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Singular limits of a two-dimensional boundary value problem arising in corrosion modelling

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Abstract

We consider the boundary value problem

$$\Delta u = 0 \text{ in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 2\lambda \sinh u \text{ on } \partial\Omega$$

where Ω is a smooth and bounded domain in $\mathbb{R}^2$ and $\lambda > 0$. We prove that for any integer $k \geq 1$ there exist at least two solutions u_λ with the property that the boundary flux satisfies up to subsequences $\lambda \to 0$,

$$2\lambda \sinh(u_\lambda) \rightharpoonup 2\pi \sum_{j=1}^{2k} (-1)^{j-1} \delta_{\xi_j},$$

where the ξ_j are points of $\partial\Omega$ ordered clockwise in j.

Key words. Singularly perturbed elliptic problem, exponential Neumann boundary condition, concentrating solutions

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. A very common boundary condition arising in corrosion modelling in a planar sample represented by Ω is associated to the names of Butler and Volmer. In its simplest form it asserts an exponential relationship between boundary voltages and boundary normal currents which takes the form

$$\frac{\partial u}{\partial \nu} = \lambda(e^{2\beta u} - e^{-2(1-\beta)u}) + g \quad \text{on } \partial\Omega$$

where the constant $0 < \beta < 1$ depends on the constituents of the electrochemical system but not on their concentrations. Here λ is a constant highly dependent on their concentrations and g an externally imposed current. Assuming the presence of no sources or sinks in Ω, the balanced situation $\beta = \frac{1}{2}$ and $g = 0$, the boundary value problem satisfied by the voltage potential becomes in ideal situation

$$\Delta u = 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = 2\lambda \sinh u \quad \text{on } \partial\Omega. \tag{1.1}$$

We refer the reader to [13] and [5] for the derivation of this and related corrosion models and references to the applied literature.

We assume throughout this paper that $\lambda > 0$. We are interested in solutions to this problem when λ assumes very small values. A surprising example of explicit solution when $\Omega = D$, the unit disk in $\mathbb{R}^2$, was exhibited by Bryan and Vogelius in [3]. Consider $2k$ points on ∂D, $\xi_1, \ldots, \xi_{2k}$

corresponding to vertices of a regular polygon, ordered clockwise, and set

$$u_\lambda(x) = \sum_{j=1}^{2k} (-1)^{j-1} \log \frac{1}{|x - \alpha_k \xi_j|^2} \tag{1.2}$$

where $\alpha_k = [(k+2\lambda)/(k-2\lambda)]^{\frac{1}{2k}}$. We observe the meaningful singular concentration behavior the nonlinear boundary condition exhibits as $\lambda \to 0$. Indeed we have that

$$2\lambda \sinh u_\lambda \rightharpoonup 2\pi \sum_{j=1}^{2k} (-1)^{j-1}\, \delta_{\xi_j}$$

in the sense of measures in ∂D, where δ_{ξ_j} denote Dirac masses at points ξ_j. In [10] possible behaviors of solutions u_λ with boundary condition $2\lambda \sinh(u_\lambda)$ of uniformly bounded mass is established: the limit of the boundary flux along subsequences is a sum of Dirac masses located at a finite set of points with weights greater than or equal to 4π potentially accompanied by a regular part which is one-signed. Solutions to the problem with this property were found by Kavian and Vogelius in [8] via Ljusternik-Schnirelmann theory, however their asymptotic behavior is only partly understood by virtue of the above result. It remains an open question if solutions of the form (1.2) exist in general two-dimensional domains.

The purpose of this paper is to show that in any domain Ω there are at least two distinct families of solutions which exhibit exactly the qualitative behavior of the explicit solution (1.2), namely with limiting boundary flux given by an array of $2k$ Dirac masses with weight 2π and alternate signs. The location of the $2k$ concentration points can be accurately characterized as special critical points of a functional φ_k defined explicitly in terms of

$G(x,y)$, the Green's function for the Neumann problem

$$\begin{cases} \Delta_x G(x,y) = 0 & \text{in } \Omega \\ \frac{\partial G}{\partial \nu_x}(x,y) = 2\pi\delta_y(x) - \frac{2\pi}{|\partial\Omega|} & \text{on } \partial\Omega \\ \int_{\partial\Omega} G(x,y)dx = 0. \end{cases} \tag{1.3}$$

We denote by $H(x,y)$ its regular part:

$$H(x,y) = G(x,y) - \log\frac{1}{|x-y|^2}. \tag{1.4}$$

For $m \geq 1$ and points $\xi_1,\ldots,\xi_m$ on $\partial\Omega$ ordered clockwise we define

$$\varphi_m(\xi_1,\ldots,\xi_m) \;=\; \sum_{l=1}^{m} H(\xi_l,\xi_l) + \sum_{j\neq l}(-1)^{l+j} G(\xi_j,\xi_l). \tag{1.5}$$

Our main result states as follows.

Theorem 1. *Let $k \geq 1$ be a positive integer. There is a number $\lambda_k > 0$ such that for any $0 < \lambda < \lambda_k$ there are two solutions $u_{\lambda 1} \neq -u_{\lambda 2}$ with*

$$\lambda\int_{\partial\Omega} |\sinh u_{\lambda l}| \to 8k\pi \quad \text{as } \lambda \to 0.$$

More precisely, given any sequence $\lambda = \lambda_n \to 0$, there is a subsequence, two arrays of $2k$ points of $\partial\Omega$ $(\xi_{l1},\xi_{l2},\ldots,\xi_{l\,2k})$ ordered clockwise and distinct modulo cyclic permurations, positive constants $\mu_j = \mu_{lj}$, for $j = 1,\ldots,2k$, and two solutions $u_{\lambda l}$ of (1.1), $l = 1,2$ such that, omitting the subindex l,

$$u_\lambda(x) = \sum_{j=1}^{2k}(-1)^{j-1}\log\frac{2\mu_j}{|x-(\xi_j+\lambda\mu_j\nu_j)|^2} \;+\; O(1)$$

where ν_j denotes unit outer normal to $\partial\Omega$ at ξ_j and

$$2\lambda\sinh u_\lambda \;\rightharpoonup\; 2\pi\sum_{j=1}^{2k}(-1)^{j-1}\,\delta_{\xi_j}\,.$$

Moreover, the $2k$*-tuples* $(\xi_{l1}, \ldots, \xi_{l\,2k})$ *are critical points of* φ_{2k}*, and the constants* μ_j *are explicitly given by*

$$\log 2\mu_j = H(\xi_j, \xi_j) + \sum_{l \neq j} (-1)^{l+j} G(\xi_j, \xi_l).$$

It is easily checked that the solutions (1.2) correspond exactly to this description in the case $\Omega = D$. Medville and Vogelius [10] have established that if the limit boundary flux has no regular part, then the concentration points $(\xi_{l1}, \ldots, \xi_{l\,2k})$ necessarily constitute a critical point of φ_{2k}, and that the weights of the delta functions are equal to 2π. They provide numerical evidence that solutions with a boundary flux having a non-trivial regular part exist but this remains an open question. Let us mention that in [11] Medville and Vogelius considered the nonlinear boundary condition

$$\frac{\partial u}{\partial \nu} = Du + 2\lambda \sinh u \quad \text{ on } \partial\Omega,$$

where $D > 0$. They analyze the difference in blow-up as λ approaches 0 from the right (pointwise blow-up) and from the left (blow-up "almost everywhere").

It is interesting to mention the analogy existing between this result and the problem $-\Delta u = \lambda e^u$ under Dirichlet boundary conditions, whose solutions with $\lambda \int_\Omega e^u$ uniformly bounded have become well understood after the works [12,2,9]. It follows from those results that concentration occurs in the form $\lambda e^u \rightharpoonup 8\pi \sum \delta_{\xi_j}$. In [1,6,7] solutions with these properties have been built. In [4] the problem

$$\Delta u - u = 0, \quad \frac{\partial u}{\partial \nu} = \lambda e^u \tag{1.6}$$

was analyzed and given any $k \geq 1$, a solution peaking in such a way that $\lambda e^u \rightharpoonup 2\pi \sum_{j=1}^k \delta_{\xi_j}$ was built up, using as basic cells (after suitable zooming-up) explicit solutions of

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^2_+ \\ \dfrac{\partial v}{\partial \nu} = e^v & \text{on } \partial\mathbb{R}^2_+, \end{cases} \tag{1.7}$$

where $\mathbb{R}^2_+$ denotes the upper half plane $\{(x_1, x_2) : x_2 > 0\}$ and ν the unit exterior normal to $\partial\mathbb{R}^2_+$, given by

$$w_{t,\,\mu}(x_1, x_2) = \log \frac{2\mu}{(x_1 - t)^2 + (x_2 + \mu)^2} \tag{1.8}$$

where $t \in \mathbb{R}$ and $\mu > 0$ are parameters. The solutions predicted in Theorem 1 are also constructed using these ones suitably scaled and projected to make it up to a good order for the boundary condition. Solutions are found as a small additive perturbation of these initial approximations. A linearization procedure leads to a finite dimensional reduction, where the reduced problem corresponds to that of adjusting variationally the location of the concentration points. An important element in the reduction procedure is the non-degeneracy of these solutions up to variations of the parameters t and μ in (1.8). Problem (1.1) has a basic difficulty in comparison with (1.6), linked to the fact that the limiting equation formally satisfied by the seeked solution u_λ is

$$\Delta u = 0, \quad \frac{\partial u}{\partial \nu} = 2\pi \sum_{j=1}^{2k} (-1)^{j-1} \delta_{\xi_j}$$

whose solution is not unique but invariant under the addition of constants. This is not such an innocent matter since this hidden limiting invariance is not present in the equation itself, and unlike the other "obvious" elements of the limiting kernel (see (2.1), (2.2) below), it is not localized near the points of concentration. To be mentioned is that the simple use of an additive constant as an extra parameter in the solution does not suffice, basically because the constant itself is not a good approximation of an element of the kernel before reaching the limit. We are able to overcome this difficulty by identifying an extra element of the approximate kernel (see (3.5) below), which introduces another parameter to be adjusted in the problem. We will devote the rest of this paper to the proof of Theorem 1.

2. Preliminaries

Let us define

$$z_0 = 1 - 2\mu \frac{x_2 + \mu}{x_1^2 + (x_2 + \mu)^2}, \tag{2.1}$$

and

$$z_1 = -2 \frac{x_1}{x_1^2 + (x_2 + \mu)^2}, \tag{2.2}$$

which correspond to derivatives of the basic solutions $w_{t,\mu}$ with respect to its parameters respectively of translation and dilation. These objects obviously lie in the kernel of the linearization of problem (1.7) at the solution $w_{0,\mu}$, namely they solve the problem

$$\begin{cases} \Delta\phi = 0 & \text{in } \mathbb{R}^2_+ \\ \dfrac{\partial\phi}{\partial\nu} - \dfrac{2\mu}{x_1^2 + \mu^2}\phi = 0 & \text{on } \partial\mathbb{R}^2_+. \end{cases} \tag{2.3}$$

Reciprocally, we have the following.

Lemma 1. *Any bounded solution of (2.3) is a linear combination of* z_0 *and* z_1.

Proof. This result was established in [4]. For the sake of self-containedness, we shall present a proof here. Let ϕ be a solution to (2.3) and set

$$w(y) = \phi\left(\frac{y}{|y|^2} - (0,\mu)\right).$$

The function w is just the Kelvin transform of ϕ about the point $(0,-\mu)$. The domain of w is the disk $D = B((0,\frac{1}{2\mu}),\frac{1}{2\mu})$ and w is a bounded function that satisfies $\Delta w = 0$ in D,

$$\frac{\partial w}{\partial \nu'} = 2\mu w \quad \text{on } \partial D \setminus \{0\}, \tag{2.4}$$

where ν' is the exterior unit normal to D. To see this observe that the map $y \mapsto K(y) = \frac{y}{|y|^2} - (0,\mu)$ is anti-conformal (preserves angles and reverses orientation) and maps the normal vector to D to a normal vector to $\partial\mathbb{R}^2_+$. More precisely, if ν' is the exterior unit normal vector to D then

$$\frac{\partial w}{\partial \nu'} = \frac{1}{|y|^2}\frac{\partial \phi}{\partial \nu}.$$

Thus on ∂D

$$\frac{\partial w}{\partial \nu'} = \frac{1}{|y|^2} e^{w_{0,\mu}(K(y))} w$$

and a calculation shows that

$$\frac{1}{|y|^2} e^{w_{0,\mu}(K(y))} = \frac{1}{|y|^2}\frac{2\mu}{\frac{y_1^2}{|y|^4} + \mu^2} = 2\mu.$$

Since w is bounded, by elliptic regularity (2.4) holds in all ∂D.

By translating in the y_2 direction we can assume that D is the disk centered at the origin with radius $\frac{1}{2\mu}$. We think of w as the real part of an analytic function w and write

$$w(y) = \sum_{k=0}^{\infty} a_k r^k e^{ik\theta}$$

with $y = re^{i\theta}$. Condition (2.4) is equivalent to

$$\mathrm{Re}\left(\sum_{k=0}^{\infty} a_k (k-1) e^{ik\theta} \right) = 0 \quad \forall \theta$$

and hence $a_0 = 0$, $a_k = 0$ for all $k > 1$. Looking at the real part of w, and recalling that we shifted in the y_2 direction we see that it is a linear combination of $y_1 = \frac{x_1}{x_1^2 + (x_2+\mu)^2}$ and $y_2 - \frac{1}{2\mu} = \frac{x_2+\mu}{x_1^2+(x_2+\mu)^2} - \frac{1}{2\mu}$. □

In what remains of this paper we fix $k \geq 1$ and denote $m = 2k$ We will provide a first approximation for the solutions of problem (1.1) predicted in Theorem 1. For $j = 1, \ldots, m$, let ξ_j be clockwise ordered points on the boundary of Ω and μ_j positive numbers. Define, for $x \in \Omega$,

$$u_j^\lambda(x) = \log \frac{2\mu_j}{|x - \xi_j - \lambda \mu_j \nu_j|^2} \tag{2.5}$$

and $H_j^\lambda(x)$ to be the unique solution of

$$\Delta H_j^\lambda = 0 \text{ in } \Omega, \tag{2.6}$$

$$\frac{\partial H_j^\lambda}{\partial \nu} = -\frac{\partial u_j^\lambda}{\partial \nu} + \lambda e^{u_j^\lambda} - \lambda \frac{1}{|\partial\Omega|} \int_{\partial\Omega} e^{u_j^\lambda} \text{ on } \partial\Omega \tag{2.7}$$

with the property that

$$\int_{\partial\Omega} H_j^\lambda(x)\, dx = -\int_{\partial\Omega} u_j^\lambda(x)\, dx. \tag{2.8}$$

We look for a solution to (1.1) of the form

$$u(x) = U(x) + \Phi(x) \tag{2.9}$$

where

$$U(x) = \sum_{j=1}^{m} (-1)^{j-1} (u_j^\lambda(x) + H_j^\lambda(x)) \tag{2.10}$$

while Φ is a lower order term with respect to U.

The function H_j^λ above resembles the shape of the regular part of the Green's function. Indeed, the following estimate for H_j^λ holds true.

Lemma 2. *For any* $0 < \alpha < 1$,

$$H_j^\lambda(x) = H(x, \xi_j) - \log 2\mu_j + O(\lambda^\alpha) \tag{2.11}$$

uniformly in $\bar{\Omega}$.

Proof.

The normal derivative of H_j^λ on the boundary of Ω can be computed explicitly, namely

$$\frac{\partial H_j^\lambda}{\partial \nu}(x) = 2\lambda\mu_j \frac{1 - \nu(\xi_j)\cdot\nu(x)}{|x - \xi_j - \lambda\mu_j\nu(\xi_j)|^2} + 2\frac{(x - \xi_j)\cdot\nu(x)}{|x - \xi_j - \lambda\mu_j\nu(\xi_j)|^2} - \lambda\frac{1}{|\partial\Omega|}\int_{\partial\Omega} e^{u_j^\lambda(x)}.$$

Thus

$$\lim_{\lambda\to 0}\frac{\partial H_j^\lambda}{\partial \nu}(x) = 2\frac{(x - \xi_j)\cdot\nu(x)}{|x - \xi_j|^2} - \frac{2\pi}{|\partial\Omega|} \qquad \forall x \neq \xi_j \tag{2.12}$$

since

$$\lambda\int_{\partial\Omega} e^{u_j^\lambda(x)} = \lambda\int_{\partial\Omega}\frac{2\mu_j}{|x - \xi_j - \lambda\mu_j\nu(\xi_j)|^2} = 2\int_{\frac{\partial\Omega - \xi_j}{\lambda\mu_j}}\frac{1}{|y - \nu(0)|^2}$$

$$= 2\left(\int_{-\infty}^{\infty}\frac{dt}{1+t^2} - O(\int_{\lambda^{-1}\mu_j^{-1}}^{\infty}\frac{dt}{1+t^2})\right) = 2\pi + O(\arctan(\lambda\mu_j)^{-1} - \frac{\pi}{2})$$

$$= 2\pi + O(\arctan(\lambda\mu_j))$$

$$= 2\pi + O(\lambda\mu_j). \tag{2.13}$$

Define $z_\lambda(x) = H_j^\lambda(x) + \log 2\mu_j - H(x,\xi_j)$. Since the regular part of the Green's function $H(x,\xi_j)$ is harmonic in Ω and satisfies the Neumann boundary condition

$$\frac{\partial H}{\partial \nu_x}(x,y) = 2\frac{(x-y)\cdot\nu(x)}{|x-y|^2} - \frac{2\pi}{|\partial\Omega|} \quad x \in \partial\Omega,$$

the difference z_λ solves the problem

$$\begin{cases} -\Delta z_\lambda = 0 & \text{in } \Omega \\ \dfrac{\partial z_\lambda}{\partial \nu} = \dfrac{\partial H_j^\lambda}{\partial \nu} - 2\dfrac{(x-\xi_j)\cdot\nu(x)}{|x-\xi_j|^2} + \dfrac{2\pi}{|\partial\Omega|} & \text{on } \partial\Omega. \end{cases}$$

Since z_λ is harmonic in Ω, for any $1 \le p \le \infty$, $z_\lambda \in W^{1,p}(\Omega)$ and, by Poincaré inequality, we get

$$\|z_\lambda - \frac{1}{|\partial\Omega|}\int_{\partial\Omega} z_\lambda\|_{L^p(\Omega)} \le \|Dz_\lambda\|_{L^p(\Omega)}.$$

Hence, by L^p theory, we have for any $0 < s < \frac{1}{p}$

$$\|z_\lambda - \frac{1}{|\partial\Omega|}\int_{\partial\Omega} z_\lambda\|_{W^{1+s,p}(\Omega)} \le C\|\frac{\partial z_\lambda}{\partial \nu}\|_{L^p(\partial\Omega)} \le C\lambda^{\frac{1}{p}}$$

where the last inequality can be obtained arguing like in Lemma 3.1, in [4], and using (2.12). This implies the existence of a constant l such that, for any $\alpha \in (0,1)$,

$$z_\lambda(x) = l + O(\lambda^\alpha)$$

uniformly in $\bar{\Omega}$, where $l = \lim_{\lambda\to 0}\frac{1}{|\partial\Omega|}\int_{\partial\Omega} z_\lambda\, dx$.

In order to get the result, we are left to show that $l = 0$. We have

$$l = \lim_{\lambda\to 0}\left[\frac{1}{|\partial\Omega|}\int_{\partial\Omega} H_j^\lambda(x)\,dx + \log 2\mu_j - \frac{1}{|\partial\Omega|}\int_{\Omega} H(x,\xi_j)\,dx\right]. \tag{2.14}$$

We directly compute from (2.8)

$$\begin{aligned}\frac{1}{|\partial\Omega|}\int_{\partial\Omega} H_j^\lambda(x)\,dx &= -\frac{1}{|\partial\Omega|}\int_{\partial\Omega}\log\frac{2\mu_j}{|x-\xi_j-\lambda\mu_j\nu(\xi_j)|^2}\,dx\\ &= -\log 2\mu_j - \frac{1}{|\partial\Omega|}\int_{\partial\Omega}\log\frac{1}{|x-\xi_j|^2}\,dx\\ &+\frac{1}{|\partial\Omega|}\int_{\partial\Omega}\log\left(1+2\lambda\mu_j\nu(\xi_j)\cdot\frac{(x-\xi_j)}{|x-\xi_j|^2}+\frac{\lambda^2\mu_j^2}{|x-\xi_j|^2}\right)dx\\ &= -\log 2\mu_j + \frac{1}{|\partial\Omega|}\int_{\partial\Omega} H(x,\xi_j)\,dx + O(\lambda)\end{aligned}$$

where the last equality is consequence of the definition of the regular part of the Green's function. Hence (2.14) yields that $l = 0$. □

By the following scaling,

$$x = \lambda y, \quad y \in \Omega_\lambda \equiv \frac{\Omega}{\lambda}, \quad v(y) = u(\lambda y)$$

solving problem (1.1) is equivalent to solving

$$\Delta v = 0 \text{ in } \Omega_\lambda, \quad \frac{\partial v}{\partial\nu} = 2\lambda^2\sinh v \text{ on } \partial\Omega_\lambda. \tag{2.15}$$

In the expanded domain Ω_λ, the main term (2.10) of the ansatz (2.9) looks now like

$$V(y) = \sum_{j=1}^m (-1)^{j-1}\left[\log\frac{2\mu_j}{|y-\xi_j'-\mu_j\nu_j'|^2} - 2\log\lambda + H_j^\lambda(\lambda y)\right] \tag{2.16}$$

where $\xi_j' = \lambda^{-1}\xi_j$ and $\nu_j' = \nu(\xi_j')$.

We call

$$v_j(y) = u_j^\lambda(\lambda y) + 2\log\lambda = \log\frac{2\mu_j}{|y-\xi_j'-\mu_j\nu_j'|^2}$$

and

$$V_j(y) = v_j(y) - 2\log\lambda + H_j^\lambda(y).$$

A function v of the form

$$v(y) = V(y) + \phi(y), \quad y \in \Omega_\lambda$$

is a solution for (2.15) if and only if ϕ solves

$$\begin{cases} \Delta\phi = 0 \ \text{ in } \Omega_\lambda, \\ \frac{\partial\phi}{\partial\nu} - W\phi = R + N(\phi) \text{ on } \partial\Omega_\lambda, \end{cases} \tag{2.17}$$

where

$$W = 2\lambda^2 \cosh V, \tag{2.18}$$

$$R = -\left[\frac{\partial V}{\partial\nu} - 2\lambda^2 \sinh V\right] \tag{2.19}$$

and

$$N(\phi) = 2\lambda^2\left[\sinh(V+\phi) - \sinh V - \cosh V\phi\right]. \tag{2.20}$$

We claim that V is a good approximation for a solution of (2.15) under the assumption that we choose the parameters μ_j's to be given by the relation

$$\log 2\mu_j = H(\xi_j, \xi_j) + \sum_{l\neq j}(-1)^{l+j} G(\xi_j, \xi_l). \tag{2.21}$$

This is the content of estimate (2.22) contained in the following Lemma

Lemma 3. *Assume (2.21) holds true. Then, for any $\alpha \in (0,1)$, there exists a positive constant C independent of λ such that, for any $y \in \Omega_\lambda$,*

$$|R(y)| \leq C\lambda^\alpha \sum_{j=1}^m \frac{1}{1+|y-\xi_j'|}, \quad \forall\ y \in \Omega_\lambda, \tag{2.22}$$

and

$$W(y)=\sum_{j=1}^{m}\frac{2\mu_j}{|y-\xi_j'-\mu_j\nu_j'|^2}(1+\theta_\lambda(y)), \tag{2.23}$$

with

$$|\theta_\lambda(y)|\le C\lambda^\alpha+C\lambda\sum_{j=1}^{m}|y-\xi_j^{'}|. \tag{2.24}$$

Proof: First we observe that a direct consequence of condition (2.21) is that for $|y-\xi_j'|\le\frac{\delta}{\lambda}$, the following expansion holds true

$$\begin{aligned}&(-1)^{j-1}H_j^\lambda(\lambda y)+\sum_{i\neq j}(-1)^{i-1}\left(\log\frac{2\mu_i}{\lambda^2|y-\xi_i'-\mu_i\nu(\xi_i')|^2}+H_i^\lambda(\lambda y)\right)\\ &=(-1)^{j-1}\left(H(\xi_j,\xi_j)-\log 2\mu_j\right)+\sum_{i\neq j}(-1)^{i-1}G(\xi_i,\xi_j)+O(\lambda^\alpha)+O(\lambda|y-\xi_j'|)\\ &=O(\lambda^\alpha)+O(\lambda|y-\xi_j'|).\end{aligned} \tag{2.25}$$

We prove (2.22). By definition

$$\begin{aligned}-R=\sum_{j=1}^{m}(-1)^{j-1}\frac{2\mu_j}{|y-\xi_j'-\mu_j\nu(\xi_j')|^2}-2\lambda^2\sinh V\\ -\frac{\lambda^2}{|\partial\Omega|}\sum_{j=1}^{m}(-1)^{j-1}\int_{\partial\Omega}e^{u_j^\lambda}.\end{aligned}$$

The last term in R can be controlled by $O(\lambda^2)$. Indeed, the following fact holds true

$$\lambda\sum_{j=1}^{m}(-1)^{j-1}\int_{\partial\Omega}e^{u_j^\lambda}=O(\lambda\sum_{j\neq i}|\mu_j-\mu_i|). \tag{2.26}$$

as a direct consequence of (2.13).

On the other hand, if $|y-\xi_j'|\le\frac{\delta}{\lambda}$,

$$\begin{aligned}2\lambda^2\sinh V=\lambda^2\Big[&\exp\Big(\sum_{j=1}^{m}(-1)^{j-1}(u_j^\lambda(\lambda y)+H_j^\lambda(\lambda y))\Big)\\ &-\exp\Big(\sum_{j=1}^{m}(-1)^{j}(u_j^\lambda(\lambda y)+H_j^\lambda(\lambda y))\Big)\Big]\end{aligned}$$

$$= \lambda^2 \left(\frac{2\mu_j}{\lambda^2 |y - \xi_j' - \mu_j \nu(\xi_j')|^2} \right)^{(-1)^{j-1}} \times$$

$$\exp \left[(-1)^{j-1} H_j^\lambda(\lambda y) + \sum_{i \neq j} (-1)^{i-1} \left(\log \frac{2\mu_i}{\lambda^2 |y - \xi_i' - \mu_i \nu(\xi_i')|^2} + H_i^\lambda(\lambda y) \right) \right]$$

$$-\lambda^2 \left(\frac{2\mu_j}{\lambda^2 |y - \xi_j' - \mu_j \nu(\xi_j')|^2} \right)^{(-1)^{j}} \times$$

$$\exp \left[(-1)^{j-1} H_j^\lambda(\lambda y) + \sum_{i \neq j} (-1)^{i-1} \left(\log \frac{2\mu_i}{\lambda^2 |y - \xi_i' - \mu_i \nu(\xi_i')|^2} + H_i^\lambda(\lambda y) \right) \right]$$

(using (2.25))

$$= \lambda^2 \left[\left(\frac{2\mu_j}{\lambda^2 |y - \xi_j' - \mu_j \nu(\xi_j')|^2} \right)^{(-1)^{j-1}} - \left(\frac{2\mu_j}{\lambda^2 |y - \xi_j' - \mu_j \nu(\xi_j')|^2} \right)^{(-1)^{j}} \right] \times$$

$$e^{O(\lambda^\alpha) + O(\lambda |y - \xi_j'|)}$$

$$= (-1)^{j-1} \frac{2\mu_j}{|y - \xi_j' - \mu_j \nu(\xi_j')|^2} \left(1 + O(\lambda^\alpha) + O(\lambda |y - \xi_j'|) \right) + O(\lambda^4). \quad (2.27)$$

Hence, we get, for $|y - \xi_j'| \leq \frac{\delta}{\lambda}$,

$$R = \sum_{j=1}^{m} (-1)^{j-1} \frac{2\mu_j}{|y - \xi_j' - \mu_j \nu(\xi_j')|^2} \left(O(\lambda^\alpha) + O(\lambda |y - \xi_j'|) \right).$$

If we are far away from the points, namely if $|y - \xi_j'| > \frac{\delta}{\lambda}$ for all j, then $R = O(\lambda^2)$. This implies (2.22).

Estimates (2.23) and (2.24) follow from the same arguments used to obtain estimate (2.27). $\square$

3. Analysis of the linearized equation

In this section we study the linear problem

$$\begin{cases} -\Delta\phi = f & \text{in } \Omega_\lambda \\ \dfrac{\partial\phi}{\partial\nu} = W\phi + h & \text{on } \partial\Omega_\lambda \end{cases} \tag{3.1}$$

together with appropriate orthogonality conditions, where W is a function that satisfies (2.23) and (2.24), and f, h are given. Throughout this section we only assume that the numbers μ_j appearing in (2.23) satisfy $\frac{1}{C} \le \mu_j \le C$ independently of λ and that the points $\xi_j \in \partial\Omega$ are uniformly separated

$$|\xi_i - \xi_j| \ge d \quad \forall i \ne j, \tag{3.2}$$

where $d > 0$ is fixed.

The orthogonality conditions mentioned above are related to the kernel of (3.1) when $\lambda \to 0$. Let us look at (3.1) with $f \equiv 0$, $h \equiv 0$ as $\lambda \to 0$ at a fixed distance from one of the points, say ξ_j', and let us translate and a rotate so that $\xi_j' = 0$ and Ω_λ converges to the upper half plane $\mathbb{R}^2_+$. Then equation (3.1) approaches (2.3). By lemma 1 we know that any bounded solution to (2.3) is a linear combination of z_0 and z_1 defined in (2.1), (2.2). We define appropriate versions of z_0 and z_1 in Ω_λ through a diffeomorphism $F_j : B_\rho(\xi_j) \to \mathcal{N}_0$ where $\rho > 0$ is fixed and $\mathcal{N}_0$ is an open neighborhood of the origin such that $F_j(\Omega \cap B_\rho(\xi_j)) = \mathbb{R}^2_+ \cap \mathcal{N}_0$, $F_j(\partial\Omega \cap B_\rho(\xi_j)) = \partial\mathbb{R}^2_+ \cap \mathcal{N}_0$, and such that F_j preserves area. We define, for $y \in \Omega_\lambda$,

$$F_j^\lambda(y) = \frac{1}{\lambda} F_j(\lambda y) \tag{3.3}$$

and

$$Z_{ij}(y) = z_{ij}(F_j^\lambda(y)) \qquad i = 0,1 \quad j = 1,\dots,m,$$

where z_{ij} denotes the function z_i with parameter $\mu = \mu_j$ ($i = 0,1$ $j = 1,\dots,m$):

$$z_{0j} = 1 - 2\mu_j \frac{x_2 + \mu_j}{x_1^2 + (x_2+\mu_j)^2}, \qquad z_{1j} = -2\frac{x_1}{x_1^2 + (x_2+\mu_j)^2}.$$

Next we fix a large constant $R_0 > 0$ and a nonnegative smooth function $\bar\chi : \mathbb{R} \to \mathbb{R}$ so that $\bar\chi(r) = 1$ for $r \le R_0$ and $\bar\chi(r) = 0$ for $r \ge R_0 + 1$, $0 \le \bar\chi \le 1$. Then set

$$\chi_j(y) = \bar\chi(|F_j^\lambda(y)|). \tag{3.4}$$

Let $0 < b < 1$ and define

$$Z(y) = \begin{cases} \min(1-\lambda^b, Z_{0j}(y)) & \text{if } |y - \xi_j'| < \frac{\delta}{\lambda}, \\ 1 - \lambda^b & \text{if } |y - \xi_j'| \ge \frac{\delta}{\lambda} \quad \forall j = 1,\dots,m. \end{cases} \tag{3.5}$$

We will establish a-priori estimates for solutions to (3.1) under the orthogonality conditions

$$\int_{\Omega_\lambda} \chi_j Z_{1j} \phi = 0 \qquad \forall j = 1,\dots,m \tag{3.6}$$

and

$$\int_{\Omega_\lambda} \chi Z \phi = 0, \tag{3.7}$$

where

$$\chi = \sum_{j=1}^m \chi_j.$$

Let us introduce the norms

$$\|h\|_* = \sup_{y\in\partial\Omega_\lambda} \frac{|h(y)|}{\sum_{j=1}^m (1+|y-\xi_j'|)^{-1-\sigma}},$$

and

$$\|f\|_{**} = \sup_{y\in\Omega_\lambda} \frac{|f(y)|}{\sum_{j=1}^m (1+|y-\xi_j'|)^{-2-\sigma}}.$$

where $\sigma > 0$ is a fixed small constant.

Proposition 1. *For fixed $d > 0$ there exist $\lambda_0 > 0$, C such that if $0 < \lambda < \lambda_0$, $\xi_j \in \partial\Omega$ $(j = 1, \dots, m)$ satisfy* (3.2) *and $\phi \in L^\infty(\Omega_\lambda)$ is a solution of* (3.1) *such that* (3.6) *and* (3.7) *hold, then*

$$\|\phi\|_{L^\infty(\Omega_\lambda)} \le C \log\frac{1}{\lambda}(\|f\|_{**} + \|h\|_*).$$

We will prove this estimate by contradiction assuming that there exists a sequence $\lambda \to 0$, points $\xi_j \in \Omega$ satisfying (3.2) (we omit the dependence on λ in the notation) and functions $h \in L^\infty(\partial\Omega_\lambda)$, $f \in L^\infty(\Omega_\lambda)$, $\phi \in L^\infty(\Omega_\lambda)$ such that

$$\begin{aligned} &\|\phi\|_{L^\infty(\Omega_\lambda)} = 1 \\ &\log\frac{1}{\lambda}\|h\|_* = o(1), \quad \log\frac{1}{\lambda}\|f\|_{**} = o(1). \end{aligned} \tag{3.8}$$

Given $0 < \alpha < 1$ fix $0 < \gamma < \beta < \alpha$ and consider the function p given by

$$p(r) = \begin{cases} 1 & \text{if } r < \lambda^{-\gamma}, \\ \frac{\log\lambda^{-\beta} - \log r}{\log\lambda^{-\beta} - \log\lambda^{-\gamma}} & \text{if } \lambda^{-\gamma} < r < \lambda^{-\beta}, \\ 0 & \text{if } r > \lambda^{-\beta}. \end{cases} \tag{3.9}$$

Define

$$\widetilde{Z}_{0j}(y) = z_{0j}(F_j^\lambda(y))p(|F_j^\lambda(y)|) \qquad j = 1, \dots, m.$$

Let

$$\tilde{\phi} = \phi - \sum_{j=1}^{m} d_j \widetilde{Z}_{0j},$$

where the numbers d_j are chosen so that $\int_{\Omega_\lambda} \chi_j Z_{0j} \tilde{\phi} = 0$ for any $j = 1, \dots, m$, namely $d_j = \frac{\int_{\Omega_\lambda} \chi_j Z_{0j} \phi}{\int_{\Omega_\lambda} \chi_j Z_{0j}^2}$. Observe that

$$d_j = O(1), \qquad \|\tilde{\phi}\|_{L^\infty(\Omega_\lambda)} = O(1).$$

Furthermore $\tilde{\phi}$ solves the problem

$$\begin{cases} -\Delta \tilde{\phi} = f + \sum_{j=1}^{m} d_j \Delta \widetilde{Z}_{0j} & \text{in } \Omega_\lambda \\ \dfrac{\partial \tilde{\phi}}{\partial \nu} = W\tilde{\phi} + h + \sum_{j=1}^{m} d_j (W\widetilde{Z}_{0j} - \dfrac{\partial \widetilde{Z}_{0j}}{\partial \nu}) & \text{on } \partial\Omega_\lambda \end{cases} \tag{3.10}$$

and satisfies

$$\int_{\Omega_\lambda} \chi_j Z_{ij} \tilde{\phi} = 0 \qquad \forall i = 0, 1 \quad \forall j = 1, \dots, m. \tag{3.11}$$

To reach a contradiction we will establish the following

Lemma 4.

$$\tilde{\phi} \to 0 \qquad \textit{uniformly in } \Omega_\lambda.$$

Lemma 5.

$$d_j \to 0 \qquad \forall j = 1, \dots, m.$$

This will prove proposition 1.

We delay the proofs of these lemmas and mention first some key steps.

Lemma 6. *For all* $j = 1, \dots, m$ *and* $R > 0$

$$\tilde\phi \to 0 \qquad \textit{uniformly in } \Omega_\lambda \cap B_R(\xi_j').$$

Proof. Assume that for some $R > 0$ and $j = 1, \dots, m$ there is $c > 0$ so that $\sup_{B_R(\xi_j')} |\tilde\phi| \ge c > 0$ for a subsequence $\lambda \to 0$. Translate and rotate Ω_λ so that $\xi_j' = 0$ and Ω_λ converges to the upper half plane $\mathbb{R}^2_+$. By elliptic estimates $\tilde\phi \to \tilde\phi_0$ uniformly on compact sets and $\tilde\phi_0$ is a nontrivial solution of (2.3). Applying proposition 1 we conclude that $\tilde\phi_0$ is a linear combination of z_0 and z_1. On the other hand, consider the limit as $\lambda \to 0$ of the orthogonality conditions (3.11). After a translation and a rotation Z_{ij} converges to z_i implying $\int_{\mathbb{R}^2_+} \bar\chi z_i \tilde\phi_0 = 0$ for $i = 0, 1$. This contradicts the fact that $\tilde\phi_0 \not\equiv 0$. $\square$

Lemma 7.

$$\bar{\tilde\phi} \equiv \frac{1}{|\partial\Omega_\lambda|} \int_{\partial\Omega_\lambda} \tilde\phi \to 0.$$

Proof. By potential theory

$$\tilde\phi(y) - \bar{\tilde\phi} = \frac{1}{2\pi} \int_{\partial\Omega_\lambda} G(\lambda y, \lambda z)(W\tilde\phi + h + \sum_j d_j (W \widetilde Z_{0j} - \frac{\partial \widetilde Z_{0j}}{\partial\nu}))\, dz$$
$$+ \frac{1}{2\pi} \int_{\Omega_\lambda} G(\lambda y, \lambda z)(f + \sum_j d_j \Delta \widetilde Z_{0j})\, dz,$$

where G is Green's function defined in (1.3).

Integrating equation (3.10) yields

$$\int_{\partial\Omega_\lambda} W\tilde{\phi} + h + \sum_j d_j (W\widetilde{Z}_{0j} - \frac{\partial \widetilde{Z}_{0j}}{\partial \nu})\, dz + \int_{\Omega_\lambda} (f + \sum_j d_j \Delta \widetilde{Z}_{0j})\, dz = 0.$$

Taking into account that $G(\lambda y, \lambda z) = \log \frac{1}{\lambda^2} + \log \frac{1}{|y-z|^2} + H(\lambda y, \lambda z)$, where H is the regular part of Green's function H (c.f. (1.4)) we have

$$\begin{aligned} \tilde{\phi}(y) - \overline{\tilde{\phi}} = \frac{1}{2\pi} \int_{\partial\Omega_\lambda} (\log \frac{1}{|y-z|^2} + H(\lambda y, \lambda z)) \times \\ (W\tilde{\phi} + h + \sum_j d_j (W\widetilde{Z}_{0j} - \frac{\partial \widetilde{Z}_{0j}}{\partial \nu}))\, dz \\ + \frac{1}{2\pi} \int_{\Omega_\lambda} (\log \frac{1}{|y-z|^2} + H(\lambda y, \lambda z))(f + \sum_j d_j \Delta \widetilde{Z}_{0j})\, dz. \end{aligned} \tag{3.12}$$

Let us sketch how the proof works postponing some of the calculations.

Since $\tilde{\phi}(y) \to 0$ uniformly on sets of the form $|y - \xi_j'| < R$, we can select a sequence $R_\lambda \to \infty$ such that

$$\tilde{\phi}(y) \to 0 \quad \text{uniformly for } |y - \xi_j'| < R_\lambda.$$

We can assume $R_\lambda \to \infty$ as slow as we need.

For each $l = 1, \ldots, m$ select a point $y_l \in \partial\Omega_\lambda$ so that $|y_l - \xi_l'| = R_\lambda$. We claim that when we evaluate (3.12) at y_l all terms in the right hand side of (3.12) converge to zero except for $\int_{\Omega_\lambda} \log \frac{1}{|y_l - z|^2} \Delta \widetilde{Z}_{0j}\, dz = 2\pi \delta_{lj} + o(1)$ (where δ_{lj} is Kronecker's delta). Thus, we claim that

$$\tilde{\phi}(y_j) - \overline{\tilde{\phi}} = d_j + o(1) \qquad \forall j = 1, \ldots, m. \tag{3.13}$$

But the orthogonality condition (3.7) implies that

$$\sum_{j=1}^m d_j a_j = 0 \quad \text{where} \quad a_j = \int_{\Omega_\lambda} \chi_j Z_{0j}^2 > 0. \tag{3.14}$$

Multiplying (3.13) by a_j, adding and using (3.14) we find

$$\sum_{j=1}^{m} a_j \tilde\phi(y_j) - a\bar{\tilde\phi} = o(1) \qquad \text{where} \qquad a = \sum_{j=1}^{m} a_j.$$

Since $\tilde\phi(y_j) \to 0$ and a is bounded away from zero we reach the conclusion

$$\bar{\tilde\phi} = o(1).$$

In what follows we will obtain the necessary estimates to prove (3.13).

Claim.

$$\int_{\Omega_\lambda} \log \frac{1}{|y_l - z|^2} \Delta \widetilde{Z}_{0j}\, dz = 2\pi\delta_{lj} + o(1) \qquad \forall j, l = 1, \ldots, m.$$

Proof. Let

$$\widetilde{z}_{0j}(x) = \widetilde{Z}_{0j}((F_j^\lambda)^{-1}(x)) = z_{0j}(x) p(|x|), \tag{3.15}$$

where p was defined in (3.9). Let us write $y = (F_j^\lambda)^{-1}(x)$. Since $p'(r)$ has a jump at $r = \lambda^{-\gamma}$ and $r = \lambda^{-\beta}$ and is otherwise smooth we see that $\Delta_x \tilde z_{0j}$ is a measure:

$$\begin{aligned}
\Delta_x \tilde z_{0j} &= 2\nabla z_{0j} \nabla p + z_{0j}[p'(\lambda^{-\gamma})]\mu_{\lambda^{-\gamma}} + z_{0j}[p'(\lambda^{-\beta})]\mu_{\lambda^{-\beta}} \\
&= 2\nabla z_{0j}\nabla p - z_{0j}\frac{\lambda^\gamma}{(\beta-\gamma)\log\frac{1}{\lambda}}\mu_{\lambda^{-\gamma}} + z_{0j}\frac{\lambda^\beta}{(\beta-\gamma)\log\frac{1}{\lambda}}\mu_{\lambda^{-\beta}},
\end{aligned}$$

where $[p'(r)] = p'(r^+) - p'(r^-)$ denotes the jump of p' at r and let μ_r is the 1-dimensional measure on the circle of radius r.

Changing variables yields

$$\int_{\Omega_\lambda} \Delta\widetilde{Z}_{0j}\varphi = \int_{\lambda^{-\gamma}<|x|<\lambda^{-\beta}} \big(2\nabla p\nabla z_{0j} + O(\lambda|x||\nabla^2\tilde{z}_{0j}|) + O(\lambda|\nabla\tilde{z}_{0j}|)\big) \times \varphi((F_j^\lambda)^{-1}(x))\,dx$$
$$-\frac{\lambda^\gamma}{(\beta-\gamma)\log\frac{1}{\lambda}}\int_{r=\lambda^{-\gamma}}(1+O(\lambda|x|))z_{0j}\varphi((F_j^\lambda)^{-1}(x))\,dx$$
$$+\frac{\lambda^\beta}{(\beta-\gamma)\log\frac{1}{\lambda}}\int_{r=\lambda^{-\beta}}(1+O(\lambda|x|))z_{0j}\varphi((F_j^\lambda)^{-1}(x))\,dx \tag{3.16}$$

for any $\varphi \in C(\overline{\Omega})$.

Let us consider first the case $l = j$:

$$\int_{\Omega_\lambda} \log\frac{1}{|y_j - z|^2}\Delta\widetilde{Z}_{0j}\,dz = \int_{\Omega_\lambda}(\log\frac{1}{|y_j-z|^2} - \log\frac{1}{|\xi_j'-z|^2})\Delta\widetilde{Z}_{0j}\,dz + \int_{\Omega_\lambda}\log\frac{1}{|\xi_j'-z|^2}\Delta\widetilde{Z}_{0j}\,dz. \tag{3.17}$$

By the previous remarks, using the fact that $z_{0j}(x) = 1 + O(|x|^{-1})$ and the expansion $(F_j^\lambda)^{-1}(x) = \xi_j' + x + O(\lambda|x|)$ (after rotation) we have

$$\frac{\lambda^\gamma}{(\beta-\gamma)\log\frac{1}{\lambda}}\int_{r=\lambda^{-\gamma}}(1+O(\lambda|x|))z_{0j}\log\frac{1}{|\xi_j'-(F_j^\lambda)^{-1}(x)|^2}\,dx$$
$$= \frac{\lambda^\gamma}{(\beta-\gamma)\log\frac{1}{\lambda}}(1+O(\lambda^{1-\gamma}))(1+O(\lambda^\gamma))\pi\lambda^{-\gamma}(2\log\lambda^{-\gamma} + O(\lambda^{1-\gamma}))$$
$$= 2\pi\frac{\gamma}{\beta-\gamma} + O(\lambda^\theta),$$

where we fix

$$0 < \theta < \min(\gamma, 1-\beta).$$

Similarly

$$\frac{\lambda^\beta}{(\beta-\gamma)\log\frac{1}{\lambda}}\int_{r=\lambda^{-\beta}} z_{0j}(1+O(\lambda|x|))\log\frac{1}{|\xi_j'-(F_j^\lambda)^{-1}(x)|^2}\,dx$$

$$=2\pi\frac{\beta}{\beta-\gamma}+O(\lambda^\theta),$$

and a calculation using $\nabla z_{0j}=O(\frac{1}{|x|})$, $\nabla^2 z_{0j}=O(\frac{1}{|x|^2})$ shows that

$$\int_{\lambda^{-\gamma}<|x|<\lambda^{-\beta}}(2\nabla p\nabla z_{0j}+O(\lambda|x||\nabla^2\tilde z_{0j}|)+(\lambda|\nabla\tilde z_{0j}|))\log\frac{1}{|\xi_j'-(F_j^\lambda)^{-1}(x)|^2}dx$$

$$=O(\lambda^\theta).$$

Therefore

$$\int_{\Omega_\lambda}\log\frac{1}{|\xi_j'-z|^2}\Delta\widetilde Z_{0j}\,dz=2\pi+O(\lambda^\theta).$$

For the first integral in the right hand side of (3.17) we can assume $R_\lambda\to\infty$ slow enough so that

$$\lambda^\gamma R_\lambda\to 0.$$

Then

$$|\log\frac{1}{|y_j-z|^2}-\log\frac{1}{|\xi_j'-z|^2}|\le C\frac{|y_j-\xi_j'|}{\lambda^{-\gamma}}$$

and it follows that

$$|\int_{\Omega_\lambda}(\log\frac{1}{|y_j-z|^2}-\log\frac{1}{|\xi_j'-z|^2})\Delta\widetilde Z_{0j}\,dz|=O(\lambda^\gamma R_\lambda).$$

Next we show that if $l\neq j$ then

$$\int_{\Omega_\lambda}\log\frac{1}{|y_l-z|^2}\Delta\widetilde Z_{0j}\,dz=o(1).$$

In fact

$$\int_{\Omega_\lambda} \log \frac{1}{|y_l - z|^2} \Delta \widetilde{Z}_{0j}\, dz = \int_{\Omega_\lambda} (\log \frac{1}{|y_l - z|^2} - \log \frac{1}{|y_l - \xi_j'|^2}) \Delta \widetilde{Z}_{0j}\, dz + \int_{\Omega_\lambda} \log \frac{1}{|y_l - \xi_j'|^2} \Delta \widetilde{Z}_{0j}\, dz.$$

We can assume that $R_\lambda < \frac{\lambda^{-\gamma}}{2}$, so that

$$|\log \frac{1}{|y_l - z|^2} - \log \frac{1}{|y_l - \xi_j'|^2}| \le C\lambda |z - \xi_j'|.$$

Thus

$$|\int_{\Omega_\lambda} (\log \frac{1}{|y_l - z|^2} - \log \frac{1}{|y_l - \xi_j'|^2}) \Delta \widetilde{Z}_{0j}\, dz| = O(\lambda^\theta).$$

Finally

$$\begin{aligned} \int_{\Omega_\lambda} \Delta \widetilde{Z}_{0j}\, dz &= -(2\pi + O(\lambda^{1-\gamma}))(1 + O(\lambda^\gamma)) \frac{1}{(\beta - \gamma) \log \frac{1}{\lambda}} \\ &\quad + (2\pi + O(\lambda^{1-\beta}))(1 + O(\lambda^\beta)) \frac{1}{(\beta - \gamma) \log \frac{1}{\lambda}} + O(\lambda^\gamma) \\ &= O(\lambda^\theta) \end{aligned} \tag{3.18}$$

so

$$\log \frac{1}{|y_l - \xi_j'|^2} \int_{\Omega_\lambda} \Delta \widetilde{Z}_{0j}\, dz = o(1).$$

□

Claim. For any $0 < \alpha < 1$

$$W(y) \widetilde{Z}_{0j}(y) - \frac{\partial \widetilde{Z}_{0j}}{\partial \nu}(y) = O(\frac{\lambda^\alpha}{1 + |y - \xi_j'|}) + O(\frac{\lambda}{\log \frac{1}{\lambda}}), \quad \text{for } |y - \xi_j'| \le \frac{\delta}{\lambda}. \tag{3.19}$$

Proof. Set

$$\widetilde{W}(x) = W((F_j^\lambda)^{-1}(x))$$

where the map F_j^λ is defined (3.3). Recall that W satisfies (2.23) and (2.24), that is

$$W(y) = \frac{2\mu_j}{|y - \xi_j' - \mu_j \nu_j'|^2}(1 + O(\lambda^\alpha(1 + |y|))).$$

Since $(F_j^\lambda)^{-1}(x) = \xi_j' + x + O(\lambda|x|)$ we find

$$\begin{aligned} \widetilde{W}(x) &= W((F_j^\lambda)^{-1}(x)) = W(\xi_j' + x + O(\lambda|x|)) \\ &= \frac{2\mu_j}{x_1^2 + \mu_j^2} + O(\frac{\lambda^\alpha}{1+|x|}) \qquad x = (x_1, 0),\ |x| < \frac{\delta}{\lambda}. \end{aligned} \tag{3.20}$$

On the other hand

$$\frac{\partial \widetilde{Z}_{0j}}{\partial \nu} = -\frac{\partial \tilde{z}_{0j}}{\partial x_2} + O(\lambda |x| |\nabla \tilde{z}_{0j}|)$$

and using the fact that p has zero normal derivative on $\partial \mathbb{R}^2_+$ we deduce

$$\begin{aligned} \frac{\partial \widetilde{Z}_{0j}}{\partial \nu} &= -p \frac{\partial z_{0j}}{\partial x_2} + O(\lambda r (|\nabla p| z_{0j} + p |\nabla z_{0j}|)) \\ &= -p \frac{\partial z_{0j}}{\partial x_2} + O(\frac{\lambda}{\log \frac{1}{\lambda}}) + O(\frac{\lambda}{1+r}) \qquad r < \frac{\delta}{\lambda}, \end{aligned} \tag{3.21}$$

where $r = |y - \xi_j'|$ (observe that $\nabla p = O(\frac{1}{r \log \frac{1}{\lambda}})$).

Using (3.20) we find

$$\frac{\partial \widetilde{Z}_{0j}}{\partial \nu}(y) - W(y) \widetilde{Z}_{0j}(y) = O(\frac{\lambda}{\log \frac{1}{\lambda}}) + O(\frac{\lambda^\alpha}{1 + |y - \xi_j'|}) \qquad |y - \xi_j'| < \frac{\delta}{\lambda}.$$

□

Claim. Similarly

$$\int_{\partial \Omega_\lambda} \log \frac{1}{|y - z|^2} (W \widetilde{Z}_{0j} - \frac{\partial \widetilde{Z}_{0j}}{\partial \nu})\, dz = O(\lambda^{1-\beta}) = o(1),$$

and this is uniformly for $y \in \partial \Omega_\lambda$.

Proof. Using (3.19) and the fact that if $|y-\xi_j'| > \frac{\delta}{\lambda}$ for all j we have $W(y) = O(\lambda^2)$ and $\frac{\partial \widetilde{Z}_{0j}}{\partial \nu} = O(\lambda^2)$, we see that

$$\int_{\partial\Omega_\lambda} |W\widetilde{Z}_{0j} - \frac{\partial \widetilde{Z}_{0j}}{\partial \nu}| = O(\lambda^{1-\beta}\frac{1}{\log\frac{1}{\lambda}}). \tag{3.22}$$

Since $\log\frac{1}{|y-z|^2} = O(\log\frac{1}{\lambda})$ for $|y-z| > R$ where $R > 0$ is fixed, and

$$\int_{\partial\Omega_\lambda \cap B_R(y)} |\log\frac{1}{|y-z|^2}|\,dz \le C$$

we conclude the validity of the assertion. □

Claim.

$$\int_{\partial\Omega_\lambda} \log\frac{1}{|y-z|^2} h(z)\,dz = o(1) \tag{3.23}$$

and

$$\int_{\Omega_\lambda} \log\frac{1}{|y-z|^2} f(z)\,dz = o(1). \tag{3.24}$$

Proof. We have $\log\frac{1}{|y-z|^2} = O(\log\frac{1}{\lambda})$ for $|y-z| > R$ where $R > 0$ is fixed, and $\int_{\partial\Omega_\lambda \cap B_R(y)} |\log\frac{1}{|y-z|^2}|\,dz \le C$ and therefore

$$|\int_{\partial\Omega_\lambda} \log\frac{1}{|y-z|^2} h\,dz| \le C\log\frac{1}{\lambda}\|h\|_* = o(1),$$

by hypothesis (3.8). The proof of the other assertion is similar. □

Claim.

$$\int_{\partial\Omega_\lambda} \log\frac{1}{|y-z|^2} W\tilde{\phi}\,dz = o(1).$$

Proof. Arguing as before, it is sufficient to show that

$$\log\frac{1}{\lambda}\int_{\partial\Omega_\lambda} W\tilde{\phi} = o(1).$$

Integrating equation (3.10) we find

$$\int_{\partial\Omega_\lambda} (W\tilde\phi + h + \sum_j d_j (W\widetilde Z_{0j} - \frac{\partial \tilde Z_{0j}}{\partial\nu})\,dz + \int_{\Omega_\lambda} (f + \sum_j d_j \Delta \widetilde Z_{0j})\,dz = 0.$$

The conclusion follows then from (3.18), (3.22), (3.8). □

Claim.

$$A \equiv \int_{\partial\Omega_\lambda} H(\lambda y, \lambda z)(W\tilde\phi + h + \sum_j d_j (W\widetilde Z_{0j} - \frac{\partial \tilde Z_{0j}}{\partial\nu}))\,dz + \int_{\Omega_\lambda} H(\lambda y, \lambda z)(f + \sum_j d_j \Delta \widetilde Z_{0j})\,dz = o(1),$$

uniformly for $y \in \partial\Omega_\lambda$.

Proof. Let

$$\zeta(r) = \begin{cases} 1 & \text{if } r < \lambda^{-1/2} \\ \frac{\log\frac{\delta}{\lambda} - \log r}{\log\frac{\delta}{\lambda} - \log\lambda^{-1/2}} & \text{if } \lambda^{-1/2} < r < \frac{\delta}{\lambda} \\ 0 & \text{if } r > \frac{\delta}{\lambda} \end{cases}$$

and set

$$\psi(z) = \sum_{j=1}^m H(\lambda y, \xi_j)\zeta(|z - \xi_j'|).$$

Multiplying (3.10) by ψ and integrating by parts we have

$$\int_{\Omega_\lambda} (f + \sum_j d_j \Delta \widetilde Z_{0j})\psi + \int_{\partial\Omega_\lambda} (W\tilde\phi + h + \sum_j d_j (W\widetilde Z_{0j} - \frac{\partial \widetilde Z_{0j}}{\partial\nu}))\psi - \int_{\partial\Omega_\lambda} \tilde\phi \frac{\partial\psi}{\partial\nu} + \int_{\Omega_\lambda} \tilde\phi \Delta\psi = 0.$$

Subtracting this from A we find

$$A = \int_{\partial\Omega_\lambda} (H(\lambda y, \lambda z) - \psi)(W\tilde{\phi} + h + \sum_j d_j (W\widetilde{Z}_{0j} - \frac{\partial \widetilde{Z}_{0j}}{\partial \nu}))\, dz$$
$$+ \int_{\Omega_\lambda} (H(\lambda y, \lambda z) - \psi)(f + \sum_j d_j \Delta \widetilde{Z}_{0j})\, dz$$
$$+ \int_{\partial\Omega_\lambda} \tilde{\phi} \frac{\partial \psi}{\partial \nu} - \int_{\Omega_\lambda} \tilde{\phi} \Delta \psi.$$

Since H and ψ are bounded

$$|\int_{\partial\Omega_\lambda} (H(\lambda y, \lambda z) - \psi) h\, dz| \le C\|h\|_* = o(1) \tag{3.25}$$

$$|\int_{\Omega_\lambda} (H(\lambda y, \lambda z) - \psi) f\, dz| \le C\|f\|_{**} = o(1). \tag{3.26}$$

A calculation shows that

$$\int_{\Omega_\lambda} \tilde{\phi} \Delta \psi = O(\frac{1}{\log \frac{\delta}{\lambda}}) = o(1), \qquad \int_{\partial\Omega_\lambda} \tilde{\phi} \frac{\partial \psi}{\partial \nu} = O(\frac{1}{\log \frac{\delta}{\lambda}}) = o(1). \tag{3.27}$$

For instance, the first integral in (3.27) can be estimated as follows

$$|\int_{\Omega_\lambda} \tilde{\phi} \Delta \psi| \le \|\tilde{\phi}\|_{L^\infty(\Omega_\lambda)} \int_{\Omega_\lambda} |\Delta \psi|.$$

But $\Delta\psi$ is a measure with support on the arcs $r = \lambda^{-1/2}$, $r = \frac{\delta}{\lambda}$ $(r = |z - \xi_j'|)$ and

$$\int_{\Omega_\lambda} |\Delta \psi| = O(\lambda^{-1/2} \frac{1}{\lambda^{-1/2} \log \frac{1}{\lambda}} + \frac{\delta}{\lambda} \frac{1}{\frac{\delta}{\lambda} \log \frac{1}{\lambda}}) = O(\frac{1}{\log \frac{1}{\lambda}}) = o(1).$$

Now, at distance greater that $\frac{\delta}{\lambda}$ from all ξ_j' we have $W = O(\lambda^2)$ and H, $\tilde{\phi}$ are bounded, thus

$$\int_{\partial\Omega_\lambda \setminus (\cup_j B_{\delta/\lambda}(\xi_j'))} (H(\lambda y, \lambda z) - \psi) W \tilde{\phi} = o(1). \tag{3.28}$$

On the other hand, at distance less than $\frac{\delta}{\lambda}$ from ξ_j' we have $H(\lambda y, \lambda z) - H(\lambda y, \xi_j) = O(\lambda|z-\xi_j'|)$ and $W = O(\frac{1}{r^2})$, $r = |z-\xi_j'|$. So

$$
\begin{aligned}
|\int_{\partial\Omega_\lambda \cap B_{\lambda^{-1/2}}(\xi_j')} & (H(\lambda y, \lambda z) - \psi(z)) W \tilde\phi \, dz| \\
&= |\int_{\partial\Omega_\lambda \cap B_{\lambda^{-1/2}}(\xi_j')} (H(\lambda y, \lambda z) - H(\lambda y, \xi_j)) W \tilde\phi \, dz| \\
&\le \lambda \int_1^{\delta/\lambda} \frac{1}{r} \, dr = O(\lambda \log \frac{1}{\lambda}) = o(1). \qquad (3.29)
\end{aligned}
$$

In the region $\lambda^{-1/2} < r = |z-\xi_j'| < \frac{\delta}{\lambda}$ we use that H, ζ, $\tilde\phi$ are bounded and that $W = O(\frac{1}{r^2})$, so

$$
\begin{aligned}
|\int_{\partial\Omega_\lambda \cap B_{\delta/\lambda}(\xi_j') \setminus B_{\lambda^{-1/2}}(\xi_j')} (H(\lambda y, \lambda z) - \psi(z)) W \tilde\phi \, dz| &\le C \int_{\lambda^{-1/2}}^{\delta/\lambda} \frac{1}{r^2} \, dr \\
&= O(\lambda^{1/2}) = o(1).
\end{aligned} \tag{3.30}
$$

Collecting (3.25)–(3.30) and recalling (3.18), (3.19) we obtain the desired conclusion. □

Proof. of lemma 4 Let $\hat\phi(x) = \tilde\phi(x/\lambda)$, $x \in \Omega$. Then $\hat\phi$ satisfies

$$
\begin{cases}
-\Delta \hat\phi = \dfrac{1}{\lambda^2}(\hat f + \displaystyle\sum_{j=1}^m d_j \Delta \hat Z_{0j}) & \text{in } \Omega \\
\dfrac{\partial \hat\phi}{\partial \nu} = \dfrac{1}{\lambda}\Big(\widehat W \hat\phi + \hat h \displaystyle\sum_{j=1}^m d_j (\widehat W \hat Z_{0j} - \dfrac{\partial \hat Z_{0j}}{\partial \nu})\Big) & \text{on } \partial\Omega
\end{cases}
$$

where $\hat f(x) = f(x/\lambda)$, $\hat h(x) = h(x/\lambda)$, $\widehat W(x) = W(x/\lambda)$ and $\hat Z_{01}(x) = Z_{01}(x/\lambda)$. For a given $\delta > 0$ let $E_\delta = \Omega \setminus \cup_{j=1}^m B_\delta(\xi_j)$. Then $\frac{1}{\lambda^2}\|\hat f\|_{L^\infty(E_\delta)} \le C\|f\|_{**} \to 0$, $\frac{1}{\lambda}\|\hat h\|_{L^\infty(\partial E_\delta)} \le C\|h\|_* \to 0$, and $\frac{1}{\lambda}\|\widehat W \hat\phi\|_{L^\infty(E_\delta)} \le C\lambda$. Furthermore, in E_δ we have that $\hat Z_{0j} \equiv 0$. We also know: $\|\hat\phi\|_{L^\infty(\Omega)} \le 1$, and

$\fint_{\partial\Omega} \hat\phi \to 0$. From this it follows that $\hat\phi \to 0$ uniformly in E_δ and this implies

$$\tilde\phi \to 0 \text{ uniformly in } \Omega_\lambda \setminus \cup_{j=1}^m B_{\delta/\lambda}(\xi_j'), \text{ for any } \delta > 0.$$

For a given $R_1 > 0$ let A_j denote the annulus

$$A_j = B_{\delta/\lambda}(\xi_j') \setminus B_{R_1}(\xi_j').$$

Given $\lambda > 0$ small enough there exist $R_1 > 0$ independent of λ and $\psi_j : \Omega_\lambda \cap A_j \to \mathbb{R}$ smooth and positive so that

$$-\Delta\psi_j \geq \frac{c}{|y-\xi_j'|^{2+\sigma}} \quad \text{in } \Omega_\lambda \cap A_j$$
$$\frac{\partial\psi_j}{\partial\nu} - W\psi_j \geq \frac{c}{|y-\xi_j'|^{1+\sigma}} \quad \text{on } \partial\Omega_\lambda \cap A_j$$
$$c \leq \psi_j \leq C \qquad \text{in } \Omega_\lambda \cap A_j$$

where C, $c > 0$ can be made independent of λ. Indeed, the function

$$\psi_j(y) = \frac{(y-\xi_j')\cdot\nu_j'}{|y-\xi_j'|^{1+\sigma}} + C_0\Big(1 - \frac{1}{|y-\xi_j'|^{\sigma}}\Big)$$

with C_0 a fixed large constant satisfies the requirements, see [4] Lemma 4.3.

Thanks to the barrier ψ_j we deduce that the following maximum principle holds in $\Omega_\lambda \cap A_j$: if $\phi \in H^1(\Omega_\lambda \cap A_j)$ satisfies

$$\begin{cases} -\Delta\phi \geq 0 & \text{in } \Omega_\lambda \cap A_j \\ \dfrac{\partial\phi}{\partial\nu} - W\phi \geq 0 & \text{on } \partial\Omega_\lambda \cap A_j \\ \phi \geq 0 & \text{on } \Omega_\lambda \cap A_j \end{cases}$$

then $\phi \geq 0$ in $\Omega_\lambda \cap A_j$. By the properties of ψ_j and this maximum principle we deduce that there exists a fixed $C > 0$ so that

$$|\phi| \leq C\psi_j\Big(\sup_{\Omega_\lambda\cap\partial B_{R_1}(\xi_j')}|\phi| + \sup_{\Omega_\lambda\cap\partial B_{\delta/\lambda}(\xi_j')}|\phi| + \|h\|_* + \|f\|_{**}\Big) \quad \text{in } \Omega_\lambda \cap A_j.$$

But $\sup_{\Omega_\lambda \cap \partial B_{R_1}(\xi_j')} |\phi| \to 0$ by lemma 6, and $\sup_{\Omega_\lambda \cap \partial B_{\delta/\lambda}(\xi_j')} |\phi| \to 0$ as shown above. This proves the result. $\square$

Proof. of lemma 5 Multiplying (3.10) by $\widetilde{Z}_{0j}$ and integrating we obtain

$$d_j \Big(\int_{\Omega_\lambda} (-\Delta \widetilde{Z}_{0j}) \widetilde{Z}_{0j} + \int_{\partial \Omega_\lambda} \widetilde{Z}_{0j} (\frac{\partial \widetilde{Z}_{0j}}{\partial \nu} - W \widetilde{Z}_{0j}) \Big) = - \int_{\partial \Omega_\lambda} \widetilde{Z}_{0j} h \\ - \int_{\Omega_\lambda} \widetilde{Z}_{0j} f + \int_{\partial \Omega_\lambda} \tilde{\phi} (\frac{\partial \widetilde{Z}_{0j}}{\partial \nu} - W \widetilde{Z}_{0j}) + \int_{\Omega_\lambda} \tilde{\phi} (-\Delta \widetilde{Z}_{0j}).$$

We claim that

$$\int_{\Omega_\lambda} (-\Delta \widetilde{Z}_{0j}) \widetilde{Z}_{0j} + \int_{\partial \Omega_\lambda} \widetilde{Z}_{0j} (\frac{\partial \widetilde{Z}_{0j}}{\partial \nu} - W \widetilde{Z}_{0j}) \geq \frac{c}{\log \frac{1}{\lambda}}, \tag{3.31}$$

for some fixed $c > 0$. Assuming this for a moment we can prove the lemma, since

$$| \int_{\partial \Omega_\lambda} \widetilde{Z}_{0j} h | \leq \|h\|_* \|\widetilde{Z}_{0j}\|_{L^\infty(\Omega_\lambda)} \leq C \log \frac{1}{\lambda} \|h\|_* \frac{1}{\log \frac{1}{\lambda}} = o(1) \frac{1}{\log \frac{1}{\lambda}}$$

and

$$\int_{\Omega_\lambda} \widetilde{Z}_{0j} f = o(1) \frac{1}{\log \frac{1}{\lambda}}.$$

Similarly the other terms can be estimated as follows

$$| \int_{\partial \Omega_\lambda} \tilde{\phi} (\frac{\partial \widetilde{Z}_{0j}}{\partial \nu} - W \widetilde{Z}_{0j}) | \leq \|\tilde{\phi}\|_{L^\infty(\Omega_\lambda)} \int_{\partial \Omega_\lambda} | \frac{\partial \widetilde{Z}_{0j}}{\partial \nu} - W \widetilde{Z}_{0j} | = O(\lambda^{1-\beta}),$$

using (3.22); and

$$| \int_{\Omega_\lambda} \tilde{\phi} (-\Delta \widetilde{Z}_{0j}) | \leq \|\tilde{\phi}\|_{L^\infty(\Omega_\lambda)} \int_{\Omega_\lambda} |\Delta \widetilde{Z}_{0j}| \leq \frac{C}{\log \frac{1}{\lambda}} \|\tilde{\phi}\|_{L^\infty(\Omega_\lambda)} = o(1) \frac{1}{\log \frac{1}{\lambda}}.$$

Let us prove (3.31) using (3.16) to compute $\int_{\Omega_\lambda} \Delta\widetilde{Z}_{0j}\widetilde{Z}_{0j}$. For the part of $\Delta\widetilde{Z}_{0j}$ supported on $r=\lambda^{-\gamma}$ we have

$$
\begin{aligned}
&\frac{\lambda^\gamma}{(\beta-\gamma)\log\frac{1}{\lambda}}\int_{r=\lambda^{-\gamma}}(1+O(\lambda|x|))z_{0j}^2 p\,dx\\
&=\frac{\lambda^\gamma}{(\beta-\gamma)\log\frac{1}{\lambda}}(1+O(\lambda^{1-\gamma}))(1+O(\lambda^\gamma))^2\pi\lambda^{-\gamma}\\
&=\frac{\pi\gamma}{(\beta-\gamma)\log\frac{1}{\lambda}}+O(\lambda^\theta)\qquad 0<\theta<\min(\gamma,1-\beta).
\end{aligned}
$$

Analogously, for the part supported on $r=\lambda^{-\beta}$ we find

$$
\frac{\lambda^\beta}{(\beta-\gamma)\log\frac{1}{\lambda}}\int_{r=\lambda^{-\beta}}(1+O(\lambda|x|))z_{0j}^2 p\,dx=0
$$

since $p(\lambda^{-\beta})=0$. Also

$$
\begin{aligned}
&\int_{\lambda^{-\gamma}<|x|<\lambda^{-\beta}}(2\nabla p\nabla z_{0j}+O(\lambda|x||\nabla^2\tilde{z}_{0j}|)+(\lambda|\nabla\tilde{z}_{0j}|))z_{0j}p\,dx\\
&=O(\lambda^\theta).
\end{aligned}
$$

Thus

$$
\int_{\Omega_\lambda}\Delta\widetilde{Z}_{0j}\widetilde{Z}_{0j}=-\frac{\pi\gamma}{(\beta-\gamma)\log\frac{1}{\lambda}}+O(\lambda^\theta).
$$

Finally, similarly as in (3.22)

$$
\int_{\partial\Omega_\lambda}\widetilde{Z}_{0j}\Big(\frac{\partial\widetilde{Z}_{0j}}{\partial\nu}-W\widetilde{Z}_{0j}\Big)=O(\lambda^{1-\beta}),
$$

and this proves (3.31). □

Proposition 2. *Let $d>0$ and m a positive even integer. Then there exists $\lambda_0>0$ such that for any $0<\lambda<\lambda_0$, any family of points $\xi_1,\dots,\xi_m\in\partial\Omega$*

satisfying (3.2) *(i.e.* $|\xi_i - \xi_j| \geq d\ \forall i \neq j$*), and any* $h \in L^\infty(\partial\Omega_\lambda)$*,* $f \in L^\infty(\Omega_\lambda)$ *there is a unique solution* $\phi \in L^\infty(\Omega_\lambda)$*,* $c_0, c_1, \ldots, c_m \in \mathbb{R}$ *to*

$$\begin{cases} -\Delta\phi = f \quad \text{in } \Omega_\lambda \\ \frac{\partial\phi}{\partial\nu} - W\phi = h + \sum_{j=1}^m c_j\chi_j Z_{1j} + c_0\chi Z \quad \text{on } \partial\Omega_\lambda \\ \int_{\Omega_\lambda} \chi_j Z_{1j}\phi = 0 \quad \forall j = 1, \ldots, m, \qquad \int_{\Omega_\lambda} \chi Z\phi = 0. \end{cases} \tag{3.32}$$

Moreover there is $C > 0$ *independent of* λ *such that*

$$\|\phi\|_{L^\infty(\Omega_\lambda)} \leq C \log\frac{1}{\lambda}(\|h\|_* + \|f\|_{**}),$$

$$\max(|c_0|, \ldots, |c_m|) \leq C(\|h\|_* + \|f\|_{**}).$$

Proof. We deal first with the following linear problem

$$\begin{cases} -\Delta\phi = f + \sum_{j=1}^m d_j\chi_j Z_{1j} + d_0\chi Z \quad \text{in } \Omega_\lambda \\ \frac{\partial\phi}{\partial\nu} - W\phi = h \quad \text{on } \partial\Omega_\lambda \\ \int_{\Omega_\lambda} \chi_j Z_{1j}\phi = 0 \quad \forall j = 1, \ldots, m, \qquad \int_{\Omega_\lambda} \chi Z\phi = 0, \end{cases} \tag{3.33}$$

where $h \in L^\infty(\partial\Omega_\lambda)$, $f \in L^\infty(\Omega_\lambda)$ are given. Let us show that for any $\phi \in L^\infty(\Omega_\lambda)$, $d_0, d_1, \ldots, d_m$ solution to (3.33) we have

$$\|\phi\|_{L^\infty(\Omega_\lambda)} \leq C \log\frac{1}{\lambda}(\|h\|_* + \|f\|_{**}) \tag{3.34}$$

$$|d_j| \leq C(\|h\|_* + \|f\|_{**}) \quad \forall j = 0, \ldots, m. \tag{3.35}$$

Since by proposition 1 we have

$$\|\phi\|_{L^\infty(\Omega_\lambda)} \leq C \log\frac{1}{\lambda}(\|h\|_* + \|f\|_{**} + \sum_{j=0}^m |d_j|), \tag{3.36}$$

it suffices to prove that (3.35) holds.

Consider a cut-off function $\bar{\eta}$ such that

$$\bar{\eta} \equiv 1 \text{ in } B_{\frac{\delta}{4\lambda}}(0),\ \bar{\eta} \equiv 0 \text{ in } \mathbb{R}^2 \setminus B_{\frac{\delta}{3\lambda}}(0)$$

$$0 \le \bar{\eta} \le 1,\ |\nabla \bar{\eta}| \le C\lambda/\delta,\ |\nabla^2 \bar{\eta}| \le C\lambda^2/\delta^2 \text{ in } \mathbb{R}^2.$$

and for $j = 1, \dots, m$ set

$$\eta_j(y) = \bar{\eta}(F_j^\lambda(y)),$$

where F_j^λ is defined in (3.3). Multiplying (3.33) by $\eta_i Z_{1i}$, $i = 1, \dots, m$ and integrating by parts we obtain

$$\begin{aligned} d_i \int_{\Omega_\lambda} \chi_i Z_{1i}^2 = & - \int_{\partial\Omega_\lambda} h\eta_i Z_{1i} - \int_{\Omega_\lambda} f\eta_i Z_{1i} + \int_{\partial\Omega_\lambda} \phi \frac{\partial \eta_i}{\partial \nu} Z_{1i} \\ & + \int_{\partial\Omega_\lambda} \phi \eta_i \Big(\frac{\partial Z_{1i}}{\partial \nu} - W Z_{1i}\Big) - \int_{\Omega_\lambda} \phi \Delta(\eta_i Z_{1i}). \end{aligned} \tag{3.37}$$

Since $Z_{1i} = O(\frac{1}{1+r})$ and $\nabla\bar{\eta} = O(\lambda)$ we have

$$\Big|\int_{\partial\Omega_\lambda} \phi \frac{\partial \eta_i}{\partial \nu} Z_{1i}\Big| \le C\|\phi\|_{L^\infty(\Omega_\lambda)} \lambda \log\frac{1}{\lambda}. \tag{3.38}$$

As in the proof of (3.19) we have

$$\frac{\partial Z_{1i}}{\partial \nu}(y) - W(y)Z_{1i}(y) = O\Big(\frac{\lambda^\alpha}{1+|y-\xi_j'|}\Big) \qquad y \in \partial\Omega_\lambda,\ |y-\xi_j'| < \frac{\delta}{\lambda},$$

and this implies

$$\int_{\partial\Omega_\lambda} \eta_i \Big|\frac{\partial Z_{1i}}{\partial \nu} - W Z_{1i}\Big| = O(\lambda^{\alpha'}), \quad i = 1, \dots, m, \tag{3.39}$$

where $0 < \alpha' < \alpha$. Since $0 < \alpha < 1$ is arbitrary so is α' and so from now on we will just write α.

We also compute

$$\begin{aligned} \Delta(\eta_i Z_{1i}) &= \Delta\eta_i Z_{1i} + 2\nabla\eta_i \nabla Z_{1i} + \eta_i \Delta Z_{1i} \\ &= O\Big(\frac{\lambda^2}{1+r}\Big) + O\Big(\frac{\lambda}{1+r^2}\Big) + \eta_i \Delta Z_{1i}. \end{aligned}$$

But $\Delta_y Z_{1i} = \Delta_x z_1 + O(\lambda|x||\nabla^2 z_1|) + O(\lambda|\nabla z_1|)$ (where we have rotated Ω appropriately and $x = F_i^\lambda(y)$). Thus

$$\Delta Z_{1i} = O(\frac{\lambda}{1+r^2}) + O(\frac{\lambda^2}{1+r})$$

and it follows that

$$\int_{\Omega_\lambda} |\Delta(\eta_i Z_{1i})| = O(\lambda \log \frac{1}{\lambda}) = O(\lambda^\alpha). \tag{3.40}$$

Combining (3.37), (3.39) and (3.40) we conclude

$$d_i \int_{\Omega_\lambda} \chi_i Z_{1i}^2 \le C(\|h\|_* + \|f\|_{**} + \lambda^\alpha \|\phi\|_{L^\infty(\Omega_\lambda)})$$

and this together with (3.36) yields

$$|d_i| \le C(\|h\|_* + \|f\|_{**} + \lambda^\alpha \sum_{j=0}^m |d_j|), \qquad i = 1, \ldots, m. \tag{3.41}$$

On the other hand, multiplying (3.33) by Z we obtain

$$d_0 \int_{\Omega_\lambda} \chi Z^2 = -\int_{\Omega_\lambda} fZ - \int_{\partial\Omega_\lambda} hZ + \int_{\partial\Omega_\lambda} \phi(\frac{\partial Z}{\partial \nu} - WZ) - \int_{\Omega_\lambda} \phi \Delta Z. \tag{3.42}$$

We estimate as before

$$|\int_{\partial\Omega_\lambda} \phi(\frac{\partial Z}{\partial \nu} - WZ)| \le \|\phi\|_{L^\infty(\Omega_\lambda)} \int_{\partial\Omega_\lambda} |\frac{\partial Z}{\partial \nu} - WZ| \le C\lambda^{b/2} \|\phi\|_{L^\infty(\Omega_\lambda)} \tag{3.43}$$

and

$$|\int_{\Omega_\lambda} \phi \Delta Z| \le \|\phi\|_{L^\infty(\Omega_\lambda)} \int_{\Omega_\lambda} |\Delta Z| \le C\lambda^b \|\phi\|_{L^\infty(\Omega_\lambda)}. \tag{3.44}$$

From (3.42) and (3.36) we see that

$$|d_0| \le C(\|h\|_* + \|f\|_{**} + \lambda^{b/2} \sum_{j=0}^m |d_j|).$$

Using this and (3.41) we deduce (3.35) and (3.34).

Now consider the Hilbert space

$$H = \left\{ \phi \in H^1(\Omega_\lambda) : \int_{\Omega_\lambda} \chi Z \phi = 0, \int_{\Omega_\lambda} \chi_j Z_{1j} \phi = 0 \; \forall j = 1, \ldots, m \right\}$$

with the norm $\|\phi\|_{H^1}^2 = \int_{\Omega_\lambda} |\nabla \phi|^2$, which is indeed a norm in H since by choosing R_0 large enough (the size of the support of the cut-off functions χ_j, c.f.(3.4)) we have $\int_{\Omega_\lambda} \chi Z \neq 0$. Equation (3.33) can be formulated as to find $\phi \in H$ such that

$$\int_{\Omega_\lambda} \nabla \phi \nabla \psi - \int_{\partial \Omega_\lambda} W \phi \psi = \int_{\Omega_\lambda} f \psi + \int_{\partial \Omega_\lambda} h \psi \qquad \forall \psi \in H.$$

By (3.34) this problem has at most one solution, and by Fredholm's alternative we deduce that given f, h indeed there exists a solution.

For convenience of notation in the rest of the proof we write

$$Z_0 = Z, \quad \chi_0 = \chi \quad \text{and} \quad Z_j = Z_{1j} \quad \forall j = 1, \ldots, m.$$

Let $Y_i \in L^\infty(\Omega_\lambda)$, $d_{ij} \in \mathbb{R}$ $i, j = 0, \ldots, m$ be the solution to (3.33) with $h = \chi_i Z_{1i}$ and $f = 0$, that is

$$\begin{cases} -\Delta Y_i = \sum_{j=0}^m d_{ij} \chi_j Z_j & \text{in } \Omega_\lambda \\ \frac{\partial Y_i}{\partial \nu} - W Y_i = -\chi_i Z_i & \text{on } \partial \Omega_\lambda \\ \int_{\Omega_\lambda} \chi_j Z_j Y_i = 0 \quad \forall j = 0, \ldots, m. \end{cases} \tag{3.45}$$

There exists a unique $Y_i \in L^\infty(\Omega_\lambda)$ solution to this equation and we have the estimates

$$\|Y_i\|_{L^\infty(\Omega_\lambda)} \leq C \log \frac{1}{\lambda}, \qquad |d_{ij}| \leq C, \tag{3.46}$$

for some constant C independent of λ. We shall show that $d_{ij} = A\delta_{ij} + O(\lambda^{b/2})$ where $A > 0$ is independent of λ. Indeed, writing $\eta_0 \equiv 1$, let us multiply (3.45) by $\eta_j Z_j$ and integrate by parts

$$
\begin{aligned}
d_{ij} \int_{\Omega_\lambda} \chi_j Z_j^2 + \delta_{ij} \int_{\partial\Omega_\lambda} \chi_j Z_j^2 = \int_{\partial\Omega_\lambda} (\frac{\partial Z_j}{\partial \nu} - W Z_j)\eta_j Y_i + \int_{\partial\Omega_\lambda} \frac{\partial \eta_j}{\partial \nu} Z_j Y_i \\
- \int_{\Omega_\lambda} Y_i \Delta(\eta_j Z_j) \\
= O(\lambda^{b/2}).
\end{aligned}
$$

To estimate the integrales in the right hand side above for the case $j = 1, \ldots, m$, we argue exactly as in (3.38), (3.39), and (3.40). For the case $j = 0$ we use (3.43) and (3.44).

It follows that the matrix D with entries d_{ij} $i, j = 0, \ldots, m$ is invertible for small λ and $\|D^{-1}\| \le C$ uniformly in λ. To prove the solvability of (3.32) let $f \in L^\infty(\Omega_\lambda)$, $h \in L^\infty(\partial\Omega_\lambda)$ be given. We find ϕ_1, $d_0, \ldots, d_m$ the solution to (3.33) and define

$$\phi = \phi_1 + \sum_{i=0}^{m} c_i Y_i$$

where c_i is such that $\sum_{i=0}^m c_i d_{ij} = -d_j$ $\forall j = 0, \ldots, m$. Then ϕ satisfies (3.32) and we have the estimate

$$
\begin{aligned}
\|\phi\|_{L^\infty(\Omega_\lambda)} \le \|\phi_1\|_{L^\infty(\Omega_\lambda)} + \sum_{i=0}^{m} |c_i| \le C \log\frac{1}{\lambda}(\|f\|_{**} + \|h\|_*) + C\sum_{i=1}^{m} |d_i| \\
\le C \log\frac{1}{\lambda}(\|f\|_{**} + \|h\|_*),
\end{aligned}
$$

by (3.35). □

The previous result implies that the unique solution $\phi = T(h)$ of (3.32) with $f = 0$ defines a continuous linear map from $L^\infty(\partial\Omega_\lambda)$ with the norm

$\|\cdot\|_*$ into $L^\infty(\Omega_\lambda)$. For fixed $h \in L^\infty(\partial\Omega)$ let us compute derivative of $\phi = T(h)$ with respect to ξ'_l. Formally $Y = \partial_{\xi'_l}\phi$ satisfies the equation

$$\Delta Y = 0 \qquad \text{in } \Omega_\lambda,$$

and on $\partial\Omega_\lambda$ the boundary condition

$$\frac{\partial Y}{\partial \nu} - WY = \partial_{\xi'_l}(W)\,\phi + c_l\,\partial_{\xi'_l}(Z_{1l}\chi_l) + \sum_{j=1}^m d_j\, Z_{1j}\chi_j + c_0\partial_{\xi'_l}(\chi Z) + d_0\chi Z$$

where (still formally) $d_j = \partial_{\xi'_l}(c_j)$, $j = 0,\ldots,m$. The orthogonality conditions now become

$$\int_{\Omega_\lambda} Z_{1j}\chi_j Y = 0, \quad \text{ if } j \neq l$$
$$\int_{\Omega_\lambda} Z_{1l}\chi_l Y = -\int_{\Omega_\lambda} \partial_{\xi'_l}(Z_{1l}\chi_l)\phi.$$
$$\int_{\Omega_\lambda} \chi Z Y = -\int_{\Omega_\lambda} \partial_{\xi'_l}(\chi Z)\phi.$$

Let us write $\tilde{Y} = Y + b_l\chi_l Z_{1l} + b_0\chi Z$ where b_0, b_l are defined through

$$b_0\int_{\Omega_\lambda}\chi^2 Z^2 \equiv \int_{\Omega_\lambda}\phi\;\partial_{\xi'_l}(\chi Z), \quad b_l\int_{\Omega_\lambda}\chi_l^2 Z_{1l}^2 \equiv \int_{\Omega_\lambda}\phi\;\partial_{\xi'_l}(\chi_l Z_{1l}).$$

Hence $\int_{\Omega_\lambda}\tilde{Y}\chi_j Z_{1j} = 0$ for all j and $\int_{\Omega_\lambda}\tilde{Y}\chi Z = 0$, $\tilde{Y}$ satisfies the equation

$$\Delta\tilde{Y} = a \quad \text{in } \Omega_\lambda$$

and the boundary condition

$$\frac{\partial\tilde{Y}}{\partial\nu} - W\tilde{Y} = b + \sum_{j=1}^m d_j\, Z_{1j}\chi_j + d_0\chi Z,$$

where

$$a = b_l\Delta(\chi_l Z_{1l}) + b_0\Delta(\chi Z)$$

and

$$b = \partial_{\xi_l'}(W)\,\phi + c_l\,\partial_{\xi_l'}(Z_{1l}\chi_l) + c_0\partial_{\xi_l'}(\chi Z) + b_l(\frac{\partial(\chi_l Z_{1l})}{\partial\nu} - W\chi_l Z_{1l})$$

$$+b_0(\frac{\partial(\chi Z)}{\partial\nu} - W\chi Z),$$

with

$$\|b\|_* \le C\log\frac{1}{\lambda}\|h\|_*, \qquad \|a\|_{**} \le C\log\frac{1}{\lambda}\|h\|_*.$$

Thus

$$\|\partial_{\xi_l'}\phi\|_{L^\infty(\Omega_\lambda)} \le C(\log\frac{1}{\lambda})^2\|h\|_*. \tag{3.47}$$

4. The nonlinear problem with constraints

Let τ be a small parameter and consider

$$V_1(y) = V(y) + \tau Z(y) \quad y\in\Omega_\lambda \tag{4.1}$$

where V is given by (2.16) and Z is the function introduced in (3.5) at the beginning of section 3.

A function v of the form

$$v(y) = V_1(y) + \tilde\phi(y), \quad y\in\Omega_\lambda$$

is a solution for (2.15) if and only if $\tilde\phi$ solves

$$\begin{cases} \Delta\tilde\phi = 0 \ \text{ in } \Omega_\lambda, \\ \frac{\partial\tilde\phi}{\partial\nu} - W_1\tilde\phi = R_1 + N_1(\tilde\phi) \text{ on } \partial\Omega_\lambda, \end{cases} \tag{4.2}$$

where

$$W_1 = 2\lambda^2\cosh V_1, \tag{4.3}$$

$$R_1 = -\left[\frac{\partial V_1}{\partial \nu} - 2\lambda^2 \sinh V_1\right] \tag{4.4}$$

and

$$N_1(\tilde{\phi}) = 2\lambda^2 \left[\sinh(V_1 + \tilde{\phi}) - \sinh V_1 - \cosh(V_1)\tilde{\phi}\right]. \tag{4.5}$$

Observe that from the definition of the function Z we see that $Z(y) = O(1)$ all over Ω_λ. This readily implies that

$$V_1(y) = V(y) + O(|\tau|) \quad \forall y \in \Omega_\lambda. \tag{4.6}$$

We consider first the following auxiliary nonlinear problem

$$\begin{cases} \Delta\phi_1 = 0 \quad \text{in } \Omega_\lambda, \\ \frac{\partial \phi_1}{\partial \nu} - W_1\phi_1 = R_1 + N_1(\phi_1) + \sum_{j=1}^m c_j\chi_j Z_{1j} + c_0\chi Z \text{ on } \partial\Omega_\lambda, \\ \int_{\Omega_\lambda} \chi_j Z_{1j}\phi_1 = 0 \; \forall j = 1, ..., m, \quad \int_{\Omega_\lambda} \chi Z\phi_1 = 0 \end{cases} \tag{4.7}$$

where W_1 is as in (4.3) and N_1, R_1 are defined in (4.5) and (4.4) respectively.

Lemma 8. *Let $m > 0$, $d > 0$. Let α be any number in the interval $(0, 1)$ and $\tau = O(\lambda^\theta)$ with $\theta > \frac{\alpha}{2}$. Then there exist $\lambda_0 > 0$, $C > 0$ such that for $0 < \lambda < \lambda_0$ and for any $\xi_1, \dots, \xi_m \in \partial\Omega$ satisfying (3.2), problem (4.7) admits a unique solution ϕ_1, $c_0, c_1, \dots, c_m$ such that*

$$\|\phi_1\|_{L^\infty(\Omega_\lambda)} \le C\lambda^\alpha. \tag{4.8}$$

Furthermore, the function $(\tau, \xi') \to \phi_1(\tau, \xi') \in C(\bar{\Omega}_\lambda)$ is C^1 and

$$\|D_{\xi'}\phi_1\|_{L^\infty(\Omega_\lambda)} \le C\,\lambda^\alpha, \qquad \|D_\tau\phi_1\|_{L^\infty(\Omega_\lambda)} \le C\,\lambda^{\theta_1}, \quad \theta_1 < \theta. \tag{4.9}$$

Proof. First we observe that

$$W_1(y) = W(y) + 2\lambda^2 \sinh(V)\tau Z + \tau^2\lambda^2 \cosh(V + \bar{\tau}Z)Z^2,$$

where $|\bar\tau| \le |\tau|$. The equation for ϕ_1 can be written as

$$\begin{cases} \Delta\phi_1 = 0 \ \text{ in } \Omega_\lambda, \\ \frac{\partial\phi_1}{\partial\nu} - W\phi_1 = \tau B\phi_1 + R_1 + N_1(\phi_1) + \sum_{j=1}^m c_j\chi_j Z_{1j} + c_0\chi Z \text{ on } \partial\Omega_\lambda, \\ \int_{\Omega_\lambda} \chi_j Z_{1j}\phi_1 = 0 \ \forall j = 1, ..., m, \quad \int_{\Omega_\lambda} \chi Z\phi_1 = 0, \end{cases} \tag{4.10}$$

where $B = 2\lambda^2 \sinh(V)Z + \tau\lambda^2 \cosh(V + \bar\tau Z)Z^2$. Remark that from (2.23) and (2.27) we have the estimate $\|B\|_* \le C$.

Let A be the operator that associates to any $\phi_1 \in L^\infty(\Omega_\lambda)$ the unique solution given by proposition 2 for $h = \tau B\phi_1 + R_1 + N_1(\phi_1)$ and $f = 0$. In terms of the operator A, equation (4.10) is equivalent to the fixed point problem

$$\phi_1 = A(\phi_1)\,. \tag{4.11}$$

Let us consider the set

$$\mathcal{F} \equiv \{\phi \in C(\bar\Omega_\lambda) \,:\, ||\phi||_{L^\infty(\Omega_\lambda)} \le \lambda^\alpha\}.$$

From proposition 2 we get

$$\|A(\phi_1)\|_{L^\infty(\Omega_\lambda)} \le C|\log\lambda| \left[|\tau| \, \|B\phi_1\|_* + \|N_1(\phi_1)\|_* + \|R_1\|_* \right].$$

Let us estimate $\|R_1\|_*$. We have

$$R_1 = -\left[\frac{\partial V}{\partial\nu} + \tau\frac{\partial Z}{\partial\nu} - 2\lambda^2\sinh(V + \tau Z)\right]$$

$$= R(y) - \tau\frac{\partial Z}{\partial\nu} + 2\lambda^2\sinh(V + \tau Z) - 2\lambda^2\sinh V$$

$$= R(y) - \tau\left[\frac{\partial Z}{\partial\nu} - WZ\right] + \lambda^2\tau^2\sinh(V + \bar\tau Z)Z^2,$$

where $|\bar\tau| \le |\tau|$. For $|y-\xi_j'| \le \frac{\delta}{\lambda}$, we have $\frac{\partial Z}{\partial \nu} - WZ = O(\frac{\lambda^a}{1+|y-\xi_j'|})$ where $0<a<1$ will be fixed shortly, while for $|y-\xi_j'| > \frac{\delta}{\lambda}$ we have $\frac{\partial Z}{\partial \nu} - WZ = O(\lambda^2)$, thus $\|\frac{\partial Z}{\partial \nu} - WZ\|_* \le C\lambda^{a-\sigma}$. Similarly $\|R\|_* \le C\lambda^{a-\sigma}$. On the other hand $\|\lambda^2 \sinh(V+\bar\tau Z)Z^2\|_* \le C$ and hence

$$\|R_1\|_* \le \|R\|_* + |\tau|\lambda^{a-\sigma} + C\tau^2 \le C(\lambda^{a-\sigma} + \lambda^{2\theta}),$$

since $\tau = O(\lambda^\theta)$. We choose $0<a<1$ and $\sigma>0$ small so that $a-\sigma>\alpha$ (σ is the number that appears in the definition of the norms $\|\cdot\|_*$ and $\|\cdot\|_{**}$).

Furthermore, $\|N_1(\phi_1)\|_* \le C\,\|\phi_1\|^2_{L^\infty(\Omega_\lambda)}$ as a direct consequence of (4.5) and

$$|\tau|\|B\phi_1\|_* \le |\tau|\|\phi_1\|_{L^\infty(\Omega_\lambda)}\|B\|_* \le C\lambda^{\alpha+\theta}.$$

We get, for any $\psi_1, \psi_2, \psi \in \mathcal{F}$, the existence of a positive constant C, such that

$$\|A(\psi)\|_{L^\infty(\Omega_\lambda)} \le C|\log\lambda| \left[\lambda^{\alpha+\theta} + \lambda^{2\alpha} + \lambda^{a-\sigma} + \lambda^{2\theta}\right],$$

$$\|A(\psi_1) - A(\psi_2)\|_{L^\infty(\Omega_\lambda)} \le C\ |\log\lambda|(\lambda^\theta + \lambda^\alpha)\,\|\psi_1 - \psi_2\|_{L^\infty(\Omega_\lambda)}.$$

It follows that for all λ sufficiently small A is a contraction mapping of $\mathcal{F}$, and therefore a unique fixed point of A exists in $\mathcal{F}$.

Let us now discuss the differentiability of ϕ_1. Since R_1 depends continuously (in the *-norm) on

$$(\tau, \xi') = (\tau, \xi_1', \dots, \xi_m'),$$

using the fixed point characterization (4.11) we deduce that the map $(\tau,\xi') \mapsto \phi_1$ is also continuous. Then, formally, for $\beta = \xi_k'$ or $\beta = \tau$,

$$-\partial_\beta N_1(\phi_1) = 2\lambda^2\Big[\Big(\cosh(V_1+\phi_1) - \cosh V_1 - \sinh V_1\phi_1\Big)\partial_\beta V_1$$

$$+\Big(\sinh(V_1+\phi_1)-\cosh V_1\Big)\partial_\beta\,\phi_1\Big]\,.$$

It can be checked that $\|\partial_\beta V_1\|_*$ is uniformly bounded, for both $\beta=\xi_k'$ and $\beta=\tau$, so we conclude

$$\|\partial_\beta N_1(\phi_1)\|_*\le C\left[\|\phi_1\|_{L^\infty(\Omega_\lambda)}+\|\partial_\beta\phi_1\|_{L^\infty(\Omega_\lambda)}\right]\|\phi_1\|_{L^\infty(\Omega_\lambda)}$$
$$\le C\left[\lambda^\alpha+\|\partial_\beta\phi_1\|_{L^\infty(\Omega_\lambda)}\right]\lambda^\alpha\,.$$

Using the notation $T(h)$ for the operator that to $h\in L^\infty(\Omega_\lambda)$ associates the solution of the linear problem (3.32) with $f=0$ we may write, for $\beta=\xi_k'$,

$$\partial_{\xi_k'}\phi_1=(\partial_{\xi_k'}T)\Big(\tau B\phi_1+N_1(\phi_1)+R_1\Big)+T\Big(\partial_{\xi_k'}\Big[\tau B\phi_1+N_1(\phi_1)+R_1\Big]\Big),$$

while for $\beta=\tau$,

$$\partial_\tau\phi_1=T(B\phi_1+\tau\partial_\tau(B\phi_1)+\partial_\tau N_1(\phi_1)+\partial_\tau R_1).$$

Thus, from proposition 2 and (3.47) we deduce for $\beta=\xi_k'$

$$\|\partial_{\xi_k'}\phi_1\|_{L^\infty(\Omega_\lambda)}\le C\,|\log\lambda|^2\|(N_1(\phi_1)+R_1)\|_*+|\log\lambda|\|\partial_{\xi_k'}N_1(\phi_1)\|_*$$
$$+\|\partial_{\xi_k'}R_1\|_*)$$
$$\le C\,\lambda^{a-\sigma}|\log\lambda|^2\le C\lambda^\alpha,$$

since it can be seen that $\|\partial_{\xi_k'}R_1\|_*\le C\lambda^\alpha$. For $\beta=\tau$ we get

$$\|\partial_\tau\phi_1\|_{L^\infty(\Omega_\lambda)}\le C|\log\lambda|(\|\phi_1\|_{L^\infty(\Omega_\lambda)}+\lambda^{1-\sigma}+\lambda^\theta)\le C\lambda^{\theta_1},\quad \theta_1<\theta$$

since $\|\partial_\tau R_1\|_*\le C(\|\frac{\partial Z}{\partial\nu}-WZ\|_*+\tau\|\lambda^2\sinh(V+\bar\tau Z)Z^2\|_*)\le C\lambda^\theta$.

The above computation can be made rigorous by using the implicit function theorem and the fixed point representation (4.11) which guarantees C^1 regularity in τ and ξ'. $\square$

Remark. It is possible to verify that given $\tau_1, \tau_2 = O(\lambda^\theta)$, with $\theta > \frac{\alpha}{2}$, the unique solutions ϕ_1, ϕ_2 of lemma 8 satisfy

$$\|\phi_1 - \phi_2\|_{L^\infty(\Omega_\lambda)} \leq C\lambda^\theta |\tau_1 - \tau_2|. \tag{4.12}$$

This follows from the fixed point characterization (4.11) of these solutions. Indeed, let $A(\tau, \phi)$ be the nonlinear operator introduced in this lemma, i.e., the one that to $\phi \in L^\infty(\Omega_\lambda)$ associates the unique solution given by proposition 2 for $h = \tau B\phi + R_1(\tau) + N_1(\phi_1)$ and $f = 0$. Using proposition 2 we see that

$$\|A(\tau_1, \phi) - A(\tau_2, \phi)\|_{L^\infty(\Omega_\lambda)} \leq C\lambda^\theta |\tau_1 - \tau_2|.$$

Lemma 9. *Let $m > 0$, $d > 0$. For any $0 < \alpha < 1$ there exist $\lambda_0 > 0$, $C > 0$ such that for $0 < \lambda < \lambda_0$, any $\xi_1, \ldots, \xi_m \in \partial\Omega$ satisfying (3.2), there exists a unique τ with $|\tau| < C\lambda^{\alpha - b/2}$ such that problem (4.7) admits a unique solution ϕ, $c_0, c_1, \ldots, c_m$ with $c_0 = 0$ and such that*

$$\|\phi\|_{L^\infty(\Omega_\lambda)} \leq C\lambda^\alpha. \tag{4.13}$$

Furthermore, the function $\xi' \to \phi(\xi')$ is C^1 and

$$\|D_{\xi'}\phi\|_{L^\infty(\Omega_\lambda)} \leq C\lambda^\alpha.$$

Proof. Given $\xi_1, \ldots, \xi_m$ in $\partial\Omega$ such that $|\xi_i - \xi_j| > d$ and $\tau = O(\lambda^\theta)$ with $\frac{\alpha}{2} < \theta < \alpha$, let ϕ_1, $c_0, c_1, \ldots, c_m$ be solutions to (4.7). Multiplying (4.7) against Z and integrating by parts, we get

$$c_0\int_{\partial\Omega_\lambda}\chi Z^2 = -\int_{\partial\Omega_\lambda}\phi_1(W_1 Z-\frac{\partial Z}{\partial\nu}) - \int_{\partial\Omega_\lambda}R_1 Z - \int_{\partial\Omega_\lambda}N_1(\phi_1)Z \\ -\int_{\Omega_\lambda}\phi_1\Delta Z - \sum_{j=1}^m c_j\int_{\partial\Omega_\lambda}\chi_j\chi Z_{1j}Z. \tag{4.14}$$

Now we have:

$$|\int_{\Omega_\lambda}\phi_1\Delta Z| \le C\|\phi_1\|_{L^\infty(\Omega_\lambda)}\lambda^b \le C\lambda^{\alpha+b};$$

$$|\int_{\partial\Omega_\lambda}N_1(\phi_1)Z| \le C\|\phi_1\|^2_{L^\infty(\Omega_\lambda)} \le C\lambda^{2\alpha};$$

$$|\int_{\partial\Omega_\lambda}\phi_1(W_1 Z-\frac{\partial Z}{\partial\nu})| \le C\lambda^\alpha\|\phi_1\|_{L^\infty(\Omega_\lambda)} \le C\lambda^{2\alpha}$$

$$|\int_{\partial\Omega_\lambda}\chi_j\chi Z_{1j}Z| \le C\lambda$$

$$\int_{\partial\Omega_\lambda}R_1 Z = \int_{\partial\Omega_\lambda}RZ + \tau\int_{\partial\Omega_\lambda}[-\frac{\partial Z}{\partial\nu}+WZ]Z + \tau^2\lambda^2\int_{\partial\Omega_\lambda}\sinh(V+\bar\tau Z)Z^3$$

Let us estimate the second integral in the right hand side. Observe that in the regions $\{Z<1-\lambda^b\}$ (which are of size $|y-\xi_j'|<\mu_j\lambda^{-b/2}$) we have by a calculation similar to (3.19)

$$\int_{|y-\xi_j'|<\mu_j\lambda^{-b/2},y\in\partial\Omega_\lambda}[-\frac{\partial Z}{\partial\nu}+WZ]Z = O(\lambda^\alpha).$$

For the rest, that is at distance $\mu_j\lambda^{-b/2}<|y-\xi_j'|<\frac{\delta}{\lambda}$ we have that Z is constant, so for a given $j=1,\dots,m$

$$\int_{\mu_j\lambda^{-b/2}<|y-\xi_j'|<\frac{\delta}{\lambda},y\in\partial\Omega_\lambda}[-\frac{\partial Z}{\partial\nu}+WZ]Z = \int_{\mu_j\lambda^{-b/2}<|y-\xi_j'|<\frac{\delta}{\lambda},y\in\partial\Omega_\lambda}WZ^2 \\ = 4\lambda^{b/2}+o(\lambda^{b/2}).$$

Hence

$$\int_{\partial\Omega_\lambda}[-\frac{\partial Z}{\partial\nu}+WZ]Z = 4m\lambda^{b/2}+o(\lambda^{b/2}).$$

We also have

$$|\int_{\partial\Omega_\lambda} RZ| \le C\lambda^\alpha$$

and from the expansion (2.27)

$$\lambda^2 \int_{\partial\Omega_\lambda} \sinh(V+\bar{\tau}Z)Z^3 = O(\lambda^{b/2}).$$

This shows that it is possible to find $\tau = O(\lambda^{\alpha-b/2})$ so that $c_0 = 0$. The uniqueness of τ can be seen also from the previous estimates. Indeed, suppose we have $\tau, \tilde{\tau} = O(\lambda^\theta)$ and solutions $\phi, \tilde{\phi}$ such that for the corresponding coefficients we have $c_0 = \tilde{c}_0 = 0$. From equation (4.14) and the estimates that follow we obtain

$$\lambda^{b/2}|\tau-\tilde{\tau}| \le C\lambda^\alpha \|\phi-\tilde{\phi}\|_{L^\infty(\Omega_\lambda)} + C|\tau-\tilde{\tau}|(|\tau|+|\tilde{\tau}|)\lambda^{b/2}$$

and using (4.12) we deduce $\tau = \tilde{\tau}$.

Let us now discuss the differentiability of ϕ with respect to ξ'. We have

$$\phi(\xi') = \phi_1(\tau(\xi'), \xi')$$

where ϕ_1 is the solution to problem (4.7) given by lemma 8 while $\tau(\xi')$ is the unique positive number so that in problem (4.7) we have $c_0 = 0$.

Hence,

$$D_{\xi'_k}\phi(\xi') = D_\tau \phi_1(\tau(\xi'), \xi')\, D_{\xi'_k}\tau(\xi') + D_{\xi'_k}\phi_1(\tau(\xi'), \xi').$$

Since from (4.14) with $c_0 = 0$ we can deduce that $|D_{\xi'_k}\tau(\xi')| \le C\lambda^\theta$, from (4.8) and (4.9) we conclude that

$$\|D_{\xi'_k}\phi(\xi')\|_{L^\infty(\Omega_\lambda)} \le C\lambda^\alpha.$$

□

5. Variational reduction

In view of lemma 8 and lemma 9, given $\xi=(\xi_1,\dots,\xi_m)\in\partial\Omega^m$ satisfying $|\xi_i-\xi_j|\ge d\ \forall i\neq j$, we define $\phi(\xi)$ and $c_j(\xi)$ to be the unique solution to (4.7) with $c_0=0$ satisfying the bounds (4.8) and (4.9).

Let

$$J_\lambda(u)=\frac{1}{2}\int_{\Omega_\lambda}|Du|^2-2\lambda^2\int_{\partial\Omega_\lambda}\cosh u\,dx.$$

Given $\xi=(\xi_1,\dots,\xi_m)\in\partial\Omega^m$, define

$$F_\lambda(\xi)=J_\lambda(V_1(\xi)+\phi(\xi)) \tag{5.1}$$

where $V_1(\xi)=V(\xi)+\tau(\xi)Z(\xi)$ with $\tau(\xi)$ given by lemma 9.

Lemma 10. *If $\xi=(\xi_1,\dots,\xi_m)\in(\partial\Omega)^m$ satisfying (3.2) is a critical point of F_λ then $v=V_1(\xi)+\phi(\xi)$ is a critical point of J_λ, that is, a solution to (2.15).*

Proof. Let $\xi'=\xi/\lambda$. Therefore

$$\frac{\partial F_\lambda}{\partial\xi_k}=\frac{1}{\lambda}\frac{\partial I_\lambda(V_1(\xi')+\phi(\xi'))}{\partial\xi_k'}=\frac{1}{\lambda}DI_\lambda(V_1(\xi')+\phi(\xi'))\Big[\frac{\partial V_1(\xi')}{\partial\xi_k'}+\frac{\partial\phi(\xi')}{\partial\xi_k'}\Big].$$

Since $v'=V_1(\xi')+\phi(\xi')$ solves (4.7) with $c_0=0$

$$\frac{\partial F_\lambda}{\partial\xi_k}=\frac{1}{\lambda}\sum_{i=1}^m c_i\int_{\partial\Omega_\lambda}\chi_i Z_{1i}\Big[\frac{\partial V_1(\xi')}{\partial\xi_k'}+\frac{\partial\phi(\xi')}{\partial\xi_k'}\Big].$$

Let us assume that $DF_\lambda(\xi)=0$. From the previous equation we conclude that

$$\sum_{i=1}^m c_i\int_{\partial\Omega_\lambda}\chi_i Z_{1i}\Big[\frac{\partial V_1(\xi')}{\partial\xi_k'}+\frac{\partial\phi(\xi')}{\partial\xi_k'}\Big]=0\qquad\forall k=1,\dots,m.$$

Since $\|\frac{\partial \phi(\xi')}{\partial \xi'_k}\|_{L^\infty(\Omega_\lambda)} \leq C\lambda^\alpha$ and $\frac{\partial V(\xi')}{\partial \xi'_k} = \pm Z_{1k} + o(1)$ where $o(1)$ is in the L^∞ norm as a direct consequence of (4.6), it follows that

$$\sum_{i=1}^{m} c_i \int_{\partial\Omega_\lambda} \chi_i Z_{1i}(\pm Z_{1k} + o(1)) = 0 \qquad \forall k = 1, \dots, m,$$

which is a strictly diagonal dominant system. This implies that $c_i = 0$ $\forall i = 1, \dots, m$.

□

In order to solve for critical points of the function F_λ, a key step is its expected closeness to the function $J_\lambda(V_1(\xi))$, which we will analyze in the next section.

Lemma 11. *Assume $\alpha \in (\frac{1}{2}, 1)$. The following expansion holds*

$$F_\lambda(\xi) = J_\lambda(V) + \theta_\lambda(\xi)\,,$$

where

$$|\theta_\lambda| \to 0,$$

uniformly on points satisfying the constraints (3.2).

Proof. We write

$$J_\lambda(V_1 + \phi) - J_\lambda(V) = [J_\lambda(V_1 + \phi) - J_\lambda(V_1)] + [J_\lambda(V_1) - J_\lambda(V)]$$

$$= A + B.$$

Let us estimate A first. Taking into account that $DJ_\lambda(V_1 + \phi)[\phi] = 0$, a Taylor expansion and an integration by parts give

$$A = \int_0^1 D^2 J_\lambda(V_1 + t\phi)[\phi]^2 \,(1-t)\, dt$$

$$= \int_0^1 \Big(2\lambda^2 \int_{\partial\Omega_\lambda} [\cosh(V_1 + t\phi) - \cosh(V_1 + \phi)]\, \phi^2 \\ + \int_{\partial\Omega_\lambda} [N_1(\phi) + R_1]\, \phi\Big)\,(1-t)\, dt, \tag{5.2}$$

so we get

$$J_\lambda(V_1 + \phi) - J_\lambda(V_1) = O(\lambda^{2\alpha})$$

taking into account that $\|\phi\|_{L^\infty(\Omega_\lambda)} \le C\lambda^\alpha$.

On the other hand

$$B = \tau J_\lambda'(V + \bar\tau Z)[Z]$$

for $|\bar\tau| \le |\tau|$, and hence, since Z is almost an element of the kernel of $J_\lambda'(V)$, we get

$$J_\lambda(V_1) - J_\lambda(V) = o(1)\tau \to 0.$$

Hence $|\theta_\lambda(\xi)| = o(1)$ uniformly on points satisfying (3.2).

The continuity in ξ of all these expressions is inherited from that of ϕ and its derivatives in ξ in the L^∞ norm. $\square$

6. Energy Computations and proof of the Theorem

In this section we compute the expansion of the energy functional J_λ evaluated at V and we give the proof of Theorem 1.

We have

Lemma 12. *Let* $m > 0$, $d > 0$. *Let* μ_j *be given by* (2.21) *and let* V *be the function defined in (2.16). Then for any* $0 < \alpha < 1$ *the following expansion holds true*

$$J_\lambda(V) = 2m\pi \log\frac{1}{\lambda} + m(\beta_0 - 2\pi + 2\pi\log 2) \tag{6.1}$$

$$-\pi\varphi_m(\xi) + O(\lambda^\alpha)$$

uniformly on points $\xi = (\xi_1, \ldots, \xi_m) \in (\partial\Omega)^m$ *such that* $|\xi_i - \xi_j| > d$ *for all* $i \neq j$. *In the previous formula,* $\varphi_m(\xi)$ *is the function introduced in (1.5), namely*

$$\varphi_m(\xi) = \varphi_m(\xi_1, \ldots, \xi_m) = \left[\sum_{l=1}^m H(\xi_l, \xi_l) + \sum_{j\neq l}(-1)^{l+j} G(\xi_j, \xi_l)\right]$$

while β_0 *is the constant defined by*

$$\beta_0 = \int_R \frac{1}{1+x^2} \log \frac{1}{1+x^2}.$$

Proof: Since $V(y) = \sum_{j=1}^m (-1)^{j-1}(u_j^\lambda(\lambda y) + H_j^\lambda(\lambda y))$ satisfies $\Delta V = 0$ in Ω_λ, we write

$$J_\lambda(V) = \frac{1}{2}\int_{\partial\Omega_\lambda} V\frac{\partial V}{\partial\nu} - 2\lambda^2\int_{\partial\Omega_\lambda}\cosh(V).$$

We compute the second term first:

$$2\lambda^2\int_{\partial\Omega_\lambda}\cosh(V) = \lambda^2\sum_{l=1}^m\int_{\partial\Omega_\lambda\cap B_{\delta/\lambda}(\xi_l')}(e^V + e^{-V}) + O(\lambda^\alpha)$$

Suppose l is odd first. Then, recalling the notation introduced in section 2 $v_l(y) = u_l^\lambda(y) + 2\log\lambda$, we get

$$\lambda^2\int_{\partial\Omega_\lambda\cap B_{\delta/\lambda}(\xi_l')}(e^V + e^{-V}) = \lambda^2\int_{\partial\Omega_\lambda\cap B_{\delta/\lambda}(\xi_l')} e^V + O(\lambda^\alpha)$$

$$= \int_{\partial\Omega_\lambda\cap B_{\delta/\lambda}(\xi_l')} e^{v_l}\ e^{(-1)^{l-1}H_l^\lambda + \sum_{j\neq l}(-1)^{j-1}(u_j^\lambda + H_j^\lambda)} + O(\lambda^\alpha)$$

$$= 2\pi + O(\lambda^\alpha).$$

Thus

$$\lambda^2\int_{\partial\Omega_\lambda\cap B_{\delta/\lambda}(\xi_l')}(e^V + e^{-V}) = 2\pi + O(\lambda^\alpha). \tag{6.2}$$

Similarly for l even, we also have (6.2). So we obtain

$$2\lambda^2 \int_{\partial\Omega_\lambda} \cosh(V) = 2m\pi + O(\lambda^\alpha). \tag{6.3}$$

It remains to compute $\int_{\partial\Omega_\lambda} V\frac{\partial V}{\partial\nu}$:

$$\begin{aligned}
\int_{\partial\Omega_\lambda} V\frac{\partial V}{\partial\nu} &= \int_{\partial\Omega_\lambda}\left(\sum_{j=1}^m(-1)^{j-1}(u_j^\lambda + H_j^\lambda)\right)\cdot \\
&\qquad\cdot\left(\sum_{i=1}^m(-1)^{i-1}e^{v_i} - \frac{1}{|\partial\Omega_\lambda|}\sum_{i=1}^m(-1)^{i-1}\int_{\partial\Omega_\lambda} e^{v_i}\right) \\
&= \int_{\partial\Omega_\lambda}\left(\sum_{j=1}^m(-1)^{j-1}(u_j^\lambda + H_j^\lambda)\right)\left(\sum_{i=1}^m(-1)^{i-1}e^{v_i}\right) \\
&\quad - \frac{1}{|\partial\Omega_\lambda|}\left(\sum_{i=1}^m(-1)^{i-1}\int_{\partial\Omega_\lambda} e^{v_i}\right)\int_{\partial\Omega_\lambda}\left(\sum_{j=1}^m(-1)^{j-1}(u_j^\lambda + H_j^\lambda)\right) \\
&= \sum_{i,j=1}^m(-1)^{i+j}\int_{\partial\Omega_\lambda} e^{v_i}(u_j^\lambda + H_j^\lambda) + O(\lambda^\alpha)
\end{aligned}$$

since by (2.26)

$$\sum_{j=1}^m(-1)^{j-1}\int_{\partial\Omega_\lambda} e^{v_j} = O(\lambda).$$

For $j \neq i$, we have

$$\int_{\partial\Omega_\lambda} e^{v_i}(u_j^\lambda + H_j^\lambda) = 2\pi G(\xi_j, \xi_i) + O(\lambda^\alpha). \tag{6.4}$$

For $j = i$, we have

$$\begin{aligned}
\int_{\partial\Omega_\lambda} e^{v_i}(u_i^\lambda + H_i^\lambda) &= \int_{\partial\Omega_\lambda}\frac{2\mu_j}{|y-\mu_j\nu_j'|^2}\Bigg((\log\frac{1}{\lambda^2}) + \log\frac{2\mu_j}{|y-\mu_j\nu_j'|^2} \\
&\qquad + H(\xi_j, \xi_j + \lambda y) - \log(2\mu_j)\Bigg) + O(\lambda^\alpha) \\
&= 2\pi\log\frac{1}{\lambda^2} + 2\pi(H(\xi_j,\xi_j) - \log 2\mu_j) + 2\beta_0 + O(\lambda^\alpha)
\end{aligned}$$

(using (2.21)) so

$$\int_{\partial\Omega_\lambda} e^{v_i}(u_i^\lambda + H_i^\lambda) = 2\pi\log\frac{1}{\lambda^2} + 2\beta_0 + 4\pi\log 2 \qquad (6.5)$$
$$+ 2\pi\Bigg(- H(\xi_j,\xi_j) - 2\sum_{i\neq j}(-1)^{i+j}G(\xi_j,\xi_i)\Bigg) + O(\lambda^\alpha)$$

Combining (6.4) and (6.5), we obtain that

$$\int_{\partial\Omega_\lambda} V\frac{\partial V}{\partial \nu} = \sum_{i,j=1}^m (-1)^{i+j}\int_{\partial\Omega_\lambda} e^{v_i}(u_j^\lambda + H_j^\lambda) + O(\lambda^\alpha)$$
$$= 2m\pi\log\frac{1}{\lambda^2} + 2m\beta_0 + 4m\pi\log 2 \qquad (6.6)$$
$$+ 2\pi\Bigg(- \sum_{i=1}^m H(\xi_i,\xi_i) - \sum_{j\neq i}(-1)^{i+j}G(\xi_j,\xi_i)\Bigg) + O(\lambda^\alpha)$$

Summing up equations (6.3) and (6.6), we finally arrive at

$$J_\lambda(V) = 2m\pi\log\frac{1}{\lambda} + m(\beta_0 - 2\pi + 2\pi\log 2)$$

$$-\pi\left[\sum_{l=1}^m H(\xi_l,\xi_l) + \sum_{j\neq l}(-1)^{l+j}G(\xi_j,\xi_l)\right] + O(\lambda^\alpha)$$

□

We now have all ingredients to give the proof of Theorem 1.

Proof of Theorem 1. Define, for $\xi = (\xi_1,\dots,\xi_m) \in (\partial\Omega)^m$ with $|\xi_i - \xi_j| \geq d$, the function

$$v(y) = V_1(\xi)(y) + \phi(\xi)(y) \quad y \in \Omega_\lambda$$

where $V_1(\xi)$ is given by (4.1) and $\phi(\xi)$ is the unique solution to problem (4.7) with $c_0 = 0$, whose existence and properties are established in Lemma 9. Then, according to Lemma 8, v is solution to (2.15) provided that ξ is a critical point of the function $F_\lambda(\xi)$ defined in (5.1), or equivalently, ξ is a critical point of

$$\tilde F_\lambda(\xi) = \frac{1}{\pi}\left(2m\pi \log\frac{1}{\lambda} + m(\beta_0 - 2\pi + 2\pi\log 2) - F_\lambda(\xi)\right).$$

Let $m = 2k$ and $\tilde\Omega_m$ be the set of points $\xi = (\xi_1, \dots, \xi_m) \in (\partial\Omega)^m$ ordered clockwise along a given connected component of $\partial\Omega$ and such that $|\xi_i - \xi_j| \ge d$ for all $i \ne j$, for some $d > 0$ sufficiently small so that all the previous results hold true. Namely, if we denote by $p : [0, 2\pi] \to \partial\Omega$ a continuous parametrization of this connected component of $\partial\Omega$, then we can write

$$\tilde\Omega_m = \{\xi = (p(\theta_1), \dots, p(\theta_m)) \in (\partial\Omega)^m \, : \, |p(\theta_i) - p(\theta_j)| \ge d \quad \text{if} \quad i \ne j\}.$$

It is not restrictive to assume that $0 \in \partial\Omega$. Lemmas 11 and 12 guarantee that for $\xi \in \tilde\Omega_m$,

$$\tilde F_\lambda(\xi) = \varphi_m(\xi) + \lambda^\alpha \Theta_\lambda(\xi) \tag{6.7}$$

where Θ_λ is uniformly bounded in the considered region as $\lambda \to 0$. We will show that $\tilde F_\lambda$ has at least two distinct critical points in this region, fact that will prove our result. The function φ_m is C^1, bounded from above in $\tilde\Omega_m$ and if two consecutive points get closer it becomes unbounded from below, which implies that

$$\varphi_m(\xi_1, \dots, \xi_m) \to -\infty \text{ as } |\xi_i - \xi_j| \to 0 \text{ for some } i \ne j.$$

Hence, since d is arbitrarily small, φ_m has an absolute maximum M_0 in $\tilde\Omega_m$, so does $\tilde F_\lambda$ whenever λ is sufficiently small. Let us call M_λ this value, so that $M_\lambda = M_0 + o(1)$ as $\lambda \to 0$. On the other hand, Ljusternik-Schnirelmann theory is applicable in our setting, so we can estimate the number of critical points of φ_m in $\tilde{\Omega}_m$ by cat $(\tilde{\Omega}_m)$, the Ljusternik-Schnirelmann category of $\tilde{\Omega}_m$ relative to $\tilde{\Omega}_m$. We claim that cat $(\tilde{\Omega}_m) > 1$. Indeed, by contradiction, assume that cat$(\tilde{\Omega}_m) = 1$. This means that $\tilde{\Omega}_m$ is contractible in itself, namely there exist a point $\xi^0 \in \tilde{\Omega}_m$ and a continuous function $\Gamma : [0,1] \times \tilde{\Omega}_m \to \tilde{\Omega}_m$ such that, for all $\xi \in \tilde{\Omega}_m$,

$$\Gamma(0,\xi) = \xi, \quad \Gamma(1,\xi) = \xi^0.$$

Let $f : S^1 \to \tilde{\Omega}_m$ be the continuous function defined by

$$f(x) = (p(\theta), p(\theta + 2\pi\frac{1}{m}), \ldots, p(\theta + 2\pi\frac{m-1}{m})), \quad x = e^{i\theta},\, \theta \in [0, 2\pi].$$

Let $\eta : [0,1] \times S^1 \to S^1$ be the well defined continuous map given by

$$\eta(t,x) = \frac{\pi_1 \circ \Gamma(t, f(x))}{\|\pi_1 \circ \Gamma(t, f(x))\|}$$

where π_1 denotes the projection on the first component. The function η is a contraction of S^1 to a point and this gives a contradiction. Thus we conclude that

$$c_0 = \sup_{C \in \Xi} \inf_{\bar{\xi} \in C} \varphi_m(\bar{\xi}), \tag{6.8}$$

where

$$\Xi = \{C \subset \tilde{\Omega}_m \,:\, C \text{ closed and cat}\,(C) \geq 2\},$$

is a finite number, and a critical level for φ_m. Call c_λ the number (6.8) with φ_m replaced by $\tilde{F}_\lambda$, so that $c_\lambda = c_0 + o(1)$. If $c_\lambda \neq M_\lambda$, we conclude that there are at least two distinct critical points for $\tilde{F}_\lambda$ (distinct up to cyclic permutations) in $\tilde{\Omega}_m$. If $c_\lambda = M_\lambda$, we get that there must be a set C, with cat $(C) \geq 2$, where the function $\tilde{F}_\lambda$ reaches its absolute maximum. In this case we conclude that there are infinitely many critical points for $\tilde{F}_\lambda$ in $\tilde{\Omega}_m$. Since cyclic permutations are only in finite number, the result is thus proven. □

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