Stationary states of nonlinear Dirac equations with general potentials

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Abstract

We establish the existence of stationary states for the following nonlinear Dirac equation

$$\begin{cases} -i\sum_{k=1}^{3}\alpha_k\partial_k u + a\beta u + M(x)u = g(x,|u|)u & \text{for } x\in\mathbb{R}^3 \\ u(x)\to 0 & \text{as } |x|\to\infty \end{cases}$$

with real matrix potential $M(x)$ and super-linearity $g(x,|u|)u$ both without periodicity assumptions, via variational methods.

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1 Introduction and main results

Of concern is the existence of solutions to the following nonlinear Dirac equations

$$(1.1)\qquad \begin{cases} -i\sum_{k=1}^{3}\alpha_k\partial_k u + a\beta u + M(x)u = g(x,|u|)u & \text{for } x\in\mathbb{R}^3, \\ u(x)\to 0 & \text{as } |x|\to\infty, \end{cases}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $u(x) \in \mathbb{C}^4$, $\partial_k = \frac{\partial}{\partial x_k}$, a is a positive constant, $\alpha_1, \alpha_2, \alpha_3$ and β are 4×4 complex matrices (in 2×2 blocks):

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$M(x)$ denotes a 4×4 real symmetric matrix valued function, and $g \in C(\mathbb{R}^3 \times \mathbb{R}^+, \mathbb{R}^+)$, $\mathbb{R}^+ := [0, \infty)$. In physics, $M(x)$ represents the external potential. See [38].

Problem (1.1) arises in the study of stationary solutions to the nonlinear Dirac equation which models extended relativistic particles in external fields and has been used as effective theories in atomic, nuclear and gravitational physics (see [9, 29, 18, 16]). Its most general form is

$$-i\hbar\partial_t\psi = ic\hbar\sum_{k=1}^{3}\alpha_k\partial_k\psi - mc^2\beta\psi - P(x)\psi + G_\psi(x, \psi). \tag{1.2}$$

Here c denotes the speed of light, $m > 0$ is the mass of the electron, $\hbar$ denotes Planck's constant, the 4×4 real symmetric matrix $P(x)$ stands for the external field, and the nonlinearity $G : \mathbb{R}^3 \times \mathbb{C}^4 \to \mathbb{R}$ represents a nonlinear self-coupling. A solution $\psi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}^4$ of (1.2), with $\psi(t, \cdot) \in L^2(\mathbb{R}^3, \mathbb{C}^4)$, is a *wave function* which represents the state of a relativistic electron. Assuming that G satisfies $G(x, e^{i\theta}\psi) = G(x, \psi)$ for all $\theta \in [0, 2\pi]$, one is finding solutions of (1.2) with the form $\psi(t, x) = e^{\frac{i\theta t}{\hbar}}u(x)$ which may be regarded as "particle-like solutions" (see [29]): they propagate without changing their shape and thus have a soliton-like behavior. Then $u : \mathbb{R}^3 \to \mathbb{C}^4$ satisfies the equation

$$-i\sum_{k=1}^{3}\alpha_k\partial_k u + a\beta u + M(x)u = \tilde{G}_u(x, u) \text{ for } x \in \mathbb{R}^3 \tag{1.3}$$

with $a = mc/\hbar$, $M(x) = P(x)/\hbar c + \theta I_4$ and $\tilde{G}_u(x, u) = G_u(x, u)/\hbar c$.

Mathematically, there are new difficulties in using the Calculus of Variations to find solutions to Problem (1.3). First, the energy functional (see (1.6) below) is strongly indefinite: it is unbounded from below and all its critical points have indefinite Morse index. The second difficulty is the lack of compactness: the Palais-Smale condition is not satisfied due to the unboundedness of the domain $\mathbb{R}^3$. The combination of the above types of difficulties poses a challenge in the Calculus of Variations. As a result, many authors have developed new methods and techniques to study (1.3). We summarize the research on the existence (and multiplicity) of solutions to problems of form (1.3) for particularly the following three cases.

Case 1 – the autonomous system, that is, $M = \omega I_4$ *(*ω *a constant and* I_4 *the* 4×4 *identity matrix), and* $\tilde{G}$ *does not depend on* x. In [4, 5, 10, 27] the authors studied the problem with $\omega \in (-a, 0)$ and $\tilde{G}$ having the form

$$(1.4) \quad \tilde{G}(u) = \tfrac{1}{2} H(\tilde{u}u) \ , \ H \in C^2(\mathbb{R}, \mathbb{R}) \ , \ H(0) = 0 \, ; \text{here } \ \tilde{u}u := (\beta u, u)_{\mathbb{C}^4} ,$$

by using shooting methods. (This is the so-called Soler model [35].) Finkelstein et al. [19] considered the nonlinearity $\tilde{G}$ of the form

$$(1.5) \qquad \tilde{G}(u) = \tfrac{1}{2}|\tilde{u}u|^2 + b|\tilde{u}\alpha u|^2 \ , \ \tilde{u}\alpha u := (\beta u, \alpha u)_{\mathbb{C}^4} \ , \ \alpha := \alpha_1\alpha_2\alpha_3$$

with $b > 0$. In [17] Esteban and Séré treated the equation with $\tilde{G}(u)$ of form (1.4) under the main additional assumption that $H'(s)\, s \geq \theta\, H(s)$ for some $\theta > 1$, and all $s \in \mathbb{R}$. [17] also considered nonlinearities of type (1.5), however with a weaker growth

$$\tilde{G}(u) = \mu|\tilde{u}u|^\tau + b|\tilde{u}\alpha u|^\sigma, \ 1 < \tau, \sigma < \tfrac{3}{2}, \ \mu, b > 0;$$

they also investigated a more general $\tilde{G}(u)$ growing likely $|u|^p, p \in (2, 3)$, as $|u| \to \infty$.

Case 2 – the periodic system, that is, $M(x)$ *and* $\tilde{G}(x, u)$ *depend periodically on* x. In [7] Bartsch and Ding investigated this case with additionally $M(x) = \beta V(x), V \in C(\mathbb{R}^3, \mathbb{R})$. They treated functions $\tilde{G}(x, u)$ which may be superquadratic or asymptotically quadratic in u as $|u| \to \infty$ and they obtained infinitely many solutions if $\tilde{G}$ is additionally even in u.

Case 3 – non periodic system but $\tilde{G}_u(x, u)$ *is asymptotically linear as* $|u| \to \infty$. Recently, in their paper [16] Ding and Ruf considered this case with either the vector potentials $M(x)$ of Coulomb-type (see (M_1) below) or the scalar potentials of the form $M(x) = \beta V(x)$ satisfying roughly $\liminf_{|x| \to \infty} \beta V(x) > 0$. Under suitable assumptions they obtained the existence and multiplicity of solutions of (1.3)

Now one of the remaining cases is that *the potential* $M(x)$ *depends explicitly on* x *without periodicity assumption and the nonlinear interaction* $\tilde{G}(x, u)$ *grows super-quadratically as* $|u| \to \infty$. Indeed, as far as we know there is no existence results on (1.3) with the Coulomb-type potential (see (M_1) below) and "power-like" interaction $|u|^p$, $p > 2$ (the so-called Soler model [35]). In the present paper we consider this case.

In the following, for convenience, any real function $U(x)$ will be regarded as the symmetric matrix $U(x)I_4$. For a symmetric real matrix function $L(x)$, let $\underline{\lambda}_L(x)$ (resp. $\overline{\lambda}_L(x)$) be the minimal (resp. the maximal) eigenvalue of $L(x)$, $\inf L := \inf_x \underline{\lambda}_L(x)$, $\sup L := \sup_x \overline{\lambda}_L(x)$, $|L(x)| := \max\{|\underline{\lambda}_L(x)|, |\overline{\lambda}_L(x)|\}$, $|L|_\infty := \operatorname{ess\,sup}_x |L(x)|$, and $L(\infty) := \lim_{|x|\to\infty} L(x)$ if and only if $|L(x) - L(\infty)| \to 0$ as $|x| \to \infty$. For two given symmetric real matrix functions $L_1(x)$ and $L_2(x)$, we write $L_1(x) \le L_2(x)$ if and only if

$$\max_{\xi\in\mathbb{C}^4, |\xi|=1} \left(L_1(x) - L_2(x)\right)\xi \cdot \bar{\xi} \le 0.$$

Associated with (1.1) is the following energy functional defined by

$$\Phi_M(u) := \int_{\mathbb{R}^3} \left(\frac{1}{2}\Big(-i\sum_{k=1}^{3} \alpha_k\partial_k + a\beta + M(x)\Big) u \cdot \overline{u} - F(x,u) \right) dx \tag{1.6}$$

where

$$F(x,u) := \int_0^{|u|} g(x,s)s\, ds.$$

Let

$$c_M := \inf\big\{\Phi_M(u) : u \neq 0 \text{ is a solution of } (1.1)\big\}.$$

A solution $u_0 \neq 0$ with $\Phi_M(u_0) = c_M$ is called a least energy solution. Let S_M denote the set of all least energy solutions of (1.1).

We start with the following typical problem:

$$\begin{cases} -i\displaystyle\sum_{k=1}^{3} \alpha_k\partial_k u + a\beta u + M(x)u = q(x)|u|^{p-2}u & \text{for } x \in \mathbb{R}^3 \\ u(x) \to 0 & \text{as } |x| \to \infty \end{cases} \tag{1.7}$$

with $p \in (2,3)$. We assume

(q_0) $q \in C(\mathbb{R}^3, \mathbb{R})$ with $q(x) \ge q_0 > 0$ for all x where $q_0 := \lim_{|y|\to\infty} q(y)$.

For the vector external potentials we consider firstly the Coulomb-type:

(M_1) M is a symmetric continuous real 4×4-matrix function on $\mathbb{R}^3 \setminus \{0\}$ with $0 > M(x) \ge -\frac{\kappa}{|x|}$ where $\kappa < \frac{1}{2}$.

Our first theorem concerns the regularity of solutions to (1.7).

Theorem 1.1. *Assume that $p \in (2,3)$ and (M_1) and (q_0) are satisfied. Then equation (1.7) has at least one least energy solution $u \in W^{1,q}(\mathbb{R}^3, \mathbb{C}^4)$ for all $q \ge 2$. Moreover, S_M is compact in $H^1(\mathbb{R}^3, \mathbb{C}^4)$.*

Theorem 1.1 applies to the physically relevant case when $q(x) \equiv 1$ and $M(x)$ is Coulomb potential: $M(x) = -\frac{\kappa}{|x|}$. We refer to Chapter 4 of [38] for discussions on external fields. The restriction on κ is technical. See [7].

Next we consider the following potential:

(M_2) M is a symmetric continuous real 4×4-matrix function on $\mathbb{R}^3$ with $|M|_\infty < a$, $M(x) < M(\infty)$ for all x, and either (i) $M(\infty) \le 0$ or (ii) $M(\infty) = m_\infty I_4$ a constant.

In what follows, for describing the exponential decay of solutions, we restrict ourselves to consider the scalar potential $M(x) = V(x)\beta$ or $M(x) = V(x)$, where $V \in W^{1,\infty}(\mathbb{R}^3, \mathbb{R})$. Denote

$$\mathcal{E}(a, M(x)) = (a + V(x))^2 + i\beta \sum_{k=1}^{3} \alpha_k \partial_k V(x) \quad \text{if } M(x) = V(x)\beta$$

and

$$\mathcal{E}(a, M(x)) = a^2 - V(x)^2 \quad \text{if } M(x) = V(x)I_4.$$

We say that $\mathcal{E}(a, M(x))$ is *real positive definite at infinity* if there exist $\tau > 0$ and $R > 0$ such that

$$\Re\big[\mathcal{E}(a, M(x))\xi \cdot \overline{\xi}\big] \ge \tau|\xi|^2 \quad \text{for all } |x| \ge R \text{ and } \xi \in \mathbb{C}^4.$$

Assume

(M_3) Either $M(x) = V(x)\beta$ or $M(x) = V(x)I_4$ with $\mathcal{E}(a, M(x))$ being real positive definite at infinity.

Conditions (M_2) and (M_3) are technical conditions which are needed for concentration-compactness. Note that the Coulomb potential satisfies (M_2)–(M_3).

Our second theorem concerns the exponential decay of solutions to (1.7).

Theorem 1.2. *Assume that $p \in (2, 3)$ and (M_2) and (q_0) are satisfied. Then*

(i) *Equation* (1.7) *has at least one least energy solution $u \in W^{1,q}(\mathbb{R}^3, \mathbb{C}^4)$ for all $q \ge 2$;*

(ii) *S_M is compact in $H^1(\mathbb{R}^3, \mathbb{C}^4)$;*

(iii) *If additionally (M_3) holds and q is of $W^{1,\infty}$ then there exist $C, c > 0$ such that*

$$|u(x)| \le C \exp(-c|x|) \quad \textit{for all } x \in \mathbb{R}^3,\ u \in S_M.$$

Finally, we consider the existence of ground states under more general nonlinearities of (1.1). Assume

- (g_1) $g(x,s) \geq 0$, $g(x,s) = o(s)$ as $s \to 0$ uniformly in x, and there exist $p \in (2,3)$, $c_1 > 0$ such that $g(x,s) \leq c_1(1+s^{p-2})$;
- (g_2) there is $\mu > 2$ such that $0 < \mu F(x,u) \leq g(x,|u|)|u|^2$ if $u \neq 0$;
- (g_3) there is $g_\infty \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ with $g'_\infty(s) > 0$ for $s > 0$ such that $g(x,s) \to g_\infty(s)$ as $|x| \to \infty$ uniformly on bounded sets of s, and $g_\infty(s) \leq g(x,s)$ for all (x,s).

Here $g'_\infty(s) = dg_\infty(s)/ds$ and $F_\infty(u) := \int_0^{|u|} g_\infty(s)s\,ds$.

Conditions $(g_1)-(g_3)$ are called Ambrosetti-Rabinowitz conditions, which are assumed in saddle-type critical point theory. In particular, the nonlinearity in the so-called Soler model [35] (see (1.4)) satisfies $(g_1)-(g_3)$.

Our last theorem gives the existence of ground states.

Theorem 1.3. *Let $(g_1)-(g_3)$ and either (M_1) or (M_2) be satisfied. Then*

- (i) *Equation (1.1) has at least one least energy solution $u \in W^{1,q}(\mathbb{R}^3, \mathbb{C}^4)$ for all $q \geq 2$;*
- (ii) *S_M is compact in $H^1(\mathbb{R}^3, \mathbb{C}^4)$;*
- (iii) *If (M_2) and (M_3) are satisfied and $g(x,s)$ is additionally of class C^1 on $\mathbb{R}^3 \times (0,\infty)$, then there exist $C, c > 0$ such that*

$$|u(x)| \leq C\exp(-c|x|) \quad \text{for all } x \in \mathbb{R}^3,\ u \in S_M.$$

It is clear that Theorems 1.1 and 1.2 are consequences of Theorem 1.3.

Our argument is variational, which can be outlined as follows. The solutions of (1.1) are obtained as critical points of the energy functional Φ_M on the space $H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$. Φ_M possesses the linking structure, however it does not satisfy the Palais-Smale condition in general. Thus we consider certain auxiliary problem related to the "limit equation" of (1.1) which is autonomous and whose least energy solutions with least energy $\hat{C}$ are known. It will be proved that Φ_M satisfies the Cerami condition $(C)_c$ at all levels $c < \hat{C}$. We then show that the minimax value ℓ_M based on the linking structure of Φ_M satisfies $0 < \ell_M < \hat{C}$ via a recent critical point theorem and obtain finally the solutions.

For the corresponding nonlinear Schrödinger equation

$$h^2\Delta u - V(x)u + f(u) = 0,\ u \in H^1(\mathbb{R}^N), \tag{1.8}$$

results similar to Theorems 1.2 and 1.3 have been established previously by Rabinowitz [28] and Sirakov [37]. The existence of spike layer solutions in the semiclassical limit (i.e., $h \to 0$) has been established under various conditions of $V(x)$. See [2], [8], [13], [14], [20], [22], [23], [24], [26] and the references therein. Due to the strong indefinite structure of Dirac operator, our results in this paper seem to be the first of such for nonlinear Dirac operators.

The paper is organized as follows. In Section 2 we formulate the variational setting and recall some critical point theorems required. We then in Section 3 discuss the least energy solutions of the associated limit equation, in particular, characterize the least energy in three versions (the results of this section seems useful also for dealing with semiclassical solutions of some singularly perturbed Dirac equations). And finally, in Section 4 we complete the proof of the main results.

2 The variational setting

In what follows by $|\cdot|_q$ we denote the usual L^q-norm, and $(\cdot,\cdot)_2$ the usual L^2-inner product. Let $H_0 = -i\sum_{k=1}^{3}\alpha_k\partial_k + a\beta$ denote the selfadjoint operator on $L^2(\mathbb{R}^3,\mathbb{C}^4)$ with domain $\mathcal{D}(H_0) = H^1(\mathbb{R}^3,\mathbb{C}^4)$. For any symmetric real matrix function M set $H_M := H_0 + M$ with its spectrum and continuous spectrum denoted by $\sigma(H_M)$ and $\sigma_c(H_M)$ respectively.

Lemma 2.1. *Let M be a symmetric real matrix function.*

1) $\sigma(H_0) = \sigma_c(H_0) = \mathbb{R}\setminus(-a,a)$;

2) *If M satisfies (M_1) then H_M is selfadjoint with $\mathcal{D}(H_M) = H^1(\mathbb{R}^3,\mathbb{C}^4)$ and $\sigma(H_M)\subset\mathbb{R}\setminus(-(1-2\kappa)a,\,(1-2\kappa)a)$;*

3) *If M satisfies (M_2) then H_M is selfadjoint with $\mathcal{D}(H_M) = H^1(\mathbb{R}^3,\mathbb{C}^4)$ and $\sigma(H_M)\subset\mathbb{R}\setminus(-a+|M|_\infty),\,a-|M|_\infty)$.*

Proof. 1) follows from a standard argument of Fourier analysis.

We now check 2). Setting $V_\kappa(x) := \kappa/|x|$, it follows from (M_1) that $|Mu|_2^2 \le |V_\kappa u|_2^2$. Since $a>0$, the Kato's inequality implies that $|V_\kappa u|_2^2 \le 4\kappa^2|\nabla u|_2^2 \le 4\kappa^2|H_0u|_2^2$ (see [12]). Since $2\kappa<1$, it follows from the Kato-Rellich theorem that H_M is selfadjoint. Furthermore,

$$|H_Mu|_2 \ge |H_0u|_2 - |Mu|_2 \ge (1-2\kappa)|H_0u|_2 \ge (1-2\kappa)a|u|_2,$$

thus, $\sigma(H_M)\subset\mathbb{R}\setminus(-(1-2\kappa)a,\,(1-2\kappa)a)$.

Similarly, one checks 3) easily. □

It follows from 1) of Lemma 2.1 that the space L^2 possesses the orthogonal decomposition:

$$L^2 = L^- \oplus L^+, \quad u = u^- + u^+$$

so that H_0 is negative definite (resp. positive definite) in L^- (resp. L^+). Let $|H_0|$ denote the absolute value, $|H_0|^{1/2}$ the squared root, and take $E = \mathcal{D}(|H_0|^{1/2})$. E is a Hilbert space equipped with the inner product

$$(u, v) = \Re(|H_0|^{1/2}u, |H_0|^{1/2}v)_2$$

and the induced norm $\|u\| = (u, u)^{1/2}$. E possesses the following decomposition

$$E = E^- \oplus E^+ \quad \text{with} \quad E^\pm = E \cap L^\pm,$$

orthogonal with respect to both $(\cdot,\cdot)_2$ and $(\cdot,\cdot)$ inner products.

The following lemma can be found in [7] or [15].

Lemma 2.2. *E embeds continuously into $H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$, hence, E embeds continuously into L^q for all $q \in [2, 3]$ and compactly into L^q_{loc} for all $q \in [1, 3)$.*

Assuming $(g_1)-(g_3)$ are satisfied and either (M_1) or (M_2) holds, we define on E the following functional

$$\Phi_M(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) + \frac{1}{2}\int_{\mathbb{R}^3} M(x)u\overline{u} - \Psi(u) \tag{2.1}$$

where

$$\Psi(u) := \int_{\mathbb{R}^3} F(x, u).$$

Then $\Phi_M \in C^1(E, \mathbb{R})$ and a standard argument shows that critical points of Φ_M are solutions of (1.1).

Using the operator H_M one may give Φ_M another representation as follows. Note that, by the 2) and 3) of Lemma 2.1, $E = \mathcal{D}(|H_M|^{1/2})$ with the equivalent inner product

$$(u, v)_M := \Re(|H_M|^{1/2}u, \, |H_M|^{1/2}v)_2$$

and norm $\|u\|_M := (u, u)_M^{1/2}$. Then as above there is a decomposition

$$E = E_M^- \oplus E_M^+,$$

and Φ_M can be represented as

$$\Phi_M(u) = \frac{1}{2}\left(\|u^+\|_M^2 - \|u^-\|_M^2\right) - \Psi(u). \tag{2.2}$$

In order to find critical points of Φ_M we will use the following abstract theorem which is taken from [6, 15].

Let E be a Banach space with direct sum decomposition $E = X \oplus Y$, $u = x + y$ and corresponding projections P_X, P_Y onto X, Y, respectively. For a functional $\Phi \in C^1(E, \mathbb{R})$ we write $\Phi_a = \{u \in E : \Phi(u) \geq a\}$. Recall that a sequence $(u_n) \subset E$ is said to be a $(C)_c$-sequence (resp. $(PS)_c$-sequence) if $\Phi(u_n) \to c$ and $(1 + \|u_n\|)\Phi'(u_n) \to 0$ (resp. $\Phi'(u_n) \to 0$). Φ is said to satisfy the $(C)_c$-condition (resp. $(PS)_c$-condition) if any $(C)_c$-sequence (resp. $(PS)_c$-sequence) has a convergent subsequence.

Now we assume that X is separable and reflexive, and we fix a countable dense subset $\mathcal{S} \subset X^*$. For each $s \in \mathcal{S}$ there is a semi-norm on E defined by

$$p_s : E \to \mathbb{R}, \quad p_s(u) = |s(x)| + \|y\| \quad \text{for } u = x + y \in X \oplus Y.$$

We denote by $\mathcal{T}_{\mathcal{S}}$ the induced topology. Let w^* denote the weak*-topology on E^*. Suppose:

(Φ_0) There exists $\zeta > 0$ such that $\|u\| < \zeta \|P_Y u\|$ for all $u \in \Phi_0$.

(Φ_1) For any $c \in \mathbb{R}$, Φ_c is $\mathcal{T}_{\mathcal{S}}$-closed, and $\Phi' : (\Phi_c, \mathcal{T}_{\mathcal{S}}) \to (E^*, w^*)$ is continuous.

(Φ_2) There exists $\rho > 0$ with $\kappa := \inf \Phi(S_\rho Y) > 0$ where $S_\rho Y := \{u \in Y : \|u\| = \rho\}$.

The following theorem is a special case of [6, Theorem 3.4]; see also [15, Theorem 4.3].

Theorem 2.3. *Let* (Φ_0)–(Φ_2) *be satisfied and suppose there are* $R > \rho > 0$ *and* $e \in Y$ *with* $\|e\| = 1$ *such that* $\sup \Phi(\partial Q) \leq \kappa$ *where* $Q = \{u = x + te : x \in X, t \geq 0, \|u\| < R\}$. *Then* Φ *has a* $(C)_c$*-sequence with* $\kappa \leq c \leq \sup \Phi(Q)$.

The following lemma is useful to verify (Φ_1) (see [6] or [15]).

Lemma 2.4. *Suppose* $\Phi \in C^1(E, \mathbb{R})$ *is of the form*

$$\Phi(u) = \frac{1}{2}\big(\|y\|^2 - \|x\|^2\big) - \Psi(u) \qquad \textit{for } u = x + y \in E = X \oplus Y$$

such that

(i) $\Psi \in C^1(E, \mathbb{R})$ *is bounded from below;*

(ii) $\Psi : (E, \mathcal{T}_w) \to \mathbb{R}$ *is sequentially lower semicontinuous, that is,* $u_n \rightharpoonup u$ *in* E *implies* $\Psi(u) \leq \liminf \Psi(u_n)$;

(iii) $\Psi' : (E, \mathcal{T}_w) \to (E^*, \mathcal{T}_{w^*})$ *is sequentially continuous.*

(iv) $\nu : E \to \mathbb{R}$, $\nu(u) = \|u\|^2$, *is* C^1 *and* $\nu' : (E, \mathcal{T}_w) \to (E^*, \mathcal{T}_{w^*})$ *is sequentially continuous.*

Then Φ *satisfies* (Φ_1).

3 Autonomous equation–limit problem

In this section we study the following autonomous equation

$$\begin{cases} -i\sum_{k=1}^{3}\alpha_k\partial_k u + a\beta u + (b+L)u = g_\infty(|u|)u \quad \text{for } x \in \mathbb{R}^3 \\ u(x) \to 0 \quad \text{as } |x| \to \infty \end{cases} \tag{3.1}$$

where g_∞ is the function from assumption (g_3), b is a real number and L is a symmetric real constant matrix with

$$b \in (-a, a) \quad \text{and} \quad b - a < L \leq 0. \tag{3.2}$$

Without loss of generality we may assume $b \geq 0$ because otherwise we consider $\tilde{b} = 0$ and $\tilde{L} = b + L$ (which ≤ 0 if $b < 0$) replacing b and L. Remark that by (3.2) the minimal eigenvalue $0 \geq \underline{\lambda}_L > b - a$, hence

$$|L| < a - b. \tag{3.3}$$

In our later application we are interested in *the situation* $b = 0$ *and* $L = 0$ *in the case* (M_1)*; the situation* $b = 0$ *and* $L = M(\infty)$ *in the case* (i) *of* (M_2)*; and the situation* $b = m_\infty$ *and* $L = 0$ *in the case* (ii) *of* (M_2). Equation (3.1) may be regarded as a *"limit equation"* related to (1.1). The main consideration services to constructing linking levels of the functional Φ_M in the proof of main results. Although the existence of stationary solutions of (3.1) is known, we would also like to provide some minimax characterization of its least energy which seems useful in studying "semiclassical solutions" of singularly perturbed Dirac equation.

Let $H_b := H_0 + b$, a selfadjoint operator in L^2 with $\mathcal{D}(H_b) = H^1$ and $\sigma(H_b) \subset \mathbb{R} \setminus (-a+b,\, a+b)$. We introduce on $E = H^{1/2}$ the equivalent inner product

$$(u, v)_b := \Re(|H_b|^{1/2}u,\ |H_b|^{1/2}v)_2 \tag{3.4}$$

with the deduced norm $\|u\|_b := ||H_b|^{1/2}u|_2$. Note that the decomposition $E = E^- \oplus E^+$ is also orthogonal with respect to the inner product $(\cdot,\cdot)_b$, and

$$\|u^\pm\|_b^2 = \|u^\pm\|^2 \pm b|u^\pm|_2^2 \quad \text{for } u^\pm \in E^\pm$$

and

$$\|u\|_b^2 \geq (a-b)|u|_2^2. \tag{3.5}$$

Setting $F_\infty(u) = F_\infty(|u|) := \int_0^{|u|} g_\infty(s)s\,ds$, it follows from the assumption on g_∞ that there is $\hat{\mu} \in (2,\mu)$ such that

$$0 < \hat{\mu}F_\infty(u) \leq g_\infty(|u|)|u|^2 \quad \text{whenever } u \neq 0. \tag{3.6}$$

Indeed, one has

$$\begin{aligned} g_\infty(|u|)|u|^2 - \hat{\mu}F_\infty(u) &= \lim_{|x|\to\infty} \Big(g(x,|u|)|u|^2 - \hat{\mu}F(x,u)\Big) \\ &= \lim_{|x|\to\infty} \Big(g(x,|u|)|u|^2 - \mu F(x,u)\Big) \\ &\quad + \lim_{|x|\to\infty} (\mu - \hat{\mu})F(x,u) \\ &> 0 \end{aligned}$$

if $u \neq 0$. By (3.6), for any $\delta > 0$ there is $c_\delta > 0$ such that

$$F_\infty(u) \geq c_\delta |u|^{\hat{\mu}} \quad \text{for all } |u| \geq \delta; \tag{3.7}$$

and, for any $\varepsilon > 0$, there exist $C_\varepsilon, c_\varepsilon > 0$ with

$$g_\infty(|u|)|u| \leq \varepsilon|u| + C_\varepsilon|u|^{p-1} \tag{3.8}$$

and

$$F_\infty(u) \leq \varepsilon|u|^2 + C_\varepsilon|u|^p \tag{3.9}$$

for all $u \in \mathbb{C}^4$. Moreover, setting $\tilde{F}_\infty(u) := \frac{1}{2}g_\infty(|u|)|u|^2 - F_\infty(u)$, there holds

$$\tilde{F}_\infty(u) \geq \frac{\hat{\mu}-2}{2\hat{\mu}} g_\infty(|u|)|u|^2 \geq \frac{\hat{\mu}-2}{2} F_\infty(u).$$

Note also that, $\sigma := p/(p-2) > 3$, and for any $\delta > 0$, if $|u| \geq \delta$ then $g_\infty(|u|) \leq c_\delta|u|^{p-2}$ so $g_\infty(|u|)^{\sigma-1} \leq c'_\delta|u|^2$ and

$$\begin{aligned} g_\infty(|u|)^\sigma &= \Big(\frac{g_\infty(|u|)|u|^2}{|u|^2}\Big)^\sigma = \frac{\big(g_\infty(|u|)|u|^2\big)^{\sigma-1}}{|u|^{2\sigma}} g_\infty(|u|)|u|^2 \\ &= \frac{g_\infty(|u|)^{\sigma-1}}{|u|^2} g_\infty(|u|)|u|^2 \leq c'_\delta g_\infty(|u|)|u|^2 \\ &\leq c''_\delta \tilde{F}_\infty(u). \end{aligned}$$

Therefore, for any $\varepsilon > 0$ there exist $\rho_\varepsilon > 0$ and $c_\varepsilon > 0$ such that

$$g_\infty(|u|) \le \varepsilon \text{ if } |u| \le \rho_\varepsilon \quad \text{and} \quad g_\infty(|u|) \le c_\varepsilon \tilde{F}_\infty(u)^{1/\sigma} \text{ if } |u| \ge \rho_\varepsilon \tag{3.10}$$

Set

$$\Psi_\infty(u) := \int_{\mathbb{R}^3} F_\infty(u)$$

and define the functional

$$\begin{aligned}\Phi_b(u) &:= \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 + \frac{1}{2}\int_{\mathbb{R}^3}(b+L)u\overline{u} - \Psi_\infty(u) \\ &= \frac{1}{2}\|u^+\|_b^2 - \frac{1}{2}\|u^-\|_b^2 + \frac{1}{2}\int_{\mathbb{R}^3} Lu\overline{u} - \Psi_\infty(u)\end{aligned}$$

for $u = u^- + u^+ \in E^- \oplus E^+$.

It follows from the assumption on g_∞ that $\Phi_b \in C^1(E, \mathbb{R})$ and its critical points are solutions of (3.1).

By (3.2), (3.7) and (3.9) it is not difficult to verify that Φ_b possesses the linking structure, that is, for any finite dimensional subspace $Z \subset E^+$,

$$\Phi_b(u) \to -\infty \quad \text{as } u \in E^- \oplus Z, \|u\| \to \infty,$$

and there are $r > 0$ and $\rho > 0$ such that

$$\Phi_b|_{B_r \cap E^+} \ge 0 \quad \text{and} \quad \Phi_b|_{\partial B_r \cap E^+} \ge \rho.$$

Let $\mathcal{K}_b := \{u \in E : \Phi_b'(u) = 0\}$ be the critical set of Φ_b. The following lemma is an easy consequence of [17] and [7].

Lemma 3.1. $\mathcal{K}_b \setminus \{0\} \ne \emptyset$ *and* $\mathcal{K}_b \subset \bigcap_{q\ge 2} W^{1,q}$.

Denote

$$c_b := \inf\{\Phi_b(u) : u \in \mathcal{K}_b \setminus \{0\}\}.$$

Lemma 3.2. $c_b > 0$. *In particular,* 0 *is an isolated critical point of* Φ_b.

Proof. If $u \in \mathcal{K}_b$, one has

$$\Phi_b(u) = \Phi_b(u) - \frac{1}{2}\Phi_b'(u)u = \int_{\mathbb{R}^3} \tilde{F}_\infty(u) \ge 0.$$

For proving $c_b > 0$, assume by contradiction that $c_b = 0$. Let $u_j \in \mathcal{K}_b \setminus \{0\}$ be such that $\Phi_b(u_j) \to 0$. Then it is not difficult to check that (u_j) is bounded. We can suppose $u_j \rightharpoonup u \in \mathcal{K}_b$. Then

$$\Phi_b(u_j) = \int_{\mathbb{R}^3} \tilde{F}_\infty(u_j) \to 0.$$

Since $0 = \Phi_b'(u_j)(u_j^+ - u_j^-)$, (3.3) and (3.5) imply that

$$\begin{aligned}\|u_j\|_b^2 &= -\int_{\mathbb{R}^3} L u_j \overline{u_j^+ - u_j^-} + \int_{\mathbb{R}^3} g_\infty(|u_j|)u_j\overline{u_j^+ - u_j^-} \\ &\le \frac{|L|}{a-b}\|u_j\|_b^2 + \int_{\mathbb{R}^3} g_\infty(|u_j|)u_j\overline{u_j^+ - u_j^-}.\end{aligned}$$

By (3.10) and using Hölder inequality ($1/\sigma + 1/\sigma' = 1$, $\sigma = p/(p-2)$), one sees

$$\begin{aligned}\Big(1 - \frac{|L|}{a-b}\Big)\|u_j\|_b^2 &\le \Big(\int_{|u_j|\le\rho_\varepsilon} + \int_{|u_j|>\rho_\varepsilon}\Big) g_\infty(|u_j|)u_j\overline{u_j^+ - u_j^-} \\ &\le \varepsilon|u_j|_2^2 + c_\varepsilon\int_{\mathbb{R}^3}\tilde{F}_\infty(u_j)^{1/\sigma}|u_j||u_j^+ - u_j^-| \\ &\le \varepsilon|u_j|_2^2 + c_1c_\varepsilon\Big(\int_{\mathbb{R}^3}\tilde{F}_\infty(u_j)\Big)^{1/\sigma}|u_j|_p^2 \\ &\le c_2\varepsilon\|u_j\|_b^2 + c_3c_\varepsilon\Phi_b(u_j)^{1/\sigma}\|u_j\|_b^2\end{aligned}$$

hence $1 \le c_4\varepsilon + o(1)$, a contradiction. □

Remark 3.3. Let $\hat{S}_b$ denote the set of all least energy solutions u with $|u(0)| = |u|_\infty$. Remark that as before (4.11) keeps true for $u \in \hat{S}_b$ with C_0 independent of x and $u \in \hat{S}_b$. Although we do not know if the least energy solution of (3.1) is unique up to translation, we can show the following a substitute (which will not be used for proving our main result):

Lemma 3.4. *$\hat{S}_b$ is compact in $H^1(\mathbb{R}^3, \mathbb{C}^4)$.*

Proof. Let $u_j \in \hat{S}_b$ with $u_j \rightharpoonup u_0$ in H^1. It is clear that $u_0 \in \mathcal{K}_b$ and $u_j \to u_0$ in L^q_{loc} for any $q < 5$.

We claim first that $u_0 \ne 0$. In fact, suppose $u_0 = 0$ and $u_j \to 0$ in L^q_{loc}. Then (4.11) implies that $u_j \to 0$ in C^0_{loc}. But this contradicts with $|u_j(0)| = |u_j|_\infty \ge \delta > 0$.

Thus $u_0 \ne 0$ and hence $\Phi_b(u_0) \ge c_b$. Since there is no nonzero critical value of Φ_b less than c_b and $u_0 \ne 0$, it is standard to show that $\Phi_b(u_j - u_0) \to c_b - \Phi_b(u_0)$, $\Phi_b'(u_j - u_0) \to 0$, and $\|u_j - u_0\|_b \to 0$ (see e.g. [15]). Denoting $A = H_0 + b + L$ and using the equation for u_j and u_0,

$$\begin{aligned}|A(u_j - u_0)|_2 &= |g_\infty(|u_j|)u_j - g_\infty(|u_0|)u_0|_2 \\ &\le |g_\infty(|u_j|)(u_j - u_0)|_2 + |(g_\infty(|u_j|) - g_\infty(|u_0|))u_0|_2.\end{aligned}$$

Since $|u_j|_\infty \le C$ and $u_j \to u_0$ in E,

$$\int_{\mathbb{R}^3} |g_\infty(|u_j|)^2|u_j - u_0|^2 \le C|u_j - u_0|_2 \to 0,$$

and since $|u_0(x)| \to 0$ as $|x| \to \infty$,

$$\begin{aligned}
&\int_{\mathbb{R}^3} |(g_\infty(|u_j|) - g_\infty(|u_0|))u_0|^2 \\
&= \left(\int_{|x|<R} + \int_{|x|\geq R}\right) |(g_\infty(|u_j|) - g_\infty(|u_0|))u_0|^2 \ \to\ 0.
\end{aligned}$$

Therefore, one sees that $|A(u_j - u_0)|_2 \to 0$ which, together with (3.3), implies $|H_0(u_j - u_0)|_2 \to 0$ i.e., $u_j \to u_0$ in H^1. This proves the first conclusion. □

Following Ackermann [1], for fixed $u \in E^+$, we introduce the functional $\phi_u : E^- \to \mathbb{R}$ by

$$\begin{aligned}
\phi_u(v) &:= \Phi_b(u+v) \\
&= \frac{1}{2}\Big(\|u\|^2 - \|v\|^2 + \int_{\mathbb{R}^3} (b+L)(u+v)\overline{u+v}\Big) - \Psi_\infty(u+v) \\
&= \frac{1}{2}\big(\|u\|_b^2 - \|v\|_b^2\big) + \frac{1}{2}\int_{\mathbb{R}^3} L(u+v)\overline{u+v} - \Psi_\infty(u+v).
\end{aligned}$$

One has

$$\begin{aligned}
\phi_u''(v)[w,w] = &\ -\|w\|_b^2 + \int_{\mathbb{R}^3} Lw\overline{w} - \Psi_\infty''(u+v)[w,w] \\
= &\ -\|w\|_b^2 + \int_{\mathbb{R}^3} Lw\overline{w} - \int_{\mathbb{R}^3} \frac{g_\infty'(|u+v|)}{|u+v|}(\Re[(u+v)\overline{w}])^2 \\
&\ - \int_{\mathbb{R}^3} g_\infty(|u+v|)|w|^2
\end{aligned}$$

for all $v, w \in E^-$, which implies that $\phi_u(\cdot)$ is strictly concave (recalling that $L \leq 0$). Moreover

$$\phi_u(v) \leq \frac{1}{2}(\|u\|_b^2 - \|v\|_b^2) \to -\infty \ \text{ as } \|v\| \to \infty.$$

Plainly, ϕ_u is weakly sequentially upper semicontinuous. Thus there is a unique strict maximum point $h_b(u)$ for $\phi_u(\cdot)$, which is also the only critical point of ϕ_u on E^- and satisfies:

$$v \neq h_b(u) \Leftrightarrow \Phi_b(u+v) < \Phi_b(u + h_b(u)) \tag{3.11}$$

$$\begin{aligned}
&- (h_b(u), w)_b + \Re\int_{\mathbb{R}^3} L(u + h_b(u))\overline{w} \\
&= \Re\int_{\mathbb{R}^3} g_\infty(|u + h_b(u)|)(u + h_b(u))\overline{w}
\end{aligned} \tag{3.12}$$

for all $u \in E^+$ and $v,\ w \in E^-$.

As [1, Lemma 5.6] we have the following

Lemma 3.5. *There hold the following:*

(i) h_b *is* $\mathbb{R}^3$*-invariant, i.e.,* $h_b(a * u) = h_b(u)$ *where* $(a * u)(x) := u(x + a)$ *for all* $a \in \mathbb{R}^3$*;*

(ii) $h_b \in C^1(E^+, E^-)$ *and* $h_b(0) = 0$*;*

(iii) h_b *is a bounded map;*

(iv) If $u_n \rightharpoonup u$ *in* E^+*, then* $h_b(u_n) - h_b(u_n - u) \to h_b(u)$ *and* $h_b(u_n) \rightharpoonup h_b(u)$*. The same is true for* $|h_b(u)|_2^2$*.*

Now we define the reduce functional $I_b : E^+ \to \mathbb{R}$ by

$$\begin{aligned} I_b(u) &:= \Phi_b(u + h_b(u)) \\ &= \frac{1}{2}\|u\|_b^2 - \frac{1}{2}\|h_b(u)\|_b^2 + \frac{1}{2}\int_{\mathbb{R}^3} L(u + h_b(u))\overline{u + h_b(u)} - \tilde{\Psi}_\infty(u). \end{aligned}$$

Observe that

$$\begin{aligned} I_b'(u)v &= (u, v)_b - (h_b(u), h_b'(u)v)_b + \Re\int_{\mathbb{R}^3} L(u + h_b(u))\overline{v + h_b'(u)v} \\ &\quad - \Re\int_{\mathbb{R}^3} g_\infty(|u + h_b(u)|)(u + h_b(u))\overline{v + h_b'(u)v} \\ &= \Phi_b'(u + h_b(u))(v + h_b(v)) \qquad\qquad \text{(by (3.12))} \\ &= (u, v)_b - (h_b(u), h_b(v))_b + \Re\int_{\mathbb{R}^3} L(u + h_b(u))\overline{v + h_b(v)} \\ &\quad - \Re\int_{\mathbb{R}^3} g_\infty(|u + h_b(u)|)(u + h_b(u))\overline{v + h_b(v)} \end{aligned}$$

for all $u, v \in E^+$, and critical points of I_b and Φ_b are in one to one correspondence via the injective map $u \to u + h_b(u)$ from E^+ into E, that is, letting

$$\mathcal{K}_b^+ := \{u \in E^+ : \ I_b'(u) = 0\},$$

one has

$$\mathcal{K}_b = \{u + h_b(u) : \ u \in \mathcal{K}_b^+\}.$$

Next we discuss the mountain pass geometry of functional I_b.

Lemma 3.6. I_b *possesses the mountain pass geometry:*

1) *There is* $\rho > 0$ *such that* $\inf I_b(E^+ \cap \partial B_\rho) > 0$*;*

2) *For any finite dimensional subspace* $X \subset E^+$*,* $I_b(u) \to -\infty$ *as* $u \in X$*,* $\|u\| \to \infty$*.*

Proof. 1) We have

$$\begin{aligned}I_b(w) &= \frac{1}{2}\|w\|_b^2 - \frac{1}{2}\|h_b(w)\|_b^2 + \frac{1}{2}\int_{\mathbb{R}^3} L(w+h_b(w))\overline{w+h_b(w)} \\ &\quad - \Psi_\infty(w+h_b(w)) \\ &= \frac{1}{2}\|w\|_b^2 + \frac{1}{2}\int_{\mathbb{R}^3} Lw\overline{w} + \big(\Phi_b(w+h_b(w)) - \Phi_b(w)\big) - \Psi_\infty(w) \\ &\geq \frac{1}{2}\|w\|_b^2 + \frac{1}{2}\int_{\mathbb{R}^3} Lw\overline{w} - \Psi_\infty(w) \\ &\geq \frac{1}{2}\|w\|_b^2 - \frac{|L|}{2}|w|_2^2 - \Psi_\infty(w)\end{aligned}$$

for all $w \in E^+$. The desired conclusion now follows from (3.3), (3.5) and (3.9).

2) Let $P : L^{\hat{\mu}} \to X$ denote the natural projection. Then there is $c_1 > 0$ such that $c_1|Pv|_{\hat{\mu}}^{\hat{\mu}} \leq |v|_{\hat{\mu}}^{\hat{\mu}}$ for all $v \in L^{\hat{\mu}}$. Let $u \in X$. One has by (3.3), (3.5) and (3.7), for any $\varepsilon > 0$,

$$\begin{aligned}I_b(u) &= \frac{1}{2}\|u\|_b^2 - \frac{1}{2}\|h(u)\|_b^2 + \frac{1}{2}\int_{\mathbb{R}^3} L(u+h_b(u))\overline{u+h_b(u)} \\ &\quad - \int_{\mathbb{R}^3} F_\infty(u+h(u)) \\ &\leq \frac{1}{2}\|u\|_b^2 - \frac{1}{2}\|h(u)\|_b^2 + \varepsilon|u+h_b(u)|_2^2 - c_\varepsilon|u+h_b(u)|_{\hat{\mu}}^{\hat{\mu}} \\ &\leq \frac{1}{2}\|u\|_b^2 - \frac{1}{2}\|h(u)\|_b^2 + \frac{\varepsilon}{a-b}\|u+h_b(u)\|_b^2 - c_1c_\varepsilon|u|_{\hat{\mu}}^{\hat{\mu}} \\ &= \Big(\frac{1}{2} + \frac{\varepsilon}{a-b}\Big)\|u\|_b^2 - \Big(\frac{1}{2} - \frac{\varepsilon}{a-b}\Big)\|h_b(u)\|_b^2 - c'_\varepsilon\|u\|_b^{\hat{\mu}},\end{aligned}$$

hence the conclusion is true. □

Lemma 3.2 implies 0 is an isolated critical point of I_b. Therefore there is a $\nu > 0$ such that $\|w\| \geq \nu$ for all nontrivial critical points w of I_b. Let

$$\mathcal{N}_b^+ := \{u \in E^+ \setminus \{0\} : I_b'(u)u = 0\}.$$

Lemma 3.7. *For each $u \in E^+ \setminus \{0\}$, there is a unique $t = t(u) > 0$ such that $tu \in \mathcal{N}_b^+$.*

Proof. See [1]. We outline its proof as follows. Observe that, if $z \in E^+ \setminus \{0\}$ with $I_b'(z)z = 0$, it is not difficult to check

$$I_b''(z)[z,z] = (\nabla^2 I_b(z)z, z)_b < 0 \tag{3.13}$$

Let now $u \in E^+ \setminus \{0\}$. Setting $f(t) = I_b(tu)$ one has $f(0) = 0$, $f(t) > 0$ for $t > 0$ sufficiently small, and $f(t) \to -\infty$ as $t \to \infty$ by Lemma 3.6. Thus there is $t(u) > 0$ such that

$$I_b(t(u)u) = \sup_{t \geq 0} I_b(tu).$$

It is clear that

$$\frac{dI_b(tu)}{dt}|_{t=t(u)} = I_b'(t(u)u)u = \frac{1}{t(u)} I_b'(t(u)u)(t(u)u) = 0$$

and consequently by (3.13)

$$I_b''(t(u)u)(t(u)u) < 0.$$

One sees that such $t(u) > 0$ is unique. $\square$

Set

$$b_1 := \inf\{I_b(u) : u \in \mathcal{N}_b^+\},$$
$$b_2 := \inf\{I_b(u) : u \in \mathcal{K}_b^+ \setminus \{0\}\},$$
$$b_3 := \inf_{\gamma \in \Gamma_b} \max_{t \in [0,1]} I_b(\gamma(t)),$$

where

$$\Gamma_b := \{\gamma \in C([0,1], E) : \gamma(0) = 0, I_b(\gamma(1)) < 0\}.$$

Lemma 3.8. $c_b := b_1 = b_2 = b_3$.

Proof. We check $b_1 \leq b_2 \leq b_3 \leq b_1$.

- $b_1 \leq b_2$. This holds because $\mathcal{K}_b^+ \setminus \{0\} \subset \mathcal{N}_b^+$.
- $b_2 \leq b_3$. Let (u_j) be a Mountain-Pass sequence: $I_b(u_j) \to b_3$ and $I_b'(u_j) \to 0$. It is not difficult to check that (u_j) is bounded in E. By the concentration compactness principle, a standard argument shows that (u_j) is non-vanishing, that is, there exist $r, \eta > 0$ and $(a_j) \subset \mathbb{R}^3$ with

$$\limsup_{j \to \infty} \int_{B_r(a_j)} |u_j|^2 \geq \eta.$$

Set $v_j := a_j * u_j$. It follows from the invariance of the norm and of the functional under the $*$-action that $I_b(v_j) \to b_3, I_b'(v_j) \to 0$. Therefore $v_j \rightharpoonup v$ in E with $v \neq 0$ and $I'(v) = 0$. Additionally, a standard argument yields that $I_b(v_j - v) \to b_3 - I_b(v), I_b'(v_j - v) \to 0$ and $b_3 - I_b(v) \geq 0$ ([15]). Therefore $b_2 \leq I_b(v) = b_3$.

• $b_3 \le b_1$. Take $U \in \hat{S}_b$ and define $\gamma(t) := tU(x)$ for $t \ge 0$. Then since $I_b'(U) = 0$ one has $t(U) = 1$. Then $\gamma \in \Gamma_b$ and

$$b_3 \le \max_{t\in[0,1]} I_b(\gamma(t)) = I_b(U) = c_b.$$

The proof is completed. □

Let $u_0 \in E^+$ be such that $I_b(u_0) < 0$, and set

$$\Gamma_0 := \{\gamma \in C([0,1], E^+) : \ \gamma(0) = 0, \ \gamma(1) = u_0\}$$

$$b_0 := \inf_{\gamma\in\Gamma_0} \max_{t\in[0,1]} I_b(\gamma(t)).$$

Lemma 3.9. *There holds $b_0 = b_3$.*

Proof. Since $\Gamma_0 \subset \Gamma_b$ it is clear that $b_3 \le b_0$. Let $\gamma \in \Gamma_b$. Then as before $I_b(t\gamma(1))$ and $I_b(tu_0)$ are strictly decreasing for $t \ge 1$, and $I_b(t\gamma(1)) \to -\infty$, $I_b(tu_0) \to -\infty$ as $t \to \infty$. Let $\ell(s)$ be a cure in the two-dimensional subspace $\text{span}\{\gamma(1), u_0\}$ jointing $\gamma(1)$ and u_0 such that $I_b(\ell(s)) < 0$ for $1 \le s \le 2$ (such a cure exists because of Lemma 3.6-2)). Define $\hat{\gamma}(t)$ by $\hat{\gamma}(t) = \gamma(2t)$ for $t \in [0, 1/2]$ and $\hat{\gamma}(t) = \ell(2t)$ for $t \in [1/2, 1]$. Then $\hat{\gamma} \in \Gamma_0$ and $\max_{t\in[0,1]} I_b(\hat{\gamma}(t)) = \max_{t\in[0,1]} I_b(\gamma(t))$. Thus $b_0 \le b_3$. □

Lemma 3.10. *Let $u \in \mathcal{K}_b^+$ be such that $I_b(u) = c_b$, and set $E_u = E^- \oplus \mathbb{R}u$. Then*

$$\sup_{w\in E_u} \Phi_b(w) = I_b(u).$$

Proof. For any $w = v + su \in E_u$, by (3.11),

$$\begin{aligned}\Phi_b(w) &= \frac{1}{2}\|su\|_b^2 - \frac{1}{2}\|v\|_b^2 + \frac{1}{2}\int_{\mathbb{R}^3} L(v+su)\overline{v+su} - \int_{\mathbb{R}^3} F_\infty(v+su) \\ &\le \Phi_b(su + h_b(su)) = I_b(su).\end{aligned}$$

Thus since $u \in \mathcal{N}_b^+$,

$$\sup_{w\in E_u} \Phi_b(w) \le \sup_{s\ge 0} I_b(su) = I_b(u).$$

□

4 Proof of the main result

We are now going to prove the main result. Observe that (g_1) and (g_2) imply that

$$F(x,u) \geq c_1|u|^\mu \ \text{ for all } |u| \geq 1; \tag{4.1}$$

$$\tilde{F}(x,u) := \frac{1}{2}g(x,|u|)|u|^2 - F(x,u) \geq \frac{\mu-2}{2\mu}g(x,|u|)|u|^2 \ \text{ for all } u \tag{4.2}$$

hence $\tilde{F}(x,u) > 0$ if $u \neq 0$ and $\tilde{F}(x,u) \to \infty$ as $|u| \to \infty$; and

$$\sigma := \frac{p}{p-2} > 3, \quad \Big(\frac{|g(x,|u|)u|}{|u|}\Big)^\sigma \leq c_2\tilde{F}(x,u) \ \text{ for all } |u| \geq 1. \tag{4.3}$$

By (g_2), for $|u| \geq 1$, $g(x,|u|) \leq a_1|u|^{p-2}$, so $g(x,|u|)^{\sigma-1} \leq a_2|u|^2$ and consequently

$$\Big(\frac{|g(x,|u|)u|}{|u|}\Big)^\sigma = g(x,|u|)^\sigma \leq a_2 g(x,|u|)|u|^2 \leq a_2\tilde{F}(x,u).$$

Furthermore, for each $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that

$$|g(x,|u|)u| \leq \varepsilon|u| + C_\varepsilon|u|^{p-1} \tag{4.4}$$

and

$$F(x,u) \leq \varepsilon|u|^2 + C_\varepsilon|u|^p \tag{4.5}$$

for all (x,u).

Now consider the functional Φ_M defined by (2.1), or equivalently (2.2). Let $\mathcal{K}_M := \{u \in E : \ \Phi'_M(u) = 0\}$ be the critical set of Φ_M and recall that

$$c_M := \inf\{\Phi_M(u) : \ u \in \mathcal{K}_M \setminus \{0\}\}.$$

Using the same iterative argument of [17, Proposition 3.2] one obtains easily the following

Lemma 4.1. *If $u \in \mathcal{K}_M$ with $|\Phi_M(u)| \leq C_1$ and $|u|_2 \leq C_2$, then, for any $q \in [2,\infty)$, $u \in W^{1,q}(\mathbb{R}^3)$ with $\|u\|_{W^{1,q}} \leq \Lambda_q$ where Λ_q depends only on C_1, C_2 and q.*

Let S_M be the set of all least energy solutions u. If $u \in S_M$ then $\Phi_M(u) = c_M$ and, by $(g_1)-(g_2)$, a standard argument shows that S_M is bounded in E, hence, $|u|_2 \le C_2$ for all $u \in S_M$, some $C_2 > 0$. Therefore, as a consequence of Lemma 4.1 we see that, for each $q \in [2, \infty)$, there is $\Lambda_q > 0$ such that

$$\|u\|_{W^{1,q}} \le \Lambda_q \quad \text{for all } u \in S_M. \tag{4.6}$$

This, together with the Sobolev embedding theorem, implies that there is $C_\infty > 0$ with

$$|u|_\infty \le C_\infty \quad \text{for all } u \in S_M. \tag{4.7}$$

Lemma 4.2. *Ψ_M is weakly sequentially lower semicontinuous and Φ'_M is weakly sequentially continuous.*

Proof. In virtue of (4.4) and (4.5) the lemma follows easily because E embeds continuously into $L^q(\mathbb{R}^3, \mathbb{C}^4)$ for $q \in [2, 3]$ and compactly into $L^q_{loc}(\mathbb{R}^3, \mathbb{C}^4)$ for $q \in [1,\ 3)$ by Lemma 2.2. □

Lemma 4.3. *There exist $r > 0$ and $\rho > 0$ such that $\Phi_M|_{B_r^+}(u) \ge 0$ and $\Phi_M|_{S_r^+} \ge \rho$ where $B_r^+ = \{u \in E^+ : \|u\| \le r\}$ and $S_r^+ = \{u \in E^+ : \|u\| = r\}$.*

Proof. We only check the Coulomb potential case because the other case can be treated similarly. Assume (M_1) is satisfied. Recall that

$$|V_\kappa u|_2^2 \le 4\kappa^2 |H_0 u|_2^2 = |(2\kappa H_0)u|_2^2,$$

thus

$$|V_\kappa^{1/2} u|_2^2 \le ||(2\kappa H_0)|^{1/2} u|_2^2 = 2\kappa ||H_0|^{1/2} u|_2^2,$$

that is

$$\int_{\mathbb{R}^3} \frac{\kappa}{|x|} u \cdot \overline{u} \le 2\kappa ||H_0|^{1/2} u|_2^2 = 2\kappa \|u\|^2.$$

By (M_1),

$$-\int_{\mathbb{R}^3} M(x) u \cdot \overline{u} \le 2\kappa ||H_0|^{1/2} u|_2^2 = 2\kappa \|u\|^2.$$

For $u \in E^+$ one has

$$\begin{aligned}
\Phi_M(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_{\mathbb{R}^3} M(x) u \cdot \overline{u} - \int_{\mathbb{R}^3} F(x, u) \\
&\ge \left(\frac{1}{2} - \kappa\right)\|u\|^2 - \int_{\mathbb{R}^3} F(x, u) \\
&\ge \left(\frac{1}{2} - \kappa\right)\|u\|^2 - \varepsilon |u|_2^2 - C_\varepsilon |u|_3^3 \\
&\ge \left(\frac{1}{2} - \kappa\right)\|u\|^2 - c_1\varepsilon \|u\|^2 - c_2 C_\varepsilon \|u\|^3
\end{aligned}$$

so the conclusion follows. □

For continuing our arguments some further notations are in order. In the sequel, for the case of (M_1), the Coulomb-type potential, we consider $b=0$ and $L=0$ in (3.1), and denote the corresponding functional by

$$\Phi_0(u) := \frac{1}{2}\big(\|u^+\|^2 - \|u^-\|^2\big) - \int_{\mathbb{R}^3} F_\infty(u),$$

the critical set by $\mathcal{K}_0 = \{u \in E : \Phi_0'(u) = 0\}$, the least energy by $\hat{C}_0 = \min\{\Phi_0(u) : u \in \mathcal{K}_0 \setminus \{0\}\}$, the least energy solution set by $\hat{S}_0 = \{u \in \mathcal{K}_0 : \Phi_0(u) = \hat{C}_0\}$, and the induced map from $E^+ \to E^-$ by h_0. In the case (M_2)-(i) we consider $b=0$ and $L = M(\infty)$ in (3.1), and denote the functional by

$$\Phi_I(u) := \frac{1}{2}\big(\|u^+\|^2 - \|u^-\|^2\big) + \frac{1}{2}\int_{\mathbb{R}^3} M(\infty)u\overline{u} - \int_{\mathbb{R}^3} F_\infty(u)$$

with the critical set $\mathcal{K}_I$, the least energy $\hat{C}_I$, the least energy solution set $\hat{S}_I$, and the induced map $h_I : E^+ \to E^-$. Similarly in the case of (M_2)-(ii) we take $b = m_\infty$ and $L=0$ in (3.1), and denote correspondingly

$$\begin{aligned}\Phi_{II}(u) &:= \frac{1}{2}\big(\|u^+\|^2 - \|u^-\|^2\big) + \frac{m_\infty}{2}|u|_2^2 - \int_{\mathbb{R}^3} F_\infty(u)\\ &= \frac{1}{2}\big(\|u^+\|^2_{m_\infty} - \|u^-\|^2_{m_\infty}\big) - \int_{\mathbb{R}^3} F_\infty(u)\end{aligned}$$

(where $\|\cdot\|_{m_\infty}$ denotes the norm given by (3.4) with $b = m_\infty$) with notations $\mathcal{K}_{II}$, $\hat{C}_{II}$, $\hat{S}_{II}$ and h_{II}. Sometimes, if no confusion arises, we shall write simply Φ, $\mathcal{K}$, $\hat{C}$, $\hat{S}$ and h standing for one of the cases.

Lemma 4.4. *There is $R>0$ such that, for any $e \in E^+$ and $E_e = E^- \oplus \mathbb{R}e$,*

$$\Phi_M(u) < 0 \quad \text{for all } u \in E_e \setminus B_R \tag{4.8}$$

Proof. This follows from the following facts: if M satisfies (M_1) then

$$\begin{aligned}\Phi_M(u) &= \frac{1}{2}\big(\|u^+\|^2 - \|u^-\|^2\big) + \frac{1}{2}\int_{\mathbb{R}^3} M(x)u\overline{u} - \int_{\mathbb{R}^3} F(x,u)\\ &\le \frac{1}{2}\big(\|u^+\|^2 - \|u^-\|^2\big) - \int_{\mathbb{R}^3} F_\infty(u) = \Phi_0(u),\end{aligned}$$

similarly if (M_2)-(i) appears then

$$\Phi_M(u) \le \Phi_I(u) + \frac{1}{2}\int_{\mathbb{R}^3} \big(M(x) - M(\infty)\big)u\overline{u} \le \Phi_I(u)$$

and if (M_2)-(ii) is satisfied then

$$\Phi_M(u) = \frac{1}{2}\big(\|u^+\|^2 - \|u^-\|^2\big) + \frac{m_\infty}{2}|u|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3}(M - m_\infty)u\overline{u} - \int_{\mathbb{R}^3}F(x,u)$$
$$\leq \Phi_{II}(u) + \frac{1}{2}\int_{\mathbb{R}^3}(M - m_\infty)u\overline{u} \leq \Phi_{II}(u),$$

and Φ_n verifies (4.8) by Lemma 3.6 for $n = 0, I$ and II. □

Let $U_n \in \hat{S}_n$ for $n = 0, I$ and II. Set $e \equiv U_n^+$ and $E_e \equiv E^- \oplus \mathbb{R}e$.

Lemma 4.5. *We have*

$$d := \sup\{\Phi_M(u) : \ u \in E_e\} < \hat{C}.$$

Proof. Observe that by Lemma 4.3 and the linking property we have $d \geq \rho$.

Assume (M_1) is satisfied. Since by (M_1), $M(x) < 0$ and $\Phi_M(u) \leq \Phi_0(u)$ for all $u = v + sU_0^+$, and

$$\Phi_0(u) = \Phi_0(v + sU_0^+) \leq \Phi_0(sU_0^+ + h_0(sU_0^+)) = \hat{C}_0,$$

hence $d \leq \hat{C}_0$. Assume by contradiction that $d = \hat{C}_0$. Let $w_j = v + s_jU_0^+ \in E_e$ be such that $d - \frac{1}{j} \leq \Phi_M(w_j) \to d$. It follows from Lemma 4.4 that w_j is bounded and we can assume $w_j \rightharpoonup w$ in E with $v_j \rightharpoonup v \in E^-$ and $s_j \to s$. It is clear that $s > 0$ (otherwise there should appear the contradiction that $d = 0$). Then

$$d - \frac{1}{j} \leq \Phi_M(w_j) \leq \Phi_0(w_j) + \frac{1}{2}\int_{\mathbb{R}^3} M(x)w_j\overline{w}_j$$
$$\leq \hat{C}_0 + \frac{1}{2}\int_{\mathbb{R}^3} M(x)w_j\overline{w}_j$$

Taking the limit yields $\hat{C}_0 \leq \hat{C}_0 + \frac{1}{2}\int_{\mathbb{R}^3} M(x)w\overline{w}$ which implies that $w = 0$, a contradiction.

Similarly, if (M_2)-(i) holds, for $u = v + sU_I^+ \in E_e$, $\Phi_M(u) \leq \Phi_I(u) \leq \Phi_I(sU_I^+ + h_I(sU_I^+))$ hence $d \leq \hat{C}_I$, and as above

$$\Phi_M(u) \leq \Phi_I(u) + \frac{1}{2}\int_{\mathbb{R}^3}(M(x) - M(\infty))u\overline{u}$$

hence $d < \hat{C}_I$; if (M_2)-(ii) appears, for $u = v + sU_{II}^+ \in E_e$, $\Phi_M(u) \leq \Phi_{II}(u) \leq \Phi_{II}(sU_{II}^+ + h_{II}(sU_{II}^+))$ and

$$\Phi_M(u) \leq \Phi_{II}(u) + \frac{1}{2}\int_{\mathbb{R}^3}(M(x) - m_\infty)u\overline{u}$$

hence $d < \hat{C}_{II}$. □

Set

$$Q_0 := \{u = u^- + sU_0^+ : u^- \in E^-, s \geq 0, \|u\| < R\},$$
$$Q_I := \{u = u^- + sU_I^+ : u^- \in E^-, s \geq 0, \|u\| < R\}$$

and

$$Q_{II} := \{u = u^- + sU_{II}^+ : u^- \in E^-, s \geq 0, \|u\| < R\}.$$

Letting Q stand for one of $Q_n, n = 0, I, II$, as a consequence of Lemma 4.5 one has the following

Lemma 4.6. $\sup \Phi_M(Q) < \hat{C}$.

We now turn to the analysis on $(C)_c$ sequences. Firstly we have

Lemma 4.7. *Any $(C)_c$ sequence for Φ_M is bounded.*

Proof. It can be shown along the way of proof of [7, Lemma 7.3] by using (4.1)-(4.3) together with the representation (2.2) of Φ_M. □

In what follows let (z_j) denote a $(C)_c$-sequence for Φ_M. By Lemma 4.7, it is bounded, hence along a subsequence denoted again by (z_j), $z_j \rightharpoonup z_M$. It is obvious that z_M is a critical point of Φ_M. Moreover there holds the following

Lemma 4.8. *Either*

(i) $z_j \to z_M$, *or*

(ii) $c \geq \hat{C}$ *and there exist a positive integer* ℓ, *points* $\overline{z}_1, \cdots, \overline{z}_\ell \in \mathcal{K} \setminus \{0\}$, *a subsequence denoted again by* (z_j), *and sequences* $(a_j^i) \subset \mathbb{Z}^3$, *such that, as* $j \to \infty$,

$$\left\| z_j - z_M - \sum_{i=1}^{\ell} (a_j^i * \overline{z}_i) \right\| \to 0,$$
$$|a_j^i| \to \infty, \quad |a_j^i - a_j^k| \to \infty \quad \text{if } i \neq k$$

and

$$\Phi_M(z_M) + \sum_{i=1}^{\ell} \Phi(\overline{z}_i) = c.$$

Proof. Remark that, since (z_j) is bounded, it is a $(PS)_c$ sequence. The proof is well known (see for example Alama-Li [3]), which can be outlined as follows.

Firstly observe that $c \geq 0$ which follows by taking the limit in

$$\Phi_M(z_j) - \frac{1}{2}\Phi_M'(z_j)z_j \geq \left(\frac{1}{2} - \frac{1}{\mu}\right) \int_{\mathbb{R}^3} g(x, |z_j|)|z_j|^2 \geq 0.$$

Assume (i) is false. It is easy to see that $z_j^1 := z_j - z_M$ is a $(PS)_{c_1}$ sequence for Φ with $c_1 = c - \Phi_M(z_M)$ and $z_j^1 \rightharpoonup 0$. Note that Φ is invariant under the $*$-action of $\mathbb{R}^3$. A standard argument of concentration compactness principle implies that there exist a sequence $a_j^1 \in \mathbb{R}^3$ with $|a_j^1| \to \infty$ and a critical point $\overline{z}_1 \neq 0$ of Φ satisfying $a_j^1 * z_j^1 \rightharpoonup \overline{z}_1$ and

$$\Phi(a_j^1 * z_j^1) \to c - \Phi_M(z_M) - \Phi(\overline{z}_1) \geq 0.$$

Since $\Phi_M(z_M) \geq 0$ and $\Phi(\overline{z}_1) \geq \hat{C}$, one sees that $c \geq \hat{C}$.

If $a_j^1 * z_j^1 \to \overline{z}_1$ then we are done. Otherwise, repeating the above argument, after at most finitely many steps we finish the proof. $\square$

As a straight consequence of Lemma 4.8 we have the following

Lemma 4.9. *Φ_M satisfies the $(C)_c$ condition for all $c < \tilde{C}$.*

We now in a position to complete the proof of Theorem 1.3.

Proof of Theorem 1.3. Firstly we prove

Existence. It is clear that Φ_M checks (Φ_0) because of the form (2.2) and because of $F(x,u) \geq 0$. The combination of Lemma 4.2 and Lemma 2.4 implies that Φ_M verifies (Φ_1). Lemma 4.3 is nothing but (Φ_2). Lemma 4.4 shows that the linking condition of Theorem 2.3 is satisfied. These, together with Lemma 4.6, yield a $(C)_c$ sequence (u_j) with $c < \hat{C}$ for Φ_M. Now by virtue of Lemma 4.9, $u_j \to u$ so that $\Phi_M'(u) = 0$ and $\Phi_M(u) \geq \rho$.

Along the same lines of proof of Lemma 3.2 it is easy to check that $c_M > 0$. Let u_j be such that $\Phi_M(u_j) \to c_M, \Phi_M'(u_j) = 0$. Since $c_M < \hat{C}$, we have $u_j \to u$ in E with $\Phi_M(u) = c_M$ and $\Phi_M'(u) = 0$, hence $S_M \neq \emptyset$.

Compactness of S_M. Now we prove that S_M is compact in H^1. Let $u_j \in S_M$: $\Phi_M(u_j) = c_M$ and $\Phi_M'(u_j) = 0$. Hence (u_j) is a $(C)_{c_M}$ sequence. Since $c_M < \hat{C}$, it follows from Lemma 4.9 that $u_j \to u$ along a subsequence in E with clearly $u \in S_M$. By

$$H_0 u = -M(x)u + g(x,|u|)u$$

one has

$$\begin{aligned}|H_0(u_j - u)|_2 &\leq |M(u_j - u)|_2 + |g(\cdot,|u_j|)u_j - g(\cdot,|u|)u|_2 \\ &\leq o(1) + |g(\cdot,|u_j|)(u_j - u)|_2 + |g(\cdot,|u_j|) - g(\cdot,|u|)u|_2.\end{aligned}$$

Since $|u_j|_\infty \leq C$ and $u_j \to u$ in E,

$$\int_{\mathbb{R}^3} |g(x,|u_j|)^2 |u_j - u|^2 \leq C|u_j - u|_2^2 \to 0,$$

and since $|u(x)| \to 0$ as $|x| \to \infty$,

$$\int_{\mathbb{R}^3} |(g(x,|u_j|) - g(x,|u|))u|^2 = \Big(\int_{|x|<R} + \int_{|x|\geq R} \Big) |(g(x,|u_j|) - g(x,|u|))u|^2. \\ \to 0,$$

Therefore, one sees that $|H_0(u_j - u)|_2 \to 0$, i.e., $u_j \to u$ in H^1.

Exponential decay. Let $(g_1) - (g_3)$ and $(M_2) - (M_3)$ be satisfied. Assume $g : \mathbb{R}^3 \times (0,\infty) \to \mathbb{R}$ is of class C^1. Write

$$D = -i \sum_{k=1}^{3} \alpha_k \partial_k \quad \text{and} \quad H_0 = D + a\beta.$$

Using the relationship $H_0^2 = -\Delta + a^2$ and $H_0 u = -Mu + g(x,|u|)u$ one gets

$$-\Delta u + a^2 u = H_0(-Mu + g(x,|u|)u).$$

Hence there holds

$$\begin{aligned} \Delta u &= a^2 u + H_0(Mu) - H_0(g(x,|u|)u) \\ &= a^2 u + D(Mu) + a\beta M u - H_0(g(x,|u|)u) \\ &= \mathcal{E}(a,M)u + r_M(x,u)u \end{aligned} \tag{4.9}$$

where if $M = V\beta$

$$r_{V\beta}(x,u) = -g(x,|u|)^2 + i \sum_{k=1}^{3} \alpha_k \Big(\frac{\partial g(x,|u|)}{\partial x_k} + g_s(x,|u|) \Re\Big[\frac{u}{|u|} \overline{\partial_k u} \Big] \Big),$$

and if $M = V$

$$r_V(x,u) = -i \sum_{k=1}^{3} \alpha_k \partial_k V + 2Vg(x,|u|) + r_{V\beta}(x,u).$$

Letting

$$\operatorname{sgn} u = \begin{cases} \dfrac{\overline{u}}{|u|} & \text{if } u \neq 0 \\ 0 & \text{if } u = 0, \end{cases}$$

by the Kato's inequality ([12]), (4.9) and the real positivity of $\mathcal{E}(a,M)$, there exist $R > 0$ and $\tau > 0$ such that

$$\begin{aligned} \Delta |u| &\geq \Re\big[\Delta u (\operatorname{sgn} u)\big] \\ &= \Re\Big[\mathcal{E}(a,M) u \frac{\overline{u}}{|u|} + r_M(x,u) u \frac{\overline{u}}{|u|} \Big] \\ &\geq \tau |u| + \Re\Big[r_M(x,u) u \frac{\overline{u}}{|u|} \Big] \end{aligned} \tag{4.10}$$

for all $|x| \geq R$.

Observe that, for $f_k : \mathbb{R}^3 \to \mathbb{R}, k = 1, 2, 3,$

$$B_f := \sum_{k=1}^{3} \alpha_k f_k = \begin{pmatrix} 0 & 0 & f_3 & f_1 - if_2 \\ 0 & 0 & f_1 + if_2 & -f_3 \\ f_3 & f_1 - if_2 & 0 & 0 \\ f_1 + if_2 & -f_3 & 0 & 0 \end{pmatrix}$$

is a Hermitian matrix. Applying to $f_k = \partial g/\partial x_k,\ g_s \Re[u\overline{u}/|u|\,],\ \partial_k V$, respectively, one sees plainly that, if $M = V\beta$

$$\Re\left[r_M(x,u)u\frac{\overline{u}}{|u|}\right] = -g(x,|u|)^2|u|,$$

and if $M = V$

$$\Re\left[r_M(x,u)u\frac{\overline{u}}{|u|}\right] = \big(2V - g(x,|u|)\big)g(x,|u|)|u|.$$

Remark that, by assumption on V we see $\mathcal{E}(a, M) \in L^\infty$, and by (4.7), $g(x, |u|)$ is bounded uniformly in $u \in S_M$. Thus the sub-solution estimate [36] implies that

$$|u(x)| \leq C_0 \int_{B_1(x)} |u(y)|\, dy \tag{4.11}$$

with C_0 independent of x and $u \in S_M$.

Since S_M is compact in H^1, $|u(x)| \to 0$ as $|x| \to \infty$ uniformly in $u \in S_M$. In fact, if not, then by (4.11) there exist $\kappa > 0$, $u_j \in S_M$ and $x_j \in \mathbb{R}^3$ with $|x_j| \to \infty$ such that $\kappa \leq |u_j(x_j)| \leq C_0 \int_{B_1(x_j)} |u_j|$. One may assume $u_j \to u \in S_M$ in H^1 and to get

$$\begin{aligned} \kappa &\leq C_0 \int_{B_1(x_j)} |u_j| \leq C_0 \int_{B_1(x_j)} |u_j - u| + C_0 \int_{B_1(x_j)} |u| \\ &\leq C'\Big(\int_{\mathbb{R}^3} |u_j - u|^2\Big)^{1/2} + C_0 \int_{B_1(x_j)} |u| \ \to\ 0, \end{aligned}$$

a contradiction. Now since $g(x, s) \to 0$ as $s \to 0$ uniformly in x, one may take $0 < \delta < \tau/2$ and $R > 0$ such that $|u(x)| \leq \delta$ and

$$\left|\Re\left[r_M(x,u)u\frac{\overline{u}}{|u|}\right]\right| \leq \frac{\tau}{2}|u|$$

for all $|x| \geq R, u \in S_M$. This, together with (4.10), implies

$$\Delta|u| \geq \delta|u| \ \text{ for all } |x| \geq R, u \in S_M.$$

Let $\Gamma(y) = \Gamma(y, 0)$ be a fundamental solution to $-\Delta + \delta$ (see, e.g., [30]-[34]). Using the uniform boundedness, one may choose Γ so that $|u(y)| \le \delta\Gamma(y)$ holds on $|y| = R$, all $u \in S_M$. Let $w = |u| - \delta\Gamma$. Then

$$\begin{aligned} \Delta w &= \Delta|u| - \delta\Delta\Gamma \\ &\ge \delta|u| - \delta^2\Gamma \\ &= \delta(|u| - \delta\Gamma) = \delta w. \end{aligned}$$

By the maximum principle we can conclude that $w(y) \le 0$ on $|y| \ge R$. It is well known that there is $C' > 0$ such that $\Gamma(y) \le C' \exp(-\sqrt{\delta}|y|)$ on $|y| \ge 1$. We see that

$$|u(y)| \le C \exp(-\sqrt{\delta}|y|)$$

for all $y \in \mathbb{R}^3$ and all $u \in S_M$.

The proof is completed.

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