MULTIPLE CONDENSATIONS FOR A NONLINEAR ELLIPTIC EQUATION WITH SUB-CRITICAL GROWTH AND CRITICAL BEHAVIOR

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ABSTRACT. We consider the following nonlinear elliptic equations

$$\Delta u + u_+^{\frac{N}{N-2}} = 0 \text{ in } \Omega$$

$$u = \mu \text{ on } \partial\Omega, \ (\mu \text{ is an unknown constant})$$

$$\int_{\partial\Omega} \left(-\frac{\partial u}{\partial n}\right) = M$$

where $u_+ = \max(u,0), M$ is a prescribed constant and Ω is a bounded and smooth domain in $R^N, N \geq 3$. It is known that for $M = M_*^{(N)}, \Omega = B_R(0)$, the above problem has a continuum of solutions. The case when $M > M_*^{(N)}$ is referred as *supercritical* in the literature. We show that for M near $KM_*^{(N)}, K > 1$, there exists solution with multiple condensations in Ω. These concentration points are nondegenerate critical points of a function related to the Green's function.

1. INTRODUCTION

Let Ω be a bounded domain in R^N with a smooth boundary. Consider the following nonlinear boundary value problem of the type

$$\begin{cases} \triangle u + g(u,x) = 0 & \text{on } \Omega \\ u = \mu & \text{on } \partial\Omega, \ \mu \text{ is an unknown constant} \\ \int\limits_{\partial\Omega} \frac{\partial u}{\partial n} = -M \end{cases} \tag{1.1}$$

where $g(\cdot,\cdot) \geq 0$ on $R^1 \times \Omega$ and $M > 0$ is a prescribed constant.

Equation (1.1) arises in several applications. In particular, if $g(t,x) = K(x)e^t, \ N = 2$ the solutions to problem (1.1) yield Riemannian metrics which are conformally equivalent to the Euclidean metrics on Ω with a given Gaussian curvature K and a prescribed total curvature $M/2$, [22], [34]. If $g(t,x) = f(t - V(x))$, where

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1991 *Mathematics Subject Classification.* 35B40, 35B45; Secondary 35J40.

Key words and phrases. nonlinear elliptic equations, free boundary, singular perturbations, critical behavior.

$V(x) \in C^0(\bar{\Omega})$ is given and $f(\cdot)$ is nondecreasing, then the solutions of (1.1) represent potentials of stationary distributions of self-gravitating clusters. M stands for the total mass of the cluster. See [33].

In this paper, we study a particular type of (1.1), namely the following

$$\begin{cases} -\triangle u = u_+^p \\ u = \mu \quad \text{on } \partial\Omega, \quad \mu \text{ is an unknown constant} \\ -\int\limits_{\partial\Omega} \frac{\partial u}{\partial n} = M \end{cases} \tag{1.2}$$

where $N \geq 3, 1 < p < 2^* := \frac{N+2}{N-2}$ and $u_+ = max(u, 0)$.

The existence of solutions to (1.2) in the case of $1 \leq p < \frac{N}{N-2} = p^*$ was first established by Berestycki and Brezis [8] (Temam [31] studied the case p=1). Wang [32] extended this result to $p^* \leq p < 2^* = \frac{N+2}{N-2}$ (for M small). Bandle and Marcus [3] studied a priori estimates for (1.2) under assumptions similar to those of [8]. For more general g, please see [34].

The exponent $p^* = \frac{N}{N-2}$ turns out to be a natural critical exponent for (1.2). Even though it has subcritical growth, it has critical behavior. Wolansky [34] introduced an energy functional to (1.2). It turns out when $M > M_*^{(N)}$, the functional becomes unbounded and noncompact and therefore it has the loss of compactness similar to nonlinear elliptic equations with critical Sobolev exponent 2^* as studied in [1], [2], [5], [10], [29], [30], [39], etc. When $\Omega = B_1(0)$, one can find a continuum of solutions to (1.2). In fact, let $\psi(r)$ be the solution of

$$\begin{cases} \frac{\partial^2}{\partial r^2}\psi + \frac{N-1}{r}\frac{\partial}{\partial r}\psi + (\psi_+)^{p^*} = 0, \\ \frac{\partial \psi}{\partial r}(0) = 0, \quad \psi(0) = 1. \end{cases} \tag{1.3}$$

Let $r_0 > 0$ be the smallest root such that $\psi(r_0) = 0$. Set

$$M_*^{(N)} := \omega_N \int_0^{r_0} s^{N-1} \psi^{p^*}(s) ds \tag{1.4}$$

where ω_N stands for the area of the unit sphere $S^{N-1} \subset R^N$.

It can easily be shown that for any $\alpha \geq 1$, the following function

$$U^\alpha(r) = \begin{cases} \alpha^{N-2} r_0^{N-2} \psi(\alpha r_0 r), \text{ for } 0 \leq r \leq \alpha^{-1}, \\ \omega_N^{-1} M_*^{(N)}(r^{2-N} - 1) - \omega_N^{-1} M_*^{(N)}(\alpha^{N-2} - 1), \text{ for } \alpha^{-1} \leq r \leq 1, \end{cases} \tag{1.5}$$

is a solution of (1.2) for $p = p^*, M = M_*^{(N)}$.

In general domain, the existence of solutions of (1.2) is related to the Green's function in Ω. Let $G(x, y)$ be the Green's function of $-\Delta$ in $H_0^1(\Omega)$ and $\Gamma(|x - y|)$

be the singular part of $G(x,y)$, i.e.,

$$\Gamma(|x-y|)=\frac{1}{(N-2)|S^{N-1}||x-y|^{N-2}},$$

where S^{N-1} is the unit sphere in R^N. The regular part of $G(x,y)$ is defined as $H(x,y)=\Gamma(|x-y|)-G(x,y)$. (The function $H(x,x)$ is called Robin function. For more properties of Robin function, see [6].)

Wolansky [34] proved the following theorem

Theorem A *Assume that there exists $x_0 \in \Omega$ for which*

$$H(x_0,y)>H(x_0,x_0), \qquad \forall\, y\in\Omega\setminus\{x_0\}. \tag{1.6}$$

Then, for any $0<M<M_*^{(N)}$ (defined by (1.4)) there exists a solution u^M of (1.2) for $p=p^*$. Moreover as $M\to M_*^{(N)}$, $\mu^M\to-\infty$.

A natural question is what happens when $M>M_*^{(N)}$? As far as we know, there are no existence results yet. The purpose of this paper is to answer this question.

Fix $\mathbf{P}=(P^1,...,P^K)\in\Omega^K:=\Omega\times\Omega\times...\times\Omega$. Set

$$F(\mathbf{P})=-\sum_{j=1}^{K}H(P^j,P^j)+\sum_{l,m,l\neq m}G(P^l,P^m). \tag{1.7}$$

We then have

Theorem 1.1. *Let $K\geq 1, K\in\mathcal{N}$. Assume that $\mathbf{P}_0=(P_0^1,\cdots,P_0^K)\in\Omega^K$ is a nondegenerate critical point of $F(\mathbf{P})$. Let $p=p^*$. Then there exists a $\epsilon_0>0$ such that for every $\epsilon\in(0,\epsilon_0)$, there exists a solution u^M to (1.2) satisfying the following properties,*

(i) $u^M|_{\partial\Omega}=\mu^M=-\epsilon^{-\frac{2}{p-1}}\to-\infty$,

(ii) *The set $\{u^M>0\}$ contains exactly K connected domains,*

(iii) $M=KM_*^{(N)}+o((-\mu^M)^{-1})$ *is a continuous function of $\epsilon\in(0,\epsilon_0)$.*

Remark: A more detailed description of the free boundary $\{u^M>0\}$ can be found in Theorem 1.3 below.

In fact, for any $1<p<2^*, p\neq p^*$, we have the following theorem

Theorem 1.2. *Assume that $\mathbf{P}_0=(P_0^1,\cdots,P_0^K)\in\Omega^K$ is a nondegenerate critical point of $F(\mathbf{P})$. Let $1<p<2^*, p\neq p^*$. Then there exist $\delta_0, C_0>0$ such that for every $M\in(0,\delta_0)$ if $p^*<p<2^*$ and $M\in(C_0,+\infty)$ if $1<p<p^*$, there exists a solution u^M to (1.2) satisfying the following properties, as $M\to 0$ for $p>p^*$ or $M\to+\infty$ for $p<p^*$,*

(i) *The set $\{u^M>0\}$ contains exactly K connected domains,*

(ii) $u^M|_{\partial\Omega} = \mu^M \to -\infty$,
(iii) $M(-\mu^M)^{\frac{2p-N(p-1)}{2}} \to \mu_0$, *where* μ_0 *is a generic constant.*

Let's now introduce the main idea of the proofs. One of the key points is the following transformation of (1.2): let $\mu = -\epsilon^{-\frac{2}{p-1}}$ and $u = |\mu|v + \mu$, then we have equation (1.2) is equivalent to finding (v, ϵ) such that

$$\begin{cases} \epsilon^2 \triangle v + (v-1)_+^{p^*} = 0, \ v > 0 \text{ in } \Omega \\ v = 0 \text{ on } \ \partial\Omega \\ \epsilon^{-\frac{2p}{p-1}} \int\limits_\Omega (v-1)_+^p = M \end{cases} \tag{1.8}$$

Note that in problem (1.8), M is fixed and $\epsilon > 0$ is undetermined.

To solve (1.8), we first solve the following elliptic equation

$$\begin{cases} \epsilon^2 \triangle v + (v-1)_+^p = 0 & \text{in } \Omega \\ v > 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{1.9}$$

for ϵ small and then we solve the following algebraic equation

$$M = \epsilon^{-\frac{2p}{p-1}} \int\limits_\Omega (v-1)_+^p. \tag{1.10}$$

Equation (1.9) can be regarded as a singular perturbation problem with a free boundary. The boundary of the core

$$A := \{v > 1\}$$

is the *free boundary* for (1.9). Problem (1.9) arises in plasma physics, [6], [13], [20]. The corresponding problem in two dimensions has been studied in [6];the case when $p = 1$ by Caffarelli and Friedman [13]. The higher dimensional case has been studied by M. Flucher and the author in [20]. For more information and background for (1.9), please see [20] and the references therein. It was proved in [20] that as $\epsilon \to 0$ the least energy solution has one single local maximum $x_\epsilon \in \Omega$ and $H(x_\epsilon, x_\epsilon) \to \min_{x\in\Omega} H(x, x)$. In this paper, we shall study solutions with multiple condensations.

Our main result for (1.9) is the following

Theorem 1.3. *Let* $1 < p < \left(\frac{N+2}{N-2}\right)_+, N \geq 3$ *and suppose that* $\mathbf{P}_0$ *is a non-degenerate critical point of* $F(\mathbf{P})$. *Then for* $\epsilon << 1$, *problem (1.9) admits a solution* u_ϵ *such that* u_ϵ *has only* K *local maximum point* $P_\epsilon^j, j = 1, ..., K$ *and*

$P_\epsilon^j \to P_0^j, j = 1, ..., K$ as $\epsilon \to 0$. The core $A_\epsilon := \{u_\epsilon > 1\}$ has exactly K connected components and

$$\cup_{j=1}^K B_{R_2\epsilon}(P_\epsilon^j) \subset A_\epsilon \subset \cup_{j=1}^K B_{R_1\epsilon}(P_\epsilon^j) \tag{1.11}$$

for some numbers $R_1 > R_2 > 0, R_1 = R_w + o(1), R_2 = R_w + o(1)$, where R_w is a generic constant defined in Section 2.

Remark: When $K = 1$, Theorem 1.3 can be considered as a converse of Theorem 4 of [20].

To prove Theorems 1.1 and 1.2, we need to have the following theorem

Theorem 1.4. *Let $\mathbf{P}_0$ be a nondegenerate critical point of $F(\mathbf{P})$ and u_ϵ be the solution constructed by Theorem 1.3. Then for ϵ small, u_ϵ is unique and we have*

$$I(\epsilon) := \epsilon^{-\frac{2p}{p-1}} \int_\Omega (u_\epsilon - 1)_+^p = KI_p + d_w F(\mathbf{P}^0)\epsilon^{N-2} + o(\epsilon^{N-2}). \tag{1.12}$$

is a continuous function of ϵ, where $I_p > 0, d_w$ are defined in Section 5. In particular, when $p = p^*, I_p = M_*^{(N)}, d_w = 0$.

Theorems 1.1 and 1.2 follow directly from Theorem 1.4.

Remarks:

1. It is interesting to note that results similar to Theorem 1.3 appear frequently in the study of problems arising from applied sciences and geometry. It turns out the Green's function is naturally associated with such phenomenon. For blow-up problems involving the Sobolev critical exponent, see [1], [10], [2], [29], [30], [39], and the references therein. For multiple concentrations in singular perturbation problems, please see [6], [7], [15], [16], [17], [18], [19], [21], [23],[27], [37], and the references therein. Our problem (1.9) can be considered as a border line case of singular perturbation problems. Most of the references cited here deal with singular perturbation problems of the kind $\epsilon^2 \Delta u + f(u) = 0, f'(0) < 0$. In our case, we have $f'(0) = 0$ which is called *zero mass* case [9]. In the zero mass case, the decay of ground state is of *algebraic* order and therefore traditional Sobolev space H^2 doesn't work. Here we use a weighted Sobolev space approach.

2. We remark that our results here in some sense are a generalization of those in [5]. In [5], Baraket and Pacard considered the following problem

$$\begin{cases} \Delta u + \rho e^u = 0 \ \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{1.13}$$

where Ω is a smooth bounded domain in R^2.

They introduced a function $\mathcal{F}(\mathbf{P})$ in R^2 which is equal to our $F(\mathbf{P})$ in R^2. Then they proved that at any nondegenerate critical point $\mathbf{P}_0 = (P_0^1, ..., P_0^K)$ of $F(\mathbf{P})$, for ρ small, there exists a solution u_ρ which blows up near $P_0^j, j = 1, ..., K$. Moreover it can be shown that $\rho \int_\Omega e^{u_\rho} \to 8\pi K$. In some sense they established the results similar to our Theorem 1.3 in the case $N = 2$. There they used a weighted Hölder space approach. However they didn't show that the solution u_ρ is unique (though uniqueness is very likely to be true) and thus they don't have results similar to our Theorem 1.1.

3. The nondegeneracy condition in Theorem 1.3 may be relaxed a little bit. For example, if we have strictly local maximum point $\mathbf{P}_0$ or strictly local minimum point of $F(\mathbf{P})$, we can use the method of minimization or maximization of the reduced energy (similar to [18] or [19]) to obtain solutions of (1.9) with multiple condensations. It seems hard to remove the nondegeneracy condition in Theorem 1.1 and Theorem 1.2. The main difficulty is to show that $I(\epsilon)$ is continuous in ϵ.

4. It is an interesting question to see if there exists a solution to (1.8) when $M = KM_*^{(N)}$. We learned from Prof. A. Chang that she and Prof. C.-S. Lin have obtained some results for this question in the case when $N = 2$, [14]. The higher dimensional case is still open.

5. Although we have just studied a particular type of problem (1.1), the results in this paper can certainly be generalized to deal with problem (1.1) with $g(u, x) = f(u - V(x))$ for some f and $V(x) \in C^2$. We shall not pursue this generality here.

The organization of the paper is the following:

In Section 2, we first introduce a projection and study the properties of the projection. Then we introduce some Sobolev spaces and study the properties of the linearized operator in these spaces.

Section 3 contains the classical Liapunov-Schmidt reduction process. Here we reduce our problem to a finite dimensional one.

In Section 4, we solve the finite dimensional problem and prove Theorem 1.3.

In Section 5, we show that the solutions constructed in Section 4 are unique.

Section 6, we return to the study of (1.2) and prove Theorem 1.1 and 1.2.

Finally we prove some technical lemmas in Section 7 and Section 8.

Throughout this paper, the letters C, c, C_i, c_i will denote various constants independent of ϵ small. δ will always denote a small constant.

2. Preliminaries

In this section, we introduce some notations and prove necessary estimates.

Let $f(u) := (u-1)_+^p$ for $1 < p < \frac{N+2}{N-2}$. It is known that the following problem in R^N

$$\begin{cases} \triangle w + (w-1)_+^p = 0 & \text{in } R^N, \\ w > 0, \quad w(0) = \max_{y \in R^N} w(y), & \\ w(y) \to 0 & \text{as } |y| \to +\infty, \end{cases} \tag{2.1}$$

has a unique *"ground state"* solution w. Moreover, we have

Lemma 2.1. *([Section 4, [20]]) $w = w(r), w'(r) < 0$ for $r \neq 0$ and $w(r) = \frac{k_w}{(N-2)|S^{N-1}|} r^{2-N}$ for $r \geq R_w$, where $k_w > 0, R_w > 0$.*

Next fix any $P \in \Omega$. We project "w" into $H_0^1(\Omega)$ by defining $w_{\epsilon,P}$ to be the unique solution of

$$\begin{cases} \epsilon^2 \triangle w_{\epsilon,P} + (w(\frac{x-P}{\epsilon}) - 1)_+^p = 0, & \text{in } \Omega \\ w_{\epsilon,P} = 0 & \text{on } \partial\Omega \end{cases} \tag{2.2}$$

Note that $w_{\epsilon,P}$ is C^2 in P. Then we have by Lemma 2.1.

Lemma 2.2.

$$w_{\epsilon,P}(x) = w(\frac{x-P}{\epsilon}) - \epsilon^{N-2} k_w H(x,P) \quad \textit{if } d(P, \partial\Omega) > R_w \epsilon, \tag{2.3}$$

and

$$w_{\epsilon,P}(x) = k_w \epsilon^{N-2} G(x,P), \ \textit{for } |x-P| > R_w \epsilon, d(P,\partial\Omega) > R_w \epsilon. \tag{2.4}$$

Let $\mathbf{P} = (P^1, \cdots, P^K) \in \Omega^K$ and $\mathbf{P}_0$ be a critical point of $F(\mathbf{P})$. From now on, we will always assume that

$$\mathbf{P} \in B_\delta(\mathbf{P}_0)$$

for some $\delta > 0$ small.

We denote P_i^j to be the $i-$th component of $P^j, j = 1, ..., K$.

Define

$$w_j(x) = w(\frac{x-P^j}{\epsilon}), w_{\epsilon,\mathbf{P}} = \sum_{j=1}^K w_{\epsilon,P^j},$$

$$\partial_i^j = \frac{\partial}{\partial P_i^j}, j = 1, ..., K, i = 1, ..., N,$$

$$\Omega_{\epsilon,P^j} := \left\{ y \mid \epsilon y + P^j \in \Omega \right\}.$$

Set

$$F_j(\mathbf{P},x)=\sum_{l\neq j}G(x,P^l)-H(x,P^j). \tag{2.5}$$

Then it is easy to see that

$$F(\mathbf{P})=\sum_{j=1}^K F_j(\mathbf{P},P^j).$$

We have the following error estimates

Lemma 2.3. *For ϵ sufficiently small, we have*

$$support(\epsilon^2\Delta w_{\epsilon,\mathbf{P}}+(w_{\epsilon,\mathbf{P}}-1)_+^p)\subset\cup_{j=1}^K B_{R_0\epsilon}(P^j),$$

where $R_0=R_w+o(1)>0$. Moreover

$$\epsilon^2\Delta w_{\epsilon,\mathbf{P}}+(w_{\epsilon,\mathbf{P}}-1)_+^p=p(w-1)_+^{p-1}\epsilon^{N-2}k_wF_j(\mathbf{P},P^j)+O(\epsilon^{N-1}), \tag{2.6}$$

for $x\in B_{R_0\epsilon}(P^j)$, and

$$\epsilon^2\Delta w_{\epsilon,\mathbf{P}}(x)+(w_{\epsilon,\mathbf{P}}-1)_+^p(x)=f_{\epsilon,P^j}(|y|) \tag{2.7}$$

$$+p(w(y)-1)_+^{p-1}k_w\sum_{k=1}^N\frac{\partial}{\partial x_k}F_j(\mathbf{P},x)|_{x=P^j}\epsilon^{N-1}y_k+O(\epsilon^N),x\in B_{R_0\epsilon}(P^j),$$

where $\epsilon y+P^j=x$, f_{ϵ,P^j} is a radial function and is supported in $B_{R_0}(0)$.

Proof: Observe that by Lemma 2.2, for $x\in B_{2R_w\epsilon}(P^j)$,

$$w_{\epsilon,\mathbf{P}}(x)=\sum_{j=1}^K w_{\epsilon,P^j}(x)=w(\frac{x-P^j}{\epsilon})+\epsilon^{N-2}k_wF_j(\mathbf{P},x).$$

(2.6) and (2.7) follow from direct computations.

□

Next we introduce some Sobolev spaces.

We first define weighted Sobolev space in R^N. For $1<t<\infty$, a nonnegative integer l and a real number β, the weighted Sobolev space $W^t_{l,\beta}(R^N)$ is defined to be the completion of $C_0^\infty(R^N)$ under the norm:

$$\sum_{|\alpha|=0}^l||<x>^{\beta+|\alpha|}\partial^\alpha u||_{L^t(R^N)}, \tag{2.8}$$

where $<x>=(1+|x|^2)^{\frac12}$. If $\beta=0=l$, then $W^t_{l,\beta}(R^N)$ is just the usual $L^t(R^N)$ space. When $l=0$, we write $W^t_{0,\beta}(R^N)$ as $L^t_\beta(R^N)$. The properties of these Sobolev spaces can be found in [12], [24], [25], [28].

For any bounded open set U, we can define $W^t_{l,\beta}(U)$ to be the space $W^{l,t}(U)$ equipped with the norm:

$$\sum_{|\alpha|=0}^{l} || < x >^{\beta+|\alpha|} \partial^\alpha u||_{L^t(U)}. \tag{2.9}$$

Similarly we can define $L^t_\beta(U)$.

Let $t^{'}$ denote, as always, the conjugate of t: $\frac{1}{t}+\frac{1}{t^{'}}=1$.

We now turn to the linear operator $L := \Delta + p(w-1)_+^{p-1} : W^t_{2,\beta}(R^N) \to L^t_{\beta+2}(R^N)$. We then have

Lemma 2.4. *If $1-\frac{N}{t} < \beta < \frac{N}{t'}-1$, then $Kernel(L) \cap W^t_{2,\beta}(R^N) = X_0 := span\left\{\frac{\partial w}{\partial y_i} \mid i=1,\cdots,N\right\}$.*

Proof: By direct computations,

$$|\frac{\partial w}{\partial y_i}| \le C|y|^{1-N}$$

for $|y| > R_w$. Hence $X_0 \subset kernel(L) \cap W^t_{2,\beta}(R^N)$ since $\beta < \frac{N}{t'}-1$. The rest follows from the same proof of Proposition 2.1 of [38], Lemma 4.2 of [32], or Lemma 6 of [20]. □

Observe that when $\beta > 1-\frac{N}{t}, uv \in L^1(R^N)$ if $u \in L^t_\beta, v \in X_0$. Hence if $1-\frac{N}{t} < \beta < \frac{N}{t'}-1$, we can decompose $W^t_{2,\beta}(R^N)$ as follows

$$W^t_{2,\beta}(R^N) = X_0 \oplus X_0^\perp$$

where $X_0^\perp = \{u \in W^t_{2,\beta}(R^N) | \int_{R^N} uvdx = 0, \forall v \in X_0\}$.

Let us denote $Y_0 := \{\frac{\partial w}{\partial y_i}, i=1,...,N\} \subset L^t_{\beta+2}(R^N)$ and

$$Y_0^\perp = \{u \in L^t_{2+\beta}(R^N) | \int_{R^N} uv = 0, \forall v \in Y_0\}.$$

Then we have

Proposition 2.1. *Suppose that $N \ge 3$ and $1-\frac{N}{t} < \beta < \frac{N}{t'}-1$. then*

$$L^{-1} : Y_0^\perp \to X_0^\perp \tag{2.10}$$

exists and is bounded.

Proof: See Proposition 2.3 of [38]. □

Proposition 2.1 will have a natural analogue in Ω. We shall discuss this in the rest of this section. Before that, we introduce some notations and some technical lemmas.

Set

$$X^{t,\beta}_{\epsilon,\mathbf{P}} = \cap_{j=1}^K W^t_{2,\beta}(\Omega_{\epsilon,P^j}) \cap W^{1,2}_0(\Omega),$$

and

$$Y^{t,\beta}_{\epsilon,\mathbf{P}} = \cap_{j=1}^K L^t_\beta(\Omega_{\epsilon,P^j}).$$

In $X^{t,\beta}_{\epsilon,\mathbf{P}}$, we define its norm as follows

$$\|u\|_{t,2,\beta} = \sum_{|\alpha|=0}^{2} (\epsilon^{-N} \int_\Omega |(\sigma_{\mathbf{P}}(x))^{\beta+|\alpha|} \partial^\alpha u|^t)^{\frac{1}{t}} \tag{2.11}$$

where $\sigma_{\mathbf{P}}(x) := \min_{j=1}^K (1 + |\frac{x-P^j}{\epsilon}|^2)^{\frac{1}{2}}$. Similarly in $Y^{t,2+\beta}_{\epsilon,\mathbf{P}}$ we define the norm as

$$\|u\|_{t,0,2+\beta} = (\epsilon^{-N} \int_\Omega |(\sigma_{\mathbf{P}}(x))^{2+\beta} u|^t)^{\frac{1}{t}}.$$

From now on, we drop the index $\epsilon, \mathbf{P}$ in $X^{t,\beta}_{\epsilon,\mathbf{P}}, Y^{t,2+\beta}_{\epsilon,\mathbf{P}}$ if there is no confusion.

We then have the following technical lemma whose proof will be given in Appendix B.

Lemma 2.5. *Suppose that $0 < \beta < \frac{N}{t'} - 1$. Let Φ satisfy the following equation*

$$\begin{cases} \epsilon^2 \Delta \Phi + f = 0 & in\ \Omega \\ \Phi = 0 \ on\ \partial\Omega. \end{cases} \tag{2.12}$$

Then we have

$$\|\Phi\|_{t,2,\beta} \le C \|f\|_{t,0,2+\beta}.$$

Remark: If $K = 1$, we just need $1 - \frac{N}{t} < \beta < \frac{N}{t'} - 1$. If $K > 1$, $\beta > 0$ is needed for technical reasons.

Set

$$L_{\epsilon,\mathbf{P}} = \epsilon^2 \Delta + p(w_{\epsilon,\mathbf{P}} - 1)_+^{p-1}. \tag{2.13}$$

From now on, we restrict the exponents t, β such that

$$\frac{N}{N-1} < t < N, 0 < \beta < \frac{N}{t'} - 1. \tag{2.14}$$

We now study the linear operator $L_{\epsilon,\mathbf{P}}$ in $X^{t,\beta}$ and prove an analogue of Proposition 2.1.

Set

$$\mathcal{K}_{\epsilon,\mathbf{P}} = span \left\{ \partial_i^j w_{\epsilon,\mathbf{P}} \middle| 1 \le j \le K, 1 \le i \le N \right\} \subset X^{t,\beta},$$
$$\mathcal{C}_{\epsilon,\mathbf{P}} = span \left\{ \partial_i^j w_{\epsilon,\mathbf{P}} \middle| 1 \le j \le K, 1 \le i \le N \right\} \subset Y^{t,2+\beta},$$
$$\mathcal{K}^\perp_{\epsilon,\mathbf{P}} = \left\{ u \in X^{t,\beta} \middle| \ \int_\Omega u \partial_i^j w_{\epsilon,\mathbf{P}} = 0, \ 1 \le j \le K, 1 \le i \le N \right\},$$
$$\mathcal{C}^\perp_{\epsilon,\mathbf{P}} = \left\{ u \in Y^{t,2+\beta} \middle| \int_\Omega u \partial_i^j w_{\epsilon,\mathbf{P}} = 0, \ 1 \le j \le K, 1 \le i \le N \right\}.$$

Let $\Pi_{\epsilon,\mathbf{P}}, \Pi^\perp_{\epsilon,\mathbf{P}}$ be the projection of $Y^{t,2+\beta}$ into $\mathcal{C}_{\epsilon,\mathbf{P}}, \mathcal{C}^\perp_{\epsilon,\mathbf{P}}$ respectively.

The following is the main estimate we need in order to apply Liapunov-Schmidt reduction method.

Proposition 2.2. *Let $0 < \beta < \frac{N}{t'} - 1$ and $\frac{N}{N-1} < t < N$. Then for $\epsilon << 1$ and $\mathbf{P} \in B_\delta(\mathbf{P}_0)$, the operator defined by*

$$\tilde{L}_{\epsilon,\mathbf{P}} := \Pi^\perp_{\epsilon,\mathbf{P}} \circ L_{\epsilon,\mathbf{P}} : \mathcal{K}^\perp_{\epsilon,\mathbf{P}} \to \mathcal{C}^\perp_{\epsilon,\mathbf{P}} \tag{2.15}$$

is one-to-one and surjective (and hence invertible). Moreover

$$\|\tilde{L}_{\epsilon,\mathbf{P}}\Phi\|_{t,0,2+\beta} \geq C\|\Phi\|_{t,2,\beta}, \qquad \forall\, \Phi \in \mathcal{K}^\perp_{\epsilon,\mathbf{P}}. \tag{2.16}$$

Proof: The proof is similar to Proposition 3.1 of [18] or Proposition 3.1 of [36]. For the sake of completeness, we give a sketch of the proof of injectivity. The surjectivity follows from injectivity easily.

We just need to prove (2.16). Suppose not. Then there exists $\epsilon_k \to 0$, $\mathbf{P}_k \to \mathbf{P} \in \bar{B}_\delta(\mathbf{P}_0)$,$\Phi_k \in \mathcal{K}^\perp_{\epsilon,\mathbf{P}_k}$ such that

$$\tilde{L}_{\epsilon,\mathbf{P}_k}\Phi_k = f_k, \qquad \|f_k\|_{t,0,2+\beta} \to 0,$$

$$\Phi_k \in \mathcal{K}^\perp_{\epsilon,\mathbf{P}_k}, \qquad \|\Phi_k\|_{t,2,\beta} = 1.$$

Let

$$h_k = -p(w_{\epsilon,\mathbf{P}_k} - 1)_+^{p-1}\Phi_k, f_{k,1} = f_k + \Pi_{\epsilon,\mathbf{P}_k} \circ L_{\epsilon,\mathbf{P}_k}(\Phi_k).$$

Then Φ_k satisfies

$$\begin{cases} \epsilon^2 \triangle \Phi_k = h_k + f_{k,1}, & \text{in } \Omega, \\ \Phi_k = 0 & \text{on } \partial\Omega \end{cases} \tag{2.17}$$

Note that

$$\epsilon^{-N}\int_\Omega L_{\epsilon_k,\mathbf{P}_k}(\Phi_k)\partial_i^j w_{\epsilon_k,\mathbf{P}_k}$$
$$= \epsilon_k^{-N}\int_\Omega [p(w_{\epsilon_k,\mathbf{P}_k} - 1)_+^{p-1}\partial_i^j w_{\epsilon_k,\mathbf{P}_k} - p(w_j - 1)_+^{p-1}\partial_i^j w_j]\Phi_k$$
$$= O(\epsilon_k^{N-2})$$

Hence

$$\|\Pi_{\epsilon_k,\mathbf{P}_k} \circ L_{\epsilon_k,\mathbf{P}_k}(\Phi_k)\|_{t,0,2+\beta} = O(\epsilon_k^{N-2}) = o(1)$$

and

$$\|f_{k,1}\|_{t,0,2+\beta} = o(1).$$

Let

$$\Phi_k^j(y) = \Phi_k(\epsilon y + P_k^j)\ \chi(x - P_k^j), j = 1, ..., K,$$

where $\chi(x) = 1$ for $|x| \leq \frac{\delta}{2}$ and $\chi(x) = 0$ for $|x| \geq \delta$.

Without loss of generality, we may assume that $j=1$. Then Φ_k^1 satisfies

$$\begin{cases} \triangle \Phi_k^1 + p(w_{\epsilon,P_k^1} + \sum_{j=2}^K w_{\epsilon,P_k^j} - 1)_+^{p-1}\Phi_k^1 = \tilde{f}_k \quad \text{in } \Omega_{\epsilon,P^1} \\ \Phi_k^1 = 0 \quad \text{on } \partial\Omega_{\epsilon,P^1} \end{cases} \tag{2.18}$$

where $\tilde{f}_k = \chi f_{k,1} - 2\epsilon^2 \nabla_x \chi \nabla_x \Phi_k - \epsilon^2 \triangle \chi \Phi_k$. Note that

$$|\epsilon^2 \nabla_x \chi \nabla_x \Phi_k| \leq \epsilon |\nabla_y \Phi_k|, |\epsilon^2 \Delta \chi \Phi_k| \leq \epsilon^2 |\Phi_k|.$$

Hence we have

$$\|\tilde{f}_k\|_{L^t_{\beta+2}(\Omega_{\epsilon,P^1})} \leq C\|f_{k,1}\|_{t,0,2+\beta} + C\|\Phi_k\|_{t,2,\beta} \leq C.$$

Applying Lemma 2.5, we have

$$\|\Phi_k^1\|_{W^t_{2,\beta}(\Omega_{\epsilon,P^1})} \leq C. \tag{2.19}$$

Let $\Phi_k^1 \to \Phi_0$ weakly in $W^t_{2,\beta}(R^N)$, $\Phi_k^1 \overset{\text{strong}}{\to} \Phi_0$ in $L^t_\beta(R^N)$. Then Φ_0 satisfies

$$\triangle \Phi_0 + p(w-1)_+^{p-1}\Phi_0 = 0, \quad \Phi_0 \in W^t_{2,\beta}(R^N)$$

By Lemma 2.4

$$\Phi_0 \in Kernel(L) = span\{\frac{\partial w}{\partial y_i} \mid i = 1, \cdots, N\}$$

However $0 = \epsilon^{1-N}\int_\Omega \Phi_k \frac{\partial w_{\epsilon,P^1}}{\partial P^1_i} \to -\int_{R^N} \Phi_0 \frac{\partial w}{\partial y_i} = 0, i = 1, \cdots, N$.(Here we have used the fact that $\frac{\partial w}{\partial y_i} \in L^{t'}_{-\beta}(R^N)$ for $\beta > 0 > 1 - \frac{N}{t}$.) Hence, $\Phi_0 \equiv 0$ and $\|\Phi_k^1\|_{L^t_\beta(U)} \to 0$ for any compact set U.

In conclusion, we have obtained that

$$\|\Phi_k\|_{L^t_\beta(U)} \to 0$$

for any bounded set $U \subset \Omega_{\epsilon,P^j}, j = 1, ..., K$.

Going back to the equation for Φ_k, we have now

$$\|h_k\|_{t,2+\beta} \leq C\sum_{j=1}^K \|(w_{\epsilon,\mathbf{P}_k} - 1)_+^{p-1}\Phi_k^j\|_{L^t_{2+\beta}(\Omega_{\epsilon,P^j})}$$

$$\leq C\sum_{j=1}^K \|(w_{\epsilon,\mathbf{P}_k} - 1)_+^{p-1}\Phi_k^j\|_{L^t_{2+\beta}(B_{2R_w(0)}(0))} \to 0.$$

By Lemma 2.5,

$$\|\Phi_k\|_{t,2,\beta} = o(1)$$

which is a contradiction to our assumption that $\|\Phi_k\|_{t,2,\beta} = 1$.

$\square$

3. Liapunov-Schmidt Reduction

In this section, we use the classical Liapunov-Schmidt reduction method to reduce problem (1.9) into a finite dimensional problem. Similar procedure can be found in [18], [19], [36], [35], etc.

Recall that $\frac{N}{N-1} < t < N, 0 < \beta < \frac{N}{t'} - 1$.

Let

$$S_\epsilon(u) := \epsilon^2 \triangle u + (u-1)_+^p \tag{3.1}$$

We first solve the following equation

$$S_\epsilon(u) = 0, u \in X_{\epsilon,\mathbf{P}}^{t,\beta} \tag{3.2}$$

Setting $u = w_{\epsilon,\mathbf{P}} + \Phi$ and substituting into $S_\epsilon(u)$, we have

$$S_\epsilon(u) = S_\epsilon(w_{\epsilon,\mathbf{P}}) + L_{\epsilon,\mathbf{P}}(\Phi) + N_{\epsilon,\mathbf{P}}(\Phi) \tag{3.3}$$

where

$$\begin{aligned} L_{\epsilon,\mathbf{P}} &= S_\epsilon'(w_{\epsilon,\mathbf{P}}) = \epsilon^2\Delta + p(w_{\epsilon,\mathbf{P}} - 1)_+^{p-1}, \\ N_{\epsilon,\mathbf{P}}(\Phi) &:= S(w_{\epsilon,\mathbf{P}} + \Phi) - S(w_{\epsilon,\mathbf{P}}) - L_{\epsilon,\mathbf{P}}(\Phi) \\ &= (w_{\epsilon,\mathbf{P}} + \Phi - 1)_+^p - (w_{\epsilon,\mathbf{P}} - 1)_+^p - p(w_{\epsilon,\mathbf{P}} - 1)_+^{p-1}\Phi. \end{aligned}$$

Set

$$\mathcal{B}_\eta = \{u \in X_{\epsilon,\mathbf{P}}^{t,\beta} | \|u\|_{t,2,\beta} < \eta\}.$$

We then have

Lemma 3.1. *For ϵ and η sufficiently small, we have*
(1) $\|N_{\epsilon,\mathbf{P}}(\Phi)\|_{t,0,2+\beta} \le C\|\Phi\|_{t,2,\beta}^{1+\sigma}, \forall \Phi \in \mathcal{B}_\eta$,
(2) $\|N_{\epsilon,\mathbf{P}}(\Phi_1) - N_{\epsilon,\mathbf{P}}(\Phi_2)\|_{t,0,2+\beta} \le C\eta^\sigma \|\Phi_1 - \Phi_2\|_{t,2,\beta}, \forall \Phi_1 \in \mathcal{B}_\eta, \Phi_2 \in \mathcal{B}_\eta$,
(3) $\|S(w_{\epsilon,\mathbf{P}})\|_{t,0,2+\beta} \le C\epsilon^{N-2}$,
where $\sigma = \min(1, p-1)$.

Proof: (3) follows from Lemma 2.3.

For (1), we note that for $t > \frac{N}{N-1} \ge \frac{N}{2}, \beta > 0$, we have by Sobolev embedding theorem

$$|\Phi|_{L^\infty(\Omega)} \le C\|\Phi\|_{t,2,\beta} \le C\eta.$$

We can choose η so small that

$$|\Phi|_{L^\infty(\Omega)} < \frac{1}{2}.$$

Hence we obtain

$$\textit{support} \ \ (w_{\epsilon,\mathbf{P}} + \Phi - 1)_+^p \subset \cup_{j=1}^K B_{R_0'\epsilon}(P^j), \tag{3.4}$$

where $R_0' > 0$ is a fixed number.

For each $j = 1, ..., K, x \in B_{R_0'\epsilon}(P^j)$, we have

$$|N_{\epsilon,\mathbf{P}}(\Phi)(x)| \leq C|\Phi(x)|^{1+\sigma} \leq C\|\Phi\|_{t,2,\beta}^{1+\sigma}.$$

Because of (3.4), we have

$$\|N_{\epsilon,\mathbf{P}}(\Phi)\|_{t,0,2+\beta} \leq C\|\Phi\|_{t,2,\beta}^{1+\sigma}.$$

In a similar way, we can prove (2).

□

We are now ready to prove the following main result in this section.

Proposition 3.1. *For ϵ and δ sufficiently small and $\mathbf{P} \in B_\delta(\mathbf{P}_0)$, the following problem*

$$\Pi_{\epsilon,\mathbf{P}}^{\perp} \circ S_\epsilon(w_{\epsilon,\mathbf{P}} + \Phi) = 0, \qquad \Phi \in \mathcal{K}_{\epsilon,\mathbf{P}}^{\perp} \tag{3.5}$$

has a unique solution $\Phi := \Phi_{\epsilon,\mathbf{P}} \in \mathcal{B}_{\epsilon^{(1-\delta)(N-2)}}$.

Proof: We write (3.5) in the following form

$$\begin{aligned} 0 &= \Pi_{\epsilon,\mathbf{P}}^{\perp} \circ (S_\epsilon(w_{\epsilon,\mathbf{P}}) + L_{\epsilon,\mathbf{P}}(\Phi) + N_{\epsilon,\mathbf{P}}(\Phi)) \\ &= \Pi_{\epsilon,\mathbf{P}}^{\perp} \circ L_{\epsilon,\mathbf{P}}(\Phi) + \Pi_{\epsilon,\mathbf{P}}^{\perp} \circ S_\epsilon(w_{\epsilon,\mathbf{P}}) + \Pi_{\epsilon,\mathbf{P}}^{\perp} \circ N_{\epsilon,\mathbf{P}}(\Phi). \end{aligned}$$

Therefore (3.5) is equivalent to the following equation

$$\Phi = G_\epsilon(\Phi) := -\tilde{L}_{\epsilon,\mathbf{P}}^{-1} \circ (\Pi_{\epsilon,\mathbf{P}}^{\perp} \circ S_\epsilon(w_{\epsilon,\mathbf{P}}) + \Pi_{\epsilon,\mathbf{P}}^{\perp} \circ N_{\epsilon,\mathbf{P}}(\Phi))$$

By Proposition 2.2 and Lemma 3.1 (1) and (3), we have for $\Phi \in \mathcal{B}_\eta$, where $\eta = \epsilon^{(1-\delta)(N-2)}$,

$$\|G_\epsilon(\Phi)\|_{t,2,\beta} \leq C\|S_\epsilon(w_{\epsilon,\mathbf{P}})\|_{t,0,2+\beta} + C\|N_\epsilon(\Phi)\|_{t,0,2+\beta}$$

$$\leq C\epsilon^{N-2} + C\epsilon^{(1+\sigma)(1-\delta)(N-2)} < \frac{1}{2}\eta.$$

Moreover by Lemma 3.1 (2)

$$\|G_\epsilon(\Phi_1) - G_\epsilon(\Phi_1)\|_{t,2,\beta} \leq C\eta^\sigma\|\Phi_1 - \Phi_2\|_{t,2,\beta} < \frac{1}{2}\|\Phi_1 - \Phi_2\|_{t,2,\beta}.$$

Thus G_ϵ is a contraction map from $\mathcal{B}_\eta$ to $\mathcal{B}_\eta$.

By contraction mapping principle (see [18] and Proposition 6.5 of [35], Lemma 3.3 of [36] for similar approach), Proposition 3.1 can be easily proved. We omit the technical details. □

We next expand $\Phi_{\epsilon,\mathbf{P}}$ in terms of ϵ^{N-2}. To this end, we set $\Phi^0_{\epsilon,\mathbf{P}}$ to be defined as follows

$$\Phi^0_{\epsilon,\mathbf{P}}(x) = k_w \sum_{j=1}^{K} F_j(\mathbf{P}, P^j)\Phi_0(\frac{x-P^j}{\epsilon}) \tag{3.6}$$

where Φ_0 is the unique solution of the following problem

$$\begin{cases} \Delta\phi_0 + p(w-1)_+^{p-1}\Phi_0 + p(w-1)_+^{p-1} = 0 \text{ in } R^N \\ \Phi_0(y) = \Phi_0(|y|), \Phi_0(|y|) \to 0 \text{ as } |y| \to +\infty. \end{cases} \tag{3.7}$$

We observe that it is easy to see that $\Phi_0(|y|) = C_0|y|^{2-N}$ for $|y| \geq R_w$, where C_0 is a constant. Remarkably, we can write Φ_0 explicitly.

Lemma 3.2. *Let Φ_0 be the unique solution of (3.7). Then we have $\Phi_0 = -w - \frac{p-1}{2}rw'(r)$. Thus $C_0 = \frac{k_w}{(N-2)|S^{N-1}|}(\frac{p-1}{2}(N-2)-1)$.*

Proof: By invariance of the equation $\Delta u + u^p = 0$, it is easy to see that $-((w-1)+\frac{p-1}{2}r(w-1)')$ satisfies $\Delta u + p(w-1)_+^{p-1}u = 0$ in $B_{R_w}(0)$. Hence $-w-\frac{p-1}{2}rw'(r)$ satisfies equation (3.7). By uniqueness, $\Phi_0 = -w - \frac{p-1}{2}rw'(r)$. The formula for C_0 follows from Lemma 2.1. □

Let

$$\tilde{\Phi}^0_{\epsilon,\mathbf{P}}(x) := k_w \sum_{j=1}^{K} F_j(\mathbf{P}, P^j)(\Phi_0(\frac{x-P^j}{\epsilon}) - C_0(N-2)|S^{N-1}|\epsilon^{N-2}H(x, P^j)), \tag{3.8}$$

be the projection of $\Phi_{\epsilon,\mathbf{P}}(x)$ into $H^1_0(\Omega)$. Note that

$$\Phi^0_{\epsilon,\mathbf{P}}(x) = \tilde{\Phi}^0_{\epsilon,\mathbf{P}}(x) + O(\epsilon^{N-2}).$$

We have

Lemma 3.3. *Let $\Phi_{\epsilon,\mathbf{P}}$ be defined by Proposition 3.1. Then*

$$\|\Phi_{\epsilon,\mathbf{P}} - \epsilon^{N-2}\tilde{\Phi}^0_{\epsilon,\mathbf{P}}\|_{t,2,\beta} = O(\epsilon^{N-2+\delta_0}) \tag{3.9}$$

where $\tilde{\Phi}^0_{\epsilon,\mathbf{P}}$ is defined by (3.8) and $\delta_0 > 0$ is a positive constant.

Proof: We first note that by (2.6),

$$S_\epsilon(w_{\epsilon,\mathbf{P}}) = pk_w(w(y)-1)_+^{p-1}F_j(\mathbf{P}, P^j)\epsilon^{N-2} + O(\epsilon^{N-1})$$

for $x \in B_{R_0\epsilon}(P^j)$. Thus

$$\Pi^\perp_{\epsilon,\mathbf{P}} \circ S_\epsilon(w_{\epsilon,\mathbf{P}}(x)) = pk_w(w-1)_+^{p-1}F_j(\mathbf{P}, P^j)\epsilon^{N-2} + O(\epsilon^{N-1}) \tag{3.10}$$

for $x \in B_{R_0\epsilon}(P^j)$. Note also that

$$\|\Pi^{\perp}_{\epsilon,\mathbf{P}} \circ N_\epsilon(\Phi_{\epsilon,\mathbf{P}})\|_{t,0,2+\beta} = O(\epsilon^{(1+\sigma)(1-\delta)(N-2)}) = O(\epsilon^{N-2+\delta_0}).$$

On the other hand, $\Phi^0_{\epsilon,\mathbf{P}} \perp X_0$, (3.9) can be proved by using equation (3.5). Please see the proof of Lemma 7.1 of Appendix A for similar proofs.

□

4. The Reduced Problem: Proof of Theorem 1.3

By Proposition 3.1, there exists a unique $\Phi_{\epsilon,\mathbf{P}}$ such that

$$S_\epsilon(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) \in C_{\epsilon,\mathbf{P}}, \quad \Phi_{\epsilon,\mathbf{P}} \in \mathcal{K}^{\perp}_{\epsilon,\mathbf{P}}.$$

In other words, we have

$$S_\epsilon(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) = \sum_{k,l} \beta^l_k(\mathbf{P}) \partial^l_k w_{\epsilon,\mathbf{P}}$$

for some constants $\beta^l_k(\mathbf{P}), k = 1, ..., N, l = 1, ..., K$. (Here the summation is for $k = 1, ..., N, l = 1, ..., K$.)

Set

$$W^l_{\epsilon,k}(\mathbf{P}) := \epsilon^{-N} \int_\Omega S_\epsilon(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) \partial^l_k w_{\epsilon,\mathbf{P}}, k = 1, ..., N, l = 1, ..., K. \tag{4.1}$$

Therefore, to solve (3.2), it is enough to find out a zero of $W^l_{\epsilon,k}(\mathbf{P})$.

We need to compute $W^l_{\epsilon,k}(\mathbf{P})$. We have

Lemma 4.1. *For $\epsilon << 1$, it holds that*

$$W^j_{\epsilon,i}(\mathbf{P}) = c_w \epsilon^{N-2} \partial^j_i F(\mathbf{P}) + O(\epsilon^{N-2+\delta_0}), i = 1, ..., N, j = 1, ..., K, \tag{4.2}$$

where $c_w \neq 0$ is a generic constant given by

$$c_w = -k_w \frac{1}{2} \int_{R^N} (w-1)^p_+. \tag{4.3}$$

Proof of Lemma 4.1: Note that for $i = 1, ..., N, j = 1, ..., K$,

$$\partial^j_i w_{\epsilon,\mathbf{P}} = \partial^j_i w_{\epsilon,P^j},$$

$$W^j_{\epsilon,i} = \epsilon^{-N} \int_\Omega S_\epsilon(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) \partial^j_i w_{\epsilon,P^j}$$

$$= \int_{\Omega_{\epsilon,P^j}} S_\epsilon(w_{\epsilon,\mathbf{P}}) \partial^j_i w_{\epsilon,P^j} + \int_{\Omega_{\epsilon,P^j}} L_{\epsilon,\mathbf{P}}(\Phi_{\epsilon,\mathbf{P}}) \partial^j_i w_{\epsilon,P^j} + \int_{\Omega_{\epsilon,P^j}} N_\epsilon(\Phi_{\epsilon,\mathbf{P}}) \partial^j_i w_{\epsilon,P^j}$$

$$= J_1 + J_2 + J_3$$

where $J_i, i = 1, 2, 3$ are defined at the last equality.

We first estimate J_2 and J_3. Note that

$$J_2 = \int_{\Omega_{\epsilon,P^j}} p[(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}} - 1)_+^{p-1} \partial_i^j w_{\epsilon,P^j} - (w_j - 1)_+^{p-1} \partial_i^j w_j] \Phi_{\epsilon,\mathbf{P}}$$
$$= O(\epsilon^{N-2+\delta_0})$$

by Lemma 2.3 and Lemma 3.3, and

$$J_3 = \int_{\Omega_{\epsilon,P^j}} N_\epsilon(\Phi_{\epsilon,\mathbf{P}}) \partial_i^j w_{\epsilon,P^j} = O(\epsilon^{N-2+\delta_0}).$$

It remains to estimate J_1. By Lemma 3.3 and the fact that

$$\partial_i^j w_{\epsilon,P^j} = \partial_i^j w(\frac{x-P^j}{\epsilon}) - \epsilon^{N-2} k_w \partial_i^j H(x,P^j),$$

we have

$$\begin{aligned}
J_1 &= \int_{\Omega_{\epsilon,P^j}} S_\epsilon(w_{\epsilon,\mathbf{P}}) \partial_i^j w_{\epsilon,P^j} \\
&= \int_{\Omega_{\epsilon,P^j}} S_\epsilon(w_{\epsilon,\mathbf{P}}) \partial_i^j w_j + O(\epsilon^{2(N-2)}) \\
&= k_w \sum_{k=1}^N \frac{\partial}{\partial x_k} F_j(\mathbf{P},x)|_{x=P^j} \epsilon^{N-2} \int_{R^N} \frac{\partial w}{\partial y_i} y_k p(w-1)_+^{p-1} dy + O(\epsilon^{N-2+\delta_0}) \\
&= k_w \frac{\partial}{\partial x_i} F_j(\mathbf{P},x)|_{x=P^j} \epsilon^{N-2} \int_{R^N} \frac{\partial w}{\partial y_i} y_i p(w-1)_+^{p-1} dy + O(\epsilon^{N-2+\delta_0}) \\
&= c_w \epsilon^{N-2} \partial_i^j F(\mathbf{P}) + O(\epsilon^{N-2+\delta_0})
\end{aligned}$$

since

$$\int_{R^N} \frac{\partial w}{\partial y_i} y_i p(w-1)_+^{p-1} = \int_{R^N} \frac{\partial (w-1)_+^p}{\partial y_i} y_i = -\int_{R^N} (w-1)_+^p,$$

and

$$\frac{\partial}{\partial x_i} F_j(\mathbf{P},x)|_{x=P^j} = \frac{1}{2} \partial_i^j F(\mathbf{P}).$$

□

Finally we prove Theorem 1.3.

Proof of Theorem 1.3:

By Proposition 3.1, we have

$$S_\epsilon(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) = \sum_{k,l} \beta_k^l(\mathbf{P}) \partial_k^l w_{\epsilon,\mathbf{P}}.$$

By Lemma 4.1, we have

$$\beta_k^l(\mathbf{P}) = (1+o(1))\epsilon^2 W_{\epsilon,k}^l(\mathbf{P}) = \epsilon^N (c_w \partial_k^l F(\mathbf{P}) + o(1))$$

Since $\mathbf{P}_0$ is a nondegenerate critical point of $F(\mathbf{P})$, by Brouwer's fixed point theorem, there exists an $\mathbf{P}_\epsilon$ such that

$$\beta_k^l(\mathbf{P}_\epsilon) = 0, k = 1, ..., N, l = 1, ..., K.$$

In other words, we have $u_\epsilon = w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}} \in X^{t,\beta}$ satisfies

$$\epsilon^2 \Delta u_\epsilon + (u_\epsilon - 1)_+^p = 0 \text{ in } \Omega$$

and $u_\epsilon = 0$ on $\partial\Omega$. By standard regularity theorem, we have $u_\epsilon \in C^2(\Omega) \cap C^0(\bar{\Omega})$.

It remains to check that $u_\epsilon > 0$ in Ω. By Maximum Principle, it is enough to show that $u_\epsilon^- \equiv 0$. Multiplying the equation by u_ϵ^- and integrating by parts we have

$$\epsilon^2 \int_\Omega |\nabla u_\epsilon^-|^2 = \int_\Omega u_\epsilon^- (u_\epsilon - 1)_+^p = 0$$

which implies that $u_\epsilon^- \equiv 0$.

By following the same arguments as in Step 13 and Step 14 of [20], we have u_ϵ has exactly K local maximum points P_ϵ^j and moreover $P_\epsilon^j \to P_0^j, j = 1, ..., K$. Furthermore the set $A_\epsilon = \{u_\epsilon > 1\}$ consists of exactly K components and (1.11) holds for $R_1 > R_2$ where $R_1 = R_w + o(1), R_2 = R_w + o(1)$.

$\square$

5. Uniqueness of u_ϵ

Let u_ϵ be the solution constructed in Section 4. In this section, we prove the uniqueness of u_ϵ.

Let $w_{\epsilon,\mathbf{P}}, \mathcal{K}_{\epsilon,\mathbf{P}}, \mathcal{C}_{\epsilon,\mathbf{P}}, \mathcal{K}_{\epsilon,\mathbf{P}}^\perp, \mathcal{C}_{\epsilon,\mathbf{P}}^\perp, S_\epsilon(u)$ be defined in Section 2.

For any $u, v \in H_0^1(\Omega)$, we define

$$< u, v >_\epsilon = \epsilon^{-N} \int_\Omega (\epsilon^2 \nabla u \cdot \nabla v), \quad \|u\|_\epsilon = < u, v >_\epsilon^{\frac{1}{2}} .$$

For any $u \in H_0^1(\Omega)$, we define the energy functional associated with (1.8)

$$J_\epsilon(u) = \epsilon^{-N} [\frac{\epsilon^2}{2} \int_\Omega |\nabla u|^2 - \int_\Omega F(u)] \tag{5.1}$$

where $F(u) = \int_0^u f(s) ds$.

Fix any $\mathbf{P} \in \Omega^K$. We define

$$K_\epsilon(\mathbf{P}) = J_\epsilon(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) \tag{5.2}$$

where $\Phi_{\epsilon,\mathbf{P}}$ is the unique solution given by Proposition 3.1.

Let $\mathbf{P}_0$ be a nondegenerate critical point of $F(\mathbf{P})$. Set

$$\Lambda = B_\delta(\mathbf{P}) \subset \Omega^K.$$

Let $K_\epsilon(\mathbf{P})$ be defined by (5.2). Then we have

Lemma 5.1. *$u_\epsilon = w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}$ is a solutions of (1.9) if and only of $\mathbf{P}$ is a critical point of $K_\epsilon(\mathbf{P})$ in Λ.*

Proof: The proof is similar to Proposition 3.5 of [18]. $\square$

Let u_ϵ be a single-condensation solution with the unique local maximum $\mathbf{P}_\epsilon \to \mathbf{P}_0$. By Lemma 5.1, we have $u_\epsilon = w_{\epsilon,\mathbf{Q}_\epsilon} + \Phi_{\epsilon,\mathbf{Q}_\epsilon}$ for some $\mathbf{Q}_\epsilon \in \Lambda, \Phi_{\epsilon,\mathbf{Q}_\epsilon} \in \mathcal{K}^{\perp}_{\epsilon,\mathbf{Q}_\epsilon}$ and $\mathbf{Q}_\epsilon$ is a critical point of $K_\epsilon(\mathbf{P})$ in Λ.

By Lemma 5.1, to prove the uniqueness of u_ϵ, we just need to show that $K_\epsilon(\mathbf{P})$ has only one critical point in Λ.

Let us define

$$\tilde{K}_\epsilon(\mathbf{P}) = K(\frac{1}{2}\int_{R^N} |\nabla w|^2 - \int_{R^N} F(w)) - c_w F(\mathbf{P})\epsilon^{N-2}. \tag{5.3}$$

It is easy to see that $\tilde{K}_\epsilon(\mathbf{P})$ has only one critical point $\mathbf{P}_0$ in Λ if δ is small.

We now compute $\nabla K_\epsilon(\mathbf{P})$. We first have

Lemma 5.2. *For $\mathbf{P} \in \Lambda$, we have*

$$\nabla K_\epsilon(\mathbf{P}) = \nabla \tilde{K}_\epsilon(\mathbf{P}) + o(\epsilon^{N-2}). \tag{5.4}$$

Proof: Observe that

$$\begin{aligned}
\partial_i^j K_\epsilon(\mathbf{P}) &=< w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}, \partial_i^j (w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) >_\epsilon - \epsilon^{-N} \int_\Omega (w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}})^p \partial_i^j (w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) \\
&= \epsilon^{-N} \int_\Omega S_\epsilon(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) \partial_i^j (w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) \\
&= \sum_{k,l} \beta_k^l(\mathbf{P}) \int_{\Omega_\epsilon} \partial_k^l w_{\epsilon,\mathbf{P}} \partial_i^j (w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) \\
&= \partial_i^j \tilde{K}_\epsilon(\mathbf{P}) + o(\epsilon^{N-2}),
\end{aligned}$$

by Lemma 4.1.

$\square$

The following lemma shows that a critical point of K_ϵ is nondegenerate.

Lemma 5.3. *Let $\mathbf{Q}_\epsilon$ be a critical point of $K_\epsilon(\mathbf{P})$ over Λ. Then we have*

$$\begin{aligned}
&\partial_i^j \partial_k^l K_\epsilon(\mathbf{P})\big|_{\mathbf{P}=\mathbf{Q}_\epsilon} \\
&= \partial_i^j \partial_k^l \tilde{K}_\epsilon(\mathbf{P})|_{\mathbf{P}=\mathbf{Q}_\epsilon} + o(\epsilon^{N-2}).
\end{aligned} \tag{5.5}$$

The proof of Lemma 5.3 is very complicated and is thus left to Appendix A. Let us now use it to prove the uniqueness of u_ϵ.

The proof of the uniqueness of u_ϵ is completed by the following lemma.

Lemma 5.4. *There exists a unique critical point of $K_\epsilon(\mathbf{P})$ over Λ.*

Proof: As we already know by Theorem 1.3 and Lemma 5.1, $K_\epsilon(\mathbf{P})$ has a critical point $\mathbf{Q}_\epsilon$ and any other critical point of $K_\epsilon(\mathbf{P})$ is in Λ.

We now show that Q_ϵ is unique.

First, by Lemma 5.3, there are only finite number of critical points of $K_\epsilon(\mathbf{P})$ in Λ. Let $\mathbf{Q}_\epsilon^i \in \Lambda, i = 1, ..., k_\epsilon$ be the critical points, where k_ϵ is the number of critical points. At each critical point $\mathbf{Q}_\epsilon^i$, we have by Lemma 5.3 and Lemma 5.2,

$$deg(\nabla K_\epsilon, B_{\delta_i}(\mathbf{Q}_\epsilon^i), 0) = (-1)^{d_i}$$

where $\delta_i > 0$ are small constants so that $B_{\delta_i}\ (\mathbf{Q}_\epsilon^i)(\subset \Lambda)$ contains only one critical point (i.e. $\mathbf{Q}_\epsilon^i$) of $\nabla K_\epsilon(\mathbf{P})$. Here d_i is Morse index of $F(\mathbf{P})$ at $\mathbf{Q}_\epsilon^i$. By Lemma 5.3, $d_i = d$, where d is the Morse index of $F(\mathbf{P})$ at $\mathbf{P}_0$.

Hence by the additivity of the degree we have

$$deg(\nabla K_\epsilon, \Lambda, 0) = \sum_{j=1}^{k_\epsilon} (-1)^{d_j} = k_\epsilon (-1)^d. \tag{5.6}$$

On the other hand, since $\tilde{K}_\epsilon(\mathbf{P})$ has only one critical point in Λ and by Lemma 5.2, $\nabla K_\epsilon(\mathbf{P}) = \nabla \tilde{K}_\epsilon(\mathbf{P}) + o(\epsilon^{N-2})$, by a continuity argument (note that $\nabla K_\epsilon(\mathbf{P}) \neq 0$ and $\nabla \tilde{K}_\epsilon(\mathbf{P}) \neq 0$ on $\partial\Lambda$), we obtain

$$deg(\nabla K_\epsilon, \Lambda, 0) = deg(\nabla \tilde{K}_\epsilon, \Lambda, 0) = (-1)^d. \tag{5.7}$$

Comparing (5.6) and (5.7), we obtain $k_\epsilon = 1$. □

6. Proofs of Theorems 1.1 and 1.2

In this section, we prove Theorems 1.1 and 1.2. By Theorem 1.3, for each $\epsilon < \epsilon_0$ sufficiently small, there exists a unique solution u_ϵ with K local maximum points $P_\epsilon^j, j = 1, ..., K$ and $P_\epsilon^j \to P_0^j$ as $\epsilon \to 0$. To prove Theorems 1.1 and 1.2, it is enough to study the following equation

$$I(\epsilon) := \epsilon^{-\frac{2p}{p-1}} \int_\Omega (u_\epsilon - 1)_+^p = M. \tag{6.1}$$

We first have

Lemma 6.1. *$I(\epsilon)$ is a continuous function for $\epsilon \in (0, \epsilon_0)$.*

Proof: This follows from the uniqueness of u_ϵ. In fact let $\epsilon \in (0, \epsilon_0)$ and $\epsilon_k \to \epsilon$ as $k \to +\infty$. Since u_{ϵ_k} is uniformly bounded for $\epsilon_k \in (0, \epsilon_0)$, we can assume that $u_{\epsilon_k} \to \tilde{u}_\epsilon$, where $\tilde{u}_\epsilon$ satisfies (1.9). Moreover, the set $A_{\epsilon_k} = \{u_{\epsilon_k} > 1\}$ approaches the set $A_\epsilon = \{\tilde{u}_\epsilon > 1\}$. Therefore we see that u_ϵ has exactly K local maximum points P_j^ϵ with $P_j^\epsilon = P_j^0 + o(1)$. (We remark here that any local maximum point of u_ϵ is nondegenerate. See [20].) By uniqueness of u_ϵ, we have $\tilde{u}_\epsilon = u_\epsilon$ and thus $\lim_{\epsilon_k \to \epsilon} I(\epsilon_k) = I(\epsilon)$. □

We now compute $I(\epsilon)$.

Lemma 6.2. *For ϵ sufficiently small, we have*

$$\int_\Omega (u_\epsilon - 1)_+^p = \epsilon^N [K \int_{R^N} (w-1)_+^p + \epsilon^{N-2} d_w F(\mathbf{P}^0) + o(\epsilon^{N-2})] \tag{6.2}$$

where $d_w = (\frac{p-1}{2}(N-2) - 1)k_w$ is a generic constant.

Remarks:

1. Combining Lemma 6.1 and Lemma 6.4, Theorem 1.4 is proved by taking $I_p := K \int_{R^N} (w-1)_+^p$.
2. Note that when $p = p^*$, $d_w = 0$.

Proof: Note that

$$u_\epsilon = w_{\epsilon, \mathbf{Q}_\epsilon} + \Phi_{\epsilon, \mathbf{Q}_\epsilon}$$

and

$$\int_\Omega (u_\epsilon - 1)_+^p = \sum_{j=1}^K \int_{B_{2R_0\epsilon}(Q_\epsilon^j)} (u_\epsilon - 1)_+^p \tag{6.3}$$

for some $R_0 > 0$. We just need to compute $\int_{B_{2R_0\epsilon}(Q_\epsilon^j)} (u_\epsilon - 1)_+^p, j = 1, ..., K$. We now compute the case when $j = 1$. The others are similar. We obtain

$$\int_{B_{2R_0\epsilon}(Q_\epsilon^1)} (u_\epsilon - 1)_+^p = \epsilon^N \int_{B_{2R_0}(0)} (u_\epsilon(Q_\epsilon^1 + \epsilon y) - 1)_+^p$$

$$= \epsilon^N \int_{B_{2R_w}(0)} (w_{\epsilon, \mathbf{Q}_\epsilon} + \Phi_{\epsilon, \mathbf{Q}_\epsilon} - 1)_+^p$$

$$= \epsilon^N [\int_{B_{2R_0}(0)} (w(y) + k_w \epsilon^{N-2} F_1(\mathbf{Q}_\epsilon, Q_\epsilon^1) + \Phi_{\epsilon, \mathbf{Q}_\epsilon} - 1)_+^p + O(\epsilon^{N-1})]$$

$$= \epsilon^N [\int_{R^N} (w-1)_+^p + p k_w \int_{R^N} (w-1)_+^{p-1} \epsilon^{N-2} F_1(\mathbf{Q}_\epsilon, Q_\epsilon^1) + p \int_{R^N} (w-1)_+^{p-1} \Phi_{\epsilon, \mathbf{Q}_\epsilon} + O(\epsilon^{N-1})] \tag{6.4}$$

By Lemma 3.3, for $y \in B_{2R_0}(0)$, we have

$$\Phi_{\epsilon, \mathbf{Q}_\epsilon} = k_w \epsilon^{N-2} F_1(\mathbf{Q}_\epsilon, Q_\epsilon^1) \Phi_0(\frac{x - Q_\epsilon^1}{\epsilon}) + o(\epsilon^{N-2}) \tag{6.5}$$

where Φ_0 is the solution of (3.7).

Thus

$$p\int_{R^N}(w-1)_+^{p-1}\Phi_{\epsilon,\mathbf{Q}_\epsilon}=p\epsilon^{N-2}\int_{R^N}(w-1)_+^{p-1}F_1(\mathbf{Q}_\epsilon,Q_\epsilon^1)k_w\Phi_0(y)+o(\epsilon^{N-2}) \tag{6.6}$$

Substituting (6.6) into (6.4) and noting that $\mathbf{Q}_\epsilon\to\mathbf{P}_0$, we obtain

$$\int_{B_{2R_0\epsilon}(Q_\epsilon^1)}(u_\epsilon-1)_+^p=\epsilon^N[\int_{R^N}(w-1)_+^p+p\int_{R^N}(w-1)_+^{p-1}k_wF_1(\mathbf{P}_0,P_0^1)\epsilon^{N-2}$$

$$+p\int_{R^{N-1}}(w-1)_+^{p-1}\Phi_0k_wF_1(\mathbf{P}_0,P_0^1)\epsilon^{N-2}+o(\epsilon^{N-2})]$$

$$=\epsilon^N[\int_{R^N}(w-1)_+^p+p\int_{R^N}(w-1)_+^{p-1}(\Phi_0+1)k_wF_1(\mathbf{P}_0,P_0^1)\epsilon^{N-2}+o(\epsilon^{N-2})].$$

From the equation for Φ_0,

$$p\int_{R^N}(w-1)_+^{p-1}(\Phi_0+1)=-\int_{\partial B_{R_w}(0)}\frac{\partial\Phi_0}{\partial\nu}$$

$$=-C_0(2-N)|\partial B_1(0)|=d_w,$$

where $\Phi_0(r)=\frac{C_0}{r^{N-2}}$ for $r\geq R_w$. By Lemma 3.2, $C_0=\frac{k_w}{(N-2)|S^{N-1}|}(\frac{p-1}{2}(N-2)-1)$.

Hence

$$\int_{B_{2R_0\epsilon}(Q_\epsilon^1)}(u_\epsilon-1)_+^p=\epsilon^N[\int_{R^N}(w-1)_+^p+d_wF_1(\mathbf{P}_0,P_0^1)\epsilon^{N-2}+o(\epsilon^{N-2})]$$

Combining the estimates of $\int_{B_{2R_0\epsilon}(Q_\epsilon^j)}(u_\epsilon-1)_+^p, j=1,...,K$, (6.2) is thus proved. □

Finally we can prove Theorems 1.1 and 1.2.

Proof of Theorems 1.1 and 1.2: Consider the equation

$$I(\epsilon)=M. \tag{6.7}$$

By Lemma 6.1,

$$I(\epsilon)=\epsilon^{N-\frac{2p}{p-1}}[K\int_{R^N}(w-1)_+^p+d_wF(\mathbf{P}^0)\epsilon^{N-2}+o(\epsilon^{N-2})].$$

It is not hard to see that

$$M_*^{(N)}=\int_{R^N}(w-1)_+^{p^*}.$$

Set

$$u^M=\epsilon^{-\frac{2}{p-1}}(u_\epsilon-1). \tag{6.8}$$

Let us now prove that u^M satisfy the properties of Theorem 1.1 and Theorem 1.2.

Properties (i), (ii) and (iii) of Theorem 1.1 follow directly from those of u_ϵ and Lemma 6.2.

To prove Theorem 1.2, we note that the relation between M and ϵ is given by

$$M = \epsilon^{N-\frac{2p}{p-1}} \{ K \int_{R^N} (w-1)_+^p + O(\epsilon^{N-2}) \}.$$

Since $-\mu^M = \epsilon^{-\frac{2}{p-1}}$, we obtain

$$M(-\mu^M)^{\frac{2p-N(p-1)}{2}} = M\epsilon^{\frac{2p}{p-1}-N} = K \int_{R^N} (w-1)_+^p + o(1),$$

which proves (iii) of Theorem 1.2 with $\mu_0 = K \int_{R^N} (w-1)_+^p$. (i) and (ii) of Theorem 1.2 follow from the properties of u_ϵ.

$\square$

7. Appendix A: proof of Lemma 5.3

In this appendix, we prove the technical lemma-Lemma 5.3. The key is to expand $\Phi_{\epsilon,\mathbf{P}}$ and $\partial_k^l \Phi_{\epsilon,\mathbf{P}}$.

Recall that $\tilde{\Phi}_{\epsilon,\mathbf{P}}^0$ is defined by (3.8). We first need

Lemma 7.1. *Let $\tilde{\Phi}_{\epsilon,\mathbf{P}}$ be defined by (3.8). We then have that $\Phi_{\epsilon,\mathbf{P}}$ is C^1 in $\mathbf{P}$. Moreover we have*

$$\Phi_{\epsilon,\mathbf{P}} = \epsilon^{N-2} \tilde{\Phi}_{\epsilon,\mathbf{P}}^0 + o(\epsilon^{N-2}) \tag{7.1}$$

and

$$\partial_i^j \Phi_{\epsilon,\mathbf{P}} = \epsilon^{N-2} \partial_i^j \tilde{\Phi}_{\epsilon,\mathbf{P}}^0 + O(\epsilon^{N-2}). \tag{7.2}$$

Proof: (7.1) has been proved in Lemma 3.3. To prove (7.2), we note that

$$\tilde{\Phi}_{\epsilon,\mathbf{P}}^0 = \Phi_{\epsilon,\mathbf{P}}^0 + O(\epsilon^{N-2}).$$

Put

$$\Psi_\epsilon := \partial_i^j \Phi_{\epsilon,\mathbf{P}} - \epsilon^{N-2} \partial_i^j \tilde{\Phi}_{\epsilon,\mathbf{P}}^0 \tag{7.3}$$

and we decompose Ψ_ϵ into two parts

$$\Psi_\epsilon = \sum_{k,l} \tilde{\alpha}_{ik}^{jl} \partial_k^l w_{\epsilon,\mathbf{P}} + \Psi_\epsilon^\perp, \Psi_\epsilon^\perp \in \mathcal{K}_{\epsilon,\mathbf{P}}^\perp,$$

where $\tilde{\alpha}_{ik}^{jl}$ are some constants.

We will show that

$$\tilde{\alpha}_{ik}^{jl} = O(\epsilon^{N-1}) \tag{7.4}$$

and

$$\|\Psi_\epsilon^\perp\|_{t,2,\beta} = O(\epsilon^{N-2}). \tag{7.5}$$

We first note that since $\Phi_{\epsilon,\mathbf{P}} \perp \mathcal{K}_{\epsilon,\mathbf{P}}$,

$$\epsilon^{-N}\int_\Omega \frac{\partial \Phi_{\epsilon,\mathbf{P}}}{\partial P_i^j}\frac{\partial w_{\epsilon,\mathbf{P}}}{\partial P_k^l} = -\epsilon^{-N}\int_\Omega \Phi_{\epsilon,\mathbf{P}}\frac{\partial^2 w_{\epsilon,\mathbf{P}}}{\partial P_i^j \partial P_k^l}$$

Hence

$$\sum_{k,l}\tilde{\alpha}_{ik}^{jl}\epsilon^{-N}\int_\Omega \frac{\partial w_{\epsilon,\mathbf{P}}}{\partial P_k^l}\frac{\partial w_{\epsilon,\mathbf{P}}}{\partial P_{k'}^{l'}}$$

$$= -\epsilon^{N-2}\epsilon^{-N}\int_\Omega \Phi_{\epsilon,\mathbf{P}}\partial_i^j\partial_{k'}^{l'}w_{\epsilon,\mathbf{P}} + \epsilon^{-N}\int_\Omega \epsilon^{N-2}\Phi^0_{\epsilon,\mathbf{P}}\partial_i^j\partial_{k'}^{l'}w_{\epsilon,\mathbf{P}} + O(\epsilon^{N-3})$$

$$= O(\epsilon^{N-3}).$$

Since $\epsilon^{-N}\int_\Omega \frac{\partial w_{\epsilon,\mathbf{P}}}{\partial P_k^l}\frac{\partial w_{\epsilon,\mathbf{P}}}{\partial P_{k'}^{l'}} = \epsilon^{-2}(\Gamma+o(1))\delta_{kk'}\delta_{ll'}$, where $\Gamma>0$, we obtain $\tilde{\alpha}_{ik}^{jl} = O(\epsilon^{N-1})$, which proves (7.4).

Next we observe that

$$S_\epsilon(w_{\epsilon,\mathbf{P}}+\Phi_{\epsilon,\mathbf{P}}) = \sum_{ij}\beta_i^j(P)\frac{\partial w_{\epsilon,\mathbf{P}}}{\partial P_i^j}$$

where $\beta_i^j(\mathbf{P}) \in C^1$.

Differentiating the above equation by $\frac{\partial}{\partial P_i^j}$, we have,

$$S_\epsilon'(w_{\epsilon,\mathbf{P}}+\Phi_{\epsilon,\mathbf{P}})(\frac{\partial w_{\epsilon,\mathbf{P}}}{\partial P_i^j}+\frac{\partial \Phi_{\epsilon,\mathbf{P}}}{\partial P_i^j}) - \sum_{k,l}\beta_k^l(P)\frac{\partial^2 w_{\epsilon,\mathbf{P}}}{\partial P_i^j\partial P_k^l} \in \mathcal{C}_{\epsilon,\mathbf{P}}.$$

At a critical point $\mathbf{P}=\mathbf{Q}_\epsilon$ of $K_\epsilon(\mathbf{P})$, we must have

$$\beta_k^l(\mathbf{Q}_\epsilon) = 0, k=1,...,N, l=1,...,K.$$

Hence at $\mathbf{P}=\mathbf{Q}_\epsilon$,

$$S_\epsilon'(w_{\epsilon,\mathbf{P}}+\Phi_{\epsilon,\mathbf{P}})(\Psi^\perp_{\epsilon,\mathbf{P}})+S_\epsilon'(w_{\epsilon,\mathbf{P}}+\Phi_{\epsilon,\mathbf{P}})(\frac{\partial(w_{\epsilon,\mathbf{P}}+\epsilon^{N-2}\Phi^0_{\epsilon,\mathbf{P}})}{\partial P_i^j}+\sum_{k,l}\tilde{\alpha}_{ik}^{jl}\frac{\partial w_{\epsilon,\mathbf{P}}}{\partial P_k^l}) \in \mathcal{C}_{\epsilon,\mathbf{P}}.$$

We now need to compute

$$E_\epsilon := S_\epsilon'(w_{\epsilon,\mathbf{P}}+\Phi_{\epsilon,\mathbf{P}})(\frac{\partial(w_{\epsilon,\mathbf{P}}+\epsilon^{N-2}\Phi^0_{\epsilon,\mathbf{P}})}{\partial P_i^j}|_{\mathbf{P}=\mathbf{Q}_\epsilon})$$

By using Lemma 3.3, a simple computation shows that

$$E_\epsilon = \partial_i^j|_{\mathbf{P}=\mathbf{Q}_\epsilon}S_\epsilon(w_{\epsilon,\mathbf{P}}+\epsilon^{N-2}\Phi^0_{\epsilon,\mathbf{P}})$$

$$= O(\epsilon^{N-2}).$$

On the other hand, by the same proof as that of Proposition 2.1, it is easy to see that

$$\Pi_{\epsilon,\mathbf{P}}^{\perp} \circ S_{\epsilon}^{'}(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) : \mathcal{K}_{\epsilon,\mathbf{P}}^{\perp} \to \mathcal{C}_{\epsilon,\mathbf{P}}^{\perp}$$

is invertible for ϵ sufficiently small. Hence (since $\Psi_{\epsilon,\mathbf{P}}^{\perp} \in \mathcal{K}_{\epsilon,\mathbf{P}}^{\perp}$)

$$\begin{aligned}\|\Psi_{\epsilon,\mathbf{P}}^{\perp}\|_{t,2,\beta} &\le C\|\Pi_{\epsilon,\mathbf{P}}^{\perp} \circ E_{\epsilon}\|_{t,2+\beta} + O(\epsilon^{N-2})\\ &\le C\epsilon^{N-2}.\end{aligned}$$

(7.2) is thus proved.

□

By using Lemma 7.1, we obtain

Proof of Lemma 5.3:

In order to have a unified approach which works for all dimension $N \ge 3$, we need to modify $w_{\epsilon,\mathbf{P}}$. We introduce the following notations

$$\bar{w}_{\epsilon,P^j} = w_{\epsilon,P^j} + k_w F_j(\mathbf{P}, P^j)(\Phi_0(\frac{x-P^j}{\epsilon}) - C_0(N-2)|S^{N-1}|\epsilon^{N-2}H(x,P^j)),$$

$$\bar{w}_{\epsilon,\mathbf{P}} = \sum_{j=1}^{K} \bar{w}_{\epsilon,P^j},$$

$$\bar{w}_{\epsilon,\mathbf{P}} + \bar{\Phi}_{\epsilon,\mathbf{P}} = w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}.$$

Then by Lemma 7.1, we have

$$\bar{\Phi}_{\epsilon,\mathbf{P}} = O(\epsilon^{N-2+\delta_0}), \partial_k^l \bar{\Phi}_{\epsilon,\mathbf{P}} = O(\epsilon^{N-2}). \tag{7.6}$$

Thus we obtain

$$\frac{\partial^2 K_\epsilon(\mathbf{P})}{\partial P_i^j \partial P_k^l}|_{\mathbf{P}=\mathbf{Q}_\epsilon}$$

$$\begin{aligned}&=< \partial_i^j(\bar{w}_{\epsilon,\mathbf{P}}+\bar{\Phi}_{\epsilon,\mathbf{P}}), \partial_k^l(\bar{w}_{\epsilon,\mathbf{P}}+\bar{\Phi}_{\epsilon,\mathbf{P}}) >_\epsilon |_{\mathbf{P}=\mathbf{Q}_\epsilon} + < \bar{w}_{\epsilon,\mathbf{P}}+\bar{\Phi}_{\epsilon,\mathbf{P}}, \partial_i^j\partial_k^l(\bar{w}_{\epsilon,\mathbf{P}}+\bar{\Phi}_{\epsilon,\mathbf{P}}) >_\epsilon |_{\mathbf{P}=\mathbf{Q}_\epsilon}\\ &\quad -\epsilon^{-N}\int_\Omega p(\bar{w}_{\epsilon,\mathbf{Q}_\epsilon} + \bar{\Phi}_{\epsilon,\mathbf{Q}_\epsilon} - 1)_+^{p-1}\partial_i^j(\bar{w}_{\epsilon,\mathbf{P}} + \bar{\Phi}_{\epsilon,\mathbf{P}})|_{\mathbf{P}=\mathbf{Q}_\epsilon}\partial_k^l(\bar{w}_{\epsilon,\mathbf{P}} + \bar{\Phi}_{\epsilon,\mathbf{P}})|_{\mathbf{P}=\mathbf{Q}_\epsilon}\\ &\quad -\epsilon^{-N}\int_\Omega (\bar{w}_{\epsilon,Q_\epsilon} + \bar{\Phi}_{\epsilon,Q_\epsilon} - 1)_+^p \partial_k^l\partial_i^j(\bar{w}_{\epsilon,\mathbf{P}} + \bar{\Phi}_{\epsilon,P})|_{\mathbf{P}=\mathbf{Q}_\epsilon}\end{aligned}$$

(since $\mathbf{P} = \mathbf{Q}_\epsilon$ is a critical point)

$$\begin{aligned}&=< \partial_i^j(\bar{w}_{\epsilon,\mathbf{P}} + \bar{\Phi}_{\epsilon,\mathbf{P}}), \partial_k^l(\bar{w}_{\epsilon,\mathbf{P}} + \bar{\Phi}_{\epsilon,\mathbf{P}}) >_\epsilon |_{\mathbf{P}=\mathbf{Q}_\epsilon}\\ &\quad -\epsilon^{-N}\int_\Omega p(\bar{w}_{\epsilon,\mathbf{Q}_\epsilon} + \bar{\Phi}_{\epsilon,\mathbf{Q}_\epsilon} - 1)_+^{p-1}\partial_i^j(\bar{w}_{\epsilon,\mathbf{P}} + \bar{\Phi}_{\epsilon,\mathbf{P}})|_{\mathbf{P}=\mathbf{Q}_\epsilon}\partial_k^l(\bar{w}_{\epsilon,\mathbf{P}} + \bar{\Phi}_{\epsilon,\mathbf{P}})|_{\mathbf{P}=\mathbf{Q}_\epsilon}\\ &=< \partial_k^l(\bar{w}_{\epsilon,\mathbf{P}}), \partial_i^j(\bar{w}_{\epsilon,\mathbf{P}}) >_\epsilon |_{\mathbf{P}=\mathbf{Q}_\epsilon} - \epsilon^{-N}\int_\Omega p(\bar{w}_{\epsilon,\mathbf{Q}_\epsilon}+\bar{\Phi}_{\epsilon,\mathbf{Q}_\epsilon}-1)_+^{p-1}\partial_i^j(\bar{w}_{\epsilon,\mathbf{P}})|_{\mathbf{P}=\mathbf{Q}_\epsilon}\partial_k^l(\bar{w}_{\epsilon,\mathbf{P}})|_{\mathbf{P}=\mathbf{Q}_\epsilon}\\ &\quad + < \partial_k^l(\bar{w}_{\epsilon,\mathbf{P}}), \partial_i^j(\bar{\Phi}_{\epsilon,\mathbf{P}}) >_\epsilon |_{\mathbf{P}=\mathbf{Q}_\epsilon} - \epsilon^{-N}\int_\Omega p(\bar{w}_{\epsilon,\mathbf{Q}_\epsilon}+\bar{\Phi}_{\epsilon,\mathbf{Q}_\epsilon}-1)_+^{p-1}\partial_i^j(\bar{\Phi}_{\epsilon,\mathbf{P}})|_{\mathbf{P}=\mathbf{Q}_\epsilon}\partial_k^l(\bar{w}_{\epsilon,\mathbf{P}})|_{\mathbf{P}=\mathbf{Q}_\epsilon}\end{aligned}$$

$$+ < \partial_k^l(\bar{\Phi}_{\epsilon,\mathbf{P}}), \partial_i^j(\bar{w}_{\epsilon,\mathbf{P}}) >_\epsilon |_{\mathbf{P}=\mathbf{Q}_\epsilon} - \epsilon^{-N}\int_\Omega p(\bar{w}_{\epsilon,\mathbf{Q}_\epsilon}+\bar{\Phi}_{\epsilon,\mathbf{Q}_\epsilon}-1)_+^{p-1}\partial_i^j(\bar{w}_{\epsilon,\mathbf{P}})|_{\mathbf{P}=\mathbf{Q}_\epsilon}\partial_k^l(\bar{\Phi}_{\epsilon,\mathbf{P}})|_{\mathbf{P}=\mathbf{Q}_\epsilon}$$

$$+ < \partial_k^l(\bar{\Phi}_{\epsilon,\mathbf{P}}), \partial_i^j(\bar{\Phi}_{\epsilon,\mathbf{P}}) >_\epsilon |_{\mathbf{P}=\mathbf{Q}_\epsilon} - \epsilon^{-N}\int_\Omega p(\bar{w}_{\epsilon,\mathbf{Q}_\epsilon}+\bar{\Phi}_{\epsilon,\mathbf{Q}_\epsilon}-1)_+^{p-1}\partial_i^j(\bar{\Phi}_{\epsilon,\mathbf{P}})|_{\mathbf{P}=\mathbf{Q}_\epsilon}\partial_k^l(\bar{\Phi}_{\epsilon,\mathbf{P}})|_{\mathbf{P}=\mathbf{Q}_\epsilon}$$

$$= I_1 + I_2 + I_3 + I_4,$$

where $I_i, i = 1, 2, 3, 4$ are defined at the last equality.

By (7.6),

$$I_4 = o(\epsilon^{N-2}),$$

$$I_2 = \int_{\Omega_{\epsilon,P^j}} S'_\epsilon(\mathbf{w}_{\epsilon,\mathbf{Q}_\epsilon})(\partial_i^j|_{\mathbf{P}=\mathbf{Q}_\epsilon}\bar{w}_{\epsilon,\mathbf{P}})(\partial_k^l|_{\mathbf{P}=\mathbf{Q}_\epsilon}\bar{\Phi}_{\epsilon,\mathbf{P}})$$

$$= o(\epsilon^{N-2}).$$

Similarly we have

$$I_3 = o(\epsilon^{N-2}).$$

It remains to compute I_1.

The computation of I_1 is as follows

$$I_1 = \int_{\Omega_\epsilon} S'_\epsilon(\bar{w}_{\epsilon,\mathbf{Q}_\epsilon})\partial_i^j|_{\mathbf{P}=\mathbf{Q}_\epsilon}\bar{w}_{\epsilon,\mathbf{P}}\partial_k^l|_{\mathbf{P}=\mathbf{Q}_\epsilon}\bar{w}_{\epsilon,\mathbf{P}}$$

$$= \int_{\Omega_\epsilon} \partial_i^j[S_\epsilon(\bar{w}_{\epsilon,\mathbf{P}})\partial_k^l\bar{w}_{\epsilon,\mathbf{P}}]|_{\mathbf{P}=\mathbf{Q}_\epsilon}$$

$$= \partial_i^j\partial_k^l J_\epsilon(\bar{w}_{\epsilon,\mathbf{P}})|_{\mathbf{P}=\mathbf{Q}_\epsilon}$$

Note that

$$J_\epsilon(\bar{w}_{\epsilon,\mathbf{P}}) = KI(w) - c_w F(\mathbf{P})\epsilon^{N-2} + o(\epsilon^{N-2}).$$

We now need to show that the above equality is true in C^2 for $\mathbf{P}$. Although intuitively it should be true, the proofs are quite complicated due to the fact that when we differentiate P^j, we are also differentiating x since the variables involved are $\frac{x-P^j}{\epsilon}$. The key is to separate these two differentiations. To this end, we introduce

$$D_i^j := \frac{\partial}{\partial P_i^j} + \frac{\partial}{\partial x_i} \tag{7.7}$$

Then we have

$$I_1 = \int_{\Omega_{\epsilon,P^j}} D_i^j[S_\epsilon(\bar{w}_{\epsilon,\mathbf{P}})\partial_k^l\bar{w}_{\epsilon,\mathbf{P}}]|_{\mathbf{P}=\mathbf{Q}_\epsilon} + o(\epsilon^{N-2}).$$

Note that

$$D_i^j g(\frac{|x-P^j|}{\epsilon}) = 0$$

for any radial function g.

So

$$S_\epsilon(\bar{w}_{\epsilon,\mathbf{P}}) = (\bar{w}_{\epsilon,\mathbf{P}} - 1)_+^p - \sum_{s=1}^K (w_s - 1)_+^p - pk_w \sum_{s=1}^K (w_s - 1)_+^{p-1} F_s(\mathbf{P}, P^s)(\Phi_0 + 1)$$

$$= \epsilon^{N-2} pk_w \sum_{s=1}^K (w_s - 1)_+^{p-1} (F_s(\mathbf{P}, x) - F_s(\mathbf{P}, P^s))(\Phi_0 + 1) + o(\epsilon^{N-2})$$

Thus

$$D_i^j [S_\epsilon(\bar{w}_{\epsilon,\mathbf{P}}) \partial_k^l \bar{w}_{\epsilon,\mathbf{P}}]$$

$$= D_i^j (\epsilon^{N-2} pk_w \sum_{s=1}^K (w_s - 1)_+^{p-1} (F_j(\mathbf{P}, x) - F_j(\mathbf{P}, P^s))(\Phi_0 + 1) \partial_k^l w_l) + o(\epsilon^{N-2})$$

$$= \epsilon^{N-2} pk_w \sum_{s=1}^K (w_s - 1)_+^{p-1} D_i^j (F_s(\mathbf{P}, x) - F_s(\mathbf{P}, P^s))(\Phi_0 + 1) \partial_k^l w_l + o(\epsilon^{N-2})$$

Hence

$$I_1 = \epsilon^{N-2} pk_w \int_{\Omega_{\epsilon, P^l}} (w - 1)_+^{p-1} \partial_k^l \partial_i^j F(\mathbf{P}) y_k \frac{\partial w}{\partial y_k} (\Phi_0 + 1) + o(\epsilon^{N-2})$$

$$= -c_w \epsilon^{N-2} \partial_k^l \partial_i^j F(\mathbf{P})|_{\mathbf{P} = \mathbf{Q}_\epsilon} + o(\epsilon^{N-2}).$$

Lemma 5.3 is thus proved.

8. Appendix B: Proof of Lemma 2.5

In this appendix, we prove the regularity result Lemma 2.5. Recall that $0 < \beta < \frac{N}{t'} - 1$.

We first prove the case when $K = 1$. In this case, we rescale the domain Ω and the functions as follows

$$x = \epsilon y + P^1, \tilde{\Phi}(y) = \Phi(\epsilon y + P^1), \tilde{f}(y) = f(\epsilon y + P^1).$$

Then $\tilde{\Phi}$ satisfies

$$\Delta \tilde{\Phi} + \tilde{f} = 0 \text{ in } \Omega_{\epsilon, P^1}$$

We extend $\tilde{\Phi}$ and $\tilde{f}$ to R^N by putting 0 outside Ω_{ϵ, P^1}.

Note that the Laplace operator Δ is an isomorphism from $W_{2,\beta}^t(R^N)$ to $L_{2+\beta}^t(R^N)$. See [12], [24], [25] and [28]. Thus we have

$$\|\Phi\|_{t,0,\beta} \le \|\tilde{\Phi}\|_{L_\beta^t(R^N)} \le C\|\tilde{f}\|_{L_{2+\beta}^t(R^N)} \le C\|f\|_{t,2+\beta}. \quad (8.1)$$

On the other hand, let

$$\bar{\Phi} = \tilde{\Phi}(y)\chi(x - P^1)$$

where $\chi(x) = 1$ for $|x| \le \delta$ and $\chi(x) = 0$ for $|x| \ge 2\delta$.

Then it is easy to see that $\bar{\Phi}$ satisfies

$$\Delta_y \bar{\Phi} + \chi \tilde{f} + 2\nabla_y \chi \nabla \tilde{\Phi} + \Delta_y \chi \tilde{\Phi} = 0, \bar{\Phi} \in W^t_{2,\beta}(R^N).$$

Hence

$$\|\bar{\Phi}\|_{W^t_{2,\beta}(R^N)} \leq C\|f\|_{t,2+\beta} + C\|\nabla_y \chi \nabla \tilde{\Phi}\|_{t,2+\beta}$$
$$\leq C\|f\|_{t,2+\beta} + \delta \|\Phi\|_{t,2,\beta}$$

for any small δ. Here we have used interpolation inequality.

But outside $B_\delta(P^1)$, we have

$$\|\Phi\|_{W^t_{2,\beta}(\Omega \backslash B_\delta(P^1))} \leq C\|f\|_{t,2+\beta}$$

Thus

$$\|\Phi\|_{t,2,\beta} \leq C\|f\|_{t,2+\beta} + \delta \|\Phi\|_{t,2+\beta}$$

Lemma 2.5 is thus proved for the case $K = 1$.

Next we prove the case when $K = 2$. The other cases $K > 2$ can be proved similarly.

We divide the domain into two parts $\Omega_1 \cup \Omega_2 = \Omega, \Omega_1 \cap \Omega_2 = \phi$. We choose χ_1 and χ_2 such that $\chi_1(x) = 1$ for $x \in B_\delta(P^1)$ and $\chi_1(x) = 0$ for $\in \Omega \backslash B_\delta(P^2)$. Similarly we require that $\chi_2(x) = 1$ for $x \in B_\delta(P^2)$ and $\chi_2(x) = 0$ for $\in \Omega \backslash B_\delta(P^1)$. Moreover we can assume that $\chi_1(x) + \chi_2(x) = 1$ for $x \in \Omega$.

Let $\Phi^i, i = 1, 2$ be the unique solution of

$$\epsilon^2 \Delta \Phi^i + \chi_i(x) f = 0 \text{ in } \Omega, \Phi^i \in H^1_0(\Omega)$$

Then we have

$$\Phi(x) = \Phi_1 + \Phi_2$$

and

$$\|\Phi_j\|_{W^t_{2,\beta}(\Omega_{\epsilon,P^j})} \leq C\|f\|_{t,2+\beta}, j = 1, 2.$$

Since $\beta > 0$, $(\sigma_{\mathbf{P}}(x))^\beta \leq (1 + \frac{|x - P^j|}{\epsilon})^\beta$,

$$\|\Phi_j\|_{t,2,\beta} \leq \|\Phi\|_{W^t_{2,\beta}(\Omega_{\epsilon,P^j})}, j = 1, 2.$$

So we have

$$\|\Phi\|_{t,2,\beta} \leq C\|\Phi_1\|_{t,2,\beta} + C\|\Phi_2\|_{t,2,\beta} \leq C\|f\|_{t,2+\beta}.$$

Lemma 2.5 is thus proved.

ACKNOWLEDGMENTS

This research is supported by an Earmarked Grant from RGC of Hong Kong.

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