HALF-SPACE THEOREMS FOR THE ALLEN-CAHN EQUATION AND RELATED PROBLEMS

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Abstract. In this paper we obtain rigidity results for a non-constant entire solution u of the Allen-Cahn equation in $\mathbb{R}^n$, whose level set $\{u = 0\}$ is contained in a half-space. If $n \leq 3$ we prove that the solution must be one-dimensional. In dimension $n \geq 4$, we prove that either the solution is one-dimensional or stays below a one-dimensional solution and converges to it after suitable translations. Some generalizations to one phase free boundary problems are also obtained.

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1. Introduction and main results

We are interested in rigidity results for classical entire solutions of the Allen-Cahn equation

$$-\Delta u = u - u^3, \quad x = (x_1, \cdots, x_n) \in \mathbb{R}^n. \tag{1}$$

In the simplest case $n = 1$, equation (1) reduces to an ODE and has a heteroclinic solution

$$H(x) = \tanh\left(\frac{x}{\sqrt{2}}\right).$$

Phase plane analysis tells us that up to a translation, H is the unique monotone increasing solution in $\mathbb{R}$. The one-dimensional solution H actually plays an important role in the theory of Allen-Cahn equation in general dimensions. Indeed, De Giorgi [10] conjectured that for $n \leq 8$, if a bounded solution u to (1) is strictly monotone in one direction, say x_n, then it must be one-dimensional, which then means that u is identically equal to $H(x \cdot e + a)$ for some unit vector e, with $e_n > 0$, and some real number a. This conjecture has been proved to be true for $n = 2$ (Berestycki, Caffarelli and Nirenberg [4], Ghoussoub and Gui [20]), $n = 3$ (Ambrosio and Cabré [3]), and for $4 \leq n \leq 8$ (Savin [28]) under the additional limiting condition

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1,$$

pointwise in $x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}$. This condition implies that the level sets $\{x \in \mathbb{R}^n : u(x) = \mu\}$, for every $\mu \in (-1, 1)$, of the function u are entire graphs with respect to the first $n - 1$ variables. On the other hand, for $n \geq 9$, Del Pino, Kowalczyk and the fifth author [13] constructed monotone solutions which are not one-dimensional, showing that the condition $n \leq 8$ in the De Giorgi conjecture cannot be relaxed.

The De Giorgi conjecture can be regarded as a rigidity result for the Allen-Cahn equation. The second rigidity result we would like to mention here is about the classification of the solutions which are global minimizers of the associated energy functional. Savin [28] proved that global minimizers are one-dimensional up to dimensions $n \leq 7$. The second, fourth and fifth authors constructed in [23] counterexamples in dimension 8, i.e. global minimizers which are not one-dimensional. For related rigidity results for the solutions to the Allen-Cahn equation, we refer to [2, 7, 17, 18] and the references therein.

In this paper, we are interested in rigidity results for the Allen-Cahn equation when the zero level set

$$\Gamma_0 := \{u = 0\} = \big\{x \in \mathbb{R}^n : u(x) = 0\big\}$$

of the solution (always understood in the classical sense) is contained in a half-space, say $\{x_n > 0\} := \{x \in \mathbb{R}^n : x_n > 0\}$ up to translation and rotation. However, we point out that we make no assumption on the monotonicity of u in a direction nor on its stability or minimizing properties.

Our first result is the following half-space rigidity result:

Theorem 1. (Weak half-space theorem) *Let $n \leq 3$ and u be a non-constant solution of* (1)*. Suppose that the zero level set $\{u = 0\}$ is contained in $\{x_n > 0\}$. Then u is one-dimensional. More precisely, there exists $a \in \mathbb{R}$ such that either $u(x) = H(-x_n + a)$ for all $x \in \mathbb{R}^n$, or $u(x) = H(x_n + a)$ for all $x \in \mathbb{R}^n$.*

Note that, in any dimension $n \geq 1$, if u is a solution of (1), then u is necessarily bounded and

$$-1 \leq u \leq 1 \ \text{ in } \mathbb{R}^n,$$

as follows from [14, Proposition 1.9]. Furthermore, if $\{u = 0\}$ is empty, then by applying [16, Theorem 1.1] or by constructing suitable comparison functions as in the first part of the proof of Theorem 3 (see Section 2 below), one knows that $u \equiv \pm 1$ in $\mathbb{R}^n$. Therefore, if u is non-constant, then $\{u = 0\} \neq \emptyset$ and $-1 < u < 1$ in $\mathbb{R}^n$ from the strong maximum principle. Moreover, if $u \geq 0$ (resp. $u \leq 0$) in $\mathbb{R}^n$ and $\{u = 0\} \neq \emptyset$, then $u \equiv 0$ in $\mathbb{R}^n$ from the strong maximum principle. Hence we can assume without loss of generality that the three sets $\{u = 0\}$, $\{u > 0\}$ and $\{u < 0\}$ are not empty.

We point out that here it is not assumed that the nodal set $\{u = 0\}$ is a graph, that is, the sets $\{u > 0\}$ or $\{u < 0\}$ are not assumed to be epigraphs. For rigidity results in the epigraph case we refer to [15, 19] and the references therein.

As an application of Theorem 1, using the classification result of stable solutions of the Allen-Cahn equation in the plane, we get the following *strong half-space theorem*:

Corollary 2. (Strong half-space theorem) *Suppose $n = 2$. Let $u_1 < u_2$ be two non-constant solutions of* (1) *in $\mathbb{R}^2$. Then u_1 and u_2 are one-dimensional, namely there exist a unit vector e and some real numbers $a < b$ such that $u_1(x) = H(x \cdot e + a)$ and $u_2(x) = H(x \cdot e + b)$ for all $x \in \mathbb{R}^2$.*

We will also generalize Theorem 1 to a free boundary problem. We refer to Section 4 for the precise statement and its proof.

Our results are inspired by analogous results in the minimal surface theory. A half-space theorem for minimal surfaces in $\mathbb{R}^3$ was proved by Hoffman and Meeks [22]. It states that connected, proper, minimal surfaces in $\mathbb{R}^3$ which are contained in a half-space are necessarily planes. A version of a half-space theorem for minimal surfaces with bounded Gaussian curvature is proved in [30]. The half-space theorem plays an important role in the understanding of the structure of minimal spaces, and there is a vast literature on this subject. It is used in the proof of the local removal singularity theorem [25]. It is also used to study the properness of minimal surfaces (see, for instance, [26]). In [9], Colding and Minicozzi proved that the plane is the only complete embedded minimal disk in $\mathbb{R}^3$, by establishing a chord-arc bound and applying Hoffman-Meeks half-space theorem.

We remark that the half-space theorem is not true for minimal hypersurfaces in $\mathbb{R}^n$ with $n \geq 4$. For example the higher dimensional catenoid provides a counterexample. However, for the Allen-Cahn equation, this question is still open in higher dimensions. In view of the construction of solutions concentrated on higher dimensional catenoid [1], we turn to believe that the half-space theorem of the Allen-Cahn equation should be true also for all $n \geq 4$. Intuitively, for solutions of the Allen-Cahn equation there are strong interations between different ends, while this is not the case for minimal surfaces.

The proof of the half-space theorem for minimal surfaces uses sweeping principle. It appears that this idea does not work for the Allen-Cahn case, although we can prove partial results along this direction. Our main result for solutions of the Allen-Cahn equation in arbitrary dimension with zero level set contained in a half-space is the following:

Theorem 3. *Let $n \geq 1$ and u be a non-constant solution of the Allen-Cahn equation (1) in $\mathbb{R}^n$. If $u < 0$ in the half-space $\{x_n < 0\}$, then there exists $a \in \mathbb{R}$ such that*

$$u(x) \leq H(x_n + a)$$

for all $x \in \mathbb{R}^n$, and either

1. *$u(x) = H(x_n + a)$ for all $x \in \mathbb{R}^n$, or*
2. *$u(x) < H(x_n + a)$ for all $x \in \mathbb{R}^n$ and there exists a sequence $(y_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^{n-1} \times \{0\}$ such that $|y_k| \to +\infty$ as $k \to +\infty$, and the functions $u(\cdot + y_k)$ converge in $C^2_{loc}(\mathbb{R}^n)$ to the function $x \mapsto H(x_n + a)$ as $k \to +\infty$.*

In Theorem 3 and throughout the paper, $|\ |$ denotes the Euclidean norm in $\mathbb{R}^n$. We also use the following notations: $B_R(x) = \{y \in \mathbb{R}^n : |y - x| < R\}$ and $B_R = B_R(0)$ for $R > 0$ and $x \in \mathbb{R}^n$, and we denote

$$\mathrm{dist}(x, E) = \inf_{y\in E} |x - y|$$

for $x \in \mathbb{R}^n$ and $E \subset \mathbb{R}^n$.

We complete the introduction by listing some corollaries following from Theorem 3.

Corollary 4. *Let $n \geq 1$ and u be a non-constant solution of the Allen-Cahn equation (1) in $\mathbb{R}^n$. Then there does not exist a non-degenerate convex cone containing $\{u = 0\}$.*

Corollary 5. *Let $n \geq 1$ and u be a non-constant solution of the Allen-Cahn equation (1) in $\mathbb{R}^n$. If there exists a closed half-space E such that $\{u = 0\} \subset E$ and $\{u = 0\} \cap \partial E \neq \emptyset$, then there is $a \in \mathbb{R}$ such that $u(x) = H(x \cdot e + a)$ for all $x \in \mathbb{R}^n$, where e is a unit vector orthogonal to ∂E.*

It also follows from Theorem 3 that, in any dimension $n \geq 1$, if the zero level set of a non-constant solution of (1) is bounded in a unit direction e, then u is identically equal to $H(\pm x \cdot e + a)$ for some $a \in \mathbb{R}$. This last result (see Corollary 11 at the end of Section 2 below) corresponds to Theorem 1.1 obtained by Farina [16]. We provide in Section 2 another proof using Theorem 3. We remark also that this result points out the difference between minimal surfaces and level sets of Allen-Cahn solutions from the point of view of half-space theorems: for $n \geq 4$ the minimal catenoid is contained in a slab, while entire solutions of the Allen-Cahn equation cannot have the zero level set contained in a slab, unless they are constant or equal to $H(x_n)$ up to translation and rotation of the variables. In particular, the zero level set of the solutions obtained in [1] is not contained in a slab, although such level set approaches the minimal catenoid in a compact region.

It should be interesting to link such kind of half-space results to the De Giorgi conjecture. In this sense, an open question is to understand if Theorem 1 can be true in all dimensions, at least with the hypothesis of the monotonicity of the solution in one direction. Note, in particular, that the zero level sets of the x_n-monotone and non-planar solutions of (1) in $\mathbb{R}^n$ with $n \geq 9$, constructed by Del Pino, Kowalczyk and the fifth author in [13], are not included in any half-space.

Remark 6. Theorem 1 and Corollary 4, as well as Corollary 11 below, do not hold good if the zero level set $\{u = 0\}$ is replaced by another level set $\{u = \mu\}$ with $\mu \neq 0$. Similarly, Theorem 3 does not hold good either if one assumes that $u < \mu$ in $\{x_n < 0\}$ for some $\mu > 0$. Indeed, equation (1) admits solutions u_L which vanish on $(L\mathbb{Z})^n$ with $L > \pi\sqrt{n}$, are $2L$-periodic with respect to each variable x_i, which satisfy $\max_{\mathbb{R}^n} |u_L| \to 0$ as $L \overset{>}{\to} \pi\sqrt{n}$, and which are not one-dimensional!

Outline of the paper. Section 2 is devoted to the proof of Theorem 3 and its corollaries. The proof of Theorem 3 is itself used in the proof of Theorem 1 and Corollary 2 done in Section 3. Lastly, Section 4 is concerned with a half-space theorem for a related free boundary problem.

2. Half-space theorems in general dimension: proof of Theorem 3 and its corollaries

The half-space theorem of minimal hypersurfaces in $\mathbb{R}^n$ is not true when $n \geq 4$, because the higher dimensional catenoids lie in a half-space. For the Allen-Cahn equation (1), we still do not know whether there is version of the half-space theorem in dimension $n \geq 4$. We have obtained partial classification results in this direction, based on the maximum principle.

We start this section with a general property holding in any dimension $n \geq 1$. This can essentially be found in [5, Lemmas 3.2 and 3.3] and [15, Lemma 2.3].

Proposition 7. *Let $n \geq 1$ and u be a non-constant solution of (1) in any dimension $n \geq 1$. Then $\{u = 0\} \neq \emptyset$ and $|u(x)| \to 1$ as $\operatorname{dist}(x, \{u = 0\}) \to +\infty$.*

Proof. The proof is standard, we briefly sketch it for the sake of completeness. We recall from the introduction that u necessarily satisfies $-1 < u < 1$ in $\mathbb{R}^n$ and $\{u = 0\} \neq \emptyset$. Consider any $R > 0$ and $x \in \mathbb{R}^n$ such that

$$\operatorname{dist}(x, \{u = 0\}) > R$$

(hence, $\overline{B(x,R)} \cap \{u = 0\} = \emptyset$).

Let λ_R be the principal eigenvalue of $-\Delta$ in the ball B_R with Dirichlet boundary condition, that is, there is a function $\varphi_R \in C^2(\overline{B_R})$ such that $-\Delta\varphi_R = \lambda_R\varphi_R$ in $\overline{B_R}$, $\varphi_R = 0$ on ∂B_R, and $\varphi_R > 0$ in B_R. From the classical radial symmetry result [21], the function φ_R is radially symmetric and decreasing with respect to the distance from the origin. Therefore, up to multiplication by a positive constant, one can assume without loss of generality that $\varphi_R(0) = 1 = \max_{\overline{B_R}} \varphi_R$. Notice also that $\lambda_R = \lambda_1/R^2$. Since the conclusion is concerned with the limit as $\operatorname{dist}(x, \{u = 0\}) \to +\infty$, one can assume without loss of generality that $R > 0$ is large enough so that $0 < \lambda_R < 1$.

In the closed ball $\overline{B(x,R)}$, the continuous function u does not vanish. Up to changing u into $-u$, let us assume without loss of generality that $u > 0$ in $\overline{B(x,R)}$. There exists then ε_0 such that $\varepsilon\varphi_R(\cdot - x) < u$ in $\overline{B(x,R)}$, for all $\varepsilon \in [0, \varepsilon_0]$. Furthermore, for every $\varepsilon \in [0, \sqrt{1-\lambda_R}]$, the function $\varepsilon\varphi_R(\cdot - x)$ satisfies

$$\Delta(\varepsilon\varphi_R(\cdot - x)) + \varepsilon\varphi_R(\cdot - x) - (\varepsilon\varphi_R(\cdot - x))^3 = \varepsilon\varphi_R(\cdot - x) \times (1 - \lambda_R - (\varepsilon\varphi_R(\cdot - x))^2) \geq 0$$

in $\overline{B(x,R)}$, since $0 \leq \varphi_R \leq 1$ in $\overline{B_R}$. As a consequence, for any such ε, the function $\varepsilon\varphi_R(\cdot - x)$ is a subsolution of (1) in the closed ball $\overline{B(x,R)}$ and it vanishes on $\partial B(x,R)$, while $u > 0$ in

$\overline{B(x,R)}$. It follows from the strong maximum principle that $\varepsilon\varphi_R(\cdot - x) < u$ in $\overline{B(x,R)}$ for every $\varepsilon \in [0, \sqrt{1-\lambda_R}]$. In particular, $u(x) > \sqrt{1-\lambda_R}$ since $\varphi_R(0) = 1$. Since $\lambda_R = \lambda_1/R^2 \to 0$ as $R \to +\infty$ and $-1 < u < 1$ in $\mathbb{R}^n$, the conclusion follows. □

Remark 8. Proposition 7 implies that $\sup_{\mathbb{R}^n} |u| = 1$ if $\sup_{x\in\mathbb{R}^n} \mathrm{dist}(x, \{u = 0\}) = +\infty$. However, the property $\sup_{x\in\mathbb{R}^n} \mathrm{dist}(x, \{u = 0\}) = +\infty$ is not always satisfied, since the Allen-Cahn equation admits non-trivial periodic solutions u oscillating around the value 0, and for which $\sup_{\mathbb{R}^n} |u| < 1$.

We are now in position to prove our Theorem 3.

Proof of Theorem 3. Throughout the proof, u is a non-constant solution of (1) such that

$$u < 0 \ \text{ in } \mathbb{R}^n_- = \{x_n < 0\}.$$

As recalled in the introduction, we know that $-1 < u < 1$ in $\mathbb{R}^n$.

By Proposition 7, we have that

$$u(x_1, \cdots, x_n) \to -1 \ \text{ as } x_n \to -\infty \text{ uniformly in } (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}. \tag{2}$$

since the function u is negative in $\mathbb{R}^n_-$.

Denote $U(x) = H(x_n) = \tanh(x_n/\sqrt{2})$ and

$$U^\omega(x) = U(x_1, \cdots, x_{n-1}, x_n + \omega) = H(x_n + \omega) = \tanh\left(\frac{x_n + \omega}{\sqrt{2}}\right)$$

for $x \in \mathbb{R}^n$ and $\omega \in \mathbb{R}$. We shall now show that $u \le U^\omega$ in $\mathbb{R}^n$ for all ω large enough. To do so, let $A > 0$ be such that

$$u \le -\frac{1}{\sqrt{3}} \ \text{ in } \mathbb{R}^{n-1}\times(-\infty, -A] \quad \text{and} \quad U \ge \frac{1}{\sqrt{3}} \ \text{ in } \mathbb{R}^{n-1}\times[A, +\infty). \tag{3}$$

We claim that

$$u \le U^\omega \ \text{ in } \mathbb{R}^n \ \text{ for all } \omega \ge 2A. \tag{4}$$

To do so, pick any $\omega \in [2A, +\infty)$. We shall prove that $u \le U^\omega$ in $\mathbb{R}^{n-1} \times (-\infty, -A]$. Assume by way of contradiction that $M := \sup_{\mathbb{R}^{n-1}\times(-\infty,-A]}(u - U^\omega) > 0$. Then there is a sequence $(z_k)_{k\in\mathbb{N}} = (z'_k, z_{k,n})_{k\in\mathbb{N}}$ in $\mathbb{R}^{n-1} \times (-\infty, -A]$ such that

$$u(z_k) - U^\omega(z_k) \to M > 0 \ \text{ as } k \to +\infty.$$

By (2), the sequence $(z_{k,n})_{k\in\mathbb{N}}$ is then bounded. Furthermore, by uniform continuity of U (or of u), property (3) and the assumption $\omega \ge 2A$ imply that

$$\limsup_{k\to+\infty} z_{k,n} < -A.$$

Therefore, there is $\zeta \in (-\infty, -A)$ such that, up to extraction of a subsequence, $z_{k,n} \to \zeta$ as $k \to +\infty$. Up to extraction of another subsequence, the functions $x \mapsto u(x' + z'_k, x_n)$ converge in $C^2_{loc}(\mathbb{R}^n)$ as $k \to +\infty$ to a solution w_∞ of (1) such that $w_\infty \le -1/\sqrt{3}$ and $w_\infty - U^\omega \le M$ in $\mathbb{R}^{n-1} \times (-\infty, -A]$, while $w_\infty(0,\zeta) - U^\omega(0,\zeta) = M$. At the (interior) point $(0,\zeta) \in \mathbb{R}^{n-1} \times (-\infty, -A)$, there holds

$$0 \ge \Delta(w_\infty - U^\omega)(0,\zeta) = -w_\infty(0,\zeta) + w_\infty(0,\zeta)^3 + U^\omega(0,\zeta) - (U^\omega(0,\zeta))^3. \tag{5}$$

But $-1 < U^\omega(0,\zeta) < U^\omega(0,\zeta)+M = w_\infty(0,\zeta) \le -1/\sqrt{3}$ and the function $s \mapsto s-s^3$ is decreasing in $[-1, -1/\sqrt{3}]$. Hence the right-hand side of (5) is positive, a contradiction. Therefore,

$$\sup_{\mathbb{R}^{n-1}\times(-\infty,-A]} (u - U^\omega) \le 0,$$

that is, $u \le U^\omega$ in $\mathbb{R}^{n-1}\times(-\infty,-A]$. Similarly, since $U^\omega \ge 1/\sqrt{3}$ in $\mathbb{R}^{n-1}\times[-A,+\infty)$ and $U^\omega \ge u$ on $\mathbb{R}^{n-1}\times\{-A\}$, while the function $s\mapsto s-s^3$ is decreasing in $[1/\sqrt{3},1]$, one can show that $u\le U^\omega$ in $\mathbb{R}^{n-1}\times[-A,+\infty)$. This proves (4).

Define now

$$a=\inf\big\{\omega\in\mathbb{R} : u\le U^\omega \text{ in } \mathbb{R}^n\big\}.$$

The previous paragraph yields $a\le 2A$. On the other hand, since $U^\omega\to -1$ as $\omega\to-\infty$ (at least) pointwise in $\mathbb{R}^n$, while $u>-1$ in $\mathbb{R}^n$, one infers that $a\in\mathbb{R}$. By continuity, there holds $u\le U^a$ in $\mathbb{R}^n$, that is,

$$u(x)\le H(x_n+a) \text{ for all } x\in\mathbb{R}^n. \tag{6}$$

This statement corresponds to the first part of the conclusion of Theorem 3.

Let us now show the second part of the conclusion. First of all, if there is a point $x^*\in\mathbb{R}^n$ such that $u(x^*)=U^a(x^*)=H(x^*_n+a)$, then the strong maximum principle implies that $u\equiv U^a$ in $\mathbb{R}^n$, that is,

$$u(x)=H(x_n+a) \text{ for all } x\in\mathbb{R}^n.$$

Let us then assume in the sequel that $u<U^a$ in $\mathbb{R}^n$, that is,

$$u(x)<U^a(x)=H(x_n+a) \text{ for all } x\in\mathbb{R}^n. \tag{7}$$

Let $B>0$ be such that $U^a\ge 2/3\,(>1/\sqrt{3})$ in $\mathbb{R}^{n-1}\times[B,+\infty)$. We claim that

$$\sup_{\mathbb{R}^{n-1}\times[-A,B]}(u-U^a)=0. \tag{8}$$

Indeed, otherwise, one would have $\sup_{\mathbb{R}^{n-1}\times[-A,B]}(u-U^a)<0$ and, by uniform continuity of U, there would exist $\omega\in(-\infty,a)$ such that $u\le U^\omega$ in $\mathbb{R}^{n-1}\times[-A,B]$ and $U^\omega\ge 1/\sqrt{3}$ in $\mathbb{R}^{n-1}\times[B,+\infty)$. With the same arguments as in the previous paragraph, one gets that $u\le U^\omega$ in $\mathbb{R}^{n-1}\times(-\infty,-A]$ and $u\le U^\omega$ in $\mathbb{R}^{n-1}\times[B,+\infty)$. As a consequence, $u\le U^\omega$ in $\mathbb{R}^n$, contradicting the minimality of a. Therefore, (8) holds.

From (8), one infers the existence of a sequence $(\xi_k)_{k\in\mathbb{N}}=(\xi'_k,\xi_{k,n})_{k\in\mathbb{N}}$ in $\mathbb{R}^{n-1}\times[-A,B]$ such that

$$u(\xi_k)-U^a(\xi_k)\to 0 \text{ as } k\to+\infty. \tag{9}$$

Up to extraction of a subsequence, one can assume that $\xi_{k,n}\to\xi_{\infty,n}$ as $k\to+\infty$, for some $\xi_{\infty,n}\in[-A,B]$. Notice that $|\xi'_k|\to+\infty$ as $k\to+\infty$, since otherwise there would exist a point $\xi\in\mathbb{R}^{n-1}\times[-A,B]$ such that $u(\xi)=U^a(\xi)$, contradicting (7). Denote

$$y_k=(\xi'_k,0)\in\mathbb{R}^{n-1}\times\{0\} \text{ and } u_k(x)=u(x+y_k) \text{ for } k\in\mathbb{N} \text{ and } x\in\mathbb{R}^n.$$

To complete the proof of Theorem 3, we just need to show that

$$u_k(x)\to H(x_n+a)$$

in $C^2_{loc}(\mathbb{R}^n)$ as $k\to+\infty$. Up to extraction of a subsequence, the functions u_k converge in $C^2_{loc}(\mathbb{R}^n)$ to a solution $u_\infty:\mathbb{R}^n\to[-1,1]$ of (1) such that $u_\infty(x)\le H(x_n+a)$ in $\mathbb{R}^n$ from (6) and the definition of y_k. Furthermore, $u_\infty(0,\xi_{\infty,n})=U^a(0,\xi_{\infty,n})=H(\xi_{\infty,n}+a)$ by (9). It then follows from the strong maximum principle that $u_\infty(x)=H(x_n+a)$ for all $x\in\mathbb{R}^n$. Furthermore, the limit of the functions u_k being independent of the subsequence, one concludes that the whole sequence $(u_k)_{k\in\mathbb{N}}$ converges to the function $x\mapsto H(x_n+a)$ in $C^2_{loc}(\mathbb{R}^n)$ as $k\to+\infty$. The proof of Theorem 3 is thereby complete. □

Remark 9. In Theorem 3, one has $-1 < u(x) < H(x_n + a) = \tanh((x_n + a)/\sqrt{2})$ for all $x \in \mathbb{R}^n$. Modica's inequality $|\nabla u|^2 \le (1 - u^2)^2/2$ (see [27]) then yields

$$|\nabla u(x)| \le 2\sqrt{2}\, e^{\sqrt{2}\,(x_n+a)} \text{ for all } x \in \mathbb{R}^n. \tag{10}$$

By changing u into $-u$ in Theorem 3, the following result immediately follows.

Theorem 10. *Let $n \ge 1$ and u be a non-constant solution of the Allen-Cahn equation (1) in $\mathbb{R}^n$. If $u > 0$ in the half-space $\{x_n < 0\}$, then there exists $a \in \mathbb{R}$ such that*

$$u(x) \ge H(-x_n + a)$$

for all $x \in \mathbb{R}^n$, and either $u(x) = H(-x_n+a)$ for all $x \in \mathbb{R}^n$, or $u(x) > H(-x_n+a)$ for all $x \in \mathbb{R}^n$ and there exists a sequence $(y_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^{n-1} \times \{0\}$ such that $|y_k| \to +\infty$ as $k \to +\infty$, and the functions $u(\cdot + y_k)$ converge in $C^2_{loc}(\mathbb{R}^n)$ to the function $x \mapsto H(-x_n + a)$ as $k \to +\infty$.

Let us now turn to the proof of Corollaries 4 and 5, which follow from Theorems 3 and 10.

Proof of Corollary 4. Assume by way of contradiction that u is a non-constant solution of (1) with $\{u = 0\}$ contained in a non-degenerate convex cone. Then, up to changing u into $-u$, and up to translation and rotation of the variables, it follows that

$$u < 0 \text{ in } \big\{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n < \beta\,|x'|\big\}, \tag{11}$$

for some $\beta > 0$. Theorem 3, together with (11), then yields the existence of a real number a such that $u(x) = H(x_n + a)$ for all $x \in \mathbb{R}^n$, which leads to a contradiction. □

Proof of Corollary 5. Up to changing u into $-u$ and up to translation and rotation of the variables, one can also assume without loss of generality that $e = (0, \cdots, 0, 1)$, that $u < 0$ in $\{x_n < 0\}$ and that $u(x', 0) = 0$ for some $x' \in \mathbb{R}^{n-1}$. Theorem 3 then implies that $u(x) \le H(x_n + a)$ in $\mathbb{R}^n$, for some $a \in \mathbb{R}$, and the other parts of the conclusion hold for that real number a. We claim that $u(x', 0) = H(a)$. Indeed, if not, then $0 = u(x', 0) < H(a)$, hence $a > 0$, while Theorem 3 also yields the existence of a sequence $(y_k)_{k\in\mathbb{N}} = (y'_k, 0)_{k\in\mathbb{N}}$ in $\mathbb{R}^{n-1} \times \{0\}$ such that $u(\cdot + y_k) \to H(x_n + a)$ in $C^2_{loc}(\mathbb{R}^n)$ as $k \to +\infty$. In particular, $u(y'_k, -a/2) \to H(a/2) > 0$ as $k \to +\infty$, hence $u(y'_k, -a/2) > 0$ for all k large enough, which is impossible since $u < 0$ in $\{x_n < 0\}$. Therefore, $u(x', 0) = H(a)$ (hence, $a = 0$) and $u(x) \equiv H(x_n)$ in $\mathbb{R}^n$ from Theorem 3. □

Lastly, as announced at the end of Section 1, we can retrieve from Theorem 3 the one-dimensional property of any non-constant solution of (1) whole zero level set is contained in a slab. This result was obtained by Farina [16, Theorem 1.1].

Corollary 11. [16, Theorem 1.1] *Let $n \ge 1$ and u be a non-constant solution of the Allen-Cahn equation (1) in $\mathbb{R}^n$. If $\{u = 0\}$ is contained in a slab $\{x \in \mathbb{R}^n : |x \cdot e| < A\}$ for some unit vector e and some real number $A > 0$, then there exists $a \in \mathbb{R}$ such that either $u(x) = H(-x \cdot e + a)$ or $u(x) = H(x \cdot e + a)$, for all $x \in \mathbb{R}^n$.*

We here give a proof using Theorem 3.

Proof. Up to changing u into $-u$ and up to translation and rotation of the variables, one can also assume without loss of generality that $e = (0, \cdots, 0, 1)$, that $u < 0$ in $\{x_n \le 0\}$ and that

$$\{u = 0\} \subset \{0 < x_n < 2A\}.$$

It then follows from Theorem 3 that there exists $b \in \mathbb{R}$ such that

$$u(x) \leq H(x_n + b)$$

for all $x \in \mathbb{R}^n$ and either $u(x) = H(x_n + b)$ for all $x \in \mathbb{R}^n$, or $u(\cdot + y_k) \to H(x_n + b)$ in $C^2_{loc}(\mathbb{R}^n)$ as $k \to +\infty$, for some sequence $(y_k)_{k \in \mathbb{N}}$ in $\mathbb{R}^{n-1} \times \{0\}$. In both cases, one has $\sup_{\mathbb{R}^{n-1} \times (-\infty, 2A]} u \leq H(2A + b) < 1$ and $\sup_{\mathbb{R}^n} u = \sup_{\mathbb{R}} H = 1$. By continuity, one infers that $u > 0$ in $\{x_n \geq 2A\}$. From Theorem 3 applied to the solution

$$x = (x', x_n) \mapsto -u(x', -x_n + 2A),$$

there exists then $c \in \mathbb{R}$ such that $-u(x', -x_n + 2A) \leq H(x_n + c)$ for all $x \in \mathbb{R}^n$, hence

$$u(x) \geq H(x_n - 2A - c)$$

for all $x \in \mathbb{R}^n$. Finally $H(x_n - 2A - c) \leq u(x) \leq H(x_n + b)$ for all $x \in \mathbb{R}^n$ and one concludes from [6, Theorem 3.1] or [15, Theorem 2.1] that $u(x) \equiv H(x_n + a)$ in $\mathbb{R}^n$, for some $a \in \mathbb{R}$. □

3. Proof of the half-space theorem in dimensions $n = 2, 3$

As we mentioned in Section 1, the proof of the half-space theorem for minimal surfaces uses the family of catenoids and the sweeping principle. In the Allen-Cahn case, the solutions are defined in the whole space, and it is not easy to apply this idea.

We remark that the $n = 2$ case of Theorem 1 can also be proven by applying the method in De Silva and Savin [11]. Our proof uses Pohozaev identity (also called balancing condition, see [12]) and is very different from theirs.

We shall prove Theorem 1 for $n = 1, 2, 3$. Notice that if $n = 1$, then the solution u is trivially one-dimensional and since it is not constant, it is then equal to $H(x_1)$ up to shifts, as follows directly from ODE analysis. The cases of $n = 2, 3$ are more complicated. Although we can deal with these two cases in a unified way, we choose to first give a simple proof when $n = 2$, because this gives us a clear geometric intuition behind the whole proof.

3.1. The case $n = 2$.

Let u be a solution whose zero level set $\{u = 0\}$ is contained in the half-space $\{x_2 > 0\}$. Up to changing u into $-u$ and/or x_1 into $-x_1$, and shifting in the direction x_2, we may assume without loss of generality that $u < 0$ in $\{x_2 < 0\}$ and, from Theorem 3, that

$$u(x_1, x_2) \leq H(x_2) \quad \text{for all } (x_1, x_2) \in \mathbb{R}^2 \tag{12}$$

and that there exists a sequence $(t^+_k)_{k \in \mathbb{N}}$ such that $t^+_k \to +\infty$ and $u(x_1 + t^+_k, x_2) \to H(x_2)$ in $C^2_{loc}(\mathbb{R}^2)$ as $k \to +\infty$.

For each $x_1 \in \mathbb{R}$, we define

$$g(x_1) = \inf \{x_2 \in \mathbb{R} : u(x_1, x_2) = 0\} \ \in [0, +\infty].$$

Note that the infimum is a minimum if $g(x_1)$ is a real number. Note also that $g(x_1)$ might a priori be $+\infty$ for some values x_1, in which case $u(x_1, x_2) < 0$ for all $x_2 \in \mathbb{R}$ (nevertheless, the conclusion $u(x_1, x_2) \equiv H(x_2)$ in $\mathbb{R}^2$ will show that, a posteriori, this case is impossible). We know at this point that g cannot be equal to $+\infty$ on $(-\infty, \xi)$ for some $\xi \in \mathbb{R}$ since otherwise the zero level set of u would be included in the quarter-plane $\{x_1 \geq \xi, \, x_2 \geq 0\}$, which is ruled out by Corollary 4. Let us set

$$\alpha = \liminf_{x_1 \to -\infty} g(x_1) \ \in [0, +\infty].$$

We emphasize that in principle it may happen that α is nonzero. For instance, the zero level set of u may look like $x_2 = \ln(-x_1)$, as x_1 goes to $-\infty$.

Let us first consider the case $0 \le \alpha < +\infty$. There exists then a sequence $(t_k^-, s_k^-)_{k\in\mathbb{N}}$ such that $u(t_k^-, s_k^-) = 0$, and $t_k^- \to -\infty$ and $s_k^- \to \alpha$ as $k \to +\infty$. Up to extraction of a subsequence, the functions $(x_1, x_2) \mapsto u(x_1 + t_k^-, x_2 + \alpha)$ converge in $C^2_{loc}(\mathbb{R}^2)$ to a classical solution u_∞ of (1) such that $u_\infty(0,0) = 0$ and $u_\infty \le 0$ in $\{x_2 \le 0\}$, owing to the definition of α. Furthermore, $u_\infty(x_1, x_2) \le H(x_2 + \alpha)$ in $\mathbb{R}^2$ from (12), hence $u_\infty(x_1, -\infty) = -1$ for each $x_1 \in \mathbb{R}$, and $u_\infty < 0$ in $\{x_2 < 0\}$ from the strong maximum principle. Corollary 5 then implies that $u_\infty(x_1, x_2) \equiv H(x_2 + b)$ in $\mathbb{R}^2$ for some real number b. Since $u_\infty(0,0) = 0$, one finally infers that $b = 0$ and $u_\infty(x_1, x_2) \equiv H(x_2)$ in $\mathbb{R}^2$.

Let us consider now the semi-infinite vertical strip

$$\Omega_k := \left\{(x_1, x_2) \in \mathbb{R}^2 : t_k^- < x_1 < t_k^+, -\infty < x_2 < 0\right\}.$$

Let $X = (0, 1)$. The balancing condition (see [12, Appendix]) tells us that

$$\int_{\partial\Omega_k} \left[\left(\frac{1}{2}|\nabla u|^2 + F(u)\right) X \cdot \nu - (\nabla u \cdot X)(\nabla u \cdot \nu)\right] d\sigma = 0, \tag{13}$$

where

$$F(s) = \frac{(1 - s^2)^2}{4}$$

and ν is the outward unit normal of the domain Ω_k (which is defined everywhere except at the corners $(t_k^\pm, 0)$). The previous formula means that

$$\int_{t_k^-}^{t_k^+} \left(\frac{|\nabla u(x_1,0)|^2}{2} + F(u(x_1,0)) - u_{x_2}^2(x_1,0)\right) dx_1 + \underbrace{\int_{-\infty}^{0} \left(u_{x_1}(t_k^-, x_2)\, u_{x_2}(t_k^-, x_2) - u_{x_1}(t_k^+, x_2)\, u_{x_2}(t_k^+, x_2)\right) dx_2}_{=:I_k} = 0,$$

where the second integral I_k converges absolutely from Remark 9. Since $u(x_1 + t_k^+, x_2) \to H(x_2)$ and $u(x_1 + t_k^-, x_2 + \alpha) \to H(x_2)$ as $k \to +\infty$ in $C^2_{loc}(\mathbb{R}^2)$, together with Remark 9, it follows that $I_k \to 0$ as $k \to +\infty$. Therefore,

$$\int_{t_k^-}^{t_k^+} \left(\frac{|\nabla u(x_1,0)|^2}{2} + F(u(x_1,0)) - u_{x_2}^2(x_1,0)\right) dx_1 \to 0 \quad \text{as } k \to +\infty.$$

On the other hand, Modica's inequality

$$F(u) \ge \frac{|\nabla u|^2}{2}$$

(see [27]) implies that

$$\frac{|\nabla u|^2}{2} + F(u) - u_{x_2}^2 \ge |\nabla u|^2 - u_{x_2}^2 = u_{x_1}^2 \ge 0,$$

for all $x_1 \in \mathbb{R}$, hence

$$F(u(x_1,0)) = \frac{u_{x_2}^2(x_1,0)}{2} = \frac{|\nabla u(x_1,0)|^2}{2}$$

for all $x_1 \in \mathbb{R}$, since $t_k^\pm \to \pm\infty$ as $k \to +\infty$. It then follows from results in [8, 27] that u is one-dimensional. Let us sketch the proof of this fact for completeness. For more details, we refer to Theorem 5.1 in [8]. Let

$$P := |\nabla u|^2 - 2F(u).$$

Then at those points where $\nabla u \neq 0$, the function P satisfies the following differential inequality (we also refer to [27] or Section 5.2.2 of [29] for the detailed computation):

$$\Delta P + \frac{1}{|\nabla u|^2} \sum_{k=1}^{2} c_k P_{x_k} \geq 0,$$

where

$$c_k = -2F'(u)\, u_{x_k} - \frac{1}{2} P_{x_k}.$$

This in particular implies that P satisfies the maximum principle in a small neighbourhood of those points where $\nabla u \neq 0$. Actually, the estimate $|\nabla u|^2 \leq 2F(u)$ (that is, $P \leq 0$) follows from this inequality. Now, the strong maximum principle applied to P then tells us that $P \equiv 0$ in the plane if P vanishes somewhere at a point q_0 (one then necessarily has $|\nabla u(q_0)| > 0$, since $F(u) = (1-u^2)^2/4 > 0$). We then define v by the relation $u = H(v)$. The function v satisfies $|\nabla v| = 1$, and

$$\Delta v = 0.$$

As a consequence, v is an affine function and u is one-dimensional. This finishes the proof.

Let us finally consider the case $\alpha = +\infty$. Here, remembering also that $u < 0$ in $\{x_2 < 0\}$, it follows from Proposition 7 that

$$\sup_{x_1 \leq -R,\, x_2 \leq 0} u(x_1, x_2) \to -1$$

as $R \to +\infty$, hence $|\nabla u(x_1, x_2)| \to 0$ as $x_1 \to -\infty$ uniformly with respect to $x_2 \leq 0$, from standard elliptic estimates. Together with Remark 9, this implies that

$$\int_{-\infty}^{0} u_{x_1}(x_1, x_2)\, u_{x_2}(x_1, x_2)\, dx_2 \to 0 \quad \text{as } x_1 \to -\infty.$$

Therefore, by applying (13) in the region

$$\left\{(x_1, x_2) \in \mathbb{R}^2 : -k < x_1 < t_k^+, -\infty < x_2 < 0\right\},$$

one gets with the same arguments as before that

$$\int_{-k}^{t_k^+} \left(\frac{|\nabla u(x_1, 0)|^2}{2} + F(u(x_1, 0)) - u_{x_2}^2(x_1, 0) \right) dx_1 \to 0 \quad \text{as } k \to +\infty.$$

This leads to the same conclusion as in the previous paragraph. The proof of Theorem 1 in the case $n = 2$ is thereby complete.

Proof of Corollary 2. Let $u_1 < u_2$ be two non-constant solutions of (1) in $\mathbb{R}^2$. Remember that $-1 < u_1 < u_2 < 1$ in $\mathbb{R}^2$. Using u_1 and u_2 as barriers, we can then construct a stable solution u_3 of (1) with

$$-1 < u_1 \leq u_3 \leq u_2 < 1 \quad \text{in } \mathbb{R}^2.$$

By stable, it is meant that the Lagrangian associated to the equation has nonnegative second-order variation at u_3, that is,

$$\int_{\mathbb{R}^2} \left(|\nabla \phi|^2 - (1 - 3u_3^2)\phi^2 \right) \geq 0$$

for every $\phi \in C_c^1(\mathbb{R}^2)$. Equivalently, this means that, for every $R > 0$, the principal eigenvalue $\lambda_R[u_3]$ of the linearized operator $-\Delta - (1 - 3u_3^2)$ in the ball B_R with Dirichlet boundary condition on ∂B_R is nonnegative. The proof of the existence of such a stable solution u_3 trapped between u_1 and u_2 is quite standard, but let us sketch it here for the sake of completeness. If u_1 is stable, then simply define $u_3 = u_1$. If u_1 is not stable, then there is $R > 0$ such that the principal eigenvalue $\lambda_R[u_1]$ of the operator $-\Delta-(1-3u_1^2)$ in B_R with Dirichlet boundary condition on ∂B_R is negative. Let $\varphi_R \in C^2(\overline{B_R})$ be a principal eigenfunction, solving

$$-\Delta\varphi_R - (1-3u_1^2)\varphi_R = \lambda_R[u_1]\varphi_R \text{ in } \overline{B_R},$$

with $\varphi_R > 0$ in B_R and $\varphi_R = 0$ on ∂B_R. There is then $\varepsilon_0 > 0$ such that

$$-\Delta(u_1+\varepsilon_0\varphi_R) - \big((u_1+\varepsilon_0\varphi_R) - (u_1+\varepsilon_0\varphi_R)^3\big) < 0 \text{ in } B_R, \tag{14}$$

together with $u_1 + \varepsilon_0\varphi_R < u_2$ in $\overline{B_R}$. Let now $v(t, x_1, x_2)$ be the solution of the Cauchy problem

$$v_t = \Delta v + v - v^3$$

for $t > 0$ and $(x_1, x_2) \in \mathbb{R}^2$, with initial condition defined by $v(0,\cdot) = u_1 + \varepsilon_0\varphi_R$ in $\overline{B_R}$ and $v(0,\cdot) = u_1$ in $\mathbb{R}^2 \setminus \overline{B_R}$. The maximum principle yields $v(t,\cdot) < u_2$ in $\mathbb{R}^2$ for all $t \geq 0$. Furthermore, since u_1 solves (1), since $u_1 + \varepsilon_0\varphi_R$ is a subsolution in B_R in the sense of (14), and since $v(0,\cdot) \geq u_1$ in $\mathbb{R}^2$ and $v(0,\cdot)$ continuous, the maximum principle also implies that v is increasing with respect to $t \geq 0$. From standard parabolic estimates, there is then a solution u_3 of (1) such that $v(t,\cdot) \to u_3$ in $C^2_{loc}(\mathbb{R}^2)$ as $t \to +\infty$, and $v(0,\cdot) < u_3 \leq u_2$ in $\mathbb{R}^2$. We now claim that the solution u_3 is stable, and then fulfills the desired conclusion. Indeed, if u_3 were not stable, there would exist $R' > 0$ such that $\lambda_{R'}[u_3] < 0$ and then there would be a principal eigenfunction $\phi_{R'} \in C^2(\overline{B_{R'}})$ of the operator $-\Delta - (1-3u_3^2)$ in $B_{R'}$ with Dirichlet boundary condition on $\partial B_{R'}$. This eigenfunction can be chosen such that $\phi_{R'} > 0$ in $B_{R'}$. There would then be $\varepsilon'_0 > 0$ such that

$$-\Delta(u_3 - \varepsilon'_0\phi_{R'}) - \big((u_3 - \varepsilon'_0\phi_{R'}) - (u_3 - \varepsilon'_0\phi_{R'})^3\big) > 0 \text{ in } B_{R'},$$

together with $u_3 - \varepsilon'_0\phi_{R'} > v(0,\cdot)$ in $\overline{B_{R'}}$. In other words, the function $u_3 - \varepsilon'_0\phi_{R'}$ would be a supersolution of (1) in $\overline{B_{R'}}$ and the maximum principle would then yield $v(t,\cdot) < u_3 - \varepsilon'_0\phi_{R'}$ in $\overline{B_{R'}}$ for all $t \geq 0$, providing a contradiction as $t \to +\infty$. As a conclusion, u_3 is a desired stable solution of (1) trapped between u_1 and u_2.

In dimension two, stable solutions are one-dimensional, as follows from [4, Theorem 1.8]. The function u_3 is then one-dimensional stable and it takes values in $(-1, 1)$, hence there exist then a unit vector e and a real number c such that $u_3(x) \equiv H(x \cdot e + c)$ in $\mathbb{R}^2$. Therefore, the nodal set of u_1 is contained in the half-space $\{x \cdot e + c \geq 0\}$ and $u_1 < 0$ in $\{x \cdot e + c < 0\}$. By the weak half-space Theorem 1, u_1 has to be one-dimensional, and more precisely there is $a \in \mathbb{R}$ such that $u_1(x) \equiv H(x \cdot e + a)$ in $\mathbb{R}^2$. The same is true for u_2, with $u_2(x) \equiv H(x \cdot e + b)$ in $\mathbb{R}^2$, for some real number b such that $b > a$ (since $u_2 > u_1$). □

3.2. **The case $n = 3$.** Next, we shall consider the case of dimension 3. The arguments in this section can also be applied in the two dimensional case, but we preferred to use the more direct proof of the previous section in the case $n = 2$.

First of all, as in the case $n = 2$, up to changing u into $-u$ and/or shifting in the direction x_3, we may assume without loss of generality that $u < 0$ in $\{x_3 < 0\}$ and, from Theorem 3, that

$$-1 < u(x_1, x_2, x_3) \leq H(x_3) \ \text{ for all } (x_1, x_2, x_3) \in \mathbb{R}^3 \tag{15}$$

and there is a sequence $(y_k)_{k \in \mathbb{N}}$ in $\mathbb{R}^2 \times \{0\}$ such that $u(\cdot + y_k) \to H(x_3)$ in $C^2_{loc}(\mathbb{R}^3)$ as $k \to +\infty$.

Now, let $A > 0$ be such that

$$\tanh\Big(-\frac{A}{\sqrt{2}}\Big) \leq -\sqrt{\frac{2}{3}}.$$

For $s>0$, let Ω_s be the half-cylinder

$$\Omega_s = \{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_1^2+x_2^2<s^2,\ x_3<-A\}.$$

Let $X=(0,0,1)$. Then, for every $s>0$, the following balancing formula (see [12]) holds in Ω_s:

$$\int_{\partial\Omega_s}\left[\left(\frac{1}{2}|\nabla u|^2+F(u)\right)X\cdot\nu-(\nabla u\cdot X)(\nabla u\cdot\nu)\right]d\sigma=0, \tag{16}$$

meaning that

$$\begin{aligned} g(s) &:= \int_{\{x_1^2+x_2^2<s^2\}}\left(\frac{|\nabla u(x_1,x_2,-A)|^2}{2}+F(u(x_1,x_2,-A))-u_{x_3}^2(x_1,x_2,-A)\right)dx_1dx_2 \\ &= s\int_{-\infty}^{-A}\int_0^{2\pi}\big(u_{x_1}(s\cos\theta,s\sin\theta,x_3)\cos\theta+u_{x_2}(s\cos\theta,s\sin\theta,x_3)\sin\theta\big) \\ &\qquad\qquad \times\ u_{x_3}(s\cos\theta,s\sin\theta,x_3)\,d\theta\,dx_3. \end{aligned} \tag{17}$$

Notice that the integrals converge absolutely from Remark 9. As in the previous section, we infer from Modica's inequality

$$F(u)\geq\frac{|\nabla u|^2}{2}$$

that

$$\frac{|\nabla u|^2}{2}+F(u)-u_{x_3}^2\geq u_{x_1}^2+u_{x_2}^2\geq 0, \tag{18}$$

in $\mathbb{R}^3$. From this, we know that the function g is nonnegative and nondecreasing in $(0,+\infty)$. Hence we can define

$$\alpha=\lim_{s\to+\infty}g(s)\ \in[0,+\infty]. \tag{19}$$

Let us also define a function $K:(0,+\infty)\times\mathbb{R}\to\mathbb{R}$ by

$$K(s,x_3)=\int_{\{x_1^2+x_2^2<s^2\}}\big(u_{x_1}^2(x_1,x_2,x_3)+u_{x_2}^2(x_1,x_2,x_3)\big)\,dx_1\,dx_2 \tag{20}$$

and notice from the definition of g in (17) and from (18) that

$$K(s,-A)\leq 2\,g(s)\ \text{ for all } s>0. \tag{21}$$

Remark 9 yields suitable exponential decay of $u_{x_1}^2+u_{x_2}^2$ as $x_3\to-\infty$, from which we get the following key-property of K.

Lemma 12. *There holds*

$$K(s,x_3)\leq 3\pi+K(s+\ln s,-A),\ \textit{for all } s\geq 1 \textit{ and } x_3\leq -A. \tag{22}$$

Proof. The functions u_{x_1} and u_{x_2} satisfy

$$-\Delta u_{x_i}+(3u^2-1)u_{x_i}=0$$

in $\mathbb{R}^3$. Thanks to (15), the function u^2 converges to 1 as $x_3\to-\infty$, hence the operator $-\Delta+(3u^2-1)$ tends to $-\Delta+2$. As a matter of fact, in $\mathbb{R}^2\times(-\infty,-A]$, one has

$$-1<u\leq H(-A)=\tanh\left(-\frac{A}{\sqrt{2}}\right)\leq-\sqrt{\frac{2}{3}},$$

thus $3u^2-1\geq 1$ in $\mathbb{R}^2\times(-\infty,-A]$.

For a fixed $s \geq 1$, consider the Lipschitz-continuous and $H^1(\mathbb{R}^2)$ function ϕ defined by

$$\phi(x_1, x_2) = \min\left(1, e^{s-\sqrt{x_1^2+x_2^2}}\right).$$

From Remark 9 and standard elliptic estimates, all first, second and third order derivatives of the function u are bounded in $\mathbb{R}^3$ and decay exponentially as $x_3 \to -\infty$, uniformly in $(x_1, x_2) \in \mathbb{R}^2$. Therefore, for $i = 1$ or 2, the function

$$t \mapsto \theta(t) := \int_{\mathbb{R}^2} (u_{x_i}(x_1, x_2, t))^2 \, \phi(x_1, x_2) \, dx_1 \, dx_2$$

is well defined and of class $C^1(\mathbb{R})$, with

$$\begin{aligned}
\theta'(t) &= \int_{\mathbb{R}^2} (u_{x_i}^2)_{x_3}(x_1, x_2, t) \, \phi(x_1, x_2) \, dx_1 \, dx_2 \\
&= \int_{\partial\{(x_1,x_2,x_3)\in\mathbb{R}^3 :\, x_3<t\}} \phi(x_1, x_2) \; \nabla(u_{x_i}^2) \cdot \underbrace{\nu}_{=(0,0,1)} \, dx_1 \, dx_2 \\
&= \int_{\{(x_1,x_2,x_3)\in\mathbb{R}^3 :\, x_3<t\}} \left(\Delta(u_{x_i}^2) \, \phi + \nabla_{(x_1,x_2)}(u_{x_i}^2) \cdot \nabla\phi\right) dx_1 \, dx_2 \, dx_3 \\
&= \int_{\{(x_1,x_2,x_3)\in\mathbb{R}^3 :\, x_3<t\}} \left[\left(2\left(3u^2 - 1\right) u_{x_i}^2 + 2\,|\nabla u_{x_i}|^2\right) \phi + \nabla_{(x_1,x_2)}(u_{x_i}^2) \cdot \nabla\phi\right] dx_1 \, dx_2 \, dx_3 \\
&\geq \int_{\{(x_1,x_2,x_3)\in\mathbb{R}^3 :\, x_3<t\}} \left[\left(2u_{x_i}^2 + 2\,|\nabla u_{x_i}|^2\right) \phi + \nabla_{(x_1,x_2)}(u_{x_i}^2) \cdot \nabla\phi\right] dx_1 \, dx_2 \, dx_3
\end{aligned}$$

for all $t \leq -A$, since $3u^2 - 1 \geq 1$ in $\mathbb{R}^2 \times (-\infty, -A]$. For $x_1^2 + x_2^2 < s^2$, one has $\nabla\phi(x_1, x_2) = (0, 0)$. For $x_1^2 + x_2^2 > s^2$ and $x_3 \in \mathbb{R}$, there holds $|\nabla\phi(x_1, x_2)| = \phi(x_1, x_2)$, hence

$$\begin{aligned}
|\nabla_{(x_1,x_2)}(u_{x_i}^2)(x_1, x_2, x_3) \cdot \nabla\phi(x_1, x_2)| &\leq 2\,|u_{x_i}(x_1, x_2, x_3)|\,|\nabla u_{x_i}(x_1, x_2, x_3)|\,|\nabla\phi(x_1, x_2)| \\
&\leq \left(u_{x_i}^2(x_1, x_2, x_3) + |\nabla u_{x_i}(x_1, x_2, x_3)|^2\right) \phi(x_1, x_2).
\end{aligned}$$

Therefore, $\theta'(t) \geq 0$ for all $t \leq -A$. Since $|\nabla u|^2 \leq (1 - u^2)^2/2 \leq 1/2$ in $\mathbb{R}^3$ from [27], it then follows that, for all $s \geq 1$ and $t \leq -A$,

$$\begin{aligned}
K(s, t) &\leq \int_{\mathbb{R}^2} \left(u_{x_1}^2(x_1, x_2, t) + u_{x_2}^2(x_1, x_2, t)\right) \phi(x_1, x_2) \, dx_1 \, dx_2 \\
&\leq \int_{\mathbb{R}^2} \left(u_{x_1}^2(x_1, x_2, -A) + u_{x_2}^2(x_1, x_2, -A)\right) \phi(x_1, x_2) \, dx_1 \, dx_2 \\
&\leq K(s + \ln s, -A) + \pi \int_{s+\ln s}^{+\infty} e^{s-\rho} \rho \, d\rho \\
&= K(s + \ln s, -A) + \pi \left(1 + \frac{\ln s}{s} + \frac{1}{s}\right) \leq K(s + \ln s, -A) + 3\pi.
\end{aligned}$$

This completes the proof. □

With a slight abuse of notation, we also write u in the polar coordinates (in the (x_1, x_2)-plane) as $u(r, \theta, x_3)$. Let us now define, for $s > 0$,

$$f(s) := \int_0^s \int_{-\infty}^{-A} \int_0^{2\pi} u_r(r, \theta, x_3) \, u_{x_3}(r, \theta, x_3) \, d\theta \, dx_3 \, dr. \tag{23}$$

Note that the above integral converges absolutely for each $s > 0$, from Remark 9, and that $f(s) \to 0$ as $s \to 0$.

Lemma 13. *The quantity α defined in (19) is such that $\alpha = 0$.*

Proof. Let us assume by way of contradiction that $\alpha > 0$. Observe first that the function f is of class C^1 in $(0, +\infty)$ and that, thanks to (17),

$$f'(s) = \frac{g(s)}{s} \ \text{ for all } s > 0.$$

Using the fact that g is nonnegative and $g(\tau) \to \alpha \in (0, +\infty]$ as $\tau \to +\infty$, we deduce that

$$f(s) = \int_0^s f'(\tau)\, d\tau = \int_0^s \frac{g(\tau)}{\tau}\, d\tau \geq \alpha' \ln s \ \text{ for all } s \text{ large enough,} \tag{24}$$

with, say, $\alpha' = \alpha/2 > 0$ if $0 < \alpha < +\infty$ and $\alpha' = 1$ if $\alpha = +\infty$. On the other hand, one infers from Remark 9, with here $a = 0$ thanks to (15), that

$$|\nabla u| \leq 2\sqrt{2}\, e^{\sqrt{2}\, x_3} \ \text{ in } \mathbb{R}^3 \tag{25}$$

and from (17) that

$$\frac{g(\tau)}{\tau} = \int_{-\infty}^{-A} \int_0^{2\pi} u_r(\tau, \theta, x_3)\, u_{x_3}(\tau, \theta, x_3)\, d\theta\, dx_3 \leq 4\sqrt{2}\,\pi \leq 6\,\pi \tag{26}$$

for all $\tau > 0$, hence

$$f(s + \ln s) - f(s) = \int_s^{s+\ln s} \frac{g(\tau)}{\tau}\, d\tau \leq 6\,\pi\, \ln s \ \text{ for all } s \geq 1. \tag{27}$$

Using again (25) and (26), together with the definition (20) of K and the decomposition of the integral (23) with respect to $r \in [0, s]$ into two integrals over $[1, s]$ and $[0, 1]$, we get that, for all $s \geq 1$,

$$\begin{aligned} f(s) &\leq \int_{-\infty}^{-A} \left[\int_1^s \int_0^{2\pi} r\, u_r^2\, d\theta\, dr \right]^{1/2} \left[\int_1^s \int_0^{2\pi} \frac{u_{x_3}^2}{r}\, d\theta\, dr \right]^{1/2} dx_3 + 6\,\pi \\ &\leq \sqrt{16\,\pi \ln s} \int_{-\infty}^{-A} \sqrt{K(s, x_3)}\, e^{\sqrt{2}\, x_3}\, dx_3 + 6\,\pi. \end{aligned}$$

Applying inequality (21) and Lemma 12, we deduce that, for all $s \geq 1$,

$$\begin{aligned} f(s) &\leq \sqrt{16\,\pi \ln s} \int_{-\infty}^{-A} \sqrt{3\pi + K(s + \ln s, -A)}\, e^{\sqrt{2}\, x_3}\, dx_3 + 6\,\pi \\ &\leq \sqrt{8\,\pi\, \ln s}\, \sqrt{3\pi + 2 g(s + \ln s)} + 6\,\pi \\ &= \sqrt{8\,\pi\, \ln s}\, \sqrt{3\pi + 2 f'(s + \ln s)\, (s + \ln s)} + 6\,\pi. \end{aligned}$$

Together with (24), it follows that $f'(t)\, t \to +\infty$ as $t \to +\infty$, and that

$$0 < f(s) \leq \sqrt{17\,\pi\, (s + \ln s)\, \ln(s + \ln s)\, f'(s + \ln s)} \ \text{ for all } s \text{ large enough.}$$

Thanks to (24) and (27), we then infer that, for all s large enough,

$$\begin{aligned} 0 < f(s + \ln s) \leq f(s) + 6\,\pi\, \ln s &\leq \left(1 + \frac{6\,\pi}{\alpha'}\right) f(s) \\ &\leq C_1 \sqrt{(s + \ln s)\, \ln(s + \ln s)\, f'(s + \ln s)} \end{aligned}$$

with

$$C_1 = \sqrt{17\,\pi} \left(1 + \frac{6\,\pi}{\alpha'}\right) > 0\,.$$

In other words, there is $t_0 > 0$ such that $f > 0$ on $[t_0, +\infty)$ and

$$\frac{f'(t)}{f(t)^2} \geq \frac{1}{C_1^2\, t \ln t} \text{ for all } t \geq t_0.$$

It follows that the function

$$t \mapsto \frac{1}{f(t)} + \frac{\ln(\ln t)}{C_1^2}$$

is nonincreasing on $[t_0, +\infty)$. But since $f > 0$ on $[t_0, +\infty)$, one has

$$\frac{1}{f(t)} + \frac{\ln(\ln t)}{C_1^2} \to +\infty$$

as $t \to +\infty$. This is a contradiction. Therefore, $\alpha = 0$ and the proof of Lemma 13 is thereby complete. □

End of the proof of Theorem 1 for $n = 3$. As in the case $n = 2$, the fact that $\alpha = 0$ in (17) and (19), together with (18), implies that

$$\begin{aligned} \frac{|\nabla u(x_1, x_2, -A)|^2}{2} &+ F(u(x_1, x_2, -A)) - u_{x_3}^2(x_1, x_2, -A) \\ &= u_{x_1}^2(x_1, x_2, -A) + u_{x_2}^2(x_1, x_2, -A) = 0 \end{aligned}$$

for all $(x_1, x_2) \in \mathbb{R}^2$. Hence

$$F(u(x_1, x_2, -A)) = \frac{u_{x_3}^2(x_1, x_2, -A)}{2} = \frac{|\nabla u(x_1, x_2, -A)|^2}{2}$$

for all $(x_1, x_2) \in \mathbb{R}^2$ and one concludes from [8, 27] that u is one-dimensional, namely $u(x_1, x_2, x_3) \equiv H(x_3)$ in $\mathbb{R}^3$ from the second paragraph of this subsection. □

Remark 14. In higher dimensions $n \geq 4$, one could still apply the balancing condition and define some functions g and f with formulas similar to (17) and (23) above. However, in (17), there would be a factor s^{n-2} instead of s in the right-hand side. Even if Lemma 12 still extends to that case (with the right-hand side of (22) replaced by $C + K(s + (n-2)\ln s, -A)$, for some constant $C > 0$ depending on n), Lemma 13 does not extend as such. In particular, one would have $f'(s) = g(s)/s^{n-2}$, and the integrability of the function $1/s^{n-2}$ at infinity does not imply that $f(+\infty) = +\infty$ if $\alpha := g(+\infty) > 0$, and then the end of the proof does not work.

4. Half-space theorems for free boundary problems

In this section, we are interested in half-space properties for free boundary problems. First of all, we consider the following classical one phase free boundary problem:

$$\begin{cases} \Delta u = 0 & \text{in } \Xi := \{u > 0\} \subset \mathbb{R}^n, \\ |\nabla u| = 1 & \text{on } \partial\Xi, \end{cases} \tag{28}$$

where u is understood in the classical sense in $\overline{\Xi}$ and $\partial\Xi$ is globally smooth.

The existence of catenoid type solutions of this problem has been proved using an Allen-Cahn approximation. We refer to [24] and the references therein for more discussion on this problem.

We have the following half-space property:

Theorem 15. *Let $n \leq 3$ and u be a solution of (28) with $|\nabla u| \leq 1$. Suppose that the positive phase Ξ is contained in the half-space $\{x_n > 0\}$. Then u is one-dimensional, namely there is $h \geq 0$ such that $\Xi = \{x_n > h\}$ and u is the one-dimensional function $u(x) \equiv x_n - h$ in $\overline{\Xi}$.*

Proof. The idea of proof is same as that of Section 3.2. We sketch the proof and list the necessary modifications. Let us only consider the case $n = 3$.

Up to shift in the x_3-direction, one can assume without loss of generality that Ξ is not contained in $\{x_3 > a\}$ for any $a > 0$. From standard elliptic estimates up to the boundary, one can fix $a > 0$ small enough such that $u_{x_3} > 0$ in $\overline{\Xi} \cap \{x_3 \le a\}$.

We still adopt the notation of Section 3.2 and, for $s > 0$, let Ω_s be the half-cylinder

$$\Omega_s := \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : r^2 = x_1^2 + x_2^2 < s^2, \quad x_3 < a\right\}.$$

For $\varepsilon > 0$, let us define

$$\Xi_\varepsilon := \{Z \in \Xi : \operatorname{dist}(Z, \partial\Xi) > \varepsilon\}.$$

Let F be half the characteristic function of the interval $(0, +\infty)$, that is, $F(\tau) = 1/2$ if $\tau > 0$ and $F(\tau) = 0$ if $\tau \le 0$. Then we have the following balancing formula, with $X = (0, 0, 1)$ and $\varepsilon \in (0, a)$:

$$\int_{\partial(\Omega_s \cap \Xi_\varepsilon)} \left[\left(\frac{1}{2}|\nabla u|^2 + F(u)\right) X \cdot \nu - (\nabla u \cdot X)(\nabla u \cdot \nu)\right] d\sigma = 0.$$

Sending ε to 0 in this identity and using the free boundary condition, we get

$$\int_{\partial\Omega_s \cap \Xi} \left[\left(\frac{1}{2}|\nabla u|^2 + F(u)\right) X \cdot \nu - (\nabla u \cdot X)(\nabla u \cdot \nu)\right] d\sigma = 0.$$

Now we extend the solution u to $\mathbb{R}^3$ such that $u = 0$ in $\mathbb{R}^3 \backslash \Xi$. Still denote it as u. Then we get

$$\int_{\partial\Omega_s} \left[\left(\frac{1}{2}|\nabla u|^2 + F(u)\right) X \cdot \nu - (\nabla u \cdot X)(\nabla u \cdot \nu)\right] d\sigma = 0.$$

Note that u is not smooth across the free boundary, but, for any $Y \in \partial\Xi$, the quantity $(|\nabla u(Z)|^2/2 + F(u(Z)))\, X \cdot \nu(Y) - (\nabla u(Z) \cdot X)(\nabla u(Z) \cdot \nu(Y))$ converges to 0 as $Z \to Y$ with $Z \in \Xi$ and it vanishes for all $Z \in \mathbb{R}^3 \setminus \overline{\Xi}$.

With the same slight abuse of notation as in Section 3.2, we define, for $s > 0$,

$$f(s) = \int_0^s \int_{-\infty}^a \int_0^{2\pi} u_r(r, \theta, x_3) u_{x_3}(r, \theta, x_3)\, d\theta\, dx_3\, dr.$$

Since $|\nabla u| \le 1$ in Ξ, one has $|\nabla u|^2/2 + 1/2 \ge u_{x_3}^2$ in Ξ, hence the function f is nonnegative, non-decreasing and differentiable with respect to s. Similarly to the proof of Section 3.2, we can show that, if

$$\lim_{s \to +\infty} \int_{\{x_1^2 + x_2^2 < s^2\}} \left(\frac{1}{2}|\nabla u|^2 + F(u) - u_{x_3}^2\right) dx_1\, dx_2 > 0,$$

then there is a positive constant $C > 0$ such that

$$f'(s) \ge \frac{C f^2(s)}{s \ln s}, \text{ for } s \text{ large.}$$

The previous inequality yields a contradiction as in Section 3.2. This then implies that $|\nabla u(x_1, x_2, a)| = 1$ and $u_{x_1}(x_1, x_2, a) = u_{x_2}(x_1, x_2, a) = 0$ for all $(x_1, x_2) \in \mathbb{R}^2$ such that $(x_1, x_2, a) \in \Xi$. Since $\Delta\left(|\nabla u|^2\right) = 2\sum_{1 \le i,j \le 3} u_{x_i x_j}^2 \ge 0$ in Ξ, we conclude that $|\nabla u| = 1$ and $u_{x_1} = u_{x_2} = 0$ in each connected component of Ξ meeting $\{x_3 = a\}$. It finally follows that $\Xi \supset \{x_3 = a_k\}$, for a sequence $(a_k)_{k \in \mathbb{N}}$ with $a_k \to 0^+$ (from the normalization made in the second paragraph of the proof) and, remembering that $u_{x_3} > 0$ in $\overline{\Xi} \cap \{x_3 \le a\}$, we easily conclude that $\Xi = \{x_3 > 0\}$ and u is the one-dimensional function $u(x_1, x_2, x_3) \equiv x_3$. $\square$

Similarly, we can consider the following double-well type free boundary problem:

$$\begin{cases} \Delta u &= 0 \text{ in } \Xi := \{|u| < 1\} \subset \mathbb{R}^n, \\ |\nabla u| &= 1 \text{ on } \partial\Xi. \end{cases} \tag{29}$$

The proof of the following result is essentially same as that of Theorem 15, and we omit the details.

Theorem 16. *Let $n \leq 3$ and u be a solution of (29) with $|\nabla u| \leq 1$. Suppose that $\{|u| < 1\}$ is contained in the half-space $\{x_n > 0\}$. Then u is one-dimensional, namely there is $h \geq 1$ such that $\Xi = \{h - 1 < x_n < h + 1\}$, and either $u(x) \equiv x_n - h$ in $\overline{\Xi}$ or $u(x) \equiv -(x_n - h)$ in $\overline{\Xi}$.*

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