PROPERTIES OF POSITIVE SOLUTIONS TO AN ELLIPTIC EQUATION WITH NEGATIVE EXPONENT

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ABSTRACT. In this paper, we study some quantitative properties of positive solutions to a singular elliptic equation with negative power on the bounded smooth domain or in the whole Euclidean space. Our model arises in the study of the steady states of thin films and other applied physics as well as differential geometry. We can get some useful local gradient estimate and L^1 lower bound for positive solutions of the elliptic equation. A uniform positive lower bound for convex positive solutions is also obtained. We show that in lower dimensions, there is no stable positive solutions in the whole space. In the whole space of dimension two, we can show that there is no positive smooth solution with finite Morse index. Symmetry properties of related integral equations are also given.

1. INTRODUCTION

In this paper, we study some properties of positive solutions of the following elliptic equation in a domain $\Omega \subset \mathbf{R}^n$

$$\Delta u = u^\tau, \quad in \ \ \Omega, \tag{1}$$

where $\tau < 0$.

Problem (1) arises in many branches of applied sciences. For example, it can be considered as steady states of thin films. Equations of the type

$$u_t = -\nabla \cdot (f(u)\nabla\Delta u) - \nabla \cdot (g(u)\nabla u) \tag{2}$$

have been used to model the dynamics of thin films of viscous fluids, where $z = u(x,t)$ is the height of the air/liquid interface. The zero set $\mathbf{\Sigma}_u = \{u = 0\}$ is the liquid/solid interface and is sometimes called set of **ruptures**. Ruptures play a very important role in the study of

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thin films. The coefficient $f(u)$ reflects surface tension effects-a typical choice is $f(u)=u^3$. The coefficient of the second-order term can reflect additional forces such as gravity $g(u)=u^3$, van der Waals interactions $g(u)=u^m$, $m<0$. For backgrounds of (2), we refer to [2], [3], [25], [26], [27], and the references therein.

In general, let us assume that $f(u)=u^p$, $g(u)=u^m$, where $p, m\in R$. Then if we consider the steady-state of (2), we see that u satisfying

$$u^p\nabla\Delta u+u^m\nabla u=\mathcal{C}$$

is a steady state of (2). Where $\mathcal{C}=(C_1,C_2,\dots,C_n)$ is some constant vector. By assuming $\mathcal{C}=\mathbf{0}$ (which prevents linear term on x), we obtain

$$\Delta u+\frac{u^\tau}{\tau}-C=0 \ \text{ in } \Omega, \tag{3}$$

where $\tau=m-p+1$ and C is some constant. (Here we have assumed that $\tau\neq 0$. If $\tau=0$, we have to replace $\frac{u^\tau}{\tau}$ by $\log u$.) Note that solutions to (1) are steady-states of (3) but the reverse is not true.

When $\Omega=\mathbf{R}^n$, the radially symmetric solutions to (3) has been studied in [24].

When $\tau=-2$, problem (1) also arises in the study of MEMS. We refer to [15], [16], [17], [12], [19] and [20] and the references therein.

When $\tau=-1$, equation (1) is related to the study of singular minimal hypersurfaces with symmetry. See [31], [35] and the references therein.

In this paper, we study *quantitative* properties of solutions to (1), including the gradient estimates, L^1-estimates, global upper bounds, Liouville properties, classification of stable and finite Morse index solutions, and symmetry properties.

Here we state the main results.

The first eight theorems concerns solutions to (1).

Theorem 1. *(*Gradient Estimates*) Let $u\in C^2(\Omega)$ be a positive solution to the equation (1) in Ω. Then for any $R>0$, $x_0\in\Omega$, and $x\in B_R(x_0)\subset\Omega$ we have absolute constant $C=C(R)$ such that*

$$|\nabla u(x)|^2\leq Cu(x)^2+u(x)^{1+\tau}. \tag{4}$$

Theorem 2. *(*L^1-estimates*) Let $u\in C^2(\Omega)$ be a positive solution to the equation (1) in Ω. Then for any $R>0$ and $x_0\in\Omega$ (with $B_R(x_0)\subset\Omega$), we have absolute constant $C(n,\tau)$ such that*

$$\int_{B_R(x_0)} u\geq C(n,\alpha)R^{n+\frac{2}{1-\tau}}. \tag{5}$$

Theorem 3. *(*Global upper bound*) Let $1 \le u \in C^2(\mathbf{R}^n)$ be a positive solution to the equation (1) in $\mathbf{R}^n$. Then we have absolute constant $C(n)$ such that*

$$u(x) \le C(n)(|x|^2 + 1), \tag{6}$$

and

$$|\nabla u(x)| \le C(n)(|x| + 1), \tag{7}$$

for all $x \in \mathbf{R}^n$.

Theorem 4. *(*Liouville Properties*) Assume that $n > 2$ and $\tau \le -1 - \frac{2n}{n-2}$. We have the* Liouville property *that there is no positive convex solution $u(x)$ to (1) both on the whole space $\mathbf{R}^n$ and on the half space $\mathbf{R}^n_+$ with $u(x) \ge 1$ everywhere.*

Theorem 5. *(*Compactness*) Assume that $n > 2$ and $\tau \le -1 - \frac{2n}{n-2}$. Let Ω be a bounded or unbounded smooth convex domain. Let u be a positive convex solution to (1) on Ω. Then, we have a uniform constant $C = C(\Omega)$ such that*

$$u(x) \ge C, \quad x \in \Omega.$$

Theorem 6. *(*Stable Solutions*) If $2 \le n < 2 + \frac{4}{1-\tau}(-\tau + \sqrt{\tau^2 - \tau})$, then there are no* stable *positive solutions to (1) in $\mathbf{R}^n$.*

Theorem 7. *(*Finite Morse index solutions*) There is no finite Morse index positive solutions to (1) in $\mathbf{R}^2$.*

Theorem 8. *(*Existence*) Assume $(-1 \ne)\tau \le 0$, and let $\Omega \subset \mathbf{R}^n$ be a bounded smooth domain. Given any smooth positive boundary data ϕ. Assume that $\underline{u}$ is a sub-solution to (1) with $\underline{u} \le \phi$ on $\partial\Omega$. Then there is smooth positive solution to (1) with boundary data ϕ.*

We remark that in the special case when $\tau = -1$ Theorem 6 has been obtained in [31].

In the next two theorems, we will study a related integral equation

$$u(x) = h(x) - \int_{\mathbf{R}^n} |x - y|^{\mu - n} u(y)^\tau dy, \tag{8}$$

with $n \ge 2$, $0 < \mu < n$, $h(x)$ is a positive smooth function, and $\tau < 0$.

We prove the following two symmetry properties

Theorem 9. *Given some $\beta > 1$ and $q > 1$. Let*

$$u^{\tau - 1} \in L^\beta(\mathbf{R}^n), \tag{9}$$

be a positive solution of equation (8) with

$$\tau < 0 \quad \text{and} \quad \beta = \frac{\tau - 1}{\frac{n-\mu}{n}\tau - 1} > \frac{2n}{n-\mu}.$$

Assume that for some plane π, we have $h(x) = h(\pi(x))$. Then $u(x)$ is symmetric to the plane π.

Theorem 10. *Assume that $h(x) = h$ is a constant function. Given some $\beta > 1$ and $q > 1$. Assume that*

$$\tau < 0 \quad and \quad \beta = \frac{\tau-1}{\frac{n-\mu}{n}\tau - 1} > \frac{2n}{n-\mu}.$$

Then, for any positive solution to (8) with

$$u^{\tau-1} \in L^{\beta}(\mathbf{R}^n), \tag{10}$$

and

$$|x|^{\mu-n}u(\frac{x}{|x|^2}) \in L^q(\mathbf{R}^n), \tag{11}$$

u is radial symmetric at zero.

The plan of our paper is follows: In last two sections, we discuss symmetry properties of related integral equations (8). We prove Theorem 1 in section 2 and prove Theorem 2 in section 3. In section 5, we prove Theorem 3. Theorem 5 is proved in section 7. The Liouville property is proved in section 6. The proof of Theorems 6 and 7 are given in section 8. As we mentioned before, we shall discuss the existence theory of positive solutions to (1) in section 9. Many consequences of Theorem 2 will be discussed in section 4.

In the following, we shall use C to denote different constants which depend only on n, τ, μ, and the solution u in varying places.

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2. PROOF OF THEOREM 1

In this section, we use the maximum principle (see [11] and [30]) to obtain the *gradient estimate* for positive solutions of the elliptic equation (1), and prove Theorem 1.

Proof. Recall the following *basic formula*. Let v be any smooth function in $\mathbb{R}^n$. Then, we have

$$\frac{1}{2}\Delta|\nabla v|^2 = (\nabla\Delta v, \nabla v) + |D^2 v|^2,$$

which can be proved by an elementary calculation.

Let us begin with the gradient estimate for positive solution u to the following general elliptic equation:

$$\Delta u = f(u), \quad in \ \ \mathbf{R}^n.$$

In our case we shall set $f(u) = u^\tau$. Set

$$w = \log u.$$

Then we have

$$\nabla w = u^{-1}\nabla u,$$

and we can get

$$\Delta w = -|\nabla w|^2 + F(w),$$

where

$$F(w) = u^{-1}f(u) = e^{-w}f(e^w).$$

In particular, $F(w) = e^{(-1+\tau)w}$ and $F'(w) = (-1+\tau)F(w)$ for our case.

Assume $R_2 > R > 0$. Let ϕ be a cut-off function in $B_0(R_2)$ with $\phi = 1$ on $B_0(R)$. Define

$$P = \phi|\nabla w|^2,$$

which is usually called the Harnack quantity for the solution u.

At the maximum point of P, we have the first order condition

$$\nabla P = 0,$$

which implies that

$$\nabla|\nabla w|^2 = -\phi^{-2}\nabla\phi P.$$

and the second order condition:

$$0 \geq \Delta P = P_0(\phi)P + \phi\Delta|\nabla w|^2, \tag{12}$$

where

$$P_0(\phi) = \Delta\phi - 2|\nabla\phi|^2\phi^{-2}.$$

Using the *basic formula*, we have

$$\phi\Delta|\nabla w|^2 = 2\phi|D^2w|^2 + 2\phi(\nabla\Delta w, \nabla w).$$

Note that

$$\phi|D^2w|^2 \geq \frac{2\phi}{n}|\Delta w|^2 = \frac{2}{n\phi}(-P + \phi F(w))^2,$$

and

$$2\phi(\nabla\Delta w, \nabla w) \geq 2F'P - 2\phi(\nabla|\nabla w|^2, \nabla w) = 2F'P - 2\phi^{-1}(\nabla\phi, \nabla w)P,$$

and then, for any $\mu > 0$,

$$2\phi(\nabla\Delta w, \nabla w) \geq 2F'P - 2\mu^{-1}\phi^{-2}|\nabla\phi|^2P - \mu\phi^{-1}P^2.$$

Choose $\mu = \frac{1}{4n}$. Then

$$2\phi(\nabla\Delta w, \nabla w) \geq 2F'P - 4n\phi^{-2}|\nabla\phi|^2P - \frac{1}{4n}\phi^{-1}P^2.$$

Hence,

$$\phi\Delta|\nabla w|^2 \geq \frac{2}{n\phi}(-P+\phi F(w))^2 + 2F'P - 4n\phi^{-2}|\nabla\phi|^2 P - \frac{1}{4n}\phi^{-1}P^2.$$

Then from (12) we have

$$A(\phi, F')P \geq \frac{2}{n}(-P+\phi F)^2 - \frac{1}{4n}P^2.$$

Here

$$A(\phi, F') = 4n\phi^{-1}|\nabla\phi|^2 - 2\phi F' - \phi P_0(\phi).$$

If $P \leq 2\phi F$, then we have

$$|\nabla w|^2 \leq 2F.$$

We remark that in this case, we have

$$|\nabla u|^2 \leq 2u^2 F = 2uf(u).$$

Otherwise, we have

$$-P+\phi F \leq -P/2 \leq 0$$

and

$$\frac{2}{n}(-P+\phi F)^2 - \frac{1}{4n}P^2 \geq \frac{1}{4n}P^2.$$

Hence, we have

$$P \leq 4nA(\phi, F').$$

In conclusion, we have on $B_0(R)$,

$$|\nabla w|^2 \leq max(4nA(\phi, F'), 2F),$$

which implies the conclusion of Theorem 1.

□

We remark that our gradient estimate can be extended to other elliptic equation like

$$-\Delta u = u^\tau.$$

As a consequence of the local gradient estimate in Theorem 1, we have the following improvement of Theorem 7.1 in [31].

Corollary 11. *For any sequence (u_j) of positive solutions to the equation (1) with $\tau = -1$ and with boundary data $\phi_j \leq M$, and for every compact sub-domain K of Ω, there is a constant $C = C(n, K, \Omega)$ such that*

$$|\nabla u_j| \leq (M+1)C, \text{ on } K. \tag{13}$$

Hence, the limit u of any convergent subsequence of (u_j) is a Lipschitz continuous weak solution to a free boundary problem of the equation

$$\Delta u = u^\tau \chi_{\{u>0\}}. \tag{14}$$

For the proof of Corollary (11), we just note that the solution u is a subharmonic function and it attains its maximum only on the boundary $\partial\Omega$. Then we use the gradient bound to get the conclusion. We remark that the result is optimal in the sense that u is not differentiable at its zero point as noticed in [31]. It is unclear how large the zero level set $\Sigma_u(u) = \{u = 0\}$ is. However, a general study was made in the paper [23] and a partial result was obtained in [19].

Note that the gradient estimate implies the Harnack inequality for positive solutions with $u(x) \geq 1$ for all $x \in B_R(x_0)$. One can use this fact in deriving compactness result. The application of such compactness result on the existence theory of positive solutions to (1) will be discussed in section 9.

3. PROOF OF THEOREM 2

Our aim in this section is to obtain an L^1-estimates for solutions of (1). We shall prove a more general version of Theorem 2.

Theorem 12. *Assume $\tau \leq 0$ and assume that $\Omega \subset \mathbf{R}^n$ is an open subset in $\mathbf{R}^n$. Let $f : \mathbf{R} \to \mathbf{R}_+$ be a positive function such that*

$$s^{\frac{\tau}{\tau-1}} f(s)^{\frac{1}{1-\tau}} \geq C_0, \ \ for\ s > 0$$

for some constant C_0. Let $u \in C^0(\Omega)$ be a positive weak solution to the equation

$$\Delta u = f(u) \tag{15}$$

in Ω. Then for any $R > 0$ and $x_0 \in \Omega$ (with $B_R(x_0) \subset \Omega$), we have absolute constant $C(n, \tau)$ such that

$$\int_{B_R(x_0)} u \geq C(n, \alpha) R^{n+\frac{2}{1-\tau}}. \tag{16}$$

Proof. Without loss of generality, we take $x_0 = 0$. Let $R_2 = 2R_1 > 0$ and let $\xi(|x|)$ be a cut-off function with its support in the ball $B_{R_2}(0)$, $\xi(|x|) = 1$ on $B_{R_1}(0)$, and

$$|\nabla \xi| \leq 4/R_1, \ \ |\Delta \xi| \leq 100/R_1^2.$$

Multiplying both sides of (1) by $|x|^2\xi$ and integrating over the ball $B_{R_2}(0)$, we then get

$$\int u\Delta(|x|^2\xi) = \int f(u)|x|^2\xi.$$

Note that the right side is bigger than

$$\int_{B_{R_1}(0)} f(u)|x|^2;$$

and the left side is less than

$$C\int_{B_{R_2}(0)} u,$$

where $C>0$ is an absolute constant depending only on the dimension n. That implies that

$$\int_{B_{R_1}(0)} f(u)|x|^2 \leq C\int_{B_{R_2}(0)} u.$$

Let $p>1$. Then we have

$$(\int_{B_{R_1}(0)} f(u)|x|^2)^{1/p}(\int_{B_{R_1}(0)} u)^{(p-1)/p} \leq C^{1/p}\int_{B_{R_2}(0)} u.$$

Using Holder's inequality to the left side, we get

$$\int_{B_{R_1}(0)} |x|^{2/p} f(u)^{\frac{1}{p}} u^{(p-1)/p} \leq C^{1/p}\int_{B_{R_2}(0)} u.$$

Choose $p=-\tau+1$. Then by our assumption on f, we have

$$\int_{B_{R_1}(0)} |x|^{2/p} \leq C^{1/p}\int_{B_{R_2}(0)} u.$$

It is elementary to compute that

$$\int_{B_{R_1}(0)} |x|^{2/p} = \frac{p\omega_n}{2+p(n-1)} R_1^{n+\frac{2}{p}},$$

ω_n is the volume of the unit ball $B_1(0)$. This implies the conclusion of Theorem 12.

□

We remark that our argument above can also be used to smooth positive solutions to the following equation

$$-\Delta u = u^\tau, \quad in \quad \Omega,$$

with $\tau \leq 0$.

Note that that Theorem 2 follows from Theorem 12 when $f(u) = u^\tau$. Another proof of Theorem 2 is to use the convexity of $f(u)$, the spherical average method, and an ODE comparison lemma to get the L^1 lower bound. However, our proof is more general and works also on manifolds.

There are many consequences of Theorem 2, and they will be discussed in section 4.

4. Consequences of Theorem 2

As an easy consequence of Theorem 2, we have

Corollary 13. *Assume $\tau \leq 0$ and $\Omega \subset \mathbf{R}^n$. Let f be as in Theorem 2 above. Let $u \in C^0(\Omega)$ be a positive weak solution to the equation to (15) in Ω. Then for any $R > 0$ and $x_0 \in \Omega$ (with $B_R(x_0) \subset \mathbf{R}^n$), we have absolute constant $C(n, \tau)$ such that*

$$max_{\partial B_R(x_0)} u = sup_{B_R(x_0)} u \geq C(n, \alpha) R^{\frac{2}{1-\tau}}. \tag{17}$$

The proof of this is direct by using the L^1 lower bound (5) since u is subharmonic and the maximum occurs only at boundary point.

Corollary 13 immediately implies the following

Corollary 14. *Assume $\tau \leq 0$ and assume that $\Omega \subset \mathbf{R}^n$ is an open subset in $\mathbf{R}^n$. Let $f : \mathbf{R} \to \mathbf{R}_+$ be a positive function such that*

$$s^{\frac{\tau}{\tau-1}} f(s)^{\frac{1}{1-\tau}} \geq C_0, \ for\ s > 0$$

for some constant C_0. Then there is a positive constant $C = C(\Omega)$ such that if the positive boundary data $\phi \leq C$ on $\partial\Omega$, the Dirichlet problem to the equation (15) in Ω with $u = \phi$ on the boundary $\partial\Omega$ has no nontrivial positive weak C^0-solution.

In fact, we take a ball $B_R(x_0)$ in the domain Ω and let $C = C(n, \alpha) R^{\frac{2}{1-\tau}}$. Then we have $sup_\Omega u = sup_{\partial\Omega} \phi > C$.

Using this result, we can find a sequence (u_j) of positive solutions to (1) in the ball $B = B_1(0)$ such that $min u_j \to 0+$ as $j \to +\infty$. Choose C large enough, we can solve (1) to get a unique positive radial solution u with $u = C$ on the boundary ∂B. Then there is a constant $M > 0$ such that $|u|_{C^3} \leq M$. Let ϕ be as in the Corollary above and let $\phi_t = tC + (1-t)\phi$, where $t \in [0, 1]$. Let $p > 1$ and let $\delta > 0$. Given a smooth function $u > 0$ in $W^{2,p}(B)$ with boundary data ϕ_t. Consider the problem

$$\Delta v = \frac{v}{u^{1-\tau}}, \ in\ B,$$

with the boundary data $v = \phi_t$ on the boundary ∂B. Set $T_t(u) = v$ and

$$\mathbf{U}(\delta) = \{v \in W^{2,p}(B); v > \delta, |v|_{W^{2,p}(B)} \leq M(\delta)\}.$$

Here $M(\delta)$ is a constant coming from the L^p estimate depending on δ. Then the fixed point of T_t in $\mathbf{U}(\delta)$ is a positive solution to (1) with boundary data $u = \phi_t$. Note that T_t is a compact operator from $\mathbf{U}(\delta) \to W^{2,p}(B)$,

$$deg(1 - T_1, \mathbf{U}(\delta), 0) = 1$$

and

$$deg(1 - T_0, \mathbf{U}(\delta), 0) = 0.$$

Hence, we have $t_0 \in (0,1)$ and $u \in \partial \mathbf{U}(\delta)$ such that $T_{t_0}(u) = u$. Choose $\delta = \delta_j \to 0$. Hence, we have a sequence $t_j \in (0,1)$ such that

$$T_{t_j}(u_j) = u_j$$

and $u_j \in \partial \mathbf{U}(\delta_j)$ (which implies that $min_B u_j = \delta_j \to 0$).

We can do the same thing to (1) in any bounded domain Ω. Hence, we have

Corollary 15. *Given a bounded regular domain Ω in $\mathbf{R}^n$ such that for sufficiently large constant $C > 0$ as the Dirichlet boundary data, the problem (1) has unique positive solution. There exists a sequence of positive solutions (u_j) to (1) in the domain Ω with*

$$min_\Omega u_j \to 0.$$

An open question is, which domain has the uniqueness property in Corollary (15).

Using Theorem 2, we can easily derive the following:

Proposition 16. *There is no positive solution to (1) in a cone-like unbounded domain Ω with the bound*

$$R^{-n-\frac{2}{1-\tau}+\sigma} \int_{B_R} u dx \leq K$$

for every ball $B_R \subset \Omega$ with $R \geq 1$, for some constant $K > 0$ and $\sigma > 0$.

Proof. Assume we have a positive solution u. Note that we can choose an arbitrary large ball B_R in the domain Ω. Then, using our Theorem 2, we get

$$R^\sigma \leq K,$$

which is not true by sending $R \to +\infty$. □

We point out that for solution u to (1), the quantity

$$R^{-n-\frac{2p}{1-\tau}} \int_{B_R(0)} u^p$$

is dimensionless. By this, we mean that if u is a solution to (1) in the ball $B_R(0)$, then the function

$$v(x) = R^{-\frac{2}{1-\tau}} u(Rx)$$

is a solution to (1) in $B_1(0)$ with

$$R^{-n-\frac{2}{1-\tau}p} \int_{B_R} u^p dx = \int_{B_1(0)} v^p.$$

Hence, in this sense, the L^1 lower bound estimate in Theorem 2 is the best one.

5. PROOF OF THEOREM 3

In this section, we derive a *global upper bound* for positive solutions to (1) in $\mathbf{R}^n$.

Assume that $u(x) \geq 1$ on $B_R(0)$ satisfies

$$\Delta u = u^\tau.$$

Then

$$0 < \Delta u \leq 1.$$

Using the Gradient estimate in Theorem 1, we can easily get that

$$u(x) \leq u(0)e^{CR}$$

for all $x \in B_R(0)$. Here C is a uniform constant. However, this estimate is too rough.

We now use the mean value property to do better. Fix $0 < r = |x| \leq R/4$. Since $u > 0$ is subharmonic, on one hand, we have

$$u(x) \leq \frac{1}{|B_r(x)|}\int_{B_r(x)} u.$$

On the other hand, let

$$w(y) = u(y) - \frac{|y|^2}{2n}.$$

Then

$$\Delta w = \Delta u - 1 \leq 0.$$

Hence w is super-harmonic, and for $R = 2r$,we have

$$1 = w(0) \geq \frac{1}{|B_R(0)|}\int_{B_R(0)} u - \frac{2|x|^2}{n(n+2)}. \tag{18}$$

Note that $B_r(x) \subset B_R(0)$ and

$$\frac{1}{|B_R(0)|}\int_{B_R(0)} u \geq \frac{|B_r(x)|}{|B_R(0)|}\frac{1}{|B_r(x)|}\int_{B_r(x)} u \geq \frac{1}{2^n}\frac{1}{|B_r(x)|}\int_{B_r(x)} u \geq \frac{u(x)}{2^n}.$$

Hence, using (18), we have (6):

$$u(x) \leq C_n(|x|^2 + 1).$$

Using the standard interpolation argument and $0 < \Delta \leq 1$, we get the gradient growth:

$$|\nabla u(x)| \leq C_n(|x| + 1).$$

□

The important consequence of Theorem 3 is the following well-known result.

Corollary 17. *Assume that $u > 0$ is a positive solution to the equation*

$$\Delta u = 1, \quad in \ \ \mathbf{R}^n.$$

Then u is a polynomial of the form

$$a_0 + \sum_{j=1}^n a_j x_j^2,$$

where $a_0 > 0$,$a_j \geq 0$ for $j = 1, ...n$, and $2\sum_j a_j = 1$.

Proof. We may assume $u(x) \geq 1$ by considering $u(x) + 1$ if necessary. Using our upper bound estimate in Theorem 3, we know u has at most quadratic growth. Consider $w(x) = u(x) - \frac{|x|^2}{2n}$. Then w is a harmonic function with quadratic growth. Hence, w and u is a quadratic polynomial, which gives the conclusion. □

6. Liouville property for **convex** solutions

We prove Theorem 4 in this section. Choose any positive number $k > 1$ and let $\Omega = \{x \in \mathbf{R}^n ; u(x) \leq k\}$. By our assumption ($u$ is **convex**), Ω is a bounded convex domain. Recall the Pohozaev identity formally. Let $g(u) = -u^\tau$ and let

$$G(u) = \frac{1}{1+\tau}[k^{1+\tau} - u^{1+\tau}].$$

Multiplying by $x\nabla u$ to the equation

$$-\Delta u = g(u),$$

we have

$$0 = div(\nabla u(x\nabla u) - x\frac{|\nabla u|^2}{2} + xG(u)) + \frac{n-2}{2}|\nabla u|^2 - nG(u).$$

Note that $1 + \tau < 0$. By integrating the equation above over Ω, we get

$$\int_\Omega \frac{n-2}{2}|\nabla u|^2 + \frac{n}{1+\tau}[u^{1+\tau} - k^{1+\tau}] + \frac{1}{2}\int_{\partial\Omega} \partial_\nu u(x \cdot \nabla u) = 0.$$

Note that by multiplying by u to the equation, we have

$$\int_\Omega |\nabla u|^2 - \int_{\partial\Omega} u\partial_\nu u = \int_\Omega u^{1+\tau}.$$

Hence, using $\partial_\nu u > 0$ on the boundary $\partial\Omega$, we have

$$[\frac{n-2}{2} + \frac{n}{1+\tau}]\int_\Omega u^{1+\tau} < \frac{nk^{1+\tau}|\Omega|}{1+\tau} < 0.$$

By this we get a contradiction, and then Theorem 4 is true. So we are done.

7. Compactness result

In this section, we study the point-wise lower bound of positive solutions to the equation (1), and prove Theorem 5.

Proof. We prove it by contradiction. For otherwise, we have a sequence of positive **convex** solutions $\{u_j\}$ and a sequence of points $\{x_j\} \subset \Omega$ such that

$$u_j(x_j) = min_\Omega u_j(x) \to 0.$$

Choose

$$\lambda_j = u_j(x_j)^{\frac{1-\tau}{2}} \to 0.$$

Set

$$v_j(x) = \lambda_j^{-\frac{2}{1-\tau}} u_j(x_j + \lambda_j x),$$
$$\Omega_j := \{x \in \mathbf{R}^n; x_j + \lambda_j x \in \Omega\},$$

and

$$B_j = B_{R\lambda_j^{-1}}(0).$$

Then it is elementary to see that

$$\Delta v_j = v_j^\tau, \quad in \quad \Omega_j$$

and

$$v_j(x) \geq v_j(0) = 1.$$

Let

$$\widehat{\Omega} = lim_j \Omega_j.$$

Assume that $\lambda_j d(x_j, \partial\Omega) \to \infty$. Then $\widehat{\Omega} = \mathbf{R}^n$ and by our Harnack gradient estimate and the standard L^p theory, we can extract a convergent subsequence in $C^2(B_r(0))$ for any $r > 0$, still denoted by $\{v_j\}$, with its limit $\bar{v}$ being a positive convex function satisfying

$$\Delta \bar{v} = \bar{v}^\tau, \quad in \quad \mathbf{R}^n, \quad \bar{v}(x) \geq 1 = v(0).$$

If $\lambda_j d(x_j, \partial\Omega) \leq C$ for some constant C, then we have $\widehat{\Omega} = \mathbf{R}^n_+$ and we can get a positive solution $v \in C^2(\mathbf{R}^n_+)$, i.e.,

$$\Delta \bar{v} = \bar{v}^\tau, \quad in \quad \mathbf{R}^n_+,$$

and

$$\bar{v}(x) \geq 1 = \bar{v}(0).$$

However, both cases give us a contradiction by our Theorem 4. Then, we have proved Theorem 5.

□

Note that the convexity property in Theorem 5 can not be removed since $u(x) = A|x|^{\frac{2}{1-\tau}}$ is a non-negative solution to (1) with

$$A = [\frac{(1-\tau)^2}{2(n+1)-2(n-1)\tau}]^{\frac{1}{1-\tau}}.$$

8. Finite Morse Index solutions

From the variational point of view, it is also very interesting to discuss positive solutions with finite Morse index to the following equation

$$\Delta u = u^\tau, \quad in \ \ \mathbf{R}^n, \tag{19}$$

where $\tau < 0$, with finite Morse index. Assume that $u \in C^2$ is a positive solution to (19). Define

$$E(\phi) = \int_{\mathbf{R}^n} (|\nabla \phi|^2 + \tau u^{\tau-1}\phi^2),$$

where $\phi \in C_0^2(\mathbf{R}^n)$. By definition, we say the positive solution u to (19) with *finite Morse index* k if there exist L^2 orthogonal nontrivial functions $\{\phi_j\}_{j=1}^k \subset C_0^2(\mathbf{R}^n)$ such that we have $E(\phi) < 0$ for $\phi \in \mathbf{W} := span\{\phi_j\} - \{0\}$, and $E(\phi) \geq 0$ for $\phi \perp \mathbf{W}$. If $k = 0$, we say that the solution u is *stable*.

Assume that u is the positive solution to (19) with finite Morse index k. Choose a large ball $B_R(0)$ which contains the supports of all ϕ_j's. Let

$$T_r = B_{R+1+r}(0) - B_{R+1}(0).$$

Then we have

$$E(\phi) \geq 0 \tag{20}$$

for all $\phi \in C_0^\infty(T_r)$. Let ξ be a smooth cut-off function with compact support in T_r. Let $\phi = u^{-q}\xi$. Then we have the following stability condition for any $\epsilon > 0$,

$$\begin{aligned}
& (-\tau)\int u^{-2q-1+\tau}\xi^2 \\
\leq & \int |u^{-1}D\xi - qu^{-q-1}\xi Du|^2 \\
\leq & (1+\frac{|q|}{2\epsilon})\int u^{-2q}|D\xi|^2 + (q^2+2|q|\epsilon)\int u^{-2q-2}\xi^2|Du|^2.
\end{aligned}$$

Using the weak form of the equation (19) with the testing function $\xi^2 u^{-\beta}$,$\beta = 2q+1 > 0$, we have

$$\beta\int u^{-\beta-1}\xi^2|Du|^2 \leq \int u^{-\beta+\tau}\xi^2 + 2\int u^{-\beta}\xi|Du||D\xi|,$$

and then we have, using the Cauchy-Schwartz inequality, for any $\delta > 0$,

$$(\beta - 2\delta)\int u^{-\beta-1}\xi^2|Du|^2 \leq \int u^{-\beta+\tau}\xi^2 + \frac{1}{2\delta}\int u^{-\beta+1}|D\xi|^2.$$

Inserting this into the stability condition we get

$$(-\tau)\int u^{-2q-1+\tau}\xi^2 \leq C(\epsilon,\delta,q)\int u^{-2q}|D\xi|^2 + (\frac{q^2+2|q|\epsilon}{2q+1-2\delta})\int u^{-2q-1+\tau}\xi^2.$$

Choose $-\frac{1}{2} < q < -\tau + \sqrt{\tau^2 - \tau}$ and ϵ, δ small enough depending on q, we can have

$$\frac{q^2+2|q|\epsilon}{2q+1-2\delta} < -\tau.$$

Hence, for some constant $C(\tau, q)$, we have

$$\int u^{-2q-1+\tau}\xi^2 \leq C(\tau,q)\int u^{-2q}|D\xi|^2.$$

Take $q > 0$ and replace ξ by $\xi^{q+\frac{1-\tau}{2}}$ to get

$$\int (\frac{\xi}{u})^{2q+1-\tau} \leq C(\tau,q)\int (\frac{\xi}{u})^{2q}\xi^{-1-\tau}|D\xi|^2.$$

Here $C(\tau, q)$ is another constant. Using the Young inequality

$$ab \leq \frac{(\epsilon a)^\alpha}{\alpha} + \frac{\alpha-1}{\alpha}(\frac{b}{\epsilon})^{\frac{\alpha}{\alpha-1}}$$

with $\alpha = \frac{q+\frac{1-\tau}{2}}{q}$, $(\frac{\alpha}{\alpha-1} = \frac{2q}{1-\tau}+1)$, $a = (\frac{\xi}{u})^2$, and $b = (\xi^{-\frac{1+\tau}{2}}|D\xi|)^2$, and choosing ϵ small, we get that

$$\int (\frac{\xi}{u})^{2q+1-\tau} \leq C(\tau,q)\int (\xi^{-\frac{1+\tau}{2}}|D\xi|)^{\frac{4q}{1-\tau}+2}. \tag{21}$$

Take $0 < q < -\tau + \sqrt{\tau^2-\tau}$ such that

$$n \leq \frac{4q}{1-\tau} + 2$$

and then we can find that

$$\int_{T_r} (\frac{1}{u})^{2q+1-\tau} \leq C(R,\tau,q),$$

for all $r > 0$. Note that the restriction of q is

$$\frac{(n-2)(1-\tau)}{4} \leq q < -\tau + \sqrt{\tau^2-\tau},$$

which implies that

$$n < 2 + \frac{4}{1-\tau}(-\tau + \sqrt{\tau^2-\tau})$$

Set q such that $n = \frac{4q}{1-\tau} + 2$ and $p = 2q + 1 - \tau$. Then $p > n$, and we use the lower bound of u to get that

$$\int_{B_r(0)} (\frac{1}{u})^p \leq C(R, \tau, q),$$

for any $r > 0$. Note that $1 - \frac{n}{p} = 1 - \frac{2}{1-\tau} = -\frac{\tau+1}{1-\tau}$. Using our equation we find that

$$|\Delta u| \in L^p(\mathbf{R}^n).$$

Using the standard L^p estimate we find that

Theorem 18. *Let u is a positive solution with finite Morse index on $\mathbf{R}^n$. We now assume that $u(x) \geq u(0) = 1$. Then,*

$$|Du(x)| \leq C|x|^{1-\frac{n}{p}} = C|x|^{-\frac{\tau+1}{1-\tau}},$$

and then we have the growth estimate

$$u(x) \leq C(1 + |x|^{-\frac{2\tau}{1-\tau}}).$$

In the case when the solution u is stable, we can take $T_r = B_r(0)$. In this case, let ξ be a cut-off function such that $\xi = 1$ on the ball $B_R(0)$,

$$\xi(x) = 2 - \frac{log|x|}{logR},$$

for $x \in B_{R^2}(0) - B_R(0)$,and $\xi = 0$ outside $B_{R^2}(0)$. Then we get from the estimate (21) that

$$\int_{B_R(0)} \frac{1}{u^n} \leq \frac{C}{(logR)^{n-1}} \to 0,$$

which is impossible.

This proves Theorem 6. □

We now prove Theorem 7.

Proof. Using the test function $\phi = \xi$ in (20), we obtain that

$$\int_{B_R(0)} u^{\tau-1} \leq C \tag{22}$$

where C is independent of $R > 1$.

We now perform the following scaling:

$$u(r, \theta) = A_\tau r^{\frac{2}{1-\tau}} v(t, \theta), t = \log r, r = |x| \tag{23}$$

where $A_\tau = (\frac{1-\tau}{2})^{\frac{2}{1-\tau}}$.

Thus we obtain that $v(t, \theta)$ satisfies

(24)

$$v_{tt} + \frac{4}{1-\tau} v_t + v_{\theta\theta} + \frac{4}{(1-\tau)^2} v - \frac{4v^\tau}{(1-\tau)^2} = 0, t \in (-\infty, +\infty),\ \theta \in S^1$$

We first claim

$$v(t,\theta) \geq C \quad \text{for } t > 2. \tag{25}$$

In fact, from (22) and (24), we obtain that

$$\int_t^{t+1} v^{\tau-1}(t,\theta) ds d\theta \to 0, \text{ as } t \to +\infty. \tag{26}$$

Let $m = v^{\tau-1}$. Then it is easy to see that m satisfies

$$m_{tt} + \frac{4}{1-\tau} m_t + m_{\theta\theta} + C_1 m^2 \geq 0 \tag{27}$$

Let us fix a point $\mathbf{x}_0 = (t_0, \theta_0) \in (1, \infty) \times S^1$. Set $\hat{m} = e^{\frac{2}{1-\tau}(t_0 - t)} m$. Then $\hat{m}$ satisfies

$$\hat{m}_{tt} + \hat{m}_{\theta\theta} + C_2 \hat{m}^2 \geq 0 \tag{28}$$

for $(t,\theta) \in [t_0 - 1, t_0 + 1] \times S^1$.

By Lemma 2.2 of [22] (see also Theorem 1.7 in [1]), we see that there exists $\eta_0 > 0$ such that for any $r > 0$ if $\int_{B_r \mathbf{x}_0} \hat{m} dx \leq \eta_0$, then

$$\hat{m}(t,\theta) \leq \frac{C}{r^2} \int_{B_r(\mathbf{x}_0} \hat{m}(x) dx \text{ for } (t,\theta) \in B_{r/2}(\mathbf{x}_0)$$

Choosing $t_0 > 8$ large enough so that

$$\int_t^{t+1} v^{\tau-1}(t,\theta) d\theta < e^{-8}\eta_0, \text{ for } t > \frac{t_0}{2}. \tag{29}$$

Then

$$\int_t^{t+1} \hat{m}(t,\theta) ds d\theta < \frac{1}{2}\eta_0, \text{ for } t > \frac{t_0}{2}. \tag{30}$$

Thus

$$\hat{m}(t,\theta) \leq C$$

for $(t,\theta) \in B_{\frac{1}{2}}((t_0,\theta_0))$, which is equivalent to that $v(t,\theta) \geq C$.

(25) implies that $v(t,\theta) \geq C$. On the other hand, it is easy to see that by the Harnack inequality, $v(t,\theta) \leq C$. By the results of L. Simon [34], $v(t,\theta) \to v(\theta)$, where $v(\theta)$ satisfies

$$v_{\theta\theta} + \frac{4}{(1-\tau)^2} v - v^\tau = 0, \; v \text{ is } 2\pi\text{-periodic.} \tag{31}$$

By Theorem 2.1 of [5], $v(\theta) \equiv \textit{constant}$ if $\tau \neq -3$, and $v(\theta) = (\lambda \cos^2\theta + \lambda^{-1}\sin^2\theta)^{1/2}$ for $\tau = -3$. This implies

$$\lim_{r \to +\infty} |x|^{-\frac{2}{1-\tau}} u(x) \geq \frac{2\mu}{(-\tau)} \tag{32}$$

for some $\mu > 0$.

Next, by explicitly solving the equation (it is an Euler equation), one finds that any non-trivial solution of

$$-k'' - \frac{1}{r}k' - (\mu/r^2)k = 0 \tag{33}$$

has infinitely many (and unbounded) positive zeros if $\mu > 0$. (Note that under the changes: $r = e^s$ and $\tilde{k}(s) = k(r)$, we see that $\tilde{k}(s)$ satisfies the equation

$$\tilde{k}''(s) + \mu\tilde{k}(s) = 0.$$

It is easily seen that $\tilde{k}(s)$ has infinitely many positive zeroes for any $\mu > 0$.) Thus, we can easily deduce that q has infinitely many positive zeros. Our claim holds.

We denote the zeroes of k as $0 < r_1 < r_2 < ... < r_k < ...$ where $r_k \to +\infty$ as $k \to +\infty$. Let k_0 be such that

$$\frac{(-\tau)}{u^{1-\tau}} \geq \frac{2\mu}{r^2}, r > r_k, k \geq k_0 \tag{34}$$

We now in the position to complete the proof of Theorem 7. Let $N > 0$ be fixed and $i \geq k_0$. Let h_i be the function defined to be $k(|x|)$ for $|x|$ between the ith and $(i+1)$th the zeros of k and to be zero otherwise. Then $h_i \in H^1(\mathbf{R}^2)$, h_i are orthogonal (in $L^2(\mathbf{R}^2)$ or $H^1(\mathbf{R}^2)$) and by multiplying (33) by h_i and integrating between these zeros we see that

$$Q(h_i) = \int_{\mathbf{R}^2} \left[|\nabla h_i|^2 + \frac{\tau}{u^{1-\tau}}h_i^2\right]$$

$$= \int_{\mathbf{R}^2} \left[\frac{\mu}{r^2} + \frac{\tau}{u^{1-\tau}}\right]h_i^2$$

is strictly negative at each h_i. Hence the span of h_i is an $(N-1)$-dimensional subspace of $C_0^\infty(\mathbf{R}^2)$ such that $Q(h) < 0$. Since h_i has compact support it follows easily that there is an $(N-1)$-dimensional subspace of $H^1(\mathbf{R}^2)$ such that

$$\int_{\mathbf{R}^2} [|\nabla h(y)|^2 + \frac{\tau}{u^{1-\tau}}h^2] < 0$$

and hence the Morse index of u must be at least N. Since N is arbitrary, the Morse index of u is infinity, a contradiction to our assumption.

□

9. Existence theory; Proof of Theorem 8

We now consider the existence problem of the problem (1). One can easily find the radial solutions to (1) in the whole space (see Theorem 3.2 in [31] for the case when $\tau = -1$ and Theorem 1.1 in [20] for $\tau < -1$).

Given a positive data ϕ on the bounded smooth domain Ω. Consider the boundary problem of positive solutions to (1) on Ω with the boundary condition $u = \phi$ on $\partial\Omega$.

If $-1 < \tau < 0$, we let

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{1+\tau}\int_\Omega u^{1+\tau}$$

on the space $\mathbf{A} = \{u \in H^1(\Omega); u = \phi \ on \ \partial\Omega\}$. Since

$$\int_\Omega u^{1+\tau} \leq |\Omega|^{-\tau}(\int_\Omega u)^{1+\tau},$$

we can get a non-negative minimizer of $J(\cdot)$ on $\mathbf{A}$. For such a minimizer, one need to handle with how large for its zero set. One may see [33] and [19] for more.

When $\tau = -1$, an existence result has been discussed in [31] by degree argument.

Theorem 8 gives another type of existence criteria. Since the proof of Theorem 8 is simple, we give it here.

Proof. Choose a large constant M such that $\overline{u} = M$ is a super-solution. Then one can use the standard super-sub solution method to get a positive solution.

We give here the variational method. Let

$$\mathbf{A} = \{u \in H^1(\Omega); \underline{u} \leq u \leq M \ in \ \Omega, u = \phi \ in \ \partial\Omega\}.$$

Define

$$I(u) = \frac{1}{2}\int |Du|^2 - \frac{1}{1+\tau}\int |u|^{1+\tau}. \tag{35}$$

It is easy to see that $I(\cdot)$ is bounded from below on the closed convex set $\mathbf{A}$. Since for any $u, w \in \mathbf{A}$,

$$|\int u^{1+\tau} - w^{1+\tau}| \leq C\int |u - w|.$$

(We get this by mean value theorem in Calculus). Then we can use the Sobolev compactness embedding theorem to get a minimizer u of the functional $I(\cdot)$ in the set $\mathbf{A}$, which is the solution to (1) with the boundary data ϕ (see [37]). □

The advantage of variational methods is that one may prove that the minimizer on the class **A** is a stable solution in the usual sense. Since we shall not use this fact, we shall not discuss it. A natural question is how to find a sub-solution on a bounded domain. The usual way is to use one-dimensional (or any lower dimensional) solution or radial solution on the whole space. One can also choose a large ball containing the bounded domain, and solve the equation on the ball to get radial solutions on the ball. Such radial solutions are the sub-solutions to the equation on the original domain if the boundary value of the radial solutions on the domain are less than the given boundary data ϕ.

In particular, as an application of Theorem 8, we have

Corollary 19. *Assume* $(-1 \neq)\tau \leq 0$, *and let* $\Omega \subset \mathbf{R}^n$ *be a bounded smooth domain. Given any positive smooth boundary data* ϕ *in* $\partial\Omega$. *Assume that there is a radial solution* $u(r)$ *in lower dimensional space* $\mathbf{R}^k$ $(k < n)$ *or the whole space* $\mathbf{R}^n$ *such that* $\phi(x) > u(|x|)$ *on* $\partial\Omega$. *Then there is smooth positive solution to (1) with boundary data* ϕ.

Proof. Here we need only to use $u(r)$ as a sub-solution to (1) on the domain Ω in Proposition 8. □

Although the argument in the proof of Corollary (19) is simple, it can be used to study the existence result of positive solutions for a large class of singular elliptic partial differential equations such as

$$\Delta u + au \log u = 0$$

with Dirichlet boundary data. One may see [18] and [30] for related results.

10. Symmetry Properties: proof of Theorem 9

In the last two sections, we consider the integral equation (8) which is closely related to the elliptic differential equation the elliptic equation:

$$(-\Delta)^{\mu/2}(u-h) = -u^{\tau}, \quad in \ \ \mathbf{R}^n.$$

It is clearly that the positive solution to (8) is bounded from above by h.

We shall use the following notation. Given any hyperplane π in $\mathbf{R}^n$. For any point $x \in \mathbf{R}^n$, let x^π be the reflection of x about the plane π and let $\pi(x) \in \pi$ be projection of x into π. Define

$$u^\pi(x) = u(x^\pi)$$

for any function $u : \mathbf{R}^n \to \mathbf{R}$.

In the famous paper of Gidas and Spruck [13], they proved that for $n > 2$ and $\mu = 2$, and $1 \leq \tau < \frac{n+2}{n-2}$, the only non-negative solution to the equation

$$-\Delta u = u^\tau, \quad in \quad \mathbf{R}^n,$$

is zero. However, the negative index τ case has not been treated before Xu's recent work [39] . So, Theorem 9 can be considered as a generalization of their result to the equation (8).

In recent years, there are important progress in the study of symmetry properties of non-negative solutions to Yamabe type equations. In particular, X.Xu [39] has obtained some related results to ours. His equation is the following

$$u(x) = \int_{\mathbf{R}^n} |x-y|^{\mu-n} u(y)^\tau dy,$$

which corresponds to the elliptic equation:

$$(-\Delta)^{\mu/2} u = u^\tau, \quad in \quad \mathbf{R}^n.$$

One should be caution about the negative sign before the Laplacian operator, which forces the equation to have no positive solution on the whole space. We will use a symmetry method (see also [6] and [8]) to prove our results—Theorem 9 and Theorem 10. This symmetry method is powerful in our case since we can use the behavior at infinity of the solution.

We now give a proof of Theorem 9: After using a rotation, we may assume that the hyperplane π is orthonormal to x_1 axis at the origin. So we may assume that $h(x) = h$ is a constant in the following argument.

For a given real number λ, we define

$$\Sigma_\lambda = \{x = (x_1, \cdots, x_n) | x_1 \geq \lambda\},$$

and let $x^\lambda = (2\lambda - x_1, x_2, \cdots, x_n)$ and $u_\lambda(x) = u(x^\lambda)$.

We can easily get the following

Lemma 20. *For any positive solution $u(x)$ of (1), we have*
(36)

$$u_\lambda(x) - u(x) = -\int_{\Sigma_\lambda} \left(\frac{1}{|x-y|^{n-\mu}} - \frac{1}{|x^\lambda - y|^{n-\mu}}\right)(u_\lambda(y)^\tau - u(y)^\tau) dy.$$

Proof. Let

$$\Sigma_\lambda^c = \{x = (x_1, \cdots, x_n) | x_1 < \lambda\}.$$

Then it is easy to see that

$$\begin{aligned} h - u(x) &= \int_{\Sigma_\lambda} \tfrac{1}{|x-y|^{n-\mu}} u(y)^\tau dy + \int_{\Sigma_\lambda^c} \tfrac{1}{|x-y|^{n-\mu}} u(y)^\tau dy \\ &= \int_{\Sigma_\lambda} \tfrac{1}{|x-y|^{n-\mu}} u(y)^\tau dy + \int_{\Sigma_\lambda} \tfrac{1}{|x-y^\lambda|^{n-\mu}} u(y^\lambda)^\tau dy \\ &= \int_{\Sigma_\lambda} \tfrac{1}{|x-y|^{n-\mu}} u(y)^\tau dy + \int_{\Sigma_\lambda} \tfrac{1}{|x^\lambda-y|^{n-\mu}} u_\lambda(y)^\tau dy. \end{aligned}$$

Here we have used the fact that $|x-y^\lambda| = |x^\lambda - y|$. Substituting x by x^λ, we get

$$h - u(x^\lambda) = \int_{\Sigma_\lambda} \frac{1}{|x^\lambda - y|^{n-\mu}} u(y)^\tau dy + \int_{\Sigma_\lambda} \frac{1}{|x-y|^{n-\mu}} u_\lambda(y)^\tau dy.$$

Thus

$$u(x^\lambda) - u(x) = -\int_{\Sigma_\lambda} (\frac{1}{|x-y|^{n-\mu}} - \frac{1}{|x^\lambda - y|^{n-\mu}})(u_\lambda(y)^\tau - u(y)^\tau) dy.$$

This implies (36). □

We shall need the following Hardy-Littlewood-Sobolev inequality (see, for example, [29])

$$\| \int V(x,y) f(y) dy \|_q \leq P_{p,n} \|f\|_p \tag{37}$$

with $V(x,y) = |x-y|^{-\nu}$ and

$$1/p + \nu/n = 1 + 1/q.$$

Proof of Theorem 9.

Define

$$\Sigma_\lambda^- = \{x | x \in \Sigma_\lambda, u(x) \geq u_\lambda(x)\},$$

and

$$\Sigma_\lambda^+ = \{x | x \in \Sigma_\lambda, u(x) < u_\lambda(x)\}.$$

Then

$$\Sigma_\lambda = \Sigma_\lambda^+ \bigcup \Sigma_\lambda^-$$

We want to show that for sufficiently positive values of λ, Σ_λ^- must be empty.

Note that for $y \in \Sigma_\lambda^-$, we have $u(y)^\tau \leq u_\lambda(y)^\tau$. Whenever $x, y \in \Sigma_\lambda$, we have that $|x-y| \leq |x^\lambda - y|$ and

$$|x-y|^{\mu-n} \geq |x^\lambda - y|^{\mu-n}.$$

Then by Lemma 20, for any $x \in \Sigma_\lambda^-$,

(38)

$$\begin{aligned} u(x) - u_\lambda(x) &= -\int_{\Sigma_\lambda} (|x-y|^{\mu-n} - |x^\lambda - y|^{\mu-n})(u(y)^\tau - u_\lambda(y)^\tau) dy \\ &\leq \int_{\Sigma_\lambda^-} |x-y|^{\mu-n} (u_\lambda(y)^\tau - u(y)^\tau) dy \\ &\leq -\tau \int_{\Sigma_\lambda^-} |x-y|^{\mu-n} [u_\lambda^{\tau-1}(u - u_\lambda)](y) dy. \end{aligned}$$

It follows first from inequality (37) and the Holder inequality that, for any $q > n/(n-\mu)$,
(39)

$$\begin{array}{rcl} \|u_\lambda - u\|_{L^q(\Sigma_\lambda^-)} & \leq & C\| \int_{\Sigma_\lambda^-} |x-y|^{\mu-n}[u_\lambda^{\tau-1}(u_\lambda - u)](y)dy\|_{L^q(\Sigma_\lambda^-)} \\ & \leq & C(\int_{\Sigma_\lambda^-} u_\lambda(y)^{(\tau-1)\beta}dy)^{1/\beta}\|u_\lambda - u\|_{L^q(\Sigma_\lambda^-)} \\ & \leq & C(\int_{\Sigma_\lambda^-} u_\lambda(y)^{(\tau-1)\beta}dy)^{1/\beta}\|u_\lambda - u\|_{L^q(\Sigma_\lambda^-)}. \end{array}$$

By condition (9), we can choose N sufficiently large, such that for $\lambda > N$, we have

$$C(\int_{\Sigma_\lambda} u_\lambda(y)^{(\tau-1)\beta}dy)^{1/\beta} \leq \frac{1}{2}.$$

Now (39) implies that

$$\|u_\lambda - u\|_{L^q(\Sigma_\lambda^-)} = 0,$$

and therefore Σ_λ^- must be measure zero, and hence empty. Then using the standard moving plane trick (see ([6]) and [8]), we know that the solution u is symmetric in the variable x_1.

So we complete the proof of Theorem 9.

□

Finally, we remark that in some cases, using the analysis of ODE, we see that there is no radially symmetric positive solution to (8) with the condition (9). However, we shall not discuss this here.

11. Proof of theorem 10

Let us now introduce the Kelvin type transform of u as follow

$$v(x) = |x|^{\mu-n}u(\frac{x}{|x|^2})$$

for any $x \neq 0$. Then by elementary calculations, one can see that (8) is transformed into the following:

$$h|x|^{\mu-n} - v(x) = \int_{\mathbf{R}^n} |x-y|^{\mu-n}|y|^{-\alpha}v(y)^\tau dy, \tag{40}$$

where $\alpha = (n+\mu) - (n-\mu)\tau > 0$. Obviously, $v(x)$ has a singularity at origin. Since u is locally bounded, it is easy to see that $v(x)$ has no singularity at infinity, i.e., for any domain Ω that is a positive distance away from the origin,

$$\int_\Omega v^\beta(y)dy < \infty. \tag{41}$$

In fact, we have

$$\begin{aligned}\int_\Omega v^\beta(y)dy &= \int_\Omega(|y|^{\mu-n}u(\tfrac{y}{|y|^2}))^\beta dy\\ &= \int_{\Omega^\star}(|z|^{n-\mu}u(z))^\beta|z|^{-2n}dz\\ &= \int_{\Omega^\star}|z|^{\beta(n-\mu)-2n}u(z)^\beta dz\\ &\le C\int_{\Omega^\star}u(z)^\beta dz\\ &< \infty.\end{aligned}$$

For the second equality, we have made the transform $y = z/|z|^2$. Since Ω is a positive distance away from the origin, $\Omega^\star$, the image of Ω under this transform, is bounded. Also note that $\beta(n-\mu)-2n > 0$. Then we get the estimate (41).

For a given real number λ, we define, as before,

$$\Sigma_\lambda = \{x = (x_1, \cdots, x_n)|x_1 \ge \lambda\},$$

and let $x^\lambda = (2\lambda - x_1, x_2, \cdots, x_n)$ and $v_\lambda(x) = v(x^\lambda)$.

We can easily get the following

Lemma 21. *For any solution $v(x)$ of (40), we have*

$$v_\lambda(x) - v(x) = h(|x^\lambda|^{\mu-n} - |x|^{\mu-n}) - \int_{\Sigma_\lambda}(\frac{1}{|x-y|^{n-\mu}} - \frac{1}{|x^\lambda-y|^{n-\mu}})(\frac{1}{|y^\lambda|^\alpha}v_\lambda(y)^\tau - \frac{1}{|y|^\alpha}v(y)^\tau)dy. \tag{42}$$

Proof. The proof is similar to Lemma 20. Let

$$\Sigma_\lambda^c = \{x = (x_1, \cdots, x_n)|x_1 < \lambda\}.$$

Then it is easy to see that

$$\begin{aligned}h|x|^{\mu-n} - v(x) &= \int_{\Sigma_\lambda}\frac{1}{|x-y|^{n-\mu}}\frac{1}{|y|^\alpha}v(y)^\tau dy + \int_{\Sigma_\lambda^c}\frac{1}{|x-y|^{n-\mu}}\frac{1}{|y|^\alpha}v(y)^\tau dy\\ &= \int_{\Sigma_\lambda}\frac{1}{|x-y|^{n-\mu}}\frac{1}{|y|^\alpha}v(y)^\tau dy + \int_{\Sigma_\lambda}\frac{1}{|x-y^\lambda|^{n-\mu}}\frac{1}{|y^\lambda|^\alpha}v(y^\lambda)^\tau dy\\ &= \int_{\Sigma_\lambda}\frac{1}{|x-y|^{n-\mu}}\frac{1}{|y|^\alpha}v(y)^\tau dy + \int_{\Sigma_\lambda}\frac{1}{|x^\lambda-y|^{n-\mu}}\frac{1}{|y^\lambda|^\alpha}v_\lambda(y)^\tau dy.\end{aligned}$$

Here we have used the fact that $|x-y^\lambda| = |x^\lambda - y|$. Substituting x by x^λ, we get

$$\begin{aligned}h|x^\lambda|^{\mu-n} - v(x^\lambda) &= \int_{\Sigma_\lambda}\frac{1}{|x^\lambda-y|^{n-\mu}}\frac{1}{|y|^\alpha}v(y)^\tau dy\\ &+ \int_{\Sigma_\lambda}\frac{1}{|x-y|^{n-\mu}}\frac{1}{|y^\lambda|^\alpha}v_\lambda(y)^\tau dy.\end{aligned}$$

Thus

$$\begin{aligned}v(x) - v(x^\lambda) &= h(|x^\lambda|^{\mu-n} - |x|^{\mu-n})\\ &- \int_{\Sigma_\lambda}(\frac{1}{|x-y|^{n-\mu}} - \frac{1}{|x^\lambda-y|^{n-\mu}})(\frac{1}{|y^\lambda|^\alpha}v_\lambda(y)^\tau - \frac{1}{|y|^\alpha}v(y)^\tau)dy.\end{aligned}$$

This implies (42). □

We shall also need the following doubly weighted Hardy-Littlewood-Sobolev inequality of Stein and Weiss (see, for example, [29])

$$\| \int V(x,y)f(y)dy\|_q \leq P_{\alpha,\beta,p,\nu,n}\|f\|_p \tag{43}$$

with $V(x,y) = |x|^{-\beta}|x-y|^{-\nu}|y|^{-\alpha}$, $0 \leq \alpha < n/p'$, $0 \leq \beta < n/q$, $1/p + 1/p' = 1$, and

$$1/p + (\nu + \alpha + \beta)/n = 1 + 1/q.$$

Proof of Theorem 10. Define

$$\Sigma_\lambda^- = \{x | x \in \Sigma_\lambda, v(x) < v_\lambda(x)\},$$

and

$$\Sigma_\lambda^+ = \{x | x \in \Sigma_\lambda, v(x) \geq v_\lambda(x)\}.$$

We want to show that for sufficiently negative values of λ, Σ_λ^- and Σ_λ^+ must be empty.

Whenever $x, y \in \Sigma_\lambda$, we have that $|x-y| \leq |x^\lambda - y|$. Moreover, since $\lambda < 0$, $|y^\lambda| \geq |y|$ for any $y \in \Sigma_\lambda$. Then by Lemma 21, for any $x \in \Sigma_\lambda^-$,

(44)

$$\begin{aligned} v_\lambda(x) - v(x) &\leq -\int_{\Sigma_\lambda} (|x-y|^{\mu-n} - |x^\lambda - y|^{\mu-n})|y|^{-\alpha}(v_\lambda(y)^\tau - v(y)^\tau)dy \\ &\leq -\int_{\Sigma_\lambda^-} |x^\lambda - y|^{\mu-n}|y|^{-\alpha}(v_\lambda(y)^\tau - v(y)^\tau)dy \\ &\leq -\tau\int_{\Sigma_\lambda^-} |x-y|^{\mu-n}|y|^{-\alpha}[v^{\tau-1}(v_\lambda - v)](y)dy. \end{aligned}$$

It follows first from inequality (43) and then the Holder inequality that, for any $q > n/(n-\mu)$, which will be used below in the form that $\tau < \mu$,

(45)

$$\begin{aligned} \|v_\lambda - v\|_{L^q(\Sigma_\lambda^-)} &\leq C\|\int_{\Sigma_\lambda^-} |x-y|^{\mu-n}|y|^{-\alpha}[v^{\tau-1}(v_\lambda - v)](y)dy\|_{L^p(\Sigma_\lambda^-)} \\ &\leq C(\int_{\Sigma_\lambda^-} v(y)^{(\tau-1)\beta}dy)^{1/\beta}\|v_\lambda - v\|_{L^q(\Sigma_\lambda^-)} \\ &\leq C(\int_{\Sigma_\lambda} v(y)^{(\tau-1)\beta}dy)^{1/\beta}\|v_\lambda - v\|_{L^q(\Sigma_\lambda^-)}. \end{aligned}$$

By condition (41), we can choose N sufficiently large, such that for $\lambda > N$, we have

$$C(\int_{\Sigma_\lambda} v(y)^{(\tau-1)\beta}dy)^{1/\beta} \leq \frac{1}{2}.$$

Now (45) implies that

$$\|v_\lambda - v\|_{L^q(\Sigma_\lambda^-)} = 0,$$

and therefore Σ_λ^- must be measure zero. Then we can use the moving plane trick as before to get that v is symmetric at zero with respect to the $x-1$ direction. This completes the proof of Theorem 10.

Appendix: Regularity Result for Weak Solution to (1)

We now discuss some regularity result for weak solution to (1). We only need to get upper bound for positive weak solutions to (1) for any $\tau < 0$ by assuming a positive lower bound. As in the proof of Theorem 2, we take $R > \rho > 0$ and a cut-off function $\xi = \xi(|x|)$ such that $|\nabla\xi| \leq \frac{4}{R-\rho}$, and $\xi = 1$ on B_ρ. Then using $u(x) \geq 1$, we have as before that

$$\int u\Delta(|x|^2\xi) = \int u^\tau |x|^2\xi \leq \int |x|^2 \simeq R^{n+2}.$$

Using $\Delta|x|^2 = 2n$ we have

$$\int_{B_\rho} u \leq \frac{AR^2}{(R-\rho)^2}\int_{T_{R,\rho}} u + BR^{n+2},$$

where A, B are uniform constants and

$$T_{R,\rho} = B_R - B_\rho.$$

We can also derive some other interesting bound without the pointwise lower bound.

Take a constant $\sigma > 0$. Then we have

$$-\int \nabla u\nabla(u^\sigma\xi) = \int u^{\sigma+\tau}\xi.$$

Using integration by part, we know that the left side is

$$-\int \nabla u\nabla(u^\sigma\xi) = -\sigma\int u^{\sigma-1}|\nabla u|^2\xi - \int u^\sigma\nabla u\nabla\xi.$$

Then

$$\sigma\int u^{\sigma-1}|\nabla u|^2\xi + \int u^{\sigma+\tau}\xi = -\int u^\sigma\nabla u\nabla\xi.$$

Note that

$$-\int u^\sigma\nabla u\nabla\xi = \frac{1}{1+\sigma}\int u^{1+\sigma}\Delta\xi \leq \frac{C}{(1+\sigma)(R-\rho)^2}\int_{T_{R,\rho}} u^{1+\sigma},$$

where $T_{R,\rho} = B_R - B_\rho$. Hence, we have

$$\sigma\int_{B_R} u^{\sigma-1}|\nabla u|^2 + \int_{B_R} u^{\sigma+\tau} \leq \frac{C}{(1+\sigma)(R-\rho)^2}\int_{T_{R,\rho}} u^{1+\sigma}. \tag{46}$$

Let first consider two cases.

Case 1. If we choose $\sigma = -\tau$, then we get

$$-\tau\int_{B_R} u^{-\tau-1}|\nabla u|^2 + |B_1(0)|\mathbf{R}^n \leq \frac{C}{(1-\tau)(R-\rho)^2}\int_{T_{R,\rho}} u^{1-\tau}.$$

Case 2. If we send $\sigma \to 0$ in (46), then we get

$$\int_{B_R} u^\tau \leq \frac{C}{(R-\rho)^2} \int_{T_{R,\rho}} u.$$

In the following, we do iteration. Let $\sigma = -\tau + p$ in (46). Then we have

(47)

$$(-\tau+p) \int_{B_R} u^{-\tau+p-1} |\nabla u|^2 + \int_{B_R} u^p \leq \frac{C}{(1-\tau+p)(R-\rho)^2} \int_{T_{R,\rho}} u^{1-\tau+p}.$$

Then we have

$$\frac{4(-\tau+p)}{(-\tau+p+1)^2} \int_{B_R} |\nabla (u^{\frac{-\tau+p+1}{2}})|^2 + \int_{B_R} u^p \leq \frac{C}{(1-\tau+p)(R-\rho)^2} \int_{T_{R,\rho}} u^{1-\tau+p}.$$

We now in the standard Nash-Moser iteration situation. Hence, for any $0 < q$, we have a uniform constant $C(q,n)$ such that

$$\sup_{B_{\theta R}} u \leq \frac{C(q,n)}{((1-\theta)R)^{n/q}} (\int_{B_R} u^q)^{1/q}.$$

Once we have a upper bound, we can use the standard Calderon-Zygmund L^p theory to conclude that u is a smooth solution.

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