Clustering layers for the Fife-Greenlee problem in $\mathbb{R}^n$

Zhuoran Du* Juncheng Wei†

Abstract

We consider the following Fife-Greene problem

$$\varepsilon^2 \Delta u + (u - a(x))(1 - u^2) = 0 \quad \text{in} \ \ \Omega \ , \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \ \ \partial\Omega \tag{1}$$

where Ω is a smooth and bounded domain in $\mathbb{R}^n$, ν the outer unit normal to $\partial\Omega$ and a a smooth function satisfying $a(x) \in (-1, 1)$ in $\overline{\Omega}$. Let K, Ω_- and Ω_+ be respectively the zero-level set of $a, \{a < 0\}$ and $\{a > 0\}$. We assume $\nabla a \neq 0$ on K. Fife-Greenlee ([21, 22]) constructed stable layered solutions while del Pino-Kowalczyk-Wei ([14]) proved the existence of one unstable layer solution provided that some gap condition is satisfied. In this paper, for each *odd* integer $m \geq 3$, we prove the existence of a sequence $\varepsilon = \varepsilon_j \to 0$, and a solution u_{ε_j} with m-transition layers near K, whose mutual distance is $O(\varepsilon \log \frac{1}{\varepsilon})$. Furthermore, u_{ε_j} converges uniformly to ± 1 on the compact sets of $\Omega_\pm$ as $j \to +\infty$.

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1 Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^n (n \geq 2)$. Of concern is the following Fife-Greenlee problem

$$\begin{cases} \varepsilon^2 \Delta u + (u - a(x))(1 - u^2) = 0 & \text{in} \ \ \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \ \ \partial\Omega, \end{cases} \tag{2}$$

where $\varepsilon > 0$ is a small parameter and ν denotes unit outer normal to $\partial\Omega$.

[0*]College of Mathematics and Econometrics, Hunan University, Changsha, PRC. E-mail: duzr@hnu.edu.cn
[0†]Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, Canada E-mail: jcwei@math.ubc.ca

The particular case $a \equiv 0$ corresponds to the standard Allen-Cahn equation (see [6])

$$\begin{cases} \varepsilon^2 \Delta u + u(1-u^2) = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

The function u represents a continuous realization of the phase present in a material confined to the region at the point x which, except for a narrow region, is expected to take values close to $+1$ or -1. Of particular interest are of course non-trivial steady state configurations in which the antiphases coexist, see for instance [4, 17, 18, 19, 20, 23, 26, 27, 32, 33, 34, 36, 37, 39, 40, 41, 42, 45, 46].

There are also many known results for the general inhomogeneous case: smooth function a satisfies $-1 < a(x) < 1$ in $\overline{\Omega}$ and $\nabla a \neq 0$ on the smooth closed hypersurface $K = \{a(x) = 0\}$, which separates the domain into two disjoint components

$$\Omega = \Omega_- \cup K \cup \Omega_+,$$

with

$$a < 0 \quad \text{in } \Omega_-, \quad a > 0 \quad \text{in } \Omega_+, \quad a = 0 \quad \text{on } K.$$

The energy functional $J_\varepsilon(u)$ corresponds to the problem (2) is

$$J_\varepsilon(u) = \frac{\varepsilon}{2} \int_\Omega |\nabla u|^2 + \frac{1}{\varepsilon} \int_\Omega W(x,u),$$

where

$$W(x,u) = \int_{-1}^{u} (\tau^2 - 1)(\tau - a(x)) d\tau.$$

Fife and Greenlee in [22] first proved the existence of an interior transition layer solution approaching $+1$ in Ω_- and -1 in Ω_+, for all ε sufficiently small. Note that $+1$ is the absolute minimizer of $W(x,\cdot)$ in the domain Ω_-, while -1 is so in its complement Ω_+. The Fife-Greenlee solution, constructed by super-sub solution method, is stable.

Super-sub solutions were later used by Angenent, Mallet-Paret and Peletier in the one dimensional case [7] for construction and classification of stable solutions. Radial solutions were found variationally by Alikakos and Simpson [5]. M. del Pino [11] extended these results to general interfaces in any dimension. Further constructions have been done by Dancer and Yan [10] and Do Nascimento [16]. In particular, it is found in [10] that this solution is precisely a minimizer of J_ε. Related results can be found in [1, 2].

On the other hand, a solution exhibiting a transition layer in the opposite direction, namely u_{ε_j} approaching to $+1$ in Ω_+ and to -1 in Ω_-, has been believed to exist for many years. Hale and Sakamoto [24] established the existence of this type of solution in the one dimensional case, while this was done in the radial case in [12], see also [9]. Such an opposite direction layer in this scalar problem is meaningful in finding transition layer solutions for reaction-diffusion systems such as Gierer-Meinhardt with saturation, see [12, 21, 38, 43, 44] and the references therein. Recently, M. del Pino, Kowalczyk and the second author constructed transition layer solutions in the opposite direction in the two-dimensional case [14]. Subsequently, Mahmoudi, Malchiodi and the second author [29] extended this result to any n-dimensional case. Yang and the second author [46] constructed $(2m+1)$-transition layers solutions in the two-dimensional case. The general high dimensional case remains an open question.

In this paper we will follow the idea in [15] and [33] to establish the existence of a clustering layers solution in any n-dimensional case. More precisely, one can look at the eigenvalues of the corresponding linearized problem as functions of ε, and to estimate their derivative with respect to ε. This can be rigorously done using a linear perturbation theorem due to T.Kato, see Section 2, and by characterizing the resonant eigenfunctions. This result gives us indeed invertibility along a suitable sequence $\varepsilon_j \to 0$, and the norm of the inverse operator along this sequence has an upper bound of order $\varepsilon_j^{-\frac{n+1}{2}}\left(\log\frac{1}{\varepsilon_j}\right)^{\frac{n-1}{2}}$.

Our main result is the following.

Theorem 1.1 *Let Ω be a smooth bounded domain in $\mathbb{R}^n (n \geq 2)$ and the smooth function $a(x) \in (-1,1)$ in $\overline{\Omega}$. Denote K, Ω_- and Ω_+ to be respectively the zero-level set of $a, \{a<0\}$ and $\{a>0\}$. We assume $\nabla a \neq 0$ on K. Then for each* odd *integer $m \geq 3$, we obtain the existence of a sequence $\varepsilon = \varepsilon_j \to 0$, and a solution u_{ε_j} with m-transition layers near K, whose mutual distance is $O(\varepsilon \log\frac{1}{\varepsilon})$. Furthermore, u_{ε_j} converges uniformly to ± 1 on the compact sets of $\Omega_\pm$ as $j \to +\infty$. More precisely, near K, we have*

$$u_{\varepsilon_j}(x) \sim \sum_{\ell=1}^{m} (-1)^{\ell+1} H\left(\frac{\bar{\zeta}}{\varepsilon_j} - f_\ell(\bar{z})\right),$$

Here we parameterize $x = (\bar{z}, \bar{\zeta})$ with $\bar{z}$ and $\bar{\zeta}$, $\bar{z} \in K$ being the closest point to x and $\bar{\zeta} = d(x, K)$, while $H(x)$ is the unique hetero-clinic solution of

$$H'' + H - H^3 = 0, \;\; H(0) = 0, \;\; H(\pm\infty) = \pm 1. \tag{4}$$

The functions f_ℓ satisfy

$$f_{\ell+1}(\bar{z}) - f_\ell(\bar{z}) = \frac{\sqrt{2}}{2}\log\frac{1}{\varepsilon} - \frac{\sqrt{2}}{2}\log\log\frac{1}{\varepsilon} + O(1), \;\; 1 \leq \ell \leq m-1, \tag{5}$$

and

$$f_1 - f_2 + f_3 - \cdots + (-1)^{\ell+1} f_\ell + \cdots + f_m = \frac{m\sqrt{2}}{2}\frac{\kappa}{\partial_{\mathbf{n}} a}(1 + o(1)), \tag{6}$$

where $\kappa(\bar{z})$ is the mean curvature of K and $\partial_{\mathbf{n}} a$ the coefficient of the first order term of the Taylor expansion of a

$$a(\varepsilon z, \varepsilon\zeta) = \partial_{\mathbf{n}} a(\varepsilon z, 0)\varepsilon\zeta + o(\varepsilon). \tag{7}$$

In the rest of the paper we will complete the proof of Theorem 1.1.

2 Preliminaries

For the odd heteroclinic solution $H(x) = \tanh(\frac{\sqrt{2}}{2}x)$ of (4) we know the asymptotic properties

$$\begin{cases} H(x) - 1 = -2e^{-\sqrt{2}x} + \mathrm{O}(e^{-2\sqrt{2}x}), & x > 1, \\ H(x) + 1 = 2e^{\sqrt{2}x} + \mathrm{O}(e^{2\sqrt{2}x}), & x < -1, \\ H'(x) = 2\sqrt{2}e^{-\sqrt{2}|x|} + \mathrm{O}(e^{-2\sqrt{2}|x|}), & |x| > 1. \end{cases} \tag{8}$$

From the equation (4), we can get $\frac{H_x^2}{2} - \frac{(1-H^2)^2}{4} \equiv 0$, which yields

$$1 - H^2(x) = \sqrt{2}H_x.$$

Hence

$$\int_{-\infty}^{\infty} H_x^2 \mathrm{d}x = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} (1 - H^2) H_x \mathrm{d}x = \frac{2\sqrt{2}}{3}. \tag{9}$$

Integrating by parts, we have

$$\int_{-\infty}^{\infty} x H_x H_{xx} \mathrm{d}x = -\frac{1}{2} \int_{-\infty}^{\infty} H_x^2 \mathrm{d}x = -\frac{\sqrt{2}}{3}. \tag{10}$$

By (4), we can also get

$$3\int_{-\infty}^{\infty} (1 - H^2) H_x e^{-\sqrt{2}x} \mathrm{d}x = -\int_{-\infty}^{\infty} (H_{xxx} - 2H_x) e^{-\sqrt{2}x} \mathrm{d}x = 8. \tag{11}$$

We need to introduce the following well-known result [35].

Lemma 2.1 *Consider the following eigenvalue problem*

$$\phi_{xx} + (1 - 3H^2)\phi = \lambda\phi, \qquad \phi \in H^1(\mathbb{R}). \tag{12}$$

Then we have

$$\lambda_1 = 0, \qquad \lambda_2 < 0, \tag{13}$$

where the $(\lambda_i)_i$ denote the eigenvalues in non-increasing order (counted with multiplicity), with corresponding eigenfunctions $(\phi_i)_i$. As a consequence (by Fredholm's alternative), given any function $g \in L^2(\mathbb{R})$ satisfying $\int_{\mathbb{R}} g H_x = 0$, the following problem has a unique solution

$$\phi_{xx} + (1 - 3H^2)\phi = g, \quad \text{in} \quad \mathbb{R}, \qquad \int_{\mathbb{R}} \phi H_x = 0. \tag{14}$$

Furthermore, there exists a positive constant C such that $\|\phi\|_{H^1(\mathbb{R})} \le C\|g\|_{L^2(\mathbb{R})}$.

Now we scale the equation (2) by ε^{-1} to obtain

$$\begin{cases} \Delta u + (u - a(\varepsilon x))(1-u^2) = 0 & \text{in } \Omega_\varepsilon, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{15}$$

where $\Omega_\varepsilon = \frac{\Omega}{\varepsilon}$. Following the same notation we also set $K_\varepsilon = \frac{K}{\varepsilon}$, and for $\tau \in (0,1)$ we define

$$U_\tau := \{x \in \Omega_\varepsilon : d(x, K_\varepsilon) < \varepsilon^{-\tau}\}.$$

To consider the scaled problem (15), it is convenient to parameterize elements $x \in U_\tau$ by using their closest point z in K_ε and their distance ζ (with sign, positive in the dilation of Ω_+). Precisely, we can choose coordinates $\bar{z}$ on K, and denote by $\mathbf{n}(\bar{z})$ the unit normal vector to K (at the point with coordinates $\bar{z}$) pointing towards Ω_+. We set $\bar{z} := \varepsilon z, \bar{\zeta} := \varepsilon\zeta$. Then we can write

$$x = z + \zeta \mathbf{n}(\varepsilon z). \tag{16}$$

In the following, we let the upper-case indices $I, J, \dots$ run from 1 to n, and the lower-case indices $i, j, \dots$ run from 1 to $n-1$. We also let $\bar{g}$ denote the metric on K (inherited from $\mathbb{R}^n$), $\bar{g}_\varepsilon$ the one on K_ε, and g_ε the flat metric of Ω_ε, which will be expressed in the above coordinates (z, ζ). If $z_1, \dots, z_{n-1}$ is a local set of coordinates on K_ε, and if $(\bar{g}_\varepsilon)_{ij}$ denote the corresponding components of the metric tensor, then we have

$$(g_\varepsilon)_{IJ} = \begin{pmatrix} (\bar{g}_\varepsilon)_{ij} + \varepsilon\zeta(A_i^l \bar{g}_{jl} + A_j^k \bar{g}_{ik}) + \varepsilon^2\zeta^2 A_i^l \bar{g}_{lk} A_j^k & 0 \\ 0 & 1 \end{pmatrix}, \tag{17}$$

where (A_i^j) are the components of the second fundamental form namely they are defined by $\frac{\partial \mathbf{n}}{\partial \bar{z}_i} = A_i^j \frac{\partial \bar{z}}{\partial \bar{z}_j}$. To obtain (17), we notice that

$$\frac{\partial x}{\partial z_i} = \frac{\partial z}{\partial z_i} + \varepsilon\zeta \frac{\partial \mathbf{n}}{\partial \bar{z}_i}; \qquad \frac{\partial x}{\partial \zeta} = \mathbf{n}.$$

Hence since $(g_\varepsilon)_{ij} = \langle \frac{\partial x}{\partial z_i}, \frac{\partial x}{\partial z_j} \rangle$, and in view of $\mathbf{n}$ is perpendicular to $\frac{\partial z}{\partial z_i}$, then we obtain immediately (17).

We denote the eigenvalues of the matrix (A_i^j) (with respect to the metric $\bar{g}$) by $\kappa_i(\varepsilon z), i = 1, \dots, n-1$, which are called principal curvatures of K. Then the mean curvature of K (scaled by a factor $n-1$) is $\kappa(\varepsilon z) = \sum_{i=1}^{n-1} \kappa_i(\varepsilon z), z \in K_\varepsilon$. We have

$$dV_{g_\varepsilon} = \sqrt{g_\varepsilon} d\zeta dz = (1 + \varepsilon\zeta\kappa(\varepsilon z)) dV_{\bar{g}_\varepsilon} d\zeta + \mathrm{O}(\varepsilon^2\zeta^2) dV_{\bar{g}_\varepsilon} d\zeta. \tag{18}$$

The Laplace-Beltrami operator is defined in local coordinates by the formula

$$\Delta_g u = \frac{1}{\sqrt{\det g}} \partial_I (g^{IJ} \sqrt{\det g} \partial_J u), \tag{19}$$

where g^{IJ} are the elements of the inverse matrix of (g_{IJ}). By (17), elementary computations (see [31]) show that

$$\Delta_g u = u_{\zeta\zeta} + \varepsilon\kappa(\varepsilon z)u_\zeta + \varepsilon^2\Delta_{K_{\varepsilon\zeta}}u + \mathrm{O}(\varepsilon^2)u_\zeta. \tag{20}$$

Here $\Delta_{K_{\varepsilon\zeta}}$ stands for the operator in (19) freezing the coordinate ζ, namely summing over $i, j = 1, \dots, n-1$

$$\Delta_{K_{\varepsilon\zeta}}u = \frac{1}{\sqrt{\det g}}\partial_i(g^{ij}\sqrt{\det g}\partial_j u).$$

This operator is nothing but the Laplace-Beltrami operator for the metric $g_{K_{\varepsilon\zeta}}$ on K_ε with coefficients $((g_\varepsilon)_{ij}(\cdot,\zeta))$ in the coordinates $z_1, \dots, z_{n-1}$. With respect to this metric, one can introduce a corresponding gradient $\nabla_{K_{\varepsilon\zeta}}$, defined by duality as

$$\langle\nabla_{K_{\varepsilon\zeta}}u, v\rangle_{\nabla_{K_{\varepsilon\zeta}}} = (g_\varepsilon)^{ij}(\cdot,\zeta)\frac{\partial u}{\partial z_i}v_j, \qquad \text{if} \quad v = v_j\frac{\partial}{\partial z_j} \in T_{K_\varepsilon}. \tag{21}$$

From the expression of g_{ij} in (17) then one can finds the estimates

$$|\nabla_{K_{\varepsilon\zeta}}u|^2 := (g_\varepsilon)^{ij}(\cdot,\zeta)\frac{\partial u}{\partial z_i}\frac{\partial u}{\partial z_j} = (1 + \mathrm{O}(\varepsilon\zeta))|\nabla_{\bar{g}_\varepsilon}u|^2, \tag{22}$$

$$-\int_{K_\varepsilon} u\Delta_{K_\zeta}v\,dV_{g_{K_\zeta}} = \int_{K_\varepsilon}\langle\nabla_{\bar{g}_\varepsilon}u, \nabla_{\bar{g}_\varepsilon}v\rangle dV_{\bar{g}_\varepsilon} + \mathrm{O}(\varepsilon\zeta)\|\nabla_{\bar{g}_\varepsilon}u\|_{L^2(K_\varepsilon)}\|\nabla_{\bar{g}_\varepsilon}v\|_{L^2(K_\varepsilon)}, \tag{23}$$

foe every $u, v \in H^1(K_\varepsilon)$. Using again (17) one obtains

$$\int_{U_\tau}|\nabla_{g_\varepsilon}u|^2 dV_{g_\varepsilon} = (1+\mathrm{O}(\varepsilon^{1-\tau}))\int_{U_\tau}|u_\zeta|^2 d\zeta dV_{\bar{g}_\varepsilon} + (1+\mathrm{O}(\varepsilon^{1-\tau}))\int_{U_\tau}|\nabla_{\bar{g}_\varepsilon}u|^2 d\zeta dV_{\bar{g}_\varepsilon}. \tag{24}$$

Now we let λ_j and φ_j be the eigenvalues (with weight $\partial_{\mathbf{n}}a$) and the eigenfunctions of

$$-\Delta_K\varphi_j = \lambda_j\partial_{\mathbf{n}}a(\bar{z},0)\varphi_j, \qquad \bar{z} \in K, \tag{25}$$

with $\int_K \partial_{\mathbf{n}}a(\bar{z},0)\varphi_i\varphi_j dV_{\bar{g}} = \delta_{ij}$. Note that $\partial_{\mathbf{n}}a > 0$, considering the previous choose of $\mathbf{n}$. Such eigenvalues can be obtained using the Rayleigh quotient. Precisely if M_j denote the family of j-dimensional subspaces of $H^1(K)$, then we have

$$\lambda_j = \inf_{M\in M_j}\sup_{\varphi\in M,\varphi\neq 0}\frac{\int_K|\nabla_K\varphi|^2 dV_{\bar{g}}}{\int_K\partial_{\mathbf{n}}a(\bar{z},0)\varphi^2 dV_{\bar{g}}}.$$

We can estimate the λ_j using a standard Weyl's asymptotic formula ([8]), one has

$$\lambda_j \simeq C_{K,\partial_{\mathbf{n}}a}j^{\frac{2}{n-1}} \qquad \text{as} \quad j \to +\infty,$$

for some constant $C_{K,\partial_{\mathbf{n}}a}$ depending only on K and $\partial_{\mathbf{n}}a$.

We finally introduce the following theorem due to T. Kato ([25]), which will be fundamental for us to obtain invertibility of the linearized equation.

Theorem 2.1 *Let $T(\varepsilon)$ be a differentiable family of operators from a Hilbert space X into itself, where ε belongs to an interval containing 0. Let $T(0)$ be a self-adjoint operator of the form Identity-compact and let $\sigma(0) = \sigma_0 \neq 1$ be an eigenvalue of $T(0)$. Then the eigenvalue $\sigma(\varepsilon)$ is differentiable at 0 with respect to ε. The derivative of σ is given by*

$$\frac{\partial \sigma}{\partial \varepsilon} = \{\textit{eigenvalues of} \quad P_{\sigma_0} \circ \frac{\partial T}{\partial \varepsilon}(0) \circ P_{\sigma_0}\},$$

where $P_{\sigma_0} : X \to X_{\sigma_0}$ denotes the projection onto the σ_0-eigenspace X_{σ_0} of $T(0)$.

3 Approximate solutions

In this section, we will construct approximate solutions. We set $U := K_\varepsilon \times (-\frac{\delta}{\varepsilon}, \frac{\delta}{\varepsilon})$, $I_\varepsilon := [-\frac{\delta}{\varepsilon}, \frac{\delta}{\varepsilon}]$. From the previous section we know that equation (2) becomes

$$\begin{cases} u_{\zeta\zeta} + \varepsilon\kappa(\varepsilon z)u_\zeta + \varepsilon^2 \Delta_{K_{\varepsilon\zeta}} u + \mathrm{O}(\varepsilon^2)u_\zeta + u(1-u^2) - a(\varepsilon x)(1-u^2) = 0 \quad (z,\zeta) \in U, \\ u(\cdot, \pm\frac{\delta}{\varepsilon}) = \pm 1. \end{cases} \tag{26}$$

For a fixed odd integer $m \geq 3$, we assume that the location of the m phase transition layers are characterized by functions $\zeta = f_\ell(\varepsilon z), 1 \leq \ell \leq m$ in the coordinates (z, ζ). These functions will be left as parameters and satisfy

$$f_1(\varepsilon z) < f_2(\varepsilon z) < \cdots < f_m(\varepsilon z),$$

and

$$f_\ell = (-1)^{\ell+1}\frac{\sqrt{2}}{2}\frac{\kappa}{\partial_{\mathbf{n}} a} + \tilde{f}_\ell, \tag{27}$$

where these $\tilde{f}_\ell$ satisfy

$$\tilde{f}_{\ell+1} - \tilde{f}_\ell = \rho_{\varepsilon,\ell} + h_\ell, \qquad |h_\ell| \leq M, \qquad 1 \leq \ell \leq m-1, \tag{28}$$

with

$$16 e^{(-1)^{\ell+1}\frac{2\kappa}{\partial_{\mathbf{n}} a}} e^{-\sqrt{2}\rho_{\varepsilon,\ell}} = \frac{4}{3}\varepsilon \partial_{\mathbf{n}} a \rho_{\varepsilon,\ell}. \tag{29}$$

From (29), one has

$$\rho_{\varepsilon,1} = \rho_{\varepsilon,3} = \cdots = \rho_{\varepsilon,m}, \qquad \rho_{\varepsilon,2} = \rho_{\varepsilon,4} = \cdots = \rho_{\varepsilon,m-1}, \qquad \rho_{\varepsilon,\ell+1} - \rho_{\varepsilon,\ell} = \mathrm{O}(1), \tag{30}$$

and

$$\rho_{\varepsilon,\ell} = \frac{\sqrt{2}}{2}\log\frac{1}{\varepsilon} - \frac{\sqrt{2}}{2}\log\log\frac{1}{\varepsilon} + \mathrm{O}(1), \tag{31}$$

which gives (5).

We now define in coordinates (z,ζ) the approximation

$$u_0(z,\zeta) := \sum_{\ell=1}^{m} H_\ell(\zeta - f_\ell(\varepsilon z)),$$

where

$$H_\ell(\tau) = (-1)^{\ell+1} H(\tau).$$

With this definition we have that $u_0(z,\zeta) \approx H_\ell(\zeta - f_\ell(\varepsilon z))$ for values of ζ close to $f_\ell(\varepsilon z)$.

We define a norm

$$\|g\|_* := \sup_{\bar z\in K, \zeta\in I_\varepsilon} |e^{\sigma\times\max\{(\zeta-f_m)_+,(-\zeta+f_1)_+\}} g(\bar z,\zeta)|, \tag{32}$$

where $0<\sigma<\sqrt{2}$ is a suitable small number and $t_+ := \max(t,0)$. Similarly, for a positive integer l we set

$$\|g\|_{*,l} := \sup_{0<|\alpha|\le l} \sup_{\bar z\in K, \zeta\in I_\varepsilon} |e^{\sigma\times\max\{(\zeta-f_m)_+,(-\zeta+f_1)_+\}} D^\alpha_{\bar z} g(\bar z,\zeta)|, \tag{33}$$

where α stands for a multi-index.

For each fixed $\ell, 1\le \ell\le m$, we define the set

$$A_\ell := \left\{(z,\zeta)\in U : -\frac{f_\ell - f_{\ell-1}}{2} \le \zeta - f_\ell(\varepsilon z) \le \frac{f_{\ell+1}-f_{\ell-1}}{2}\right\}.$$

For convenience of the notation we will set

$$f_0 = -\frac{\delta}{\varepsilon} + f_1 \qquad \text{and} \qquad f_{m+1} = \frac{\delta}{\varepsilon} + f_m.$$

Fix z, we let

$$I_{\varepsilon,z,\ell} := \{\zeta : (z,\zeta)\in A_\ell\} \tag{34}$$

and we also replace $I_{\varepsilon,z,\ell}$ by I_ℓ for brevity.

In the rest of this section, we consider the solvability of the following problem

$$\begin{cases} u_{\zeta\zeta} + \varepsilon\kappa(\varepsilon z)u_\zeta + \mathrm{O}(\varepsilon^2)u_\zeta + u(1-u^2) - a(\varepsilon x)(1-u^2) = \varepsilon^2 g(\bar z,\zeta) & \zeta\in I_\varepsilon, \\ u(\pm\frac{\delta}{\varepsilon}) = \pm 1. \end{cases} \tag{35}$$

We define

$$S(u) := u_{\zeta\zeta} + \varepsilon\kappa(\varepsilon z)u_\zeta + \mathrm{O}(\varepsilon^2)u_\zeta + u(1-u^2) - a(\varepsilon x)(1-u^2) - \varepsilon^2 g(\bar z,\zeta).$$

For each fixed ℓ, we write $t = \zeta - f_\ell(\varepsilon z)$ and estimate the error of approximation $S(u_0)(z, t+f_\ell(\varepsilon z))$ in the range I_ℓ. Let us consider first the case $2\le\ell\le m-1$.

As in [15], we get

$$\begin{aligned} S(u_0) &= 6(-1)^{\ell+1}(1-H^2(t))\left[e^{-\sqrt{2}(f_\ell-f_{\ell-1})}e^{-\sqrt{2}t}-e^{-\sqrt{2}(f_{\ell+1}-f_\ell)}e^{\sqrt{2}t}\right] \\ &\quad +\varepsilon\kappa(-1)^{\ell+1}H'(t)-\varepsilon\partial_{\mathbf{n}}a(t+f_\ell)(1-H^2(t))+\Theta_\ell, \end{aligned} \tag{36}$$

where $\Theta_\ell = \mathrm{O}(\varepsilon^{1+\mu}e^{-\sigma|t|})$ for some $0<\sigma<\sqrt{2}$ and $\mu \le \frac{1}{2}\left(1-\frac{\sigma}{\sqrt{2}}\right)$.

The above expression also holds for $\ell = 1, \ell = m$. The only difference is that the term $[e^{-\sqrt{2}(f_\ell-f_{\ell-1})}e^{-\sqrt{2}t}-e^{-\sqrt{2}(f_{\ell+1}-f_\ell)}e^{\sqrt{2}t}]$ is respectively replaced by

$$-e^{-\sqrt{2}(f_2-f_1)}e^{\sqrt{2}t} \qquad \text{and} \qquad e^{-\sqrt{2}(f_m-f_{m-1})}e^{-\sqrt{2}t}.$$

We define a function in $\Omega_\varepsilon\backslash K_\varepsilon$ as

$$\mathbb{W}(x)=\begin{cases} 1 & \text{if } \ x\in\Omega_+, \\ -1 & \text{if } \ x\in\Omega_-. \end{cases} \tag{37}$$

We also let $\eta(\theta)$ be a smooth cut-off function with $\eta(\theta)=1$ for $\theta<\frac{\delta}{4}$ and $\eta(\theta)=0$ for $\theta>\frac{\delta}{2}$. Now we define our further approximation $\bar{u}_0$ as

$$\bar{u}_0 := \eta(|\varepsilon\zeta|)u_0+(1-\eta(|\varepsilon\zeta|))\mathbb{W}=\begin{cases} \eta(|\varepsilon\zeta|)[u_0-1]+1 & \text{if } x\in\Omega_+, \\ \eta(|\varepsilon\zeta|)[u_0+1]-1 & \text{if } x\in\Omega_-. \end{cases} \tag{38}$$

The error of further approximation is simply computed as

$$S(\bar{u}_0)=\eta(|\varepsilon\zeta|)S(u_0)+\tilde{\Theta}, \tag{39}$$

where $\tilde{\Theta}$ has exponential size $\mathrm{O}(e^{-\frac{c}{\varepsilon}})$ inside its support, and hence the contribution of this error to the entire error is essentially negligible.

We also need to introduce two groups of smooth cut-off functions, for given $z\in K_\varepsilon$, as following

$$\xi_{\ell\alpha,z}(\zeta)=\begin{cases} 1 & \text{if } |\zeta-f_\ell(\varepsilon z)|\le \frac{|I_\ell|}{2}-2\alpha^{-1}\log\log\frac{1}{\varepsilon}, \\ 0 & \text{if } |\zeta-f_\ell(\varepsilon z)|\ge \frac{|I_\ell|}{2}-\alpha^{-1}\log\log\frac{1}{\varepsilon}, \end{cases} \tag{40}$$

where $\alpha=1,2$. We replace $\xi_{\ell\alpha,z}$ by $\xi_{\ell\alpha}$ for brevity. Notice that

$$\xi_{\ell1}\xi_{\ell2}=\xi_{\ell1}, \tag{41}$$

and

$$|\xi'_{\ell\alpha}|=\mathrm{O}\left(\frac{1}{\log\log\frac{1}{\varepsilon}}\right), \qquad |\xi''_{\ell\alpha}|=\mathrm{O}\left(\frac{1}{(\log\log\frac{1}{\varepsilon})^2}\right). \tag{42}$$

We define

$$S_\ell(\bar{u}_0):=\xi_{\ell1}S(\bar{u}_0),$$

then from this and (36), (39), (5) we obtain

$$\|S_\ell(\bar{u}_0)\|_* \le C\varepsilon \log \frac{1}{\varepsilon}. \tag{43}$$

We consider the linearized problem

$$\begin{cases} \mathbb{L}_\ell(\phi) := \phi_{\zeta\zeta} + \varepsilon\kappa(\varepsilon z)\phi_\zeta + \mathrm{O}(\varepsilon^2)\phi_\zeta + (1-3H_\ell^2)\phi + 2a(\varepsilon x)H_\ell\phi = g + c_{\ell,\varepsilon}\xi_{\ell 1}H'_\ell, \\ \int_{I_\varepsilon} \xi_{\ell 1}\phi H'_\ell = 0. \end{cases} \tag{44}$$

Lemma 3.1 *Let $(\phi, g, c_{\ell,\varepsilon})$ satisfy (44) with the boundary conditions $\phi(\pm\frac{\delta}{\varepsilon}) = 0$. Then for ε sufficiently small we have*

$$\|\phi\|_* + |c_{\ell,\varepsilon}| \le C\|g\|_*. \tag{45}$$

Proof. We prove this lemma by contradiction. Suppose that there exists $(\phi, g, c_{\ell,\varepsilon})$ such that $\|g\|_* = \mathrm{o}(1)$ and $\|\phi\|_* + |c_{\ell,\varepsilon}| = 1$ as $\varepsilon \to 0$. Multiplying (44) by H'_ℓ and integrating over I_ε, using the equation satisfied by H' and integrating by parts we obtain

$$|c_{\ell,\varepsilon}| = \mathrm{o}(1),$$

which yields $\|g + c_{\ell,\varepsilon}\xi_{\ell 1}H'_\ell\|_* = \mathrm{o}(1)$. Next we first show that $\|\phi\|_{H^1(I_\varepsilon)} = \mathrm{o}(1)$. To show this we rewrite (44) as

$$\phi_{\zeta\zeta} + (1-3H_\ell^2)\phi = G_{\varepsilon,h}(g,\phi), \tag{46}$$

where

$$G_{\varepsilon,h}(g,\phi) := g - \varepsilon\kappa(\varepsilon z)\phi_\zeta + \mathrm{O}(\varepsilon^2)\phi_\zeta - 2a(\varepsilon x)H_\ell\phi + c_{\ell,\varepsilon}\xi_{\ell 1}H'_\ell.$$

Note that $\|G_{\varepsilon,h}\|_{L^2(I_\varepsilon)} = \mathrm{o}(1) + \mathrm{O}(1)c_{\ell,\varepsilon} + \mathrm{o}(1)\|\phi\|_{H^1(I_\varepsilon)}$ as $\varepsilon \to 0$. Hence Lemma 2.1 and the contraction mapping theorem give a solution $(\phi, c_{\ell,\varepsilon})$ of (44) for which $\|\phi\|_{H^1(I_\varepsilon)} + |c_{\ell,\varepsilon}| = \mathrm{o}(1)$. Then the estimate in the $\|\cdot\|_*$ (and hence (45)) follows from standard regularity results. The proof of this lemma is complete.

Remark 1 *In fact, we can proved the following estimate*

$$\|\phi\|_{H^2(I_\varepsilon)} + |c_{\ell,\varepsilon}| \le C\|g\|_{L^2(I_\varepsilon)}.$$

Lemma 3.2 *There exists a unique solution $\varphi_{\varepsilon,\mathbf{h}}$ of*

$$S(\bar{u}_0 + \varphi_{\varepsilon,\mathbf{h}}) = \sum_{\ell=1}^{m} c_{\ell,\varepsilon}\xi_{\ell 1}H'_\ell(\zeta - f_\ell), \qquad \int_{I_\varepsilon} \xi_{\ell 1}\varphi_{\varepsilon,\mathbf{h}}H'_\ell = 0, \qquad \ell = 1,\dots,m \tag{47}$$

for some constants $c_{\ell,\varepsilon}$. Moreover, $\varphi_{\varepsilon,\mathbf{h}}$ is unique, differentiable in z and satisfies

$$\|\varphi_{\varepsilon,\mathbf{h}}\|_* \le C\varepsilon \log \frac{1}{\varepsilon}. \tag{48}$$

Proof. We shall look for such $\varphi_{\varepsilon,\mathbf{h}}$ in the following

$$\varphi_{\varepsilon,\mathbf{h}}(x) = \sum_{\ell=1}^{m} \xi_{\ell 2}(\zeta)\phi_{\varepsilon,\ell}(x) + \psi(x).$$

We set

$$N_1(\phi) := -3\bar{u}_0\phi^2 - \phi^3 \quad \text{and} \quad N_2(\phi) := a\bar{u}_0\phi^2. \tag{49}$$

Elementary computations show that

$$\begin{aligned} & S(\bar{u}_0 + \varphi_{\varepsilon,\mathbf{h}}) = S(\bar{u}_0 + \sum_{\ell=1}^{m} \xi_{\ell 2}\phi_{\varepsilon,\ell} + \psi) \\ = & \sum\nolimits_{\ell=1}^{m} \xi_{\ell 2}[\phi''_{\varepsilon,\ell} + \varepsilon\kappa\phi'_{\varepsilon,\ell} + \mathrm{O}(\varepsilon^2)\phi'_{\varepsilon,\ell} + (1-3\bar{u}_0^2)\phi_{\varepsilon,\ell} + 2a\bar{u}_0\phi_{\varepsilon,\ell} \\ + & 3\xi_{\ell 1}(1-\bar{u}_0^2)\psi + \xi_{\ell 1}(N_1(\psi + \phi_{\varepsilon,\ell}) + N_2(\psi+\phi_{\varepsilon,\ell})) + \xi_{\ell 1}S(\bar{u}_0)] \\ + & \psi'' + \varepsilon\kappa\psi' + \mathrm{O}(\varepsilon^2)\psi' - 2(1-a\bar{u}_0)\psi \\ + & \left(1 - \sum_{\ell=1}^{m}\xi_{\ell 1}\right)\left\{3(1-\bar{u}_0^2)\psi + N_1\left(\psi + \sum_{\ell=1}^{m}\xi_{\ell 2}\phi_{\varepsilon,\ell}\right) + N_2\left(\psi + \sum_{\ell=1}^{m}\xi_{\ell 2}\phi_{\varepsilon,\ell}\right) + S(\bar{u}_0)\right\} \\ + & \sum_{\ell=1}^{m}[\phi_{\varepsilon,\ell}\xi''_{\ell 2} + 2\phi'_{\varepsilon,\ell}\xi'_{\ell 2}] + (\varepsilon\kappa + \mathrm{O}(\varepsilon^2))\sum_{\ell=1}^{m}\xi'_{\ell 2}\phi_{\varepsilon,\ell}, \end{aligned} \tag{50}$$

where $\phi'_{\varepsilon,\ell}, \phi''_{\varepsilon,\ell}$ denote respectively $\frac{\partial \phi_{\varepsilon,\ell}}{\partial \zeta}, \frac{\partial^2 \phi_{\varepsilon,\ell}}{\partial \zeta^2}$. Then the problem (47) is equivalent to the following system

$$\begin{aligned} \phi''_{\varepsilon,\ell} \; + & \; \varepsilon\kappa\phi'_{\varepsilon,\ell} + \mathrm{O}(\varepsilon^2)\phi'_{\varepsilon,\ell} + (1-3\bar{u}_0^2)\phi_{\varepsilon,\ell} + 2a\bar{u}_0\phi_{\varepsilon,\ell} + 3\xi_{\ell 1}(1-\bar{u}_0^2)\psi \\ + & \; \xi_{\ell 1}(N_1(\psi+\phi_{\varepsilon,\ell}) + N_2(\psi+\phi_{\varepsilon,\ell})) + S_\ell(\bar{u}_0) \\ = & \; c_{\ell,\varepsilon}\xi_{\ell 1}H'_\ell, \quad \zeta \in I_\ell, \quad \ell = 1,\dots,m, \end{aligned} \tag{51}$$

$$\int_{I_\ell} \xi_{\ell 1}(\phi_{\varepsilon,\ell} + \psi)H'_\ell = 0, \quad \ell = 1,\dots,m, \tag{52}$$

and

$$\begin{aligned} & \psi'' - 2(1-a\bar{u}_0)\psi + \varepsilon\kappa\psi' + \mathrm{O}(\varepsilon^2)\psi' \\ & = -\left(1 - \sum_{\ell=1}^{m}\xi_{\ell 1}\right)\left\{3(1-\bar{u}_0^2)\psi + N_1\left(\psi + \sum_{\ell=1}^{m}\xi_{\ell 2}\phi_{\varepsilon,\ell}\right) + N_2\left(\psi + \sum_{\ell=1}^{m}\xi_{\ell 2}\phi_{\varepsilon,\ell}\right) + S(\bar{u}_0)\right\} \\ & -\sum_{\ell=1}^{m}[\phi_{\varepsilon,\ell}\xi''_{\ell 2} + 2\phi'_{\varepsilon,\ell}\xi'_{\ell 2}] - (\varepsilon\kappa + \mathrm{O}(\varepsilon^2))\sum_{\ell=1}^{m}\xi'_{\ell 2}\phi_{\varepsilon,\ell}. \end{aligned} \tag{53}$$

Observe that the orthogonality condition in (52) is satisfied for $\phi_{\varepsilon,\ell} + \psi$ rather than $\phi_{\varepsilon,\ell}$, hence we introduce new variable $\tilde{\phi}_{\varepsilon,\ell} = \phi_{\varepsilon,\ell} + \psi$. Then from (51) and (52) we obtain

$$\begin{aligned} & \tilde{\phi}''_{\varepsilon,\ell} + \varepsilon\kappa\tilde{\phi}'_{\varepsilon,\ell} + \mathrm{O}(\varepsilon^2)\tilde{\phi}'_{\varepsilon,\ell} + (1-3\bar{u}_0^2)\tilde{\phi}_{\varepsilon,\ell} + 2a\bar{u}_0\tilde{\phi}_{\varepsilon,\ell} \\ = \; - & \; 3\xi_{\ell 1}(1-\bar{u}_0^2)\psi - \xi_{\ell 1}(N_1(\tilde{\phi}_{\varepsilon,\ell}) + N_2(\tilde{\phi}_{\varepsilon,\ell})) - S_\ell(\bar{u}_0) \\ + & \; \psi'' + (\varepsilon\kappa + \mathrm{O}(\varepsilon^2))\psi' + (1 - 3\bar{u}_0^2 + 2a\bar{u}_0)\psi + c_{\ell,\varepsilon}\xi_{\ell 1}H'_\ell, \quad \zeta \in I_\ell, \end{aligned} \tag{54}$$

$$\int_{I_\ell} \xi_{\ell 1} \tilde{\phi}_{\varepsilon,\ell} H'_\ell = 0, \qquad \ell = 1, \ldots, m, \tag{55}$$

Given small $\tilde{\Phi}_{\varepsilon,\ell}$ with $\|\tilde{\Phi}_{\varepsilon,\ell}\|_{H^2(I_\ell)} \leq C\varepsilon \log \frac{1}{\varepsilon}, \ell = 1, \ldots, m$, we solve problem (53) for ψ. Observe that since $|a(x)| < 1$ and $|\bar{u}_0| \leq 1$, we have $\min_{x\in\bar{\Omega}} 2(1 - a\bar{u}_0) > 0$. Then by a fixed point argument we have

$$\begin{aligned} \|\psi\|_{H^2(I_\varepsilon)} &\leq C\left(\varepsilon \log \frac{1}{\varepsilon} + \sum_{\ell=1}^m \|\tilde{\Phi}_{\varepsilon,\ell}\|^2_{H^2(I_\ell)} + \left(\varepsilon + \frac{1}{\log\log \frac{1}{\varepsilon}}\right) \sum_{\ell=1}^m \|\tilde{\Phi}_{\varepsilon,\ell}\|_{H^2(I_\ell)}\right) \\ &\leq C\varepsilon \log \frac{1}{\varepsilon}, \end{aligned} \tag{56}$$

where we have used (42). Next from Remark 1 we can solve (54)-(55) for $\tilde{\phi}_{\varepsilon,\ell}$ which in addition satisfies

$$\|\tilde{\phi}_{\varepsilon,\ell}\|_{H^2(I_\ell)} \leq C\left(\varepsilon \log \frac{1}{\varepsilon} + \|\Psi\|_{H^2(I_\varepsilon)} + \|\tilde{\phi}_{\varepsilon,\ell}\|^2_{H^2(I_\ell)}\right) \qquad \ell = 1, \ldots, m.$$

Combining this with (56), taking ε small, and applying a fixed point argument again we get a solution to (54)-(55) satisfying $\sum_{\ell=1}^m \|\tilde{\phi}_{\varepsilon,\ell}\|_{H^2(I_\ell)} \leq C\varepsilon \log \frac{1}{\varepsilon}, \ell = 1, \ldots, m$. The proof is now complete.

Next we show that we can choose $\mathbf{h} = (h_1, \ldots, h_m)$ such that the coefficients in (47) $\mathbf{c}_\varepsilon := (c_{1,\varepsilon}, \ldots, c_{m,\varepsilon}) = 0$.

Lemma 3.3 *For ε sufficiently small, there exists a solution $u_\varepsilon(\bar{z}, \zeta; g)$ to (35) satisfying*

$$u_\varepsilon(\bar{z}, \zeta; g) = \hat{u}_\varepsilon(\bar{z}, \zeta) + O(\varepsilon^{1+\mu}), \tag{57}$$

in the $\|\cdot\|_*$, where

$$\hat{u}_\varepsilon(\bar{z}, \zeta) = u_0 + \varepsilon \log \frac{1}{\varepsilon}\left[\sum_{\ell=1}^m \xi_{\ell 2} \hat{\varphi}_{\ell,0} + \hat{\psi}\right]. \tag{58}$$

Here for every ℓ, $\hat{\varphi}_{\ell,0}$ satisfies

$$\begin{aligned} \hat{\varphi}''_{\ell,0} + (1 - 3H_\ell^2)\hat{\varphi}_{\ell,0} &= \left(\log \frac{1}{\varepsilon}\right)^{-1} \partial_{\mathbf{n}} a(t + \tilde{f}_\ell)(1 - H_\ell^2) \\ - \ 6\left(\varepsilon \log \frac{1}{\varepsilon}\right)^{-1} &(-1)^{\ell+1}(1 - H_\ell^2)\left[e^{-\sqrt{2}(f_\ell - f_{\ell-1})} e^{-\sqrt{2}t} - e^{-\sqrt{2}(f_{\ell+1} - f_\ell)} e^{\sqrt{2}t}\right], \end{aligned} \tag{59}$$

and $\hat{\psi}$ satisfies

$$\begin{aligned} \hat{\psi}'' - 2(1 - a\bar{u}_0)\hat{\psi} &= \left(1 - \sum_{\ell=1}^m \xi_{\ell 1}\right)\{\left(\log \frac{1}{\varepsilon}\right)^{-1} \partial_{\mathbf{n}} a(t + \tilde{f}_\ell)(1 - H_\ell^2) \\ - \ 6\left(\varepsilon \log \frac{1}{\varepsilon}\right)^{-1} &(-1)^{\ell+1}(1 - H_\ell^2)\left[e^{-\sqrt{2}(f_\ell - f_{\ell-1})} e^{-\sqrt{2}t} - e^{-\sqrt{2}(f_{\ell+1} - f_\ell)} e^{\sqrt{2}t}\right]\}. \end{aligned} \tag{60}$$

Proof. Multiplying (47) by $H'_\ell(\zeta - f_\ell)$ and integrating over I_ℓ we obtain

$$c_{\ell,\varepsilon}\int_{I_\ell}\xi_{\ell 1}(H'_\ell)^2 = \int_{I_\ell} S(u_0)H'_\ell + \int_{I_\ell}[\varphi''_{\varepsilon,\mathbf{h}} + (1-3u_0^2)\varphi_{\varepsilon,\mathbf{h}}]H'_\ell + \mathrm{O}(\varepsilon^{\frac{3}{2}}\log\frac{1}{\varepsilon}), \tag{61}$$

and we have

$$\int_{I_\ell}[\varphi''_{\varepsilon,\mathbf{h}} + (1-3u_0^2)\varphi_{\varepsilon,\mathbf{h}}]H'_\ell = \int_{I_\ell}[H'''_\ell + (1-3u_0^2)H'_\ell]\varphi_{\varepsilon,\mathbf{h}} + \mathrm{O}(\varepsilon^{\frac{3}{2}}\log\frac{1}{\varepsilon}) = \mathrm{O}(\varepsilon^{\frac{3}{2}}\log\frac{1}{\varepsilon}).$$

The left hand side of (61) can be estimated as

$$c_{\ell,\varepsilon}\int_{I_\ell}\xi_{\ell 1}(H'_\ell)^2 = \frac{2\sqrt{2}}{3}c_{\ell,\varepsilon}(1+\mathrm{o}(1)),$$

while for the first term in the right hand side we can use (36) to obtain

$$\begin{aligned}
&\int_{I_\ell} S(u_0)H'_\ell = \varepsilon\kappa\int_{I_\ell}(H'_\ell)^2 - \varepsilon\partial_{\mathbf{n}}a\int_{I_\ell}(t+f_\ell)(1-H^2)H'_\ell \\
+ \;\; & 6(-1)^{\ell+1}\int_{I_\ell}H'_\ell(1-H_\ell^2)\left[e^{-\sqrt{2}(f_\ell - f_{\ell-1})}e^{-\sqrt{2}t} - e^{-\sqrt{2}(f_{\ell+1}-f_\ell)}e^{\sqrt{2}t}\right] + \mathrm{O}(\varepsilon^{1+\mu}) \\
= \;\; & 16\left[e^{-\sqrt{2}(f_\ell - f_{\ell-1})} - e^{-\sqrt{2}(f_{\ell+1}-f_\ell)}\right] + \frac{2\sqrt{2}}{3}\varepsilon\kappa - \frac{4}{3}\varepsilon\partial_{\mathbf{n}}a(-1)^{\ell+1}f_\ell + \mathrm{O}(\varepsilon^{1+\mu}),
\end{aligned} \tag{62}$$

where we have used (9) and (11). Hence we obtain, for $2 \le \ell \le m-1$

$$\begin{aligned}
16 \;\; & \left[e^{-\sqrt{2}(f_\ell - f_{\ell-1})} - e^{-\sqrt{2}(f_{\ell+1}-f_\ell)}\right] - \frac{4}{3}\varepsilon\partial_{\mathbf{n}}a(-1)^{\ell+1}f_\ell \\
+ \;\; & \frac{2\sqrt{2}}{3}\varepsilon\kappa = \frac{2\sqrt{2}}{3}c_{\ell,\varepsilon}(1+\mathrm{o}(1)) + \mathrm{O}(\varepsilon^{1+\mu}).
\end{aligned} \tag{63}$$

Similarly, for $\ell = 1$ and $\ell = m$, we can get respectively

$$-16e^{-\sqrt{2}(f_2-f_1)} - \frac{4}{3}\varepsilon\partial_{\mathbf{n}}af_1 + \frac{2\sqrt{2}}{3}\varepsilon\kappa = \frac{2\sqrt{2}}{3}c_{1,\varepsilon}(1+\mathrm{o}(1)) + \mathrm{O}(\varepsilon^{1+\mu}), \tag{64}$$

$$16e^{-\sqrt{2}(f_m-f_{m-1})} - \frac{4}{3}\varepsilon\partial_{\mathbf{n}}af_m + \frac{2\sqrt{2}}{3}\varepsilon\kappa = \frac{2\sqrt{2}}{3}c_{m,\varepsilon}(1+\mathrm{o}(1)) + \mathrm{O}(\varepsilon^{1+\mu}). \tag{65}$$

From (63)-(65), we derive that $(c_{1,\varepsilon},\dots,c_{m,\varepsilon}) = 0$ if and only if the following system hold

$$\begin{cases}
-16e^{-\sqrt{2}(f_2-f_1)} - \frac{4}{3}\varepsilon\partial_{\mathbf{n}}af_1 + \frac{2\sqrt{2}}{3}\varepsilon\kappa = \mathrm{O}(\varepsilon^{1+\mu}), \\
16\left[e^{-\sqrt{2}(f_\ell - f_{\ell-1})} - e^{-\sqrt{2}(f_{\ell+1}-f_\ell)}\right] - \frac{4}{3}\varepsilon\partial_{\mathbf{n}}a(-1)^{\ell+1}f_\ell + \frac{2\sqrt{2}}{3}\varepsilon\kappa \\
\qquad\qquad\qquad\qquad = \mathrm{O}(\varepsilon^{1+\mu}),\ 2 \le \ell \le m-1, \\
16e^{-\sqrt{2}(f_m-f_{m-1})} - \frac{4}{3}\varepsilon\partial_{\mathbf{n}}af_m + \frac{2\sqrt{2}}{3}\varepsilon\kappa = \mathrm{O}(\varepsilon^{1+\mu}).
\end{cases} \tag{66}$$

Substituting (27) into (66) we obtain

$$\begin{cases} -16be^{-\sqrt{2}(\tilde{f}_2-\tilde{f}_1)} - \frac{4}{3}\varepsilon\partial_{\mathbf{n}}a\tilde{f}_1 = \mathrm{O}(\varepsilon^{1+\mu}), \\ 16\left[b^{(-1)^\ell}e^{-\sqrt{2}(\tilde{f}_\ell-\tilde{f}_{\ell-1})} - b^{(-1)^{\ell+1}}e^{-\sqrt{2}(\tilde{f}_{\ell+1}-\tilde{f}_\ell)}\right] - \frac{4}{3}\varepsilon\partial_{\mathbf{n}}a(-1)^{\ell+1}\tilde{f}_\ell \\ \qquad\qquad = \mathrm{O}(\varepsilon^{1+\mu}),\ 2\le\ell\le m-1, \\ 16b^{(-1)^m}e^{-\sqrt{2}(\tilde{f}_m-\tilde{f}_{m-1})} - \frac{4}{3}\varepsilon\partial_{\mathbf{n}}a\tilde{f}_m = \mathrm{O}(\varepsilon^{1+\mu}), \end{cases} \tag{67}$$

where

$$b := e^{\frac{2\kappa}{\partial_{\mathbf{n}}a}}.$$

We add all equations in (67) and obtain

$$\tilde{f}_1 - \tilde{f}_2 + \tilde{f}_3 - \cdots + (-1)^{\ell+1}\tilde{f}_\ell + \cdots + \tilde{f}_m = \mathrm{O}(\varepsilon^\mu). \tag{68}$$

Combining this with (28), to find $\tilde{f}_\ell, \ell = 1, \ldots, m$ (hence f_ℓ from (27)), we only need to find $h_\ell, \ell = 1, \ldots, m-1$. To this end, we add every adjoint two equations in (67) and get

$$\begin{cases} -16b^{-1}e^{-\sqrt{2}(\tilde{f}_3-\tilde{f}_2)} - \frac{4}{3}\varepsilon\partial_{\mathbf{n}}a(-1)^{2+1}(\tilde{f}_2-\tilde{f}_1) = \mathrm{O}(\varepsilon^{1+\mu}), \\ 16b^{(-1)^\ell}\left[e^{-\sqrt{2}(\tilde{f}_\ell-\tilde{f}_{\ell-1})} - e^{-\sqrt{2}(\tilde{f}_{\ell+2}-\tilde{f}_{\ell+1})}\right] - \frac{4}{3}\varepsilon\partial_{\mathbf{n}}a(-1)^{\ell+2}(\tilde{f}_{\ell+1}-\tilde{f}_\ell) \\ \qquad\qquad = \mathrm{O}(\varepsilon^{1+\mu}),\ 2\le\ell\le m-2, \\ 16be^{-\sqrt{2}(\tilde{f}_{m-1}-\tilde{f}_{m-2})} - \frac{4}{3}\varepsilon\partial_{\mathbf{n}}a(-1)^{m+1}(\tilde{f}_m-\tilde{f}_{m-1}) = \mathrm{O}(\varepsilon^{1+\mu}). \end{cases} \tag{69}$$

Substituting (28) into (69) and using (29) we obtain

$$\begin{cases} -e^{-\sqrt{2}h_2} - (-1)^{2+1} - (-1)^{2+1}\frac{h_1}{\rho_{\varepsilon,2}} = \mathrm{o}(\varepsilon^\mu), \\ e^{-\sqrt{2}h_{\ell-1}} - e^{-\sqrt{2}h_{\ell+1}} - (-1)^{\ell+2} - (-1)^{\ell+2}\frac{h_\ell}{\rho_{\varepsilon,\ell-1}} = \mathrm{o}(\varepsilon^\mu),\ 2\le\ell\le m-2, \\ e^{-\sqrt{2}h_{m-2}} - (-1)^{m+1} - (-1)^{m+1}\frac{h_{m-1}}{\rho_{\varepsilon,m-2}} = \mathrm{o}(\varepsilon^\mu), \end{cases} \tag{70}$$

where we have used (30) and (31).

We write the $(m-1)\times(m-1)$ matrix

$$\mathbf{A} = \begin{bmatrix} 0 & -1 & 0 & 0 & & & \cdots & 0 \\ 1 & 0 & -1 & 0 & & & \cdots & 0 \\ 0 & 1 & 0 & -1 & & & \cdots & 0 \\ 0 & 0 & 1 & 0 & & & \cdots & 0 \\ \vdots & & & & \ddots & \ddots & & \vdots \\ 0 & \cdots & & & \ddots & 0 & -1 & 0 \\ 0 & \cdots & & & & 1 & 0 & -1 \\ 0 & \cdots & & & & 0 & 1 & 0 \end{bmatrix}$$

and denote

$$\bar{\mathbf{h}} = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_{m-1} \end{pmatrix}, \qquad \mathbf{a} = \begin{pmatrix} (-1)^{1+2} \\ (-1)^{2+2} \\ \vdots \\ (-1)^{m-1+2} \end{pmatrix}, \qquad \mathbf{f}(\bar{\mathbf{h}}) = \begin{pmatrix} (-1)^{1+2} \frac{h_1}{\rho_{\varepsilon,2}} \\ (-1)^{2+2} \frac{h_2}{\rho_{\varepsilon,1}} \\ \vdots \\ (-1)^{m-1+2} \frac{h_{m-1}}{\rho_{\varepsilon,m-2}} \end{pmatrix}.$$

Furthermore, we set

$$\mathbb{T}(\bar{\mathbf{h}}) = \mathbf{A} \begin{bmatrix} e^{-\sqrt{2}h_1} \\ e^{-\sqrt{2}h_2} \\ \vdots \\ e^{-\sqrt{2}h_{m-1}} \end{bmatrix}.$$

Then (70) can be written as

$$\mathbb{T}(\bar{\mathbf{h}}) - \mathbf{a} - \mathbf{f}(\bar{\mathbf{h}}) = \mathrm{o}(\varepsilon^{\mu}). \tag{71}$$

For matrix $\mathbf{A}$, if we denote

$$\mathbf{B} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{F} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \tag{72}$$

then

$$\mathbf{A} = \begin{bmatrix} \mathbf{D} & \mathbf{B} & 0 & 0 & & & \cdots & 0 \\ \mathbf{F} & \mathbf{D} & \mathbf{B} & 0 & & & \cdots & 0 \\ 0 & \mathbf{F} & \mathbf{D} & \mathbf{B} & & & \cdots & 0 \\ 0 & 0 & \mathbf{F} & \mathbf{D} & & & \cdots & 0 \\ \vdots & & & & \ddots & \ddots & & \vdots \\ 0 & \cdots & & & \ddots & \mathbf{D} & \mathbf{B} & 0 \\ 0 & \cdots & & & & \mathbf{F} & \mathbf{D} & \mathbf{B} \\ 0 & \cdots & & & & 0 & \mathbf{F} & \mathbf{D} \end{bmatrix},$$

Elementary calculations show that

$$\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{D}^{-1} & \mathbf{F} & \mathbf{F} & \mathbf{F} & & & \cdots & \mathbf{F} \\ \mathbf{B} & \mathbf{D}^{-1} & \mathbf{F} & \mathbf{F} & & & \cdots & \mathbf{F} \\ \mathbf{B} & \mathbf{B} & \mathbf{D}^{-1} & \mathbf{F} & & & \cdots & \mathbf{F} \\ \mathbf{B} & \mathbf{B} & \mathbf{B} & \mathbf{D}^{-1} & & & \cdots & \mathbf{F} \\ \vdots & & & & \ddots & \ddots & & \vdots \\ \mathbf{B} & \cdots & & & \ddots & \mathbf{D}^{-1} & \mathbf{F} & \mathbf{F} \\ \mathbf{B} & \cdots & & & & \mathbf{B} & \mathbf{D}^{-1} & \mathbf{F} \\ \mathbf{B} & \cdots & & & & \mathbf{B} & \mathbf{B} & \mathbf{D}^{-1} \end{bmatrix}, \tag{73}$$

where

$$\mathbf{D^{-1}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{74}$$

We introduce the norm

$$\|\bar{\mathbf{h}}\|_\infty := \max_{1\le i\le m-1} |h_i|.$$

For a given $\mathbf{b} \in \mathbb{R}^{m-1}$ we first solve the problem

$$\mathbb{T}(\bar{\mathbf{h}}) - \mathbf{a} - \mathbf{f}(\mathbf{b}) = 0. \tag{75}$$

Note that

$$\|\mathbf{f}(\mathbf{b})\|_\infty = \mathrm{O}(\frac{1}{\log\frac{1}{\varepsilon}}).$$

By this and (72)-(74), we know that (75) exists a unique solution

$$e^{-\sqrt{2}h_{2j+1}} = \frac{m-1}{2} - j + \mathrm{O}(\frac{1}{\log\frac{1}{\varepsilon}}), \qquad e^{-\sqrt{2}h_{2j+2}} = j+1+\mathrm{O}(\frac{1}{\log\frac{1}{\varepsilon}}), \qquad 0\le j\le \frac{m-3}{2}.$$

Hence

$$h_{2j+1} = -\frac{\sqrt{2}}{2}\log(\frac{m-1}{2} - j) + \mathrm{O}(\frac{1}{\log\frac{1}{\varepsilon}}), \qquad h_{2j+2} = -\frac{\sqrt{2}}{2}\log(j+1) + \mathrm{O}(\frac{1}{\log\frac{1}{\varepsilon}}).$$

We denote

$$\bar{\mathbf{h}} = \mathbb{T}^{-1}(\mathbf{a} + \mathbf{f}(\mathbf{b})).$$

Then solving problem (71) is equivalent to solving the following fixed point problem

$$\bar{\mathbf{h}} = \mathbb{T}^{-1}(\mathbf{a} + \mathbf{f}(\bar{\mathbf{h}}) + \mathrm{o}(\varepsilon^\mu)) =: \mathbb{G}(\bar{\mathbf{h}}). \tag{76}$$

Clearly, for sufficiently large $M > 0$, $\mathbb{G}$ is a contraction operator in the set $\{\bar{\mathbf{h}} : \ \|\bar{\mathbf{h}}\|_\infty \le M\}$. Indeed, we have

$$\|\mathbb{G}(\bar{\mathbf{h}}^1) - \mathbb{G}(\bar{\mathbf{h}}^2)\|_\infty \le \frac{C}{\log\frac{1}{\varepsilon}}\|\bar{\mathbf{h}}^1 - \bar{\mathbf{h}}^2\|_\infty.$$

Hence the contraction mapping principle shows that problem (76) exists a solution $\bar{\mathbf{h}}$.

To show that u_ε has the expansion (57), we use the equation satisfied by $\varphi_{\varepsilon,\mathbf{h}}$. Let $\varphi_{\varepsilon,\mathbf{h}} = \varepsilon\log\frac{1}{\varepsilon}\left[\sum_{\ell=1}^m \xi_{\ell 2}\hat{\varphi}_{\ell,0} + \hat{\psi}\right] + \mathrm{O}(\varepsilon^{1+\mu})$. By (51), (53) and (36), we deduce that $\hat{\varphi}_{\ell,0}$ and $\hat{\psi}$ satisfy respectively

$$\begin{aligned} &\hat{\varphi}_{\ell,0}'' + (1-3H_\ell^2)\hat{\varphi}_{\ell,0} \\ = \ &\left(\log\frac{1}{\varepsilon}\right)^{-1}\partial_{\mathbf{n}} a(t+f_\ell)(1-H^2(t)) - \left(\log\frac{1}{\varepsilon}\right)^{-1}\kappa(-1)^{\ell+1}H'(t) \\ - \ &6\left(\varepsilon\log\frac{1}{\varepsilon}\right)^{-1}(-1)^{\ell+1}(1-H^2(t))\left[e^{-\sqrt{2}(f_\ell - f_{\ell-1})}e^{-\sqrt{2}t} - e^{-\sqrt{2}(f_{\ell+1}-f_\ell)}e^{\sqrt{2}t}\right], \end{aligned}$$

and

$$\hat{\psi}'' - 2(1 - a\bar{u}_0)\hat{\psi} = \left(1 - \sum_{\ell=1}^{m} \xi_{\ell 1}\right) \{ \left(\log \frac{1}{\varepsilon}\right)^{-1} \partial_{\mathbf{n}} a(t + f_\ell)(1 - H_\ell^2)$$

$$- \left(\log \frac{1}{\varepsilon}\right)^{-1} \kappa(-1)^{\ell+1} H'(t)$$

$$- \quad 6\left(\varepsilon \log \frac{1}{\varepsilon}\right)^{-1} (-1)^{\ell+1}(1 - H_\ell^2) \left[e^{-\sqrt{2}(f_\ell - f_{\ell-1})} e^{-\sqrt{2}t} - e^{-\sqrt{2}(f_{\ell+1} - f_\ell)} e^{\sqrt{2}t}\right] \}.$$

These and (27) yield (57). We complete the proof of this lemma.

Using the solution u_ε obtained in the previous lemma, we can define the operator

$$\mathbb{L}(\phi) := \phi_{\zeta\zeta} + \varepsilon\kappa(\varepsilon z)\phi_\zeta + \mathrm{O}(\varepsilon^2)\phi_\zeta + (1 - 3u_\varepsilon^2)\phi + 2a(\varepsilon x)u_\varepsilon\phi.$$

Lemma 3.4 *The solution u_ε constructed in Lemma 3.3 is unique. Indeed, the eigenvalues for the following problem*

$$\mathbb{L}(\phi_{\ell,0}) + \lambda_{\ell,\varepsilon}\phi_{\ell,0} = 0 \tag{77}$$

satisfy

$$\lambda_{\ell,\varepsilon} = -\varepsilon \log \frac{1}{\varepsilon} \gamma_\ell \partial_{\mathbf{n}} a(1 + o(1)) \ (\ell = 1, \ldots, m), \quad \lambda_{m+1,\varepsilon} \geq \gamma_{m+1} > 0, \tag{78}$$

for some positive constants γ_ℓ, γ_{m+1}. Furthermore, if

$$\mathbb{L}(\phi) = \psi, \tag{79}$$

then we have

$$\phi = \sum_{\ell=1}^{m} c_{\ell,\varepsilon} H_\ell' + \phi^\perp, \tag{80}$$

where

$$\|\phi^\perp\|_* = O(\|\psi\|_*), \quad \sum_{\ell=1}^{m} |c_{\ell,\varepsilon}| = \frac{1}{\varepsilon \log \frac{1}{\varepsilon}} O\left(\sum_{\ell=1}^{m} \left| \int_{I_\varepsilon} \psi H_\ell' \right| \right), \tag{81}$$

hence

$$\|\phi\|_* \leq \frac{C}{\varepsilon \log \frac{1}{\varepsilon}} \|\mathbb{L}\phi\|_*. \tag{82}$$

Proof. We first show (78). Let $(\lambda_{\ell,\varepsilon}, \phi_{\ell,0})$ satisfy (77). By Lemma 2.1 it is easy to see that either $\lambda_{\ell,\varepsilon} \to 0$, or $\lambda_{\ell,\varepsilon} \geq \gamma > 0$. We discuss the first case decomposing $\phi_{\ell,0}$ as

$$\phi_{\ell,0} = c_{\ell,\varepsilon} H_\ell' + \phi_{\ell,0}^\perp, \qquad \int_{I_\varepsilon} \phi_{\ell,0}^\perp H_\ell' = 0. \tag{83}$$

Then we have

$$\mathbb{L}(\phi_{\ell,0}^{\perp}) + \lambda_{\ell,\varepsilon}\phi_{\ell,0}^{\perp} = -c_{\ell,\varepsilon}\mathbb{L}(H_\ell') - c_{\ell,\varepsilon}\lambda_{\ell,\varepsilon}H_\ell', \tag{84}$$

where

$$\mathbb{L}(H_\ell') = 3(H_\ell^2 - u_\varepsilon^2)H_\ell' + \varepsilon\kappa H_\ell'' + 2au_\varepsilon H_\ell' + \mathrm{O}(\varepsilon^{\frac{3}{2}}).$$

Since $\lambda_{\ell,\varepsilon} \to 0$ and $\int_{I_\varepsilon} \phi_{\ell,0}^{\perp} H_\ell' = 0$, from Lemma 2.1 we obtain that

$$\|\phi_{\ell,0}^{\perp}\|_* \le C|c_{\ell,\varepsilon}| \left(\varepsilon \log\frac{1}{\varepsilon} + |\lambda_{\ell,\varepsilon}|\right). \tag{85}$$

Now multiplying (84) by H_ℓ' , $\ell = 1, \ldots, m$, respectively and integrating over I_ε, we have

$$\int_{I_\varepsilon} \mathbb{L}(\phi_{\ell,0}^{\perp})H_\ell' = -c_{\ell,\varepsilon}\left[\int_{I_\varepsilon} \mathbb{L}(H_\ell')H_\ell' + \lambda_{\ell,\varepsilon}\int_{I_\varepsilon}(H_\ell')^2\right]. \tag{86}$$

For the left-hand side, we have

$$\begin{aligned} \int_{I_\varepsilon} \mathbb{L}(\phi_{\ell,0}^{\perp})H_\ell' &= \int_{I_\varepsilon} [H_\ell''' + (1-3u_\varepsilon^2)H_\ell']\phi_{\ell,0}^{\perp} + \mathrm{O}\left(\varepsilon\left(\varepsilon\log\frac{1}{\varepsilon} + |\lambda_{\ell,\varepsilon}|\right)\right) \\ &= \int_{I_\varepsilon} 3[H_\ell^2 - u_\varepsilon^2]H_\ell'\phi_{\ell,0}^{\perp} + \mathrm{O}\left(\varepsilon\left(\varepsilon\log\frac{1}{\varepsilon} + |\lambda_{\ell,\varepsilon}|\right)\right) \\ &= \mathrm{O}\left(\varepsilon\left(\varepsilon\log\frac{1}{\varepsilon} + |\lambda_{\ell,\varepsilon}|\right)\right), \end{aligned} \tag{87}$$

while for the the first integral of the right-hand side we have

$$\begin{aligned} \int_{I_\varepsilon} \mathbb{L}(H_\ell')H_\ell' &= \int_{I_\varepsilon} 3(H_\ell^2 - u_\varepsilon^2)(H_\ell')^2 + \mathrm{O}(\varepsilon) \\ &= -6\int_{\mathbb{R}} (-1)^{\ell-1}H(t)[(-1)^{\ell-2}(H(t+f_\ell - f_{\ell-1}) - 1) \\ &\quad +(-1)^\ell(H(t+f_\ell - f_{\ell+1}) + 1)](H'(t))^2dt + \mathrm{O}(\varepsilon) \\ &= -6e^{-\sqrt{2}(f_\ell - f_{\ell-1})}\int_{\mathbb{R}} H(t)(H'(t))^2 e^{-\sqrt{2}t}dt \\ &\quad +6e^{-\sqrt{2}(f_{\ell+1}-f_\ell)}\int_{\mathbb{R}} H(t)(H'(t))^2 e^{\sqrt{2}t}dt + \mathrm{O}(\varepsilon) \\ &=: \varepsilon\log\frac{1}{\varepsilon}\tilde{\gamma}_\ell \partial_{\mathbf{n}} a(1+\mathrm{o}(1)) + \mathrm{O}(\varepsilon). \end{aligned} \tag{88}$$

Note that

$$\tilde{\gamma}_\ell > 0,$$

since

$$\int_{\mathbb{R}} H(t)(H'(t))^2 e^{-\sqrt{2}t}dt < 0 \ \text{ and } \ \int_{\mathbb{R}} H(t)(H'(t))^2 e^{\sqrt{2}t}dt > 0.$$

Clearly

$$\lambda_{\ell,\varepsilon}\int_{I_\varepsilon}(H_\ell')^2=\lambda_{\ell,\varepsilon}\left(\frac{2\sqrt{2}}{3}+\mathrm{o}(1)\right). \tag{89}$$

From (86)-(89) we obtain (78), where $\gamma_\ell=\frac{3}{2\sqrt{2}}\tilde{\gamma}_\ell>0$. The proof of (80), (81) follows from similar argument. The uniqueness of u_ε can be deduced from (78). We complete the proof of this lemma.

By using Lemma 3.4 we can obtain the following estimates.

Lemma 3.5 *If $\|g\|_{*,l}\le C$ for some integer l, then*

$$\|u_\varepsilon(\bar{z},\zeta;g)\|_{*,l}\le C. \tag{90}$$

Proof. We only consider the simplest case: $D^\alpha_{\bar{z}_1}=\frac{\partial}{\partial\bar{z}_1}$, since the higher-order derivatives case can be deal with similarly. Differentiating (35) with respect to $\bar{z}_1$ and letting $v:=D^\alpha_{\bar{z}_1}u_\varepsilon(\bar{z},\zeta;g)$, we have

$$\mathbb{L}v+\varepsilon D^\alpha_{\bar{z}_1}\kappa(\bar{z})u_{\varepsilon,\zeta}+D^\alpha_{\bar{z}_1}a(\varepsilon x)(1-u_\varepsilon^2)+\mathrm{O}(\varepsilon^2)=0$$

in the norm $\|\cdot\|_{*,l-1}$. By (82) and the fact that $D^\alpha_{\bar{z}_1}a(\varepsilon x)=\mathrm{O}(\varepsilon\log\frac{1}{\varepsilon})$, (90) follows immediately.

Lemma 3.6 *If $\|g_i\|_*\le C, i=1,2$ and if $u_\varepsilon(\bar{z},\zeta;g_i)$ are the corresponding solutions of (35), then we have the following estimate*

$$\|u_\varepsilon(\bar{z},\zeta;g_1)-u_\varepsilon(\bar{z},\zeta;g_2)\|_*\le C\varepsilon\|g_1-g_2\|_*. \tag{91}$$

More precisely, following the notations in the proof of Lemma 3.4, the following estimate holds true

$$u_\varepsilon(\bar{z},\zeta;g_1)-u_\varepsilon(\bar{z},\zeta;g_2)=\sum_{\ell=1}^m d_{\ell,0}H_\ell'+\psi_0, \tag{92}$$

where

$$\sum_{\ell=1}^m|d_{\ell,0}|=O(\varepsilon\|g_1-g_2\|_*),\quad \|\psi_0\|_*=O(\varepsilon^2\|g_1-g_2\|_*). \tag{93}$$

Proof. Let $w=u_\varepsilon(\bar{z},\zeta;g_1)-u_\varepsilon(\bar{z},\zeta;g_2)$. Then by (57) we have $\|w\|_*=\mathrm{O}(\varepsilon)$ and

$$\mathbb{L}^{(2)}w-3u_\varepsilon(\bar{z},\zeta;g_2)w^2+a(\varepsilon x)w^2+\mathrm{O}(\|w\|_*^3)+\varepsilon^2(g_1-g_2)=0$$

in the norm $\|\cdot\|_*$, where $\mathbb{L}^{(2)}w=w_{\zeta\zeta}+\varepsilon\kappa w_\zeta+\mathrm{O}(\varepsilon^2)w_\zeta+(1-3u_\varepsilon(\bar{z},\zeta;g_2)^2)w+2a(\varepsilon x)u_\varepsilon(\bar{z},\zeta;g_2)w$.

By (80), (81), we have

$$\|\psi_0\|_*=\mathrm{O}(\varepsilon^2\|g_1-g_2\|_*)$$

and

$$
\begin{aligned}
&\sum_{\ell=1}^{m}|d_{\ell,0}|\\
&=\frac{1}{\varepsilon\log\frac{1}{\varepsilon}}\mathrm{O}\left(\sum_{j=1}^{m}\left|\int_{I_\varepsilon}(a(\varepsilon x)-3u_\varepsilon(\bar z,\zeta;g_2))\left[\sum_{\ell=1}^{m}d_{\ell,0}H'_\ell+\psi_0\right]^2H'_j\right|\right)+\mathrm{O}(\varepsilon\|g_1-g_2\|_*).
\end{aligned}
$$

Observe that $a=\mathrm{O}(\varepsilon\log\frac{1}{\varepsilon})$ near f_ℓ and $\int_{\mathbb{R}}H(H')^3=0$. The similar argument as in the proof of Lemma 3.4 yields (93).

Lemma 3.7 *If $\|g_i\|_*\le C, i=1,2$ and $u_\varepsilon(\bar z,\zeta;g_i)$ are as in the previous lemma, then the following estimate holds true*

$$\|u_\varepsilon(\bar z,\zeta;g_1)-u_\varepsilon(\bar z,\zeta;g_2)\|_{*,l}\le C\varepsilon(\|g_1-g_2\|_*+\|g_1-g_2\|_{*,l}). \tag{94}$$

More precisely, for any multi-index α with $|\alpha|\le l$, we have

$$D^\alpha_{\bar z}(u_\varepsilon(\bar z,\zeta;g_1)-u_\varepsilon(\bar z,\zeta;g_2))=\sum_{\ell=1}^{m}d_{\ell,\alpha}H'_\ell+\psi_\alpha, \tag{95}$$

where

$$\sum_{\ell=1}^{m}|d_{\ell,\alpha}| = O(\varepsilon(\|g_1-g_2\|_*+\|g_1-g_2\|_{*,l})), \tag{96}$$

$$\|\psi_\alpha\|_* = O\left(\varepsilon^2(\|g_1-g_2\|_*+\|g_1-g_2\|_{*,l})\right). \tag{97}$$

Proof. As before, we set $w=u_\varepsilon(\bar z,\zeta;g_1)-u_\varepsilon(\bar z,\zeta;g_2)$. Then $D_{\bar z}w$ satisfies

$$
\begin{aligned}
&\mathbb{L}^{(2)}D_{\bar z}w+\varepsilon D_{\bar z}\kappa w_\zeta+\mathrm{O}(\varepsilon^2)w_\zeta-6u_\varepsilon(\bar z,\zeta;g_2)wD_{\bar z}w-3w^2D_{\bar z}u_\varepsilon(\bar z,\zeta;g_2)\\
&-6u_\varepsilon(\bar z,\zeta;g_2)D_{\bar z}u_\varepsilon(\bar z,\zeta;g_2)w+D_{\bar z}a(\varepsilon x)w^2+2awD_{\bar z}w\\
&+2D_{\bar z}au_\varepsilon(\bar z,\zeta;g_2)w+2aD_{\bar z}u_\varepsilon(\bar z,\zeta;g_2)w+\mathrm{O}(\|w\|_*^2)D_{\bar z}w+\varepsilon^2D_{\bar z}(g_1-g_2)=0.
\end{aligned}
$$

As before, we decompose $D_{\bar z}w$ as

$$D_{\bar z}w=\sum_{\ell=1}^{m}d_{\ell,1}H'_\ell+\psi_1.$$

The same argument as in Lemma 3.4 gives (96), (97). By induction in the length of α, we obtain the desired estimate.

From the results in Lemmas 3.3-3.7, we have obtained the following Theorem.

Theorem 3.1 *Assume*

$$\|g(\bar{z},\zeta)\|_{*,l} < C, \quad l \in \mathbb{N}. \tag{98}$$

Then there exists $\varepsilon_0 > 0$ *such that for* $\varepsilon < \varepsilon_0$ *and* g *satisfying (98), there exists a unique solution* $u_\varepsilon(\bar{z},\zeta;g)$ *to the problem (35), which satisfies*

$$u_\varepsilon(\bar{z},\zeta;g) = \hat{u}_\varepsilon(\bar{z},\zeta) + O(\varepsilon^{1+\mu}),$$

in the $\|\cdot\|_*$, *where*

$$\hat{u}_\varepsilon(\bar{z},\zeta) = u_0 + \varepsilon \log\frac{1}{\varepsilon}\left[\sum_{\ell=1}^{m} \xi_{\ell 2}\hat{\varphi}_{\ell,0} + \hat{\psi}\right].$$

The functions $\hat{\varphi}_{\ell,0}$ *and* $\hat{\psi}$ *satisfy respectively (59) and (60).*

Moreover, we have

$$\|u_\varepsilon(\bar{z},\zeta;g)\|_{*,l} \le C,$$

and if g_1, g_2 *satisfy (98), then*

$$\|u_\varepsilon(\bar{z},\zeta;g_1) - u_\varepsilon(\bar{z},\zeta;g_2)\|_{*,l} \le C\varepsilon(\|g_1-g_2\|_* + \|g_1-g_2\|_{*,l}).$$

By Theorem 3.1, using an iteration procedure, we can easily obtain the main result of this section, concerning existence of approximate solutions to (15).

Theorem 3.2 *For each fixed integer* $J \ge 3$, *there exists an approximate solution* u_ε^J *satisfying (57) and*

$$\|u^J_{\zeta\zeta} + \varepsilon\kappa(\varepsilon z)u^J_\zeta + \varepsilon^2\Delta_{K_{\varepsilon\zeta}}u^J + O(\varepsilon^2)u^J_\zeta + u(1-u^2) - a(\varepsilon x)(1-(u^J)^2)\|_{*,2} \le C\varepsilon^J. \tag{99}$$

Proof. We set

$$u^2_\varepsilon(\bar{z},\zeta) := u_\varepsilon(\bar{z},\zeta;0), \quad g_2 := 0,$$

and

$$u^j_\varepsilon(\bar{z},\zeta) := u_\varepsilon(\bar{z},\zeta;g_j), \quad g_j := -\Delta_{K_{\varepsilon\zeta}}u^{j-1}_\varepsilon,$$

where $j = 3,\dots,J$.

We first consider the case $J=3$. Observe that u^2_ε satisfies

$$u^2_{\zeta\zeta} + \varepsilon\kappa(\varepsilon z)u^2_\zeta + \mathrm{O}(\varepsilon^2)u^2_\zeta + u^2(1-(u^2)^2) - a(\varepsilon x)(1-(u^2)^2) = 0,$$

while u^3_ε satisfies

$$u^3_{\zeta\zeta} + \varepsilon\kappa(\varepsilon z)u^3_\zeta + \mathrm{O}(\varepsilon^2)u^3_\zeta + u^3(1-(u^3)^2) - a(\varepsilon x)(1-(u^3)^2) + \varepsilon^2\Delta_{K_{\varepsilon\zeta}}u^2_\varepsilon = 0.$$

By (90), for any $l \in \mathbb{N}$ we have

$$\|u^2_\varepsilon\|_{*,l} \le C,$$

and by (94)

$$\|u^3_\varepsilon - u^2_\varepsilon\|_{*,l-2} \le C\varepsilon,$$

which implies that u^3_ε satisfies

$$\|u^3_{\zeta\zeta} + \varepsilon\kappa(\varepsilon z)u^3_\zeta + \varepsilon^2\Delta_{K_{\varepsilon\zeta}}u^3 + \mathrm{O}(\varepsilon^2)u^3_\zeta + u^3(1-(u^3)^2) - a(\varepsilon x)(1-(u^3)^2)\|_{*,l-4} \le C\varepsilon^3.$$

For $J > 3$ (choosing l in the initial step sufficiently large depending on J), we can prove (99) using an induction argument.

Remark 2 *The approximate solution u^J_ε constructed in Theorem 3.2 is actually unique (since the solution in Theorem 3.1 is unique), and smooth in ε.*

Finally, we consider the dependence of u^J_ε in ε. It is convenient to scale the function u^J_ε to Ω defining $\bar{u}^J_\varepsilon(\varepsilon x) := u^J_\varepsilon(x)$. Then for $J > 2$ the derivative of u^J_ε with respect to ε, namely $v^J_\varepsilon(x) = \frac{\partial \bar{u}^J_\varepsilon}{\partial\varepsilon}(\varepsilon x)$, satisfies

$$\begin{aligned} v^J_{\varepsilon,\zeta\zeta} \quad &+ \quad \varepsilon\kappa(\varepsilon z)v^J_{\varepsilon,\zeta} + \mathrm{O}(\varepsilon^2)v^J_{\varepsilon,\zeta} + (1-3(u^J_\varepsilon)^2)v^J_\varepsilon + 2a(\varepsilon x)u^J_\varepsilon v^J_\varepsilon \\ &+\frac{\partial a}{\partial\varepsilon}(\varepsilon x)((u^J_\varepsilon))^2 - 1) + \frac{2}{\varepsilon}[((u^J_\varepsilon))^3 - u^J_\varepsilon) - a((u^J_\varepsilon))^2 - 1)] = \mathrm{O}(\varepsilon^2), \end{aligned} \tag{100}$$

in the $\|\cdot\|_*$ norm.

Remark 3 *The eigenvalue estimates in Lemma 3.4 also hold when we replace u_ε by u^J_ε. Furthermore, the eigenfunctions $\phi_{\ell,0}, \ell = 1, \ldots, m$ in (77) satisfies regularity estimates similar to those in (90).*

4 Invertibility of the linearized operator

First we need to characterize the eigenfunctions of the linearized equation corresponding to small eigenvalues. We study the eigenfunctions of the operator

$$L_\varepsilon\phi := \mathbb{L}\phi + \Delta_{K_\zeta}\phi$$

corresponding to suitably small eigenvalues. The reason is that in order to apply Theorem 2.1, it is necessary to consider the projection onto the eigenspace of σ_0. Precisely, the eigenvalues of $P_{\sigma_0} \circ \frac{\partial T}{\partial\varepsilon}(0) \circ P_{\sigma_0}$ can be found by using the Rayleigh quotient

$$\rho(u) = \frac{(P_{\sigma_0} \circ \frac{\partial T}{\partial\varepsilon}(0) \circ P_{\sigma_0} u, u)_X}{(u,u)_X}, \qquad u \in X,\ u \neq 0.$$

Lemma 4.1 *Suppose the function ϕ satisfies (see the notation in Lemma 3.4)*

$$L_\varepsilon\phi + \lambda\partial_{\mathbf{n}}a\phi = 0, \qquad \|\phi\|_{L^2(U_\tau)} = 1, \tag{101}$$

with $\lambda = O(\varepsilon\log\frac{1}{\varepsilon})$ as $\varepsilon \to 0$. We decompose

$$\phi = \sum_{\ell=1}^{m} \psi_\ell(z)\phi_{\ell,0}(z,\zeta) + \phi^\perp,$$

where $\phi_{\ell,0}(z,\zeta)$ is the eigenfunctions (normalized in $L^2([-\varepsilon^{-\tau},\varepsilon^{-\tau}])$ with respect to the volume form of g_ε) of $\mathbb{L}$ and where $\phi^\perp$ satisfies

$$\int_{[-\varepsilon^{-\tau},\varepsilon^{-\tau}]} \phi^\perp(z,\zeta)\phi_{\ell,0}(z,\zeta)d\zeta = 0, \quad \forall z \in K_\varepsilon, \quad \ell = 1,\dots,m.$$

Then, as $\varepsilon \to 0$, writing $\psi_\ell(z) = \sum_j \alpha_{\ell,j}\varphi_j(\varepsilon z)$, we have the following estimate

$$\|\phi^\perp\|^2_{H^1(U_\tau)} \le \frac{C}{\varepsilon^{n-1}} \sum_{\ell=1}^m \sum_j \alpha^2_{\ell,j} \left(\varepsilon^4 + \varepsilon^4 j^{\frac{2}{n-1}}\right), \tag{102}$$

for some constant C.

Proof. We multiply the eigenvalue equation in (101) by $\phi^\perp$ and integrate on U_τ. From the definition of $L_\varepsilon = \mathbb{L} + \Delta_{K_\zeta}$ and the uniform invertibility of $\mathbb{L}$ on $\phi^\perp$, see Lemma 3.4 (we are actually substituting $[-\frac{\delta}{\varepsilon},\frac{\delta}{\varepsilon}]$ with $[-\varepsilon^{-\tau},\varepsilon^{-\tau}]$, but this not affects the eigenvalue estimates), we find that

$$\int_{U_\tau} \phi^\perp \mathbb{L}\phi^\perp dV_{g_\varepsilon} \le -C[\|\phi^\perp\|^2_{L^2(U_\tau)} + \|\phi^\perp_\zeta\|^2_{L^2(U_\tau)}]. \tag{103}$$

We also obtain from (23) that

$$-\int_{K_\varepsilon} \phi^\perp \Delta_{K_\zeta}\phi^\perp dV_{g_{K_\zeta}} = (1 + \mathrm{O}(\varepsilon\zeta)) \int_{K_\varepsilon} |\nabla_{\bar{g}_\varepsilon}\phi^\perp|^2 dV_{\bar{g}_\varepsilon}. \tag{104}$$

From (103), (104) and (24) we deduce that

$$\int_{U_\tau} \phi^\perp L_\varepsilon \phi^\perp dV_{g_\varepsilon} \le -C\|\phi^\perp\|^2_{H^1(U_\tau)},$$

and therefore

$$\begin{aligned} C\|\phi^\perp\|^2_{H^1(U_\tau)} \quad &\le \left| \int_{U_\tau} \phi^\perp \sum_{\ell=1}^m (\psi_\ell \mathbb{L}\phi_{\ell,0}) dV_{g_\varepsilon} + \int_{U_\tau} \phi^\perp \sum_{\ell=1}^m (\phi_{\ell,0}\Delta_{K_\zeta}\psi_\ell) dV_{g_\varepsilon} \right| \\ &+ \left| \int_{U_\tau} \phi^\perp \sum_{\ell=1}^m (\psi_\ell \Delta_{K_\zeta}\phi_{\ell,0}) dV_{g_\varepsilon} \right| \\ &+ \left| 2\int_{U_\tau} \phi^\perp \sum_{\ell=1}^m \langle \nabla_{K_\zeta}\psi_\ell, \nabla_{K_\zeta}\phi_{\ell,0}\rangle dV_{g_\varepsilon} \right| + C|\lambda| \|\phi^\perp\|^2_{L^2(U_\tau)}. \end{aligned}$$

From the orthogonality conditions on $\phi^\perp$ and from the fact that these functions $\phi_{\ell,0}, \ell = 1,\dots,m$ are eigenfunctions for $\mathbb{L}$ (up to a small error), the first term on the right-hand side vanishes. Since $\phi_{\ell,0}, \ell = 1, \dots, m$ satisfy a decay estimate with respect to ζ as in (90), from (18) and (22) we obtain the following estimate

$$\|\phi^{\perp}\|_{H^1(U_\tau)} \leq C\varepsilon^2 \sum_{\ell=1}^{m} \|\psi_\ell\|_{L^2(K_\varepsilon)} + C\varepsilon \sum_{\ell=1}^{m} \|\nabla_{K_\zeta}\psi_\ell\|_{L^2(K_\varepsilon)},$$

where we have used the that that $\lambda = \mathrm{O}(\varepsilon \log \frac{1}{\varepsilon})$. By $\psi_\ell(z) = \sum_j \alpha_{\ell,j}\varphi_j(\varepsilon z)$, the asymptotic formula for λ_j and a change of variables we find

$$\int_{K_\varepsilon} |\psi_\ell(z)|^2 dV_{\bar{g}_\varepsilon} \leq C \int_{K_\varepsilon} \partial_{\mathbf{n}} a |\psi_\ell(z)|^2 dV_{\bar{g}_\varepsilon} \leq \frac{C}{\varepsilon^{n-1}} \sum_j \alpha_{\ell,j}^2$$

and

$$\int_{K_\varepsilon} |\nabla_{K_\zeta}\psi_\ell(z)|^2 dV_{\bar{g}_\varepsilon} \leq \frac{C}{\varepsilon^{n-1}} \varepsilon^2 \sum_j j^{\frac{2}{n-1}} \alpha_{\ell,j}^2.$$

Hence (102) follows from the last three formulas.

Lemma 4.2 *Suppose the same assumptions of Lemma 4.1 hold. Then, as* $\varepsilon \to 0$ *we have* $\|\phi^{\perp}\|_{H^1(U_\tau)} = O(\varepsilon^{\frac{3}{2}} \log \frac{1}{\varepsilon})$.

Proof. We rewrite the eigenvalue equation in (101) as

$$\begin{aligned} L_\varepsilon \phi &= \sum_{\ell=1}^{m} [\phi_{\ell,0}\Delta_{K_\zeta}\psi_\ell(z) + \psi_\ell(z)\mathbb{L}\phi_{\ell,0} + \psi_\ell(z)\Delta_{K_\zeta}\phi_{\ell,0} + 2\langle \nabla_{K_\zeta}\psi_\ell(z), \nabla_{K_\zeta}\phi_{\ell,0}\rangle] + L_\varepsilon\phi^{\perp} \\ &= -\lambda\partial_{\mathbf{n}} a \phi^{\perp} - \lambda \partial_{\mathbf{n}} a \sum_{\ell=1}^{m} \psi_\ell(z)\phi_{\ell,0}. \end{aligned}$$

Using the facts that $\mathbb{L}\phi_{\ell,0} = \varepsilon \log \frac{1}{\varepsilon} \gamma_\ell \partial_{\mathbf{n}} a(1 + \mathrm{o}(1))\phi_{\ell,0}$ $(\ell = 1, \dots, m)$, we have

$$\begin{aligned} L_\varepsilon \phi = & \sum_{\ell=1}^{m} [\phi_{\ell,0}(\Delta_{K_\zeta}\psi_\ell(z) + \varepsilon \log \frac{1}{\varepsilon} \gamma_\ell \partial_{\mathbf{n}} a(1 + \mathrm{o}(1))\psi_\ell) + \psi_\ell(z)\Delta_{K_\zeta}\phi_{\ell,0} \\ & +2\langle \nabla_{K_\zeta}\psi_\ell(z), \nabla_{K_\zeta}\phi_{\ell,0}\rangle] + L_\varepsilon\phi^{\perp} = -\lambda\partial_{\mathbf{n}} a\phi^{\perp} - \lambda\partial_{\mathbf{n}} a \sum_{\ell=1}^{m} \psi_\ell(z)\phi_{\ell,0}. \end{aligned} \tag{105}$$

Writing still $\psi_\ell(z) = \sum_j \alpha_{\ell,j}\varphi_j(\varepsilon z)$, we let j_ε be the first integer j such that $\varepsilon^2\lambda_j > \varepsilon$. For each ℓ, we multiply then the last equation by $\sum_{j \geq j_\varepsilon} \alpha_{\ell,j}\varphi_j(\varepsilon z)\phi_{\ell,0}$ respectively and integrate in U_τ, and then sum for $\ell = 1, \dots, m$. Using the orthogonality of $\phi^{\perp}$ to $\phi_{\ell,0}$, the self-adjointness of L_ε and integrating by parts we obtain

$$\frac{1}{\varepsilon^{n-1}} \sum_{\ell=1}^{m} \sum_{j \geq j_\varepsilon} \varepsilon^2 \alpha_{\ell,j}^2 \lambda_j \quad \leq C(\varepsilon \log \frac{1}{\varepsilon} + |\lambda|) \left(\frac{1}{\varepsilon^{n-1}} \sum_{\ell=1}^{m} \sum_{j \geq j_\varepsilon} \alpha_{\ell,j}^2 \right)^{\frac{1}{2}} \left(\frac{1}{\varepsilon^{n-1}} \sum_{\ell=1}^{m} \sum_j \alpha_{\ell,j}^2 \right)^{\frac{1}{2}}$$

$$+C\varepsilon\left(\frac{1}{\varepsilon^{n-1}}\sum_{\ell=1}^{m}\sum_{j}\alpha_{\ell,j}^{2}\right)^{\frac{1}{2}}\left(\frac{1}{\varepsilon^{n-1}}\sum_{\ell=1}^{m}\sum_{j\geq j_\varepsilon}\varepsilon^{2}\alpha_{\ell,j}^{2}\lambda_j\right)^{\frac{1}{2}}$$
$$+\left|\int_{U_\tau}\phi^{\perp}L_\varepsilon\left(\sum_{\ell=1}^{m}\sum_{j\geq j_\varepsilon}\alpha_{\ell,j}\varphi_j\phi_{\ell,0}\right)dV_{g_\varepsilon}\right|.$$

From (105), the last term can be evaluated as

$$\begin{aligned}\left|\int_{U_\tau}\phi^{\perp}L_\varepsilon\left(\sum_{\ell=1}^{m}\sum_{j\geq j_\varepsilon}\alpha_{\ell,j}\varphi_j\phi_{\ell,0}\right)dV_{g_\varepsilon}\right| &\leq C\varepsilon\sum_{\ell=1}^{m}\|\nabla_{K_\zeta}\psi_\ell\|_{L^2(K_\varepsilon)}\|\phi^{\perp}\|_{L^2(U_\tau)}\\ &\leq C\varepsilon\|\phi^{\perp}\|_{L^2(U_\tau)}\left(\frac{1}{\varepsilon^{n-1}}\sum_{\ell=1}^{m}\sum_{j\geq j_\varepsilon}\varepsilon^{2}\alpha_{\ell,j}^{2}\lambda_j\right)^{\frac{1}{2}}.\end{aligned}$$

Hence from the last two formulas and from the fact that $\lambda_j \gg 1$ for $j \geq j_\varepsilon$ we get

$$\left(\frac{1}{\varepsilon^{n-1}}\sum_{\ell=1}^{m}\sum_{j\geq j_\varepsilon}\varepsilon^{2}\alpha_{\ell,j}^{2}\lambda_j\right)^{\frac{1}{2}}\leq C\varepsilon^{\frac{1}{2}}\log\frac{1}{\varepsilon}\left(\left(\frac{1}{\varepsilon^{n-1}}\sum_{\ell=1}^{m}\sum_{j}\alpha_{\ell,j}^{2}\right)^{\frac{1}{2}}+\|\phi^{\perp}\|_{L^2(U_\tau)}\right). \quad (106)$$

We also notice that by the L^2 normalization of ϕ one has

$$\frac{1}{\varepsilon^{n-1}}\sum_{\ell=1}^{m}\sum_{j}\alpha_{\ell,j}^{2}+\|\phi^{\perp}\|_{L^2(U_\tau)}^{2}\leq C.$$

Then from Lemma 4.1, (dividing the j's into $\{j < j_\varepsilon\}$ and $\{j \geq j_\varepsilon\}$), recalling our definition of j_ε and (106) we have

$$\begin{aligned}\|\phi^{\perp}\|_{H^1(U_\tau)} &\leq C\varepsilon^{2}+C\varepsilon^{\frac{3}{2}}+C\varepsilon\left(\frac{1}{\varepsilon^{n-1}}\sum_{\ell=1}^{m}\sum_{j\geq j_\varepsilon}\varepsilon^{2}\alpha_{\ell,j}^{2}\lambda_j\right)^{\frac{1}{2}}\\ &\leq C\varepsilon^{\frac{3}{2}}+C\varepsilon^{\frac{3}{2}}\log\frac{1}{\varepsilon}(1+\|\phi^{\perp}\|_{H^1(U_\tau)}),\end{aligned}$$

which yields the desired result.

From (25) we have

$$\varepsilon^{2}\int_{K}|\nabla_K\varphi_j|^{2}-\varepsilon\log\frac{1}{\varepsilon}\gamma_\ell\int_{K}\partial_{\mathbf{n}}a\varphi_j^{2}=\varepsilon^{2}\lambda_j-\varepsilon\log\frac{1}{\varepsilon}\gamma_\ell=:\lambda_{\ell,j}. \quad (107)$$

Now we differentiate some suitably small eigenvalues of L_ε with respect to the parameter ε. As an application we will obtain the invertibility of L_ε for a quite large family of ε. Then, as in [30], Proposition 7.3, using Kato's theorem one can prove the following result.

Proposition 4.1 *The eigenvalues* λ *of the problem*

$$L_\varepsilon u + \lambda \partial_{\mathbf{n}} a u = 0, \qquad in\ U_\tau \tag{108}$$

are differentiable with respect to ε*, and they satisfy the following estimates*

$$M^1_{\lambda,\varepsilon} \leq \frac{\partial \lambda}{\partial \varepsilon} \leq M^2_{\lambda,\varepsilon}, \tag{109}$$

where

$$M^1_{\lambda,\varepsilon} = \inf_{u \in H_\lambda, u \neq 0} \frac{\int_{U_\tau} (\frac{2}{\varepsilon}|\nabla_{g_\varepsilon} u|^2 + 6u^J_\varepsilon v^J_\varepsilon u^2 - 2a v^J_\varepsilon u^2 - 2\partial_\varepsilon a u^J_\varepsilon u^2) dV_{g_\varepsilon}}{\int_{U_\tau} \partial_{\mathbf{n}} a u^2 dV_{g_\varepsilon}}$$

and

$$M^2_{\lambda,\varepsilon} = \sup_{u \in H_\lambda, u \neq 0} \frac{\int_{U_\tau} (\frac{2}{\varepsilon}|\nabla_{g_\varepsilon} u|^2 + 6u^J_\varepsilon v^J_\varepsilon u^2 - 2a v^J_\varepsilon u^2 - 2\partial_\varepsilon a u^J_\varepsilon u^2) dV_{g_\varepsilon}}{\int_{U_\tau} \partial_{\mathbf{n}} a u^2 dV_{g_\varepsilon}}$$

Lemma 4.3 *Suppose the assumptions of Lemma 4.1 hold, except that we now use the normalization* $\|\phi\|_{H^1(U_\tau)} = 1$*. Then, if* $|\lambda| = O(\varepsilon^{\frac{3}{2}} \log \frac{1}{\varepsilon})$ *we have*

$$\frac{1}{\varepsilon^{n-1}} \sum_{\ell=1}^{m} \sum_{|\lambda_{\ell,j}| \geq \varepsilon^{\frac{5}{4}}} \alpha^2_{\ell,j} = O\left(\varepsilon^{\frac{1}{4}} \log \frac{1}{\varepsilon}\right),$$

and

$$\frac{1}{\varepsilon^{n-1}} \sum_{\ell=1}^{m} \sum_{|\lambda_{\ell,j}| \geq \varepsilon^{\frac{5}{4}}} |\lambda_{\ell,j}| \alpha^2_{\ell,j} = O\left(\varepsilon^{\frac{3}{2}} \log \frac{1}{\varepsilon}\right).$$

Proof. We define the sets

$$E_{\ell,1} := \{j \in \mathbb{N} : \lambda_{\ell,j} < -\varepsilon^{\frac{5}{4}}\}, \qquad E_{\ell,2} := \{j \in \mathbb{N} : \lambda_{\ell,j} > \varepsilon^{\frac{5}{4}}\},$$

and the functions

$$\bar{\psi}_{\ell,1}(z) = \sum_{j \in E_{\ell,1}} \alpha_{\ell,j} \varphi_j(\varepsilon z), \qquad \bar{\psi}_{\ell,2}(z) = \sum_{j \in E_{\ell,2}} \alpha_{\ell,j} \varphi_j(\varepsilon z),$$

$$\phi_1 = \sum_{\ell=1}^{m} \bar{\psi}_{\ell,1}(z) \phi_{\ell,0}, \qquad \phi_2 = \sum_{\ell=1}^{m} \bar{\psi}_{\ell,2}(z) \phi_{\ell,0}.$$

As one can easily see from the orthogonality of $\bar{\psi}_{\ell,1}(z)$ and $\bar{\psi}_{\ell,2}(z)$, $\|\phi_1\|_{H^1(U_\tau)}$, $\|\phi_2\|_{H^1(U_\tau)}$ and $\| \sum_{\ell=1}^m \psi_\ell \phi_{\ell,0}\|_{L^2(U_\tau)}$ stay uniformly bounded as ε tends to zero. We multiply next the equation in (101) by ϕ_1 and integrate

$$\begin{aligned}
\mathrm{O}\left(\varepsilon^{\frac{3}{2}}\log\frac{1}{\varepsilon}\right) &= \int_{U_\tau}\phi_1 L_\varepsilon\phi dV_{g_\varepsilon} = \int_{U_\tau}\left(\sum_{\ell=1}^m \psi_\ell\phi_{\ell,0}+\phi^\perp\right)L_\varepsilon\phi_1 dV_{g_\varepsilon}\\
&= \mathrm{O}\left(\varepsilon^{\frac{3}{2}}\log\frac{1}{\varepsilon}\right)\|\phi_1\|_{H^1(U_\tau)} + \int_{U_\tau}\sum_{\ell=1}^m \psi_\ell\phi_{\ell,0}L_\varepsilon\phi_1 dV_{g_\varepsilon}\\
&= \mathrm{O}\left(\varepsilon^{\frac{3}{2}}\log\frac{1}{\varepsilon}\right) + \int_{U_\tau}\sum_{\ell=1}^m \psi_\ell\phi_{\ell,0}L_\varepsilon\phi_1 dV_{g_\varepsilon}.
\end{aligned}$$

From the expression of L_ε we have

$$\begin{aligned}
\mathrm{O}\left(\varepsilon^{\frac{3}{2}}\log\frac{1}{\varepsilon}\right) = {} & \int_{U_\tau}\sum_{\ell=1}^m \psi_\ell\phi_{\ell,0}\{\sum_{j=1}^m[\phi_{j,0}\Delta_{K_\zeta}\bar\psi_{j,1}(z) + \varepsilon\log\frac{1}{\varepsilon}\gamma_\ell\partial_{\mathbf{n}}a(1+\mathrm{o}(1)\phi_{j,0}\bar\psi_{j,1}(z)\\
& +\bar\psi_{j,1}(z)\Delta_{K_\zeta}\phi_{j,0} + 2\langle\nabla_{K_\zeta}\bar\psi_{j,1}(z),\nabla_{K_\zeta}\phi_{j,0}\rangle]\}\\
= {} & -\frac{1+\mathrm{o}(1)}{\varepsilon^{n-1}}\sum_{\ell=1}^m\sum_{j\in E_{\ell,1}}\lambda_{\ell,j}\alpha_{\ell,j}^2\\
& +\mathrm{O}(\varepsilon^2)\left(\frac{1}{\varepsilon^{n-1}}\sum_{\ell=1}^m\sum_{j\in E_{\ell,1}}\alpha_{\ell,j}^2\right)^{\frac{1}{2}}\sum_{\ell=1}^m\|\bar\psi_{\ell,1}\|_{L^2(K_\varepsilon)}\\
& +\mathrm{O}(\varepsilon)\left(\frac{1}{\varepsilon^{n-1}}\sum_{\ell=1}^m\sum_{j\in E_{\ell,1}}\alpha_{\ell,j}^2\lambda_j\right)^{\frac{1}{2}}\sum_{\ell=1}^m\|\bar\psi_{\ell,1}\|_{L^2(K_\varepsilon)}.
\end{aligned}$$

Then we have

$$\frac{1}{\varepsilon^{n-1}}\sum_{\ell=1}^m\sum_{j\in E_{\ell,1}}\alpha_{\ell,j}^2|\lambda_{\ell,j}| \le C\varepsilon^{\frac{3}{2}}\log\frac{1}{\varepsilon}.$$

Still from the fact that $|\lambda_{\ell,j}| > \varepsilon^{\frac{5}{4}}$ for $j\in E_{\ell,1}$, one also deduces

$$\frac{1}{\varepsilon^{n-1}}\sum_{\ell=1}^m\sum_{j\in E_{\ell,1}}\alpha_{\ell,j}^2 \le C\varepsilon^{\frac{1}{4}}\log\frac{1}{\varepsilon}.$$

A similar argument, replacing $E_{\ell,1}$ with $E_{\ell,2}$ gives similar estimates, so we obtain the conclusion.

As an application of the above lemma, we obtain the following estimates of the derivatives of small eigenvalues of L_ε.

Lemma 4.4 *Suppose λ is as in Lemma 4.1, and assume that $|\lambda| = O(\varepsilon^{\frac{3}{2}}\log\frac{1}{\varepsilon})$. Then, for ε sufficiently small the eigenvalue λ is differentiable with respect to ε, and satisfies*

$$\frac{\partial\lambda}{\partial\varepsilon} > 0.$$

Proof. Suppose u is an eigenfunction of L_ε with eigenvalue λ. Using the eigenvalue equation and Proposition 4.1, we see that the numerator in Kato's formula can be substituted by the expression

$$\int_{U_\tau}\left(\frac{2}{\varepsilon}[(1-3(u_\varepsilon^J)^2)u^2+2au_\varepsilon^J u^2]+6u_\varepsilon^J\frac{\partial\bar{u}_\varepsilon^J}{\partial\varepsilon}u^2-2a\frac{\partial\bar{u}_\varepsilon^J}{\partial\varepsilon}u^2-2\partial_\varepsilon a u_\varepsilon^J u^2\right)dV_{g_\varepsilon}+\mathrm{O}(\varepsilon)\|u\|_{H^1}^2.$$

By Lemmas 4.2 and 4.3 we can evaluate the latter integrand substituting to u the function

$$u=\sum_{\ell=1}^m\phi_{\ell,0}\bar{\psi}_\ell:=\sum_{\ell=1}^m\sum_{|\lambda_{\ell,j}|\le\varepsilon^{\frac{5}{4}}}\alpha_{\ell,j}\phi_{\ell,0}\varphi_j(\varepsilon z).$$

We normalize u so that

$$\int_{U_\tau}\partial_{\mathbf{n}}a\left(\sum_{\ell=1}^m\phi_{\ell,0}\bar{\psi}_\ell\right)^2dV_{g_\varepsilon}=1.$$

We have

$$\frac{\partial\lambda}{\partial\varepsilon}=\int_{K_\varepsilon}\int_{-\varepsilon^{-\tau}}^{\varepsilon^{-\tau}}(1+\varepsilon\zeta\kappa)\left(\frac{2}{\varepsilon}[(1-3(u_\varepsilon^J)^2)+2au_\varepsilon^J]+6u_\varepsilon^J\frac{\partial\bar{u}_\varepsilon^J}{\partial\varepsilon}-2a\frac{\partial\bar{u}_\varepsilon^J}{\partial\varepsilon}-2\partial_\varepsilon a u_\varepsilon^J\right)\left(\sum_{\ell=1}^m\phi_{\ell,0}\bar{\psi}_\ell\right)^2+\mathrm{o}(1).$$

We claim

$$\begin{aligned}&\int_{-\varepsilon^{-\tau}}^{\varepsilon^{-\tau}}(1+\varepsilon\zeta\kappa)\left(\frac{2}{\varepsilon}[(1-3(u_\varepsilon^J)^2)+2au_\varepsilon^J]+6u_\varepsilon^J\frac{\partial\bar{u}_\varepsilon^J}{\partial\varepsilon}-2a\frac{\partial\bar{u}_\varepsilon^J}{\partial\varepsilon}-2\partial_\varepsilon a u_\varepsilon^J\right)\phi_{\ell,0}^2\\&=\partial_{\mathbf{n}}a\left(\frac{2}{3}+2\int_{\mathbb{R}}t^2(H'(t))^3dt+2f_\ell^2\int_{\mathbb{R}}(H'(t))^3dt\right)(1+\mathrm{O}(\varepsilon^{1-\tau})).\end{aligned}\tag{110}$$

Indeed, from [33], we know

$$\int_{\mathbb{R}}[2(1-3H^2)-6tHH'](H')^2dt=0,$$

hence we have

$$\int_{-\varepsilon^{-\tau}}^{\varepsilon^{-\tau}}(1+\varepsilon\zeta\kappa)\left(\frac{2}{\varepsilon}(1-3(u_\varepsilon^J)^2)+6u_\varepsilon^J\frac{\partial\bar{u}_\varepsilon^J}{\partial\varepsilon}\right)\phi_{\ell,0}^2=\mathrm{O}(\varepsilon^{1-\tau}),\tag{111}$$

where we have used the facts that $\frac{\partial\bar{u}_\varepsilon^J}{\partial\varepsilon}\simeq\left(-\frac{\zeta}{\varepsilon}-\frac{\partial f_\ell}{\partial\varepsilon}\right)H'$ near f_ℓ and $\phi_{\ell,0}=c_{\ell,\varepsilon}H'_\ell+\phi_{\ell,0}^\perp$. We also have

$$\begin{aligned}&\int_{-\varepsilon^{-\tau}}^{\varepsilon^{-\tau}}(1+\varepsilon\zeta\kappa)\left(\frac{4}{\varepsilon}au_\varepsilon^J-2a\frac{\partial\bar{u}_\varepsilon^J}{\partial\varepsilon}-2\partial_\varepsilon a u_\varepsilon^J\right)\phi_{\ell,0}^2\\&=\partial_{\mathbf{n}}a\int_{\mathbb{R}}[2tH(H')^2+2(t+f_\ell)^2(H')^3]dt(1+\mathrm{O}(\varepsilon^{1-\tau}))\\&=\partial_{\mathbf{n}}a\left(\frac{2}{3}+2\int_{\mathbb{R}}t^2(H'(t))^3dt+2f_\ell^2\int_{\mathbb{R}}(H'(t))^3dt\right)(1+\mathrm{O}(\varepsilon^{1-\tau})).\end{aligned}\tag{112}$$

(111) and (112) give (110). By (110) we can obtain the result of this lemma.

In the rest of this section we prove our main theorem, showing that the operator L_ε is invertible for a suitable sequence $\varepsilon_j \to 0$.

Theorem 4.1 *For $J \geq 3$, let u_ε^J and L_ε be as above. Then for a suitable sequence $\varepsilon_j \to 0$, $L_{\varepsilon_j} : H^2(U_\tau) \to L^2(U_\tau)$ is invertible and the inverse operator satisfies*

$$\|L_{\varepsilon_j}^{-1}\| \leq C\varepsilon_j^{-\frac{n+1}{2}} \left(\log \frac{1}{\varepsilon_j}\right)^{\frac{n-1}{2}}, \qquad \textit{for all } j \in \mathbb{N}.$$

Proof. First of all we give an asymptotic estimate on the number N_ε of negative eigenvalues of L_ε. We denote the eigenvalues of L_ε by $\tilde{\lambda}_{j,\varepsilon}$ in non-decreasing order and counting them with multiplicity. From the Courant-Fisher characterization we can write $\tilde{\lambda}_{j,\varepsilon}$ in two different ways

$$-\tilde{\lambda}_{j,\varepsilon} = \sup_{M \in M_j} \inf_{u \in M, u \neq 0} \frac{\int_{U_\tau} u L_\varepsilon u dV_{g_\varepsilon}}{\int_{U_\tau} \partial_{\mathbf{n}} a u^2 dV_{g_\varepsilon}}, \qquad -\tilde{\lambda}_{j,\varepsilon} = \inf_{M \in M_{j-1}} \sup_{u \perp M, u \neq 0} \frac{\int_{U_\tau} u L_\varepsilon u dV_{g_\varepsilon}}{\int_{U_\tau} \partial_{\mathbf{n}} a u^2 dV_{g_\varepsilon}} \qquad (113)$$

Here M_j (resp. M_{j-1}) represents the family of j-dimensional (resp. $j-1$ dimensional) subspaces of $H^2(U_\tau)$, and the symbol $\perp$ denotes orthogonality with respect to the L^2 scalar product with weight $\partial_{\mathbf{n}} a$.

Using the first formula in (113) one can plug-in functions of the form $u = \sum_{\ell=1}^m \phi_{\ell,0}\psi_\ell$ so that (see (105))

$$L_\varepsilon u = \sum_{\ell=1}^m [\phi_{\ell,0}(\Delta_{K_\zeta}\psi_\ell(z) + \varepsilon \log \frac{1}{\varepsilon} \gamma_\ell \partial_{\mathbf{n}} a(1+\mathrm{o}(1))\psi_\ell) + \psi_\ell(z)\Delta_{K_\zeta}\phi_{\ell,0} + 2\langle \nabla_{K_\zeta}\psi_\ell(z), \nabla_{K_\zeta}\phi_{\ell,0}\rangle].$$

From the decay estimates of $\phi_{\ell,0}, \ell = 1, \ldots, m$ with respect to ζ and the Weyl's asymptotic formula we can obtain the lower bound

$$N_\varepsilon \geq (1 + \mathrm{o}(1))C_\Omega \left(\varepsilon_j^{-1} \log \frac{1}{\varepsilon_j}\right)^{\frac{n-1}{2}}.$$

The similar argument as in [33], we can get the upper bound

$$N_\varepsilon \leq (1 + \mathrm{o}(1))C_\Omega \left(\varepsilon_j^{-1} \log \frac{1}{\varepsilon_j}\right)^{\frac{n-1}{2}},$$

with the same constant as before. In conclusion we have

$$N_\varepsilon \sim C_\Omega \left(\varepsilon_j^{-1} \log \frac{1}{\varepsilon_j}\right)^{\frac{n-1}{2}}, \qquad \text{as } \varepsilon \to 0. \qquad (114)$$

Now for $l \in \mathbb{N}$, we let $\varepsilon_l = 2^{-l}$. Then from (114) we have

$$\begin{aligned} N_{\varepsilon_{l+1}} - N_\varepsilon &\sim C_\Omega \left(2^{(l+1)\frac{n-1}{2}} (\log 2^{l+1})^{\frac{n-1}{2}} - 2^{l\frac{n-1}{2}} (\log 2^l)^{\frac{n-1}{2}} \right) \\ &= C_\Omega \left(2^{\frac{n-1}{2}} \left(\frac{l+1}{l} \right)^{\frac{n-1}{2}} - 1 \right) \left(\varepsilon_l^{-1} \log \frac{1}{\varepsilon_l} \right)^{\frac{n-1}{2}}. \end{aligned} \tag{115}$$

By Lemma 4.4, the eigenvalues of L_ε bounded in absolute value by $\mathrm{o}(\varepsilon)$ are increasing in ε. Equivalently, by the last equation, the number if eigenvalues which become negative, when ε decrease from ε_l to ε_{l+1}, is of order $\left(\varepsilon_l^{-1} \log \frac{1}{\varepsilon_l} \right)^{\frac{n-1}{2}}$. We define

$$B_l := \{ \varepsilon \in (\varepsilon_{l+1}, \varepsilon_l) : \ker L_\varepsilon \neq \varnothing \}, \quad \tilde{B}_l := (\varepsilon_{l+1}, \varepsilon_l) \setminus B_l.$$

By (115) and the monotonicity in ε of the small eigenvalues, we deduce that

$$\mathrm{card}(B_l) \leq N_{\varepsilon_{l+1}} - N_\varepsilon \leq C \left(\varepsilon_l^{-1} \log \frac{1}{\varepsilon_l} \right)^{\frac{n-1}{2}},$$

and hence there exists an interval (a_l, b_l) such that

$$(a_l, b_l) \subseteq B_l, \qquad |b_l - a_l| \geq C \frac{\mathrm{meas}(B_l)}{\mathrm{card}(B_l)} \geq C \varepsilon_l^{\frac{n+1}{2}} \left(\log \frac{1}{\varepsilon_l} \right)^{-\frac{n-1}{2}}.$$

From Lemma 4.4 we deduce that $L_{\frac{a_l+b_l}{2}}$ is invertible and

$$\| L^{-1}_{\frac{a_l+b_l}{2}} \| \leq C \varepsilon_l^{-\frac{n+1}{2}} \left(\log \frac{1}{\varepsilon_l} \right)^{\frac{n-1}{2}}.$$

Now it is sufficient to set $\varepsilon_j = \frac{a_l+b_l}{2}$. The proof is completed.

We consider now the problem in the whole domain Ω_ε, and not only in the strip U_τ. Precisely, we first choose a cutoff function $\eta_\varepsilon(\theta)$ which is identically equal to 1 for $\theta \leq \frac{\varepsilon^{-\tau}}{2}$, and which is identically equal to 0 for $\theta \geq \frac{3\varepsilon^{-\tau}}{4}$. We then define the function $\hat{u}_\varepsilon^J$ by

$$\hat{u}_\varepsilon^J(z, \zeta) := \eta_\varepsilon(|\zeta|) u_\varepsilon^J(z, \zeta) + (1 - \eta_\varepsilon(|\zeta|)) \mathbb{W},$$

where $\mathbb{W}$ is defined in (37). It is easy to verify that, by the exponential convergence to ± 1 of u_ε^J in the compact sets of $\Omega_\pm$ (and also by the decay of its derivative), that

$$\| S_\varepsilon(\hat{u}_\varepsilon^J) \|_{L^2(\Omega_\varepsilon)} \leq C \varepsilon^{J - \frac{n-1}{2}}, \quad \| S_\varepsilon(\hat{u}_\varepsilon^J) \|_{L^\infty(\Omega_\varepsilon)} \leq C \varepsilon^J,$$

where

$$S_\varepsilon(u) := u_{\zeta\zeta} + \varepsilon \kappa(\varepsilon z) u_\zeta + \varepsilon^2 \Delta_{K_{\varepsilon\zeta}} u + \mathrm{O}(\varepsilon^2) u_\zeta + u(1 - u^2) - a(\varepsilon x)(1 - u^2).$$

We consider next the eigenvalue problem

$$\Delta u + 3(1-(\hat{u}_\varepsilon^J)^2)u - 2(1-a\hat{u}_\varepsilon^J)u + \lambda\partial_{\mathbf{n}} a u = 0,$$

and we denote the eigenvalues by $\hat{\lambda}_{j,\varepsilon}$, counted in non-decreasing order with their multiplicity.

As one can easily check, if λ is bounded from above, the corresponding eigenfunctions decay exponentially away from K_ε. Therefore, reasoning as for [30], Proposition 5.6, one finds that there exists a constant C such that

$$|\hat{\lambda}_{j,\varepsilon} - \tilde{\lambda}_{j,\varepsilon}| \leq Ce^{-\frac{C}{\varepsilon}} \quad \text{provided } \hat{\lambda}_{j,\varepsilon} \leq 1 \text{ or } \tilde{\lambda}_{j,\varepsilon} \leq 1.$$

Hence, by Theorem 4.1 and the last formula we obtain the following result.

Corollary 1 *For $J \in \mathbb{N}$, let $\hat{\lambda}_{j,\varepsilon}$ be as above, and define the operator $\hat{L}_\varepsilon(u) := \Delta u + 3(1-(\hat{u}_\varepsilon^J)^2)u - 2(1-a\hat{u}_\varepsilon^J)u$. Then for a suitable sequence $\varepsilon_j \to 0$, $\hat{L}_{\varepsilon_j} : H^2(\Omega_\varepsilon) \to L^2(\Omega_\varepsilon)$ is invertible and the inverse operator satisfies*

$$\|\hat{L}_{\varepsilon_j}^{-1}\| \leq C\varepsilon_j^{-\frac{n+1}{2}} \left(\log\frac{1}{\varepsilon_j}\right)^{\frac{n-1}{2}}, \qquad \text{for all } j \in \mathbb{N}.$$

5 Proof of the main theorem

Finally we prove Theorem 1.1 by applying the contraction mapping theorem.

Proof of Theorem 1.1 Let ε_j be as in Corollary 1. We set

$$u_\varepsilon = \hat{u}_\varepsilon^J + \phi, \qquad \phi \in H^2(\Omega_\varepsilon).$$

Since $\hat{L}_{\varepsilon_j}$ is invertible,

$$S_\varepsilon(\hat{u}_\varepsilon^J + \phi) = 0 \tag{116}$$

can be written as

$$\phi = T_\varepsilon(\phi) := -\hat{L}_{\varepsilon_j}[S_\varepsilon(\hat{u}_\varepsilon^J) - 3\hat{u}_\varepsilon^J\phi^2 - \phi^3 + a\phi^2].$$

For $\rho > 0$, we introduce the set

$$\Lambda_\rho := \{\phi \in H^2(\Omega_\varepsilon) \cap L^\infty(\Omega_\varepsilon) : |||\phi||| \leq \rho\},$$

where $|||\phi||| := \|\phi\|_{H^2(\Omega_\varepsilon)} + \|\phi\|_{L^\infty(\Omega_\varepsilon)}$.

By standard elliptic regularity results and by Corollary 1 we know that there exists a positive constant $C(n,\Omega)$ such that

$$|||T_\varepsilon(\phi)||| \leq C(n,\Omega)\varepsilon^{-\frac{n+1}{2}} \left(\log\frac{1}{\varepsilon}\right)^{\frac{n-1}{2}} [\varepsilon^{J-\frac{n-1}{2}} + |||\phi|||^2],$$

and

$$|||T_\varepsilon(\phi_1) - T_\varepsilon(\phi_2)||| \le C(n,\Omega)\varepsilon^{-\frac{n+1}{2}} \left(\log\frac{1}{\varepsilon}\right)^{\frac{n-1}{2}} (|||\phi_1||| + |||\phi_2|||)(|||\phi_1 - \phi_2|||),$$

for $\varepsilon = \varepsilon_j$ and $\phi, \phi_1, \phi_2 \in H^2(\Omega_\varepsilon) \cap L^\infty(\Omega_\varepsilon)$. Now, letting $\rho = \varepsilon^l$, choosing first l sufficiently large, then T_ε is contractive in Λ_ρ. Furthermore, we choose sufficiently large J, then $T_\varepsilon(\phi) \in \Lambda_\rho$ for any $\phi \in \Lambda_\rho$. Then by contraction mapping theorem we find a solution of (116), which completes the proof of Theorem 1.1.

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References

[1] N. Alikakos and P. W. Bates, *On the singular limit in a phase field model of phase transitions,* Ann. Inst. H. Poincaré Anal. Non Linéaire 5(1988), no. 2, 141-178.

[2] N. Alikakos, P. W. Bates and X. Chen, *Periodic traveling waves and locating oscillating patterns in multidimensional domains,* Trans. Amer. Math. Soc. 351(1999), no. 7, 2777-2805.

[3] N. Alikakos, P. W. Bates and G. Fusco, *Solutions to the nonautonomous bistable equation with specified Morse index. I. Existence,* Trans. Amer. Math. Soc. 340(1993), no. 2, 641-654.

[4] N. Alikakos, X. Chen and G. Fusco, *Motion of a droplet by surface tension along the boundary,* Cal. Var. PDE 11(2000), 233-306.

[5] N. Alikakos and H. C. Simpson, *A variational approach for a class of singular perturbation problems and applications,* Proc. Roy. Soc. Edinburgh Sect. A 107(1987), no. 1-2, 27-42.

[6] S. Allen and J. W. Cahn, *A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening,* Acta. Metall. 27(1979), 1084-1095.

[7] S. Angenent, J. Mallet-Paret and L. A. Peletier, *Stable transition layers in a semilinear boundary value problem,* J. Diff. Eqns. 67(1987), 212-242.

[8] I. Chavel, *Riemannian Geometry-A Mordern Introduction,* Cambridge Tracts in Math. 108, Cambridge Univ. press, Cambridge, 1993.

[9] E. N. Dancer and S. Yan, *multi-layer solutions for an elliptic problem,* J. Diff. Eqns. 194(2003), 382-405.

[10] E. N. Dancer and S. Yan, *Construction of various types of solutions for an elliptic problem,* Calc. Var. Partial Differential Equations 20(2004), no. 1, 93-118.

[11] M. del Pino, *Layers with nonsmooth interface in a semilinear elliptic problem,* Comm. Partial Differential Equations, 17(1992), no. 9-10, 1695-1708.

[12] M. del Pino, *Radially symmetric internal layers in a semilinear elliptic system,* Trans. Amer. Math. Soc. 347(1995), no. 12, 4807-4837.

[13] M. del Pino, M. Kowalczyk and J. Wei, *Concentration on curves for nonlinear Schrödinger equations,* Comm. Pure Appl. Math. 70(2007), 113-146.

[14] M. del Pino, M. Kowalczyk and J. Wei, *Resonance and interior layers in an inhomogenous phase transition model,* SIAM J. Math. Anal.38(2007), no.5, 1542-1564.

[15] M. del Pino, M. Kowalczyk, J. Wei and J. Yang, *Interface foliation near minimal submanifolds in Riemannian manifolds with positive Ricci curvature,* Geom. Funct. Anal., 20(2010), no. 4, 918-957.

[16] A. S. Do Nascimento, *Stable transition layers in a semilinear diffusion equation with spatial inhomogeneities in N-dimensional domains,* J. Diff. Eqns. 190(2003), no.1, 16-38.

[17] Y. Du and K. Nakashima, *Morse index of layered solutions to the heterogeneous Allen-Cahn equation,* J. Diff. Eqns. 238(2007), no. 1, 87-117.

[18] Z. Du and C. Gui, *Interior layers for an inhomogeneous Allen-Cahn equation,* J. Diff. Eqns. 249(2010), 215-239.

[19] Z. Du and B. Lai, *Transition layers for an inhomogeneous Allen-Cahn equation in Riemannian manifolds,* Discrete Contin. Dynam. Systems, A 33(2013), no. 4, 1407-1429.

[20] Z. Du and L. Wang, *Interface foliation for an inhomogeneous Allen-Cahn equation in Riemannian manifolds,* Calc. Var. Partial Differential Equations 47(2013), no. 1, 343-381.

[21] P. C. Fife, *Boundary and interior transition layer phenomena for pairs of second-order differential equations,* J. Math. Anal. Appl. 54(1976), no. 2, 497-521.

[22] P. C. Fife and W. M. Greenlee, *Interior transition layers for elliptic boundary value problems with a small parameter,* Russian Math. Surveys. 29:4 (1974), 103-131.

[23] G. Flores and P. Padilla, *Higher energy solutions in the theory of phase transitions: a variational approach,* J. Diff. Eqns. 169(2001), 190-207.

[24] J. Hale and K. Sakamoto, *Existence and stability of transition layers,* Japan J. Appl. Math. 5(1988), no. 3, 367-405.

[25] T. Kato, *Perturbation theory for linear operators,* Classics in Mathematics. Springer-Verlag, Berlin, 1995.

[26] R.V.Kohn and P. Sternberg, *Local minimizers and singular perturbations,* Proc. Royal Soc. Edinburgh 11A(1989), 69-84.

[27] M. Kowalczyk, *On the existence and Morse index of solutions to the Allen-Cahn equation in two dimensions,* Annali di Matematica Pura et Aplicata, (4) 184(2005), no. 1, 17-52.

[28] F. Mahmoudi, R. Mazzeo and F. Pacard, *Constant mean curvature hypersurfaces condensing on a submanifold,* Geom. Funct. Anal. 16(2006), no. 4, 924-958.

[29] F. Mahmoudi, A. Malchiodi and J. Wei, *Transition layer for the Heterogeneous Allen-Cahn equation,* Ann. Inst. H. Poincaré Anal. Non Linéaire, 25(2008), no.3, 609-631.

[30] A. Malchiodi and M. Montenegro, *Boundary concentration phenomenena for a singularly perturbed elliptic problem,* Comm. Pure Appl. Math. 55(2002), 1507-1568.

[31] A. Malchiodi and M. Montenegro, *Multidimensional boundary layers for a singularly perturbed Neumann problem,* Duke Math. J. 124(2004), no. 1, 105-143.

[32] A. Malchiodi, W.-M. Ni and J. Wei, *Boundary clustered interfaces for the Allen-Cahn equation,* Pacific J. Math., Vol. 229, No.2,(2007), 447-468.

[33] A. Malchiodi and J. Wei, *Boundary interface for the Allen-Cahn equation,* J. Fixed Point Theory Appl. 1(2007), no. 2, 305-336.

[34] L. Modica, *The gradient theory of phase transitions and the minimal interface criterion,* Arch. Rat. Mech. Anal. 98(1987), 357-383.

[35] S. Muller, *Singular perturbations as a selection criterion for periodic minimizing sequences,* Cal. Var. Partial Differential Equations. 1(1993), no. 2, 169-204.

[36] Kimie Nakashima, *Multi-layered stationary solutions for a spatially inhomogeneous Allen-Cahn equation,* J. Diff. Eqns. 191(2003), 234-276.

[37] K. Nakashima and K. Tanaka, *Clustering layers and boundary layers in spatially inhomogenerous phase transition problems,* Ann. Inst. H. Poincaré Anal. Non Linéaire 20(2003), no. 1, 107-143.

[38] Y. Nishiura and H. Fujii, *Stability of singularity perturbed solutions to systems of reaction-diffusion equations,* SIAM J. Math. Anal. 18(1987), 1726-1770.

[39] F. Pacard and M. Ritoré, *From constant mean curvature hypersurfaces to the gradient theory of phase transitions,* J. Diff. Geom. 64(2003), 359-423.

[40] P. Padilla and Y. Tonegawa, *On the convergence of stable phase transitiona,* Comm. Pure Appl. Math. 51(1998), 551-579.

[41] P. H. Rabinowitz and E. Stredulinsky, *Mixed states for an Allen-Cahn type equation, I,* Commun. Pure Appl. Math. 56(2003), 1078-1134.

[42] P. H. Rabinowitz and E. Stredulinsky, *Mixed states for an Allen-Cahn type equation, II,* Calc. Var. Partial Differential Equations. 21(2004), 157-207.

[43] K. Sakamoto, *Construction and stability analysis of transition layer solutions in reaction-diffusion systems,* Tohoku Math. J. (2)42(1990), no. 1, 17-44.

[44] K. Sakamoto, *Infinitely many fine modes bifurcating from radilly symmetric internal layers,* Asymptot. Anal., 42(2005), no. 1-2, 55-104.

[45] P. Sternberg and K. Zumbrun, *Connectivity of phase boundaries in strictly convex domains,* Arch. Rational Mech. Anal. 141(1998), no. 4, 375-400.

[46] J. Wei and J. Yang, *Toda system and cluster phase transition layers in an inhomogeneous phase transition model,* Asymptot. Anal., 69(2010), no. 3-4, 175-218.

[47] J. Yang and X. Yang, *Clustered interior phase transition layers for an inhomogeneous Allen-Cahn equation on higher dimensional domain,* to appear.