tilde{U}_2}.
\end{cases} \tag{2.3}$$

From now on, we shall work with (2.3). For simplicity of notations, we still denote the variable by z instead of $\tilde{z}$.

2.2 First Approximate Solution

When $\varepsilon = 0$, (2.3) becomes

$$\begin{cases}
\Delta\tilde{U}_1 + |z|^{2N_1} e^{2\tilde{U}_1-\tilde{U}_2} = 0 \\
\Delta\tilde{U}_2 + |z|^{2N_2} e^{2\tilde{U}_2-\tilde{U}_1} = 0
\end{cases} \tag{2.4}$$

whose solutions can be completely classified, thanks to [16].

Defining

$$\begin{pmatrix} \tilde{U}_1 \\ \tilde{U}_2 \end{pmatrix} = \begin{pmatrix} w_1 - 2\alpha_1 \ln|z| \\ w_2 - 2\alpha_2 \ln|z| \end{pmatrix}, \tag{2.5}$$

where $\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \mathbf{A}_2^{-1} \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}$, we get an equivalent form of (2.4):

$$\begin{cases} \Delta w_1 + e^{2w_1 - w_2} = 4\pi\alpha_1\delta_0, \\ \Delta w_2 + e^{2w_2 - w_1} = 4\pi\alpha_2\delta_0, \\ \int_{\mathbb{R}^2} e^{2w_1 - w_2} < +\infty \ , \ \int_{\mathbb{R}^2} e^{2w_2 - w_1} < +\infty. \end{cases} \tag{2.6}$$

For this system, we have the following classification result by [16]:

Lemma 2.1. *Let (w_1, w_2) be a solution of (2.6). Then we have*

- *(Classification) The solution space of (2.6) is eight dimensional. More precisely, all the solutions of (2.6) satisfy that*

$$e^{-w_1} = \frac{c_1 + c_2|z^{N_1+1} - a|^2 + c_3|z^{N_1+N_2+2} - bz^{N_1+1} - d|^2}{|z|^{2\alpha_1}},$$
$$e^{-w_2} = \frac{c_1' + c_2'|z^{N_2+1} - a'|^2 + c_3'|z^{N_1+N_2+2} - b'z^{N_2+1} - d'|^2}{|z|^{2\alpha_2}},$$

  with $(\mu_1, \mu_2, a, b, d) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{C} \times \mathbb{C} \times \mathbb{C}$ and

$$c_1 = \frac{(N_2+1)\mu_1}{4\Gamma}, \ c_2 = \frac{(N_1+N_2+2)\mu_2}{4\Gamma}, \ c_3 = \frac{N_1+1}{4\Gamma\mu_1\mu_2},$$
$$\Gamma = (N_1+1)(N_2+1)(N_1+N_2+2), \quad c_1c_2c_3\Gamma^2 = \frac{1}{64},$$
$$a' = \frac{N_1+1}{N_1+N_2+2}b, \quad b' = \frac{N_1+N_2+2}{N_2+1}a, \quad d' = -\frac{N_1+1}{N_2+1}(d+ab),$$
$$c_1' = \frac{(N_1+1)\mu_1\mu_2}{4\Gamma} = 4c_1c_2(N_1+1)^2,$$
$$c_2' = \frac{N_1+N_2+2}{4\Gamma\mu_2} = 4c_1c_3(N_1+N_2+2)^2,$$
$$c_3' = \frac{N_2+1}{4\Gamma\mu_1} = 4c_2c_3(N_2+1)^2.$$

By (2.5), all the solutions of (2.4) are of the form

$$\begin{cases} e^{-\tilde{U}_1} = e^{-w_1}|z|^{2\alpha_1} = c_1 + c_2|z^{N_1+1} - a|^2 + c_3|z^{N_1+N_2+2} - bz^{N_1+1} - d|^2 \\ e^{-\tilde{U}_2} = e^{-w_2}|z|^{2\alpha_2} = c_1' + c_2'|z^{N_2+1} - a'|^2 + c_3'|z^{N_1+N_2+2} - b'z^{N_2+1} - d'|^2, \end{cases} \tag{2.7}$$

where we have eight parameters $(\mathbf{a}, \mu) = (a_1, a_2, b_1, b_2, d_1, d_2, \mu_1, \mu_2)$.

When $\mathbf{a} = 0$, we obtain the radially symmetric solution

$$\begin{aligned} e^{-\tilde{U}_{1,0}} &= \rho_1^{-1} = c_1 + c_2|z^{N_1+1}|^2 + c_3|z^{N_1+N_2+2}|^2, \\ e^{-\tilde{U}_{2,0}} &= \rho_2^{-1} = c_1' + c_2'|z^{N_2+1}|^2 + c_3'|z^{N_1+N_2+2}|^2. \end{aligned} \tag{2.8}$$

Observe that the radial solution $(\tilde{U}_{1,0}, \tilde{U}_{2,0})$ depends on two scaling parameters (μ_1, μ_2). Later we shall choose (μ_1, μ_2) in different settings.

Next we have the following non-degeneracy result from [16]:

Lemma 2.2. *(Non-degeneracy) The above solutions of (2.6) are non-degenerate, i.e., the set of solutions corresponding to the linearized operator is exactly eight dimensional. More precisely, if* $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ *satisfies* $|\phi(z)| \leq C(1 + |z|)^\alpha$ *for some* $0 \leq \alpha < 1$*, and*

$$\begin{cases} \Delta\phi_1 + |z|^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}}(2\phi_1 - \phi_2) = 0 \\ \Delta\phi_2 + |z|^{2N_2} e^{2\tilde{U}_{2,0} - \tilde{U}_{1,0}}(2\phi_2 - \phi_1) = 0, \end{cases} \tag{2.9}$$

then ϕ *belongs to the following linear space* $\mathcal{K}$ *: the span of*

$$\{Z_{\mu_1}, Z_{\mu_2}, Z_{a_1}, Z_{a_2}, Z_{b_1}, Z_{b_2}, Z_{d_1}, Z_{d_2}\},$$

where

$$
\begin{aligned}
Z_{\mu_1} &= \begin{pmatrix} Z_{\mu_1,1} \\ Z_{\mu_1,2} \end{pmatrix} = \begin{pmatrix} \partial_{\mu_1}\tilde{U}_{1,0} \\ \partial_{\mu_1}\tilde{U}_{2,0} \end{pmatrix} = \begin{pmatrix} \rho_1(N_2+1-\frac{N_1+1}{\mu_1^2\mu_2}r^{2(N_1+N_2+2)}) \\ \rho_2((N_1+1)\mu_2-\frac{N_2+1}{\mu_1^2}r^{2(N_1+N_2+2)}) \end{pmatrix}, \\
Z_{\mu_2} &= \begin{pmatrix} Z_{\mu_2,1} \\ Z_{\mu_2,2} \end{pmatrix} = \begin{pmatrix} \partial_{\mu_2}\tilde{U}_{1,0} \\ \partial_{\mu_2}\tilde{U}_{2,0} \end{pmatrix} = \begin{pmatrix} \rho_1((N_1+N_2+2)r^{2(N_1+1)}-\frac{N_1+1}{\mu_1\mu_2^2}r^{2(N_1+N_2+2)}) \\ \rho_2((N_1+1)\mu_1-\frac{N_1+N_2+2}{\mu_2^2}r^{2(N_2+1)}) \end{pmatrix}, \\
Z_{a_1} &= \begin{pmatrix} Z_{a_1,1} \\ Z_{a_1,2} \end{pmatrix} = \begin{pmatrix} \partial_{a_1}\tilde{U}_{1,0} \\ \partial_{a_1}\tilde{U}_{2,0} \end{pmatrix} = \begin{pmatrix} c_2\rho_1 r^{N_1+1}\cos(N_1+1)\theta \\ c_3'\rho_2\frac{\partial b'}{\partial a_1}r^{N_1+2N_2+3}\cos(N_1+1)\theta \end{pmatrix}, \\
Z_{a_2} &= \begin{pmatrix} Z_{a_2,1} \\ Z_{a_2,2} \end{pmatrix} = \begin{pmatrix} \partial_{a_2}\tilde{U}_{1,0} \\ \partial_{a_2}\tilde{U}_{2,0} \end{pmatrix} = \begin{pmatrix} c_2\rho_1 r^{N_1+1}\sin(N_1+1)\theta \\ c_3'\rho_2\frac{\partial b'}{\partial a_2}r^{N_1+2N_2+3}\sin(N_1+1)\theta \end{pmatrix}, \\
Z_{b_1} &= \begin{pmatrix} Z_{b_1,1} \\ Z_{b_1,2} \end{pmatrix} = \begin{pmatrix} \partial_{b_1}\tilde{U}_{1,0} \\ \partial_{b_1}\tilde{U}_{2,0} \end{pmatrix} = \begin{pmatrix} c_3\rho_1 r^{2N_1+N_2+3}\cos(N_2+1)\theta \\ c_2'\frac{\partial a'}{\partial b_1}\rho_2 r^{N_2+1}\cos(N_2+1)\theta \end{pmatrix}, \\
Z_{b_2} &= \begin{pmatrix} Z_{b_2,1} \\ Z_{b_2,2} \end{pmatrix} = \begin{pmatrix} \partial_{b_2}\tilde{U}_{1,0} \\ \partial_{b_2}\tilde{U}_{2,0} \end{pmatrix} = \begin{pmatrix} c_3\rho_1 r^{2N_1+N_2+3}\sin(N_2+1)\theta \\ c_2'\frac{\partial a'}{\partial b_2}\rho_2 r^{N_2+1}\sin(N_2+1)\theta \end{pmatrix}, \\
Z_{d_1} &= \begin{pmatrix} Z_{d_1,1} \\ Z_{d_1,2} \end{pmatrix} = \begin{pmatrix} \partial_{d_1}\tilde{U}_{1,0} \\ \partial_{d_1}\tilde{U}_{2,0} \end{pmatrix} = \begin{pmatrix} c_3\rho_1 r^{N_1+N_2+2}\cos(N_1+N_2+2)\theta \\ c_3'\frac{\partial d'}{\partial d_1}\rho_2 r^{N_1+N_2+2}\cos(N_1+N_2+2)\theta \end{pmatrix}, \\
Z_{d_2} &= \begin{pmatrix} Z_{d_2,1} \\ Z_{d_2,2} \end{pmatrix} = \begin{pmatrix} \partial_{d_2}\tilde{U}_{1,0} \\ \partial_{d_2}\tilde{U}_{2,0} \end{pmatrix} = \begin{pmatrix} c_3\rho_1 r^{N_1+N_2+2}\sin(N_1+N_2+2)\theta \\ c_3'\frac{\partial d'}{\partial d_2}\rho_2 r^{N_1+N_2+2}\sin(N_1+N_2+2)\theta \end{pmatrix},
\end{aligned}
$$

and ρ_1, ρ_2 *are defined in (2.8).*

For simplicity of notations, we also denote by $(Z_1, Z_2, \cdots, Z_8)$ the kernels $(Z_{\mu_1}, Z_{\mu_2}, \cdots, Z_{d_2})$.

From the above definitions, we have

$$\int_{\mathbb{R}^2}\Delta Z_{\mu_i}\cdot Z_{a_j} = \int_{\mathbb{R}^2}\Delta Z_{\mu_i}\cdot Z_{b_j} = \int_{\mathbb{R}^2}\Delta Z_{\mu_i}\cdot Z_{d_j} = \int_{\mathbb{R}^2}\Delta Z_{a_i}\cdot Z_{d_j} = \int_{\mathbb{R}^2}\Delta Z_{b_i}\cdot Z_{d_j} = 0$$

for $i, j = 1, 2$, and

$$\int_{\mathbb{R}^2}\Delta Z_{a_i}\cdot Z_{a_j} = \int_{\mathbb{R}^2}\Delta Z_{b_i}\cdot Z_{b_j} = \int_{\mathbb{R}^2}\Delta Z_{d_i}\cdot Z_{d_j} = 0$$

for $i \neq j$. By Hölder's inequality,

$$
\begin{aligned}
(\int_{\mathbb{R}^2}\Delta Z_{\mu_1}\cdot Z_{\mu_2})^2 = (\int_{\mathbb{R}^2}\nabla Z_{\mu_1}\cdot\nabla Z_{\mu_2})^2 &< (\int_{\mathbb{R}^2}|\nabla Z_{\mu_1}|^2)(\int_{\mathbb{R}^2}|\nabla Z_{\mu_2}|^2) \\
&= (\int_{\mathbb{R}^2}\Delta Z_{\mu_1}\cdot Z_{\mu_1})(\int_{\mathbb{R}^2}\Delta Z_{\mu_2}\cdot Z_{\mu_2}),
\end{aligned}
$$

$$
\begin{aligned}
(\int_{\mathbb{R}^2} \Delta Z_{a_i} \cdot Z_{b_i})^2 = (\int_{\mathbb{R}^2} \nabla Z_{a_i} \cdot \nabla Z_{b_i})^2 &< (\int_{\mathbb{R}^2} |\nabla Z_{a_i}|^2)(\int_{\mathbb{R}^2} |\nabla Z_{b_i}|^2) \\
&= (\int_{\mathbb{R}^2} \Delta Z_{a_i} \cdot Z_{a_i})(\int_{\mathbb{R}^2} \Delta Z_{b_i} \cdot Z_{b_i})
\end{aligned}
$$

for $i = 1, 2$. So we obtain

$$
\det[(\int_{\mathbb{R}^2} \Delta Z_i \cdot Z_j)_{i,j=1,\cdots,8}] \neq 0. \tag{2.10}
$$

Corollary 2.1. *If* $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ *satisfies* $|\phi(z)| \leq C(1+|z|)^\alpha$ *for some* $0 \leq \alpha < 1$, *and*

$$
\begin{cases} \Delta\phi_1 + 2|z|^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} \phi_1 - |z|^{2N_2} e^{2\tilde{U}_{2,0} - \tilde{U}_{1,0}} \phi_2 = 0 \\ \Delta\phi_2 + 2|z|^{2N_2} e^{2\tilde{U}_{2,0} - \tilde{U}_{1,0}} \phi_2 - |z|^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} \phi_1 = 0, \end{cases} \tag{2.11}
$$

then ϕ *belongs to the following linear space* $\mathcal{K}^*$: *the span of*

$$
\{Z^*_{\mu_1}, Z^*_{\mu_2}, Z^*_{a_1}, Z^*_{a_2}, Z^*_{b_1}, Z^*_{b_2}, Z^*_{d_1}, Z^*_{d_2}\},
$$

where

$$
Z^*_i = \begin{pmatrix} Z^*_{i,1} \\ Z^*_{i,2} \end{pmatrix} = \begin{pmatrix} 2Z_{i,1} - Z_{i,2} \\ 2Z_{i,2} - Z_{i,1} \end{pmatrix}. \tag{2.12}
$$

Since

$$
\int_{\mathbb{R}^2} Z^*_{a_i} \cdot Z^*_{d_j} = \int_{\mathbb{R}^2} Z^*_{b_i} \cdot Z^*_{d_j} = 0
$$

for $i, j = 1, 2$, and

$$
\int_{\mathbb{R}^2} Z^*_{a_i} \cdot Z^*_{a_j} = \int_{\mathbb{R}^2} Z^*_{b_i} \cdot Z^*_{b_j} = \int_{\mathbb{R}^2} Z^*_{d_i} \cdot Z^*_{d_j} = \int_{\mathbb{R}^2} Z^*_{a_i} \cdot Z^*_{b_j} = 0
$$

for $i \neq j$. By Hölder inequality

$$
(\int_{\mathbb{R}^2} Z^*_{a_i} \cdot Z^*_{b_i})^2 < (\int_{\mathbb{R}^2} Z^*_{a_i} \cdot Z^*_{a_i}) \cdot (\int_{\mathbb{R}^2} Z^*_{b_i} \cdot Z^*_{b_i})
$$

for $i = 1, 2$.

We have

$$
\det[(\int_{\mathbb{R}^2} Z^*_i \cdot Z^*_j)_{i,j=3,\cdots,8}] \neq 0. \tag{2.13}
$$

We will choose the first approximate solution to be $\begin{pmatrix} \tilde{U}_{1,(\mu,\mathbf{a})} \\ \tilde{U}_{2,(\mu,\mathbf{a})} \end{pmatrix}$, where the parameters $\mu, \mathbf{a}$ satisfy

$$|\mathbf{a}| := |a| + |b| + |d| \leq C_0\varepsilon, \quad |\mu| = O(1) \tag{2.14}$$

for some fixed constant $C_0 > 0$.

We want to look for solutions of the form

$$\tilde{U}_1 = \tilde{U}_{1,(\mu,\mathbf{a})} + \varepsilon\Psi_1 + \varepsilon^2\phi_1, \quad \tilde{U}_2 = \tilde{U}_{2,(\mu,\mathbf{a})} + \varepsilon\Psi_2 + \varepsilon^2\phi_2, \tag{2.15}$$

where $\mathbf{a} = O(\varepsilon)$, and μ fixed. For simplicity of notation, we will denote by $\begin{pmatrix} \tilde{U}_{1,(\mu,\mathbf{a})} \\ \tilde{U}_{2,(\mu,\mathbf{a})} \end{pmatrix} = \begin{pmatrix} \tilde{U}_{1,\mathbf{a}} \\ \tilde{U}_{2,\mathbf{a}} \end{pmatrix}$. One should always keep in mind that the radial solutions depend on two parameters μ_1, μ_2.

To obtain the next order term, we need to study the linearized operator around the solution $\begin{pmatrix} \tilde{U}_{1,0} \\ \tilde{U}_{2,0} \end{pmatrix}$.

2.3 Invertibility of Linearized Operator

Now we consider the invertibility of the linearized operator in some suitable Sobolev spaces. To this end, we use the technical framework introduced by Chae-Imanuvilov [2]. Let $\alpha \in (0,1)$ and

$$X_\alpha = \{u \in L^2_{loc}(\mathbb{R}^2), \int_{\mathbb{R}^2} (1 + |x|^{2+\alpha})u^2 dx < +\infty\}, \tag{2.16}$$

$$Y_\alpha = \{u \in W^{2,2}_{loc}(\mathbb{R}^2), \int_{\mathbb{R}^2} (1 + |x|^{2+\alpha})|\Delta u|^2 + \frac{u^2}{1 + |x|^{2+\alpha}} < +\infty\}. \tag{2.17}$$

On X_α and Y_α, we equip with two norms respectively:

$$\|f\|_{**} = \sup_{y\in\mathbb{R}^2} (1 + |y|)^{2+\alpha}|f(y)|, \quad \|h\|_* = \sup_{y\in\mathbb{R}^2} (\log(2 + |y|))^{-1}|h(y)|. \tag{2.18}$$

For $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$, we denote by $\langle f, g\rangle = \int_{\mathbb{R}^2} f \cdot g dx$.

The main lemma on solvability is the following:

Lemma 2.3. *Assume that* $h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in X_\alpha$ *is such that*

$$\langle Z_i^*, h\rangle = 0, \ \textit{for}\ i = 3, \cdots, 8. \tag{2.19}$$

Then one can find a unique solution $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = T^{-1}(h) \in Y_\alpha$ *such that*

$$\begin{cases} \Delta\phi_1 + |z|^{2N_1} e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2\phi_1 - \phi_2) = h_1 \\ \Delta\phi_2 + |z|^{2N_1} e^{2\tilde{U}_{2,0}-\tilde{U}_{1,0}}(2\phi_2 - \phi_1) = h_2 \\ \quad \langle \phi, \Delta Z_j\rangle = 0, \ \textit{for}\ j = 1, \cdots, 8 \end{cases}, \ \ \|\phi\|_* \le C\|h\|_{**}. \tag{2.20}$$

Moreover, the map $h \xrightarrow{T} \phi$ *can be made continuous and smooth .*

Proof:

First, we show that there exists a solution to

$$\begin{cases} \Delta\phi_1^i + |z|^{2N_1} e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2\phi_1^i - \phi_2^i) = \Delta Z_{i,1}^* \\ \Delta\phi_2^i + |z|^{2N_2} e^{2\tilde{U}_{2,0}-\tilde{U}_{1,0}}(2\phi_2^i - \phi_1^i) = \Delta Z_{i,2}^* \end{cases} \ \text{for}\ i = 1, 2, \tag{2.21}$$

and $\phi^i \in Y_\alpha$. To this end, we solve this system in radial coordinates. We only consider ϕ^1. (ϕ^2 can be obtained in the same way.) Multiplying the first equation of (2.21) by $Z_{i,1}^*$, and the second equation by $Z_{i,2}^*$, for $i = 1, 2$, and recalling the equations that are satisfied by $Z_{i,1}, Z_{i,2}$, we have

$$\begin{cases} \Delta\phi_1^1(2Z_{i,1} - Z_{i,2}) - \Delta Z_{i,1}(2\phi_1^1 - \phi_2^1) = \Delta Z_{1,1}^* Z_{i,1}^* \\ \Delta\phi_2^1(2Z_{i,2} - Z_{i,1}) - \Delta Z_{i,2}(2\phi_2^1 - \phi_1^1) = \Delta Z_{1,2}^* Z_{i,2}^* \end{cases}. \tag{2.22}$$

Integrating the above equations over a ball B_r, and integrating by parts, we obtain

$$\begin{aligned} & 2\int_{\partial B_r} \frac{\partial \phi_1^1}{\partial\nu} Z_{i,1} - \frac{\partial Z_{i,1}}{\partial\nu}\phi_1^1 + \frac{\partial \phi_2^1}{\partial\nu} Z_{i,2} - \frac{\partial Z_{i,2}}{\partial\nu}\phi_2^1 \\ & - \int_{\partial B_r} \frac{\partial \phi_1^1}{\partial\nu} Z_{i,2} - \frac{\partial Z_{i,2}}{\partial\nu}\phi_1^1 + \frac{\partial \phi_2^1}{\partial\nu} Z_{i,1} - \frac{\partial Z_{i,1}}{\partial\nu}\phi_2^1 \\ & + \int_{B_r} \nabla Z_{i,1}^* \nabla Z_{1,1}^* + \nabla Z_{i,2}^* \nabla Z_{1,2}^* \\ & - \int_{\partial B_r} \frac{\partial Z_{1,1}^*}{\partial\nu} Z_{i,1}^* + \frac{\partial Z_{1,2}^*}{\partial\nu} Z_{i,2}^* = 0, \ \text{for}\ i = 1, 2. \end{aligned}$$

So $\phi^1 = \begin{pmatrix} \phi_1^1 \\ \phi_2^1 \end{pmatrix}$ will satisfy the following ODE system:

$$\phi^{1\prime}(r) = H(r)\phi^1 + f(r), \tag{2.23}$$

where

$$H = \begin{pmatrix} 2Z_{1,1} - Z_{1,2} & 2Z_{1,2} - Z_{1,1} \\ 2Z_{2,1} - Z_{2,2} & 2Z_{2,2} - Z_{2,1} \end{pmatrix}^{-1} \begin{pmatrix} 2Z'_{1,1} - Z'_{1,2} & 2Z'_{1,2} - Z'_{1,1} \\ 2Z'_{2,1} - Z'_{2,2} & 2Z'_{2,2} - Z'_{2,1} \end{pmatrix} = H_1^{-1}H_2, \tag{2.24}$$

and

$$f = -H_1^{-1} \begin{pmatrix} \frac{1}{|\partial B_r|}(\int_{B_r} |\nabla Z^*_{1,1}|^2 + |\nabla Z^*_{1,2}|^2 - \int_{\partial B_r} \frac{\partial Z^*_{1,1}}{\partial \nu} Z^*_{1,1} + \frac{\partial Z^*_{1,2}}{\partial \nu} Z^*_{1,2}) \\ \frac{1}{|\partial B_r|}(\int_{B_r} \nabla Z^*_{2,1}\nabla Z^*_{1,1} + \nabla Z^*_{2,2}\nabla Z^*_{1,2} - \int_{\partial B_r} \frac{\partial Z^*_{1,1}}{\partial \nu} Z^*_{2,1} + \frac{\partial Z^*_{1,2}}{\partial \nu} Z^*_{2,2}) \end{pmatrix}. \tag{2.25}$$

We know that the fundamental solutions of this ODE system are given by $\begin{pmatrix} Z_{1,1} \\ Z_{1,2} \end{pmatrix}, \begin{pmatrix} Z_{2,1} \\ Z_{2,2} \end{pmatrix}$, and we can get that $f = O(\frac{1}{r})$. Hence we have

$$\phi^1 = t_1 Z_1 + t_2 Z_2, \tag{2.26}$$

where

$$t_i = O(\log(1+r)). \tag{2.27}$$

So we can solve (2.21) in the radial coordinate, and we can get that the asymptotic growth of ϕ^i is at most $\log|z|$. Thus $\phi^i \in Y_\alpha$.

We let L be the linearized operator:

$$L\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \Delta\phi_1 + |z|^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}}(2\phi_1 - \phi_2) \\ \Delta\phi_2 + |z|^{2N_1} e^{2\tilde{U}_{2,0} - \tilde{U}_{1,0}}(2\phi_2 - \phi_1) \end{pmatrix}. \tag{2.28}$$

Since both coefficients of $(2\phi_1 - \phi_2)$ and $(2\phi_2 - \phi_1)$ decay like $O(|z|^{-2(2+N_2)})$ and $O(|z|^{-2(2+N_1)})$ at ∞, it is standard to show that $L : Y_\alpha \to X_\alpha$ is a bounded linear operator and L has closed range in X_α for $\alpha \in (0, \frac{1}{2})$. For a proof, we refer to [2] (See Proposition 2.1 in [2]). Therefore

$$X_\alpha = ImL \oplus (ImL)^\perp.$$

Let $\phi \in (ImL)^\perp$. Then $(Lu, \phi)_{X_\alpha} = 0$, $\forall u \in Y_\alpha$, or equivalently,

$$(Lu, \varphi)_{L^2(\mathbb{R}^2)} = 0,$$

where $\varphi = (1+|z|^{2+\alpha})\phi(z)$. Thus, $L^*\varphi = 0$ in $\mathbb{R}^2$, where φ satisfies

$$\begin{cases} \Delta\varphi_1 + 2|z|^{2N_1}e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}\varphi_1 - |z|^{2N_2}e^{2\tilde{U}_{2,0}-\tilde{U}_{1,0}}\varphi_2 = 0 \\ \Delta\varphi_2 + 2|z|^{2N_2}e^{2\tilde{U}_{2,0}-\tilde{U}_{1,0}}\varphi_2 - |z|^{2N_1}e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}\varphi_1 = 0 \end{cases} \text{in } \mathbb{R}^2,$$

with

$$\int_{\mathbb{R}^2} \frac{\varphi^2(z)}{(1+|z|^{2+\alpha})}dz < +\infty.$$

From here, we can apply the Green formulas to show that

$$|\varphi(z)| = O(1+\log|z|).$$

Thus, by Lemma 2.2, we have that $\varphi(z)$ belongs to the 8-dimensional kernel space of L^*. Thus $\varphi(z) \in \text{span}\{Z_i^* |\, i = 1, 2, \ldots, 8\}$.

However, our previous result shows that ΔZ_1^* and $\Delta Z_2^* \in ImL$. Then it implies $\varphi(z) \in \text{span}\{Z_i^* |\, i = 3, \ldots, 8\}$. So $(ImL)^\perp \subseteq \text{span}\{Z_i^* |\, i = 3, \ldots, 8\}$. By the integration by part and the fact that $Z_i^* \to 0$ as $|z| \to +\infty$, $i = 3, \ldots, 8$, we have $Z_i^* \subseteq (ImL)^\perp$. Therefore we prove

$$ImL = \text{span}\{Z_i^* |\, i = 3, \ldots, 8\}^\perp.$$

Since by (2.10), $\det(\langle \Delta \mathrm{Z_i}, \mathrm{Z_j}\rangle) = \det(-\langle \nabla \mathrm{Z_i}, \nabla \mathrm{Z_j}\rangle) \neq 0$, there exists a unique ϕ to (2.20) and $\langle \Delta Z_i, \phi\rangle = 0$, $i = 1, 2, \ldots, 8$, provided that $\langle h, Z_i^*\rangle = 0$, $i = 3, \ldots, 8$.

It remains to prove the estimate in (2.20). We prove (2.20) by contradiction. Assuming that there exist (ϕ_n, h_n) satisfying the system(2.20) such that $\|\phi_n\|_* = 1, \|h_n\|_{**} = o(1)$. By blowing-up analysis, we obtain a limiting solution to

$$\begin{cases} \Delta\phi_1 + |z|^{2N_1}e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2\phi_1 - \phi_2) = 0, \\ \Delta\phi_2 + |z|^{2N_1}e^{2\tilde{U}_{2,0}-\tilde{U}_{1,0}}(2\phi_2 - \phi_1) = 0, \\ \langle \begin{pmatrix} |z|^{2N_1}e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2Z_{i,1} - Z_{i,2}) \\ |z|^{2N_2}e^{2\tilde{U}_{2,0}-\tilde{U}_{1,0}}(2Z_{i,2} - Z_{i,1}) \end{pmatrix}, \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \rangle = 0, \text{ for } i = 1, \cdots, 8, \end{cases} \tag{2.29}$$

with $\|\phi\|_* \leq 1$. By the non-degeneracy result Lemma 2.2, we conclude that $\phi \equiv 0$. Hence $\phi_n \to 0$ in $C^1_{loc}(\mathbb{R}^2)$.

Next we write $\Delta\phi_n = \tilde{h}_n$ with $\tilde{h}_n = -\begin{pmatrix} |z|^{2N_1}e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2\phi_{n,1} - \phi_{n,2}) \\ |z|^{2N_2}e^{2\tilde{U}_{2,0}-\tilde{U}_{1,0}}(2\phi_{n,2} - \phi_{n,1}) \end{pmatrix} + \begin{pmatrix} h_{n,1} \\ h_{n,2} \end{pmatrix}$. Since $\phi_n \to 0$ in $C^1_{loc}(\mathbb{R}^2)$, $\tilde{h}_n(y) = \frac{o(1)}{1+|y|^{2+\alpha}}$. Hence by the Green's representation formula, we have

$$\phi_n(z)-\phi_n(0)=\int_{\mathbb{R}^2}\log\frac{|y|}{|y-z|}\tilde{h}_n(y)dy=o(1)\log(2+|z|),\ \phi_n(0)=o(1).$$

This is a contradiction to the assumption that $\|\phi_n\|_*=1$.

□

2.4 Improvements of the Approximate Solution: $O(\varepsilon)$ Term

For the order $O(\varepsilon)$ term, one need to solve equation of the form (2.32), the right hand side of the equation only involves $\cos\theta$ and $\sin\theta$ and hence can be solved.

First denote by

$$f(\varepsilon,z)=\Pi_{j=1}^{N_1}|z-\varepsilon p_j|^2,\quad g(\varepsilon,z)=\Pi_{j=1}^{N_2}|z-\varepsilon q_j|^2. \tag{2.30}$$

Then by Taylor's expansion, we have

$$f(\varepsilon,z)=f(0,z)+\varepsilon f_\varepsilon(0,z)+\frac{\varepsilon^2}{2}f_{\varepsilon\varepsilon}(0,z)+O(\varepsilon^3)$$

where

$$f(0,z)=|z|^{2N_1},\, f_\varepsilon(0,z)=-2|z|^{2N_1-2}<\sum_{j=1}^{N_1}p_j,z>. \tag{2.31}$$

Similarly we can get the expansions for $g(\varepsilon,z)$.

Let $\begin{pmatrix}\Psi_{0,1}\\ \Psi_{0,2}\end{pmatrix}$ be the solution of

$$\begin{cases} \quad\Delta\Psi_{0,1}+|z|^{2N_1}e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2\Psi_{0,1}-\Psi_{0,2}) \\ =2|z|^{2N_1-1}e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}\sum_{j=1}^{N_1}(p_{j1}\cos\theta+p_{j2}\sin\theta) \\ \quad\Delta\Psi_{0,2}+|z|^{2N_2}e^{2\tilde{U}_{2,0}-\tilde{U}_{1,0}}(2\Psi_{0,2}-\Psi_{0,1}) \\ =2|z|^{2N_2-1}e^{2\tilde{U}_{2,0}-\tilde{U}_{1,0}}\sum_{j=1}^{N_2}(q_{j1}\cos\theta+q_{j2}\sin\theta). \end{cases} \tag{2.32}$$

Note that the existence of the solutions to (2.32) is guaranteed by Lemma 2.3. In fact, since $h(r)\cos\theta,\ h(r)\sin\theta\perp Z_i^*$ for $i=3,\cdots,8$, there exists a unique solution $\begin{pmatrix} \Psi_{0,1} \\ \Psi_{0,2} \end{pmatrix}$ of equation (2.32) such that $\langle \Psi_0, \Delta Z_i \rangle = 0$ for $i = 1, \cdots, 8$. Moreover, it is the linear combination of functions of the form $h_1(r)\cos\theta$ and $h_1(r)\sin\theta$.

Next, if we use

$$\bar{V} = \begin{pmatrix} \bar{V}_1 \\ \bar{V}_2 \end{pmatrix} = \begin{pmatrix} \tilde{U}_{1,\mathbf{a}} + \varepsilon\Psi_{0,1} \\ \tilde{U}_{2,\mathbf{a}} + \varepsilon\Psi_{0,2} \end{pmatrix}, \tag{2.33}$$

we incur an error between $O(\varepsilon)$ term and $\tilde{U}_{i,\mathbf{a}} - \tilde{U}_{i,0}$. To get rid of this term, we need to solve

$$\begin{cases} \Delta\Psi_{i,1} + |z|^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2\Psi_{i,1} - \Psi_{i,2}) \\ = -|z|^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2\Psi_{0,1} - \Psi_{0,2})(2Z_{i,1} - Z_{i,2}) - f_\varepsilon(0,z) e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2Z_{i,1} - Z_{i,2}) \\ \\ \Delta\Psi_{i,2} + |z|^{2N_2} e^{2\tilde{U}_{2,0} - \tilde{U}_{1,0}} (2\Psi_{i,2} - \Psi_{i,1}) \\ = -|z|^{2N_2} e^{2\tilde{U}_{2,0} - \tilde{U}_{1,0}} (2\Psi_{0,2} - \Psi_{0,1})(2Z_{i,2} - Z_{i,1}) - g_\varepsilon(0,z) e^{2U_{2,0} - U_{1,0}} (2Z_{i,2} - Z_{i,1}), \end{cases} \tag{2.34}$$

for $i = 3, \cdots, 8$.

The solvability of (2.34) depends on (N_1, N_2).Since the right hand side are linear combinations of functions of the form $h(r)\cos(N_i\theta), h(r)\cos(N_i + 2)\theta, h(r)\cos(N_1+N_2+1)\theta, h(r)\cos(N_1+N_2+3)\theta, h(r)\sin(N_i\theta), h(r)\sin(N_i+2)\theta, h(r)\sin(N_1 + N_2 + 1)\theta, h(r)\sin(N_1 + N_2 + 3)\theta$, one can easily see that if $|N_1 - N_2| \neq 1$, the right hand side of equation (2.34) is orthogonal to Z_i^* for $i = 3, \cdots, 8$. Similarly by Lemma 2.3, there exists a unique solution $\begin{pmatrix} \Psi_{i,1} \\ \Psi_{i,2} \end{pmatrix}$ of (2.34) such that $\langle \Psi_i, \Delta Z_j \rangle = 0$ for $j = 1, \cdots, 8$.

Thus the approximate solution including $O(\varepsilon)$ term is

$$\tilde{V} = \begin{pmatrix} \tilde{V}_1 \\ \tilde{V}_2 \end{pmatrix} = \begin{pmatrix} U_{1,\mathbf{a}} + \varepsilon(\Psi_{0,1} + \sum_{i=3}^{8} \Psi_{i,1}\mathbf{a}_i) \\ U_{2,\mathbf{a}} + \varepsilon(\Psi_{0,2} + \sum_{i=3}^{8} \Psi_{i,2}\mathbf{a}_i) \end{pmatrix}, \tag{2.35}$$

where we use the notation $\mathbf{a} = (\mathbf{a}_3, \mathbf{a}_4, \cdots, \mathbf{a}_8) = (a_1, a_2, b_1, b_2, d_1, d_2)$.

Remark 2.1. *Note that when $\sum_{i=1}^{N_1} p_i = \sum_{j=1}^{N_2} q_j$, with a shift of origin, we can assume that $\sum_{i=1}^{N_1} p_i = \sum_{j=1}^{N_2} q_j = 0$, then the $O(\varepsilon)$ vanishes, i.e. Ψ_0 and Ψ_i are all zero for $i = 3, \cdots, 8$.*

2.5 Next Improvement of the Approximate Solution: $O(\varepsilon^2)$ Term

The next order term, which is of $O(\varepsilon^2)$, is crucial.

Let the first correction of $O(\varepsilon^2)$ be $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$:

$$\begin{cases} \Delta\psi_1 + |z|^{2N_1} e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2\psi_1 - \psi_2) = 2|z|^{4N_1} e^{4\tilde{U}_{1,0}-2\tilde{U}_{2,0}} - |z|^{2(N_1+N_2)} e^{\tilde{U}_{1,0}+\tilde{U}_{2,0}} \\ -\frac{1}{2}|z|^{2N_1} e^{2U_{1,0}-U_{2,0}}(2\Psi_{0,1} - \Psi_{0,2})^2 - f_\varepsilon(0,z) e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2\Psi_{0,1} - \Psi_{0,2}) - \frac{f_{\varepsilon\varepsilon}(0,z)}{2} e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}} \\ \\ \Delta\psi_2 + |z|^{2N_2} e^{2\tilde{U}_{2,0}-\tilde{U}_{1,0}}(2\psi_2 - \psi_1) = 2|z|^{4N_2} e^{4\tilde{U}_{2,0}-2\tilde{U}_{1,0}} - |z|^{2(N_1+N_2)} e^{\tilde{U}_{1,0}+\tilde{U}_{2,0}} \\ -\frac{1}{2}|z|^{2N_1} e^{2U_{2,0}-U_{1,0}}(2\Psi_{0,2} - \Psi_{0,1})^2 - g_\varepsilon(0,z) e^{2\tilde{U}_{2,0}-\tilde{U}_{1,0}}(2\Psi_{0,2} - \Psi_{0,1}) - \frac{g_{\varepsilon\varepsilon}(0,z)}{2} e^{2\tilde{U}_{2,0}-\tilde{U}_{1,0}}. \end{cases} \tag{2.36}$$

Observe that the right hand side of (2.36) consists of functions which are either radially symmetric or radially symmetric multiplying $\cos(2\theta)$ and $\sin(2\theta)$. Thus if $N_1, N_2 > 1$, the right hand side of (2.36) is orthogonal to Z_i^* for $i = 3, \cdots, 8$. By Lemma 2.3, we can find a unique solution $\psi_0 = \begin{pmatrix} \psi_1^0 \\ \psi_2^0 \end{pmatrix}$ such that $\langle \psi_0, \Delta Z_i \rangle = 0$ for $i = 1, \cdots, 8$.

When $N_1 = N_2$ and $\sum_{j=1}^{N_1} p_j = \sum_{j=1}^{N} q_j$, we are able to deal with (2.36) like in [2] by choosing $\mu_1 = \mu_2 = 1$. However in the general case we do not have explicit formulas for (2.36). Here comes our new idea. Instead of solving (2.36), we observe that adding the two radially symmetric kernels Z_{μ_1} and Z_{μ_2} gives another solution to (2.36). Thus the solution we will use later is $\psi = \psi_0 + \xi_1 Z_{\mu_1} + \xi_2 Z_{\mu_2}$ where ξ_1, ξ_2 are two constants independent of $\mathbf{a}$ and will be determined later.

Finally, the approximate solution with all the $O(\varepsilon)$ and $O(\varepsilon^2)$ terms is

$$V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} \tilde{U}_{1,\mathbf{a}} + \varepsilon(\Psi_{0,1} + \sum_{i=3}^{8} \Psi_{i,1}\mathbf{a}_i) + \varepsilon^2\psi_1 \\ \tilde{U}_{2,\mathbf{a}} + \varepsilon(\Psi_{0,2} + \sum_{i=3}^{8} \Psi_{i,2}\mathbf{a}_i) + \varepsilon^2\psi_2 \end{pmatrix}. \tag{2.37}$$

Then $\begin{pmatrix} V_1 + \varepsilon^2 v_1 \\ V_2 + \varepsilon^2 v_2 \end{pmatrix}$ is a solution of (2.3) if $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ satisfies

$$\begin{cases} \Delta v_1 + |z|^{2N_1} e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2v_1 - v_2) = G_1 \\ \Delta v_2 + |z|^{2N_2} e^{2\tilde{U}_{2,0}-\tilde{U}_{1,0}}(2v_2 - v_1) = G_2, \end{cases} \tag{2.38}$$

where

$$
\begin{aligned}
G_1 &= E_1 + N_{11}(v) + N_{12}(v), && (2.39)\\
G_2 &= E_2 + N_{21}(v) + N_{22}(v), && (2.40)\\
N_{11}(v) &= 2\Pi|z-\varepsilon p_j|^4(e^{4\tilde{U}_1-2\tilde{U}_2} - e^{4V_1-2V_2}) && (2.41)\\
&\quad -\Pi|z-\varepsilon p_j|^2\Pi|z-\varepsilon q_j|^2(e^{\tilde{U}_1+\tilde{U}_2} - e^{V_1+V_2}),\\
N_{12}(v) &= \frac{-f(\varepsilon,z)e^{2\tilde{U}_1-\tilde{U}_2} + f(\varepsilon,z)e^{2V_1-V_2}}{\varepsilon^2} && (2.42)\\
&\quad +f(0,z)e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2v_1-v_2),\\
N_{21}(v) &= 2\Pi|z-\varepsilon q_j|^4(e^{4\tilde{U}_2-2\tilde{U}_1} - e^{4V_2-2V_1}) && (2.43)\\
&\quad -\Pi|z-\varepsilon p_j|^2\Pi|z-\varepsilon q_j|^2(e^{\tilde{U}_1+\tilde{U}_2} - e^{V_1+V_2}),\\
N_{22}(v) &= \frac{-g(\varepsilon,z)e^{2\tilde{U}_2-\tilde{U}_1} + g(\varepsilon,z)e^{2V_2-V_1}}{\varepsilon^2} && (2.44)\\
&\quad +g(0,z)e^{2\tilde{U}_{2,0}-\tilde{U}_{1,0}}(2v_2-v_1),
\end{aligned}
$$

and E_i are the errors:

$$
\begin{aligned}
&E_1 = 2\Pi|z-\varepsilon p_j|^4 e^{4V_1-2V_2} - \Pi|z-\varepsilon p_j|^2\Pi|z-\varepsilon q_j|^2 e^{V_1+V_2} - 2|z|^{4N_1}e^{4\tilde{U}_{1,0}-2\tilde{U}_{2,0}}\\
&+|z|^{2N_1+2N_2}e^{\tilde{U}_{1,0}+\tilde{U}_{2,0}} + \frac{E_{11}}{\varepsilon^2}\\
&+f(0,z)e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2\psi_1-\psi_2)\\
&+\frac{f(0,z)}{2}e^{2U_{1,0}-U_{2,0}}(2\Psi_{0,1}-\Psi_{0,2})^2 + f_\varepsilon(0,z)e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2\Psi_{0,1}-\Psi_{0,2}) + \frac{f_{\varepsilon\varepsilon}(0,z)}{2}e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}},
\end{aligned}
$$

$$
\begin{aligned}
&E_2 = 2\Pi|z-\varepsilon q_j|^4 e^{4V_2-2V_1} - \Pi|z-\varepsilon p_j|^2\Pi|z-\varepsilon q_j|^2 e^{V_1+V_2} - 2|z|^{4N_2}e^{4\tilde{U}_{2,0}-2\tilde{U}_{1,0}}\\
&+|z|^{2N_1+2N_2}e^{\tilde{U}_{1,0}+\tilde{U}_{2,0}} + \frac{E_{22}}{\varepsilon^2}\\
&+g(0,z)e^{2\tilde{U}_{2,0}-\tilde{U}_{1,0}}(2\psi_2-\psi_1)\\
&+\frac{g(0,z)}{2}e^{2U_{2,0}-U_{1,0}}(2\Psi_{0,2}-\Psi_{0,1})^2 + g_\varepsilon(0,z)e^{2\tilde{U}_{2,0}-\tilde{U}_{1,0}}(2\Psi_{0,2}-\Psi_{0,1}) + \frac{g_{\varepsilon\varepsilon}(0,z)}{2}e^{2\tilde{U}_{2,0}-\tilde{U}_{1,0}}.
\end{aligned}
$$

Here

$$E_{11} =$$
$$-f(\varepsilon, z)e^{2V_1 - V_2} + f(0, z)e^{2\tilde{U}_{1,\mathbf{a}} - \tilde{U}_{2,\mathbf{a}}} + \varepsilon f(0, z)e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}}(2\Psi_{0,1} - \Psi_{0,2}) + \varepsilon f_\varepsilon(0, z)e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}}$$
$$+ \sum_{i=3}^{8} \varepsilon(f(0, z)e^{2U_{1,0} - U_{2,0}}(2\Psi_{i,1} - \Psi_{i,2}) + f(0, z)e^{2U_{1,0} - U_{2,0}}(2\Psi_{0,1} - \Psi_{0,2})(2Z_{i,1} - Z_{i,2}))\mathbf{a}_i$$
$$+ \sum_{i=3}^{8} \varepsilon f_\varepsilon(0, z)e^{2U_{1,0} - U_{2,0}}(2Z_{i,1} - Z_{i,2}))\mathbf{a}_i$$

and

$$E_{22} =$$
$$-g(\varepsilon, z)e^{2V_2 - V_1} + g(0, z)e^{2\tilde{U}_{2,\mathbf{a}} - \tilde{U}_{1,\mathbf{a}}} + \varepsilon g(0, z)e^{2\tilde{U}_{2,0} - \tilde{U}_{1,0}}(2\Psi_{0,2} - \Psi_{0,1}) + \varepsilon g_\varepsilon(0, z)e^{2\tilde{U}_{2,0} - \tilde{U}_{1,0}}$$
$$+ \sum_{i=3}^{8} \varepsilon(g(0, z)e^{2U_{2,0} - U_{1,0}}(2\Psi_{i,2} - \Psi_{i,1}) + g(0, z)e^{2U_{2,0} - U_{1,0}}(2\Psi_{0,2} - \Psi_{0,1})(2Z_{i,2} - Z_{i,1}))\mathbf{a}_i$$
$$+ \sum_{i=3}^{8} \varepsilon g_\varepsilon(0, z)e^{2U_{2,0} - U_{1,0}}(2Z_{i,2} - Z_{i,1}))\mathbf{a}_i.$$

By Taylor's expansion, we have

$$
\begin{aligned}
&E_1 = -f(0,z)e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}\{\tfrac{1}{\varepsilon}\sum_{i=3}^{8}(2\Psi_{i,1}-\Psi_{i,2})\mathbf{a}_i\sum_{i=3}^{8}(2Z_{i,1}-Z_{i,2})\mathbf{a}_i \\
&+(2\psi_1-\psi_2)\sum_{i=3}^{8}(2Z_{i,1}-Z_{i,2})\mathbf{a}_i+\frac{1}{2}(2\Psi_{0,1}-\Psi_{0,2})^2\sum_{i=3}^{8}(2Z_{i,1}-Z_{i,2})\mathbf{a}_i \\
&+(2\Psi_{0,1}-\Psi_{0,2})\sum_{i=3}^{8}(2\Psi_{i,1}-\Psi_{i,2})\mathbf{a}_i+(2\Psi_{0,1}-\Psi_{0,2})\sum_{i=3}^{8}(2Z_{i,1}-Z_{i,2})\mathbf{a}_i\} \\
&-\tfrac{f(0,z)}{2\varepsilon}\sum_{i,j=3}^{8}\partial_{\mathbf{a}_i\mathbf{a}_j}(e^{(2\tilde{U}_{1,0}-\tilde{U}_{2,0})})\,(2\Psi_{0,1}-\Psi_{0,2})\,\mathbf{a}_i\mathbf{a}_j \\
&-f_\varepsilon(0,z)\{e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}((2\Psi_{0,1}-\Psi_{0,2})\sum_{i=3}^{8}(2Z_{i,1}-Z_{i,2})\mathbf{a}_i+e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}\sum_{i=3}^{8}(2\Psi_{i,1}-\Psi_{i,2})\mathbf{a}_i \\
&+\tfrac{1}{2\varepsilon}\sum_{i,j=3}^{8}\partial_{\mathbf{a}_i\mathbf{a}_j}(e^{(2U_{1,0}-U_{2,0})})\mathbf{a}_i\mathbf{a}_j\}-\frac{1}{2}f_{\varepsilon\varepsilon}(0,z)e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}\sum_{i=3}^{8}(2Z_{i,1}-Z_{i,2})\mathbf{a}_i \\
&+4f(0,z)^2e^{4\tilde{U}_{1,0}-2\tilde{U}_{2,0}}\sum_{i=3}^{8}(2Z_{i,1}-Z_{i,2})\mathbf{a}_i-f(0,z)g(0,z)\sum_{i=3}^{8}(Z_{i,1}+Z_{i,2})\mathbf{a}_ie^{\tilde{U}_{1,0}+\tilde{U}_{2,0}} \\
&+O(\varepsilon)+O(\varepsilon^2+|\mathbf{a}|^2),
\end{aligned}
\tag{2.45}
$$

where $O(\varepsilon)$ denotes all items only involving with ε, not with $\mathbf{a}$. We have a similar formula for E_2.

From the Taylor's expansions above, we can obtain that E can be expressed as

$$E = \frac{1}{\varepsilon}\mathbf{A}\mathbf{a}\cdot\mathbf{a}+\mathbf{Q}\mathbf{a}+O(|\mathbf{a}|^2)+O(\varepsilon). \tag{2.46}$$

Remark 2.2. *Since the case $\sum_{i=1}^{N_1}p_i=\sum_{j=1}^{N_2}q_j$, $N_1=N_2$ is the reminiscent of the scalar equation, we will use $\psi=\psi_0$ in this case. In the other cases, we use $\psi=\psi_0+\xi_1Z_{\mu_1}+\xi_2Z_{\mu_2}$ to be the second approximate solution instead of ψ_0. This is the main difference from [2]. This plays an important role in the proof. It turns out that because of the two free parameters ξ_1,ξ_2, in the proof of the main theorem in Section 2.8 and 2.9, we can reduce our problem to the invertibility of a 6×6 matrix $\mathbf{Q}$ defined in Lemma 2.7. This definitely simplifies our proof. Unlike in [2], we do not need to calculate all the other integrals in $\int E\cdot Z_i^*$.*

Remark 2.3. *Let us recall the conditions we need for the $O(\varepsilon)$ and $O(\varepsilon^2)$ improvements of the approximate solutions. For the solvability of the $O(\varepsilon)$ term, we need $\sum_{i=1}^{N_1} p_i = \sum_{j=1}^{N_2} q_j$ or $\sum_{i=1}^{N_1} p_i \neq \sum_{j=1}^{N_2} q_j, |N_1 - N_2| \neq 1$. For the solvability of the $O(\varepsilon^2)$ term, we need $N_1, N_2 > 1$.*

2.6 A Nonlinear Projected Problem

We have the following proposition:

Proposition 2.1. *For* $\mathbf{a}$ *satisfying (2.14), there exists a solution (v, m_i) to the following system*

$$\left\{\begin{array}{c} \Delta v_1 + |z|^{2N_1} e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2v_1 - v_2) = G_1 + \sum_{i=3}^{8} m_i(v) Z_{i,1}^* \\ \Delta v_2 + |z|^{2N_2} e^{2\tilde{U}_{2,0}-\tilde{U}_{1,0}}(2v_2 - v_1) = G_2 + \sum_{i=3}^{8} m_i(v) Z_{i,2}^* \\ \langle \Delta Z_i, v\rangle = 0, \ \text{for } i = 1, \cdots, 8, \end{array}\right. \tag{2.47}$$

where $m_i(v)$ is determined by

$$\langle G + \sum_{i=3}^{8} m_i(v) Z_i^*, Z_j^*\rangle = 0, \ \text{for } j = 3, \ldots, 8, \tag{2.48}$$

for $G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$. *Furthermore, v satisfies the following estimate*

$$\|v\|_* \leq C\varepsilon, \tag{2.49}$$

for some constant C independent of ε, where $\|\cdot\|_*$ is defined in (2.18) .

Proof:

This can be proved by Lemma 2.3 and a contraction mapping. In fact, v is a solution of this system if and only if $v = T^{-1}(\tilde{G}(v))$, $\tilde{G}(v) = G + \sum_{i=3}^{8} m_i Z_i^*$, where $m_i(v)$ satisfies (2.48). For C large enough, we define

$$\mathbb{B} = \{v \in Y_\alpha, \|v\|_* \leq C\varepsilon, \langle \Delta Z_i, v\rangle = 0 \text{ for } i = 1, \cdots, 8\}. \tag{2.50}$$

Since

$$e^{\tilde{U}_{1,\mathbf{a}}} = O(\frac{1}{1 + |z|^{2(N_1+N_2+1)}}) \text{ as } |z| \to \infty, \tag{2.51}$$

from (2.45), we can get that

$$\|E\|_{**} \leq C\varepsilon. \tag{2.52}$$

Since

$$\begin{aligned}
&N_{11}(v) = 2\Pi|z-\varepsilon p_j|^4(e^{4\tilde{U}_1-\tilde{U}_2} - e^{4V_1-2V_2}) - \Pi|z\\
&-\varepsilon p_j|^2\Pi|z-\varepsilon q_j|^2(e^{\tilde{U}_1+\tilde{U}_2} - e^{V_1+V_2})\\
&= 2\Pi|z-\varepsilon p_j|^4 e^{4V_1-2V_2}(\varepsilon^2(4v_1-2v_2) + O(\varepsilon^4)|v|^2)\\
&-\Pi|z-\varepsilon p_j|^2\Pi|z-\varepsilon q_j|^2 e^{V_1+V_2}(\varepsilon^2(v_1+v_2)+O(\varepsilon^4)|v|^2),
\end{aligned}$$

by (2.51), we have

$$|N_{11}(v)| \leq C\left(\frac{1}{1+|z|^{4(N_2+2)}} + \frac{1}{1+|z|^{2(N_1+N_2+4)}}\right)(\varepsilon^2|v| + \varepsilon^4|v|^2), \tag{2.53}$$

Thus

$$\|N_{11}(v)\|_{**} \leq C\varepsilon^2\|v\|_*, \tag{2.54}$$

and

$$\|N_{11}(v^{(1)}) - N_{11}(v^{(2)})\|_{**} \leq C\varepsilon^2\|v^{(1)} - v^{(2)}\|_*. \tag{2.55}$$

Similarly, we have

$$\begin{aligned}
&N_{12}(v) = \frac{-f(\varepsilon,z)e^{2\tilde{U}_1-\tilde{U}_2} + f(\varepsilon,z)e^{2V_1-V_2}}{\varepsilon^2} + f(0,z)e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2v_1-v_2)\\
&= -f(0,z)(e^{2U_{1,\mathbf{a}}-U_{2,\mathbf{a}}} - e^{2U_{1,0}-U_{2,0}})(2v_1-v_2) + \varepsilon f(0,z)e^{2U_{1,\mathbf{a}}-U_{2,\mathbf{a}}}(2v_1-v_2)\\
&+\varepsilon f_\varepsilon(0,z)e^{2U_{1,\mathbf{a}}-U_{2,\mathbf{a}}})(2v_1-v_2) + O(\varepsilon^2)|v|^2(f(0,z)e^{2U_{1,0}-U_{2,0}}),
\end{aligned}$$

again by (2.51), we have

$$|N_{12}(v)| \leq C\varepsilon\frac{1}{1+|z|^{2N_2+2}}(|v| + \varepsilon|v|^2), \tag{2.56}$$

so we can get

$$\|N_{12}(v)\|_{**} \leq C\varepsilon\|v\|_*, \tag{2.57}$$

and

$$\|N_{12}(v^{(1)}) - N_{12}(v^{(2)})\|_{**} \leq C\varepsilon\|v^{(1)} - v^{(2)}\|_*. \tag{2.58}$$

From (2.52), (2.54) and (2.57), we can also prove

$$|m_i(v)| \leq C\varepsilon$$

and

$$|m_i(v_1) - m_i(v_2)| \le C\varepsilon \|v_1 - v_2\|_*.$$

So for $v^{(1)}, v^{(2)} \in \mathbb{B}$, we have

$$\|T^{-1}(\tilde{G}(v))\|_* \le C\varepsilon, \|T^{-1}(\tilde{G}(v^{(1)})) - T^{-1}(\tilde{G}(v^{(2)}))\|_* \le \frac{1}{2}\|v^{(1)} - v^{(2)}\|_*.$$

Thus by the contraction mapping theorem , there exist a solution v of (2.47).

□

By Proposition 2.1, the full solvability for (2.3) is reduced to achieving $m_i = 0$ for $i = 3, \cdots, 8$. Since by (2.13), $\det(\langle \mathrm{Z}_\mathrm{i}^*, \mathrm{Z}_\mathrm{j}^* \rangle_{\mathrm{i,j}=3,\cdots,8}) \ne 0$, and recalling the definition of m_i in (2.48), $m_i = 0$ is equivalent to

$$\int_0^{+\infty} \int_0^{2\pi} G \cdot Z_i^* r d\theta dr = 0 \quad \text{for } i = 3, \cdots, 8, \tag{2.59}$$

where $G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$ given in (2.39) and (2.40).

We have the following lemma:

Lemma 2.4. *Let* $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ *be a solution of (2.47). Then we have the following estimates:*

$$\int_{\mathbb{R}^2} (N_{11}(v) + N_{12}(v)) Z_{i,1}^* + (N_{21}(v) + N_{22}(v)) Z_{i,2}^* dx = O(\varepsilon^2), \tag{2.60}$$

for $i = 3, \cdots, 8$.

Proof:

By proposition 2.1, (2.54) and (2.57), we have

$$\begin{aligned}
&|\int_{\mathbb{R}^2} (N_{11}(v) + N_{12}(v)) Z_{i,1}^* + (N_{21}(v) + N_{22}(v)) Z_{i,2}^* dx| \\
&\le c \int_{\mathbb{R}^2} \|N(v)\|_{**} \frac{1}{1 + |x|^{2+\alpha}} |Z_i^*| dx \\
&\le c\varepsilon \int_{\mathbb{R}^2} \|v\|_* \frac{1}{1 + |x|^{2+\alpha}} |Z_i^*| dx \\
&\le c\varepsilon^2.
\end{aligned}$$

Thus the proof is finished.

□

2.7 Proof of Theorem 1.1 for $\mathbf{A}_2$ under Assumption (i)

IWe assume that $\sum_{i=1}^{N_1} p_i = \sum_{j=1}^{N_2} q_j$ and $N_1 = N_2$, and we choose $(\xi_1, \xi_2) = (0,0)$ in this section. This case is the reminiscent of $SU(2)$ case, even though, the proof is considerably harder since there are eight dimensional kernels instead of three for the $SU(2)$ case. We have the following estimates for the error projection:

Lemma 2.5. *Let* $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ *be a solution of (2.47). The following estimate holds:*

$$\begin{aligned}&(\langle E, Z^*_{a_1}\rangle, \langle E, Z^*_{a_2}\rangle, \langle E, Z^*_{b_1}\rangle, \langle E, Z^*_{b_2}\rangle, \langle E, Z^*_{d_1}\rangle, \langle E, Z^*_{d_2}\rangle)^t \\ &= \mathcal{T}(\mathbf{a}) + O(|\mathbf{a}|^2) + O(\varepsilon),\end{aligned} \tag{2.61}$$

where $\mathcal{T}$ *is a* 6×6 *matrix defined in (2.69). Moreover,* $\mathcal{T}$ *is non-degenerate.*

Proof:

Without loss of generality, we may assume that $\sum_{j=1}^{N_1} p_j = \sum_{j=1}^{N_2} q_j = 0$ and $N_1 = N_2 = N$. Now we choose the parameters $(\mu_1, \mu_2) = (1,1)$ so that we have

$$e^{\tilde{U}_{1,0}} = e^{\tilde{U}_{2,0}} = \rho = \frac{8(N+1)^2}{(1+r^{2N+2})^2}. \tag{2.62}$$

Since $\sum_{i=1}^{N} p_i = \sum_{i=1}^{N} q_i = 0$, we have $f_\varepsilon(0,z) = 0$, $\Psi_{0,1} = \Psi_{0,2} = 0$. By (2.34), we have $\Psi_{i,1} = \Psi_{i,2} = 0$, $i = 3, \ldots, 8$. First let us recall that

$$E = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}, \tag{2.63}$$

where by (2.45),

$$\begin{aligned} E_1 &= -|z|^{2N}\rho(2\psi_1 - \psi_2)\sum_{i=3}^{8}(2Z_{i,1} - Z_{i,2})\mathbf{a}_i \\ &- \frac{1}{2}f_{\varepsilon\varepsilon}(0,z)\rho\sum_{i=3}^{8}(2Z_{i,1} - Z_{i,2})\mathbf{a}_i \\ &+ 4|z|^{4N}\rho^2\sum_{i=3}^{8}(2Z_{i,1} - Z_{i,2})\mathbf{a}_i - |z|^{4N}\rho^2\sum_{i=3}^{8}(Z_{i,1} + Z_{i,2})\mathbf{a}_i, \end{aligned}$$

and $f_{\varepsilon\varepsilon}(0,z)$ is

$$
\begin{aligned}
f_{\varepsilon\varepsilon}(0,z) &= 2|z|^{2(N-1)}(\sum_i |p_i|^2 + 2\sum_{i\neq j}(p_{i1}\cos\theta + p_{i2}\sin\theta)(p_{j1}\cos\theta + p_{j2}\sin\theta)) \\
&= 2|z|^{2(N-1)}(|\sum_i p_i|^2 + \sum_{i\neq j}(p_{i1}p_{j1} - p_{i2}p_{j2})\cos 2\theta + (p_{i1}p_{j2} + p_{i2}p_{j1})\sin 2\theta) \\
&= 2|z|^{2(N-1)}(\sum_{i\neq j}(p_{i1}p_{j1} - p_{i2}p_{j2})\cos 2\theta + (p_{i1}p_{j2} + p_{i2}p_{j1})\sin 2\theta). \qquad (2.64)
\end{aligned}
$$

We have a similar formula for E_2.

Since

$$
\begin{aligned}
&\int h(r)\cos 2\theta(2Z_{i,1} - Z_{i,2})(2Z_{j,1} - Z_{j,2})rdrd\theta \\
= &\int h(r)\sin 2\theta(2Z_{i,1} - Z_{i,2})(2Z_{j,1} - Z_{j,2})rdrd\theta = 0,
\end{aligned}
$$

for $i,j = 3,\cdots,8$, from (2.64), we have

$$
\int_0^\infty \int_0^{2\pi} f_{\varepsilon\varepsilon}(0,z)e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2Z_{i,1} - Z_{i,2})(2Z_{j,1} - Z_{j,2})rdrd\theta = 0, \qquad (2.65)
$$

for $i,j = 3,\cdots,8$. Note that $f_{\varepsilon\varepsilon}(0,z) = 0$ if $N = 1$.

Another important observation is the following:

$$
\begin{aligned}
&\int_0^\infty \int_0^{2\pi} r^{2N}e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2\psi_1 - \psi_2)(2Z_{i,1} - Z_{i,2})(2Z_{j,1} - Z_{j,2})rdrd\theta \quad (2.66) \\
= &\int_0^\infty \int_0^{2\pi} r^{2N}e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2\psi_{0,1} - \psi_{0,2})(2Z_{i,1} - Z_{i,2})(2Z_{j,1} - Z_{j,2})rdrd\theta,
\end{aligned}
$$

where $(\psi_{0,1},\psi_{0,2})$ is the radial solution of the following system:

$$
\begin{cases}
\Delta\psi_1 + |z|^{2N_1}e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2\psi_1 - \psi_2) = 2|z|^{4N_1}e^{4\tilde{U}_{1,0}-2\tilde{U}_{2,0}} - |z|^{2(N_1+N_2)}e^{\tilde{U}_{1,0}+\tilde{U}_{2,0}} \\
\Delta\psi_2 + |z|^{2N_2}e^{2\tilde{U}_{2,0}-\tilde{U}_{1,0}}(2\psi_2 - \psi_1) = 2|z|^{4N_2}e^{4\tilde{U}_{2,0}-2\tilde{U}_{1,0}} - |z|^{2(N_1+N_2)}e^{\tilde{U}_{1,0}+\tilde{U}_{2,0}}.
\end{cases}
\qquad (2.67)
$$

Now we choose $\psi_{0,1} = \psi_{0,2} = \psi$ such that it is the solution of the following ODE:

$$
\Delta\psi + \frac{8(N+1)^2r^{2N}}{(1+r^{2N+2})^2}\psi = r^{4N}(\frac{8(N+1)^2}{(1+r^{2N+2})^2})^2. \qquad (2.68)
$$

Combining (2.65) and (2.66), we have

$$
\begin{aligned}
& \int E \cdot Z_k^* r dr d\theta \\
= \; & \int_0^\infty \int_0^{2\pi} \sum_{i=3}^{8} \Big(\Big[2|z|^{4N_1} e^{4\tilde{U}_{1,0} - 2\tilde{U}_{2,0}} (4Z_{i,1} - 2Z_{i,2}) \mathbf{a}_i - |z|^{2N_1+2N_2} e^{\tilde{U}_{1,0}+\tilde{U}_{2,0}} (Z_{i,1} + Z_{i,2}) \mathbf{a}_i \\
- \; & f(0,z) e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2\psi_{0,1} - \psi_{0,2})(2Z_{i,1} - Z_{i,2}) \mathbf{a}_i \Big] Z_{k,1}^* \\
+ \; & \Big[2|z|^{4N_2} e^{4\tilde{U}_{2,0} - 2\tilde{U}_{1,0}} (4Z_{i,2} - 2Z_{i,1}) \mathbf{a}_i - |z|^{2N_1+2N_2} e^{\tilde{U}_{1,0}+\tilde{U}_{2,0}} (Z_{i,1} + Z_{i,2}) \mathbf{a}_i \\
- \; & g(0,z) e^{2\tilde{U}_{2,0} - \tilde{U}_{1,0}} (2\psi_{0,2} - \psi_{0,1})(2Z_{i,2} - Z_{i,1}) \mathbf{a}_i \Big] Z_{k,2}^* \Big) r dr d\theta \\
+ \; & O(|\mathbf{a}|^2 + |\varepsilon|^2) + O(\varepsilon),
\end{aligned}
$$

where $O(\varepsilon)$ is independent of $\mathbf{a}$. Replacing the Z_k^* term in the above expression by $Z_{a_i}^*$, we have for $i = 1, 2$

$$
\begin{aligned}
& \int E \cdot Z_{a_i}^* r dr d\theta \\
& = \int_0^\infty \int_0^{2\pi} \Big(r^{4N} \rho^2 (19 Z_{a_i,1}^2 + 19 Z_{a_i,2}^2 - 34 Z_{a_i,1} Z_{a_i,2}) a_i \\
& + r^{4N} \rho^2 (19 Z_{a_i,1} Z_{b_i,1} + 19 Z_{a_i,2} Z_{b_i,2} - 17 Z_{a_i,1} Z_{b_i,2} - 17 Z_{a_i,2} Z_{b_i,1}) b_i \\
& - \psi \Big[r^{2N} \rho (5 Z_{a_i,1}^2 + 5 Z_{a_i,2}^2 - 8 Z_{a_i,1} Z_{a_i,2}) a_i \\
& + r^{2N} \rho (5 Z_{a_i,1} Z_{b_i,1} + 5 Z_{a_i,2} Z_{b_i,2} - 4 Z_{a_i,1} Z_{b_i,2} - 4 Z_{a_i,2} Z_{b_i,1}) b_i \Big] \Big) r dr d\theta \\
& = \pi \int_0^\infty \Big(r^{4N} \rho^4 (76 r^{2(N+1)} + 76 r^{6(N+1)} - 136 r^{4(N+1)}) a_i \\
& + r^{4N} \rho^4 (76 r^{4(N+1)} - 34 r^{2(N+1)} - 34 r^{6(N+1)}) b_i \\
& - \psi \Big[r^{2N} \rho^3 (20 r^{2(N+1)} + 20 r^{6(N+1)} - 32 r^{4(N+1)}) a_i \\
& + r^{2N} \rho^3 (20 r^{4(N+1)} - 8 r^{2(N+1)} - 8 r^{6(N+1)}) b_i \Big] \Big) r dr \\
& + O(|\mathbf{a}|^2 + O(\varepsilon)^2) + O(\varepsilon) \\
& = (76 J_1 - 20 J_2 - 136 J_3 + 32 J_4) a_i - (34 J_1 - 8 J_2 - 76 J_3 + 20 J_4) b_i \\
& + O(\varepsilon) + O(|\mathbf{a}|^2 + \varepsilon^2),
\end{aligned}
$$

and replacing the Z_k^* term by $Z_{b_i}^*$, we have for $i = 1, 2$

$$
\begin{aligned}
&\int E \cdot Z_{b_i}^* r dr d\theta \\
&= \int_0^\infty \int_0^{2\pi} \Big(r^{4N} \rho^2 (19 Z_{b_i,1}^2 + 19 Z_{b_i,2}^2 - 34 Z_{b_i,1} Z_{b_i,2}) b_i \\
&+ r^{4N} \rho^2 (19 Z_{a_i,1} Z_{b_i,1} + 19 Z_{a_i,2} Z_{b_i,2} - 17 Z_{a_i,1} Z_{b_i,2} - 17 Z_{a_i,2} Z_{b_i,1}) a_i \\
&- \psi \Big[r^{2N} \rho (5 Z_{b_i,1}^2 + 5 Z_{b_i,2}^2 - 8 Z_{b_i,1} Z_{b_i,2}) b_i \\
&+ r^{2N} \rho (5 Z_{a_i,1} Z_{b_i,1} + 5 Z_{a_i,2} Z_{b_i,2} - 4 Z_{a_i,1} Z_{b_i,2} - 4 Z_{a_i,2} Z_{b_i,1}) a_i \Big] \Big) r dr d\theta \\
&= \pi \int_0^\infty \Big(r^{4N} \rho^4 (19 r^{2(N+1)} + 19 r^{6(N+1)} - 34 r^{4(N+1)}) b_i \\
&+ r^{4N} \rho^4 (76 r^{4(N+1)} - 34 r^{2(N+1)} - 34 r^{6(N+1)}) a_i \\
&- \psi \Big[r^{2N} \rho^3 (5 r^{2(N+1)} + 5 r^{6(N+1)} - 8 r^{4(N+1)}) b_i \\
&+ r^{2N} \rho^3 (20 r^{4(N+1)} - 8 r^{2(N+1)} - 8 r^{6(N+1)}) a_i \Big] \Big) r dr \\
&+ O(|\mathbf{a}|^2 + \varepsilon^2) + O(\varepsilon) \\
&= (19 J_1 - 5 J_2 - 34 J_3 + 8 J_4) b_i + (76 J_3 - 20 J_4 - 34 J_1 + 8 J_2) a_i \\
&+ O(\varepsilon) + O(\varepsilon^2 + |a|^2).
\end{aligned}
$$

Similarly, we have for $i = 1, 2$

$$
\begin{aligned}
&\int E \cdot Z_{d_i}^* r dr d\theta \\
&= \int_0^\infty \int_0^{2\pi} \Big(r^{4N} \rho^2 (19 Z_{d_i,1}^2 + 19 Z_{d_i,2}^2 - 34 Z_{d_i,1} Z_{d_i,2}) d_i \\
&- \psi \Big[r^{2N} \rho (5 Z_{d_i,1}^2 + 5 Z_{d_i,2}^2 - 8 Z_{d_i,1} Z_{d_i,2}) d_i \Big] \Big) r dr d\theta \\
&= \pi \int_0^\infty \Big(72 r^{4N} \rho^4 r^{4(N+1)} d_i - \psi \Big[18 r^{2N} \rho^3 r^{4(N+1)} d_i \Big] \Big) r dr \\
&+ O(|\mathbf{a}|^2) + O(\varepsilon) \\
&= (72 J_3 - 18 J_4) d_i + O(|\mathbf{a}|^2) + O(\varepsilon),
\end{aligned}
$$

where

$$J_1 = \pi \int_0^\infty r^{4N}\rho^4(r^{2(N+1)} + r^{6(N+1)})rdr,$$
$$J_2 = \pi \int_0^\infty \psi r^{2N+1}\rho^3(r^{2(N+1)} + r^{6(N+1)})dr,$$
$$J_3 = \pi \int_0^\infty r^{4N+1}\rho^4 r^{4(N+1)}dr,$$
$$J_4 = \pi \int_0^\infty \psi r^{2N+1}\rho^3 r^{4(N+1)}dr.$$

So we get that

$$(\langle E, Z^*_{a_1}\rangle, \langle E, Z^*_{a_2}\rangle, \langle E, Z^*_{b_1}\rangle, \langle E, Z^*_{b_2}\rangle, \langle E, Z^*_{d_1}\rangle, \langle E, Z^*_{d_2}\rangle)^t$$
$$= \mathcal{T}(\mathbf{a}) + O(|\mathbf{a}|^2) + O(\varepsilon),$$

where

$$\mathcal{T} = \begin{pmatrix} T_1 & 0 & T_2 & 0 & 0 & 0 \\ 0 & T_1 & 0 & T_2 & 0 & 0 \\ T_3 & 0 & T_4 & 0 & 0 & 0 \\ 0 & T_3 & 0 & T_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & T_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & T_5 \end{pmatrix}, \tag{2.69}$$

and

$$T_1 = 76J_1 - 20J_2 - 136J_3 + 32J_4, \ T_2 = -(34J_1 - 8J_2 - 76J_3 + 20J_4),$$
$$T_3 = 76J_3 - 20J_4 - 34J_1 + 8J_2, \quad T_4 = 19J_1 - 5J_2 - 34J_3 + 8J_4,$$
$$T_5 = 72J_3 - 18J_4.$$

The determinant of the matrix $\mathcal{T}$ is

$$(T_1T_4 - T_2T_3)^2T_5^2$$
$$= [36(2J_1 - J_2 + 2(2J_3 - J_4))(4J_1 - J_2 - 2(4J_3 - J_4))(4J_3 - J_4)]^2.$$

Next we prove that the matrix $\mathcal{T}$ is non-degenerate, i.e. the determinant of $\mathcal{T}$ is nonzero. For this purpose, we need to calculate the integrals J_1, J_2, J_3 and J_4. But these involves the function ψ for which the expression is unknown. In order to get rid of ψ, we use integration by parts and a key observation is that for any ϕ satisfying $\phi(\infty) = 0$, we have

$$\int_0^\infty [(\Delta + \frac{8(N+1)^2 r^{2N}}{(1+r^{2N+2})^2})\psi]\phi r dr = \int_0^\infty [(\Delta + \frac{8(N+1)^2 r^{2N}}{(1+r^{2N+2})^2})\phi]\psi r dr. \quad (2.70)$$

So we need to find solutions of

$$\Delta\phi_1 + \frac{8(N+1)^2 r^{2N}}{(1+r^{2N+2})^2}\phi_1 = r^{2N}\rho^3(r^{2(N+1)} + r^{6(N+1)}), \quad (2.71)$$

and

$$\Delta\phi_2 + \frac{8(N+1)^2 r^{2N}}{(1+r^{2N+2})^2}\phi_2 = r^{2N}\rho^3 r^{4(N+1)}. \quad (2.72)$$

For this purpose, let us consider a general form of this ODE first:

$$\Delta\phi + \frac{8(N+1)^2 r^{2N}}{(1+r^{2N+2})^2}\phi = \rho^3 r^{tN+t-2}. \quad (2.73)$$

Letting $s = r^{N+1}$, and $\bar{\phi}(s) = \phi(r)$, then $\bar{\phi}$ will satisfy

$$\Delta_s\bar{\phi} + \frac{8}{(1+s^2)^2}\bar{\phi} = \frac{512(N+1)^4 s^{t-2}}{(1+s^2)^6}. \quad (2.74)$$

By stereographical projection, letting $s = \tan\frac{\theta}{2}$, and $\tilde{\phi}(\theta) = \bar{\phi}(s)$, we get that the equation satisfied by $\tilde{\phi}(\theta)$ is the following:

$$\Delta_{S^2}\tilde{\phi} + 2\tilde{\phi} = 8(N+1)^4(1+\cos\theta)^4 \tan^{t-2}\frac{\theta}{2}, \quad (2.75)$$

where Δ_{S^2} is the Laplace-Betrami operator on S^2. The equation for ϕ_1, ϕ_2 are reduced to

$$\Delta_{S^2}\tilde{\phi}_2 + 2\tilde{\phi}_2 = 8(N+1)^4(\frac{5}{4} - \cos 2\theta - \frac{1}{4}\cos 4\theta), \quad (2.76)$$

and

$$\Delta_{S^2}\tilde{\phi}_2 + 2\tilde{\phi}_2 = 8(N+1)^4(\frac{3}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta). \quad (2.77)$$

By direct calculation, we can get that

$$\tilde{\phi}_1 = 8(N+1)^4(\frac{7}{8} + \frac{2}{9}\cos 2\theta + \frac{1}{72}\cos 4\theta),$$

and

$$\tilde{\phi}_2 = 8(N+1)^4(\frac{5}{16} + \frac{5}{36}\cos 2\theta - \frac{1}{144}\cos 4\theta).$$

So we have

$$\phi_1 = 8(N+1)^4(\frac{7}{8} + \frac{2(1-6r^{2N+2}+r^{4N+4})}{9(1+r^{2N+2})^2} + \frac{1}{72}(2\frac{(1-6r^{2N+2}+r^{4N+4})^2}{(1+r^{2N+2})^4} - 1) + \frac{10}{9}\varphi_0), \tag{2.78}$$

and

$$\phi_2 = 8(N+1)^4(\frac{5}{16} + \frac{5(1-6r^{2N+2}+r^{4N+4})}{36(1+r^{2N+2})^2} - \frac{1}{144}(2\frac{(1-6r^{2N+2}+r^{4N+4})^2}{(1+r^{2N+2})^4} - 1) + \frac{4}{9}\varphi_0), \tag{2.79}$$

where $\varphi_0(r) = \frac{1-r^{2N+2}}{1+r^{2N+2}}$ satisfies

$$\Delta\varphi_0 + \frac{8(N+1)^2 r^{2N}}{(1+r^{2N+2})^2}\varphi_0 = 0.$$

Note that we add the terms containing φ_0 to ensure that $\phi_i(\infty) = 0$ for $i = 1, 2$.

After simplification, we have

$$\begin{aligned}
\phi_1 &= \frac{32(N+1)^4}{9}\frac{(5+10r^{2N+2}+9r^{4N+4})}{(1+r^{2N+2})^4}, \\
\phi_2 &= \frac{64(N+1)^4}{9}\frac{2r^{2N+2}+1}{(1+r^{2N+2})^4}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
J_1 &= \int_0^\infty r^{4N+1}\rho^4(r^{2N+2}+r^{6N+6})dr \\
&= \frac{(8(N+1)^2)^4}{2(N+1)}\int_0^\infty \frac{t^\eta}{(1+t)^5} - \frac{3t^\eta}{(1+t)^6} + \frac{4t^\eta}{(1+t)^7} - \frac{2t^\eta}{(1+t)^8}dt,
\end{aligned}$$

where $\eta = \frac{N}{1+N}$. By direct computation, we have

$$\int_0^\infty \frac{t^\eta}{(1+t)^k}dt = \frac{k}{k-1-\eta}\int_0^\infty \frac{t^\eta}{(1+t)^{k+1}}dt,$$

so

$$J_1 = \frac{(8(N+1)^2)^4}{2(N+1)}(((\frac{5}{4-\eta} - 3)\frac{6}{5-\eta} + 4)\frac{7}{6-\eta} - 2)\int_0^\infty \frac{t^\eta}{(1+t)^8}dt.$$

By (2.70), we have

$$\begin{aligned}
J_2 &= \int_0^\infty (\frac{8(N+1)^2 r^{2N}}{(1+r^{2N+2})^2})^2 \phi_1 r dr \\
&= \frac{(8(N+1)^2)^4}{2(N+1)} \int_0^\infty \frac{1}{18}\frac{t^\eta(5+10t+9t^2)}{(1+t)^8}dt \\
&= (8(N+1))^4 \int_0^\infty \frac{1}{18}(\frac{9t^\eta}{(1+t)^6} - \frac{8t^\eta}{(1+t)^7} + \frac{4t^\eta}{(1+t)^8})dt \\
&= \frac{(8(N+1)^2)^4}{36(N+1)}((\frac{54}{5-\eta} - 8)\frac{7}{6-\eta} + 4)\int_0^\infty \frac{t^\eta}{(1+t)^8}dt,
\end{aligned}$$

$$\begin{aligned}
J_3 &= \int_0^\infty r^{4N+1}\rho^4 r^{4N+4}dr \\
&= \frac{(8(N+1)^2)^4}{2(N+1)} \int_0^\infty \frac{t^{2+\eta}}{(1+t)^8}dt \\
&= \frac{(8(N+1)^2)^4}{2(N+1)} \int_0^\infty \frac{t^\eta}{(1+t)^6} - \frac{2t^\eta}{(1+t)^7} + \frac{t^\eta}{(1+t)^8})dt \\
&= \frac{(8(N+1)^2)^4}{2(N+1)}((\frac{6}{5-\eta} - 2)\frac{7}{6-\eta} + 1)\int_0^\infty \frac{t^\eta}{(1+t)^8}dt,
\end{aligned}$$

and

$$\begin{aligned}
J_4 &= \int_0^\infty (\frac{8(N+1)^2 r^{2N}}{(1+r^{2N+2})^2})^2 \phi_2 r dr \\
&= \frac{(8(N+1)^2)^4}{2(N+1)} \int_0^\infty \frac{1}{9}\frac{(2t+1)t^\eta}{(1+t)^8}dt \\
&= \frac{(8(N+1)^2)^4}{2(N+1)} \int_0^\infty \frac{1}{9}(\frac{2t^\eta}{(1+t)^7} - \frac{t^\eta}{(1+t)^8})dt \\
&= \frac{(8(N+1)^2)^4}{18(N+1)}(\frac{14}{6-\eta} - 1)\int_0^\infty \frac{t^\eta}{(1+t)^8}dt.
\end{aligned}$$

So we can get that

$$
\begin{aligned}
(2J_1 - J_2) + 2(2J_3 - J_4) &= \frac{(8(N+1)^2)^4}{2(N+1)} \frac{105\eta}{(6-\eta)(5-\eta)(4-\eta)} \int_0^\infty \frac{t^\eta}{(1+t)^8} dt \neq 0,\\
(4J_1 - J_2) - 2(4J_3 - J_4) &= \frac{(8(N+1)^2)^4}{2(N+1)} \frac{244 + 53\eta - 148\eta^2 + 148\eta^3}{9(6-\eta)(5-\eta)(4-\eta)} \int_0^\infty \frac{t^\eta}{(1+t)^8} dt \neq 0,\\
4J_3 - J_4 &= \frac{(8(N+1)^2)^4}{2(N+1)} \frac{32 + 111\eta + 37\eta^2}{9(6-\eta)(5-\eta)} \int_0^\infty \frac{t^\eta}{(1+t)^8} dt \neq 0,
\end{aligned}
$$

for $\eta = \frac{N}{N+1}$. This shows that $\mathcal{T}$ is non-degenerate.

□

As a consequence of Lemma 2.4 and Lemma 2.5, the coefficients m_i varnish if and only if the parameters $\mathbf{a}$ satisfy

$$
\mathcal{T}(\mathbf{a}) + O(|\mathbf{a}|^2) + O(\varepsilon) = 0, \tag{2.80}
$$

where $\mathcal{T}$ is defined in (2.69). Obviously, (2.80) can be solved immediately from (2.69) with $|\mathbf{a}| \leq C\varepsilon$, for some C large but fixed.

□

Remark 2.4. *Note that even* $\sum_{i=1}^{N_1} p_i = \sum_{i=1}^{N_2} q_i$ *and* $N_1 = N_2$, u_1 *and* u_2 *may be different from each other.*

2.8 Proof of Theorem 1.1 for $\mathbf{A}_2$ under Assumption (ii)

In this section, we are going to prove Theorem 1.1 for $\mathbf{A}_2$ under **Assumption (ii)**. This situation is more complicated than the previous one, since the $O(\varepsilon)$ approximation and $O(\varepsilon^2)$ approximation induce several difficulties. The problem is that we cannot obtain the explicit expressions for these terms. In this case, we will see that the two free parameters ξ_1, ξ_2 we introduced in Section 2.3 for the improvement of the $O(\varepsilon^2)$ approximate solution play an important role. A key observation is that we only need to consider the terms involving ξ_1 and ξ_2. This is contained in the following lemma.

Lemma 2.6. *Let* $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ *be a solution of (2.47). The following estimates* *hold:*

$$\langle E, Z_{a_i}^* \rangle \tag{2.81}$$
$$= \xi_1(\mathcal{A}_1 a_i + \mathcal{B}_1 b_i) + \xi_2(\mathcal{A}_2 a_i + \mathcal{B}_2 b_i) + \mathcal{T}_{1i}(\mathbf{a}) + O(\frac{|\mathbf{a}|^2}{\varepsilon}) + O((1+|\xi|)|\mathbf{a}|^2) + O(\varepsilon),$$
$$\langle E, Z_{b_i}^* \rangle \tag{2.82}$$
$$= \xi_1(\mathcal{D}_1 a_i + \mathcal{C}_1 b_i) + \xi_2(\mathcal{D}_2 a_i + \mathcal{C}_2 b_i) + \mathcal{T}_{2i}(\mathbf{a}) + O(\frac{|\mathbf{a}|^2}{\varepsilon}) + O((1+|\xi|)|\mathbf{a}|^2) + O(\varepsilon),$$
$$\langle E, Z_{d_i}^* \rangle \tag{2.83}$$
$$= \xi_1 \mathcal{E}_1 d_i + \xi_2 \mathcal{E}_2 d_i + \mathcal{T}_{3i}(\mathbf{a}) + O(\frac{|\mathbf{a}|^2}{\varepsilon}) + O((1+|\xi|)|\mathbf{a}|^2) + O(\varepsilon),$$

for $i = 1, 2$, where

$$\begin{aligned} \mathcal{A}_j &= \int_0^{+\infty}\int_0^{2\pi} \Big[r^{2N_1} e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}} (2Z_{j,1} - Z_{j,2})(2Z_{a_1,1} - Z_{a_1,2})^2 \\ &\quad + r^{2N_2} e^{2\tilde{U}_{2,0}-\tilde{U}_{1,0}} (2Z_{j,2} - Z_{j,1})(2Z_{a_1,2} - Z_{a_1,1})^2 \Big] r dr d\theta, \end{aligned} \tag{2.84}$$

$$\begin{aligned} \mathcal{B}_j = \int_0^{+\infty}\int_0^{2\pi} \Big[r^{2N_1} e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}} (2Z_{j,1} - Z_{j,2})(2Z_{b_1,1} - Z_{b_1,2})(2Z_{a_1,1} - Z_{a_1,2}) \\ + r^{2N_2} e^{2\tilde{U}_{2,0}-\tilde{U}_{1,0}} (2Z_{j,2} - Z_{j,1})(2Z_{b_1,2} - Z_{b_1,1})(2Z_{a_1,2} - Z_{a_1,1}) \Big] r dr d\theta, \end{aligned} \tag{2.85}$$

$$\begin{aligned} \mathcal{C}_j &= \int_0^{+\infty}\int_0^{2\pi} \Big[r^{2N_1} e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}} (2Z_{j,1} - Z_{j,2})(2Z_{b_1,1} - Z_{b_1,2})^2 \\ &\quad + r^{2N_2} e^{2\tilde{U}_{2,0}-\tilde{U}_{1,0}} (2Z_{j,2} - Z_{j,1})(2Z_{b_1,2} - Z_{b_1,1})^2 \Big] r dr d\theta, \end{aligned} \tag{2.86}$$

$$\begin{aligned} \mathcal{D}_j = \int_0^{+\infty}\int_0^{2\pi} \Big[r^{2N_1} e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}} (2Z_{j,1} - Z_{j,2})(2Z_{b_1,1} - Z_{b_1,2})(2Z_{a_1,1} - Z_{a_1,2}) \\ + r^{2N_2} e^{2\tilde{U}_{2,0}-\tilde{U}_{1,0}} (2Z_{j,2} - Z_{j,1})(2Z_{b_1,2} - Z_{b_1,1})(2Z_{a_1,2} - Z_{a_1,1}) \Big] r dr d\theta, \end{aligned} \tag{2.87}$$

$$\begin{aligned} \mathcal{E}_j &= \int_0^{+\infty}\int_0^{2\pi} \Big[r^{2N_1} e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}} (2Z_{j,1} - Z_{j,2})(2Z_{d_1,1} - Z_{d_1,2})^2 \\ &\quad + r^{2N_2} e^{2\tilde{U}_{2,0}-\tilde{U}_{1,0}} (2Z_{j,2} - Z_{j,1})(2Z_{d_1,2} - Z_{d_1,1})^2 \Big] r dr d\theta, \end{aligned} \tag{2.88}$$

for $j = 1, 2$, and $\mathcal{T}_{ij}$ are 6×1 vectors which are uniformly bounded as ε tends to 0, and are independent of ξ_1, ξ_2.

Proof:

By (2.45), E is of the form

$$\frac{1}{\varepsilon}(...)\mathbf{a}\cdot\mathbf{a}+((...)\mathbf{a})+O((1+|\xi|)|\mathbf{a}|^2)+O(\varepsilon). \tag{2.89}$$

Recall that $\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=\begin{pmatrix}\psi_1^0\\ \psi_2^0\end{pmatrix}+\xi_1 Z_{\mu_1}+\xi_2 Z_{\mu_2}$. In the following computations, we only want to compute those items involving ξ_1 and ξ_2, since all other terms are independent of ξ_1 and ξ_2. Obviously, we have

$$< h(r)Z_{a_2,i}, Z_{a_1,j} >=< h(r)Z_{b_1,i}, Z_{b_2,j} >=< h(r)Z_{d_1,i}, Z_{d_2,j} >= 0.$$

Hence we obtain

$$\begin{aligned}
&-\int_0^{+\infty}\int_0^{2\pi} E\cdot Z_{a_1}^* r d\theta dr\\
&=\int_0^{\infty}\int_0^{2\pi}\Big[r^{2N_1}e^{2\tilde U_{1,0}-\tilde U_{2,0}}\Big(\xi_1(2Z_{1,1}-Z_{1,2})+\xi_2(2Z_{2,1}-Z_{2,2})\Big)(2Z_{a_1,1}-Z_{a_1,2})a_1 Z_{a_1,1}^*\\
&+r^{2N_2}e^{2\tilde U_{2,0}-\tilde U_{1,0}}\Big(\xi_1(2Z_{1,2}-Z_{1,1})+\xi_2(2Z_{2,2}-Z_{2,1})\Big)(2Z_{a_1,2}-Z_{a_1,1})a_1 Z_{a_1,2}^*\\
&+r^{2N_1}e^{2\tilde U_{1,0}-\tilde U_{2,0}}\Big(\xi_1(2Z_{1,1}-Z_{1,2})+\xi_2(2Z_{2,1}-Z_{2,2})\Big)(2Z_{b_1,1}-Z_{b_1,2})b_1 Z_{a_1,1}^*\\
&+r^{2N_2}e^{2\tilde U_{2,0}-\tilde U_{1,0}}\Big(\xi_1(2Z_{1,2}-Z_{1,1})+\xi_2(2Z_{2,2}-Z_{2,1})\Big)(2Z_{b_1,2}-Z_{b_1,1})b_1 Z_{a_1,2}^*\Big] r dr d\theta\\
&+\mathcal{T}_{11}(\mathbf{a})+O(\varepsilon)+O((1+|\xi|)|\mathbf{a}|^2+\varepsilon^2),
\end{aligned}$$

where $\mathcal{T}_{11}(\mathbf{a})$ is the remaining terms which is a linear combinations of $\mathbf{a}$, and the coefficients of the linear combination are uniformly bounded which are independent of $\xi_1, \xi_2, \mathbf{a}$.

Thus

$$
\begin{aligned}
&-\int_0^{+\infty}\int_0^{2\pi} E\cdot Z^*_{a_1} r d\theta dr\\
&=\xi_1\Big[\Big(\int_0^{+\infty}\int_0^{2\pi}\{r^{2N_1}e^{2\tilde U_{1,0}-\tilde U_{2,0}}(2Z_{1,1}-Z_{1,2})(2Z_{a_1,1}-Z_{a_1,2})^2\\
&+r^{2N_2}e^{2\tilde U_{2,0}-\tilde U_{1,0}}(2Z_{1,2}-Z_{1,1})(2Z_{a_1,2}-Z_{a_1,1})^2\}rdrd\theta\Big)a_1\\
&+\Big(\int_0^{+\infty}\int_0^{2\pi}\{r^{2N_1}e^{2\tilde U_{1,0}-\tilde U_{2,0}}(2Z_{1,1}-Z_{1,2})(2Z_{a_1,1}-Z_{a_1,2})(2Z_{b_1,1}-Z_{b_1,2})\\
&+r^{2N_2}e^{2\tilde U_{2,0}-\tilde U_{1,0}}(2Z_{1,2}-Z_{1,1})(2Z_{a_1,2}-Z_{a_1,1})(2Z_{b_1,2}-Z_{b_1,1})\}rdrd\theta\Big)b_1\Big]\\
&+\xi_2\Big[\Big(\int_0^{+\infty}\int_0^{2\pi}\{r^{2N_1}e^{2\tilde U_{1,0}-\tilde U_{2,0}}(2Z_{2,1}-Z_{2,2})(2Z_{a_1,1}-Z_{a_1,2})^2\\
&+r^{2N_2}e^{2\tilde U_{2,0}-\tilde U_{1,0}}(2Z_{2,2}-Z_{2,1})(2Z_{a_1,2}-Z_{a_1,1})^2\}rdrd\theta\Big)a_1\\
&+\Big(\int_0^{+\infty}\int_0^{2\pi}\{r^{2N_1}e^{2\tilde U_{1,0}-\tilde U_{2,0}}(2Z_{2,1}-Z_{2,2})(2Z_{a_1,1}-Z_{a_1,2})(2Z_{b_1,1}-Z_{b_1,2})\\
&+r^{2N_2}e^{2\tilde U_{2,0}-\tilde U_{1,0}}(2Z_{a_1,2}-Z_{a_1,1})(2Z_{b_1,2}-Z_{b_1,1})Z_{b_1,1}Z_{a_1,2})\}rdrd\theta\Big)b_1\Big]\\
&+\mathcal{T}_{11}(\mathbf{a})+O(\varepsilon)+O(\frac{|\mathbf{a}|^2}{\varepsilon})+O((1+|\xi|)|\mathbf{a}|^2+\varepsilon^2)\\
&=\xi_1(\mathcal{A}_1a_1+\mathcal{B}_1b_1)+\xi_2(\mathcal{A}_2a_1+\mathcal{B}_2b_1)+\mathcal{T}_{11}(\mathbf{a})+O(\varepsilon)+O(\frac{|\mathbf{a}|^2}{\varepsilon})+O((1+|\xi|)|\mathbf{a}|^2+\varepsilon^2),
\end{aligned}
$$

where $\mathcal{A}_1,\mathcal{A}_2,\mathcal{B}_1,\mathcal{B}_2$ is in (2.84) and (2.85).

Similarly, we can get the other estimates.

□

From Lemma 2.4 and 2.6, we have the following result:

Lemma 2.7. *Let* $\begin{pmatrix} v_1\\ v_2\end{pmatrix}$ *be a solution of (2.47). Then the coefficients* m_i *varnish if and only if the parameters* $\mathbf{a}$ *satisfy*

$$\mathbf{Q}(\mathbf{a})=-\mathcal{T}(\mathbf{a})+O(\frac{|\mathbf{a}|^2}{\varepsilon})+O((1+|\xi|)|\mathbf{a}|^2)+O(\varepsilon), \tag{2.90}$$

where $\mathbf{Q}$

$$\mathbf{Q} = \xi_1 \begin{pmatrix} \mathcal{A}_1 & 0 & \mathcal{B}_1 & 0 & 0 & 0 \\ 0 & \mathcal{A}_1 & 0 & \mathcal{B}_1 & 0 & 0 \\ \mathcal{D}_1 & 0 & \mathcal{C}_1 & 0 & 0 & 0 \\ 0 & \mathcal{D}_1 & 0 & \mathcal{C}_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{E}_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{E}_1 \end{pmatrix} + \xi_2 \begin{pmatrix} \mathcal{A}_2 & 0 & \mathcal{B}_2 & 0 & 0 & 0 \\ 0 & \mathcal{A}_2 & 0 & \mathcal{B}_2 & 0 & 0 \\ \mathcal{D}_2 & 0 & \mathcal{C}_2 & 0 & 0 & 0 \\ 0 & \mathcal{D}_2 & 0 & \mathcal{C}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{E}_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{E}_2 \end{pmatrix}$$
$$= \xi_1 \mathbf{Q}_1 + \xi_2 \mathbf{Q}_2, \tag{2.91}$$

and $\mathcal{T}$ *is a* 6×6 *matrix which is uniformly bounded and independent of* ξ_1, ξ_2.

Proof of Theorem 1.1 for $\mathbf{A}_2$ under Assumption (ii): Under the **Assumptions (ii)**, we will choose $\mu_1 = 1$ and μ_2 large. Even in this simplified situation, the computations of $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are very complicated. So we put the computations in the appendix.

If $N_1 \neq N_2$, we might assume $N_1 > N_2$. Otherwise, we can exchange u_1 and u_2. First, obviously under the assumption on N_1 and N_2, $\mathcal{B}_i = \mathcal{D}_i = 0$ due to the orthogonality of $\cos k\theta$ and $\cos l\theta$ for $k \neq l$. Secondly, the computations in the appendix give us:

$$\mathcal{A}_1 = \gamma_1 \mu_2^{-\frac{N_1+1}{N_1+N_2+2}} + o(\mu_2^{-\frac{N_1+1}{N_1+N_2+2}}),$$
$$\mathcal{C}_1 = \gamma_2 \mu_2^{-\frac{6N_1+4N_2+10}{N_1+N_2+2}} + o(\mu_2^{-\frac{6N_1+4N_2+10}{N_1+N_2+2}}),$$
$$\mathcal{E}_1 = \gamma_3 \mu_2^{-\frac{3N_1+2N_2+5}{N_1+N_2+2}} + o(\mu_2^{-\frac{3N_1+2N_2+5}{N_1+N_2+2}}).$$

If $N_1 = N_2$, the computation in the appendix gives us:

$$\mathcal{A}_1\mathcal{C}_1 - \mathcal{B}_1^2 = \gamma_4^2 \mu_2^{-4} + o(\mu_2^{-4}),$$
$$\mathcal{E}_1 = \gamma_5 \mu_2^{-4} \ln \mu_2 + o(\mu_2^{-4}),$$

where γ_i for $i = 1, \cdots, 5$ are non zero constants. Therefore, we choose ξ_1 large and $\xi_2 = 0$ to conclude that $\mathbf{Q}(\xi_1, \xi_2) - \mathcal{T}$ is non-degenerate. After fixing (μ_1, μ_2), (ξ_1, ξ_2), it is easy to see (2.90) can be solved with $\mathbf{a} = O(\varepsilon)$.

□

2.9 Proof of Theorem 1.1 for $\mathbf{A}_2$ under Assumption (iii)

We are left to prove the theorem for $\sum_{i=1}^{N_1} p_i = \sum_{j=1}^{N_2} q_j, N_1 \neq N_2$ and one of N_i is 1. Without loss of generality, assume $N_1 = 1$ and $\sum_{i=1}^{N_1} p_i = \sum_{j=1}^{N_2} q_j = 0$. In this case, for the improvement of approximate solution of the $O(\varepsilon^2)$ term, we can not solve equation (2.36) in Section 2.5. Instead of solving (2.36), we first find a unique solution of the following equations which is guaranteed by Lemma 2.3:

$$\begin{cases} \Delta\psi_1 + |z|^{2N_1} e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2\psi_1 - \psi_2) = 2|z|^{4N_1} e^{4\tilde{U}_{1,0}-2\tilde{U}_{2,0}} - |z|^{2(N_1+N_2)} e^{\tilde{U}_{1,0}+\tilde{U}_{2,0}} \\ \Delta\psi_2 + |z|^{2N_2} e^{2\tilde{U}_{2,0}-\tilde{U}_{1,0}}(2\psi_2 - \psi_1) = 2|z|^{4N_2} e^{4\tilde{U}_{2,0}-2\tilde{U}_{1,0}} - |z|^{2(N_1+N_2)} e^{\tilde{U}_{1,0}+\tilde{U}_{2,0}}. \end{cases} \tag{2.92}$$

We use this unique solution as the new ψ_0, and proceed as before. Then by checking the previous proof, we can get that in this case, the error $\|E\|_* \leq C_0$ and we can get a solution v of (2.47) which satisfies

$$\|v\|_* \leq C_0, \tag{2.93}$$

for some positive constant C_0, and the following estimates hold:

$$\int_{\mathbb{R}^2} (N_{11}(v) + N_{12}(v))Z_{i,1}^* + (N_{21}(v) + N_{22}(v))Z_{i,2}^* dx = O(\varepsilon), \tag{2.94}$$

for $i = 3, \cdots, 8$.

Then the reduced problem we get is

$$\mathbf{Q}(\mathbf{a}) + \mathcal{T}(\mathbf{a}) + O((1 + |\xi|)|\mathbf{a}|^2) + O(1) + O(\varepsilon) = 0, \tag{2.95}$$

where the $O(1)$ term comes from the $O(1)$ term of the error E since we use the solution of (2.92) instead of (2.36) as the $O(\varepsilon^2)$ improvement. Recalling that $\mathbf{Q} = \mathbf{Q}(\xi_1, \xi_2)$ depend on two free parameters ξ_1, ξ_2, arguing as before, we can choose ξ_1 large enough, then it is easy to get a solution of (2.95) with $\mathbf{a} = O(\xi_1^{-\alpha})$ for any $0 < \alpha < 1$.

3 Proof of Theorem 1.1 in the B_2 Case

In this section, we consider the following $\mathbf{B}_2$ system in $\mathbb{R}^2$:

$$\begin{cases} \Delta u_1 + 2e^{u_1} - e^{u_2} = 4e^{2u_1} - 2e^{2u_2} + 4\pi \sum_{j=1}^{N_1} \delta_{p_j} \\ \Delta u_2 + 2e^{u_2} - 2e^{u_1} = 4e^{2u_2} - 4e^{2u_1} - 2e^{u_1+u_2} + 4\pi \sum_{j=1}^{N_2} \delta_{q_j}. \end{cases} \tag{3.1}$$

We follow the same idea as the proof for the $\mathbf{A}_2$ case. Since the idea is similar, without confusion, we will use the same notation as in Section 2.

3.1 Functional Formulation of the Problem for B_2 Case

As in Section 2.1, we let

$$u_1 = \sum_{j=1}^{N_1} \ln|z - p_j|^2 + \tilde{u}_1, \quad u_2 = \sum_{j=1}^{N_2} \ln|z - q_j|^2 + \tilde{u}_2,$$

and $z = \frac{\tilde{z}}{\varepsilon}$, and let

$$\tilde{u}_1(z) = U_1(\tilde{z}) + (2N_1 + 2)\ln\varepsilon, \quad \tilde{u}_2(z) = U_2(\tilde{z}) + (2N_2 + 2)\ln\varepsilon,$$

and

$$\begin{pmatrix} \tilde{U}_1 \\ \tilde{U}_2 \end{pmatrix} = \mathbf{B}_2^{-1} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix},$$

where $\mathbf{B}_2$ is the Cartan matrix $\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$. Then $(\tilde{U}_1, \tilde{U}_2)$ will satisfy

$$\begin{cases} \Delta\tilde{U}_1 + \Pi_{j=1}^{N_1}|\tilde{z} - \varepsilon p_j|^2 e^{2\tilde{U}_1 - \tilde{U}_2} \\ = 2\varepsilon^2\Pi_{j=1}^{N_1}|\tilde{z} - \varepsilon p_j|^4 e^{4\tilde{U}_1 - 2\tilde{U}_2} - \varepsilon^2\Pi_{j=1}^{N_1}|\tilde{z} - \varepsilon p_j|^2\Pi_{j=1}^{N_2}|\tilde{z} - \varepsilon q_j|^2 e^{\tilde{U}_2}, \\ \Delta\tilde{U}_2 + \Pi_{j=1}^{N_2}|\tilde{z} - \varepsilon q_j|^2 e^{2\tilde{U}_2 - 2\tilde{U}_1} \\ = 2\varepsilon^2\Pi_{j=1}^{N_2}|\tilde{z} - \varepsilon q_j|^4 e^{4\tilde{U}_2 - 4\tilde{U}_1} - 2\varepsilon^2\Pi_{j=1}^{N_2}|\tilde{z} - \varepsilon q_j|^2\Pi_{j=1}^{N_1}|\tilde{z} - \varepsilon p_j|^2 e^{\tilde{U}_2}. \end{cases} \tag{3.2}$$

From now on, we shall work with (3.2). For simplicity of notations, we still denote the variable by z instead of $\tilde{z}$.

3.2 Classification and Non-degeneracy for B_2 Toda system

When $\varepsilon = 0$, (3.2) becomes

$$\begin{cases} \Delta\tilde{U}_1 + |z|^{2N_1} e^{2\tilde{U}_1 - \tilde{U}_2} = 0 \\ \Delta\tilde{U}_2 + |z|^{2N_2} e^{2\tilde{U}_2 - 2\tilde{U}_1} = 0 \end{cases} \tag{3.3}$$

whose solutions can be completely classified. See Theorem 4.1.

Defining

$$\begin{pmatrix} \tilde{U}_1 \\ \tilde{U}_2 \end{pmatrix} = \begin{pmatrix} w_1 - 2\alpha_1 \ln|z| \\ w_2 - 2\alpha_2 \ln|z| \end{pmatrix}, \tag{3.4}$$

where $\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \mathbf{B}_2^{-1} \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}$, we get an equivalent form of (3.3):

$$\begin{cases} \Delta w_1 + e^{2w_1 - w_2} = 4\pi\alpha_1\delta_0 \\ \Delta w_2 + e^{2w_2 - 2w_1} = 4\pi\alpha_2\delta_0 \\ \int_{\mathbb{R}^2} e^{2w_1 - w_2} < +\infty, \quad \int_{\mathbb{R}^2} e^{2w_2 - 2w_1} < +\infty. \end{cases} \tag{3.5}$$

For this system, we have a classification result, see Theorem 4.1. From Theorem 4.1, we see that all the solutions of (3.5) depend on ten parameters $(c_{21}, c_{30}, c_{31}, c_{32}, \lambda_0, \lambda_1) \in \mathbb{C}^4 \times (\mathbb{R}^+)^2$. So all the solutions of (3.3) are of the form

$$e^{-\tilde{U}_1} = e^{-w_1}|z|^{2\alpha_1} = (\lambda_0 + \sum_{i=1}^{3} \lambda_i |P_i(z)|^2), \tag{3.6}$$

where

$$(\lambda, \mathbf{a}) = (\lambda_0, \lambda_1, c_{21,1}, c_{21,2}, c_{30,1}, c_{30,2}, c_{31,1}, c_{31,2}, c_{32,1}, c_{32,2}). \tag{3.7}$$

When $\mathbf{a} = 0$, we obtain the radially symmetric solution of (3.3)

$$\begin{aligned} e^{-\tilde{U}_{1,0}} &= \rho_{1,B}^{-1} && (3.8)\\ &= \lambda_0 + \lambda_1|z|^{2\alpha} + \lambda_2|z|^{2(\alpha+\beta)} + \lambda_3|z|^{2(2\alpha+\beta)}, \\ e^{-\tilde{U}_{2,0}} &= \rho_{2,B}^{-1} && (3.9)\\ &= 4[\lambda_0\lambda_1\alpha^2 + \lambda_0\lambda_2(\alpha+\beta)^2|z|^{2\beta} + (\lambda_0\lambda_3(2\alpha+\beta)^2 + \lambda_1\lambda_2\beta^2)|z|^{2(\alpha+\beta)} \\ &+ \lambda_1\lambda_3(\alpha+\beta)^2|z|^{2(2\alpha+\beta)} + \lambda_2\lambda_3\alpha^2|z|^{4(\alpha+\beta)}], \end{aligned}$$

where $\alpha = N_1 + 1, \ \ \beta = N_2 + 1$. Observe that the radial solution $(\tilde{U}_{1,0}, \tilde{U}_{2,0})$ depends on two scaling parameters (λ_0, λ_1). Later we shall choose (λ_0, λ_1) in different ways.

Next we have the following non-degeneracy result (see Corollary 4.1):

Lemma 3.1. *(Non-degeneracy) The previous solutions of (3.3) are non-degenerate, i.e., the set of solutions corresponding to the linearized operator is exactly ten dimensional. More precisely, if* $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ *satisfies* $|\phi(z)| \le C(1+|z|)^\alpha$ *for some* $0 \le \alpha < 1$*, and*

$$\begin{cases} \Delta\phi_1 + |z|^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}}(2\phi_1 - \phi_2) = 0 \\ \Delta\phi_2 + |z|^{2N_2} e^{2\tilde{U}_{2,0} - 2\tilde{U}_{1,0}}(2\phi_2 - 2\phi_1) = 0, \end{cases} \tag{3.10}$$

then ϕ *belongs to the following linear space* $\mathcal{K}$*: the span of*

$$\{Z_{\lambda_0}, Z_{\lambda_1}, Z_{c_{21,1}}, Z_{c_{21,2}}, Z_{c_{30,1}}, Z_{c_{30,2}}, Z_{c_{31,1}}, Z_{c_{31,2}}, Z_{c_{32,1}}, Z_{c_{32,2}}\},$$

where

$$Z_{\lambda_0} = \begin{pmatrix} Z_{\lambda_0,1} \\ Z_{\lambda_0,2} \end{pmatrix} = \begin{pmatrix} \partial_{\lambda_0}\tilde{U}_{1,0} \\ \partial_{\lambda_0}\tilde{U}_{2,0} \end{pmatrix}, \quad Z_{\lambda_1} = \begin{pmatrix} Z_{\lambda_1,1} \\ Z_{\lambda_1,2} \end{pmatrix} = \begin{pmatrix} \partial_{\lambda_1}\tilde{U}_{1,0} \\ \partial_{\lambda_1}\tilde{U}_{2,0} \end{pmatrix},$$
$$Z_{c_{21,i}} = \begin{pmatrix} Z_{c_{21,i},1} \\ Z_{c_{21,i},2} \end{pmatrix} = \begin{pmatrix} \partial_{c_{21,i}}\tilde{U}_{1,0} \\ \partial_{c_{21,i}}\tilde{U}_{2,0} \end{pmatrix}, \quad Z_{c_{30,i}} = \begin{pmatrix} Z_{c_{30,i},1} \\ Z_{c_{30,i},2} \end{pmatrix} = \begin{pmatrix} \partial_{c_{30,i}}\tilde{U}_{1,0} \\ \partial_{c_{30,i}}\tilde{U}_{2,0} \end{pmatrix},$$
$$Z_{c_{31,i}} = \begin{pmatrix} Z_{c_{31,i},1} \\ Z_{c_{31,i},2} \end{pmatrix} = \begin{pmatrix} \partial_{c_{31,i}}\tilde{U}_{1,0} \\ \partial_{c_{31,i}}\tilde{U}_{2,0} \end{pmatrix}, \quad Z_{c_{32,i}} = \begin{pmatrix} Z_{c_{32,i},1} \\ Z_{c_{32,i},2} \end{pmatrix} = \begin{pmatrix} \partial_{c_{32,i}}\tilde{U}_{1,0} \\ \partial_{c_{32,i}}\tilde{U}_{2,0} \end{pmatrix},$$

for $i = 1, 2$*, and*

$$\begin{cases} \partial_{\lambda_0}\tilde{U}_{1,0} = -\rho_{1,B}\Big[1 - \frac{r^{2(2\alpha+\beta)}}{(8\alpha(\alpha+\beta)(2\alpha+\beta))^2\lambda_0^2}\Big], \\ \partial_{\lambda_0}\tilde{U}_{2,0} = -4\rho_{2,B}\Big[\alpha^2\lambda_1 + \frac{r^{2\beta}}{(8\alpha\beta)^2\lambda_1} - \frac{\lambda_1 r^{2(2\alpha+\beta)}}{(8\alpha(2\alpha+\beta))^2\lambda_0^2} \\ \qquad\qquad - \frac{r^{4(\alpha+\beta)}}{(8(\alpha+\beta)(2\alpha+\beta))^2(8\alpha\beta(\alpha+\beta))^2\lambda_1\lambda_0^2}\Big], \end{cases} \tag{3.11}$$

$$\begin{cases} \partial_{\lambda_1}\tilde{U}_{1,0} = -\rho_{1,B}\Big[1 - \frac{r^{2\beta}}{(8\alpha\beta(\alpha+\beta))^2\lambda_1^2}]r^{2\alpha}\Big], \\ \partial_{\lambda_1}\tilde{U}_{2,0} = -4\rho_{2,B}\Big[\alpha^2\lambda_0 - \frac{\lambda_0 r^{2\beta}}{(8\alpha\beta)^2\lambda_1^2} + \frac{r^{2(2\alpha+\beta)}}{(8\alpha(2\alpha+\beta))^2\lambda_0} \\ \qquad\qquad - \frac{r^{4(\alpha+\beta)}}{(8(\alpha+\beta)(2\alpha+\beta))^2(8\alpha\beta(\alpha+\beta))^2\lambda_1^2\lambda_0}\Big], \end{cases} \tag{3.12}$$

and

$$
\begin{cases}
\partial_{c_{21,1}}\tilde{U}_{1,0} = -\rho_{1,B}\lambda_2 r^{2\alpha+\beta}\cos\beta\theta, \\
\partial_{c_{21,1}}\tilde{U}_{2,0} = -4f_2^{-1}\lambda_2\alpha(\alpha+\beta)[\lambda_0 r^{\beta} + \lambda_3 r^{4\alpha+3\beta}]\cos\beta\theta,
\end{cases} \tag{3.13}
$$

$$
\begin{cases}
\partial_{c_{21,2}}\tilde{U}_{1,0} = -\rho_{1,B}\lambda_2 r^{2\alpha+\beta}\sin\beta\theta, \\
\partial_{c_{21,2}}\tilde{U}_{2,0} = -4f_2^{-1}\lambda_2\alpha(\alpha+\beta)[\lambda_0 r^{\beta} + \lambda_3 r^{4\alpha+3\beta}]\sin\beta\theta,
\end{cases} \tag{3.14}
$$

$$
\begin{cases}
\partial_{c_{30,1}}\tilde{U}_{1,0} = -\rho_{1,B}\lambda_3 r^{2\alpha+\beta}\cos(2\alpha+\beta)\theta, \\
\partial_{c_{30,1}}\tilde{U}_{2,0} = 4\rho_{2,B}\lambda_3\alpha(\alpha+\beta)[\lambda_1 r^{2\alpha+\beta} + \lambda_2 r^{2\alpha+3\beta}]\cos(2\alpha+\beta)\theta,
\end{cases} \tag{3.15}
$$

$$
\begin{cases}
\partial_{c_{30,2}}\tilde{U}_{1,0} = -\rho_{1,B}\lambda_3 r^{2\alpha+\beta}\sin(2\alpha+\beta)\theta, \\
\partial_{c_{30,2}}\tilde{U}_{2,0} = 4\rho_{2,B}\lambda_3\alpha(\alpha+\beta)[\lambda_1 r^{2\alpha+\beta} + \lambda_2 r^{2\alpha+3\beta}]\sin(2\alpha+\beta)\theta,
\end{cases} \tag{3.16}
$$

$$
\begin{cases}
\partial_{c_{31,1}}\tilde{U}_{1,0} = -\rho_{1,B}\Big[\lambda_3 r^{3\alpha+\beta} - \frac{\lambda_2\beta}{(2\alpha+\beta)} r^{\alpha+\beta}\Big]\cos(\alpha+\beta)\theta, \\
\partial_{c_{31,1}}\tilde{U}_{2,0} = -4\rho_{2,B}\Big[(\lambda_0\lambda_3\alpha(2\alpha+\beta) + \frac{\lambda_1\lambda_2\alpha\beta^2}{(2\alpha+\beta)})r^{\alpha+\beta} \\
\qquad -2\lambda_2\lambda_3\alpha\beta r^{3\alpha+3\beta}\Big]\cos(\alpha+\beta)\theta,
\end{cases} \tag{3.17}
$$

$$
\begin{cases}
\partial_{c_{31,2}}\tilde{U}_{1,0} = -\rho_{1,B}(\lambda_3 r^{3\alpha+\beta} - \frac{\lambda_2\beta}{(2\alpha+\beta)} r^{\alpha+\beta})\sin(\alpha+\beta)\theta, \\
\partial_{c_{31,2}}\tilde{U}_{2,0} = -4\rho_{2,B}\Big[(\lambda_0\lambda_3\alpha(2\alpha+\beta) + \frac{\lambda_1\lambda_2\alpha\beta^2}{(2\alpha+\beta)})r^{\alpha+\beta} \\
\qquad -2\lambda_2\lambda_3\alpha\beta r^{3\alpha+3\beta}\Big]\sin(\alpha+\beta)\theta,
\end{cases} \tag{3.18}
$$

$$
\begin{cases}
\partial_{c_{32,1}}\tilde{U}_{1,0} = -\rho_{1,B}(\frac{\lambda_1\beta}{(2\alpha+\beta)} r^{\alpha} + \lambda_3 r^{3\alpha+2\beta})\cos\alpha\theta, \\
\partial_{c_{32,1}}\tilde{U}_{2,0} = -4\rho_{2,B}\Big[(\lambda_0\lambda_3(\alpha+\beta)(2\alpha+\beta) + \frac{\lambda_1\lambda_2\beta^2(\alpha+\beta)}{(2\alpha+\beta)})r^{\alpha+2\beta} \\
\qquad +2\lambda_1\lambda_3\beta(\beta+\alpha) r^{3\alpha+2\beta}\Big]\cos\alpha\theta,
\end{cases} \tag{3.19}
$$

$$
\begin{cases}
\partial_{c_{32,2}}\tilde{U}_{1,0} = -\rho_{1,B}(\frac{\lambda_1\beta}{(2\alpha+\beta)} r^{\alpha} + \lambda_3 r^{3\alpha+2\beta})\sin\alpha\theta, \\
\partial_{c_{32,2}}\tilde{U}_{2,0} = -4\rho_{2,B}\Big[(\lambda_0\lambda_3(\alpha+\beta)(2\alpha+\beta) + \frac{\lambda_1\lambda_2\beta^2(\alpha+\beta)}{(2\alpha+\beta)})r^{\alpha+2\beta} \\
\qquad +2\lambda_1\lambda_3\beta(\beta+\alpha) r^{3\alpha+2\beta}\Big]\sin\alpha\theta,
\end{cases} \tag{3.20}
$$

and $\rho_{1,B}, \rho_{2,B}$ *are defined in (3.8) and (3.9).*

For simplicity of notations, we also denote by $(Z_1, Z_2, \cdots, Z_{10})$ the kernels $(Z_{\lambda_0}, Z_{\lambda_1}, \cdots, Z_{c_{32,2}})$. Because $\{Z_i\}$ are linearly independent, we have

$$\det[(\int_{\mathbb{R}^2} \Delta Z_i \cdot Z_j)_{i,j=1,\cdots,10}] \neq 0. \tag{3.21}$$

We have the following corollary:

Corollary 3.1. *If* $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ *satisfies* $|\phi(z)| \leq C(1+|z|)^\alpha$ *for some* $0 \leq \alpha < 1$*, and*

$$\begin{cases} \Delta\phi_1 + 2|z|^{2N_1} e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}\phi_1 - 2|z|^{2N_2} e^{2\tilde{U}_{2,0}-2\tilde{U}_{1,0}}\phi_2 = 0 \\ \Delta\phi_2 + 2|z|^{2N_2} e^{2\tilde{U}_{2,0}-2\tilde{U}_{1,0}}\phi_2 - |z|^{2N_1} e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}\phi_1 = 0, \end{cases} \tag{3.22}$$

then ϕ *belongs to the following linear space* $\mathcal{K}^*$*: the span of*

$$\{Z^*_{\lambda_0}, Z^*_{\lambda_1}, Z^*_{c_{21,1}}, Z^*_{c_{21,2}}, Z^*_{c_{30,1}}, Z^*_{c_{30,2}}, Z^*_{c_{31,1}}, Z^*_{c_{31,2}}, Z^*_{c_{32,1}}, Z^*_{c_{32,2}}\},$$

where

$$Z^*_i = \begin{pmatrix} Z^*_{i,1} \\ Z^*_{i,2} \end{pmatrix} = \begin{pmatrix} 2Z_{i,1} - Z_{i,2} \\ Z_{i,2} - Z_{i,1} \end{pmatrix}. \tag{3.23}$$

We have

$$\det[(\int_{\mathbb{R}^2} Z^*_i \cdot Z^*_j)_{i,j=3,\cdots,10}] \neq 0. \tag{3.24}$$

We will choose the first approximate solution to be $\begin{pmatrix} \tilde{U}_{1,(\lambda,c_{21},c_{30},c_{31},c_{32})} \\ \tilde{U}_{2,(\lambda,c_{21},c_{30},c_{31},c_{32})} \end{pmatrix}$, where the parameters $\lambda, c_{21}, c_{30}, c_{31}, c_{32}$ satisfy

$$|\mathbf{a}| := |c_{21}| + |c_{30}| + |c_{31}| + |c_{32}| \leq C_0\varepsilon, \quad |\lambda| = O(1) \tag{3.25}$$

for some fixed constant $C_0 > 0$.

We want to look for solutions of the form

$$\tilde{U}_1 = \tilde{U}_{1,(\lambda,c_{21},c_{30},c_{31},c_{32})} + \varepsilon\Psi_1 + \varepsilon^2\phi_1, \quad \tilde{U}_2 = \tilde{U}_{2,(\lambda,c_{21},c_{30},c_{31},c_{32})} + \varepsilon\Psi_2 + \varepsilon^2\phi_2, \tag{3.26}$$

where $(c_{21}, c_{30}, c_{31}, c_{32}) = O(\varepsilon)$, and λ fixed. For simplicity of notations, we will denote by $\begin{pmatrix} \tilde{U}_{1,(\lambda,c_{21},c_{30},c_{31},c_{32})} \\ \tilde{U}_{2,(\lambda,c_{21},c_{30},c_{31},c_{32})} \end{pmatrix} = \begin{pmatrix} \tilde{U}_{1,\mathbf{a}} \\ \tilde{U}_{2,\mathbf{a}} \end{pmatrix}$.

To obtain the next order term, we need to study the linearized operator around the solution $\begin{pmatrix} \tilde{U}_{1,0} \\ \tilde{U}_{2,0} \end{pmatrix}$.

3.3 Invertibility of the Linearized Operator

In this section, X_α, Y_α and the $\|\cdot\|_*, \|\cdot\|_{**}$ norms are defined as in Section 2.3. Using the non-degeneracy result we get in Section 3.2 and following the argument as in the proof of Lemma 2.3, we have the following:

Lemma 3.2. *Assume that* $h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in X_\alpha$ *be such that*

$$\langle Z_i^*, h\rangle = 0, \text{ for } i = 3, \cdots, 10. \tag{3.27}$$

Then one can find a unique solution $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = T^{-1}(h) \in Y_\alpha$ *satisfying*

$$\begin{cases} \Delta\phi_1 + |z|^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}}(2\phi_1 - \phi_2) = h_1 \\ \Delta\phi_2 + |z|^{2N_1} e^{2\tilde{U}_{2,0} - 2\tilde{U}_{1,0}}(2\phi_2 - 2\phi_1) = h_2 \end{cases}, \ \|\phi\|_* \le C\|h\|_{**}, \tag{3.28}$$

such that $< \Delta Z_i, \phi >= 0$ *for* $i = 1, \cdots, 10$. *Moreover, the map* $h \xrightarrow{T} \phi$ *can be made continuous and smooth .*

In the next two subsections, using Taylor's expansions, we improve our approximate solution up to $O(\varepsilon^2)$.

3.4 Improvements of the Approximate Solution

Similar to the $\mathbf{A}_2$ case, we need to find the $O(\varepsilon)$ and $O(\varepsilon^2)$ improvement of our approximate solutions. So we need to find solutions of the following equations: Let $\begin{pmatrix} \Psi_{0,1} \\ \Psi_{0,2} \end{pmatrix}$ be the solution of

$$\begin{cases} \Delta\Psi_{0,1} + |z|^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}}(2\Psi_{0,1} - \Psi_{0,2}) \\ = 2|z|^{2N_1 - 1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} \sum_{j=1}^{N_1} (p_{j1}\cos\theta + p_{j2}\sin\theta) \\ \Delta\Psi_{0,2} + |z|^{2N_2} e^{2\tilde{U}_{2,0} - 2\tilde{U}_{1,0}}(2\Psi_{0,2} - 2\Psi_{0,1}) \\ = 2|z|^{2N_2 - 1} e^{2\tilde{U}_{2,0} - 2\tilde{U}_{1,0}} \sum_{j=1}^{N_2} (q_{j1}\cos\theta + q_{j2}\sin\theta). \end{cases} \tag{3.29}$$

Let $\begin{pmatrix} \Psi_{i,1} \\ \Psi_{i,2} \end{pmatrix}$ be the solution of

$$\begin{cases} \Delta\Psi_{i,1} + |z|^{2N_1} e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2\Psi_{i,1} - \Psi_{i,2}) = -|z|^{2N_1} e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2\Psi_{0,1} - \Psi_{0,2})(2Z_{i,1} - Z_{i,2}) \\ \qquad\qquad -f_\varepsilon(0,z) e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2Z_{i,1} - Z_{i,2}) \\ \\ \Delta\Psi_{i,2} + |z|^{2N_2} e^{2\tilde{U}_{2,0}-2\tilde{U}_{1,0}}(2\Psi_{i,2} - 2\Psi_{i,1}) = -|z|^{2N_2} e^{2\tilde{U}_{2,0}-2\tilde{U}_{1,0}}(2\Psi_{0,2} - 2\Psi_{0,1})(2Z_{i,2} - 2Z_{i,1}) \\ \qquad\qquad -g_\varepsilon(0,z) e^{2U_{2,0}-2U_{1,0}}(2Z_{i,2} - 2Z_{i,1}) \end{cases} \tag{3.30}$$

for $i = 3, \cdots, 10$.

Let $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ be the solution of

$$\begin{cases} \Delta\psi_1 + |z|^{2N_1} e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2\psi_1 - \psi_2) = 2|z|^{4N_1} e^{4\tilde{U}_{1,0}-2\tilde{U}_{2,0}} - |z|^{2(N_1+N_2)} e^{\tilde{U}_{2,0}} \\ -\frac{1}{2}|z|^{2N_1} e^{2U_{1,0}-U_{2,0}}(2\Psi_{0,1} - \Psi_{0,2})^2 - f_\varepsilon(0,z) e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2\Psi_{0,1} - \Psi_{0,2}) - \frac{f_{\varepsilon\varepsilon}(0,z)}{2} e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}} \\ \\ \Delta\psi_2 + |z|^{2N_2} e^{2\tilde{U}_{2,0}-2\tilde{U}_{1,0}}(2\psi_2 - 2\psi_1) = 2|z|^{4N_2} e^{4\tilde{U}_{2,0}-4\tilde{U}_{1,0}} - 2|z|^{2(N_1+N_2)} e^{\tilde{U}_{2,0}} \\ -|z|^{2N_2} e^{2U_{2,0}-2U_{1,0}}(2\Psi_{0,2} - 2\Psi_{0,1})^2 - g_\varepsilon(0,z) e^{2\tilde{U}_{2,0}-2\tilde{U}_{1,0}}(2\Psi_{0,2} - 2\Psi_{0,1}) - \frac{g_{\varepsilon\varepsilon}(0,z)}{2} e^{2\tilde{U}_{2,0}-2\tilde{U}_{1,0}}. \end{cases} \tag{3.31}$$

Using Lemma 3.2, we know that there exists a unique solution $\begin{pmatrix} \psi_{0,1} \\ \psi_{0,2} \end{pmatrix} \in Y_\alpha$ of (3.29) such that $\langle \Psi_0, \Delta Z_j \rangle = 0$ for $j = 1, \cdots, 10$. While the solvability of (3.30) depends on (N_1, N_2). Note that if $\sum_{j=1}^{N_1} p_j = \sum_{j=1}^{N_2} q_j$, we might assume they are zero. Then $\Psi_0 \equiv 0$ and $\Psi_i \equiv 0$ for $i = 3, \cdots, 10$. But if $\sum_{j=1}^{N_1} p_j \neq \sum_{j=1}^{N_2} q_j$, then $|N_1 - N_2| \neq 1$ implies that the right hand side of equation (3.30) is orthogonal to Z_i^* for $i = 3, \cdots, 10$. Similarly by Lemma 3.2, there exists a unique solution $\begin{pmatrix} \Psi_{i,1} \\ \Psi_{i,2} \end{pmatrix} \in Y_\alpha$ of (3.30) such that $\langle \Psi_i, \Delta Z_j \rangle = 0$ for $j = 1, \cdots, 10$. And if $N_1, N_2 > 1$, the right hand side of (3.31) is orthogonal to Z_i^* for $i = 3, \cdots, 10$. By Lemma 3.2, we can find a unique solution $\psi_0 = \begin{pmatrix} \psi_{0,1} \\ \psi_{0,2} \end{pmatrix} \in Y_\alpha$ such that $\langle \psi_0, \Delta Z_i \rangle = 0$ for $i = 1, \cdots, 10$.

Similar to the $\mathbf{A}_2$ case, the solution we will use later is $\psi = \psi_0 + \xi_1 Z_{\lambda_0} + \xi_2 Z_{\lambda_1}$ where ξ_1, ξ_2 are two constants independent of $c_{21}, c_{30}, c_{31}, c_{32}$ and will be determined later.

Finally, the approximate solution with all the terms of $O(\varepsilon)$ and $O(\varepsilon^2)$ is

$$V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} \tilde{U}_{1,\mathbf{a}} + \varepsilon(\Psi_{0,1} + \sum_{i=3}^{10} \Psi_{i,1}\mathbf{a}_i) + \varepsilon^2\psi_1 \\ \tilde{U}_{2,\mathbf{a}} + \varepsilon(\Psi_{0,2} + \sum_{i=3}^{10} \Psi_{i,2}\mathbf{a}_i) + \varepsilon^2\psi_2 \end{pmatrix}, \tag{3.32}$$

we use the notation

$$\mathbf{a} = (\mathbf{a}_3, \mathbf{a}_4, \cdots, \mathbf{a}_{10}) = (c_{21,1}, c_{21,2}, c_{30,1}, c_{31,1}, c_{31,2}, c_{32,1}, c_{32,2}).$$

Then $\begin{pmatrix} V_1 + \varepsilon^2 v_1 \\ V_2 + \varepsilon^2 v_2 \end{pmatrix}$ is a solution of (3.2) if $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ satisfies

$$\begin{cases} \Delta v_1 + |z|^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}}(2v_1 - v_2) = G_1 \\ \Delta v_2 + |z|^{2N_2} e^{2\tilde{U}_{2,0} - 2\tilde{U}_{1,0}}(2v_2 - 2v_1) = G_2, \end{cases} \tag{3.33}$$

where

$$G_1 = E_1 + N_{11}(v) + N_{12}(v), \tag{3.34}$$

$$G_2 = E_2 + N_{21}(v) + N_{22}(v), \tag{3.35}$$

$$\begin{aligned}
N_{11}(v) &= 2\Pi|z - \varepsilon p_j|^4 (e^{4\tilde{U}_1 - 2\tilde{U}_2} - e^{4V_1 - 2V_2}) \\
&\quad - \Pi|z - \varepsilon p_j|^2 \Pi|z - \varepsilon q_j|^2 (e^{\tilde{U}_2} - e^{V_2}), \\
N_{12}(v) &= \frac{-f(\varepsilon, z) e^{2\tilde{U}_1 - \tilde{U}_2} + f(\varepsilon, z) e^{2V_1 - V_2}}{\varepsilon^2} \\
&\quad + f(0, z) e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}}(2v_1 - v_2), \\
N_{21}(v) &= 2\Pi|z - \varepsilon q_j|^4 (e^{4\tilde{U}_2 - 4\tilde{U}_1} - e^{4V_2 - 4V_1}) \\
&\quad - 2\Pi|z - \varepsilon p_j|^2 \Pi|z - \varepsilon q_j|^2 (e^{\tilde{U}_2} - e^{V_2}), \\
N_{22}(v) &= \frac{-g(\varepsilon, z) e^{2\tilde{U}_2 - 2\tilde{U}_1} + g(\varepsilon, z) e^{2V_2 - 2V_1}}{\varepsilon^2} \\
&\quad + g(0, z) e^{2\tilde{U}_{2,0} - 2\tilde{U}_{1,0}}(2v_2 - 2v_1),
\end{aligned}$$

and E_i are the errors:

$$\begin{aligned}
E_1 &= 2\Pi|z - \varepsilon p_j|^4 e^{4V_1 - 2V_2} - \Pi|z - \varepsilon p_j|^2 \Pi|z - \varepsilon q_j|^2 e^{V_2} - 2|z|^{4N_1} e^{4\tilde{U}_{1,0} - 2\tilde{U}_{2,0}} \\
&+ |z|^{2N_1 + 2N_2} e^{\tilde{U}_{2,0}} + \frac{E_{11}}{\varepsilon^2} \\
&+ f(0, z) e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}}(2\psi_1 - \psi_2) \\
&+ \frac{f(0, z)}{2} e^{2U_{1,0} - U_{2,0}}(2\Psi_{0,1} - \Psi_{0,2})^2 + f_\varepsilon(0, z) e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}}(2\Psi_{0,1} - \Psi_{0,2}) + \frac{f_{\varepsilon\varepsilon}(0, z)}{2} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}},
\end{aligned}$$

$$E_2 = 2\Pi|z-\varepsilon q_j|^4 e^{4V_2-4V_1} - 2\Pi|z-\varepsilon p_j|^2\Pi|z-\varepsilon q_j|^2 e^{V_2} - 2|z|^{4N_2}e^{4\tilde{U}_{2,0}-4\tilde{U}_{1,0}}$$
$$+2|z|^{2N_1+2N_2}e^{\tilde{U}_{2,0}} + \frac{E_{22}}{\varepsilon^2}$$
$$+g(0,z)e^{2\tilde{U}_{2,0}-2\tilde{U}_{1,0}}(2\psi_2 - 2\psi_1)$$
$$+\frac{g(0,z)}{2}e^{2U_{2,0}-2U_{1,0}}(2\Psi_{0,2}-2\Psi_{0,1})^2 + g_\varepsilon(0,z)e^{2\tilde{U}_{2,0}-2\tilde{U}_{1,0}}(2\Psi_{0,2}-2\Psi_{0,1})$$
$$+\frac{g_{\varepsilon\varepsilon}(0,z)}{2}e^{2\tilde{U}_{2,0}-2\tilde{U}_{1,0}}.$$

Here

$$E_{11} =$$
$$-f(\varepsilon,z)e^{2V_1-V_2} + f(0,z)e^{2\tilde{U}_{1,\mathbf{a}}-\tilde{U}_{2,\mathbf{a}}} + \varepsilon f(0,z)e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2\Psi_{0,1}-\Psi_{0,2}) + \varepsilon f_\varepsilon(0,z)e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}$$
$$+\sum_{i=3}^{10}\varepsilon(f(0,z)e^{2U_{1,0}-U_{2,0}}(2\Psi_{i,1}-\Psi_{i,2}) + f(0,z)e^{2U_{1,0}-U_{2,0}}(2\Psi_{0,1}-\Psi_{0,2})(2Z_{i,1}-Z_{i,2}))\mathbf{a}_i$$
$$+\sum_{i=3}^{10}\varepsilon f_\varepsilon(0,z)e^{2U_{1,0}-U_{2,0}}(2Z_{i,1}-Z_{i,2})\mathbf{a}_i$$

and

$$E_{22} =$$
$$-g(\varepsilon,z)e^{2V_2-2V_1} + g(0,z)e^{2\tilde{U}_{2,\mathbf{a}}-2\tilde{U}_{1,\mathbf{a}}} + \varepsilon g(0,z)e^{2\tilde{U}_{2,0}-2\tilde{U}_{1,0}}(2\Psi_{0,2}-2\Psi_{0,1})$$
$$+\varepsilon g_\varepsilon(0,z)e^{2\tilde{U}_{2,0}-2\tilde{U}_{1,0}}$$
$$+\sum_{i=3}^{10}\varepsilon(g(0,z)e^{2U_{2,0}-2U_{1,0}}(2\Psi_{i,2}-2\Psi_{i,1}) + g(0,z)e^{2U_{2,0}-2U_{1,0}}(2\Psi_{0,2}-2\Psi_{0,1})(2Z_{i,2}-2Z_{i,1}))\mathbf{a}_i$$
$$+\sum_{i=3}^{10}\varepsilon g_\varepsilon(0,z)e^{2U_{2,0}-2U_{1,0}}(2Z_{i,2}-2Z_{i,1})\mathbf{a}_i.$$

By Taylor's expansion, we have

$$
\begin{aligned}
E_1 = &-f(0,z)e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}\{\tfrac{1}{\varepsilon}\sum_{i=3}^{10}(2\Psi_{i,1}-\Psi_{i,2})\mathbf{a}_i\sum_{i=3}^{10}(2Z_{i,1}-Z_{i,2})\mathbf{a}_i \\
&+(2\psi_1-\psi_2)\sum_{i=3}^{10}(2Z_{i,1}-Z_{i,2})\mathbf{a}_i+\frac{1}{2}(2\Psi_{0,1}-\Psi_{0,2})^2\sum_{i=3}^{10}(2Z_{i,1}-Z_{i,2})\mathbf{a}_i \\
&+(2\Psi_{0,1}-\Psi_{0,2})\sum_{i=3}^{10}(2\Psi_{i,1}-\Psi_{i,2})\mathbf{a}_i+(2\Psi_{0,1}-\Psi_{0,2})\sum_{i=3}^{10}(2Z_{i,1}-Z_{i,2})\mathbf{a}_i\} \\
&-\tfrac{f(0,z)}{2\varepsilon}\sum_{i,j=3}^{10}\partial^2_{\mathbf{a}_i\mathbf{a}_j}(e^{(2\tilde{U}_{1,0}-\tilde{U}_{2,0})})\,(2\Psi_{0,1}-\Psi_{0,2})\,\mathbf{a}_i\mathbf{a}_j \\
&-f_\varepsilon(0,z)\{e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}((2\Psi_{0,1}-\Psi_{0,2})\sum_{i=3}^{10}(2Z_{i,1}-Z_{i,2})\mathbf{a}_i+e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}\sum_{i=3}^{10}(2\Psi_{i,1}-\Psi_{i,2})\mathbf{a}_i \\
&+\tfrac{1}{2\varepsilon}\sum_{i,j=3}^{10}\partial^2_{\mathbf{a}_i\mathbf{a}_j}(e^{(2\tilde{U}_{1,0}-\tilde{U}_{2,0})})\mathbf{a}_i\mathbf{a}_j\}-\frac{1}{2}f_{\varepsilon\varepsilon}(0,z)e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}\sum_{i=3}^{10}(2Z_{i,1}-Z_{i,2})\mathbf{a}_i \\
&+4f(0,z)^2e^{4\tilde{U}_{1,0}-2\tilde{U}_{2,0}}\sum_{i=3}^{10}(2Z_{i,1}-Z_{i,2})\mathbf{a}_i-f(0,z)g(0,z)\sum_{i=3}^{10}e^{\tilde{U}_{2,0}}Z_{i,2}\mathbf{a}_i \\
&+O(\varepsilon)+O(\varepsilon^2+|\mathbf{a}|^2),
\end{aligned}
\tag{3.36}
$$

and

$$
\begin{aligned}
E_2 = &-g(0,z)e^{2\tilde{U}_{2,0}-2\tilde{U}_{1,0}}\{\tfrac{1}{\varepsilon}\sum_{i=3}^{10}(2\Psi_{i,2}-2\Psi_{i,1})\mathbf{a}_i\sum_{i=3}^{10}(2Z_{i,2}-2Z_{i,1})\mathbf{a}_i \\
&+(2\psi_2-2\psi_1)\sum_{i=3}^{10}(2Z_{i,2}-2Z_{i,1})\mathbf{a}_i+\frac{1}{2}(2\Psi_{0,2}-2\Psi_{0,1})^2\sum_{i=3}^{10}(2Z_{i,2}-2Z_{i,1})\mathbf{a}_i \\
&+(2\Psi_{0,2}-2\Psi_{0,1})\sum_{i=3}^{10}(2\Psi_{i,2}-2\Psi_{i,1})\mathbf{a}_i+(2\Psi_{0,2}-2\Psi_{0,1})\sum_{i=3}^{10}(2Z_{i,2}-2Z_{i,1})\mathbf{a}_i\} \\
&-\tfrac{g(0,z)}{2\varepsilon}\sum_{i,j=3}^{10}\partial^2_{\mathbf{a}_i\mathbf{a}_j}(e^{(2\tilde{U}_{2,0}-2\tilde{U}_{1,0})})\,(2\Psi_{0,2}-2\Psi_{0,1})\,\mathbf{a}_i\mathbf{a}_j
\end{aligned}
\tag{3.37}
$$

$$
\begin{aligned}
&-g_\varepsilon(0,z)\{e^{2\tilde U_{2,0}-2\tilde U_{1,0}}((2\Psi_{0,2}-2\Psi_{0,1})\sum_{i=3}^{10}(2Z_{i,2}-2Z_{i,1})\mathbf{a}_i+e^{2\tilde U_{2,0}-2\tilde U_{1,0}}\sum_{i=3}^{10}(2\Psi_{i,2}-2\Psi_{i,1})\mathbf{a}_i\\
&+\tfrac{1}{2\varepsilon}\sum_{i,j=3}^{10}\partial^2_{\mathbf{a}_i\mathbf{a}_j}(e^{(2\tilde U_{2,0}-2\tilde U_{1,0})})\mathbf{a}_i\mathbf{a}_j\}-\frac{1}{2}g_{\varepsilon\varepsilon}(0,z)e^{2\tilde U_{2,0}-2\tilde U_{1,0}}\sum_{i=3}^{10}(2Z_{i,2}-2Z_{i,1})\mathbf{a}_i\\
&+4g(0,z)^2e^{4\tilde U_{2,0}-4\tilde U_{1,0}}\sum_{i=3}^{10}(2Z_{i,2}-2Z_{i,1})\mathbf{a}_i-2f(0,z)g(0,z)\sum_{i=3}^{10}e^{\tilde U_{2,0}}Z_{i,2}\mathbf{a}_i\\
&+O(\varepsilon)+O(\varepsilon^2+|\mathbf{a}|^2),
\end{aligned}
$$

where $O(\varepsilon)$ denotes all items only involving with ε, and not with $\mathbf{a}$.

From the Taylor expansions above, we obtain that $E=\begin{pmatrix}E_1\\E_2\end{pmatrix}$ can be expressed as

$$
E=\frac{1}{\varepsilon}\tilde{\mathbf{A}}\mathbf{a}\cdot\mathbf{a}+\tilde{\mathbf{Q}}\mathbf{a}+O(|\mathbf{a}|^2)=O(\varepsilon). \tag{3.38}
$$

Similar to Proposition 2.1, we have the following result:

Proposition 3.1. *For* $\mathbf{a}$ *satisfying (3.25), there exists a solution* $(v,\{m_i\})$ *to the following system*

$$
\begin{cases}
\Delta v_1+|z|^{2N_1}e^{2\tilde U_{1,0}-\tilde U_{2,0}}(2v_1-v_2)=G_1+\sum_{i=3}^{10}m_i(v)Z^*_{i,1}\\
\Delta v_2+|z|^{2N_2}e^{2\tilde U_{2,0}-2\tilde U_{1,0}}(2v_2-2v_1)=G_2+\sum_{i=3}^{10}m_i(v)Z^*_{i,2}\\
\langle\Delta Z_i,v\rangle=0,\ \textit{for}\ i=1,\cdots,10,
\end{cases} \tag{3.39}
$$

where $G=\begin{pmatrix}G_1\\G_2\end{pmatrix}$ *and* $m_i(v)$ *can be determined by*

$$
\langle G+\sum_{i=3}^{10}m_i(v)Z^*_i,Z^*_j\rangle=0,\ \textit{for}\ j=3,\dots,10. \tag{3.40}
$$

Furthermore, v *satisfies the following estimate*

$$
\|v\|_*\le C\varepsilon, \tag{3.41}
$$

for some constant C *independent of* ε.

By Proposition 3.1, the full solvability for (3.2) is reduced to $m_i=0$ for $i=3,\cdots,10$. Since by (3.24), $\det(\langle \mathrm{Z}^*_\mathrm{i},\mathrm{Z}^*_\mathrm{j}\rangle_{\mathrm{i,j=3,\cdots,10}})\neq 0$, and recall the definition of m_i in (3.40), $m_i=0$ is equivalent to

$$
\int_0^{+\infty}\int_0^{2\pi}G\cdot Z^*_i r d\theta dr=0\ \ \text{for}\ i=3,\cdots,10. \tag{3.42}
$$

We have the following lemma:

Lemma 3.3. *Let* $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ *be a solution of (3.39). Then we have the following estimates:*

$$\int_{\mathbb{R}^2} (N_{11}(v) + N_{12}(v))Z_{i,1}^* + (N_{21}(v) + N_{22}(v))Z_{i,2}^* dx = O(\varepsilon^2), \tag{3.43}$$

for $i = 3, \cdots, 10$.

Proof:

The proof is similar to that of Proposition 2.1.

□

3.5 Proof of Theorem 1.1 for B$_2$ under Assumption (i)

We assume that $\sum_{i=1}^{N_1} p_i = \sum_{j=1}^{N_2} q_j$ and $N_1 = N_2$, and we choose $(\xi_1, \xi_2) = (0, 0)$ in this section. This case is the reminiscent of $SU(2)$ case, even though, the proof is considerably harder since there are ten dimensional kernels instead of a three-dimensional one for the $SU(2)$ case.

Lemma 3.4. *Let* $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ *be a solution of (3.39). The following estimates hold:*

$$\begin{aligned} &(\langle E, Z_{c_{21,1}}^* \rangle, \langle E, Z_{c_{21,2}}^* \rangle, \langle E, Z_{c_{30,1}}^* \rangle, \langle E, Z_{c_{30,2}}^* \rangle, && (3.44)\\ &\qquad \langle E, Z_{c_{31,1}}^* \rangle, \langle E, Z_{c_{31,2}}^* \rangle, \langle E, Z_{c_{32,1}}^* \rangle, \langle E, Z_{c_{32,2}}^* \rangle)^t \\ &= \tilde{\mathcal{T}}(\mathbf{a}) + O(|\mathbf{a}|^2) + O(\varepsilon), && (3.45) \end{aligned}$$

where $\tilde{\mathcal{T}}$ *is an* 8×8 *matrix defined in (3.55). Moreover,* $\tilde{\mathcal{T}}$ *is non-degenerate.*

Proof:

Without loss of generality , we may assume that $\sum_{j=1}^{N_1} p_j = \sum_{j=1}^{N_2} q_j = 0$ and $N_1 = N_2 = N$, and denote by $\mu = N+1$. Now we choose the parameters $(\lambda_0, \lambda_1, \xi_1, \xi_2) = (\frac{1}{48\mu^3}, \frac{1}{16\mu^3}, 0, 0)$ so that we have

$$e^{\tilde{U}_{1,0}} = \rho_{1,B} = \frac{3 \times 2^4 \mu^3}{(1 + r^{2\mu})^3}, \quad e^{\tilde{U}_{2,0}} = \rho_{2,B} = \frac{3 \times 2^6 \mu^4}{(1 + r^{2\mu})^4}. \tag{3.46}$$

Since $\sum_{i=1}^N p_i = \sum_{i=1}^N q_i = 0$, we have $f_\varepsilon(0,z) = 0$, $\Psi_{0,1} = \Psi_{0,2} = 0$. By (3.30), we have $\Psi_{i,1} = \Psi_{i,2} = 0$, $i = 3, \dots, 10$. Recall that

$$E = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}, \tag{3.47}$$

where by (3.36), and (3.37), we have

$$\begin{aligned} E_1 &= -|z|^{2N} \frac{\rho_{1,B}^2}{\rho_{2,B}} (2\psi_1 - \psi_2) \sum_{i=3}^{10} (2Z_{i,1} - Z_{i,2}) \mathbf{a}_i - \frac{1}{2} f_{\varepsilon\varepsilon}(0,z) \frac{\rho_{1,B}^2}{\rho_{2,B}} \sum_{i=3}^{10} (2Z_{i,1} - Z_{i,2}) \mathbf{a}_i \\ &+ 4|z|^{4N} \frac{\rho_{1,B}^4}{\rho_{2,B}^2} \sum_{i=3}^{10} (2Z_{i,1} - Z_{i,2}) \mathbf{a}_i - |z|^{4N} \rho_{2,B} \sum_{i=3}^{10} Z_{i,2} \mathbf{a}_i, \end{aligned}$$

$$\begin{aligned} E_2 &= -|z|^{2N} \frac{\rho_{2,B}^2}{\rho_{1,B}^2} (2\psi_2 - 2\psi_1) \sum_{i=3}^{10} (2Z_{i,2} - 2Z_{i,1}) \mathbf{a}_i - \frac{1}{2} g_{\varepsilon\varepsilon}(0,z) \frac{\rho_{2,B}^2}{\rho_{1,B}^2} \sum_{i=3}^{10} (2Z_{i,2} - 2Z_{i,1}) \mathbf{a}_i \\ &+ 4|z|^{4N} \frac{\rho_{2,B}^4}{\rho_{1,B}^4} \sum_{i=3}^{10} (2Z_{i,2} - 2Z_{i,1}) \mathbf{a}_i - 2|z|^{4N} \rho_{2,B} \sum_{i=3}^{10} Z_{i,2} \mathbf{a}_i, \end{aligned}$$

where $f_{\varepsilon\varepsilon}(0,z)$ is

$$\begin{aligned} & f_{\varepsilon\varepsilon}(0,z) = 2|z|^{2(N-1)} (\sum_i |p_i|^2 + 2 \sum_{i \neq j} (p_{i1} \cos\theta + p_{i2} \sin\theta)(p_{j1} \cos\theta + p_{j2} \sin\theta)) \\ &= 2|z|^{2(N-1)} (|\sum_i p_i|^2 + \sum_{i \neq j} (p_{i1} p_{j1} - p_{i2} p_{j2}) \cos 2\theta + (p_{i1} p_{j2} + p_{i2} p_{j1}) \sin 2\theta) \\ &= 2|z|^{2(N-1)} (\sum_{i \neq j} (p_{i1} p_{j1} - p_{i2} p_{j2}) \cos 2\theta + (p_{i1} p_{j2} + p_{i2} p_{j1}) \sin 2\theta). \end{aligned} \tag{3.48}$$

Similarly we can get the expression for $g_{\varepsilon\varepsilon}(0,z)$.

Since

$$\begin{aligned} & \int h(r) \cos 2\theta (2Z_{i,1} - Z_{i,2})(2Z_{j,1} - Z_{j,2}) r dr d\theta \\ = & \int h(r) \sin 2\theta (2Z_{i,1} - Z_{i,2})(2Z_{j,1} - Z_{j,2}) r dr d\theta = 0, \end{aligned}$$

and

$$\int h(r)\cos 2\theta(Z_{i,2}-Z_{i,1})(Z_{j,2}-Z_{j,1})rdrd\theta$$
$$=\int h(r)\sin 2\theta(Z_{i,2}-Z_{i,1})(Z_{j,2}-Z_{j,1})rdrd\theta=0,$$

for $i,j=3,\cdots,10$, from (3.48), we have

$$\int_0^\infty\int_0^{2\pi} f_{\varepsilon\varepsilon}(0,z)e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2Z_{i,1}-Z_{i,2})(2Z_{j,1}-Z_{j,2})rdrd\theta=0, \quad (3.49)$$

and

$$\int_0^\infty\int_0^{2\pi} g_{\varepsilon\varepsilon}(0,z)e^{2\tilde{U}_{2,0}-2\tilde{U}_{1,0}}(2Z_{i,2}-2Z_{i,1})(2Z_{j,2}-2Z_{j,1})rdrd\theta=0, \quad (3.50)$$

for $i,j=3,\cdots,8$. Note that $f_{\varepsilon\varepsilon}(0,z)=g_{\varepsilon\varepsilon}(0,z)=0$ if $N=1$.

Another important observation is the following:

$$\int_0^\infty\int_0^{2\pi} r^{2N}e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2\psi_1-\psi_2)(2Z_{i,1}-Z_{i,2})(2Z_{j,1}-Z_{j,2})rdrd\theta \quad (3.51)$$
$$=\int_0^\infty\int_0^{2\pi} f(0,z)e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2\psi_{0,1}-\psi_{0,2})(2Z_{i,1}-Z_{i,2})(2Z_{j,1}-Z_{j,2})rdrd\theta,$$

$$\int_0^\infty\int_0^{2\pi} r^{2N}e^{2\tilde{U}_{2,0}-2\tilde{U}_{1,0}}(\psi_2-\psi_1)(Z_{i,2}-Z_{i,1})(Z_{j,2}-Z_{j,1})rdrd\theta \quad (3.52)$$
$$=\int_0^\infty\int_0^{2\pi} g(0,z)e^{2\tilde{U}_{2,0}-2\tilde{U}_{1,0}}(\psi_{0,2}-\psi_{0,1})(Z_{i,2}-Z_{i,1})(Z_{j,2}-Z_{j,1})rdrd\theta,$$

where $(\psi_{0,1},\psi_{0,2})$ is the radial solution of the following system:

$$\begin{cases} \Delta\psi_1+|z|^{2N_1}e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2\psi_1-\psi_2)=2|z|^{4N_1}e^{4\tilde{U}_{1,0}-2\tilde{U}_{2,0}}-|z|^{2(N_1+N_2)}e^{\tilde{U}_{2,0}} \\ \Delta\psi_2+|z|^{2N_2}e^{2\tilde{U}_{2,0}-2\tilde{U}_{1,0}}(2\psi_2-2\psi_1)=2|z|^{4N_2}e^{4\tilde{U}_{2,0}-4\tilde{U}_{1,0}}-2|z|^{2(N_1+N_2)}e^{\tilde{U}_{2,0}}. \end{cases} \quad (3.53)$$

Because of this observation, when dealing with the $O(\varepsilon^2)$ approximation, we only need to consider the radial part of the solutions.

In fact we can choose $\psi_{0,1}=\psi,\ \ \psi_{0,2}=\frac{4}{3}\psi$ such that ψ is the solution of the following ODE:

$$\Delta\psi+\frac{8(N+1)^2r^{2N}}{(1+r^{2N+2})^2}\psi=r^{4N}\frac{96(N+1)^4}{(1+r^{2N+2})^4}. \quad (3.54)$$

Combining (3.49), (3.50), (3.51) and (3.52), one can get the following:

$$
\begin{aligned}
& \int E \cdot Z_k^* r dr d\theta \\
= & \int_0^\infty \int_0^{2\pi} \sum_{i=3}^{10} \Big(\Big[2|z|^{4N_1} e^{4\tilde{U}_{1,0} - 2\tilde{U}_{2,0}} (4Z_{i,1} - 2Z_{i,2}) \mathbf{a}_i - |z|^{2N_1+2N_2} e^{\tilde{U}_{2,0}} Z_{i,2} \mathbf{a}_i \\
- & f(0,z) e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2\psi_{0,1} - \psi_{0,2})(2Z_{i,1} - Z_{i,2}) \mathbf{a}_i \Big] Z_{k,1}^* \\
+ & \Big[4|z|^{4N_2} e^{4\tilde{U}_{2,0} - 4\tilde{U}_{1,0}} (2Z_{i,2} - 2Z_{i,1}) \mathbf{a}_i - 2|z|^{2N_1+2N_2} e^{\tilde{U}_{2,0}} Z_{i,2} \mathbf{a}_i \\
- & g(0,z) e^{2\tilde{U}_{2,0} - 2\tilde{U}_{1,0}} (2\psi_{0,2} - 2\psi_{0,1})(2Z_{i,2} - 2Z_{i,1}) \mathbf{a}_i \Big] Z_{k,2}^* \Big) r dr d\theta \\
& + O(|\mathbf{a}|^2 + |\varepsilon|^2) + O(\varepsilon),
\end{aligned}
$$

where $O(\varepsilon)$ is independent of $\mathbf{a}$. Replacing the Z_k^* term in the above expression by $Z_{c_{21,i}}^*$, we have for $i = 1, 2$

$$
\begin{aligned}
& \int E \cdot Z_{c_{21,i}}^* r dr d\theta \\
= & \int \Big(4r^{4N} \frac{\rho_1^4}{\rho_2^2} \Big[(2Z_{c_{21,i},1} - Z_{c_{21,i},2})^2 c_{21,i} + (2Z_{c_{32,i},1} - Z_{c_{32,i},2})(2Z_{c_{21,i},1} - Z_{c_{21,i},2}) c_{32,i} \Big] \\
- & r^{4N} \rho_2 \Big[Z_{c_{21,i},2} (2Z_{c_{21,i},1} - Z_{c_{21,i},2}) c_{21,i} + Z_{c_{32,i},2} (2Z_{c_{21,i},1} - Z_{c_{21,i},2}) c_{32,i} \Big] \\
- & r^{2N} \frac{\rho_1^2}{\rho_2} (2\psi_{1,0} - \psi_{2,0}) \Big[(2Z_{c_{21,i},1} - Z_{c_{21,i},2})^2 c_{21,i} + (2Z_{c_{32,i},1} - Z_{c_{32,i},2})(2Z_{c_{21,i},1} - Z_{c_{21,i},2}) c_{32,i} \Big] \\
+ & 4r^{4N} \frac{\rho_2^4}{\rho_1^4} \Big[2(Z_{c_{21,i},2} - Z_{c_{21,i},1})^2 c_{21,i} + 2(Z_{c_{32,i},2} - Z_{c_{32,i},1})(Z_{c_{21,i},2} - Z_{c_{21,i},1}) c_{32,i} \Big] \\
- & 2r^{4N} \rho_2 \Big[Z_{c_{21,i},2} (Z_{c_{21,i},2} - Z_{c_{21,i},1}) c_{21,i} + Z_{c_{32,i},2} (Z_{c_{21,i},2} - Z_{c_{21,i},1}) c_{32,i} \Big] \\
- & r^{2N} \frac{\rho_2^2}{\rho_1^2} (2\psi_{2,0} - 2\psi_{1,0}) \Big[2(Z_{c_{21,i},2} - Z_{c_{21,i},1})^2 c_{21,i} \\
+ & 2(Z_{c_{32,i},2} - Z_{c_{32,i},1})(Z_{c_{21,i},2} - Z_{c_{21,i},1}) c_{32,i} \Big] \Big) r dr d\theta \\
= & \int \Big(\frac{16\mu^4 r^{4N+1}}{(1 + r^{2N+2})^4} \Big[(240 Z_{c_{21,i},1}^2 + 120 Z_{c_{21,i},2}^2 - 336 Z_{c_{21,i},1} Z_{c_{21,i},2}) c_{21,i} \\
+ & (240 Z_{c_{21,i},1} Z_{c_{32,i},1} + 120 Z_{c_{21,i},2} Z_{c_{32,i},2} - 168 Z_{c_{21,i},1} Z_{c_{32,i},2} - 168 Z_{c_{21,i},2} Z_{c_{32,i},1}) c_{32,i} \Big]
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{2\mu^2 r^{2N+1}}{3(1+r^{2N+2})^2}\psi\Big[(80Z^2_{c_{21,i},1}+44Z^2_{c_{21,i},2}-112Z_{c_{21,i},1}Z_{c_{21,i},2})c_{21,i}\\
&+ (80Z_{c_{21,i},1}Z_{c_{32,i},1}+44Z_{c_{21,i},2}Z_{c_{32,i},2}-56Z_{c_{21,i},1}Z_{c_{32,i},2}-56Z_{c_{21,i},2}Z_{c_{32,i},1})c_{32,i}\Big]\Big)drd\theta\\
&+ O(\varepsilon)+O(|\mathbf{a}|^2+\varepsilon^2)\\
&= (J_1-J_3)c_{21,i}+(J_2-J_4)c_{32,i}\\
&+ O(\varepsilon)+O(|\mathbf{a}|^2+\varepsilon^2),
\end{aligned}
$$

similarly, we can get that

$$
\begin{aligned}
&\int E\cdot Z^*_{c_{32,i}}rdrd\theta\\
&= \int\Big(4r^{4N}\frac{\rho_1^4}{\rho_2^2}\Big[(2Z_{c_{32,i},1}-Z_{c_{32,i},2})^2c_{32,i}+(2Z_{c_{21,i},1}-Z_{c_{21,i},2})(2Z_{c_{32,i},1}-Z_{c_{32,i},2})c_{21,i}\Big]\\
&- r^{4N}\rho_2\Big[Z_{c_{32,i},2}(2Z_{c_{32,i},1}-Z_{c_{32,i},2})c_{32,i}+Z_{c_{21,i},2}(2Z_{c_{32,i},1}-Z_{c_{32,i},2})c_{21,i}\Big]\\
&- r^{2N}\frac{\rho_1^2}{\rho_2}(2\psi_{1,0}-\psi_{2,0})\Big[(2Z_{c_{32,i},1}-Z_{c_{32,i},2})^2c_{32,i}+(2Z_{c_{21,i},1}-Z_{c_{21,i},2})(2Z_{c_{32,i},1}-Z_{c_{32,i},2})c_{21,i}\Big]\\
&+ 4r^{4N}\frac{\rho_2^4}{\rho_1^4}\Big[2(Z_{c_{32,i},2}-Z_{c_{32,i},1})^2c_{32,i}+2(Z_{c_{21,i},2}-Z_{c_{21,i},1})(Z_{c_{32,i},2}-Z_{c_{32,i},1})c_{21,i}\Big]\\
&- 2r^{4N}\rho_2\Big[Z_{c_{32,i},2}(Z_{c_{32,i},2}-Z_{c_{32,i},1})c_{32,i}+Z_{c_{21,i},2}(Z_{c_{32,i},2}-Z_{c_{32,i},1})c_{21,i}\Big]\\
&- r^{2N}\frac{\rho_2^2}{\rho_1^2}(2\psi_{2,0}-2\psi_{1,0})\Big[2(Z_{c_{32,i},2}-Z_{c_{32,i},1})^2c_{32,i}\\
&+ 2(Z_{c_{21,i},2}-Z_{c_{21,i},1})(Z_{c_{32,i},2}-Z_{c_{32,i},1})c_{21,i}\Big]\Big)rdrd\theta\\
&= \int\Big(\frac{16\mu^4 r^{4N+1}}{(1+r^{2N+2})^4}\Big[(240Z^2_{c_{32,i},1}+120Z^2_{c_{32,i},2}-336Z_{c_{32,i},1}Z_{c_{32,i},2})c_{32,i}\\
&+ (240Z_{c_{21,i},1}Z_{c_{32,i},1}+120Z_{c_{21,i},2}Z_{c_{32,i},2}-168Z_{c_{21,i},1}Z_{c_{32,i},2}-168Z_{c_{21,i},2}Z_{c_{32,i},1})c_{21,i}\Big]\\
&- \frac{2\mu^2 r^{2N+1}}{3(1+r^{2N+2})^2}\psi\Big[(80Z^2_{c_{32,i},1}+44Z^2_{c_{32,i},2}-112Z_{c_{32,i},1}Z_{c_{32,i},2})c_{32,i}\\
&+ (80Z_{c_{21,i},1}Z_{c_{32,i},1}+44Z_{c_{21,i},2}Z_{c_{32,i},2}-56Z_{c_{21,i},1}Z_{c_{32,i},2}-56Z_{c_{21,i},2}Z_{c_{32,i},1})c_{21,i}\Big]\Big)drd\theta\\
&+ O(\varepsilon)+O(|\mathbf{a}|^2+\varepsilon^2)\\
&= (J_5-J_7)c_{32,i}+(J_6-J_8)c_{21,i}\\
&+ O(\varepsilon)+O(|\mathbf{a}|^2+\varepsilon^2),
\end{aligned}
$$

and

$$
\begin{aligned}
& \int E \cdot Z^*_{c_{31,i}} r dr d\theta \\
= & \int \Big(4r^{4N} \frac{\rho_1^4}{\rho_2^2} [(2Z_{c_{31,i},1} - Z_{c_{31,i},2})^2 c_{31,i}] - r^{4N} \rho_2 [Z_{c_{31,i},2}(2Z_{c_{31,i},1} - Z_{c_{31,i},2}) c_{31,i}] \\
- & r^{2N} \frac{\rho_1^2}{\rho_2} (2\psi_{1,0} - \psi_{2,0}) [(2Z_{c_{31,i},1} - Z_{c_{31,i},2})^2 c_{31,i}] \\
+ & 4r^{4N} \frac{\rho_2^4}{\rho_1^4} [2(Z_{c_{31,i},2} - Z_{c_{31,i},1})^2 c_{31,i}] - 2r^{4N} \rho_2 [Z_{c_{31,i},2}(Z_{c_{31,i},2} - Z_{c_{31,i},1}) c_{31,i}] \\
- & r^{2N} \frac{\rho_2^2}{\rho_1^2} (2\psi_{2,0} - 2\psi_{1,0}) [2(Z_{c_{31,i},2} - Z_{c_{31,i},1})^2 c_{31,i}] \Big) r dr d\theta \\
= & \int \Big(\frac{16\mu^4 r^{4N+1}}{(1+r^{2N+2})^4} \Big[(240 Z^2_{c_{31,i},1} + 120 Z^2_{c_{31,i},2} - 336 Z_{c_{31,i},1} Z_{c_{31,i},2}) c_{31,i} \Big] \\
- & \frac{2\mu^2 r^{2N+1}}{3(1+r^{2N+2})^2} \psi \Big[(80 Z^2_{c_{31,i},1} + 44 Z^2_{c_{31,i},2} - 112 Z_{c_{31,i},1} Z_{c_{31,i},2}) c_{31,i} \Big] \Big) dr d\theta \\
+ & O(\varepsilon) + O(|\mathbf{a}|^2 + \varepsilon^2) \\
= & (J_9 - J_{10}) c_{31,i} + O(\varepsilon) + O(|\mathbf{a}|^2 + \varepsilon^2),
\end{aligned}
$$

$$
\begin{aligned}
& \int E \cdot Z^*_{c_{30,i}} r dr d\theta \\
= & \int \Big(4r^{4N} \frac{\rho_1^4}{\rho_2^2} [(2Z_{c_{30,i},1} - Z_{c_{30,i},2})^2 c_{30,i}] - r^{4N} \rho_2 [Z_{c_{30,i},2}(2Z_{c_{30,i},1} - Z_{c_{30,i},2}) c_{30,i}] \\
- & r^{2N} \frac{\rho_1^2}{\rho_2} (2\psi_{1,0} - \psi_{2,0}) [(2Z_{c_{30,i},1} - Z_{c_{30,i},2})^2 c_{30,i}] \\
+ & 4r^{4N} \frac{\rho_2^4}{\rho_1^4} [2(Z_{c_{30,i},2} - Z_{c_{30,i},1})^2 c_{30,i}] \\
- & 2r^{4N} \rho_2 [Z_{c_{30,i},2}(Z_{c_{30,i},2} - Z_{c_{30,i},1}) c_{30,i}] - r^{2N} \frac{\rho_2^2}{\rho_1^2} (2\psi_{2,0} - 2\psi_{1,0}) [2(Z_{c_{30,i},2} - Z_{c_{30,i},1})^2 c_{30,i}] \Big) r dr d\theta \\
= & \int \Big(\frac{16\mu^4 r^{4N+1}}{(1+r^{2N+2})^4} \Big[(240 Z^2_{c_{30,i},1} + 120 Z^2_{c_{30,i},2} - 336 Z_{c_{30,i},1} Z_{c_{30,i},2}) c_{30,i} \Big] \\
- & \frac{2\mu^2 r^{2N+1}}{3(1+r^{2N+2})^2} \psi \Big[(80 Z^2_{c_{30,i},1} + 44 Z^2_{c_{30,i},2} - 112 Z_{c_{30,i},1} Z_{c_{30,i},2}) c_{30,i} \Big] \Big) dr d\theta \\
+ & O(\varepsilon) + O(|\mathbf{a}|^2 + \varepsilon^2) \\
= & (J_{11} - J_{12}) c_{30,i} + O(\varepsilon) + O(|\mathbf{a}|^2 + \varepsilon^2),
\end{aligned}
$$

where J_i for $i = 1, \cdots, 12$ are given in the above expressions.

So we get that

$$
\begin{aligned}
&(\langle E, Z^*_{c21,1}\rangle, \langle E, Z^*_{c21,2}\rangle, \langle E, Z^*_{c30,1}\rangle, \langle E, Z^*_{c30,2}\rangle,\\
&\quad \langle E, Z^*_{c31,1}\rangle, \langle E, Z^*_{c31,2}\rangle, \langle E, Z^*_{c32,1}\rangle, \langle E, Z^*_{c32,2}\rangle)^t\\
&= \tilde{\mathcal{T}}(\mathbf{a}) + O(|\mathbf{a}|^2) + O(\varepsilon),
\end{aligned}
$$

and

$$
\tilde{\mathcal{T}} = \begin{pmatrix}
T_1 & 0 & 0 & 0 & 0 & 0 & T_2 & 0\\
0 & T_1 & 0 & 0 & 0 & 0 & 0 & T_2\\
0 & 0 & T_3 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & T_3 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & T_4 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & T_4 & 0 & 0\\
T_5 & 0 & 0 & 0 & 0 & 0 & T_6 & 0\\
0 & T_5 & 0 & 0 & 0 & 0 & 0 & T_6
\end{pmatrix}, \tag{3.55}
$$

where

$$
\begin{aligned}
&T_1 = J_1 - J_3,\ T_2 = J_2 - J_4,\\
&T_3 = J_{11} - J_{12}, \quad T_4 = J_9 - J_{10},\\
&T_5 = J_6 - J_8, \quad T_6 = J_5 - J_7.
\end{aligned}
$$

The determinant of the matrix $\tilde{\mathcal{T}}$ is

$$
\begin{aligned}
&(T_1T_6 - T_2T_5)^2 T_3^3 T_4^2\\
&= [(J_1 - J_3)(J_5 - J_7) - (J_2 - J_4)(J_6 - J_8)]^2 (J_9 - J_{10})^2 (J_{11} - J_{12})^2.
\end{aligned}
$$

Next we prove that the matrix $\tilde{\mathcal{T}}$ is non-degenerate, i.e. the determinant of $\tilde{\mathcal{T}}$ is nonzero. For this purpose, we need to calculate the integrals J_1 to J_{12}. But in the integrals, there is the function ψ for which the expression is unknown. In order to get rid of ψ, we use integration by parts. The key observation is that for any ϕ satisfying $\phi(\infty) = 0$, we have

$$
\int_0^\infty [(\Delta + \frac{8(N+1)^2 r^{2N}}{(1+r^{2N+2})^2})\psi]\phi r dr = \int_0^\infty [(\Delta + \frac{8(N+1)^2 r^{2N}}{(1+r^{2N+2})^2})\phi]\psi r dr. \tag{3.56}
$$

So we need to find solutions of the following ODEs:

$$
\Delta\phi_1 + \frac{8(N+1)^2 r^{2N}}{(1+r^{2N+2})^2}\phi = \frac{r^{4N+2} + r^{12N+10}}{(1+r^{2N+2})^8}, \tag{3.57}
$$

$$\Delta\phi_2 + \frac{8(N+1)^2 r^{2N}}{(1+r^{2N+2})^2}\phi = \frac{r^{6N+4} + r^{10N+8}}{(1+r^{2N+2})^8}, \tag{3.58}$$

and

$$\Delta\phi_3 + \frac{8(N+1)^2 r^{2N}}{(1+r^{2N+2})^2}\phi = \frac{r^{8N+6}}{(1+r^{2N+2})^8}. \tag{3.59}$$

Let us first consider a general form of this ODE:

$$\Delta\phi + \frac{8(N+1)^2 r^{2N}}{(1+r^{2N+2})^2}\phi = \frac{r^{tN+t-2}}{(1+r^{2N+2})^8}, \tag{3.60}$$

for $t = 4, 6, 8, 10, 12$. Let $s = r^{N+1}$, and $\bar{\phi}(s) = \phi(r)$. Then $\bar{\phi}$ satisfies

$$\Delta_s\bar{\phi} + \frac{8}{(1+s^2)^2}\bar{\phi} = \frac{s^{t-2}}{(N+1)^2(1+s^2)^8}. \tag{3.61}$$

By stereo-graphical projection, letting $s = \tan\frac{\theta}{2}$, and $\tilde{\phi}(\theta) = \bar{\phi}(s)$, we get that the equation satisfied by $\tilde{\phi}(\theta)$ is

$$\Delta_{S^2}\tilde{\phi} + 2\tilde{\phi} = \frac{1}{2^8(N+1)^2}(1+\cos\theta)^6\tan^{t-2}\frac{\theta}{2}, \tag{3.62}$$

where Δ_{S^2} is the Laplace-Betrami operator on S^2. By direct calculation, we have

$$\tilde{\phi}_1 = \frac{1}{2^8\times 32(N+1)^2}\left(\frac{99}{2} + \frac{59}{36}\cos(2\theta) + \frac{257}{90}\cos(4\theta) + \frac{1}{20}\cos(6\theta)\right), \tag{3.63}$$

and

$$\tilde{\phi}_2 = \frac{1}{2^8\times 32(N+1)^2}\left(\frac{45}{2} + \frac{325}{36}\cos(2\theta) - \frac{17}{90}\cos(4\theta) - \frac{1}{20}\cos(6\theta)\right), \tag{3.64}$$

and

$$\tilde{\phi}_3 = \frac{1}{2^8\times 32(N+1)^2}\left(\frac{35}{4} + \frac{35}{8}\cos(2\theta) - \frac{7}{20}\cos(4\theta) + \frac{1}{40}\cos(6\theta)\right). \tag{3.65}$$

From the relation between ϕ and $\tilde{\phi}$, we can get

$$\begin{aligned}
\phi_1 &= \frac{128(19+76r^{2\mu}+275r^{4\mu}+400r^{6\mu}+275r^{8\mu}+76r^{10\mu}+19r^{12\mu})}{2^8\times 32(N+1)^2\times 45(1+r^{2\mu})^6} \\
&- \frac{128\times 19}{45\times 2^8\times 32(N+1)^2}\varphi_0, \\
\phi_2 &= \frac{128(11+44r^{2\mu}+55r^{4\mu}+80r^{6\mu}+55r^{8\mu}+44r^{10\mu}+11r^{12\mu})}{2^8\times 32(N+1)^2\times 45(1+r^{2\mu})^6} \\
&- \frac{128\times 11}{45\times 2^8\times 32(N+1)^2}\varphi_0, \\
\phi_3 &= \frac{64(1+4r^{2\mu}+5r^{4\mu}+5r^{8\mu}+4r^{10\mu}+r^{12\mu})}{2^8\times 32(N+1)^2\times 5(1+r^{2\mu})^6} \\
&- \frac{64}{5\times 2^8\times 32(N+1)^2}\varphi_0, \qquad (3.66)
\end{aligned}$$

where $\varphi_0(r) = \frac{1-r^{2N+2}}{1+r^{2N+2}}$ satisfies

$$\Delta\varphi_0 + \frac{8(N+1)^2 r^{2N}}{(1+r^{2N+2})^2}\varphi_0 = 0.$$

We add the terms containing φ_0, to ensure that $\phi_i(\infty) = 0$ for $i = 1, 2, 3$ so that the integration by parts makes sense.

So by direct calculation, we have

$$J_3 = \int \frac{96\mu^4 r^{4N+1}}{(1+r^{2N+2})^4} \times \frac{32(11\phi_1 - 64\phi_2 + 120\phi_3)}{3}\mu^2 dr, \qquad (3.67)$$

$$J_4 = J_8 = -\int \frac{96\mu^4 r^{4N+1}}{(1+r^{2N+2})^4} \times \frac{32(7\phi_1 - 44\phi_2 + 78\phi_3)}{3}\mu^2 dr, \qquad (3.68)$$

$$J_7 = \int \frac{96\mu^4 r^{4N+1}}{(1+r^{2N+2})^4} \times \frac{32(5\phi_1 - 28\phi_2 + 54\phi_3)}{3}\mu^2 dr, \qquad (3.69)$$

$$J_{10} = \int \frac{96\mu^4 r^{4N+1}}{(1+r^{2N+2})^4} \times 320(\phi_2 - 2\phi_3)\mu^2 dr, \qquad (3.70)$$

$$J_{12} = \int \frac{96\mu^4 r^{4N+1}}{(1+r^{2N+2})^4} \times 320\phi_3\mu^2 dr. \qquad (3.71)$$

Since all the terms in the integrals are explicit now, by direct calculation, we get

$$
\begin{aligned}
&J_9 - J_{10} \\
&= -\int \frac{1}{3(1+r^{2\mu})^{10}}(256r^{8\mu-3}\mu^4(-261+16384r^{6\mu}\mu^2+4096r^{8\mu}\mu^2+r^{4\mu}(-261+20480\mu^2) \\
&+r^{2\mu}(522+81920\mu^2)))dr \\
&= -\frac{128\mu^3}{3}\int(-\frac{1044}{(1+t)^{10}}+\frac{3132}{(1+t)^9}-\frac{3393}{(1+t)^8}+\frac{1566}{(1+t)^7}-\frac{261}{(1+t)^6}-\frac{73728\mu^2}{(1+t)^{10}} \\
&+\frac{221184\mu^2}{(1+t)^9}-\frac{225280\mu^2}{(1+t)^8}+\frac{81920\mu^2}{(1+t)^7}-\frac{8192\mu^2}{(1+t)^5}+\frac{4096\mu^2}{(1+t)^4})t^\eta dt,
\end{aligned}
$$

$$
\begin{aligned}
&J_{11} - J_{12} \\
&= -768\mu^4\int r^{10\mu-3}\frac{-29+5120\mu^2r^{2\mu}+4096\mu^2r^{4\mu}+1024\mu^2r^{6\mu}}{(1+r^{2\mu})^10}dr \\
&= -384\mu^3\int(\frac{29}{(1+t)^{10}}-\frac{87}{(1+t)^9}+\frac{87}{(1+t)^8}-\frac{29}{(1+t)^7}+\frac{2048\mu^2}{(1+t)^{10}}-\frac{6144\mu^2}{(1+t)^9} \\
&+\frac{5120\mu^2}{(1+t)^8}-\frac{2048\mu^2}{(1+t)^5}+\frac{1024\mu^2}{(1+t)^4})t^\eta dt,
\end{aligned}
$$

$$
\begin{aligned}
&J_1 - J_3 \\
&= -\int \frac{1}{(1+r^{2\mu})^{10}}(256r^{6\mu-3}\mu^4(-30+1024r^{10\mu}\mu^2+r^{8\mu}(-30+4096\mu^2)+6r^{6\mu}(31+4608\mu^2) \\
&-r^{4\mu}(351+8192\mu^2)+2r^{2\mu}(93+11264\mu^2)))dr \\
&= -128\mu^3\int(\frac{783}{(1+t)^{10}}-\frac{2349}{(1+t)^9}+\frac{2655}{(1+t)^8}-\frac{1395}{(1+t)^7}+\frac{336}{(1+t)^6}-\frac{30}{(1+t)^5}+\frac{55296\mu^2}{(1+t)^{10}} \\
&-\frac{165888\mu^2}{(1+t)^9}+\frac{187392\mu^2}{(1+t)^8}-\frac{98304\mu^2}{(1+t)^7}+\frac{22528\mu^2}{(1+t)^6}-\frac{2048\mu^2}{(1+t)^5}+\frac{1024\mu^2}{(1+t)^4})t^\eta dt,
\end{aligned}
$$

$$
\begin{aligned}
&J_5 - J_7 \\
&= -\frac{256\mu^4}{3}\int \frac{1}{(1+r^{2\mu})^10}(r^{-3+6\mu}(-45+2048r^{10\mu}\mu^2+12r^{2\mu}(21+2560\mu^2) \\
&-2r^{4\mu}(225+4096\mu^2)+r^{8\mu}(-45+8192\mu^2)+4r^{6\mu}(63+10240\mu^2)))dr \\
&= -\frac{128\mu^3}{3}\int(\frac{1044}{(1+t)^{10}}-\frac{3132}{(1+t)^9}+\frac{3564}{(1+t)^8}-\frac{1908}{(1+t)^7}+\frac{477}{(1+t)^6}-\frac{45}{(1+t)^5}+\frac{73728\mu^2}{(1+t)^{10}} \\
&-\frac{221184\mu^2}{(1+t)^9}+\frac{249856\mu^2}{(1+t)^8}-\frac{131072\mu^2}{(1+t)^7}+\frac{30720\mu^2}{(1+t)^6}-\frac{4096\mu^2}{(1+t)^5}+\frac{2048\mu^2}{(1+t)^4})t^\eta dt,
\end{aligned}
$$

$$
\begin{aligned}
&J_2 - J_4 = J_6 - J_8 \\
&= 256\mu^4\int \frac{1}{(1+r^{2\mu})^{10}}(r^{-3+6\mu}(-21-21r^{8\mu}+14r^{2\mu}(9+1024\mu^2)+14r^{6\mu}(9+1024\mu^2) \\
&-4r^{4\mu}(57+2048\mu^2)))dr \\
&= 128\mu^3\int(\frac{522}{(1+t)^{10}}-\frac{1566}{(1+t)^9}+\frac{1776}{(1+t)^8}-\frac{942}{(1+t)^7}+\frac{231}{(1+t)^6}-\frac{21}{(1+t)^5}+\frac{36864\mu^2}{(1+t)^{10}} \\
&-\frac{110592\mu^2}{(1+t)^9}+\frac{124928\mu^2}{(1+t)^8}-\frac{65536\mu^2}{(1+t)^7}+\frac{14336\mu^2}{(1+t)^6})t^\eta dt,
\end{aligned}
$$

where $\eta = \frac{N}{1+N}$. By direct computation, we have

$$
\int_0^\infty \frac{t^\eta}{(1+t)^k}dt = \frac{k}{k-1-\eta}\int_0^\infty \frac{t^\eta}{(1+t)^{k+1}}dt.
$$

So one can get that

$$
\begin{aligned}
&(J_1 - J_3)(J_5 - J_7) - (J_2 - J_4)(J_6 - J_8) \\
&= \frac{(128\mu^3)^2}{((3+2N)^2(4+3N)^2(5+4N)^2(6+5N)^2(7+6N)^2(8+7N)^2)} \\
&\times(1296(1+N)^4(1+2N)^2(111153029448+N(1155840790846+N(5383345380791 \\
&+4N(3698970614035+3N(2215580694621+128N(21256160723+N(18071934337 \\
&+224N(46903715+32N(557767+96N(1307+132N)))))))))) \\
&\times(\int \frac{t^\eta}{(1+t)^{10}})^2 \neq 0,
\end{aligned}
$$

$$
\begin{aligned}
&J_9 - J_{10}\\
&= -\frac{128\mu^3}{3(3+2N)(4+3N)(5+4N)(6+5N)(7+6N)(8+7N)}\\
&\times(36(2+7N+6N^2)(613356+3574985N+8639370N^2+11096430N^3+7999140N^4\\
&+3072000N^5+491520N^6))\int\frac{t^\eta}{(1+t)^{10}}dt \neq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&J_{11} - J_{12}\\
&= -\frac{384\mu^3}{((3+2N)(4+3N)(5+4N)(6+5N)(7+6N)(8+7N))}\\
&\times((1+2N)(2+3N)(3+4N)(612660+3090671N+6206694N^2+6206792N^3\\
&+3092480N^4+614400N^5))\int\frac{t^\eta}{(1+t)^{10}}dt \neq 0.
\end{aligned}
$$

Therefore, $\tilde{\mathcal{T}}$ is non-degenerate, and then Lemma 3.4 is proved.

□

In the next section, we assume the **Assumption (ii)** holds. We are going to prove Theorem 1.1 for $\mathbf{B}_2$.

3.6 Proof of Theorem 1.1 for $\mathbf{B}_2$ under Assumption (ii)

Similar to the A_2 case in Section 2, we will see that in this case, the two free parameters ξ_1, ξ_2 we introduced in Section 3.4 for the improvement of the $O(\varepsilon^2)$ approximate solutions play an important role. It turns out that we only need to consider the terms involving ξ_1 and ξ_2. See Lemma 3.5 below. Thus we reduce our problem to the non-degeneracy of an 8×8 matrix $\tilde{\mathbf{Q}}$, which is relatively easy to handle. See Lemma 3.6.

Lemma 3.5. *Let* $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ *be a solution of (3.39). The following estimates* *hold:*

$$\langle E, Z^*_{c_{21,i}}\rangle \tag{3.72}$$
$$= \xi_1(\tilde{\mathcal{A}}_1 c_{21,i} + \tilde{\mathcal{B}}_1 c_{32,i}) + \xi_2(\tilde{\mathcal{A}}_2 c_{21,i} + \tilde{\mathcal{B}}_2 c_{32,i}) + \tilde{\mathcal{T}}_{1i}(\mathbf{a}) + O((1+|\xi|)|\mathbf{a}|^2) + O(\varepsilon),$$

$$\langle E, Z^*_{c_{32,i}}\rangle \tag{3.73}$$
$$= \xi_1(\tilde{\mathcal{C}}_1 c_{32,i} + \tilde{\mathcal{D}}_1 c_{21,i}) + \xi_2(\tilde{\mathcal{C}}_2 c_{32,i} + \tilde{\mathcal{D}}_2 c_{21,i}) + \tilde{\mathcal{T}}_{2i}(\mathbf{a}) + O((1+|\xi|)|\mathbf{a}|^2) + O(\varepsilon),$$

$$\langle E, Z^*_{c_{30,i}}\rangle \tag{3.74}$$
$$= \xi_1\tilde{\mathcal{E}}_1 c_{30,i} + \xi_2\tilde{\mathcal{E}}_2 c_{30,i} + \tilde{\mathcal{T}}_{3i}(\mathbf{a}) + O((1+|\xi|)|\mathbf{a}|^2) + O(\varepsilon),$$

$$\langle E, Z^*_{c_{31,i}}\rangle \tag{3.75}$$
$$= \xi_1\tilde{\mathcal{F}}_1 c_{31,i} + \xi_2\tilde{\mathcal{F}}_2 c_{31,i} + \tilde{\mathcal{T}}_{4i}(\mathbf{a}) + O((1+|\xi|)|\mathbf{a}|^2) + O(\varepsilon),$$

for $i = 1, 2$, *where*

$$\begin{aligned}\tilde{\mathcal{A}}_j &= \int_0^{+\infty}\int_0^{2\pi}\Big[r^{2N_1}e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2Z_{j,1}-Z_{j,2})(2Z_{c_{21,1},1}-Z_{c_{21,1},2})^2 \\ &+ 4r^{2N_2}e^{2\tilde{U}_{2,0}-2\tilde{U}_{1,0}}(Z_{j,2}-Z_{j,1})(Z_{c_{21,1},2}-Z_{c_{21,1},1})^2\Big]rdrd\theta, \end{aligned} \tag{3.76}$$

$$\begin{aligned}\tilde{\mathcal{B}}_j = \int_0^{+\infty}\int_0^{2\pi}\Big[r^{2N_1}e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2Z_{j,1}-Z_{j,2})(2Z_{c_{32,1},1}-Z_{c_{32,1},2})(2Z_{c_{21,1},1}-Z_{c_{21,1},2}) \\ +4r^{2N_2}e^{2\tilde{U}_{2,0}-2\tilde{U}_{1,0}}(Z_{j,2}-Z_{j,1})(Z_{c_{32,1},2}-Z_{c_{32,1},1})(Z_{c_{21,1},2}-Z_{c_{21,1},1})\Big]rdrd\theta,\end{aligned} \tag{3.77}$$

$$\begin{aligned}\tilde{\mathcal{C}}_j &= \int_0^{+\infty}\int_0^{2\pi}\Big[r^{2N_1}e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2Z_{j,1}-Z_{j,2})(2Z_{c_{32,1},1}-Z_{c_{32,1},2})^2 \\ &+ 4r^{2N_2}e^{2\tilde{U}_{2,0}-2\tilde{U}_{1,0}}(Z_{j,2}-Z_{j,1})(Z_{c_{32,1},2}-Z_{c_{32,1},1})^2\Big]rdrd\theta, \end{aligned} \tag{3.78}$$

$$\begin{aligned}\tilde{\mathcal{D}}_j = \int_0^{+\infty}\int_0^{2\pi}\Big[r^{2N_1}e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2Z_{j,1}-Z_{j,2})(2Z_{c_{21,1},1}-Z_{c_{21,1},2})(2Z_{c_{32,1},1}-Z_{c_{32,1},2}) \\ +4r^{2N_2}e^{2\tilde{U}_{2,0}-2\tilde{U}_{1,0}}(Z_{j,2}-Z_{j,1})(Z_{c_{21,1},2}-Z_{c_{21,1},1})(Z_{c_{32,1},2}-Z_{c_{32,1},1})\Big]rdrd\theta,\end{aligned} \tag{3.79}$$

$$\begin{aligned}\tilde{\mathcal{E}}_j &= \int_0^{+\infty}\int_0^{2\pi}\Big[r^{2N_1}e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2Z_{j,1}-Z_{j,2})(2Z_{c_{30,1},1}-Z_{c_{30,1},2})^2 \\ &+ 4r^{2N_2}e^{2\tilde{U}_{2,0}-2\tilde{U}_{1,0}}(Z_{j,2}-Z_{j,1})(Z_{c_{30,1},2}-Z_{c_{30,1},1})^2\Big]rdrd\theta, \end{aligned} \tag{3.80}$$

$$\tilde{\mathcal{F}}_j = \int_0^{+\infty}\int_0^{2\pi}\Big[r^{2N_1}e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2Z_{j,1}-Z_{j,2})(2Z_{c_{31,1},1}-Z_{c_{31,1},2})^2 + 4r^{2N_2}e^{2\tilde{U}_{2,0}-2\tilde{U}_{1,0}}(Z_{j,2}-2Z_{j,1})(Z_{c_{31,1},2}-Z_{c_{31,1},1})^2\Big]rdrd\theta, \quad (3.81)$$

for $j = 1, 2$, and $\tilde{\mathcal{T}}_{ij}$ are 8×1 vectors which are uniformly bounded as ε tends to 0, and are independent of ξ_1, ξ_2.

Proof:

By (3.36) and (3.37), E is of the form

$$\frac{1}{\varepsilon}(...)\mathbf{a}\cdot\mathbf{a} + ((...)\mathbf{a}) + O(|\mathbf{a}|^2) + O(\varepsilon). \quad (3.82)$$

Recall that $\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}\psi_1^0\\ \psi_2^0\end{pmatrix} + \xi_1 Z_{\lambda_0} + \xi_2 Z_{\lambda_1}$. In the following computations, we only need to consider the terms involving ξ_1 and ξ_2, since all other terms are independent of ξ_1 and ξ_2. By the orthogonality of $\cos(k\theta)$ and $\cos(l\theta)$ for $k \neq l$, we obtain

$$\begin{aligned}
&- \int_0^{+\infty}\int_0^{2\pi} E\cdot Z^*_{c_{21,1}} rd\theta dr \\
&= \int_0^{\infty}\int_0^{2\pi}\Big[r^{2N_1}e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}\Big(\xi_1(2Z_{1,1}-Z_{1,2}) + \xi_2(2Z_{2,1}-Z_{2,2})\Big) \\
&\times (2Z_{c_{21,1},1}-Z_{c_{21,1},2})c_{21,1}Z^*_{c_{21,1},1} \\
&+ r^{2N_2}e^{2\tilde{U}_{2,0}-2\tilde{U}_{1,0}}\Big(\xi_1(2Z_{1,2}-2Z_{1,1}) + \xi_2(2Z_{2,2}-2Z_{2,1})\Big) \\
&\times (2Z_{c_{21,1},2}-2Z_{c_{21,1},1})c_{21,1}Z^*_{c_{21,1},2} \\
&+ r^{2N_1}e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}\Big(\xi_1(2Z_{1,1}-Z_{1,2}) + \xi_2(2Z_{2,1}-Z_{2,2})\Big) \\
&\times (2Z_{c_{32,1},1}-Z_{c_{32,1},2})c_{32,1}Z^*_{c_{21,1},1} \\
&+ r^{2N_2}e^{2\tilde{U}_{2,0}-2\tilde{U}_{1,0}}\Big(\xi_1(2Z_{1,2}-2Z_{1,1}) + \xi_2(2Z_{2,2}-2Z_{2,1})\Big) \\
&\times (2Z_{c_{32,1},2}-2Z_{c_{32,1},1})c_{32,1}Z^*_{c_{21,1},2}\Big]rdrd\theta \\
&+ \tilde{\mathcal{T}}_{11}(\mathbf{a}) + O(\frac{|\mathbf{a}|^2}{\varepsilon}) + O(\varepsilon) + O((1+|\xi|)|\mathbf{a}|^2 + \varepsilon^2),
\end{aligned}$$

where $\tilde{\mathcal{T}}_{11}(\mathbf{a})$ is the remaining terms which is a linear combinations of $\mathbf{a}$ which comes from the remaining terms of $O(\varepsilon^2)$ of E, and the coefficients of the linear combinations are uniformly bounded and are independent of $\xi_1, \xi_2, \mathbf{a}$. The $O(\frac{|\mathbf{a}|^2}{\varepsilon})$ terms comes from the $O(\frac{|\mathbf{a}|^2}{\varepsilon})$ term of E which is independent of ξ.

Thus

$$
\begin{aligned}
&-\int_0^{+\infty}\int_0^{2\pi} E\cdot Z^*_{c_{21,1}} rd\theta dr \\
&=\xi_1\Big[\Big(\int_0^{+\infty}\int_0^{2\pi}\Big\{r^{2N_1}e^{2\tilde U_{1,0}-\tilde U_{2,0}}(2Z_{1,1}-Z_{1,2})(2Z_{c_{21,1},1}-Z_{c_{21,1},2})^2 \\
&+4r^{2N_2}e^{2\tilde U_{2,0}-2\tilde U_{1,0}}(Z_{1,2}-Z_{1,1})(Z_{c_{21,1},2}-Z_{c_{21,1},1})^2\Big\}rdrd\theta\Big)c_{21,1} \\
&+\Big(\int_0^{+\infty}\int_0^{2\pi}\Big\{r^{2N_1}e^{2\tilde U_{1,0}-\tilde U_{2,0}}(2Z_{1,1}-Z_{1,2})(2Z_{c_{21,1},1}-Z_{c_{21,1},2})(2Z_{c_{32,1},1}-Z_{c_{32,1},2}) \\
&+4r^{2N_2}e^{2\tilde U_{2,0}-2\tilde U_{1,0}}(Z_{1,2}-Z_{1,1})(Z_{c_{21,1},2}-Z_{c_{21,1},1})(Z_{c_{32,1},2}-Z_{c_{32,1},1})\Big\}rdrd\theta\Big)c_{32,1}\Big] \\
&+\xi_2\Big[\Big(\int_0^{+\infty}\int_0^{2\pi}\Big\{r^{2N_1}e^{2\tilde U_{1,0}-\tilde U_{2,0}}(2Z_{2,1}-Z_{2,2})(2Z_{c_{21,1},1}-Z_{c_{21,1},2})^2 \\
&+4r^{2N_2}e^{2\tilde U_{2,0}-2\tilde U_{1,0}}(Z_{2,2}-Z_{2,1})(Z_{c_{21,1},2}-Z_{c_{21,1},1})^2\Big\}rdrd\theta\Big)c_{21,1} \\
&+\Big(\int_0^{+\infty}\int_0^{2\pi}\Big\{r^{2N_1}e^{2\tilde U_{1,0}-\tilde U_{2,0}}(2Z_{2,1}-Z_{2,2})(2Z_{c_{21,1},1}-Z_{c_{21,1},2})(2Z_{c_{32,1},1}-Z_{c_{32,1},2}) \\
&+4r^{2N_2}e^{2\tilde U_{2,0}-2\tilde U_{1,0}}(Z_{2,2}-Z_{2,1})(Z_{c_{21,1},2}-Z_{c_{21,1},1})(Z_{c_{32,1},2}-Z_{c_{32,1},1})\Big\}rdrd\theta\Big)c_{32,1}\Big] \\
&+\tilde{\mathcal{T}}_{11}(\mathbf{a})+O(\frac{|\mathbf{a}|^2}{\varepsilon})+O(\varepsilon)+O((1+|\xi|)|\mathbf{a}|^2+\varepsilon^2) \\
&=\xi_1(\tilde{\mathcal{A}}_1c_{21,1}+\tilde{\mathcal{B}}_1c_{32,1})+\xi_2(\tilde{\mathcal{A}}_2c_{21,1}+\tilde{\mathcal{B}}_2c_{32,1}) \\
&+\tilde{\mathcal{T}}_{11}(\mathbf{a})+O(\frac{|\mathbf{a}|^2}{\varepsilon})+O(\varepsilon)++O((1+|\xi|)|\mathbf{a}|^2+\varepsilon^2),
\end{aligned}
$$

where $\tilde{\mathcal{A}}_1, \tilde{\mathcal{A}}_2, \tilde{\mathcal{B}}_1, \tilde{\mathcal{B}}_2$ is in (3.76) and (3.77).

Similarly, we can get the other estimates.

□

From the above lemma, we have the following result:

Lemma 3.6. *Let* $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ *be a solution of (3.39). Then the coefficients* $m_i = 0$ *if and only if the parameters* $\mathbf{a}$ *satisfy*

$$\tilde{\mathbf{Q}}(\mathbf{a}) = \tilde{\mathcal{T}}(\mathbf{a}) + O(\frac{|\mathbf{a}|^2}{\varepsilon}) + O((1+|\xi|)|\mathbf{a}|^2) + O(\varepsilon), \tag{3.83}$$

where $\tilde{\mathbf{Q}}$

$$\begin{aligned}\tilde{\mathbf{Q}} &= \xi_1 \begin{pmatrix} \tilde{\mathcal{A}}_1 & 0 & 0 & 0 & 0 & 0 & \tilde{\mathcal{B}}_1 & 0 \\ 0 & \tilde{\mathcal{A}}_1 & 0 & 0 & 0 & 0 & 0 & \tilde{\mathcal{B}}_1 \\ 0 & 0 & \tilde{\mathcal{E}}_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{\mathcal{E}}_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{\mathcal{F}}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tilde{\mathcal{F}}_1 & 0 & 0 \\ \tilde{\mathcal{D}}_1 & 0 & 0 & 0 & 0 & 0 & \tilde{\mathcal{C}}_1 & 0 \\ 0 & \tilde{\mathcal{D}}_1 & 0 & 0 & 0 & 0 & 0 & \tilde{\mathcal{C}}_1 \end{pmatrix} + \xi_2 \begin{pmatrix} \tilde{\mathcal{A}}_2 & 0 & 0 & 0 & 0 & 0 & \tilde{\mathcal{B}}_2 & 0 \\ 0 & \tilde{\mathcal{A}}_2 & 0 & 0 & 0 & 0 & 0 & \tilde{\mathcal{B}}_2 \\ 0 & 0 & \tilde{\mathcal{E}}_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{\mathcal{E}}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{\mathcal{F}}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tilde{\mathcal{F}}_2 & 0 & 0 \\ \tilde{\mathcal{D}}_2 & 0 & 0 & 0 & 0 & 0 & \tilde{\mathcal{C}}_2 & 0 \\ 0 & \tilde{\mathcal{D}}_2 & 0 & 0 & 0 & 0 & 0 & \tilde{\mathcal{C}}_2 \end{pmatrix} \\ &= \xi_1 \tilde{\mathbf{Q}}_1 + \xi_2 \tilde{\mathbf{Q}}_2, \end{aligned} \tag{3.84}$$

and $\tilde{\mathcal{T}}$ *is a* 8×8 *matrix which is uniformly bounded and independent of* ξ_1, ξ_2.

Proof of Theorem 1.1 for $\mathbf{B}_2$ under Assumption (ii): Under the **Assumptions (ii)**, we will choose $\lambda_0 = 1$ and λ_1 large. The computations in the third part of the appendix show that

$$\begin{aligned} \tilde{\mathcal{A}}_1 &= \gamma_1 \lambda_1^{-2} + o(\lambda_1^{-2}), \\ \tilde{\mathcal{C}}_1 &= \gamma_2 \lambda_1^{-\frac{N_1+1}{N_1+N_2+2}} + o(\lambda_1^{-\frac{N_1+1}{N_1+N_2+2}}), \\ \tilde{\mathcal{E}}_1 &= \gamma_3 \lambda_1^{-\frac{2N_1+N_2+3}{N_1+N_2+2}} + o(\lambda_1^{-\frac{2N_1+N_2+3}{N_1+N_2+2}}), \\ \tilde{\mathcal{F}}_1 &= \gamma_4 \lambda_1^{-\frac{3N_1+2N_2+5}{N_1+N_2+2}} + o(\lambda_1^{-\frac{3N_1+2N_2+5}{N_1+N_2+2}}), \end{aligned}$$

and

$$\tilde{\mathcal{B}}_1 = \tilde{\mathcal{D}}_1 = \begin{cases} \lambda_1^{-1}\gamma_5 + O(\lambda_1^{-2}) \text{ if } N_1 = N_2, \\ 0 \text{ if } N_1 \neq N_2, \end{cases} \tag{3.85}$$

where $\gamma_1, \gamma_2, \gamma_3$ and γ_4 are non zero constants. So we have

$$\begin{aligned} \tilde{\mathcal{A}}_1 \tilde{\mathcal{C}}_1 - \tilde{\mathcal{B}}_1 \tilde{\mathcal{D}}_1 &= \begin{cases} \gamma_1 \gamma_2 \lambda_1^{-\frac{3N_1+2N_2+5}{N_1+N_2+2}} + o(\lambda_1^{-\frac{3N_1+2N_2+5}{N_1+N_2+2}}) \text{ if } \gamma_5 = 0 \\ \gamma_5^2 \lambda_1^{-2} + o(\lambda_1^{-2}) \text{ if } \gamma_5 \neq 0 \end{cases} \\ &\neq 0, \end{aligned}$$

and $\tilde{\mathcal{E}}_1, \tilde{\mathcal{F}}_1$ are both non-zero. Therefore, we choose ξ_1 large and $\xi_2 = 0$ to conclude that $\tilde{\mathbf{Q}}(\xi_1, \xi_2) - \tilde{\mathcal{T}}$ is non-degenerate. After fixing (λ_0, λ_1), (ξ_1, ξ_2), it is easy to see (3.76) can be solved with $\mathbf{a} = O(\varepsilon)$.

□

In the next section, we will prove our main theorem under the **Assumption (iii)**, i.e.,

$$\sum_{i=1}^{N_1} p_i = \sum_{j=1}^{N_2} q_j, N_1 = 1, N_2 \neq 1. \tag{3.86}$$

3.7 Proof of Theorem 1.1 for B_2 under Assumption (iii)

We are left to prove the theorem for $\sum_{i=1}^{N_1} p_i = \sum_{j=1}^{N_2} q_j, N_1 \neq N_2$ and one of N_i is 1. Without loss of generality, assume $N_1 = 1$ and $\sum_{i=1}^{N_1} p_i = \sum_{j=1}^{N_2} q_j = 0$. In this case, for the improvement of approximate solution in the $O(\varepsilon^2)$ term, we can not solve equation (3.31) in Section 3.4. Instead of solving (3.31), we can find a unique solution of the following equations which is guaranteed by Lemma 3.2:

$$\begin{cases} \Delta\psi_1 + |z|^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}}(2\psi_1 - \psi_2) = 2|z|^{4N_1} e^{4\tilde{U}_{1,0} - 2\tilde{U}_{2,0}} - |z|^{2(N_1+N_2)} e^{\tilde{U}_{2,0}} \\ \Delta\psi_2 + |z|^{2N_2} e^{2\tilde{U}_{2,0} - 2\tilde{U}_{1,0}}(2\psi_2 - 2\psi_1) = 2|z|^{4N_2} e^{4\tilde{U}_{2,0} - 4\tilde{U}_{1,0}} - 2|z|^{2(N_1+N_2)} e^{\tilde{U}_{2,0}}. \end{cases} \tag{3.87}$$

We use this unique solution as the new ψ_0, and proceed as before. Then by checking the previous proof, we can get that in this case, the error $\|E\|_* \leq C_0$ and we can get a solution v of (3.39) which satisfies

$$\|v\|_* \leq C_0, \tag{3.88}$$

for some positive constant C_0, and the following estimates hold:

$$\int_{\mathbb{R}^2} (N_{11}(v) + N_{12}(v))Z_{i,1}^* + (N_{21}(v) + N_{22}(v))Z_{i,2}^* dx = O(\varepsilon), \tag{3.89}$$

for $i = 3, \cdots, 10$.

Then the reduced problem we get is

$$\tilde{\mathbf{Q}}(\mathbf{a}) + \tilde{\mathcal{T}}(\mathbf{a}) + O((1 + |\xi|)|\mathbf{a}|^2) + O(1) + O(\varepsilon) = 0, \tag{3.90}$$

where the $O(1)$ term comes from the $O(1)$ term of the error E since we use the solution of (3.87) instead of (3.31) as the $O(\varepsilon^2)$ improvement. Recalling that

$\tilde{\mathbf{Q}} = \tilde{\mathbf{Q}}(\xi_1, \xi_2)$ depend on two free parameters ξ_1, ξ_2 and arguing as before, we can choose ξ_1 large enough. Then it is easy to get a solution of (3.90) with $\mathbf{a} = O(\xi_1^{-\alpha})$ for any $0 < \alpha < 1$.

$\square$

4 Appendix

This appendix contains three parts. In the first part, we show the classification and non-degeneracy results of the $\mathbf{B}_2$ Toda system with singular sources. In the second part, we give the technical calculations of the matrix $\mathbf{Q}$ while in the third part, we give the technical calculations of the matrix $\tilde{\mathbf{Q}}$.

4.1 Classification and Non-degeneracy of $\mathbf{B}_2$ Toda System with Singular Sources

In [16], Lin-Wei-Ye obtained the classification and non-degeneracy of solutions for $SU(n+1)$ Toda system with one single source

$$-\Delta u_i = \sum_{j=1}^{n} a_{ij} e^{u_j} + 4\pi\gamma_i\delta_0, \text{ in } \mathbb{R}^2, i = 1, ..., n \tag{4.1}$$

where $A = (a_{ij})$ is the Cartan matrix for $SU(n+1)$, given by

$$A := (a_{ij}) = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & \dots & -1 & 2 & -1 \\ 0 & \dots & & -1 & 2 \end{pmatrix}. \tag{4.2}$$

In this subsection, we consider the $\mathbf{B}_2$ Toda system:

$$\begin{cases} \triangle u + 2e^u - e^v = 4\pi\gamma_1\delta_0 & \text{in } \mathbb{R}^2 \\ \triangle v + 2e^v - 2e^u = 4\pi\gamma_2\delta_0 & \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^u < +\infty, \ \int_{\mathbb{R}^2} e^v < +\infty. \end{cases} \tag{4.3}$$

An important observation is that Toda system with $\mathbf{B}_2$ can be embedded into the $\mathbf{A_3}$Toda system

$$\begin{cases} \triangle u_1 + 2e^{u_1} - e^{u_2} = 4\pi\gamma_1^{'}\delta_0 \quad \text{in } \mathbb{R}^2 \\ \triangle u_2 + 2e^{u_2} - e^{u_1} - e^{u_3} = 4\pi\gamma_2^{'}\delta_0 \quad \text{in } \mathbb{R}^2 \\ \triangle u_3 + 2e^{u_3} - e^{u_2} = 4\pi\gamma_3^{'}\delta_0 \quad \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^{u_i} < +\infty, i = 1, 2, 3. \end{cases} \tag{4.4}$$

The transformation from (4.4) to (4.3) is the following:

$$u_1 = u, u_2 = v, u_3 = u; \gamma_1^{'} = \gamma_1, \gamma_2^{'} = \gamma_2, \gamma_3 = \gamma_1^{'}.$$

In other words, Toda system with $\mathbf{B}_2$ corresponds to solutions of Toda system with $\mathbf{A}_3$ under the following group action

$$u_1 = u_3. \tag{4.5}$$

As a consequence, we just need to take the solutions of Lin-Wei-Ye [16] in A_3 case with $\gamma_1 = \gamma_3$ and compute the solutions under the group action (4.5).

We define

$$\begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{pmatrix} = \mathbf{B}_2^{-1} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{4.6}$$

Then the system (4.3) is transformed to

$$\begin{cases} \Delta\tilde{w}_1 + e^{2\tilde{w}_1 - \tilde{w}_2} = 4\pi\alpha_1\delta_0 \\ \Delta\tilde{w}_2 + e^{2\tilde{w}_2 - 2\tilde{w}_1} = 4\pi\alpha_2\delta_0 \\ \int_{\mathbb{R}^2} e^{2w_1 - w_2} < +\infty, \int_{\mathbb{R}^2} e^{2w_2 - 2w_1} < +\infty \end{cases} \tag{4.7}$$

where $\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \mathbf{B}_2^{-1} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}$. We introduce the notation $(w_1, w_2, w_3)^t = \mathbf{A}_3^{-1}(u_1, u_2, u_3)^t$, and $(\alpha_1, \alpha_2, \alpha_3)^t = \mathbf{A}_3^{-1}(\gamma_1', \gamma_2', \gamma_3')^t$. Then (4.4) is transformed to

$$\Delta w_i + e^{u_i} = 4\pi\alpha_i\delta_0, \text{ in } \mathbb{R}^2, \quad \text{where } \alpha_i = \sum_{j=1}^{3} a^{ij}\gamma_j'. \tag{4.8}$$

In order to find the solution of (4.7), we need to find the solutions w of (4.8) under the group action $w_1 = w_3$, and then $\begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$. The following theorem gives a complete classification.

Theorem 4.1. *Let $\tilde{w}$ be a solution of (4.7), then $\tilde{w}_1$ can be expressed as*

$$e^{-\tilde{w}_1} = |z|^{-2\alpha_1}(\lambda_0 + \sum_{i=1}^{3} \lambda_i |P_i(z)|^2), \tag{4.9}$$

where

$$P_i(z) = z^{\mu'_1+\cdots+\mu'_i} + \sum_{j=0}^{i-1} c_{ij} z^{\mu'_1+\cdots+\mu'_j}, \tag{4.10}$$

$\mu'_i = \gamma'_i + 1$, and c_{ij} are complex numbers and $\lambda_i > 0$, satisfy

$$\begin{cases} \lambda_2 = \frac{1}{(8\mu'_1\mu'_2(\mu'_1+\mu'_2))^2\lambda_1}, \\ \lambda_3 = \frac{1}{(8\mu'_1(\mu'_1+\mu'_2)(2\mu'_1+\mu'_2))^2\lambda_0}, \\ c_{10} = \frac{\mu'_2}{2\mu'_1+\mu'_2}c_{32}, \;\; c_{20} = \frac{-(c_{31}-c_{21}c_{32})\mu'_2}{(2\mu'_1+\mu'_2)}. \end{cases} \tag{4.11}$$

Moreover,

- *if $\gamma_1, \gamma_2 \in \mathbb{N}$, the solution space is a ten dimensional smooth manifold;*
- *if $\gamma_1 \in \mathbb{N}, \gamma_2 \notin \mathbb{N}$, then $c_{21} = c_{30} = c_{31} = 0$, the solution manifold is four dimensional;*
- *if $\gamma_1 \notin \mathbb{N}, \gamma_2 \in \mathbb{N}$, and $2\gamma_1 \in \mathbb{N}$, then $c_{31} = c_{32} = 0$, the solution manifold is six dimensional;*
- *if $\gamma_1 \notin \mathbb{N}, \gamma_2 \in \mathbb{N}$, but $2\gamma_1 \notin \mathbb{N}$, then $c_{30} = c_{31} = c_{32} = 0$, the solution manifold is four dimensional;*
- *if $\gamma_1 \notin \mathbb{N}, \gamma_2 \notin \mathbb{N}$, and $\gamma_1 + \gamma_2 \in \mathbb{N}$, then $c_{21} = c_{30} = c_{32} = 0$, the solution manifold is four dimensional;*
- *if $\gamma_1 \notin \mathbb{N}, \gamma_2 \notin \mathbb{N}$, and $2\gamma_1 + \gamma_2 \in \mathbb{N}$, then $c_{21} = c_{31} = c_{32} = 0$, the solution manifold is four dimensional;*
- *if $\gamma_1 \notin \mathbb{N}, \gamma_2 \notin \mathbb{N}$, and $2\gamma_1 + \gamma_2 \notin \mathbb{N}, \gamma_1 + \gamma_2 \notin \mathbb{N}$, then $c_{21} = c_{30} = c_{31} = c_{32} = 0$, the solution manifold is two dimensional. All the solutions must be radial.*

Remark 4.1. *The maximal dimension of the space of the solutions is 10, which coincides with the dimension of the Lie algebra associated with $\mathbf{B}_2$.*

Proof:

By Theorem 1.1 of Lin-Wei-Ye [16] with $n = 3$, the solution of (4.8) can be expressed as

$$e^{-w_1} = |z|^{-2\alpha_1}(\lambda_0 + \sum_{i=1}^{3} \lambda_i |P_i(z)|^2), \tag{4.12}$$

where

$$P_i(z) = z^{\mu'_1+\cdots+\mu'_i} + \sum_{j=0}^{i-1} c_{ij} z^{\mu'_1+\cdots+\mu'_j}, \tag{4.13}$$

$\mu'_i = \gamma'_i + 1$, and c_{ij} are complex numbers and $\lambda_i > 0$, satisfying

$$\lambda_0\lambda_1\lambda_2\lambda_3 = 2^{-12}\Pi_{1\leq i\leq j\leq 3}(\sum_{k=i}^{j} \mu'_k)^{-2}. \tag{4.14}$$

Using formula (5.16) in Lin-Wei-Ye [16], we have

$$e^{-w_3} = |z|^{-2\alpha_1}(\lambda'_0 + \sum_{i=1}^{3} \lambda'_i |P'_i(z)|^2), \tag{4.15}$$

where

$$P'_i(z) = z^{\mu'_1+\cdots+\mu'_i} + \sum_{j=0}^{i-1} c'_{ij} z^{\mu'_1+\cdots+\mu'_j}. \tag{4.16}$$

If we denote by $L_i = \sqrt{\lambda_i}, L'_i = \sqrt{\lambda'_i}$, we have

$$\begin{aligned}
L'_0 &= 8L_0L_1L_2\mu'_1\mu'_2(\mu'_1 + \mu'_2), \\
L'_1 &= 8L_0L_1L_3\mu'_1(\mu'_1 + \mu'_2)(2\mu'_1 + \mu'_2), \\
L'_2 &= 8L_0L_2L_3\mu'_1(\mu'_1 + \mu'_2)(2\mu'_1 + \mu'_2), \\
c'_{10} &= \frac{c_{32}\mu'_2}{(2\mu'_1 + \mu'_2)}, \\
c'_{20} &= \frac{-(c_{31} - c_{21}c_{32})\mu'_2}{(2\mu'_1 + \mu'_2)}, \ c'_{21} = c_{21}, \\
c'_{30} &= (c_{30} - c_{10}c_{31} - c_{20}c_{32} + c_{10}c_{21}c_{32}), \\
c'_{31} &= \frac{-(c_{20} - c_{10}c_{21})(2\mu'_1 + \mu'_2)}{\mu'_2}, \ c'_{32} = \frac{c_{10}(2\mu'_1 + \mu'_2)}{\mu'_2}.
\end{aligned}$$

Since $w_1 = w_3$, we obtain the following relations:

$$L_2 = \frac{1}{8L_1\mu_1'\mu_2'(\mu_1' + \mu_2')}, \quad L_3 = \frac{1}{8L_0\mu_1'(\mu_1' + \mu_2')(2\mu_1' + \mu_2')}, \quad (4.17)$$

$$c_{10} = \frac{\mu_2'}{2\mu_1' + \mu_2'}c_{32}, \quad c_{20} = \frac{-(c_{31} - c_{21}c_{32})\mu_2'}{(2\mu_1' + \mu_2')}. \quad (4.18)$$

We conclude that w_1 satisfies (4.9) and (4.11). The other parts of the theorem follow from [16].

□

From Theorem 4.1,we can get that the solutions of (4.7) depend on 10 parameters $\lambda_0, \lambda_1, c_{21}, c_{30}, c_{31}$ and c_{32}. By formula (5.16) in [16], we get the radial solution of this system can be written as

$$\begin{aligned}
\rho_{1,B}^{-1} &= r^{2\alpha_1}e^{-w_{1,0}} \\
&= \lambda_0 + \lambda_1 r^{2\mu_1'} + \lambda_2 r^{2(\mu_1'+\mu_2')} + \lambda_3 r^{2(2\mu_1'+\mu_2')}, \\
\rho_{2,B}^{-1} &= r^{2\alpha_2}e^{-w_{2,0}} \\
&= 4\Big[\lambda_0\lambda_1\mu_1'^2 + \lambda_0\lambda_2(\mu_1' + \mu_2')^2 r^{2\mu_2'} + (\lambda_0\lambda_3(2\mu_1' + \mu_2')^2 + \lambda_1\lambda_2\mu_2'^2)r^{2(\mu_1'+\mu_2')} \\
&+ \lambda_1\lambda_3(\mu_1' + \mu_2')^2 r^{2(2\mu_1'+\mu_2')} + \lambda_2\lambda_3\mu_1'^2 r^{4(\mu_1'+\mu_2')}\Big],
\end{aligned}$$

where the parameters are defined in (4.11). We state the following non-degeneracy result which may be useful in constructing non-topological solutions of the Chern-Simons system:

Corollary 4.1. *(Non-degeneracy) Assume $\gamma_1, \gamma_2 \in \mathbb{N}$. The set of solutions corresponding to the linearized operator of (4.7) is exactly ten dimensional. More precisely, if $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ satisfies $|\phi(z)| \le C(1 + |z|)^\alpha$ for some $0 \le \alpha < 1$, and*

$$\begin{cases} \Delta\phi_1 + e^{2w_{1,0}-w_{2,0}}(2\phi_1 - \phi_2) = 0 \\ \Delta\phi_2 + e^{2w_{2,0}-2w_{1,0}}(2\phi_2 - 2\phi_1) = 0, \end{cases} \quad (4.19)$$

then ϕ belongs to the following linear space $\mathcal{K}$: the span of

$$\{w_{\lambda_0}, w_{\lambda_1}, w_{c_{21,1}}, w_{c_{21,2}}, w_{c_{30,1}}, w_{c_{30,2}}, w_{c_{31,1}}, w_{c_{31,2}}, w_{c_{32,1}}, w_{c_{32,2}}\}, \quad (4.20)$$

where

$$\begin{cases} w_{1,\lambda_0} = -\rho_{1,B}\Big[1 - \frac{r^{2(2\mu_1'+\mu_2')}}{(8\mu_1'(\mu_1'+\mu_2')(2\mu_1'+\mu_2'))^2\lambda_0^2}\Big], \\ w_{2,\lambda_0} = -4\rho_{2,B}\Big[\mu'^2_1\lambda_1 + \frac{r^{2\mu_2'}}{(8\mu_1'\mu_2')^2\lambda_1} \\ \qquad -\frac{r^{4(\mu_1'+\mu_2')}}{(8(\mu_1'+\mu_2')(2\mu_1'+\mu_2'))^2(8\mu_1'\mu_2'(\mu_1'+\mu_2'))^2\lambda_1\lambda_0^2} - \frac{\lambda_1 r^{2(2\mu_1'+\mu_2')}}{(8\mu_1'(2\mu_1'+\mu_2'))^2\lambda_0^2}\Big], \end{cases}$$

$$\begin{cases} w_{1,\lambda_1} = -\rho_{1,B} r^{2\mu_1'}\Big[1 - \frac{r^{2\mu_2'}}{(8\mu_1'\mu_2'(\mu_1'+\mu_2'))^2\lambda_1^2}\Big], \\ w_{2,\lambda_1} = -4\rho_{2,B}\Big[\mu'^2_1\lambda_0 - \frac{\lambda_0 r^{2\mu_2'}}{(8\mu_1'\mu_2')^2\lambda_1^2} + \frac{r^{2(2\mu_1'+\mu_2')}}{(8\mu_1'(2\mu_1'+\mu_2'))^2\lambda_0} \\ \qquad -\frac{r^{4(\mu_1'+\mu_2')}}{(8(\mu_1'+\mu_2')(2\mu_1'+\mu_2'))^2(8\mu_1'\mu_2'(\mu_1'+\mu_2'))^2\lambda_1^2\lambda_0}\Big], \end{cases}$$

and

$$\begin{cases} w_{1,c_{21,1}} = -\rho_{1,B}\lambda_2 r^{2\mu_1'+\mu_2'}\cos\mu_2'\theta, \\ w_{2,c_{21,1}} = -4\rho_{2,B}\lambda_2\mu_1'(\mu_1'+\mu_2')[\lambda_0 r^{\mu_2'} + \lambda_3 r^{4\mu_1'+3\mu_2'}]\cos\mu_2'\theta, \end{cases}$$

$$\begin{cases} w_{1,c_{30,1}} = -\rho_{1,B}\lambda_3 r^{2\mu_1'+\mu_2'}\cos(2\mu_1'+\mu_2')\theta, \\ w_{2,c_{30,1}} = 4\rho_{2,B}\lambda_3\mu_1'(\mu_1'+\mu_2')[\lambda_1 r^{2\mu_1'+\mu_2'} + \lambda_2 r^{2\mu_1'+3\mu_2'}]\cos(2\mu_1'+\mu_2')\theta, \end{cases}$$

$$\begin{cases} w_{1,c_{31,1}} = -\rho_{1,B}(\lambda_3 r^{3\mu_1'+\mu_2'} - \frac{\lambda_2\mu_2'}{(2\mu_1'+\mu_2')} r^{\mu_1'+\mu_2'})\cos(\mu_1'+\mu_2')\theta, \\ w_{2,c_{31,1}} = -4\rho_{2,B}\Big[(\lambda_0\lambda_3\mu_1'(2\mu_1'+\mu_2') + \frac{\lambda_1\lambda_2\mu_1'\mu'^2_2}{(2\mu_1'+\mu_2')}) r^{\mu_1'+\mu_2'} \\ \qquad -2\lambda_2\lambda_3\mu_1'\mu_2' r^{3\mu_1'+3\mu_2'}\Big]\cos(\mu_1'+\mu_2')\theta, \end{cases}$$

$$\begin{cases} w_{1,c_{32,1}} = -\rho_{1,B}(\frac{\lambda_1\mu_2'}{(2\mu_1'+\mu_2')} r^{\mu_1'} + \lambda_3 r^{3\mu_1'+2\mu_2'})\cos\mu_1'\theta, \\ w_{2,c_{32,1}} = -4\rho_{2,B}\Big[(\lambda_0\lambda_3(\mu_1'+\mu_2')(2\mu_1'+\mu_2') + \frac{\lambda_1\lambda_2\mu'^2_2(\mu_1'+\mu_2')}{(2\mu_1'+\mu_2')}) r^{\mu_1'+2\mu_2'} \\ \qquad +2\lambda_1\lambda_3\mu_2'(\mu_2'+\mu_1') r^{3\mu_1'+2\mu_2'}\Big]\cos\mu_1'\theta, \end{cases}$$

and by replacing the cos *terms by* sin, *we get* $w_{i,c_{jk,2}}$.

Finally, using Theorem 1.3 of [16], we have the following quantization result:

Corollary 4.2. *Suppose* $u = (u_1, u_2)$ *is the solution of (4.3). Then the following hold:*

$$\int_{\mathbb{R}^2} e^{u_1} dx = 4\pi(2\gamma_1 + \gamma_2 + 3), \ \int_{\mathbb{R}^2} e^{u_2} dx = 8\pi(\gamma_1 + \gamma_2 + 2) \qquad (4.21)$$

and $u_1(z) = -(4+2\gamma_1)\log|z| + O(1),\ u_2(z) = -(4+2\gamma_2)\log|z| + O(1)$ *as* $|z| \to \infty$.

4.2 The Calculations of the Matrix $\mathbf{Q}_1$

In this section, we give the computations of $\mathbf{Q}_1$. As mentioned in Section 2, we set $\mu_1 = 1$ and let μ_2 large enough. For simplicity, we denote by $\alpha = N_1 + 1$, $\beta = N_2 + 1$. We will show that for μ_2 large, the elements of the matrix $\mathbf{Q}_1$ and $\mathbf{Q}_2$ can not both be zero.

Case I: The calculations of $\mathbf{Q}_1$ when $N_1 \neq N_2$

Suppose $N_1 \neq N_2$, we have $\langle Z_i, Z_j \rangle = 0$ for $i \neq j$ and $i, j = 3, \ldots, 8$. So the coefficient matrix is the same as above with $\mathcal{B}_i = \mathcal{D}_i = 0$, i.e the matrix is diagonal. Without loss of generality, we assume $\alpha > \beta$. The asymptotic behavior of the coefficient matrix in the case of $(\mu_1, \mu_2) = (1, \mu_2)$ as $\mu_2 \to +\infty$ can be estimated as follows.

Recall that

$$\begin{aligned} \rho_1 &= \frac{1}{c_1 + c_2|z^\alpha|^2 + c_3|z^{\alpha+\beta}|^2}, \\ \rho_2 &= \frac{1}{c_1' + c_2'|z^\beta|^2 + c_3'|z^{\alpha+\beta}|^2}, \end{aligned}$$

where after ignoring a common constant $\frac{1}{4\Gamma}$,

$$\begin{aligned} &c_1 = \beta,\ c_2 = (\alpha+\beta)\mu_2,\ c_3 = \frac{\alpha}{\mu_2}, \\ &c_1' = \alpha\mu_2,\ c_2' = \frac{\alpha+\beta}{\mu_2},\ c_3' = \beta. \end{aligned}$$

First by the change of variable $t = r^2$, from (2.84), one can get that

$$\begin{aligned} \frac{\mathcal{A}_1}{\pi} &= \frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} \Big[t^{\alpha-1}\frac{\rho_1^2}{\rho_2}(2Z_{1,1} - Z_{1,2})(2Z_{a_1,1} - Z_{a_1,2})^2 \\ &+ t^{\beta-1}\frac{\rho_2^2}{\rho_1}(2Z_{1,2} - Z_{1,1})(2Z_{a_1,2} - Z_{a_1,1})^2 \Big] dt d\theta \\ &= A_1 + A_2 + A_3 + \cdots + A_{12}, \end{aligned}$$

where $A_i, i = 1, \cdots, 12$ are defined below.

First we consider A_1:

$$\begin{aligned}
A_1 &= \frac{1}{\pi}\int_0^{2\pi}\int_0^\infty 8t^{\alpha-1}\frac{\rho_1^2}{\rho_2}Z_{1,1}Z_{a_1,1}^2 dtd\theta \\
&= \int_0^\infty 8(\alpha+\beta)^2 t^{2\alpha-1}\frac{\rho_1^5}{\rho_2}\mu_2^2(\beta-\frac{\alpha}{\mu_2}t^{\alpha+\beta})dt \\
&= \int_0^\infty \frac{8(\alpha+\beta)^2 t^{2\alpha-1}\mu_2^2(\beta-\frac{\alpha}{\mu_2}t^{\alpha+\beta})(\alpha\mu_2+\frac{\alpha+\beta}{\mu_2}t^\beta+\beta t^{\alpha+\beta})}{(\beta+(\alpha+\beta)\mu_2 t^\alpha+\frac{\alpha}{\mu_2}t^{\alpha+\beta})^5}dt \\
&\quad (t=\mu_2^{-\frac{1}{\alpha}}s) \\
&= \mu_2\int_0^\infty \frac{8(\alpha+\beta)^2\alpha\beta s^{2\alpha-1}}{(\beta+(\alpha+\beta)s^\alpha)^5}ds+\mu_2^{-\frac{\alpha+\beta}{\alpha}}\int_0^\infty \frac{8(\alpha+\beta)^3 s^{2\alpha+\beta-1}(\beta+(\beta-\alpha)s^\alpha)}{(\beta+(\alpha+\beta)s^\alpha)^5}ds+h.o.t.
\end{aligned}$$

Then we consider A_2:

$$\begin{aligned}
A_2 &= \frac{2}{\pi}\int_0^{2\pi}\int_0^\infty t^{\alpha-1}\frac{\rho_1^2}{\rho_2}Z_{1,1}Z_{a_1,2}^2 dtd\theta \\
&= \int_0^\infty 2(\alpha+\beta)^2 t^{2\alpha+2\beta-1}\rho_1^3\rho_2(\beta-\frac{\alpha}{\mu_2}t^{\alpha+\beta})dt \\
&= \int_0^\infty \frac{2(\alpha+\beta)^2 t^{2\alpha+2\beta-1}(\beta-\frac{\alpha}{\mu_2}t^{\alpha+\beta})}{(\beta+(\alpha+\beta)\mu_2 t^\alpha+\frac{\alpha}{\mu_2}t^{\alpha+\beta})^3(\alpha\mu_2+\frac{\alpha+\beta}{\mu_2}t^\beta+\beta t^{\alpha+\beta})}dt.
\end{aligned}$$

We divide the estimates of A_2 into three cases.

If $\alpha > 2\beta$, we use $t = \mu_2^{-\frac{1}{\alpha}}s$ to obtain that

$$A_2 = \mu_2^{-\frac{3\alpha+2\beta}{\alpha}}\int_0^\infty \frac{2(\alpha+\beta)^2\beta s^{2\alpha+2\beta-1}}{\alpha(\beta+(\alpha+\beta)s^\alpha)^3}ds+h.o.t.$$

If $\alpha < 2\beta$, we use $t = \mu_2^{\frac{2}{\beta}}s$ to obtain that

$$A_2 = -\mu_2^{-\frac{2\alpha}{\beta}}\int_0^\infty \frac{2(\alpha+\beta)^2\alpha s^{2\beta-\alpha-1}}{\beta(\alpha+\beta+\alpha s^\beta)^3}ds+h.o.t.$$

If $\alpha = 2\beta$, one can get that

$$\begin{aligned} A_2 &= \int_0^\infty \frac{2(\alpha+\beta)^2 t^{6\beta-1}(\beta - \frac{\alpha}{\mu_2}t^{3\beta})}{\beta(\beta+(\alpha+\beta)\mu_2 t^{2\beta} + \frac{\alpha}{\mu_2}t^{3\beta})^3(\alpha\mu_2 + \frac{\alpha+\beta}{\mu_2}t^\beta + \beta t^{3\beta})}dt \\ & (x = t^\beta) \\ &= \int_0^\infty \frac{2(\alpha+\beta)^2 x^5(\beta - \frac{\alpha}{\mu_2}x^3)}{(\beta+(\alpha+\beta)\mu_2 x^2 + \frac{\alpha}{\mu_2}x^3)^3(\alpha\mu_2 + \frac{\alpha+\beta}{\mu_2}x + \beta x^3)}dx. \end{aligned}$$

In this case, we divide the integral into two parts:

$$\begin{aligned} A_2 &= \int_0^{\mu_2^2} + \int_{\mu_2^2}^\infty \frac{2(\alpha+\beta)^2 x^5(\beta - \frac{\alpha}{\mu_2}x^3)}{(\beta+(\alpha+\beta)\mu_2 x^2 + \frac{\alpha}{\mu_2}x^3)^3(\alpha\mu_2 + \frac{\alpha+\beta}{\mu_2}x + \beta x^3)}dx \\ &= A_{21} + A_{22}. \end{aligned}$$

For A_{21}, we use the change of variables $x = \mu_2^{-\frac{1}{2}}s$, and obtain that

$$\begin{aligned} A_{21} &= \mu_2^{-4}\int_0^{\mu_2^{\frac{5}{2}}} \frac{2(\alpha+\beta)^2\beta s^5}{\alpha(\beta+(\alpha+\beta)s^2)^3}ds + h.o.t \\ \tilde{s} = s^2 & \\ &= \mu_2^{-4}\int_0^{\mu_2^5} \frac{(\alpha+\beta)^2\beta\tilde{s}^2}{\alpha(\beta+(\alpha+\beta)\tilde{s})^3}d\tilde{s} + h.o.t \\ &= \frac{5\beta}{\alpha(\alpha+\beta)}\mu_2^{-4}\ln\mu_2 + h.o.t, \end{aligned}$$

while for A_{22}, we have

$$\begin{aligned} |A_{22}| &\leq c\int_{\mu_2^2}^\infty \frac{x^5\cdot\mu_2^{-1}\cdot x^3}{\mu_1^{-3}x^9\cdot x^3}dx \\ &\leq c\mu_2^{-4}. \end{aligned}$$

Combining the above two estimates, we get that for $\alpha = 2\beta$

$$\begin{aligned} A_2 &= O(\mu_2^{-4}\ln\mu_2) \\ &= o(\mu_2^{-\frac{\alpha}{\alpha+\beta}}). \end{aligned}$$

Next we estimate the remaining integral $A_3 - A_{12}$:

$$
\begin{aligned}
A_3 &= -\frac{8}{\pi}\int_0^{2\pi}\int_0^{\infty} t^{\alpha-1}\frac{\rho_1^2}{\rho_2} Z_{1,1}Z_{a_1,1}Z_{a_1,2}dtd\theta \\
&= -8\int_0^{\infty} t^{2\alpha+\beta-1}\rho_1^4(\beta - \frac{\alpha}{\mu_2}t^{\alpha+\beta})(\alpha+\beta)^2\mu_2 dt \\
&= -8\int_0^{\infty} \frac{t^{2\alpha+\beta-1}(\beta - \frac{\alpha}{\mu_2}t^{\alpha+\beta})(\alpha+\beta)^2\mu_2}{(\beta+(\alpha+\beta)\mu_2 t^{\alpha} + \frac{\alpha}{\mu_2}t^{\alpha+\beta})^4}dt \\
&\quad (t = \mu_2^{-\frac{1}{\alpha}}s) \\
&= \mu_2^{-\frac{\alpha+\beta}{\alpha}}\int_0^{\infty}\frac{-8(\alpha+\beta)^2\beta s^{2\alpha+\beta-1}}{(\beta+(\alpha+\beta)s^{\alpha})^4}ds + h.o.t,
\end{aligned}
$$

$$
\begin{aligned}
A_4 &= -\frac{4}{\pi}\int_0^{2\pi}\int_0^{\infty} t^{\alpha-1}\frac{\rho_1^2}{\rho_2} Z_{1,2}Z_{a_1,1}^2 dtd\theta \\
&= -4\int_0^{\infty}(\alpha+\beta)^2\rho_1^4\mu_2^2 t^{2\alpha-1}(\alpha\mu_2 - \beta t^{\alpha+\beta})dt \\
&= -4\int_0^{\infty}\frac{(\alpha+\beta)^2\mu_2^2 t^{2\alpha-1}(\alpha\mu_2 - \beta t^{\alpha+\beta})}{(\beta+(\alpha+\beta)\mu_2 t^{\alpha} + \frac{\alpha}{\mu_2}t^{\alpha+\beta})^4}dt \\
&\quad (t = \mu_2^{-\frac{1}{\alpha}}s) \\
&= \mu_2\int_0^{\infty}\frac{-4(\alpha+\beta)^2\alpha s^{2\alpha-1}}{(\beta+(\alpha+\beta)s^{\alpha})^4}ds + \mu_2^{-\frac{\alpha+\beta}{\alpha}}\int_0^{\infty}\frac{4(\alpha+\beta)^2\beta s^{3\alpha+\beta-1}}{(\beta+(\alpha+\beta)s^{\alpha})^4}ds + h.o.t,
\end{aligned}
$$

$$
\begin{aligned}
A_5 &= -\frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty} t^{\alpha-1}\frac{\rho_1^2}{\rho_2} Z_{1,2}Z_{a_1,2}^2 dtd\theta \\
&= -\int_0^{\infty}(\alpha+\beta)^2\rho_1^2\rho_2^2 t^{2\alpha+2\beta-1}(\alpha\mu_2 - \beta t^{\alpha+\beta})dt \\
&= -\int_0^{\infty}\frac{(\alpha+\beta)^2 t^{2\alpha+2\beta-1}(\alpha\mu_2 - \beta t^{\alpha+\beta})}{(\beta+(\alpha+\beta)\mu_2 t^{\alpha} + \frac{\alpha}{\mu_2}t^{\alpha+\beta})^2(\alpha\mu_2 + \frac{\alpha+\beta}{\mu_2}t^{\beta} + \beta t^{\alpha+\beta})^2}dt \\
&\quad (t = \mu_2^{\frac{1}{\alpha+\beta}}s) \\
&= -\mu_2^{-\frac{3\alpha+\beta}{\alpha+\beta}}\int_0^{\infty}\frac{(\alpha - \beta s^{\alpha+\beta})s^{2\beta-1}}{(\alpha+\beta s^{\alpha+\beta})^2}ds + h.o.t,
\end{aligned}
$$

$$
\begin{aligned}
A_6 &= \frac{4}{\pi}\int_0^{2\pi}\int_0^{\infty} t^{\alpha-1}\frac{\rho_1^2}{\rho_2} Z_{1,2}Z_{a_1,1}Z_{a_1,2}dtd\theta \\
&= \int_0^{\infty} 4t^{2\alpha+\beta-1}\rho_1^3\rho_2(\alpha+\beta)^2\mu_2(\alpha\mu_2-\beta t^{\alpha+\beta})dt \\
&= \int_0^{\infty} \frac{4t^{2\alpha+\beta-1}(\alpha+\beta)^2\mu_2(\alpha\mu_2-\beta t^{\alpha+\beta})}{(\beta+(\alpha+\beta)\mu_2 t^{\alpha}+\frac{\alpha}{\mu_2}t^{\alpha+\beta})^3(\alpha\mu_2+\frac{\alpha+\beta}{\mu_2}t^{\beta}+\beta t^{\alpha+\beta})}dt \\
&\quad (t=\mu_2^{-\frac{1}{\alpha}}s) \\
&= \mu_2^{-\frac{\alpha+\beta}{\alpha}}\int_0^{\infty}\frac{4(\alpha+\beta)^2 s^{2\alpha+\beta-1}}{(\beta+(\alpha+\beta)s^{\alpha})^3}ds + h.o.t,
\end{aligned}
$$

$$
\begin{aligned}
A_7 &= \frac{8}{\pi}\int_0^{2\pi}\int_0^{\infty} t^{\beta-1}\frac{\rho_2^2}{\rho_1} Z_{1,2}Z_{a_1,2}^2 dtd\theta \\
&= \int_0^{\infty} 8t^{\alpha+3\beta-1}\frac{\rho_2^5}{\rho_1}(\alpha\mu_2-\beta t^{\alpha+\beta})(\alpha+\beta)^2 dt \\
&= \int_0^{\infty} \frac{8t^{\alpha+3\beta-1}(\alpha\mu_2-\beta t^{\alpha+\beta})(\alpha+\beta)^2(\beta+(\alpha+\beta)\mu_2 t^{\alpha}+\frac{\alpha}{\mu_2}t^{\alpha+\beta})}{(\alpha\mu_2+\frac{\alpha+\beta}{\mu_2}t^{\beta}+\beta t^{\alpha+\beta})^5}dt \\
&\quad (t=\mu_2^{\frac{1}{\alpha+\beta}}s) \\
&= \mu_2^{-\frac{\alpha}{\alpha+\beta}}\int_0^{\infty}\frac{8(\alpha+\beta)^3 s^{2\alpha+3\beta-1}(\alpha-\beta s^{\alpha+\beta})}{(\alpha+\beta s^{\alpha+\beta})^5}ds + h.o.t,
\end{aligned}
$$

$$
\begin{aligned}
A_8 &= \frac{2}{\pi}\int_0^{2\pi}\int_0^{\infty} t^{\beta-1}\frac{\rho_2^2}{\rho_1} Z_{1,2}Z_{a_1,1}^2 dtd\theta \\
&= \int_0^{\infty} 2t^{\alpha+\beta-1}\rho_2^3\rho_1(\alpha\mu_2-\beta t^{\alpha+\beta})(\alpha+\beta)^2\mu_2^2 dt \\
&= \int_0^{\infty} \frac{2t^{\alpha+\beta-1}(\alpha\mu_2-\beta t^{\alpha+\beta})(\alpha+\beta)^2\mu_2^2}{(\alpha\mu_2+\frac{\alpha+\beta}{\mu_2}t^{\beta}+\beta t^{\alpha+\beta})^3(\beta+(\alpha+\beta)\mu_2 t^{\alpha}+\frac{\alpha}{\mu_2}t^{\alpha+\beta})}dt \\
&\quad (t=\mu_2^{\frac{1}{\alpha+\beta}}s) \\
&= \mu_2^{-\frac{\alpha}{\alpha+\beta}}\int_0^{\infty}\frac{2(\alpha+\beta)s^{\beta-1}(\alpha-\beta s^{\alpha+\beta})}{(\alpha+\beta s^{\alpha+\beta})^3}ds + h.o.t,
\end{aligned}
$$

$$\begin{aligned}
A_9 &= -\frac{8}{\pi}\int_0^{2\pi}\int_0^{\infty} t^{\beta-1}\frac{\rho_2^2}{\rho_1}Z_{1,1}Z_{a_1,1}Z_{a_1,2}dtd\theta \\
&= -\int_0^{\infty} 8t^{\alpha+2\beta-1}\rho_2^3\rho_1(\beta-\frac{\alpha}{\mu_2}t^{\alpha+\beta})(\alpha+\beta)^2\mu_2 dt \\
&= -\int_0^{\infty}\frac{8t^{\alpha+2\beta-1}(\beta-\frac{\alpha}{\mu_2}t^{\alpha+\beta})(\alpha+\beta)^2\mu_2}{(\alpha\mu_2+\frac{\alpha+\beta}{\mu_2}t^{\beta}+\beta t^{\alpha+\beta})^3(\beta+(\alpha+\beta)\mu_2 t^{\alpha}+\frac{\alpha}{\mu_2}t^{\alpha+\beta})}dt \\
& \quad (t=\mu_2^{\frac{1}{\alpha+\beta}}s) \\
&= -8\mu_2^{-\frac{3\alpha+\beta}{\alpha+\beta}}\int_0^{\infty}\frac{(\alpha+\beta)s^{2\beta-1}(\beta-\alpha s^{\alpha+\beta})}{(\alpha+\beta s^{\alpha+\beta})^3}ds+h.o.t,
\end{aligned}$$

$$\begin{aligned}
A_{10} &= -\frac{4}{\pi}\int_0^{2\pi}\int_0^{\infty} t^{\beta-1}\frac{\rho_2^2}{\rho_1}Z_{1,1}Z_{a_1,2}^2 dtd\theta \\
&= -4\int_0^{\infty} t^{\alpha+3\beta-1}\rho_2^4(\beta-\frac{\alpha}{\mu_2}t^{\alpha+\beta})(\alpha+\beta)^2 dt \\
&= -4\int_0^{\infty}\frac{t^{\alpha+3\beta-1}(\beta-\frac{\alpha}{\mu_2}t^{\alpha+\beta})(\alpha+\beta)^2}{(\alpha\mu_2+\frac{\alpha+\beta}{\mu_2}t^{\beta}+\beta t^{\alpha+\beta})^4}dt \\
& \quad (t=\mu_2^{\frac{1}{\alpha+\beta}}s) \\
&= \mu_2^{-\frac{3\alpha+\beta}{\alpha+\beta}}\int_0^{\infty}\frac{-4(\alpha+\beta)^2 s^{\alpha+3\beta-1}(\beta-\alpha s^{\alpha+\beta})}{(\alpha+\beta s^{\alpha+\beta})^4}ds+h.o.t,
\end{aligned}$$

$$\begin{aligned}
A_{11} &= -\frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty} t^{\beta-1}\frac{\rho_2^2}{\rho_1}Z_{1,1}Z_{a_1,1}^2 dtd\theta \\
&= -\int_0^{\infty}(\alpha+\beta)^2 t^{\alpha+\beta-1}\rho_1^2\rho_2^2(\beta-\frac{\alpha}{\mu_2}t^{\alpha+\beta})\mu_2^2 dt \\
&= -\int_0^{\infty}\frac{(\alpha+\beta)^2 t^{\alpha+\beta-1}(\beta-\frac{\alpha}{\mu_2}t^{\alpha+\beta})\mu_2^2}{(\alpha\mu_2+\frac{\alpha+\beta}{\mu_2}t^{\beta}+\beta t^{\alpha+\beta})^2(\beta+(\alpha+\beta)\mu_2 t^{\alpha}+\frac{\alpha}{\mu_2}t^{\alpha+\beta})^2}dt \\
& \quad (t=\mu_2^{-\frac{1}{\alpha}}s) \\
&= -\mu_2^{-\frac{\alpha+\beta}{\alpha}}\int_0^{\infty}\frac{(\alpha+\beta)^2\beta s^{\alpha+\beta-1}}{\alpha^2(\beta+(\alpha+\beta)s^{\alpha})^2}ds+h.o.t,
\end{aligned}$$

$$
\begin{aligned}
A_{12} &= \frac{4}{\pi}\int_0^{2\pi}\int_0^{\infty} t^{\beta-1}\frac{\rho_2^2}{\rho_1}Z_{1,1}Z_{a_1,1}Z_{a_1,2}dtd\theta \\
&= \int_0^{\infty} 4t^{\alpha+2\beta-1}\rho_2^3\rho_1(\alpha+\beta)^2\mu_2(\beta-\frac{\alpha}{\mu_2}t^{\alpha+\beta})dt \\
&= \int_0^{\infty}\frac{4t^{\alpha+2\beta-1}(\alpha+\beta)^2\mu_2(\beta-\frac{\alpha}{\mu_2}t^{\alpha+\beta})}{(\alpha\mu_2+\frac{\alpha+\beta}{\mu_2}t^{\beta}+\beta t^{\alpha+\beta})^3(\beta+(\alpha+\beta)\mu_2 t^{\alpha}+\frac{\alpha}{\mu_2}t^{\alpha+\beta})}dt \\
& (t=\mu_2^{\frac{1}{\alpha+\beta}}s) \\
&= \mu_2^{-\frac{3\alpha+\beta}{\alpha+\beta}}\int_0^{\infty}\frac{4(\alpha+\beta)s^{2\beta-1}(\beta-\alpha s^{\alpha+\beta})}{(\alpha+\beta s^{\alpha+\beta})^3}ds+h.o.t.
\end{aligned}
$$

Note that

$$
\int_0^{\infty}\frac{8(\alpha+\beta)^2\alpha\beta s^{2\alpha-1}}{(\beta+(\alpha+\beta)s^{\alpha})^5}ds+\int_0^{\infty}\frac{-4(\alpha+\beta)^2\alpha s^{2\alpha-1}}{(\beta+(\alpha+\beta)s^{\alpha})^4}ds=0,
$$

by summing up the above 12 expansions , we can get that for μ_2 sufficiently large, $\mathcal{A}_1$ has the following asymptotic expansion:

$$
\begin{aligned}
\frac{\mathcal{A}_1}{\pi} &= \mu_2^{-\frac{\alpha}{\alpha+\beta}}\int_0^{\infty}\{\frac{8(\alpha+\beta)^3 s^{2\alpha+3\beta-1}(\alpha-\beta s^{\alpha+\beta})}{(\alpha+\beta s^{\alpha+\beta})^5}+\frac{2(\alpha+\beta)s^{\beta-1}(\alpha-\beta s^{\alpha+\beta})}{(\alpha+\beta s^{\alpha+\beta})^3}\}ds+h.o.t. \\
&= r_a\mu_2^{-\frac{\alpha}{\alpha+\beta}}+h.o.t
\end{aligned}
$$

Next let us consider $\mathcal{C}_1$. From (2.86), one can get that

$$
\begin{aligned}
\frac{\mathcal{C}_1}{\pi} &= \frac{1}{2\pi}\int_0^{\infty}\int_0^{2\pi}\Big[t^{\alpha-1}\frac{\rho_1^2}{\rho_2}(2Z_{1,1}-Z_{1,2})(2Z_{b_1,1}-Z_{b_1,2})^2 \\
&+ t^{\beta-1}\frac{\rho_2^2}{\rho_1}(2Z_{1,2}-Z_{1,1})(2Z_{b_1,2}-Z_{b_1,1})^2\Big]dtd\theta \\
&= (I_1+I_2+I_3+\cdots+I_{12}),
\end{aligned}
$$

where I_i for $i=1,\cdots,12$ are given below.

First we consider I_1:

$$\begin{aligned}
I_1 &= \frac{2}{\pi}\int_0^\infty\int_0^{2\pi} t^{\alpha-1}\frac{\rho_1^2}{\rho_2}Z_{11}Z_{b_1,1}^2 dtd\theta\\
&= \int_0^\infty 8t^{3\alpha+\beta-1}\frac{\rho_1^5}{\rho_2}\alpha^2\mu_2^{-2}(\beta-\frac{\alpha}{\mu_2}t^{\alpha+\beta})dt\\
&\quad (t=\mu_2^{-\frac{1}{\alpha}}s)\\
&= \mu_2^{-\frac{4\alpha+\beta}{\alpha}}\int_0^\infty \frac{8\alpha^3\beta s^{3\alpha+\beta-1}}{(\beta+(\alpha+\beta)s^\alpha)^5}ds + O(\mu_2^{-\frac{6\alpha+2\beta}{\alpha}}).
\end{aligned}$$

Similarly we get

$$\begin{aligned}
I_2 &= \frac{8}{\pi}\int_0^\infty\int_0^{2\pi} t^{\alpha-1}\frac{\rho_1^2}{\rho_2}Z_{11}Z_{b_1,2}^2 dtd\theta\\
&= \mu_2^{-\frac{4\alpha+\beta}{\alpha}}\int_0^\infty \frac{2\alpha\beta s^{\alpha+\beta-1}}{(\beta+(\alpha+\beta)s^\alpha)^3}ds + O(\mu_2^{-\frac{6\alpha+2\beta}{\alpha}}),\\
I_3 &= -\frac{8}{\pi}\int_0^\infty\int_0^{2\pi} t^{\alpha-1}\frac{\rho_1^2}{\rho_2}Z_{11}Z_{\beta_1}Z_{b_1,2} dtd\theta\\
&= \mu_2^{-\frac{4\alpha+\beta}{\alpha}}\int_0^\infty \frac{-8\alpha^2\beta s^{2\alpha+\beta-1}}{(\beta+(\alpha+\beta)s^\alpha)^4}ds + O(\mu_2^{-\frac{6\alpha+2\beta}{\alpha}}),\\
I_4 &= \frac{-4}{\pi}\int_0^\infty\int_0^{2\pi} t^{\alpha-1}\frac{\rho_1^2}{\rho_2}Z_{12}Z_{b_1,1}^2 dtd\theta\\
&= \mu_2^{-\frac{4\alpha+\beta}{\alpha}}\int_0^\infty \frac{-4\alpha^3 s^{3\alpha+\beta-1}}{(\beta+(\alpha+\beta)s^\alpha)^4}ds + O(\mu_2^{-\frac{6\alpha+2\beta}{\alpha}}),\\
I_5 &= \frac{-1}{\pi}\int_0^\infty\int_0^{2\pi} t^{\alpha-1}\frac{\rho_1^2}{\rho_2}Z_{12}Z_{b_1,2}^2 dtd\theta\\
&= \mu_2^{-\frac{4\alpha+\beta}{\alpha}}\int_0^\infty \frac{-\alpha s^{\alpha+\beta-1}}{(\beta+(\alpha+\beta)s^\alpha)^2}ds + O(\mu_2^{-\frac{6\alpha+2\beta}{\alpha}}),\\
I_6 &= \frac{4}{\pi}\int_0^\infty\int_0^{2\pi} t^{\alpha-1}\frac{\rho_1^2}{\rho_2}Z_{12}Z_{b_1,1}Z_{\beta_2} dtd\theta\\
&= \mu_2^{-\frac{4\alpha+\beta}{\alpha}}\int_0^\infty \frac{4\alpha^2 s^{2\alpha+\beta-1}}{(\beta+(\alpha+\beta)s^\alpha)^3}ds + O(\mu_2^{-\frac{6\alpha+2\beta}{\alpha}}),
\end{aligned}$$

$$
\begin{aligned}
I_7 &= \frac{8}{\pi}\int_0^\infty\int_0^{2\pi} t^{\beta-1}\frac{\rho_2^2}{\rho_1}Z_{12}Z_{b_1,2}^2 dtd\theta \\
&= \mu_2^{-\frac{4\alpha+3\beta}{\alpha+\beta}}\int_0^\infty \frac{8(\alpha+\beta)\alpha^2 s^{\alpha+2\beta-1}(\alpha-\beta s^{\alpha+\beta})}{(\alpha+\beta s^{\alpha+\beta})^5}ds \\
&+ \mu_2^{-\frac{6\alpha+4\beta}{\alpha+\beta}}\int_0^\infty [\frac{8\alpha^2 t^{2\beta-1}(\alpha-\beta s^{\alpha+\beta})(\beta+\alpha s^{\alpha+\beta})}{(\alpha+\beta s^{\alpha+\beta})^5} \\
&- \frac{40(\alpha+\beta)^2\alpha^2 s^{\alpha+3\beta-1}(\alpha-\beta s^{\alpha+\beta})}{(\alpha+\beta s^{\alpha+\beta})^6}]ds + h.o.t,
\end{aligned}
$$

$$
\begin{aligned}
I_8 &= \frac{2}{\pi}\int_0^\infty\int_0^{2\pi} t^{\beta-1}\frac{\rho_2^2}{\rho_1}Z_{12}Z_{b_1,1}^2 dtd\theta \\
&= \mu_2^{-\frac{4\alpha+3\beta}{\alpha+\beta}}\int_0^\infty \frac{2\alpha^2 s^{\alpha+2\beta-1}(\alpha-\beta s^{\alpha+\beta})}{(\alpha+\beta)(\alpha+\beta s^{\alpha+\beta})^3}ds \\
&+ \mu_2^{-\frac{6\alpha+4\beta}{\alpha+\beta}}\int_0^\infty [\frac{-2\alpha^2 s^{2\beta-1}(\alpha-\beta s^{\alpha+\beta})(\beta+\alpha s^{\alpha+\beta})}{(\alpha+\beta)^2(\alpha+\beta s^{\alpha+\beta})^3} \\
&- \frac{6\alpha^2 s^{\alpha+3\beta-1}(\alpha-\beta s^{\alpha+\beta})}{(\alpha+\beta s^{\alpha+\beta})^4}]ds + h.o.t,
\end{aligned}
$$

$$
\begin{aligned}
I_9 &= \frac{-8}{\pi}\int_0^\infty\int_0^{2\pi} t^{\beta-1}\frac{\rho_2^2}{\rho_1}Z_{12}Z_{b_1,1}Z_{b_1,2} dtd\theta \\
&= \mu_2^{-\frac{4\alpha+3\beta}{\alpha+\beta}}\int_0^\infty \frac{-8\alpha^2(\alpha-\beta s^{\alpha+\beta})s^{\alpha+2\beta-1}}{(\alpha+\beta s^{\alpha+\beta})^4}ds \\
&+ \mu_2^{-\frac{6\alpha+4\beta}{\alpha+\beta}}\int_0^\infty \frac{32\alpha^2(\alpha+\beta)(\alpha-\beta s^{\alpha+\beta})s^{\alpha+3\beta-1}}{(\alpha+\beta s^{\alpha+\beta})^5}ds + h.o.t,
\end{aligned}
$$

$$
\begin{aligned}
I_{10} &= \frac{-4}{\pi}\int_0^\infty\int_0^{2\pi} t^{\beta-1}\frac{\rho_2^2}{\rho_1}Z_{11}Z_{b_1,2}^2 dtd\theta \\
&= \mu_2^{-\frac{6\alpha+4\beta}{\alpha+\beta}}\int_0^\infty \frac{-4\alpha^2(\beta-\alpha s^{\alpha+\beta})s^{2\beta-1}}{(\alpha+\beta s^{\alpha+\beta})^4}ds + h.o.t,
\end{aligned}
$$

$$
\begin{aligned}
I_{11} &= \frac{-1}{\pi}\int_0^\infty\int_0^{2\pi} t^{\beta-1}\frac{\rho_2^2}{\rho_1}Z_{11}Z_{b_1,1}^2 dtd\theta \\
&= \mu_2^{-\frac{6\alpha+4\beta}{\alpha+\beta}}\int_0^\infty \frac{-\alpha^2 s^{2\beta-1}(\beta-\alpha s^{\alpha+\beta})}{(\alpha+\beta)^2(\alpha+\beta s^{\alpha+\beta})^2}ds + h.o.t,
\end{aligned}
$$

$$
\begin{aligned}
I_{12} &= \frac{4}{\pi}\int_0^\infty\int_0^{2\pi} t^{\beta-1}\frac{\rho_2^2}{\rho_1}Z_{11}Z_{b_1,1}Z_{b_1,2} dtd\theta \\
&= \mu_2^{-\frac{6\alpha+4\beta}{\alpha+\beta}}\int_0^\infty \frac{\alpha^2 s^{2\beta-1}(\beta-\alpha s^{\alpha+\beta})}{(\alpha+\beta)(\alpha+\beta s^{\alpha+\beta})^3}ds + h.o.t.
\end{aligned}
$$

So by summing up the above expansions and combining the terms of the same order of μ_2, one can get that

$$\frac{\mathcal{C}_1}{\pi} = r_{b1}\mu_2^{-\frac{4\alpha+3\beta}{\alpha+\beta}} + r_{b2}\mu_2^{-\frac{4\alpha+\beta}{\alpha}} + r_{b3}\mu_2^{-\frac{6\alpha+4\beta}{\alpha+\beta}} + h.o.t,$$

where r_{b1}, r_{b2}, r_{b3} are given in the above expansions.

Similarly, one can get

$$\begin{aligned}
\frac{\mathcal{E}_1}{\pi} &= \mu_2^{-\frac{3\alpha+2\beta}{\alpha+\beta}} \int_0^\infty \{\frac{8\alpha^2 s^{\alpha+2\beta-1}(\alpha-\beta s^{\alpha+\beta})}{(\alpha+\beta s^{\alpha+\beta})^4} + \frac{8\alpha^2 s^{\alpha+2\beta-1}(\alpha-\beta s^{\alpha+\beta})(\beta+\alpha s^{\alpha+\beta})}{(\alpha+\beta s^{\alpha+\beta})^5} \\
&- \frac{4\alpha^2(\beta-\alpha s^{\alpha+\beta})s^{\alpha+2\beta-1}}{(\alpha+\beta s^{\alpha+\beta})^4} - \frac{\alpha^2 s^{\beta-1}(\alpha-\beta s^{\alpha+\beta})}{(\alpha+\beta)^2(\alpha+\beta s^{\alpha+\beta})^2}\}ds + h.o.t. \\
&= r_e\mu_2^{-\frac{3\alpha+2\beta}{\alpha+\beta}} + h.o.t.
\end{aligned}$$

For the above integrations, we can calculate them explicitly using Mathematica, and we can get that

$$\begin{aligned}
r_a &= \int_0^\infty \left[\frac{8(\alpha+\beta)^3 s^{2\alpha+3\beta-1}(\alpha-\beta s^{\alpha+\beta})}{(\alpha+\beta s^{\alpha+\beta})^5} + \frac{2(\alpha+\beta)s^{\beta-1}(\alpha-\beta s^{\alpha+\beta})}{(\alpha+\beta s^{\alpha+\beta})^3}\right]ds \\
&= \frac{4(\alpha-\beta)\beta(\frac{\beta}{\alpha})^{-\frac{\alpha+2\beta}{\alpha+\beta}}\pi\csc(\frac{\pi\beta}{\alpha+\beta})}{3\alpha^2(\alpha+\beta)^2} \neq 0 \text{ if } \alpha\neq\beta,
\end{aligned}$$

$$r_{b1} = r_{b2} = 0,$$

and

$$\begin{aligned}
r_{b3} &= \int_0^\infty \left[\frac{8\alpha^2(\alpha-\beta s^{\alpha+\beta})(\beta+\alpha s^{\alpha+\beta})s^{2\beta-1}}{(\alpha+\beta s^{\alpha+\beta})^5} - \frac{40(\alpha+\beta)^2\alpha^2(\alpha-\beta s^{\alpha+\beta})s^{\alpha+3\beta-1}}{(\alpha+\beta s^{\alpha+\beta})^6}\right. \\
&- 2\frac{\alpha^2(\alpha-\beta s^{\alpha+\beta})(\beta+\alpha s^{\alpha+\beta})s^{2\beta-1}}{(\alpha+\beta)^2(\alpha+\beta s^{\alpha+\beta})^3} - 6\frac{\alpha^2(\alpha-\beta s^{\alpha+\beta})s^{\alpha+3\beta-1}}{(\alpha+\beta s^{\alpha+\beta})^4} \\
&+ \frac{32\alpha^2(\alpha+\beta)(\alpha-\beta s^{\alpha+\beta})s^{\alpha+3\beta-1}}{(\alpha+\beta s^{\alpha+\beta})^5} - 4\frac{\alpha^2(\beta-\alpha s^{\alpha+\beta})s^{2\beta-1}}{(\alpha+\beta s^{\alpha+\beta})^4} \\
&- \left.\frac{\alpha^2(\beta-\alpha s^{\alpha+\beta})s^{2\beta-1}}{(\alpha+\beta s^{\alpha+\beta})^2(\alpha+\beta)^2} + \frac{2\alpha^2(\alpha-\beta s^{\alpha+\beta})s^{\alpha+2\beta-1}}{(\alpha+\beta s^{\alpha+\beta})^3(\alpha+\beta)}\right]ds \\
&= \frac{\beta^3(\frac{\beta}{\alpha})^{-\frac{2(\alpha+2\beta)}{\alpha+\beta}}(5\alpha+\beta)\pi\csc(\frac{2\beta\pi}{\alpha+\beta})}{\alpha^2(\alpha+\beta)^4} \neq 0 \text{ if } \alpha\neq\beta,
\end{aligned}$$

and

$$
\begin{aligned}
r_e &= \int_0^\infty \left[\frac{8\alpha^2 s^{\alpha+2\beta-1}(\alpha-\beta s^{\alpha+\beta})}{(\alpha+\beta s^{\alpha+\beta})^4} + \frac{8\alpha^2 s^{\alpha+2\beta-1}(\alpha-\beta s^{\alpha+\beta})(\beta+\alpha s^{\alpha+\beta})}{(\alpha+\beta s^{\alpha+\beta})^5}\right. \\
&- \left.\frac{4\alpha^2(\beta-\alpha s^{\alpha+\beta})s^{\alpha+2\beta-1}}{(\alpha+\beta s^{\alpha+\beta})^4} - \frac{\alpha^2 s^{\beta-1}(\alpha-\beta s^{\alpha+\beta})}{(\alpha+\beta)^2(\alpha+\beta s^{\alpha+\beta})^2}\right] ds \\
&= \frac{(\frac{\beta}{\alpha})^{-\frac{3\alpha+4\beta}{\alpha+\beta}}(\alpha-\beta)\beta^2(2\alpha^2+5\alpha\beta+2\beta^2)\pi\csc(\frac{\alpha\pi}{\alpha+\beta})}{3\alpha^3(\alpha+\beta)^4} \neq 0 \text{ if } \alpha\neq\beta.
\end{aligned}
$$

Therefore, $\mathcal{A}_1$, $\mathcal{C}_1$ and $\mathcal{E}_1$ do not vanish for μ_2 large if $N_1 \neq N_2$. So we get that for μ_2 large enough, the matrix $\mathbf{Q}_1$ is non degenerate. Since $\mathcal{T}$ is a matrix with fixed coefficients, for μ_2 large enough, we can choose ξ_1, ξ_2 such that the matrix $\mathbf{Q} - \mathcal{T}$ is non-degenerate.

□

Case II: The calculation of $\mathbf{Q}_1$ when $N_1 = N_2$

In this case, the matrix $\mathbf{Q}_1$ and $\mathbf{Q}_2$ may not be diagonal, so we need to consider $\mathcal{B}_i$ and $\mathcal{D}_i$, and the calculation is more complicated. As we see from the above expansions, when $\alpha = \beta$, we need to use different change of variables. First we consider $\mathcal{A}_1$. Follow the notations in the previous section,

$$
\frac{\mathcal{A}_1}{\pi} = A_1 + \cdots + A_{12}.
$$

Then we calculate the 12 terms one by one:

$$
\begin{aligned}
A_1 &= \int_0^\infty \frac{8(\alpha+\beta)^2\mu_2^2 t^{2\alpha-1}(\beta-\frac{\alpha}{\mu_2}t^{\alpha+\beta})(\alpha\mu_2+\frac{\alpha+\beta}{\mu_2}t^\beta+\beta t^{\alpha+\beta})}{(\beta+(\alpha+\beta)\mu_2 t^\alpha+\frac{\alpha}{\mu_2}t^{\alpha+\beta})^5} dt \\
&\quad (t^\alpha = x) \\
&= \int_0^\infty \frac{8(\alpha+\beta)^2\mu_2^2 x(\beta-\frac{\alpha}{\mu_2}x^2)(\alpha\mu_2+\frac{\alpha+\beta}{\mu_2}x+\beta x^2)}{\alpha(\beta+(\alpha+\beta)\mu_2 x+\frac{\alpha}{\mu_2}x^2)^5} dx \\
&\quad (x = \mu_2^{-1}s) \\
&= \mu_2 \int_0^\infty \frac{8(\alpha+\beta)^2\alpha\beta s}{\alpha(\beta+(\alpha+\beta)s)^5} ds \\
&+ \mu_2^{-2} \int_0^\infty \frac{8(\alpha+\beta)^2 s(\beta((\alpha+\beta)s+\beta s^2)-\alpha^2 s^2)}{\alpha(\beta+(\alpha+\beta)s)^5} - \frac{40(\alpha+\beta)^2\alpha^2\beta s^3}{\alpha(\beta+(\alpha+\beta)s)^6} ds + h.o.t,
\end{aligned}
$$

$$\begin{aligned}
A_2 &= \int_0^\infty \frac{2(\alpha+\beta)^2 t^{4\alpha-1}(\beta - \frac{\alpha}{\mu_2}t^{2\alpha})}{(\beta+(\alpha+\beta)dt^\alpha + \frac{\alpha}{\mu_2}t^{2\alpha})^3(\alpha\mu_2 + \frac{\alpha+\beta}{\mu_2}t^\alpha + \beta t^{2\alpha})}dt \\
&\quad (t^\alpha = x) \\
&= \int_0^\infty \frac{2(\alpha+\beta)^2 x^3(\beta - \frac{\alpha}{\mu_2}x^2)}{\alpha(\beta+(\alpha+\beta)\mu_2 x + \frac{\alpha}{\mu_2}x^2)^3(\alpha\mu_2 + \frac{\alpha+\beta}{\mu_2}x + \beta x^2)}dx \\
&\quad (x = \mu_2^2 s) \\
&= \mu_2^{-2}\int_0^\infty \frac{-2(\alpha+\beta)^3\alpha}{\alpha\beta(\alpha+\beta+\alpha s)^3}ds + h.o.t, \\
A_3 &= \int_0^\infty \frac{-8t^{3\alpha-1}(\beta - \frac{\alpha}{\mu_2}t^{2\alpha})(\alpha+\beta)^2\mu_2}{(\beta+(\alpha+\beta)\mu_2 t^\alpha + \frac{\alpha}{\mu_2}t^{2\alpha})^4}dt \\
&\quad (t^\alpha = x) \\
&= \int_0^\infty \frac{-8x^2(\beta - \frac{\alpha}{\mu_2}x^2)(\alpha+\beta)^2\mu_2}{\alpha(\beta+(\alpha+\beta)\mu_2 x + \frac{\alpha}{\mu_2}x^2)^4}dx \\
&\quad (x = \mu_2^2 s) \\
&= \mu_2^{-2}\int_0^\infty \frac{8\alpha^2(\alpha+\beta)^2}{(\alpha+\beta+\alpha s)^4}ds + h.o.t, \\
A_4 &= \int_0^\infty \frac{-4(\alpha+\beta)^2\mu_2^2 t^{2\alpha-1}(\alpha\mu_2 - \beta t^{\alpha+\beta})}{(\beta+(\alpha+\beta)\mu_2 t^\alpha + \frac{\alpha}{\mu_2}t^{2\alpha})^4}dt \\
&\quad (t^\alpha = x) \\
&= \int_0^\infty \frac{-4(\alpha+\beta)^2\mu_2^2 x(\alpha\mu_2 - bx^2)}{\alpha(\beta+(\alpha+\beta)\mu_2 x + \frac{\alpha}{\mu_2}x^2)^4}dx \\
&\quad (x = \mu_2^{-1} s) \\
&= \mu_2\int_0^\infty \frac{-4(\alpha+\beta)^2 s}{(\beta+(\alpha+\beta)s)^4}ds + \mu_2^{-2}\int_0^\infty \frac{4(\alpha+\beta)^2 s^2}{(\beta+(\alpha+\beta)s)^4} + \frac{16(\alpha+\beta)^2\alpha s^3}{(\beta+(\alpha+\beta)s)^5}ds + h.o.t,
\end{aligned}$$

Next we estimate A_5:

$$\begin{aligned} A_5 &= \int_0^\infty \frac{-(\alpha+\beta)^2 t^{4\alpha-1}(\alpha\mu_2-\beta t^{2\alpha})}{(\beta+(\alpha+\beta)\mu_2 t^\alpha+\frac{\alpha}{\mu_2}t^{2\alpha})^2(\alpha\mu_2+\frac{\alpha+\beta}{\mu_2}t^\alpha+\beta t^{2\alpha})^2}dt \\ & (t^\alpha = x) \\ &= \int_0^\infty \frac{-(\alpha+\beta)^2 x^3(\alpha\mu_2-\beta x^2)}{\alpha(\beta+(\alpha+\beta)\mu_2 x+\frac{\alpha}{\mu_2}x^2)^2(\alpha\mu_2+\frac{\alpha+\beta}{\mu_2}x+\beta x^2)^2}dx. \end{aligned}$$

We then divide the integral into three parts:

$$\begin{aligned} A_5 &= \int_0^{\mu_2^{\frac{1}{2}}} + \int_{\mu_2^{\frac{1}{2}}}^{\mu_2^2} + \int_{\mu_2^2}^\infty \frac{-(\alpha+\beta)^2 x^3(\alpha\mu_2-\beta x^2)}{\alpha(\beta+(\alpha+\beta)\mu_2 x+\frac{\alpha}{\mu_2}x^2)^2(\alpha\mu_2+\frac{\alpha+\beta}{\mu_2}x+\beta x^2)^2}dx \\ &= i_{51}+i_{52}+i_{53}. \end{aligned}$$

First consider i_{51}, we have

$$\begin{aligned} i_{51} &= \int_0^{\mu_2^{\frac{1}{2}}} \frac{-(\alpha+\beta)^2 x^3(\alpha\mu_2-\beta x^2)}{\alpha(\beta+(\alpha+\beta)\mu_2 x+\frac{\alpha}{\mu_2}x^2)^2(\alpha\mu_2+\frac{\alpha+\beta}{\mu_2}x+\beta x^2)^2}dx \\ (x &= \mu_2^{\frac{1}{2}}s) \\ &= \mu_2^{-2}\int_0^1 \frac{(\alpha-\beta s^2)s}{\alpha(\alpha+\beta s^2)^2}ds + h.o.t. \end{aligned}$$

For i_{52}, we use a different change of variable and get that

$$\begin{aligned} i_{52} &= \int_{\mu_2^{\frac{1}{2}}}^{\mu_2^2} \frac{-(\alpha+\beta)^2 x^3(\alpha\mu_2-\beta x^2)}{\alpha(\beta+(\alpha+\beta)\mu_2 x+\frac{\alpha}{\mu_2}x^2)^2(\alpha\mu_2+\frac{\alpha+\beta}{\mu_2}x+\beta x^2)^2}dx \\ (x &= \mu_2^2 s) \\ &= \mu_2^{-2}\int_{\mu_2^{-\frac{3}{2}}}^1 \frac{(\alpha+\beta)^2}{\alpha\beta(\alpha+\beta+\alpha s)^2 s}ds + h.o.t \\ &= \frac{3}{2\beta^2}\mu_2^{-2}\ln\mu_2 + h.o.t. \end{aligned}$$

As to i_{53}, we can easily get the following estimate:

$$\begin{aligned} i_{53} &\le c\int_{\mu_2^2}^\infty \frac{x^3\cdot x^2}{\mu_2^{-2}x^4\cdot x^4} \\ &\le c\mu_2^{-2}. \end{aligned}$$

So combining the above three estimates, we have

$$A_5 = \frac{3}{2\beta^2}\mu_2^{-2}\ln\mu_2 + O(\mu_2^{-2}).$$

Next let's consider A_6:

$$\begin{aligned} A_6 &= \int_0^\infty \frac{4t^{3\alpha-1}(\alpha+\beta)^2\mu_2(\alpha\mu_2-\beta t^{2\alpha})}{(\beta+(\alpha+\beta)\mu_2 t^\alpha+\frac{\alpha}{\mu_2}t^{2\alpha})^3(\alpha\mu_2+\frac{\alpha+\beta}{\mu_2}t^\alpha+\beta t^{2\alpha})}dt \\ &\quad (t^\alpha = x) \\ &= \int_0^\infty \frac{4x^2(\alpha+\beta)^2\mu_2(\alpha\mu_2-\beta x^2)}{\alpha(\beta+(\alpha+\beta)\mu_2 x+\frac{\alpha}{\mu_2}x^2)^3(\alpha\mu_2+\frac{\alpha+\beta}{\mu_2}x+\beta x^2)}dx. \end{aligned}$$

Similar to A_5, we also divide A_6 into three parts,

$$\begin{aligned} A_6 &= \int_0^{\mu_2^{\frac{1}{2}}} + \int_{\mu_2^{\frac{1}{2}}}^{\mu_2^2} + \int_{\mu_2^2}^\infty \frac{4x^2(\alpha+\beta)^2\mu_2(\alpha\mu_2-\beta x^2)}{\alpha(\beta+(\alpha+\beta)\mu_2 x+\frac{\alpha}{\mu_2}x^2)^3(\alpha\mu_2+\frac{\alpha+\beta}{\mu_2}x+\beta x^2)}dx \\ &= i_{61}+i_{62}+i_{63}. \end{aligned}$$

For i_{61}, we use $(x=\mu_2^{-1}s)$ to obtain

$$\begin{aligned} i_{61} &= \int_0^{\mu_2^{\frac{1}{2}}} \frac{4x^2(\alpha+\beta)^2\mu_2(\alpha\mu_2-\beta x^2)}{\alpha(\beta+(\alpha+\beta)\mu_2 x+\frac{\alpha}{\mu_2}x^2)^3(\alpha\mu_2+\frac{\alpha+\beta}{\mu_2}x+\beta x^2)}dx \\ &\quad (x=\mu_2^{-1}s) \\ &= \mu_2^{-2}\int_0^{\mu_2^{\frac{3}{2}}} \frac{4(\alpha+\beta)^2 s^2}{\alpha(\beta+(\alpha+\beta)s)^3}ds + h.o.t \\ &= \frac{6}{\alpha(\alpha+\beta)}\mu_2^{-2}\ln\mu_2 + O(\mu_2^{-2}), \end{aligned}$$

while for i_{62}, we have

$$\begin{aligned} i_{62} &= \int_{\mu_2^{\frac{1}{2}}}^{\mu_2^2} \frac{4x^2(\alpha+\beta)^2\mu_2(\alpha\mu_2-\beta x^2)}{\alpha(\beta+(\alpha+\beta)\mu_2 x+\frac{\alpha}{\mu_2}x^2)^3(\alpha\mu_2+\frac{\alpha+\beta}{\mu_2}x+\beta x^2)}dx \\ &\quad (x=\mu_2^2 s) \\ &= \mu_2^{-2}\int_{\mu_2^{-\frac{3}{2}}}^1 \frac{4(\alpha+\beta)^2}{\alpha(\alpha+\beta+\alpha s)^3 s}ds + h.o.t \\ &= -\frac{6}{\alpha(\alpha+\beta)}\mu_2^{-2}\ln\mu_2 + O(\mu_2^{-2}), \end{aligned}$$

it is easy to see that

$$\begin{aligned} i_{63} &\leq c\int_{\mu_2^2}^{\infty} \frac{dx^2\cdot x^2}{\mu_2^{-3}x^6\cdot x^2} \\ &\leq c\mu_2^{-4}. \end{aligned}$$

So combining the above three estimates, we have

$$A_6 = O(\mu_2^{-2}).$$

By checking the estimates in the previous subsection, we can get that the expansions for A_7, A_8, A_9, A_{10} and A_{12} hold for $\alpha = \beta$, so we have

$$A_7 + A_8 = O(\mu_2^{-2}),\ A_9 = O(\mu_2^{-2}),\ A_{10} = O(\mu_2^{-2}),\ A_{12} = O(\mu_2^{-2}).$$

While for A_{11}, we have

$$\begin{aligned} A_{11} &= \int_0^{\infty} \frac{-(\alpha+\beta)^2 t^{2\alpha-1}(\beta - \frac{\alpha}{\mu_2}t^{2\alpha})\mu_2^2}{(\beta+(\alpha+\beta)\mu_2 t^{\alpha} + \frac{\alpha}{\mu_2}t^{2\alpha})^2(\alpha\mu_2 + \frac{\alpha+\beta}{\mu_2}t^{\alpha} + \beta t^{2\alpha})^2}dt \\ &\quad (t^{\alpha} = x) \\ &= \int_0^{\infty} \frac{-(\alpha+\beta)^2 x(\beta - \frac{\alpha}{\mu_2}x^2)\mu_2^2}{\alpha(\beta+(\alpha+\beta)\mu_2 x + \frac{\alpha}{\mu_2}x^2)^2(\alpha\mu_2 + \frac{\alpha+\beta}{\mu_2}x + \beta x^2)^2}dx \\ &= \int_0^{\mu_2^{\frac{1}{2}}} + \int_{\mu_2^{\frac{1}{2}}}^{\mu_2^2} + \int_{\mu_2^2}^{\infty} \frac{-(\alpha+\beta)^2 x(\beta - \frac{\alpha}{\mu_2}x^2)\mu_2^2}{\alpha(\beta+(\alpha+\beta)\mu_2 x + \frac{\alpha}{\mu_2}x^2)^2(\alpha\mu_2 + \frac{\alpha+\beta}{\mu_2}x + \beta x^2)^2}dx \\ &= i_{111} + i_{112} + i_{113}, \end{aligned}$$

where

$$\begin{aligned} i_{111} &= \int_0^{\mu_2^{\frac{1}{2}}} \frac{-(\alpha+\beta)^2 x(\beta - \frac{\alpha}{\mu_2}x^2)\mu_2^2}{\alpha(\beta+(\alpha+\beta)\mu_2 x + \frac{\alpha}{\mu_2}x^2)^2(\alpha\mu_2 + \frac{\alpha+\beta}{\mu_2}x + \beta x^2)^2}dx \\ &\quad (x = \mu_2^{-1}s) \\ &= \mu_2^{-2}\int_0^{\mu_2^{\frac{3}{2}}} \frac{-(\alpha+\beta)^2\beta s}{\alpha^3(\beta+(\alpha+\beta)s)^2}ds + h.o.t \\ &= -\frac{3}{2\alpha^2}\mu_2^{-2}\ln\mu_2 + O(\mu_2^{-2}), \end{aligned}$$

$$
\begin{aligned}
i_{112} &= \int_{\mu_2^{\frac{1}{2}}}^{\mu_2^2} \frac{-(\alpha+\beta)^2 x(\beta - \frac{\alpha}{\mu_2}x^2)\mu_2^2}{\alpha(\beta+(\alpha+\beta)\mu_2 x + \frac{\alpha}{\mu_2}x^2)^2(\alpha\mu_2 + \frac{\alpha+\beta}{\mu_2}x + \beta x^2)^2} dx \\
& (x = \mu_2^2 s) \\
&= \mu_2^{-5} \int_{\mu_2^{-\frac{3}{2}}}^{1} \frac{(\alpha+\beta)^2}{\beta^3 s^3(\alpha+\beta+\alpha s)^3} ds \\
&\leq c\mu_2^{-5} \int_{\mu_2^{-\frac{3}{2}}}^{1} \frac{1}{s^3} ds \\
&\leq c\mu_2^{-2},
\end{aligned}
$$

and

$$
\begin{aligned}
i_{113} &\leq c\int_{\mu_2^2}^{\infty} \frac{\mu_2^2 x \cdot \mu_2^{-1} \cdot x^2}{\mu_2^{-2} x^4 \cdot x^4} \\
&\leq c\mu_2^{-5}.
\end{aligned}
$$

So one can get that

$$
A_{11} = -\frac{3}{2\alpha^2}\mu_2^{-2}\ln\mu_2 + O(\mu_2^{-2}).
$$

So by summing the above 12 terms, we can obtain

$$
\mathcal{A}_1 = O(\mu_2^{-2}).
$$

Similarly,

$$
\mathcal{C}_1 = O(\mu_2^{-5}).
$$

Next we consider $\mathcal{B}_1$. From (2.85), we have

$$
\begin{aligned}
\frac{\mathcal{B}_1}{\pi} &= \frac{1}{2\pi}\int_0^{\infty}\int_0^{2\pi} \Big[t^{\alpha-1}\frac{\rho_1^2}{\rho_2}(2Z_{1,1}-Z_{1,2})(2Z_{a_1,1}-Z_{a_1,2})(2Z_{b_1,1}-Z_{b_1,2}) \\
&+ t^{\beta-1}\frac{\rho_2^2}{\rho_1}(2Z_{1,2}-Z_{1,1})(2Z_{a_1,2}-Z_{a_1,1})(2Z_{b_1,2}-Z_{b_1,1})\Big] dt d\theta \\
&= (B_1 + B_2 + B_3 + \cdots + B_{16}),
\end{aligned}
$$

where $B_i, i = 1, \cdots, 16$ are as follows:

$$
\begin{aligned}
B_1 &= \frac{1}{\pi}\int_0^\infty\int_0^{2\pi} 8t^{\alpha-1}\frac{\rho_1^2}{\rho_2}Z_{11}Z_{a_1,1}Z_{b_1,1}dtd\theta \\
&= \int_0^\infty 8t^{3\alpha-1}\frac{\rho_1^5}{\rho_2}\alpha(\alpha+\beta)(\beta-\frac{\alpha}{\mu_2}t^{2\alpha})dt \\
&\quad (t^\alpha = x) \\
&= \int_0^\infty \frac{8x^2\alpha(\alpha+\beta)(\beta-\frac{\alpha}{\alpha\mu_2}x^2)(\alpha\mu_2+\frac{\alpha+\beta}{\mu_2}x+\beta x^2)}{(\beta+(\alpha+\beta)\mu_2 x+\frac{\alpha}{\mu_2}x^2)^5}dx \\
&\quad (x=\mu_2^{-1}s) \\
&= \mu_2^{-2}\int_0^\infty \frac{8\alpha^2\beta(\alpha+\beta)s^2}{\alpha(\beta+(\alpha+\beta)s)^5}ds + h.o.t.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
B_2 &= \frac{1}{\pi}\int_0^\infty\int_0^{2\pi} -4t^{\alpha-1}\frac{\rho_1^2}{\rho_2}Z_{11}Z_{a_1,1}Z_{b_1,2}dtd\theta \\
&= \mu_2^{-2}\int_0^\infty \frac{-4\alpha^2\beta(\alpha+\beta)s}{(\beta+(\alpha+\beta)s)^4}ds + h.o.t,
\end{aligned}
$$

$$
\begin{aligned}
B_3 &= \frac{1}{\pi}\int_0^\infty\int_0^{2\pi} -4t^{\alpha-1}\frac{\rho_1^2}{\rho_2}Z_{11}Z_{a_1,2}Z_{b_1,1}dtd\theta \\
&= \mu_2^{-2}\int_0^\infty \frac{4\alpha^2(\alpha+\beta)s}{\alpha(\alpha+\beta+\alpha s)^4}ds + h.o.t.
\end{aligned}
$$

Next let us consider B_4:

$$
\begin{aligned}
B_4 &= \frac{1}{\pi}\int_0^\infty\int_0^{2\pi} 2t^{\alpha-1}\frac{\rho_1^2}{\rho_2}Z_{11}Z_{a_1,2}Z_{b_1,2}dtd\theta \\
&= \int_0^\infty 2t^{3\alpha-1}\rho_1^3\rho_2(\beta-\frac{\alpha}{\mu_2}t^{2\alpha})\alpha(\alpha+\beta)\mu_2^{-1}dt \\
&\quad (t^\alpha = x) \\
&= \int_0^\infty \frac{2x^2(\beta-\frac{\alpha}{\mu_2}x^2)\alpha(\alpha+\beta)\mu_2^{-1}}{\alpha(\beta+(\alpha+\beta)\mu_2 x+\frac{\alpha}{\mu_2}x^2)^3(\alpha\mu_2+\frac{\alpha+\beta}{\mu_2}x+\beta x^2)}dx \\
&= \int_0^{\mu_2^{\frac{1}{2}}} + \int_{\mu_2^{\frac{1}{2}}}^{\mu_2^2} + \int_{\mu_2^2}^\infty \frac{2x^2(\beta-\frac{\alpha}{\mu_2}x^2)\alpha(\alpha+\beta)\mu_2^{-1}}{\alpha(\beta+(\alpha+\beta)\mu_2 x+\frac{\alpha}{\mu_2}x^2)^3(\alpha\mu_2+\frac{\alpha+\beta}{\mu_2}x+\beta x^2)}dx \\
&= i_{41}+i_{42}+i_{43},
\end{aligned}
$$

where

$$
\begin{aligned}
i_{41} &= \int_0^{\mu_2^{\frac{1}{2}}} \frac{2x^2(\beta - \frac{\alpha}{\mu_2}x^2)\alpha(\alpha+\beta)\mu_2^{-1}}{\alpha(\beta+(\alpha+\beta)\mu_2 x + \frac{\alpha}{\mu_2}x^2)^3(\alpha\mu_2 + \frac{\alpha+\beta}{\mu_2}x + \beta x^2)} dx \\
& \quad (x = \mu_2^{-1}s) \\
&= \mu_2^{-5} \int_0^{\mu_2^{\frac{3}{2}}} \frac{2\beta(\alpha+\beta)s^2}{\alpha(\beta+(\alpha+\beta)s)^3} ds + h.o.t \\
&= \frac{3}{(\alpha+\beta)^2}\mu_2^{-5}\ln\mu_2 + h.o.t,
\end{aligned}
$$

$$
\begin{aligned}
i_{42} &= \int_{\mu_2^{\frac{1}{2}}}^{\mu_2^2} \frac{2x^2(\beta - \frac{\alpha}{\mu_2}x^2)\alpha(\alpha+\beta)\mu_2^{-1}}{\alpha(\beta+(\alpha+\beta)\mu_2 x + \frac{\alpha}{\mu_2}x^2)^3(\alpha\mu_2 + \frac{\alpha+\beta}{\mu_2}x + \beta x^2)} dx \\
& \quad (x = \mu_2^2 s) \\
&= \mu_2^{-5} \int_{\mu_2^{-\frac{3}{2}}}^{1} \frac{-2\alpha^2(\alpha+\beta)}{\alpha\beta(\alpha+\beta+\alpha s)^3 s} ds + h.o.t \\
&= -\frac{3}{(\alpha+\beta)^2}\mu_2^{-5}\ln\mu_2 + h.o.t,
\end{aligned}
$$

and

$$
i_{43} \le cd^{-5}.
$$

So from the above three expansions, we can get the expansion for B_4,

$$
B_4 = O(\mu_2^{-5}).
$$

Next we consider B_5:

$$\begin{aligned}
B_5 &= \frac{1}{\pi}\int_0^\infty\int_0^{2\pi} -4t^{\alpha-1}\frac{\rho_1^2}{\rho_2}Z_{12}Z_{a_1,1}Z_{b_1,1}dtd\theta \\
&= \int_0^\infty -4t^{3\alpha-1}\rho_1^3\rho_2(\alpha\mu_2-\beta t^{2\alpha})\alpha(\alpha+\beta)dt \\
&\quad (t^\alpha = x) \\
&= \int_0^\infty \frac{-4x^2\alpha(\alpha+\beta)(\alpha\mu_2-\beta x^2)}{\alpha(\beta+(\alpha+\beta)dx+\frac{\alpha}{\mu_2}x^2)^3(\alpha\mu_2+\frac{\alpha+\beta}{\mu_2}x+\beta x^2)}dx \\
&= \int_0^{\mu_2^{\frac{1}{2}}} + \int_{\mu_2^{\frac{1}{2}}}^{\mu_2^2} + \int_{\mu_2^2}^\infty \frac{-4x^2\alpha(\alpha+\beta)(\alpha\mu_2-\beta x^2)}{\alpha(\beta+(\alpha+\beta)dx+\frac{\alpha}{\mu_2}x^2)^3(\alpha\mu_2+\frac{\alpha+\beta}{\mu_2}x+\beta x^2)}dx \\
&= i_{51}+i_{52}+i_{53},
\end{aligned}$$

where

$$\begin{aligned}
i_{51} &= \int_0^{\mu_2^{\frac{1}{2}}} \frac{-4x^2\alpha(\alpha+\beta)(\alpha\mu_2-\beta x^2)}{\alpha(\beta+(\alpha+\beta)dx+\frac{\alpha}{\mu_2}x^2)^3(\alpha\mu_2+\frac{\alpha+\beta}{\mu_2}x+\beta x^2)}dx \\
&\quad (x=\mu_2^{-1}s) \\
&= \mu_2^{-3}\int_0^{\mu_2^{\frac{3}{2}}} \frac{-4\alpha(\alpha+\beta)s^2}{\alpha(\beta+(\alpha+\beta)s)^3}ds + h.o.t \\
&= \frac{-6}{(\alpha+\beta)^2}\mu_2^{-3}\ln\mu_2 + h.o.t,
\end{aligned}$$

$$\begin{aligned}
i_{52} &= \int_{\mu_2^{\frac{1}{2}}}^{\mu_2^2} \frac{-4x^2\alpha(\alpha+\beta)(\alpha\mu_2-\beta x^2)}{\alpha(\beta+(\alpha+\beta)dx+\frac{\alpha}{\mu_2}x^2)^3(\alpha\mu_2+\frac{\alpha+\beta}{\mu_2}x+\beta x^2)}dx \\
&\quad (x=\mu_2^2 s) \\
&= \mu_2^{-3}\int_{\mu_2^{-\frac{3}{2}}}^1 \frac{4\alpha(\alpha+\beta)}{\alpha(\alpha+\beta+\alpha s)^3 s}ds + h.o.t \\
&= \frac{6}{(\alpha+\beta)^2}\mu_2^{-3}\ln\mu_2 + h.o.t,
\end{aligned}$$

and

$$i_{53} \le c\mu_2^{-3}.$$

So

$$B_5 = O(\mu_2^{-3}).$$

Next we consider $B_6 - B_{16}$:

$$\begin{aligned}
B_6 &= \frac{1}{\pi}\int_0^\infty \int_0^{2\pi} 2t^{\alpha-1}\frac{\rho_1^2}{\rho_2} Z_{12} Z_{a_1,1} Z_{b_1,2} dt d\theta \\
&= \mu_2^{-2}\int_0^\infty \frac{2\alpha(\alpha+\beta)s}{\alpha(\beta+(\alpha+\beta)s)^3} ds + h.o.t,
\end{aligned}$$

$$\begin{aligned}
B_7 &= \frac{1}{\pi}\int_0^\infty \int_0^{2\pi} 2t^{\alpha-1}\frac{\rho_1^2}{\rho_2} Z_{12} Z_{a_1,2} Z_{b_1,1} dt d\theta \\
&= \mu_2^{-2}\int_0^\infty \frac{-2\alpha(\alpha+\beta)}{(\alpha+\beta+\alpha s)^3} ds + h.o.t,
\end{aligned}$$

$$\begin{aligned}
B_8 &= \frac{1}{\pi}\int_0^\infty -t^{\alpha-1}\frac{\rho_1^2}{\rho_2} Z_{12} Z_{a_1,2} Z_{b_1,2} dt d\theta \\
&= \mu_2^{-\frac{7}{2}}\int_0^\infty \frac{-\alpha(\alpha-\beta s^2)}{\alpha(\alpha+\beta s^2)^2} ds + h.o.t,
\end{aligned}$$

$$\begin{aligned}
B_9 &= \frac{1}{\pi}\int_0^\infty \int_0^{2\pi} 8t^{\beta-1}\frac{\rho_2^2}{\rho_1} Z_{12} Z_{a_1,2} Z_{b_1,2} dt d\theta \\
&= \mu_2^{-2}\int_0^\infty \frac{8\alpha(\alpha+\beta)^2 s^3(\alpha-\beta s^2)}{\alpha(\alpha+\beta s^2)^5} ds + h.o.t,
\end{aligned}$$

$$\begin{aligned}
B_{10} &= \frac{1}{\pi}\int_0^\infty \int_0^{2\pi} -4t^{\beta-1}\frac{\rho_2^2}{\rho_1} Z_{12} Z_{a_1,2} Z_{b_1,1} dt d\theta \\
&= \mu_2^{-2}\int_0^\infty \frac{-4\alpha(\alpha+\beta) s^3(\alpha-\beta s^2)}{\alpha(\alpha+\beta s^2)^4} ds + h.o.t,
\end{aligned}$$

$$\begin{aligned}
B_{11} &= \frac{1}{\pi}\int_0^\infty \int_0^{2\pi} -4t^{\beta-1}\frac{\rho_2^2}{\rho_1} Z_{12} Z_{a_1,1} Z_{b_1,2} dt d\theta \\
&= \mu_2^{-2}\int_0^\infty \frac{-4\alpha(\alpha+\beta) s(\alpha-\beta s^2)}{\alpha(\alpha+\beta s^2)^4} ds + h.o.t,
\end{aligned}$$

$$\begin{aligned}
B_{12} &= \frac{1}{\pi}\int_0^\infty \int_0^{2\pi} 2t^{\beta-1}\frac{\rho_2^2}{\rho_1} Z_{12} Z_{a_1,1} Z_{b_1,1} \\
&= \mu_2^{-2}\int_0^\infty \frac{2\alpha s(\alpha-\beta s^2)}{\alpha(\alpha+\beta s^2)^3} ds + h.o.t,
\end{aligned}$$

$$
\begin{aligned}
B_{13} &= \frac{1}{\pi}\int_0^\infty\int_0^{2\pi} -4t^{\beta-1}\frac{\rho_2^2}{\rho_1}Z_{11}Z_{a_1,2}Z_{b_1,2}dtd\theta \\
&= \mu_2^{-\frac{7}{2}}\int_0^\infty \frac{-4\alpha(\alpha+\beta)s^2(\beta-\alpha s^2)}{\alpha(\alpha+\beta s^2)^4}ds + h.o.t,
\end{aligned}
$$

$$
\begin{aligned}
B_{14} &= \frac{1}{\pi}\int_0^\infty\int_0^{2\pi} 2t^{\beta-1}\frac{\rho_2^2}{\rho_1}Z_{11}Z_{a_1,2}Z_{b_1,1}dtd\theta \\
&= \mu_2^{-\frac{7}{2}}\int_0^\infty \frac{2\alpha s^2(\beta-\alpha s^2)}{\alpha(\alpha+\beta s^2)^3}ds + h.o.t,
\end{aligned}
$$

$$
\begin{aligned}
B_{15} &= \frac{1}{\pi}\int_0^\infty\int_0^{2\pi} 2t^{\beta-1}\frac{\rho_2^2}{\rho_1}Z_{11}Z_{a_1,1}Z_{b_1,2}dtd\theta \\
&= \mu_2^{-\frac{7}{2}}\int_0^\infty \frac{2\alpha(\beta-\alpha s^2)}{\alpha(\alpha+\beta s^2)^3}ds + h.o.t,
\end{aligned}
$$

$$
\begin{aligned}
B_{16} &= \frac{1}{\pi}\int_0^\infty\int_0^{2\pi} -t^{\beta-1}\frac{\rho_2^2}{\rho_1}Z_{11}Z_{a_1,1}Z_{b_1,1}dtd\theta \\
&= \mu_2^{-\frac{7}{2}}\int_0^\infty \frac{-\alpha(\beta-\alpha s^2)}{\alpha(\alpha+\beta)(\alpha+bs^2)^2}ds + h.o.t.
\end{aligned}
$$

So by combining the above 16 expansions, we have

$$\mathcal{B}_1 = \gamma_4\mu_2^{-2} + h.o.t,$$

where γ_4 is given in the above expansions. By direct calculation, one can get

$$
\begin{aligned}
\gamma_4 &= \frac{1}{\alpha}\int_0^\infty [\frac{16s^2}{(1+2s)^5} - \frac{8s}{(1+2s)^4} + \frac{8s}{(2+s)^4} \\
&+ \frac{4s}{(1+2s)^3} - \frac{4}{(2+s)^3} + \frac{32(1-s^2)s^3}{(1+s^2)^5} \\
&- \frac{8(1-s^2)s^3}{(1+s^2)^4} - \frac{8s(1-s^2)}{(1+s^2)^4} + \frac{2(1-s^2)s}{(1+s^2)^3}]ds \\
&= \frac{1}{3\alpha} \neq 0.
\end{aligned}
$$

We have

$$\mathcal{A}_1\mathcal{C}_1 - \mathcal{B}_1^2 = \gamma_4^2\mu_2^{-2} + h.o.t \neq 0, \tag{4.22}$$

for μ_2 large enough.

Similarly, we have

$$\mathcal{E}_1 = \frac{1}{4\alpha}\mu_2^{-4}\ln\mu_2 + h.o.t \neq 0.$$

So for μ_2 large, when $N_1 = N_2$, we can still get that the matrix $\mathbf{Q}_1$ is non-degenerate. Thus, we first fix $\mu_1 = 1$, μ_2 large, we can choose $\xi_1 = 0$, ξ_2 large such that $\mathbf{Q}$ is non-degenerate.

4.3 The Calculations of the Matrix $\tilde{\mathbf{Q}}_1$

In this subsection, we give the computations of $\tilde{\mathbf{Q}}_1$. As mentioned in Section 3, we set $\lambda_0 = 1$ and let λ_1 large enough. If we denote by $\alpha = N_1 + 1$ and $\beta = N_2 + 1$. We will show that for λ_1 large, the elements of the matrix $\tilde{\mathbf{Q}}_1$ and $\tilde{\mathbf{Q}}_2$ can not both be zero.

For simplicity of notation, let us first recall

$$\begin{aligned}
\rho_{1,B}^{-1} &= 1 + \lambda_1 r^{2\alpha} + \frac{1}{(8\alpha\beta(\alpha+\beta))^2\lambda_1}r^{2(\alpha+\beta)} + \frac{1}{(8\alpha(\alpha+\beta)(2\alpha+\beta))^2}r^{2(2\alpha+\beta)} \\
&= 1 + \lambda_1 r^{2\alpha} + \frac{a_1}{\lambda_1}r^{2(\alpha+\beta)} + a_2 r^{2(2\alpha+\beta)}, \\
\rho_{2,B}^{-1} &= 4\Big[\lambda_1\alpha^2 + \frac{1}{(8\alpha\beta)^2\lambda_1}r^{2\beta} + \frac{2}{(8\alpha(\alpha+\beta))^2}r^{2(\alpha+\beta)} + \frac{\lambda_1}{(8\alpha(2\alpha+\beta))^2}r^{2(2\alpha+\beta)} \\
&+ \frac{1}{(8\beta(\alpha+\beta))^2(8\alpha(\alpha+\beta)(2\alpha+\beta))^2\lambda_1}r^{4(\alpha+\beta)}\Big] \\
&= b_1\lambda_1 + \frac{b_2}{\lambda_1}r^{2\beta} + b_3 r^{2(\alpha+\beta)} + b_4\lambda_1 r^{2(2\alpha+\beta)} + \frac{b_5}{\lambda_1}r^{4(\alpha+\beta)}, \\
Z_{1,1} &= \rho_{1,B}\Big(1 - \frac{1}{(8\alpha(\alpha+\beta)(2\alpha+\beta))^2}r^{2(2\alpha+\beta)}\Big) \\
&= \rho_{1,B}(1 + c_1 r^{2(2\alpha+\beta)}),
\end{aligned}$$

$$\begin{aligned}
Z_{1,2} &= 4\rho_{2,B}\Big[\alpha^2\lambda_1 + \frac{1}{(8\alpha\beta)^2\lambda_1}r^{2\beta} - \frac{\lambda_1}{(8\alpha(2\alpha+\beta))^2}r^{2(2\alpha+\beta)} \\
&- \frac{1}{(8(\alpha+\beta)(2\alpha+\beta))^2(8\alpha\beta(\alpha+\beta))^2\lambda_1}r^{4(\alpha+\beta)}\Big] \\
&= \rho_{2,B}\Big(d_1\lambda_1 + \frac{d_2}{\lambda_1}r^{2\beta} + d_3\lambda_1 r^{2(2\alpha+\beta)} + \frac{d_4}{\lambda_1}r^{4(\alpha+\beta)}\Big),
\end{aligned}$$

$$
\begin{aligned}
Z_{c_{32},1} &= \rho_{1,B}\left(\frac{\beta\lambda_1}{(2\alpha+\beta)}r^{\alpha} + \frac{1}{(8\alpha(\alpha+\beta)(2\alpha+\beta))^2}r^{3\alpha+2\beta}\right) \\
&= \rho_{1,B}(e_1\lambda_1 r^{\alpha} + e_2 r^{3\alpha+2\beta}), \\
Z_{c_{32},2} &= 4\rho_{2,B}\Big[\frac{2}{(8\alpha)^2(\alpha+\beta)(2\alpha+\beta)}r^{\alpha+2\beta} + \frac{2\beta\lambda_1}{(8\alpha(2\alpha+\beta))^2(\alpha+\beta)}r^{3\alpha+2\beta}\Big] \\
&= \rho_{2,B}(f_1 r^{\alpha+2\beta} + f_2\lambda_1 r^{3\alpha+2\beta}), \\
Z_{c_{31},1} &= \rho_{1,B}\left(-\frac{1}{(8\alpha(\alpha+\beta))^2\beta(2\alpha+\beta)\lambda_1}r^{\alpha+\beta} + \frac{1}{(8\alpha(\alpha+\beta)(2\alpha+\beta))^2}r^{3\alpha+\beta}\right) \\
&= \rho_{1,B}\left(\frac{g_1}{\lambda_1}r^{\alpha+\beta} + g_2 r^{3\alpha+\beta}\right), \\
Z_{c_{31},2} &= 4\rho_{2,B}\Big[\frac{2r^{\alpha+\beta}}{(8(\alpha+\beta))^2\alpha(2\alpha+\beta)} - \frac{2r^{3\alpha+3\beta}}{(8(\alpha+\beta))^2(8\alpha(\alpha+\beta)(2\alpha+\beta))^2\alpha\beta\lambda_1}\Big] \\
&= \rho_{2,B}\left(h_1 r^{\alpha+\beta} + \frac{h_2}{\lambda_1}r^{3\alpha+3\beta}\right), \\
Z_{c_{30},1} &= \frac{1}{(8\alpha(\alpha+\beta)(2\alpha+\beta))^2}\rho_{1,B}r^{2\alpha+\beta} \\
&= \rho_{1,B}k_1 r^{2\alpha+\beta}, \\
Z_{c_{30},2} &= -4\rho_{2,B}\Big[\frac{\lambda_1 r^{2\alpha+\beta}}{(8(2\alpha+\beta))^2\alpha(\alpha+\beta)} + \frac{r^{2\alpha+3\beta}}{(8\alpha\beta(\alpha+\beta))^2(8(2\alpha+\beta))^2\alpha(\alpha+\beta)\lambda_1}\Big] \\
&= \rho_{2,B}\left(l_1\lambda_1 + \frac{l_2}{\lambda_1}r^{2\beta}\right)r^{2\alpha+\beta}, \\
Z_{c_{21},1} &= \frac{1}{(8\alpha\beta(\alpha+\beta))^2\lambda_1}\rho_{1,B}r^{2\alpha+\beta} \\
&= \rho_{1,B}\frac{m_1}{\lambda_1}r^{2\alpha+\beta}, \\
Z_{c_{21},2} &= 4\rho_{2,B}\Big[\frac{r^{\beta}}{(8\beta)^2\alpha(\alpha+\beta)\lambda_1} + \frac{r^{\beta}}{(8\beta)^2\alpha(\alpha+\beta)(8\alpha(\alpha+\beta)(2\alpha+\beta))^2\lambda_1}r^{2(2\alpha+\beta)}\Big] \\
&= \rho_{2,B}\left(\frac{n_1}{\lambda_1} + \frac{n_2}{\lambda_1}r^{2(2\alpha+\beta)}\right)r^{\beta}.
\end{aligned}
$$

The asymptotic behavior of the coefficient matrix in the case of $(\lambda_0, \lambda_1) = (1, \lambda_1)$ as $\lambda_1 \to +\infty$ can be estimated as follows, let us first consider $\tilde{\mathcal{C}}_1$:

To start with, first, by a change of variable $t = r^2$, from (3.78), one can get that

$$
\begin{aligned}
\frac{\tilde{\mathcal{C}}_1}{\pi} &= \frac{1}{2\pi}\int_0^\infty\int_0^{2\pi}\Big[t^{\alpha-1}\frac{\rho_{1,B}^2}{\rho_{2,B}}(2Z_{1,1}-Z_{1,2})(2Z_{c_{32,1},1}-Z_{c_{32,1},2})^2 \\
&+ 4t^{\beta-1}\frac{\rho_{2,B}^2}{\rho_{1,B}^2}(Z_{1,2}-Z_{1,1})(Z_{c_{32,1},2}-Z_{c_{32,1},1})^2\Big]dtd\theta \\
&= (I_1+I_2+I_3+\cdots+I_{12}),
\end{aligned}
$$

where $I_i, i=1,\cdots,12$ are defined below.

First let us estimate I_1, we will use some change of variable to get the asymptotic expansion:

$$
\begin{aligned}
I_1 &= \frac{1}{\pi}\int_0^{2\pi}\int_0^\infty 8t^{\alpha-1}\frac{\rho_{1,B}^2}{\rho_{2,B}}Z_{1,1}Z_{c_{32},1}^2dtd\theta \\
&= \int_0^\infty 8\frac{\rho_{1,B}^5}{\rho_{2,B}}t^{2\alpha-1}(e_1\lambda_1+e_2t^{\alpha+\beta})^2(1+c_1t^{2\alpha+\beta})dt \\
&= \int_0^\infty \frac{1}{(1+\lambda_1t^\alpha+\frac{a_1}{\lambda_1}t^{\alpha+\beta}+a_2t^{2\alpha+\beta})^5}\Big(8t^{2\alpha-1}(b_1\lambda_1+\frac{b_2}{\lambda_1}t^\beta+b_3t^{\alpha+\beta}+b_4\lambda_1t^{2\alpha+\beta} \\
&+ \frac{b_5}{\lambda_1}t^{2(\alpha+\beta)})(e_1\lambda_1+e_2t^{\alpha+\beta})^2(1+c_1t^{2\alpha+\beta})\Big)dt \\
&\quad (t=\lambda_1^{\frac{1}{\alpha+\beta}}s) \\
&= \lambda_1^{-\frac{\alpha}{\alpha+\beta}}\int_0^\infty\frac{8b_4c_1s^{\alpha+2\beta-1}(e_1+e_2s^{\alpha+\beta})^2}{(1+a_2s^{\alpha+\beta})^5}ds+O(\lambda_1^{-\frac{3\alpha+\beta}{\alpha+\beta}}),
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &= -\frac{4}{\pi}\int_0^{2\pi}\int_0^\infty t^{\alpha-1}\frac{\rho_{1,B}^2}{\rho_{2,B}}Z_{1,2}Z_{c_{32},1}^2dtd\theta \\
&= -4\int_0^\infty t^{2\alpha-1}\rho_{1,B}^4(e_1\lambda_1+e_2t^{\alpha+\beta})^2(d_1\lambda_1+\frac{d_2}{\lambda_1}t^\beta+d_3\lambda_1t^{2\alpha+\beta}+\frac{d_4}{\lambda_1}t^{2(\alpha+\beta)})dt \\
&= -4\int_0^\infty\frac{t^{2\alpha-1}(e_1\lambda_1+e_2t^{\alpha+\beta})^2(d_1\lambda_1+\frac{d_2}{\lambda_1}t^\beta+d_3\lambda_1t^{2\alpha+\beta}+\frac{d_4}{\lambda_1}t^{2(\alpha+\beta)})}{(1+\lambda_1t^\alpha+\frac{a_1}{\lambda_1}t^{\alpha+\beta}+a_2t^{2\alpha+\beta})^4}dt \\
&\quad (t=\lambda_1^{\frac{1}{\alpha+\beta}}s) \\
&= -\lambda_1^{-\frac{\alpha}{\alpha+\beta}}\int_0^\infty\frac{4d_3(e_1+e_2s^{\alpha+\beta})^2s^{\beta-1}}{(1+a_2s^{\alpha+\beta})^4}ds+O(\lambda_1^{-\frac{3\alpha+\beta}{\alpha+\beta}}).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
I_3 &= -\frac{8}{\pi}\int_0^{2\pi}\int_0^{\infty} t^{\alpha-1}\frac{\rho_{1,B}^2}{\rho_{2,B}} Z_{1,1}Z_{c32,1}Z_{c32,2}dtd\theta \\
&= -8\int_0^{\infty} t^{\alpha-1}\rho_{1,B}^4(e_1\lambda_1+e_2t^{\alpha+\beta})t^{\frac{\alpha}{2}}(f_1t^{\frac{\alpha+2\beta}{2}}+f_2\lambda_1t^{\frac{3\alpha+2\beta}{2}})(1+c_1t^{2\alpha+\beta})dt \\
&= -8\int_0^{\infty} \frac{t^{\alpha-1}(e_1\lambda_1+e_2t^{\alpha+\beta})t^{\frac{\alpha}{2}}(f_1t^{\frac{\alpha+2\beta}{2}}+f_2\lambda_1t^{\frac{3\alpha+2\beta}{2}})(1+c_1t^{2\alpha+\beta})}{(1+\lambda_1t^{\alpha}+\frac{a_1}{\lambda_1}t^{\alpha+\beta}+a_2t^{2\alpha+\beta})^4}dt \\
&\quad (t=\lambda_1^{\frac{1}{\alpha+\beta}}s) \\
&= -\lambda_1^{-\frac{\alpha}{\alpha+\beta}}\int_0^{\infty}\frac{8c_1f_2s^{\alpha+2\beta-1}(e_1+e_2s^{\alpha+\beta})}{(1+a_2s^{\alpha+\beta})^4}ds+O(\lambda_1^{-\frac{3\alpha+\beta}{\alpha+\beta}}),
\end{aligned}
$$

$$
\begin{aligned}
I_4 &= \frac{4}{\pi}\int_0^{2\pi}\int_0^{\infty} t^{\alpha-1}\frac{\rho_{1,B}^2}{\rho_{2,B}} Z_{1,2}Z_{c32,1}Z_{c32,2}dtd\theta \\
&= 4\int_0^{\infty} t^{\alpha-1}\rho_{1,B}^3\rho_{2,B}(e_1\lambda_1+e_2t^{\alpha+\beta})t^{\frac{\alpha}{2}}(f_1t^{\frac{\alpha+2\beta}{2}}+f_2\lambda_1t^{\frac{3\alpha+2\beta}{2}})(d_1\lambda_1+\frac{d_2}{\lambda_1}t^{\beta} \\
&+ d_3\lambda_1t^{2\alpha+\beta}+\frac{d_4}{\lambda_1}t^{2(\alpha+\beta)})dt \\
&= 4\int_0^{\infty}(1+\lambda_1t^{\alpha}+\frac{a_1}{\lambda_1}t^{\alpha+\beta}+a_2t^{2\alpha+\beta})^{-3} \\
&\times (b_1\lambda_1+\frac{b_2}{\lambda_1}t^{\beta}+b_3t^{\alpha+\beta}+b_4\lambda_1t^{2\alpha+\beta}+\frac{b_5}{\lambda_1}t^{2(\alpha+\beta)})^{-1} \\
&\times t^{\alpha-1}\rho_{1,B}^3\rho_{2,B}(e_1\lambda_1+e_2t^{\alpha+\beta})t^{\frac{\alpha}{2}}(f_1t^{\frac{\alpha+2\beta}{2}}+f_2\lambda_1t^{\frac{3\alpha+2\beta}{2}}) \\
&\times (d_1\lambda_1+\frac{d_2}{\lambda_1}t^{\beta}+d_3\lambda_1t^{2\alpha+\beta}+\frac{d_4}{\lambda_1}t^{2(\alpha+\beta)})dt \\
&\quad (t=\lambda_1^{\frac{1}{\alpha+\beta}}s) \\
&= \lambda_1^{-\frac{\alpha}{\alpha+\beta}}\int_0^{\infty}\frac{4s^{\beta-1}(e_1+e_2s^{\alpha+\beta})d_3f_2}{b_4(1+a_2s^{\alpha+\beta})^3}ds+O(\lambda_1^{-\frac{3\alpha+\beta}{\alpha+\beta}}),
\end{aligned}
$$

$$
\begin{aligned}
I_5 &= \frac{2}{\pi}\int_0^{2\pi}\int_0^{\infty} t^{\alpha-1}\frac{\rho_{1,B}^2}{\rho_{2,B}} Z_{1,1}Z_{c_{32},2}^2 dtd\theta \\
&= \int_0^{\infty} 2t^{\alpha-1}\rho_{1,B}^3\rho_{2,B}(f_1 t^{\frac{\alpha+2\beta}{2}} + f_2\lambda_1 t^{\frac{3\alpha+2\beta}{2}})^2(1+c_1 t^{2\alpha+\beta}) \\
&= \int_0^{\infty}(1+\lambda_1 t^{\alpha} + \frac{a_1}{\lambda_1}t^{\alpha+\beta} + a_2 t^{2\alpha+\beta})^{-3} \\
&\times\ (b_1\lambda_1 + \frac{b_2}{\lambda_1}t^{\beta} + b_3 t^{\alpha+\beta} + b_4\lambda_1 t^{2\alpha+\beta} + \frac{b_5}{\lambda_1}t^{2(\alpha+\beta)})^{-1} \\
&\times\ 2t^{\alpha-1}(f_1 t^{\frac{\alpha+2\beta}{2}} + f_2\lambda_1 t^{\frac{3\alpha+2\beta}{2}})^2(1+c_1 t^{2\alpha+\beta})dt \\
&\quad (t = \lambda_1^{\frac{1}{\alpha+\beta}}) \\
&= \lambda_1^{-\frac{\alpha}{\alpha+\beta}}\int_0^{\infty}\frac{2c_1 f_2^2 s^{\alpha+2\beta-1}}{b_4(1+a_2 s^{\alpha+\beta})^3}ds + O(\lambda_1^{-\frac{3\alpha+\beta}{\alpha+\beta}}),
\end{aligned}
$$

$$
\begin{aligned}
I_6 &= -\frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty} t^{\alpha-1}\frac{\rho_{1,B}^2}{\rho_{2,B}} Z_{1,2}Z_{c_{32},2}^2 dtd\theta \\
&= -\int_0^{\infty} t^{\alpha-1}\rho_{1,B}^2\rho_{2,B}^2(f_1 + f_1 t^{\frac{\alpha+2\beta}{2}} + f_2\lambda_1 t^{\frac{3\alpha+2\beta}{2}})^2(d_1\lambda_1 + \frac{d_2}{\lambda_1}t^{\beta} + d_3\lambda_1 t^{2\alpha+\beta} \\
&+\ \frac{d_4}{\lambda_1}t^{2(\alpha+\beta)})dt \\
&= \int_0^{\infty}(1+\lambda_1 t^{\alpha} + \frac{a_1}{\lambda_1}t^{\alpha+\beta} + a_2 t^{2\alpha+\beta})^{-2} \\
&\times\ (b_1\lambda_1 + \frac{b_2}{\lambda_1}t^{\beta} + b_3 t^{\alpha+\beta} + b_4\lambda_1 t^{2\alpha+\beta} + \frac{b_5}{\lambda_1}t^{2(\alpha+\beta)})^{-2} \\
&\times\ t^{\alpha-1}(f_1 + f_1 t^{\frac{\alpha+2\beta}{2}} + f_2\lambda_1 t^{\frac{3\alpha+2\beta}{2}})^2(d_1\lambda_1 + \frac{d_2}{\lambda_1}t^{\beta} + d_3\lambda_1 t^{2\alpha+\beta} + \frac{d_4}{\lambda_1}t^{2(\alpha+\beta)})dt \\
&\quad (t = \lambda_1^{\frac{1}{\alpha+\beta}}) \\
&= -\lambda_1^{-\frac{\alpha}{\alpha+\beta}}\int_0^{\infty}\frac{d_3 f_2^2 s^{\beta-1}}{b_4^2(1+a_2 s^{\alpha+\beta})^2}ds + O(\lambda_1^{-\frac{3\alpha+\beta}{\alpha+\beta}}).
\end{aligned}
$$

Next we consider $I_7 - I_{12}$, as we see from below that , we can get the asymptotic expansion directly from the integral without any change of variables which definitely simplify our computations:

$$
\begin{aligned}
I_7 &= \frac{1}{\pi}\int_0^{2\pi}\int_0^\infty 4t^{\beta-1}\frac{\rho_{2,B}^2}{\rho_{1,B}^2}Z_{1,2}Z_{c_{32},2}^2 dtd\theta \\
&= \int_0^\infty 4t^{\alpha+3\beta-1}\frac{\rho_{2,B}^5}{\rho_{1,B}^2}(f_1+f_2\lambda_1 t^\alpha)^2(d_1\lambda_1+\frac{d_2}{\lambda_1}t^\beta+d_3\lambda_1 t^{2\alpha+\beta}+\frac{d_4}{\lambda_1}t^{2\alpha+2\beta})dt \\
&= \int_0^\infty \frac{4f_2^2 t^{5\alpha+3\beta-1}(d_1+d_3t^{2\alpha+\beta})}{(b_1+b_4t^{2\alpha+\beta})^5}dt-\lambda_1^{-1}\int_0^\infty \frac{20f_2^2b_3t^{6\alpha+4\beta-1}(d_1+d_3t^{2\alpha+\beta})}{(b_1+b_4t^{2\alpha+\beta})^6}dt \\
&+ \lambda_1^{-1}\int_0^\infty \frac{8t^{3\alpha+2\beta-1}(d_1+d_3t^{2\alpha+\beta})}{(b_1+b_4t^{2\alpha+\beta})^5}[f_2t^{\frac{\alpha}{2}}f_1t^{\frac{\alpha}{2}+\beta}+f_2^2t^{\alpha+\beta}(1+a_2t^{2\alpha+\beta})]dt \\
&+ O(\lambda_1^{-2}), \\
I_8 &= -\frac{1}{\pi}\int_0^{2\pi}\int_0^\infty 4t^{\beta-1}\frac{\rho_{2,B}^2}{\rho_{1,B}^2}Z_{1,1}Z_{c_{32},2}^2 dtd\theta \\
&= -4\int_0^\infty t^{\beta-1}\frac{\rho_{2,B}^4}{\rho_{1,B}}(f_1t^{\frac{\alpha+2\beta}{2}}+f_2\lambda_1t^{\frac{3\alpha+2\beta}{2}})^2(1+c_1t^{2\alpha+\beta})dt \\
&= -4\int_0^\infty (b_1\lambda_1+\frac{b_2}{\lambda_1}t^\beta+b_3t^{\alpha+\beta}+b_4\lambda_1t^{2\alpha+\beta}+\frac{b_5}{\lambda_1}t^{2(\alpha+\beta)})^{-4} \\
&\times (1+\lambda_1t^\alpha+\frac{a_1}{\lambda_1}t^{\alpha+\beta}+a_2t^{2\alpha+\beta})t^{\beta-1}(f_1t^{\frac{\alpha+2\beta}{2}}+f_2\lambda_1t^{\frac{3\alpha+2\beta}{2}})^2(1+c_1t^{2\alpha+\beta})dt \\
&= -4\lambda_1^{-1}\int_0^\infty \frac{f_2^2t^{4\alpha+3\beta-1}(1+c_1t^{2\alpha+\beta})}{(b_1+b_4t^{2\alpha+\beta})^4}dt+O(\lambda_1^{-2}), \\
I_9 &= -\frac{8}{\pi}\int_0^{2\pi}\int_0^\infty t^{\beta-1}\frac{\rho_{2,B}^2}{\rho_{1,B}^2}Z_{1,2}Z_{c_{32},1}Z_{c_{32},2}dtd\theta \\
&= -8\int_0^\infty t^{\beta-1}\frac{\rho_{2,B}^4}{\rho_{1,B}}(f_1+f_2\lambda_1t^\alpha)(e_1\lambda_1+e_2t^{\alpha+\beta})(d_1\lambda_1+\frac{d_2}{\lambda_1}t^\beta+d_3\lambda_1t^{2\alpha+\beta} \\
&+ \frac{d_4}{\lambda_1}t^{2\alpha+2\beta})t^{\alpha+\beta}dt \\
&= -8\int_0^\infty \frac{f_2e_1t^{3\alpha+2\beta-1}(d_1+d_3t^{2\alpha+\beta})}{(b_1+b_4t^{2\alpha+\beta})^4}dt+8\lambda_1^{-1}\int_0^\infty \frac{4f_2e_1b_3t^{4\alpha+3\beta-1}(d_1+d_3t^{2\alpha+\beta})}{(b_1+b_4t^{2\alpha+\beta})^5}dt \\
&- 8\lambda_1^{-1}\int_0^\infty \frac{(d_1+d_3t^{2\alpha+\beta})}{(b_1+b_4t^{2\alpha+\beta})^4}[e_1t^{\frac{3\alpha}{2}+\beta-1}f_1t^{\frac{\alpha+2\beta}{2}}+f_2e_2t^{4\alpha+3\beta-1} \\
&+ f_2t^{2\alpha+2\beta-1}(1+a_2t^{2\alpha+\beta})]dt+O(\lambda_1^{-2}).
\end{aligned}
$$

$$
\begin{aligned}
I_{10} &= \frac{8}{\pi}\int_0^{2\pi}\int_0^{\infty} t^{\beta-1}\frac{\rho_{2,B}^2}{\rho_{1,B}^2} Z_{1,1}Z_{c_{32},1}Z_{c_{32},2}dtd\theta \\
&= 8\int_0^{\infty} t^{\alpha+2\beta-1}\rho_{2,B}^3(f_1+\lambda_1 f_2 t^{\beta})(e_1\lambda_1+e_2t^{\alpha+\beta})(1+c_1t^{2\alpha+\beta})dt \\
&= 8\int_0^{\infty}(b_1\lambda_1+\frac{b_2}{\lambda_1}t^{\beta}+b_3t^{\alpha+\beta}+b_4\lambda_1t^{2\alpha+\beta}+\frac{b_5}{\lambda_1}t^{2(\alpha+\beta)})^{-3} \\
&\times\ t^{\alpha+2\beta-1}(f_1+\lambda_1 f_2t^{\beta})(e_1\lambda_1+e_2t^{\alpha+\beta})(1+c_1t^{2\alpha+\beta})dt \\
&= \lambda_1^{-1}\int_0^{\infty}\frac{8t^{2\alpha+2\beta-1}f_2e_1(1+c_1t^{2\alpha+\beta})}{(b_1+b_4t^{2\alpha+\beta})^3}dt+O(\lambda_1^{-2}),
\end{aligned}
$$

$$
\begin{aligned}
I_{11} &= \frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty} 4t^{\beta-1}\frac{\rho_{2,B}^2}{\rho_{1,B}^2} Z_{1,2}Z_{c_{32},1}^2dtd\theta \\
&= 4\int_0^{\infty} t^{\alpha+\beta-1}\rho_{2,B}^3(e_1\lambda_1+e_2t^{\alpha+\beta})^2(d_1\lambda_1+\frac{d_2}{\lambda_1}t^{\beta}+d_3\lambda_1t^{2\alpha+\beta}+\frac{d_4}{\lambda_1}t^{2\alpha+2\beta})dt \\
&= 4\int_0^{\infty}(b_1\lambda_1+\frac{b_2}{\lambda_1}t^{\beta}+b_3t^{\alpha+\beta}+b_4\lambda_1t^{2\alpha+\beta}+\frac{b_5}{\lambda_1}t^{2(\alpha+\beta)})^{-3} \\
&\times\ t^{\alpha+\beta-1}(e_1\lambda_1+e_2t^{\alpha+\beta})^2(d_1\lambda_1+\frac{d_2}{\lambda_1}t^{\beta}+d_3\lambda_1t^{2\alpha+\beta}+\frac{d_4}{\lambda_1}t^{2\alpha+2\beta})dt \\
&= \int_0^{\infty}\frac{4e_1^2t^{\alpha+\beta-1}(d_1+d_3t^{2\alpha+\beta})}{(b_1+b_4t^{2\alpha+\beta})^3}-\lambda_1^{-1}\int_0^{\infty}\frac{12e_1^2b_3t^{2\alpha+2\beta-1}(d_1+d_3t^{2\alpha+\beta})}{(b_1+b_4t^{2\alpha+\beta})^4}dt \\
&+\ \lambda_1^{-1}\int_0^{\infty}\frac{8e_1e_2t^{2\alpha+2\beta-1}(d_1+d_3t^{2\alpha+\beta})}{(b_1+b_4t^{2\alpha+\beta})^3}+O(\lambda_1^{-2}),
\end{aligned}
$$

$$
\begin{aligned}
I_{12} &= -\frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty} 4t^{\beta-1}\frac{\rho_{2,B}^2}{\rho_{1,B}^2} Z_{1,1}Z_{c_{32},1}^2dtd\theta \\
&= -4\int_0^{\infty} t^{\alpha+\beta-1}\rho_{2,B}^2\rho_{1,B}(e_1\lambda_1+e_2t^{\alpha+\beta})^2(1+c_1t^{2\alpha+\beta})dt \\
&= -4\int_0^{\infty}(b_1\lambda_1+\frac{b_2}{\lambda_1}t^{\beta}+b_3t^{\alpha+\beta}+b_4\lambda_1t^{2\alpha+\beta}+\frac{b_5}{\lambda_1}t^{2(\alpha+\beta)})^{-2} \\
&\times\ (1+\lambda_1t^{\alpha}+\frac{a_1}{\lambda_1}t^{\alpha+\beta}+a_2t^{2\alpha+\beta})^{-1}t^{\alpha+\beta-1}\rho_{2,B}^2\rho_{1,B}(e_1\lambda_1+e_2t^{\alpha+\beta})^2(1+c_1t^{2\alpha+\beta})dt \\
&= \lambda_1^{-1}\int_0^{\infty}\frac{-4e_1^2t^{\beta-1}(1+c_1t^{2\alpha+\beta})}{(b_1+b_4t^{2\alpha+\beta})^2}dt+O(\lambda_1^{-2}).
\end{aligned}
$$

Note that the $O(1)$ terms are zero, so by summing up the above 12 expansions, we can get the asymptotic behavior of $\tilde{\mathcal{C}}_1$ as $\lambda_1 \to \infty$ is

$$\begin{aligned}
\frac{\tilde{\mathcal{C}}_1}{\pi} &= \lambda_1^{-\frac{\alpha}{\alpha+\beta}} \int_0^\infty \Big[\frac{8b_4c_1s^{\alpha+2\beta-1}(e_1+e_2s^{\alpha+\beta})^2}{(1+a_2s^{\alpha+\beta})^5} - \frac{4d_3(e_1+e_2s^{\alpha+\beta})^2s^{\beta-1}}{(1+a_2s^{\alpha+\beta})^4} \\
&- \frac{8c_1f_2s^{\alpha+2\beta-1}(e_1+e_2s^{\alpha+\beta})}{(1+a_2s^{\alpha+\beta})^4} + \frac{4s^{\beta-1}(e_1+e_2s^{\alpha+\beta})d_3f_2}{b_4(1+a_2s^{\alpha+\beta})^3} \\
&+ \frac{2c_1f_2^2s^{\alpha+2\beta-1}}{b_4(1+a_2s^{\alpha+\beta})^3} - \frac{d_3f_2^2s^{\beta-1}}{b_4^2(1+a_2s^{\alpha+\beta})^2}\Big]dt + h.o.t \\
&= \lambda_1^{-\frac{\alpha}{\alpha+\beta}} \frac{2^{-1+\frac{6\beta}{\alpha+\beta}}\alpha\beta^2(\alpha(\alpha+\beta)(2\alpha+\beta))^{-2\frac{\alpha}{\alpha+\beta}}\pi\csc(\frac{\pi\alpha}{\alpha+\beta})}{(\alpha+\beta)^2} + h.o.t \\
&= \lambda_1^{-\frac{\alpha}{\alpha+\beta}}\gamma_2 + h.o.t \neq 0.
\end{aligned}$$

Similarly, one can get that

$$\tilde{\mathcal{B}}_1 = \tilde{\mathcal{D}}_1 = \begin{cases} \gamma_5\lambda_1^{-1} + O(\lambda_1^{-2}), \text{ if } N_1 = N_2, \\ 0, \text{ if } N_1 \neq N_2, \end{cases}$$

where γ_5 is a constant.

$$\begin{aligned}
\tilde{\mathcal{A}}_1 &= -\lambda_1^{-2}\int \frac{4n_2^2c_1a_2t^{2\beta-1}}{(b_4+b_5t^\beta)^4}dt + h.o.t \\
&= \lambda_1^{-2}\gamma_1 + h.o.t \neq 0,
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\mathcal{E}}_1 &= \lambda_1^{-\frac{2\alpha+\beta}{\alpha+\beta}} \int_0^\infty \Big[\frac{8k_1^2c_1b_4t^{2\alpha+3\beta-1}}{(1+a_2t^{\alpha+\beta})^5} - \frac{4k_1^2d_3t^{\alpha+2\beta-1}}{(1+a_2t^{\alpha+\beta})^4} - \frac{8k_1l_1c_1t^{\alpha+2\beta-1}}{(1+a_2t^{\alpha+\beta})^4} \\
&+ \frac{4k_1l_1d_3t^{\beta-1}}{b_4(1+a_2t^{\alpha+\beta})^3} + \frac{2l_1^2c_1t^{\beta-1}}{b_4(1+a_2t^{\alpha+\beta})^3}\Big]dt + h.o.t \\
&= \lambda_1^{-\frac{2\alpha+\beta}{\alpha+\beta}} \frac{2^{-4-\frac{6\alpha}{\alpha+\beta}}\alpha(\alpha(\alpha+\beta)(2\alpha+\beta))^{-1-\frac{2\alpha}{\alpha+\beta}}\pi\csc(\frac{\pi\alpha}{\alpha+\beta})}{(\alpha+\beta)^3} + h.o.t \\
&= \gamma_3\lambda_1^{-\frac{2\alpha+\beta}{\alpha+\beta}} + h.o.t \neq 0,
\end{aligned}$$

and

$$
\begin{aligned}
\tilde{\mathcal{F}}_1 &= \lambda_1^{-\frac{3\alpha+2\beta}{\alpha+\beta}} \int_0^\infty \Big[\frac{8}{(1+a_2 s^{\alpha+\beta})^5}\Big((g_2^2 c_1 s^{\alpha+2\beta-1}(b_1 + b_3 s^{\alpha+\beta} \\
&+ b_5 s^{2(\alpha+\beta)}) + 2g_1 g_2 c_1 b_4 s^{2\alpha+3\beta-1} + b_4 g_2^2 s^{\alpha+2\beta-1})\Big) \\
&- \frac{4(2g_1 g_2 d_3 s^{\alpha+2\beta-1} + g_2^2(d_1 + d_4 s^{2(\alpha+\beta)})s^{\beta-1})}{(1+a_2 s^{\alpha+\beta})^4} \\
&- \frac{40 b_4 c_1 g_2^2 s^{2\alpha+3\beta-1}(1+a_1 s^{\alpha+\beta})}{(1+a_2 s^{\alpha+\beta})^6} + \frac{16 g_2^2 d_3 s^{\alpha+2\beta-1}(1+a_1 s^{\alpha+\beta})}{(1+a_2 s^{\alpha+\beta})^5} \\
&- \frac{8 g_2 c_1 s^{\alpha+2\beta-1}(h_1 + h_2 s^{\alpha+\beta})}{(1+a_2 s^{\alpha+\beta})^4} + \frac{4 g_2 d_3 s^{\beta-1}(h_1 + h_2 s^{\alpha+\beta})}{b_4(1+a_2 s^{\alpha+\beta})^3} \\
&- \frac{4 g_2^2 c_1 s^{\beta-1}}{b_4^2(1+a_2 s^{\alpha+\beta})}\Big] ds + h.o.t \\
&= -\lambda_1^{-\frac{3\alpha+2\beta}{\alpha+\beta}} \frac{2^{-3-\frac{6\alpha}{\alpha+\beta}}(\alpha-\beta)\alpha(\alpha+\beta)(2\alpha+\beta)^{-\frac{2\alpha}{\alpha+\beta}}(\alpha+2\beta)\pi \csc(\frac{\pi\alpha}{\alpha+\beta})}{3\beta(\alpha+\beta)^6} \\
&+ h.o.t \\
&= \gamma_4 \lambda_1^{-\frac{3\alpha+2\beta}{\alpha+\beta}} + h.o.t \neq 0.
\end{aligned}
$$

So we can get that

$$
\tilde{\mathcal{A}}_1\tilde{\mathcal{C}}_1 - \tilde{\mathcal{B}}_1\tilde{\mathcal{D}}_1 = \begin{cases} \lambda_1^{-\frac{3\alpha+\beta}{\alpha+\beta}}\gamma_1\gamma_2 + h.o.t, \text{ if } \gamma_5 = 0, \\ \lambda_1^{-2}\gamma_5^2, \text{ if } \gamma_5 \neq 0, \end{cases}
$$

In any case, this determinant is nonzero. Therefore, $\tilde{\mathcal{A}}_1\tilde{\mathcal{C}}_1 - \tilde{\mathcal{B}}_1\tilde{\mathcal{D}}_1$, $\tilde{\mathcal{E}}_1$ and $\tilde{\mathcal{F}}_1$ do not vanish for λ_1 large. So we get that the matrix $\tilde{\mathbf{Q}}_1$ is non-degenerate. Thus, we first fix $\lambda_0 = 1$, λ_1 large, and then choose ξ_1 large, $\xi_2 = 0$ such that $\tilde{\mathbf{Q}}$ is non-degenerate. □

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References

[1] L. Caffarelli and Y.S. Yang, Vortex condensation in the Chern-Simons Higgs model: An existence theorem, *Commun. Math. Phys.* 168(1995), 321-336.

[2] D. Chae and O.Y. Imanuvilov, The existence of non-topological multi-vortex solutions in the relativistic self-dual Chern-Simons theory, *Comm. Math. Phys.* 215(2000), 119-142.

[3] H. Chan, C.C. Fu and C.S. Lin, Non-topological multi-vortex solution to the self-dual Chern-Simons-Higgs equation, *Comm. Math. Phys.*, 231(2002), 189-221.

[4] J.L. Chern, Z.-Y. Chen and C.S. Lin, Uniqueness of topological solutions and the structure of solutions for the Chern-Simons system with two Higgs particles, *Comm. Math. Phys.*, 296(2010), 323-351.

[5] K. Choe, N. Kim and C.S. Lin, Existence of self-dual non-topological solutions in the Chern-Simons-Higgs model, *Ann. Inst. H. Poincaŕe Anal. Non Linéaire*, 28(2011), 837-852.

[6] A. Doliwa, Holomorphic curves and Toda systems, *Lett. Math. Phys.* 39(1997),21-32.

[7] G. Dunne, Mass degeneracies in self-dual models, *Phys. Lett.*, B 345(1995), 452-457.

[8] G. Dunne, Self-dual Chern-Simons Theories. *Lecture Notes in Physics*, vol. m36(1995), Berlin-New York, Spring-Verlag.

[9] G. Dunne, Vacuum mass spectra for $SU(N)$ self-dual Chern-Simons-Higgs, *Nucl. Phys.*, B 433(1995), 333-348.

[10] P. Esposito, M. Del Pino and M. Musso, Nondegeneracy of entire solutions of a singular Liouville equation, *Proc. Amer. Math. Soc.*, to appear (2011).

[11] J. Hong, Y. Kim and P.Y. Pac, Multivortex solutions of the Abelian Chern-Simons theory, *Phys. Rev. Lett.*, 64(1990), 2230-2233.

[12] R. Jackiw and E.J. Weinberg, Self-dual Chern-Simons vortices, *Phys. Rev. Lett.*, 64(1990), 2234-2237.

[13] H. Kao and K. Lee, Selfsual $SU(3)$ Chern-Simons-Higgs systems, *Phys. Rev.*, D 50(1994), 6626-6635.

[14] K. Lee, Relativistic nonabelian Chern-Simons systems, *Phys. Lett.*, B 225(1991), 381-384.

[15] K. Lee, Selfdual nonabelian Chern-Simons solitons, *Phys. Rev. Lett.*, 66(1991), 553-555.

[16] C.S. Lin, J. Wei and D. Ye, On classification and nondegeneracy of $SU(n+1)$ Toda system with singular sources, *Inventiones Mathematicae* 190(2012), no.1, 169-207.

[17] C.S. Lin, S. Yan, Bubbling solutions for relativistic abelian Chern-Simons model on a torus, *Comm. Math. Phys.*, 297(2010), 733-758.

[18] C.S. Lin, S. Yan, Bubbling solutions for the $SU(3)$ Chern-Simons model on a torus, *Comm. Pure Appl. Math.*, to appear.

[19] A. Malchiodi, D. Ruiz, New improved Moser-Trudinger inequalities and singular Liouville equations on compact surfaces, *Geom. Funct. Anal.*, to appear.

[20] A. Malchiodi, D. Ruiz, A variational analysis of the Toda-system on compact surfaces, *Comm. Pure Appl. Math.*, to appear.

[21] M. Nolasco, G. Tarantello, Double vortex condensaties in the Chern-Simons-Higgs theory, *Calc. Var. Par. Diff. Eqns.*, 9(1999), 31-94.

[22] M. Nolasco, G. Tarantello, Vortex condensates for the $SU(3)$ Chern-Simons theory, *Comm. Math. Phys.*, 213(2000), 599-639.

[23] J. Prajapat and G. Tarantello, On a class of elliptic problems in $\mathbb{R}^2$: symmetry and uniqueness resluts, *Proc. Royal Society Edin.* **131A** (2001), 967-985.

[24] J. Spruck and Y. Yang, Topological solutions in the self-dual Chern-Simons theory: Existence and approximation, *Ann. Inst. H. P*, 12(1995), 75-97.

[25] J. Spruck and Y. Yang, The existence of non-topological solitions in the self-dual Chern-Simons theory, *Comm. Math. Phys.*, 149(1992), 361-376.

[26] G. Tarantello, Multiple condensate solutions for the Chern-Simons Higgs theory, *J. Math. Phys.*, 37(1996), 3769-3796.

[27] G. Tarantello, Uniqueness of self-dual penodic Chern-Simons vortices of topological type, *Calc. Var. Par. Diff. Eqns.*, 29(2007), 191-217.

[28] R. Wang, The existence of Chern-Simons vortices, *Comm. Math. Phys*, 137(1991), 587-597.

[29] J. Wei, CY Zhao and F. Zhou, On non-degeneracy of solutions to $SU(3)$ Toda system, *CRAS.* 349(2011), no.3-4, 185-190.

[30] G. Wang and L. Zhang, Non-topological solutions of the relativistic SU(3) Chern-Simons Higgs model, *Comm. Math. Phys.*, 202(1999), no.3, 501-515.

[31] Y. Yang, *Solitons in field theory and nonlinear analysis,* Springer Monographs in Mathematics, Springer, New York, 2001.

[32] Y. Yang, The relativistic non-Abelian Chern-Simons equations, *Comm. Math. Phys.*, 186(1997), 199-218.