SCHRÖDINGER-POISSON SYSTEMS IN THE 3-SPHERE

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Abstract. We investigate nonlinear Schrödinger-Poisson systems in the 3-sphere. We prove existence results for these systems and discuss the question of the stability of the systems with respect to their phases. While, in the subcritical case, we prove that all phases are stable, we prove in the critical case that there exists a sharp explicit threshold below which all phases are stable and above which resonant frequencies and multi-spikes blowing-up solutions can be constructed. Solutions of the Schrödinger-Poisson systems are standing waves solutions of the electrostatic Maxwell-Schrödinger system. Stable phases imply the existence of a priori bounds on the amplitudes of standing waves solutions. Unstable phases give rise to resonant states.

We investigate in this paper nonlinear Schrödinger-Poisson systems in the 3-sphere. These are electrostatic versions of the Maxwell-Schrödinger system which describes the evolution of a charged nonrelativistic quantum mechanical particle interacting with the electromagnetic field it generates. We adopt here the Proca formalism. Then the particle interacts via the minimum coupling rule

$$\partial_t \to \partial_t + i\frac{q}{\hbar}\varphi \ , \ \ \nabla \to \nabla - i\frac{q}{\hbar}A$$

with an external massive vector field (φ, A) which is governed by the Maxwell-Proca Lagrangian. In particular, we recover as part of the full system the massive modified Maxwell equations in SI units, which are hereafter explicitly written down:

$$\begin{aligned} &\nabla.E = \rho/\varepsilon_0 - \mu^2\varphi \ , \\ &\nabla \times H = \mu_0\left(J + \varepsilon_0\frac{\partial E}{\partial t}\right) - \mu^2 A \ , \\ &\nabla \times E + \frac{\partial H}{\partial t} = 0 \text{ and } \nabla.H = 0 \ . \end{aligned} \tag{0.1}$$

These massive Maxwell equations, as modified to Proca form, appear to have been first written in modern format by Schrödinger [25]. The Proca formalism a priori breaks Gauge invariance. Gauge invariance can be restored by the Stueckelberg trick, as pointed out by Pauli [21], and then by the Higgs mechanism. We refer to Goldhaber and Nieto [14, 15], Luo, Gillies and Tu [20], and Ruegg and Ruiz-Altaba [24] for very complete references on the Proca approach. In the electrostatic case of the Maxwell-Schrödinger system, looking for standing waves solutions, we are led to the nonlinear Schrödinger-Poisson system we investigate in this paper. It is stated as follows:

$$\begin{cases} \frac{\hbar^2}{2m_0^2}\Delta_g u + \omega^2 u + qvu = u^{p-1} \\ \Delta_g v + m_1^2 v = 4\pi q u^2 \ , \end{cases} \tag{0.2}$$

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where $\omega \in \mathbb{R}$, $p \in (4,6]$, $\Delta_g = -\mathrm{div}_g\nabla$ is the Laplace-Beltrami operator, the constants $\hbar$, m_0, m_1 and q are positive, and $u, v \geq 0$ in S^3. Solutions of (0.2) are standing wave solutions $\psi(x,t) = u(x)e^{i\omega^2 t/\hbar}$, with purely electrostatic field v, of the Maxwell-Schrödinger system in Proca form we mentioned above. The system (0.2) is energy critical when $p = 6$. We refer to the temporal frequency ω as the phase and investigate both the question of the existence of one or more solutions to (0.2), and the question of the stability of phases in (0.2). Stability of a phase implies compactness of the set of associated solutions of (0.2). We define the stability of a phase as follows.

Definition 0.1. *Let (S^3, g) be the unit 3-sphere, and $p \in (4,6]$. A phase $\omega \in \mathbb{R}$ is stable if for any sequence $\psi_\alpha(x,t) = u_\alpha(x)e^{i\omega_\alpha^2 t/\hbar}$ of standing waves, with purely electrostatic field v_α, solutions of*

$$\begin{cases} \frac{\hbar^2}{2m_0^2}\Delta_g u_\alpha + \omega_\alpha^2 u_\alpha + q v_\alpha u_\alpha = u_\alpha^{p-1} \\ \Delta_g v_\alpha + m_1^2 v_\alpha = 4\pi q u_\alpha^2 \end{cases}$$

for all $\alpha \in \mathbb{N}$, the convergence $\omega_\alpha \to \omega$ in $\mathbb{R}$ as $\alpha \to +\infty$ implies that, up to a subsequence, the u_α's and v_α's converge in $C^2(S^3)$ to solutions u and v of (0.2) as $\alpha \to +\infty$.

In particular, if ω is stable then we get an upper bound on the L^∞-norm of the amplitude of arbitray standing waves with phases close to ω. The first result we prove addresses the subcritical case $p \in (4,6)$ in (0.2). The mountain pass solutions we obtain in our theorems are precisely defined in Section 2. These are variational solutions which inherit an additional ground state structure in the Nehari setting.

Theorem 0.1 (Subcritical case). *Let (S^3, g) be the unit 3-sphere, $\hbar, m_0, m_1 > 0$, and $q > 0$. Let $p \in (4,6)$. For any ω there exists a mountain pass solution of (0.2). Moreover, all phases $\omega \in \mathbb{R}$ are stable.*

As an interesting remark it can be noted that both the bounds 4 and 6 on the nonlinearity are sharp with respect to the stability issue in the theorem. Stability as in Theorem 0.1 is indeed false in general when $p = 4$ (see Section 3). As shown in Theorem 0.2, it is also false when $p = 6$ and ω takes specific (sufficiently) large values. When $p = 6$, a critical threshold for ω appears. In the case of S^3 that we consider in this paper this can be made very explicit. We let $\Lambda(m_0)$ be given by

$$\Lambda(m_0) = \frac{\sqrt{3}\hbar}{2\sqrt{2}m_0} . \tag{0.3}$$

The theorem we prove in the critical case answers positively the question of existence of special solutions and of stability of phases in the range $(-\Lambda(m_0), +\Lambda(m_0))$, and asserts that resonant frequencies appear in the complementary range.

Theorem 0.2 (Critical case). *Let (S^3, g) be the unit 3-sphere, $\hbar, m_0, m_1 > 0$, and $q > 0$. Let $p = 6$. For any $\omega \in (-\Lambda(m_0), +\Lambda(m_0))$ there exists a mountain pass solution of (0.2) and the solution is nonconstant when $m_1 \ll q$. Moreover:*

(i) all phases $\omega \in (-\Lambda(m_0), +\Lambda(m_0))$ are stable,

(ii) there exists an increasing sequence $(\omega_k)_{k\geq 1}$ of phases such that $\omega_1 = \Lambda(m_0)$, $\omega_k \to +\infty$ as $k \to +\infty$, and both all the $-\omega_k$'s and ω_k's are unstable.

In particular, resonant frequencies appear outside $(-\Lambda(m_0), +\Lambda(m_0))$, starting with $\pm\Lambda(m_0)$, and the threshold $\Lambda(m_0)$ is critical.

The mountain pass solution we obtain in Theorem 0.2 comes in addition to the constant solution when $m_1 \ll q$ and we thus get two solutions in that case. As already mentioned, the stability of phases implies the existence of an upper bound for the amplitude of standing waves $\psi(x,t) = u(x)e^{i\omega^2 t/\hbar}$ when ω is in compact subsets of $(-\Lambda(m_0), +\Lambda(m_0))$. The resonant frequencies ω_k break this upper bound. As we will see when proving the second part of Theorem 0.2, they come with blowing-up sequences of multi-spike solutions.

1. Coupling NLS with a massive field

The nonlinear focusing Schrödinger equation (NLS) is written as

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{\hbar^2}{2m_0^2}\Delta_g\psi - |\psi|^{p-2}\psi \ .$$

By coupling (NLS) with a gauge vector field (φ, A) governed by the Maxwell-Proca theory, the coupling being made via the minimum substitution rule,

$$\partial_t \to \partial_t + i\frac{q}{\hbar}\varphi \ , \ \ \nabla \to \nabla - i\frac{q}{\hbar}A \ ,$$

we get a system of particle-electromagnetic field describing the interactions of a matter scalar field ψ with its electromagnetic field (φ, A). Here, $\hbar$ is the reduced Planck's constant, $m_0 > 0$ represents the mass of ψ, q its charge, and $m_1 > 0$ represents the mass of (φ, A) in the Maxwell-Proca theory. To be more precise, let

$$\mathcal{L}_{NLS} = \frac{1}{2}\left(i\hbar\frac{\partial\psi}{\partial t}\overline{\psi} - q\varphi|\psi|^2 - \frac{\hbar^2}{2m_0^2}|\nabla\psi - i\frac{q}{\hbar}A\psi|^2\right) + \frac{1}{p}|\psi|^p \ , \text{ and}$$

$$\mathcal{L}_{MP}(\varphi, A) = \frac{1}{8\pi}\left|\frac{\partial A}{\partial t} + \nabla\varphi\right|^2 - \frac{1}{8\pi}|\nabla\times A|^2 + \frac{m_1^2}{8\pi}|\varphi|^2 - \frac{m_1^2}{8\pi}|A|^2 \ ,$$

where $\nabla\times$ is the curl operator, and define $\mathcal{S} = \int\int(\mathcal{L}_{NLS} + \mathcal{L}_{MP})\,dv_g dt$ to be the total action functional. Writing $\psi = ue^{\frac{iS}{\hbar}}$ in polar form, $u \geq 0$, and taking the variation of $\mathcal{S}$ with respect to u, S, φ, and A, we get that

$$\begin{cases} \frac{\hbar^2}{2m_0^2}\Delta_g u + \left(\frac{\partial S}{\partial t} + q\varphi + \frac{1}{2m_0^2}|\nabla S - qA|^2\right)u = u^{p-1} \\ \frac{\partial u^2}{\partial t} + \frac{1}{m_0^2}\nabla.\left((\nabla S - qA)u^2\right) = 0 \\ -\frac{1}{4\pi}\nabla.\left(\frac{\partial A}{\partial t} + \nabla\varphi\right) + \frac{m_1^2}{4\pi}\varphi = qu^2 \\ \frac{1}{4\pi}\left(\nabla\times(\nabla\times A) + \frac{\partial}{\partial t}\left(\frac{\partial A}{\partial t} + \nabla\varphi\right)\right) + \frac{m_1^2}{4\pi}A = \frac{q}{m_0^2}\left(\nabla S - qA\right)u^2 \ . \end{cases} \tag{1.1}$$

Letting $E = -\frac{1}{4\pi}\left(\frac{\partial A}{\partial t} + \nabla\varphi\right)$, $H = \frac{1}{4\pi}\nabla\times A$, $\rho = qu^2$, and $J = \frac{q}{m_0^2}\left(\nabla S - qA\right)u^2$, we recover the Maxwell-Proca equations (0.1) with the two last equations in (1.1), where $\mu^2 = m_1^2/(4\pi)$ and we normalize such that $\varepsilon_0 = 1$ and $\mu_0 = 1$ (the last two equations in (0.1) are automaticaly satisfied due to the choice of E and H). The second equation in (1.1) then reads as the charge continuity equation $\frac{\partial\rho}{\partial t} + \nabla.J = 0$. We assume in what follows that A and φ depend on the sole spatial variables, thus we restrict our attention to the static case of (1.1), and look for standing waves solutions of (1.1), namely

$$\psi(x,t) = u(x)e^{\frac{i\omega^2 t}{\hbar}} \ .$$

The fourth equation in (1.1) then implies that $A \equiv 0$, while the second equation in (1.1) is automatically satisfied since $S = \omega^2 t$. The first and third equations in (1.1) are rewritten as

$$\begin{cases} \frac{\hbar^2}{2m_0^2}\Delta_g u + \omega^2 u + qvu = u^{p-1} \\ \Delta_g \varphi + m_1^2 \varphi = 4\pi q u^2 \ . \end{cases} \tag{1.2}$$

Letting $\varphi = v$, the system (1.2) is precisely the system (0.2) we investigate in this paper. Solutions of (0.2) are standing wave solutions of (1.1) in the static (or purely electrostatic) case.

2. Functional setting and existence of mountain pass solutions

Let $m_0, m_1 > 0$, $\omega \in \mathbb{R}$, and $q > 0$. We aim in getting mountain pass solutions for (0.2). For this purpose we define an auxiliary functional $\Phi : H^1 \to H^1$ by letting $\Phi(u)$ be the unique solution of

$$\Delta_g \Phi(u) + m_1^2 \Phi(u) = 4\pi q u^2 \tag{2.1}$$

for $u \in H^1$. Then Φ is C^1 and its differential Φ_u at u, when computed over $\varphi \in H^1$, solves an equation like (2.1) with a right hand side like $8\pi q u \varphi$. In particular, $\mathcal{H}(u) = \int u^2 \Phi(u)$ is C^1 with $\mathcal{H}_u(\varphi) = 4 \int u \Phi(u) \varphi$ for $\varphi \in H^1$. For $p \in (4, 6]$, we define $I_p : H^1 \to \mathbb{R}$ by

$$\begin{aligned} I_p(u) = & \frac{\hbar^2}{4m_0^2} \int_{S^3} |\nabla u|^2 dv_g + \frac{\omega^2}{2} \int_{S^3} u^2 dv_g \\ & + \frac{q}{4} \int_{S^3} u^2 \Phi(u) dv_g - \frac{1}{p} \int_{S^3} (u^+)^p dv_g \ , \end{aligned} \tag{2.2}$$

where $u^+ = \max(u, 0)$. If $u \geq 0$ is a critical point of I_p, then $(u, \Phi(u))$ solves (0.2). As is easily seen, $\Phi(tu) = t^2 \Phi(u)$ for all t and all u, and $\Phi(u) \geq 0$ for all u. We prove the existence part of Theorem 0.2 in what follows. We say that $\big(u, \Phi(u)\big)$ is a *mountain pass solution* of (0.2) when u is obtained from I_p by the mountain pass lemma from 0 to an endpoint u_1 such that $I_p(u_1) < 0$. Existence in the subcritical case somehow follows from a direct application of the mountain pass lemma.

Proof of Theorem 0.1 - Existence part. Let $p \in (4, 6)$ and $u_0 \in H^1$ be such that $u_0^+ \not\equiv 0$. There holds $I_p(0) = 0$, and there exists $T_0 = T(u_0)$ such that $I_p(T_0 u_0) < 0$. For any $0 < \delta \ll 1$, there exists $\varepsilon_\delta > 0$ such that $\Phi(u) \geq \varepsilon_\delta$ for all $u \in H^1$ satisfying that $\|u\|_{H^1} = 1$ and $\|u\|_{L^2} \geq \delta$. It follows that there exists $\varepsilon_0 > 0$ such that $\int_{S^3} \big(|\nabla u|^2 + \Phi(u) u^2\big) dv_g \geq \varepsilon_0$ for all $u \in H^1$ satisfying that $\|u\|_{H^1} = 1$. Since $\Phi(tu) = t^2 \Phi(u)$, we then get that there exist $C_1, C_2 > 0$ such that

$$I_p(u) \geq C_1 \|u\|_{H^1}^4 - C_2 \|u\|_{H^1}^p$$

for all u such that $\|u\|_{H^1} \leq 1$. In particular the mountain pass lemma can be applied since $p > 4$. Let

$$c_p = \inf_{P \in \mathcal{P}} \max_{u \in P} I_p(u) \ , \tag{2.3}$$

where $\mathcal{P}$ is the set of continuous paths from 0 to $T_0 u_0$. Since $\int u^2 \Phi(u) \leq C \|u\|_{H^1}^4$, mountain pass sequences associated to c_p are bounded in H^1. Standard arguments then give the existence of $u_p \geq 0$ such that $I_p(u_p) = c_p$, and such that u_p and $v_p = \Phi(u_p)$ solve (0.2). Then u_p, v_p are smooth and by the maximum principle $u_p, v_p > 0$ in S^3. This ends the proof of the existence part in Theorem 0.1. □

From now on we assume $p = 6$. Let $x_0 \in S^3$, $(\beta_\alpha)_\alpha$ be a sequence such that $\beta_\alpha > 1$ for all α and $\beta_\alpha \to 1$ as $\alpha \to +\infty$, and define

$$\varphi_\alpha(x) = \frac{\left(3(\beta_\alpha^2 - 1)\right)^{1/4}}{\sqrt{2(\beta_\alpha - \cos r)}} \ , \tag{2.4}$$

where $r = d_g(x_0, x)$. The φ_α's are 1-spike solutions of

$$\Delta_g \varphi_\alpha + \frac{3}{4}\varphi_\alpha = \varphi_\alpha^5 \tag{2.5}$$

and they satisfy the energy estimates

$$\int_{S^3} |\nabla \varphi_\alpha|^2 dv_g = \frac{1}{K_3^3} + o(1) \ \ , \ \ \int_{S^3} \varphi_\alpha^6 dv_g = \frac{1}{K_3^3} \ , \tag{2.6}$$

where $K_3 = \frac{2}{\sqrt{3}|S^3|^{1/3}}$ is the sharp constant for the Euclidean Sobolev inequality $\|u\|_{L^6} \le K_3 \|\nabla u\|_{L^2}$. The proof of the existence part in Theorem 0.2 is as follows.

Proof of Theorem 0.2 - Existence part. There holds $\varphi_\alpha \to 0$ in L^p for $p < 6$. Hence $\Phi(\varphi_\alpha) \to 0$ in $H^{2,p}$ for $p < 3$, and we get that $\|\Phi(\varphi_\alpha)\|_{L^\infty} \to 0$ as $\alpha \to +\infty$. By (2.6) there exists $T \gg 1$ such that $I_6(T\varphi_\alpha) < 0$ for all $\alpha \gg 1$. Let

$$\mathcal{H}_\varepsilon(u) = \int_{S^3} |\nabla u|^2 dv_g + (\frac{3}{4} - \varepsilon) \int_{S^3} u^2 dv_g - \frac{2m_0^2}{3\hbar^2} \int_{S^3} |u|^6 dv_g \ .$$

Since $|\omega| < \Lambda(m_0)$ and $\|\Phi(\varphi_\alpha)\|_{L^\infty} \to 0$ as $\alpha \to +\infty$ we get that there exists $\varepsilon_0 > 0$ such that

$$\begin{aligned} \max_{0 \le t \le T} I_6(t\varphi_\alpha) \ &\le \ \frac{\hbar^2}{4m_0^2} \max_{0 \le t \le T} \mathcal{H}_{\varepsilon_0}(t\varphi_\alpha) \\ &\le \ \frac{\sqrt{2}\hbar^3}{12 m_0^3} \left(\frac{\int_{S^3} |\nabla \varphi_\alpha|^2 dv_g + (\frac{3}{4} - \varepsilon_0) \int_{S^3} \varphi_\alpha^2 dv_g}{\left(\int_{S^3} \varphi_\alpha^6 dv_g\right)^{1/3}} \right)^{3/2} \end{aligned}$$

for all $\alpha \gg 1$. There also exist $C_1, C_2 > 0$ such that

$$I_6(u) \ge C_1 \|u\|_{H^1}^4 - C_2 \|u\|_{H^1}^6$$

for all u such that $\|u\|_{H^1} \le 1$. We let $u_0 = \varphi_\alpha$ for $\alpha \gg 1$ sufficiently large, $T_0 = T$, and we define

$$c_6 = \inf_{P \in \mathcal{P}} \max_{u \in P} I_6(u) \ , \tag{2.7}$$

where $\mathcal{P}$ is the set of continuous paths from 0 to $T_0 u_0$. According to the above and by (2.5), there exist $\delta_0 > 0$ and $\varepsilon_1 > 0$ such that

$$\delta_0 \le c_6 \le \frac{1}{3K_3^3} \left(\frac{\hbar}{\sqrt{2} m_0} \right)^3 - \varepsilon_1 \ , \tag{2.8}$$

where c_6 is as in (2.7). Since $I_6(0) = 0$, the mountain pass lemma can be applied. We obtain the existence of a Palais-Smale sequence $(u_\alpha)_\alpha$ such that $I_6(u_\alpha) \to c_6$ and $I_6'(u_\alpha) \to 0$ as $\alpha \to +\infty$. Noting that $\int u^2 \Phi(u) \le C\|u\|_{L^6}^4$, the u_α's are bounded in H^1. In particular, there exists $u \in H^1$ such that, up to a subsequence, $u_\alpha \rightharpoonup u$ in H^1, $u_\alpha \to u$ a.e., and $u_\alpha \to u$ in L^p for $p < 6$. Then $\Phi(u_\alpha) \to \Phi(u)$ in $H^{2,p}$ for $p < 3$, and we get that $\Phi(u_\alpha) \to \Phi(u)$ in $C^{0,\theta}$ for some $0 < \theta \ll 1$. Mimicking the argument in Brézis and Nirenberg [4], it follows from (2.8) that $u \not\equiv 0$, that $u_\alpha \to u$ in H^1, that $u > 0$ in M, and that $\mathcal{U} = (u, \Phi(u))$ solves (0.2). In particular, $I_6(u) = c_6$. □

The mountain pass solution u we obtain in the critical case is such that $I_6(u) = c_6$. As is easily checked, (0.2) always possesses a constant solution $\mathcal{U}_0 = (u_0, v_0)$ which, in the critical case, is given by $v_0 = \frac{4\pi q}{m_1^2} u_0^2$ and

$$u_0^4 = \frac{4\pi q^2}{m_1^2} u_0^2 + \omega^2 . \tag{2.9}$$

Then

$$\begin{aligned} I_6(u_0) &= \frac{1}{2}\left(\omega^2 + qv_0\right)^{1/2} |S^3| \left(\omega^2 + \frac{q}{2}v_0 - \frac{1}{3}(\omega^2 + qv_0)\right) \\ &= \frac{1}{2}\left(\omega^2 + qv_0\right)^{1/2} |S^3| \left(\frac{2}{3}\omega^2 + \frac{q}{6}v_0\right) \end{aligned}$$

and we get that

$$I_6(u_0) \geq \frac{1}{12}|S^3|(qv_0)^{3/2} . \tag{2.10}$$

Let $\varepsilon = \left(\frac{m_1^2}{4\pi q}\right)^2$. Then $\varepsilon v_0^2 = u_0^4$ and by (2.9) we get that $2\varepsilon v_0 = q + \sqrt{q^2 + 4\varepsilon\omega^2}$. In particular, coming back to (2.10),

$$I_6(u_0) \geq \frac{(4\pi)^3 |S^3|}{12} \left(\frac{q}{m_1}\right)^6$$

and by (2.8) the mountain pass solution we obtain is nonconstant when $m_1 \ll q$.

Let $\mathcal{N}_p$ be the Nehari manifold associated to (0.2). By definition

$$\mathcal{N}_p = \left\{u \in H^1, u \not\equiv 0, \text{ s.t. } I_p'(u).u = 0\right\} . \tag{2.11}$$

Following an idea due to Rabinowitz, see Willem [28] for a presentation in book form, there holds that

$$c_p = \inf_{u \in \mathcal{N}_p} I_p(u) \tag{2.12}$$

for all $p \in (4, 6]$, where $\mathcal{N}_p$ is as in (2.11), and c_p is as in (2.3) and (2.7). In particular, the solutions we obtain are ground states in the sense of Willem [28]. We get (2.12) by noting that for any $u \in H^1$, $u^+ \not\equiv 0$, there is one and only one $t = t_0(u)$, where $t > 0$, such that $I_p'(tu).(tu) = 0$.

3. Stability in the subcritical case

Stability of the phases in the subcritical case follows from (and can actually be reformulated into) the general theorem below, where we prove the existence of uniform bounds for arbitrary solutions of (0.2). Let $\mathcal{S}_p(\omega)$ be the set of all positive solutions $\mathcal{U} = (u, v)$, $u, v > 0$, of (0.2). Given $\theta \in (0, 1)$ we define $\|\mathcal{U}\|_{C^{2,\theta}} = \|u\|_{C^{2,\theta}} + \|v\|_{C^{2,\theta}}$ for all $\mathcal{U} = (u, v)$. The following theorem holds true.

Theorem 3.1. *Let (S^3, g) be the unit 3-sphere, $\hbar, m_0, m_1 > 0$, and $q > 0$. Let $p \in (4, 6)$. For any $\theta \in (0, 1)$, and any $\Lambda > 0$, there exists $C > 0$ such that $\|\mathcal{U}\|_{C^{2,\theta}} \leq C$ for all $\mathcal{U} \in \mathcal{S}_p(\omega)$ and all $|\omega| \leq \Lambda$.*

Let $p \in (4, 6)$ and let $(\omega_\alpha)_\alpha$ be a sequence of phases such that $\omega_\alpha \to \omega$ as $\alpha \to +\infty$ for some $\omega \in \mathbb{R}$, and let $\mathcal{U}_\alpha = (u_\alpha, v_\alpha)$ be positive solutions of

$$\begin{cases} \frac{\hbar^2}{2m_0^2}\Delta_g u_\alpha + \omega_\alpha^2 u_\alpha + qv_\alpha u_\alpha = u_\alpha^{p-1} \\ \Delta_g v_\alpha + m_1^2 v_\alpha = 4\pi q u_\alpha^2 . \end{cases} \tag{3.1}$$

Up to a subsequence we can assume that $\omega_\alpha \to \omega$ as $\alpha \to +\infty$ for some $\omega \in \mathbb{R}$. The proof of the existence of a priori bounds in Theorem 3.1 reduces to proving that the u_α's and v_α's are automatically bounded in $C^{2,\theta}(S^3)$, $0 < \theta < 1$.

Proof of Theorem 3.1 - Existence of a priori bounds. We divide the first equation in (3.1) by u_α and integrate over S^3. Then

$$\begin{aligned} q\int_{S^3} v_\alpha dv_g &= \frac{\hbar^2}{2m_0^2}\int_{S^3}\frac{|\nabla u_\alpha|^2}{u_\alpha^2}dv_g + \int_{S^3} u_\alpha^{p-2}dv_g - \omega_\alpha^2|S^3| \\ &\geq \int_{S^3} u_\alpha^{p-2}dv_g - \omega_\alpha^2|S^3|\,. \end{aligned} \tag{3.2}$$

Integrating the second equation in (3.1) there also holds that $m_1^2\int v_\alpha = 4\pi q\int u_\alpha^2$. By (3.2) and Hölder's inequality we then get that

$$\int_{S^3} u_\alpha^{p-2}dv_g \leq C_1 + C_2\left(\int_{S^3} u_\alpha^{p-2}dv_g\right)^{2/(p-2)}$$

for all α, where $C_1, C_2 < 0$ are independent of α. Then the u_α's are bounded in $L^{p-2}(S^3)$, and by the second equation in (3.1), the v_α's turn out to be bounded in $H^{2,(p-2)/2}(S^3)$. By the Sobolev embedding's theorem we then get that the v_α's are bounded in $L^q(S^3)$ when $p \in (4,5)$, where $q = \frac{3(p-2)}{2(5-p)}$, and in $C^{0,\theta}(S^3)$ for some $\theta \in (0,1)$ when $p \in (5,6)$. In particular, they are bounded in $L^3(S^3)$. From now on we assume by contradiction that we can choose (u_α, v_α) such that

$$\max_M u_\alpha \to +\infty \tag{3.3}$$

as $\alpha \to +\infty$. Let $x_\alpha \in M$ and $\mu_\alpha > 0$ be such that $u_\alpha(x_\alpha) = \|u_\alpha\|_{L^\infty} = \mu_\alpha^{-2/(p-2)}$. By (3.3), $\mu_\alpha \to 0$ as $\alpha \to +\infty$. Let $\tilde{u}_\alpha$ be given by

$$\tilde{u}_\alpha(x) = \mu_\alpha^{\frac{2}{p-2}} u_\alpha\left(\exp_{x_\alpha}(\mu_\alpha x)\right)$$

for $x \in \mathbb{R}^3$. Let also $\tilde{g}_\alpha(x) = \left(\exp_{x_\alpha}^\star g\right)(\mu_\alpha x)$ and $\hat{v}_\alpha(x) = v_\alpha\left(\exp_{x_\alpha}(\mu_\alpha x)\right)$. There holds

$$\frac{\hbar^2}{2m_0^2}\Delta_{\tilde{g}_\alpha}\tilde{u}_\alpha + \omega_\alpha^2\mu_\alpha^2\tilde{u}_\alpha + q\mu_\alpha^2\hat{v}_\alpha\tilde{u}_\alpha = \tilde{u}_\alpha^{p-1} \tag{3.4}$$

and there also holds that $0 \leq \tilde{u}_\alpha \leq 1$, $\tilde{u}_\alpha(0) = 1$, and $\tilde{g}_\alpha \to \xi$ in $C^2_{loc}(\mathbb{R}^3)$, where ξ is the Euclidean metric. Then there exists $C > 0$ such that for any compact subset $K \subset \mathbb{R}^3$,

$$\int_K \left(\mu_\alpha\hat{v}_\alpha\tilde{u}_\alpha\right)^3 dx \leq C$$

for all $\alpha \gg 1$ since the v_α's are bounded in L^3. By elliptic theory it follows that $\tilde{u}_\alpha \to \tilde{u}$ in $C^{0,\theta}_{loc}(\mathbb{R}^3)$ as $\alpha \to +\infty$, where $\tilde{u}$ satisfies $0 \leq \tilde{u} \leq 1$ and $\tilde{u}(0) = 1$. Moreover, by (3.4), we have that $\frac{\hbar^2}{2m_0^2}\Delta\tilde{u} = \tilde{u}^{p-1}$, a contradiction with the Liouville result of Gidas and Spruck [13]. Hence, (3.3) cannot happen, and for any $(\omega_\alpha)_\alpha$ such that $\omega_\alpha \to \omega$ as $\alpha \to +\infty$, and any (u_α, v_α) solutions of (4.1), there exists $C > 0$ such that $\|u_\alpha\|_{L^\infty} \leq C$. By the second equation in (3.1) it follows that $\|u_\alpha\|_{L^\infty} + \|v_\alpha\|_{L^\infty} \leq C$ for all α, and by standard elliptic theory, a $C^{2,\theta}$-bound holds as well. This proves the existence of a priori bounds in Theorem 0.2. □

As an interesting remark it is necessary in the above proof to assume that $p > 4$. Indeed, let $p = 4$, $\omega_\alpha = 0$ for all α, and $4\pi q^2 = m_1^2$. Then $u_\alpha = \alpha$ and $v_\alpha = \alpha^2/q$ solve (3.1) and, obiously, $\|u_\alpha\|_{L^\infty} \to +\infty$, $\|v_\alpha\|_{L^\infty} \to +\infty$ as $\alpha \to +\infty$. It is independently necessary to assume a bound on the ω_α's since if not we get counter examples by the constant solutions which satisfy $u_\alpha \geq \omega_\alpha^{2/(p-2)}$. As a remark, Theorem 3.1 is true on arbitrary compact Riemannian 3-manifolds.

4. Stability in the critical case

Stability in the critical case is a consequence of, and is actually equivalent to, the following theorem where the existence of uniform bounds is obtained for phases in compact subsets of $(-\Lambda(m_0), +\Lambda(m_0))$.

Theorem 4.1. *Let (S^3, g) be the unit 3-sphere, $\hbar, m_0, m_1 > 0$, and $q > 0$. Let $p = 6$. For any $\theta \in (0,1)$, and any $\varepsilon > 0$, there exists $C > 0$ such that $\|\mathcal{U}\|_{C^{2,\theta}} \leq C$ for all $\mathcal{U} \in \mathcal{S}_6(\omega)$ and all $|\omega| \leq \Lambda(m_0) - \varepsilon$.*

By the analysis in Druet and Hebey [9], we refer also to Druet and Laurain [12] for a related reference, Theorem 4.1 can be extended to the case of arbitrary compact 3-dimensional manifolds. The result holds true as long as $\Lambda(m_0) \leq \min \Lambda$, where Λ is such that $\Delta_g + \Lambda$ has a nonnegative mass. By the positive mass theorem, assuming the Yamabe invariant of g is positive, $\Lambda \geq \frac{1}{8} S_g$, where S_g is the scalar curvature of g. In both cases we recover (0.3) when the manifold is the 3-sphere. The proof we present is a shortcut with respect to the analysis in Druet and Hebey [9]. We mix in our analysis ideas from Li and Zhang [19], Druet and Hebey [8], Hebey and Robert [16], and Hebey, Robert and Wen [17]. The proof extends almost as it is to compact conformally flat manifolds of positive scalar curvature. The 4-dimensional analogue of Theorem 4.1 for the Klein-Gordon equation is established in Hebey and Truong [18].

In what follows we let $(\omega_\alpha)_\alpha$ be a sequence of phases such that $\omega_\alpha \to \omega$ as $\alpha \to +\infty$ for some $\omega \in \mathbb{R}$, and let $\mathcal{U}_\alpha = (u_\alpha, v_\alpha)$ be positive solutions of

$$\begin{cases} \frac{\hbar^2}{2m_0^2} \Delta_g u_\alpha + \omega_\alpha^2 u_\alpha + q v_\alpha u_\alpha = u_\alpha^5 \\ \Delta_g v_\alpha + m_1^2 v_\alpha = 4\pi q u_\alpha^2 \ . \end{cases} \tag{4.1}$$

Dividing the first equation in (4.1) by u_α and integrating over S^3 we get as in Section 3 that

$$\int_{S^3} u_\alpha^4 dv_g \leq C_1 + C_2 \left(\int_{S^3} u_\alpha^4 dv_g \right)^{1/2}$$

for all α, where $C_1, C_2 < 0$ are independent of α. Then the u_α's are bounded in L^4, and by the second equation in (4.1), the v_α's are in turn bounded in H^2. By the Sobolev embedding theorem we thus get that there exists $v \in C^{0,\theta}(S^3)$, $0 < \theta < 1$, such that, up to a subsequence,

$$v_\alpha \to v \ \text{ in } C^{0,\theta}(S^3) \tag{4.2}$$

as $\alpha \to +\infty$. By standard elliptic theory, an L^∞-bound on the u_α's implies the $C^{2,\theta}$-bound we are looking for in the theorem. We define

$$h_\alpha = \omega_\alpha^2 + q v_\alpha \ , \tag{4.3}$$

and assume by contradiction that we can choose (u_α, v_α) such that

$$\max_{S^3} u_\alpha \to +\infty \tag{4.4}$$

as $\alpha \to +\infty$. By (4.2) the h_α's converge in $C^{0,\theta}$. The following lemma directly follows from the analysis in Li and Zhang [19].

Lemma 4.1 (Li-Zhang [19]). *Let $\hat{u}_\alpha > 0$ be a smooth positive solution of*

$$\frac{\hbar^2}{2m_0^2}\Delta \hat{u}_\alpha + \hat{h}_\alpha \hat{u}_\alpha = \hat{u}_\alpha^5 \tag{4.5}$$

in $\mathbb{R}^3$, where Δ is the Euclidean Laplacian and $(\hat{h}_\alpha)_\alpha$ is a converging sequence of functions in $C^0_{loc}(\mathbb{R}^3)$. There exist $C, \delta > 0$ such that, up to a subsequence,

$$\sup_{B_0(\varepsilon)} \hat{u}_\alpha \times \inf_{B_0(4\varepsilon)} \hat{u}_\alpha \le \frac{C}{\varepsilon} \tag{4.6}$$

for all $0 < \varepsilon < \delta$, and all α, where $B_0(\varepsilon)$ and $B_0(4\varepsilon)$ are the Euclidean balls of center 0 and radii ε and 4ε.

Proof of Lemma 4.1. We very briefly sketch the proof and refer to Li-Zhang [19] for more details. By contradiction we assume there exists $(\varepsilon_\alpha)_\alpha$ and $(\Lambda_\alpha)_\alpha$, $\varepsilon_\alpha > 0$ for all α, $\varepsilon_\alpha \to 0$ and $\Lambda_\alpha \to +\infty$ as $\alpha \to +\infty$, such that

$$\max_{\overline{B_0(\varepsilon_\alpha)}} \hat{u}_\alpha \times \min_{\overline{B_0(4\varepsilon_\alpha)}} \hat{u}_\alpha \ge \frac{\Lambda_\varepsilon}{\varepsilon_\alpha} \tag{4.7}$$

for all α. Let $\overline{x}_\alpha \in \overline{B_0(\varepsilon_\alpha)}$ be a point where $\hat{u}_\alpha$ attains its maximum in $\overline{B_0(\varepsilon_\alpha)}$. There exist $x_\alpha \in B_{\overline{x}_\alpha}(\varepsilon_\alpha/2)$ and $\sigma_\alpha \in (0, \varepsilon_\alpha/4)$ such that $\hat{u}_\alpha(x_\alpha) \ge \hat{u}_\alpha(\overline{x}_\alpha)$ for all α, $\hat{u}_\alpha(x) \le C\hat{u}_\alpha(x_\alpha)$ for all α and all $x \in B_{x_\alpha}(\sigma_\alpha)$, and $\hat{u}_\alpha(x_\alpha)^2\sigma_\alpha \to +\infty$ as $\alpha \to +\infty$. Let $\mu_\alpha = \hat{u}_\alpha(x_\alpha)^{-2}$, and define $\hat{v}_\alpha$ by

$$\hat{v}_\alpha(x) = \mu_\alpha^{1/2}\hat{u}_\alpha(x_\alpha + \mu_\alpha x) \ . \tag{4.8}$$

There holds $\sigma_\alpha\mu_\alpha^{-1} \to +\infty$ by (4.7). By standard elliptic theory,

$$\hat{v}_\alpha \to \hat{v} \ \text{ in } C^2_{loc}(\mathbb{R}^3) \ , \tag{4.9}$$

where $\hat{v} > 0$ satisfies $\frac{\hbar^2}{2m_0^2}\Delta\hat{v} = \hat{v}^5$ and is given by the Caffarelli-Gidas-Spruck [5] classification. Given $\lambda > 0$ and $x \in \mathbb{R}^3$, we let

$$\hat{v}_\alpha^{\lambda,x}(y) = \frac{\lambda}{|y-x|}\hat{v}_\alpha\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right) \tag{4.10}$$

and $\Sigma_\alpha^{\lambda,x} = B_x(\varepsilon_\alpha\mu_\alpha^{-1})\backslash\overline{B_x(\lambda)}$, where $\hat{v}_\alpha$ is as in (4.8). Let $w_\alpha^{\lambda,x} = \hat{v}_\alpha - \hat{v}_\alpha^{\lambda,x}$, and for $C > 0$ let

$$h_{\alpha,C}^{\lambda,x}(y) = -C\lambda\mu_\alpha^2\left(|y-x| - \lambda\right) \ . \tag{4.11}$$

For any $\lambda_1 \gg 1$ and any x, there exists $C > 0$ such that $w_\alpha^{\lambda,x} + h_{\alpha,C}^{\lambda,x} \ge 0$ in Σ_λ for all $0 < \lambda \le \lambda_1$ and all α, where $h_{\alpha,C}^{\lambda,x}$ is as in (4.11). Letting $\alpha \to +\infty$ it follows that $\hat{v} \ge \hat{v}^{\lambda,x}$ for all $|y-x| \ge \lambda > 0$, where $\hat{v}$ is as in (4.9) and $\hat{v}^{\lambda,x}$ is built on $\hat{v}$ as in (4.10). This implies that $\hat{v}$ is constant, and we get a contradiction with the equation for $\hat{v}$. This ends the proof of the lemma. □

Thanks to the estimates in Lemma 4.1, as noticed by Chen and Lin [6], the following holds true.

Lemma 4.2. *There exists $C > 0$ such that $\|u_\alpha\|_{H^1} \le C$ for all α, where u_α is as in* (4.1).

Proof of Lemma 4.2. Let $x \in S^3$ be any point in S^3. By the stereographic projection of pole $-x$, there exists $\phi > 0$ smooth and positive such that $\hat{g} = \phi^4 g$ is flat in $S^3\backslash\{-x\}$, the set $S^3\backslash\{-x\}$ can be assimilated with $\mathbb{R}^3$, x with 0, and $\hat{g}$ with the Euclidean metric, and such that

$$\frac{\hbar^2}{2m_0^2}\Delta_{\hat{g}}\hat{u}_\alpha + \hat{h}_\alpha\hat{u}_\alpha = \hat{u}_\alpha^5 , \tag{4.12}$$

where $\hat{u}_\alpha = \phi^{-1}u_\alpha$, $\phi^4\hat{h}_\alpha = \frac{2m_0^2}{\hbar^2}h_\alpha - \frac{3}{4}$, and $h_\alpha = \omega_\alpha^2 + qv_\alpha$. By (4.2), $\hat{h}_\alpha \to \hat{h}$ in $C^{0,\theta}_{loc}$. Given $\delta > 0$, let $\lambda > 0$ be such that $\hat{h} < \lambda$ in $B_0(R)$, $R \gg \delta$. Let $\hat{G}$ be the Green's function of $\frac{\hbar^2}{2m_0^2}\Delta_{\hat{g}} + \lambda$ with zero Dirichlet boundary condition in $B_0(5\delta)$. Let also $\hat{v}_\alpha$ solve

$$\begin{cases} \frac{\hbar^2}{2m_0^2}\Delta_{\hat{g}}\hat{v}_\alpha + \lambda\hat{v}_\alpha = \hat{u}_\alpha^5 \text{ in } B_0(5\delta) \\ \hat{v}_\alpha = 0 \text{ on } \partial B_0(5\delta) . \end{cases}$$

By the maximum principle, $\hat{v}_\alpha \le \hat{u}_\alpha$ in $B_0(5\delta)$ for $\alpha \gg 1$. Let $y_\alpha \in \overline{B}_0(4\delta)$ be such that $\hat{u}_\alpha(y_\alpha) = \inf_{B_0(4\delta)} \hat{u}_\alpha$. By standard estimates on G, see Robert [23], following Chen and Lin [6], we can write thanks to (4.23) and the estimates in Lemma 4.1 that

$$\begin{aligned} \int_{B_x(\delta)} u_\alpha^6 dv_g &\le C_\delta \int_{B_0(\delta)} \hat{u}_\alpha^6(y)dy \\ &\le C_\delta\delta\left(\sup_{B_0(\delta)} \hat{u}_\alpha\right)\int_{B_0(\delta)} G(y_\alpha,y)\hat{u}_\alpha^5(y)dy \\ &\le C_\delta\delta\left(\sup_{B_0(\delta)} \hat{u}_\alpha\right)\int_{B_0(5\delta)} G(y_\alpha,y)\left(\frac{\hbar^2}{2m_0^2}\Delta\hat{v}_\alpha + \lambda\hat{v}_\alpha\right)(y)dy \\ &\le C_\delta\delta \sup_{B_0(\delta)} \hat{u}_\alpha \times \inf_{B_0(4\delta)} \hat{u}_\alpha \le C_\delta \end{aligned}$$

for all $\alpha \gg 1$ and $\delta > 0$ sufficiently small, where $C_\delta > 0$ does not depend on α and change values from line to line in the above inequalities. In particular, since x is arbitrary, there exists $C > 0$ such that $\int_{S^3} u_\alpha^6 dv_g \le C$ for all α. By (4.1) this proves Lemma 4.2. ☐

By Lemma 4.2 the u_α's have bounded energy and Struwe's decomposition [26] can be applied. In particular, up to a subsequence,

$$u_\alpha = u_\infty + \sum_{i=1}^k B_{i,\alpha} + \mathcal{R}_\alpha , \tag{4.13}$$

where $\mathcal{R}_\alpha \to 0$ in H^1 as $\alpha \to +\infty$, $k \in \mathbb{N}$, $u_\alpha \to u_\infty$ a.e., and

$$B_{i,\alpha}(x) = \left(\frac{2m_0^2}{\hbar^2}\right)^{1/4}\left(\frac{\mu_{i,\alpha}}{\mu_{i,\alpha}^2 + \frac{d_g(x_{i,\alpha},x)^2}{3}}\right)^{1/2} \tag{4.14}$$

for some converging sequence $(x_{i,\alpha})_\alpha$ in S^3 and a sequence $(\mu_{i,\alpha})_\alpha$ of positive real numbers such that $\mu_{i,\alpha} \to 0$ as $\alpha \to +\infty$. Moreover, there holds that

$$\mu_{i,\alpha}^{\frac{1}{2}} u_\alpha \left(\exp_{x_{i,\alpha}}(\mu_{i,\alpha} x)\right) \to \left(\frac{2m_0^2}{\hbar^2}\right)^{1/4} \sqrt{\frac{1}{1+\frac{|x|^2}{3}}} \tag{4.15}$$

in $C^2_{loc}(\mathbb{R}^3 \backslash \mathcal{S}_i)$ for all i, where $\mathcal{S}_i$ consists of the limits of the $\frac{1}{\mu_{i,\alpha}} \exp^{-1}_{x_{i,\alpha}}(x_{j,\alpha})$'s as $\alpha \to +\infty$ for $j \in I_i$, and I_i stands for the set of the j's which are such that $d_g(x_{i,\alpha}, x_{j,\alpha}) = O(\mu_{i,\alpha})$ and $\mu_{j,\alpha} = o(\mu_{i,\alpha})$. Let $D_\alpha, \hat{D}_\alpha : S^3 \to \mathbb{R}^+$ be defined by

$$\begin{aligned} D_\alpha(x) &= \min_{i=1,\dots,k} d_g(x_{i,\alpha}, x) \text{ and }, \\ \hat{D}_\alpha(x) &= \min_{i=1,\dots,k} \left(d_g(x_{i,\alpha}, x) + \mu_{i,\alpha}\right) . \end{aligned} \tag{4.16}$$

There holds that $D_\alpha \le \hat{D}_\alpha$ and by the analysis in Druet and Hebey [8], since (4.2) holds true, we can write that

$$\hat{D}_\alpha^{\frac{1}{2}} \left| u_\alpha - u_\infty - \sum_{i=1}^k B_{i,\alpha} \right| \to 0 \text{ in } L^\infty(S^3) \tag{4.17}$$

as $\alpha \to +\infty$. In particular, if $\mathcal{S}$ stands for the set consisting of the limits of the $x_{i,\alpha}$'s as $\alpha \to +\infty$, then $u_\alpha \to u_\infty$ in $L^\infty_{loc}(S^3 \backslash \mathcal{S})$.

Lemma 4.3. *Let $G_\alpha : S^3 \times S^3 \backslash D \to \mathbb{R}$ be the Green's function of $\frac{\hbar^2}{2m_0^2}\Delta_g + h_\alpha$, where h_α is given by* (4.3)*, and D is the diagonal in $S^3 \times S^3$. Suppose $\omega = 0$ and $v = 0$, where v is as in* (4.2)*. Then $inf_{S^3 \times S^3 \backslash D} G_\alpha \to +\infty$ as $\alpha \to +\infty$.*

Proof of Lemma 4.3. Let $\varepsilon_\alpha = \|h_\alpha\|_{L^\infty}$ and $k_\alpha \in \mathbb{R}$ be such that $k_\alpha \to +\infty$ and $\varepsilon_\alpha k_\alpha \to 0$ as $\alpha \to +\infty$. Let $\hat{G}_\alpha$ be the Green's function of $\frac{\hbar^2}{2m_0^2}\Delta_g + \varepsilon_\alpha$. By the maximum principle, $G_\alpha \ge \hat{G}_\alpha$ in $S^3 \times S^3 \backslash D$ and we can use the specific form of $\hat{G}_\alpha$ in S^3 or use the following more general S^3-free argument. We let $G \ge 0$ be a Green's function of $\frac{\hbar^2}{2m_0^2}\Delta_g$. For any $x \in S^3$, if $G_x = G(x, \cdot)$, there holds that $\frac{\hbar^2}{2m_0^2}\Delta_g G_x = \delta_x - \frac{1}{|S^3|}$. Let $x \in S^3$ and V_α solve

$$\frac{\hbar^2}{2m_0^2}\Delta_g V_\alpha + \varepsilon_\alpha V_\alpha = \varepsilon_\alpha G_x .$$

There holds $\int V_\alpha = \int G_x$ so that, by Poincaré's inequality and standard estimates on G, V_α is bounded in H^1 uniformly with respect to x. By standard elliptic properties and standard estimates on G, it follows that $\|V_\alpha\|_{L^\infty} \le C$ for all α with a bound which is uniform with respect to x. Let $\Phi_\alpha = \hat{G}_\alpha(x, \cdot) - G_x - k_\alpha + V_\alpha$. Then

$$\frac{\hbar^2}{2m_0^2}\Delta_g \Phi_\alpha + \varepsilon_\alpha \Phi_\alpha \ge \frac{1}{|S^3|} - \varepsilon_\alpha k_\alpha$$

for all α, and by the maximum principle and the above estimates it follows that $\hat{G}_\alpha(x, \cdot) \ge k_\alpha - C$ for all α and all x, where C is independent of α and x. This proves the lemma. □

The following key estimate is established in Druet and Hebey [8] (see also Druet, Hebey and Robert [10]). A slight difference here is that we need to handle the noncoercive case where $\omega = 0$ and $v = 0$. We handle this case thanks to Lemma 4.3.

Lemma 4.4 (Step 5.2 in Druet and Hebey [8]). *There exists $C > 0$ such that, up to a subsequence,*

$$u_\alpha \le C\left(\mu_\alpha^{\frac{1}{2}} D_\alpha^{-1} + \|u_\infty\|_{L^\infty}\right) \tag{4.18}$$

in S^3, for all α, where $\mu_\alpha = \max_i \mu_{i,\alpha}$.

Proof of Lemma 4.4. We briefly sketch the proof and refer to Druet and Hebey [8] for more details. Given $\delta > 0$ we define

$$\eta_\alpha(\delta) = \max_{M\backslash \cup_{i=1}^k B_{x_{i,\alpha}}(\delta)} u_\alpha\ . \tag{4.19}$$

Let G be the Green's function of $\frac{\hbar^2}{2m_0^2}\Delta_g + 1$. Given $\varepsilon \in (0, \frac{1}{2})$, we let $\Psi_{\alpha,\varepsilon}$ be given by

$$\Psi_{\alpha,\varepsilon}(x) = \mu_\alpha^{\frac{1}{2}(1-2\varepsilon)} \sum_{i=1}^N G\left(x_{i,\alpha}, x\right)^{1-\varepsilon} + \eta_\alpha\left(\delta\right) \sum_{i=1}^k G\left(x_{i,\alpha}, x\right)^\varepsilon\ ,$$

and let $\Omega_\alpha = M\backslash \bigcup_{i=1}^k B_{x_{i,\alpha}}\left(R\mu_{i,\alpha}\right)$. We define $y_\alpha \in \Omega_\alpha$ be such that $u_{i,\alpha}\Psi_{\alpha,\varepsilon}^{-1}$ is maximum in Ω_α at y_α. Up to choosing $\delta > 0$ sufficiently small, and $R \gg 1$ sufficiently large, $y_\alpha \in \partial\Omega_\alpha$ or $D_\alpha(y_\alpha) > \delta$ for $\alpha \gg 1$. By (4.15) and standard properties of the Green's function it follows that for any $\varepsilon \in (0, \frac{1}{2})$, there exist $R_\varepsilon \gg 1$, $0 < \delta_\varepsilon \ll 1$, and $C_\varepsilon > 0$ such that, up to a subsequence,

$$u_\alpha(x) \le C_\varepsilon\left(\mu_\alpha^{\frac{1}{2}(1-2\varepsilon)} D_\alpha(x)^{(2-n)(1-\varepsilon)} + \eta_\alpha(\delta_\varepsilon) D_\alpha(x)^{(2-n)\varepsilon}\right) \tag{4.20}$$

for all α and all $x \in M\backslash \bigcup_{i=1}^k B_{x_{i,\alpha}}\left(R_\varepsilon\mu_{i,\alpha}\right)$. Now we claim that there exists $\delta > 0$ small such that for any sequence $(y_\alpha)_\alpha$ of points in S^3,

$$\limsup_{\alpha\to+\infty} \frac{u_\alpha(y_\alpha)}{\mu_\alpha^{1/2} D_\alpha(y_\alpha)^{-1} + \eta_\alpha(\delta)} < +\infty\ . \tag{4.21}$$

By the definition of $\eta_\alpha(\delta)$ and (4.17) we can assume that $D_\alpha(y_\alpha) \le \delta$ and that $\mu_\alpha^{-1} D_\alpha(y_\alpha) \to +\infty$ as $\alpha \to +\infty$. Let $0 < \lambda \ll 1$ be such that $\lambda \notin \mathrm{Sp}(\frac{\hbar^2}{2m_0^2}\Delta_g)$, where $\mathrm{Sp}(\frac{\hbar^2}{2m_0^2}\Delta_g)$ is the spectrum of $\frac{\hbar^2}{2m_0^2}\Delta_g$, and let G be the Green's function of $\frac{\hbar^2}{2m_0^2}\Delta_g - \lambda$. There exist $C_1 > 1$, $C_2, C_3 > 0$ such that

$$\frac{1}{C_1} d_g(x,y)^{-1} - C_2 \le G(x,y) \le C_3 d_g(x,y)^{-1}$$

and $|\nabla G(x,y)| \le C_3 d_g(x,y)^{-2}$ for all $x, y \in S^3$, $x \ne y$. We choose $\delta > 0$ small such that $d_g(x,y) \ge 4\delta$ for all $x, y \in \mathcal{S}$, $x \ne y$, and such that $4\delta C_1 C_2 \le 1$. Let $x_0 \in \mathcal{S}$ be such that $d_g(x_0, y_\alpha) \le \delta + o(1)$. By the Green's representation formula and the above estimates on G, there exists $C > 0$ such that

$$\begin{aligned} u_\alpha(y_\alpha) &= \int_{B_{x_0}(2\delta)} G_{y_\alpha}\left(\frac{\hbar^2}{2m_0^2}\Delta_g u_\alpha - \lambda u_\alpha\right) dv_g + O\left(\eta_\alpha(\delta)\right) \\ &\le C\int_{S^3} d_g(y_\alpha, x)^{-1} u_\alpha^5(x) dv_g(x) + O\left(\eta_\alpha(\delta)\right) \end{aligned}$$

for all $\alpha \gg 1$, since $G_{y_\alpha} \ge 0$ in $B_{y_\alpha}(2\delta)$ for α large by our choice of δ. By (4.20), letting $\varepsilon > 0$ be small, we get that

$$\int_{S^3} d_g(y_\alpha, x)^{-1} u_\alpha^5(x) dv_g(x) = O\left(\mu_\alpha^{1/2} D_\alpha(y_\alpha)^{-1}\right) + O\left(\eta_\alpha(\delta_\varepsilon)\right)\ .$$

Choosing $\delta \in (0, \delta_\varepsilon)$, $\delta \ll 1$, we get that (4.21) holds true. Now it remains to prove that if $u_\infty \equiv 0$, then $\eta_\alpha(\delta) = O(\mu_\alpha^{1/2})$. As a consequence of (4.21), assuming by contradiction that $\eta_\alpha(\delta)\mu_\alpha^{-1/2} \to +\infty$ as $\alpha \to +\infty$, we get by standard elliptic theory that $\eta_\alpha(\delta)^{-1}u_\alpha \to H$ in $C^2_{loc}(S^3\backslash\mathcal{S})$ as $\alpha \to +\infty$, where $\Delta_g H + hH = 0$ and $|H| \le C$ in $S^3\backslash\mathcal{S}$, $H \not\equiv 0$, and $h = \omega + qv$. Then H is in the kernel of $\Delta_g + h$ and we get a contradiction if $h > 0$. In case $h = 0$, and thus in case $\omega = 0$ and $v \equiv 0$, we get thanks to Lemma 4.1, that we apply around a point where u_α is maximum, that $\max_M u_\alpha \times \min_M u_\alpha \le C$ for some $C > 0$ and all α. Independently, by Lemma 4.3, if $\tilde{x}_\alpha$ is a point where u_α is minimum, and G_α is the Green function of $\frac{\hbar^2}{2m_0^2}\Delta_g + h_\alpha$, then

$$\begin{aligned}\max_M u_\alpha \times \min_M u_\alpha &\ge \max_M u_\alpha \int_S^3 G_\alpha(\tilde{x}_\alpha, \cdot)u_\alpha^5 dv_g \\ &\ge \int_{S^3} G_\alpha(\tilde{x}_\alpha, \cdot)u_\alpha^6 dv_g \ge \Lambda_\alpha \int_{S^3} u_\alpha^6 dv_g\ ,\end{aligned}$$

where $\Lambda_\alpha \to +\infty$ as $\alpha \to +\infty$. Then $\int_{S^3} u_\alpha^6 dv_g \to 0$ as $\alpha \to +\infty$ and we get a contradiction with (4.4) since if $k = 0$, then $u_\alpha \to u_\infty$ uniformly in S^3. In other words, $\eta_\alpha(\delta) = O(\mu_\alpha^{1/2})$ holds true, and by (4.21), this ends the proof of the lemma. □

Up to now we did not use the assumption that $|\omega| < \Lambda(m_0)$ neither $|\omega_\alpha| < \Lambda(m_0)$. The conclusion of the proof does use this assumption.

Proof of Theorem 4.1 - Existence of a priori bounds. We can assume that, up to a subsequence, $\mu_\alpha = \mu_{1,\alpha}$ for all α, where μ_α is as in Lemma 4.4. In what follows we let $x_\alpha = x_{1,\alpha}$ for all α. First we claim that $u_\infty \equiv 0$. In order to prove this we proceed by contradiction and assume that $u_\infty \not\equiv 0$. Then $v > 0$ in S^3, where v is as in (4.2), since

$$\Delta_g v + m_1^2 v = 4\pi q u_\infty^2 \tag{4.22}$$

in S^3. In particular, since $h_\alpha = \omega_\alpha^2 + qv_\alpha$ by (4.3), there holds that $h > 0$ in S^3, where h is the limit of the h_α's. Let $\theta > 1$ be given, and let G_θ be the Green's function of $\frac{\hbar^2}{2m_0^2}\Delta_g + \theta h$. By the maximum principle, $G_\alpha \ge G_\theta$ for $\alpha \gg 1$, where G_α is as in Lemma 4.3. Let $\mathcal{S} = \{x_1, \dots, x_m\}$ be the set consisting of the limits of the $x_{i,\alpha}$'s and $a \in S^3\backslash\mathcal{S}$. Let $\delta > 0$ be such that $B_a(\delta) \subset M\backslash\mathcal{S}$. Then, for any $x \in S^3$,

$$\begin{aligned}u_\alpha(x) &= \int_{S^3} G_\alpha(x,y)u_\alpha^5(y)dv_g(y) \\ &\ge \int_{B_a(\delta)} G_\theta(x,y)u_\alpha^5(y)dv_g(y) \\ &\ge \int_{B_a(\delta)} G_\theta(x,y)u_\infty^5(y)dv_g(y) + o(1)\end{aligned}$$

since $u_\alpha \to u_\infty$ in $L^\infty_{loc}(S^2\backslash\mathcal{S})$. In particular, there exists $\varepsilon_0 > 0$ such that $u_\alpha \ge \varepsilon_0$ in S^3 for all α. Let $y_\alpha \in S^3$, given by (4.4), be such that $u_\alpha(y_\alpha) \to +\infty$ as $\alpha \to +\infty$. Up to a subsequence, $y_\alpha \to y$ as $\alpha \to +\infty$. Coming back to the beginning of the proof of Lemma 4.2, by the stereographic projection of pole $-y$, there exists $\phi > 0$

smooth and positive such that $\hat{g} = \phi^4 g$ is flat in $S^3\backslash\{-y\}$, the set $S^3\backslash\{-y\}$ can be assimilated with $\mathbb{R}^3$, y with 0, and $\hat{g}$ with the Euclidean metric, and such that

$$\frac{\hbar^2}{2m_0^2}\Delta_{\hat{g}}\hat{u}_\alpha + \hat{h}_\alpha\hat{u}_\alpha = \hat{u}_\alpha^5 \tag{4.23}$$

in $\mathbb{R}^3$, where $\hat{u}_\alpha = \phi^{-1}u_\alpha$, and $\phi^4\hat{h}_\alpha = \frac{2m_0^2}{\hbar^2}h_\alpha - \frac{3}{4}$. By construction we have that $\sup_{B_0(\varepsilon)}\hat{u}_\alpha \to +\infty$ for all $\varepsilon > 0$ as $\alpha \to +\infty$. Since, according to the above, $\inf_{B_0(4\varepsilon)}\hat{u}_\alpha \geq \tilde{\varepsilon}_0 > 0$ as long as $u_\infty \not\equiv 0$, we get a contradiction with Lemma 4.1. This proves that $u_\infty \equiv 0$. In particular, by (4.22), there holds that $v \equiv 0$. We assume in what follows that $|\omega| < \Lambda$, where Λ is as in (0.3). By Lemma 4.4, for any $K \subset\subset S^3\backslash\mathcal{S}$, there exists $C_K > 0$ such that $\mu_\alpha^{-1/2}u_\alpha \leq C_K$ in K for all α. There holds

$$\frac{\hbar^2}{2m_0^2}\Delta_g(\mu_\alpha^{-1/2}u_\alpha) + h_\alpha(\mu_\alpha^{-1/2}u_\alpha) = \mu_\alpha^2(\mu_\alpha^{-1/2}u_\alpha)^5 ,$$

where $h_\alpha = \omega_\alpha^2 + qv_\alpha$. By standard elliptic theory it follows that $\mu_\alpha^{-1/2}u_\alpha \to \mathcal{U}$ in $C^1_{loc}(S^3\backslash S)$ as $\alpha \to +\infty$. Splitting S^3 into the two subsets $\{D_\alpha \leq R\mu_\alpha\}$ and $\{D_\alpha \geq R\mu_\alpha\}$, using (4.15) around x_α, thanks to Lemma 4.4 and since $u_\infty \equiv 0$, there exists $A > 0$ such that

$$\int_{S^3} u_\alpha^5 dv_g = (A + o(1))\,\mu_\alpha^{1/2} . \tag{4.24}$$

By the Green's representation formula we then get that

$$u_\alpha(x) \geq \left(\inf_{S^3\times S^3\backslash D} G_\alpha\right)\int_{S^3} u_\alpha^5 dv_g ,$$

and since $v \equiv 0$, it follows from the bound $\mu_\alpha^{-1/2}u_\alpha \leq C_K$, from (4.24), and from Lemma 4.3, that $\omega \neq 0$. In particular $\frac{\hbar^2}{2m_0^2}\Delta_g + \omega^2$ is coercive. Let G be its Green's function. Then, as in Hebey and Robert [16], we can write that

$$\mathcal{U}(x) = \frac{\sqrt{3}\hbar^2\omega_2}{2m_0^2}\sum_{i=1}^m \mu_i^{1/2}G(x_i, x)$$

for all $x \in S^3\backslash\mathcal{S}$, and $\mathcal{U}$ satisfies that

$$\frac{\hbar^2}{2m_0^2}\Delta_g\mathcal{U} + \omega^2\mathcal{U} = \frac{\sqrt{3}\hbar^2\omega_2}{2m_0^2}\sum_{i=1}^m \mu_i^{1/2}\delta_{x_i}$$

in the sense of distributions, where $\mu_i \geq 0$ for all i, and $\mu_1 > 0$ by (4.15). Sharper estimates would give that $\mu_i\mu_\alpha = (1 + o(1))\,\mu_{i,\alpha}$. There holds that

$$G(x, y) = \frac{\eta(x, y)}{\omega_2 d_g(x, y)} + R(x, y) ,$$

where $R : S^3 \times S^3 \to \mathbb{R}$ is continuous, and $\eta : S^3 \times S^3 \to \mathbb{R}$ is smooth such that $0 \leq \eta \leq 1$, $\eta(x, y) = 1$ if $d_g(x, y) \leq \delta$, and $\eta(x, y) = 0$ if $d_g(x, y) \geq 2\delta$ for $\delta > 0$ sufficiently small. Moreover, $d_g(x, y)|\nabla R_x(y)| \leq C$ for all $y \in S^3\backslash\{x\}$, where $R_x(y) = R(x, y)$, and

$$\delta_\alpha \max_{y\in\partial B_x(\delta_\alpha)} |\nabla R_x(y)| = o(1) , \tag{4.25}$$

where $\delta_\alpha \to 0$ as $\alpha \to +\infty$. At last, since $|\omega| < \Lambda$, it follows from the maximum principle that $R(x, x) > 0$ for all $x \in S^3$. Let x_1 be the limit of the x_α's. Let also

$\phi > 0$ smooth be such that $\phi^4\xi = g$ in a neighbourhood Ω of x_1, and $\phi(x_1) = 1$, where ξ is the Euclidean metric. Define $\hat{u}_\alpha = \phi u_\alpha$. There holds

$$d_g(0,x) = |x|\left(1 + (\nabla\phi(0), x) + o(|x|^2)\right) ,$$

where we assimilate x_1 with 0 and $(\cdot,\cdot)$ stands for the Euclidean scalar product. Assume $\Omega \cap \mathcal{S} = \{x_1\}$. Then $\mu_\alpha^{-1/2}\hat{u}_\alpha \to H$ in $C^1_{loc}(\Omega\backslash\{0\})$ as $\alpha \to +\infty$, where

$$H(x) = \frac{\sqrt{3}\hbar^2\omega_2}{2m_0^2}\left(\frac{\mu_1^{1/2}}{\omega_2|x|} + K(x)\right) , \tag{4.26}$$

$K \in C^0(\Omega)$, and $K(0) = \mu_1^{1/2}R(x_1,x_1) + \sum_{j\neq 1}\mu_j^{1/2}G(x_1,x_j)$. Given $\delta > 0$ sufficiently small, let

$$A_\delta = -\int_{\partial B_0(\delta)}(x^k\partial_k H)\partial_\nu H d\sigma + \frac{1}{2}\int_{\partial B_0(\delta)}(x,\nu)|\nabla H|^2 d\sigma - \frac{1}{2}\int_{\partial B_0(\delta)} H\partial_\nu H d\sigma ,$$

where ν is the outward unit normal. By (4.25) and (4.26), we get that

$$\lim_{\delta\to 0} A_\delta = \frac{3}{2}\left(\frac{\hbar^2}{2m_0^2}\right)^2\omega_2^2\mu_1^{1/2}K(0) . \tag{4.27}$$

By the Pohozaev identity applied to $\hat{u}_\alpha$ in $B_0(\delta)$, there also holds that

$$\int_{B_0(\delta)}\left((x^k\partial_k\hat{u}_\alpha) + \frac{1}{2}\hat{u}_\alpha\right)\Delta\hat{u}_\alpha dx = (A_\delta + o(1))\,\mu_\alpha . \tag{4.28}$$

It remains to handle the left hand side in (4.28), the difficulty here being that we only have a $C^{0,\theta}$-convergence for the h_α's and not a C^1-convergence. We claim that there exists $C > 0$ such that

$$|\nabla v_\alpha| \le C\mu_\alpha D_\alpha^{-1} \tag{4.29}$$

in S^3, for all α. In order to prove (4.29), let G be the Green's function of $\Delta_g + m_1^2$. Then

$$v_\alpha(x) = 4\pi q\int_{S^3} G(x,y)u_\alpha^2(y)dv_g(y)$$

for all $x \in S^3$ and all α, and by standard properties of the Green's functions, see Druet, Hebey and Robert [10], by Lemma 4.4, and since $u_\infty \equiv 0$, we get that

$$|\nabla v_\alpha(x)| \le C\mu_\alpha\sum_{i=1}^k\int_{S^3}\frac{dv_g(y)}{d_g(x,y)^2 d_g(x_{i,\alpha},y)^2} .$$

In particular, (4.29) follows from Giraud's lemma, and this ends the proof of (4.29). There holds that

$$\frac{\hbar^2}{2m_0^2}\Delta\hat{u}_\alpha + \hat{h}_\alpha\hat{u}_\alpha = \hat{u}_\alpha^5$$

for all α, where

$$\hat{h}_\alpha = \left(\frac{2m_0^2}{\hbar^2}h_\alpha - \frac{3}{4}\right)\phi^4 .$$

By (4.29), and since $|\nabla v_\alpha|$ is bounded in $H^{1,3}$, we can write with Lemma 4.4 that

$$\begin{aligned}\int_{B_0(\delta)} |\nabla v_\alpha| u_\alpha^2 dv_g &\le o(\mu_\alpha) + \int_{B_0(\delta)\backslash \cup_{i=1}^k B_{x_{i,\alpha}}(\mu_\alpha)} |\nabla v_\alpha| u_\alpha^2 dv_g \\ &\le o(\mu_\alpha) + C\mu_\alpha^2 \sum_{i=1}^k \int_{B_0(\delta)\backslash B_{x_{i,\alpha}}(\mu_\alpha)} \frac{1}{d(x_{i,\alpha},x)^3} dx \\ &\le o(\mu_\alpha) + C\mu_\alpha^2 \sum_{i\in\tilde{I}} \int_{B_{x_{i,\alpha}}(2\delta)\backslash B_{x_{i,\alpha}}(\mu_\alpha)} \frac{1}{d(x_{i,\alpha},x)^3} dx \\ &= o(\mu_\alpha) + O\left(\mu_\alpha^2 \ln \frac{1}{\mu_\alpha}\right) ,\end{aligned}$$

where $\tilde{I}$ is the subset of $\{1,\dots,k\}$ consisting of the i's which are such that $x_{i,\alpha} \to x_1$ as $\alpha \to +\infty$, d is the Euclidean metric, and $\delta > 0$ is sufficiently small. Integrating by parts the left hand side in (4.28), we then get that

$$\lim_{\delta\to 0} \lim_{\alpha\to+\infty} \frac{1}{\mu_\alpha} \int_{B_0(\delta)} \left((x^k \partial_k \hat{u}_\alpha) + \frac{1}{2}\hat{u}_\alpha \right) \Delta \hat{u}_\alpha dx = 0 . \tag{4.30}$$

Combining (4.27), (4.28), and (4.30), the contradiction follows since $G \ge 0$ and $R(x_1, x_1) > 0$ when $|\omega| < \Lambda$. As a consequence, $\|u_\alpha\|_{L^\infty} + \|v_\alpha\|_{L^\infty} \le C$ for some $C > 0$. By standard elliptic theory a $C^{2,\theta}$-bound holds as well. Up to a subsequence we can pass to a C^2-limit . This proves the existence of a priori bounds in Theorem 4.1. □

5. Unstable phases and resonant frequencies - Affine estimates

The goal in this section and in the following one is to prove the second part of Theorem 0.2 by constructing multi-spikes solutions to (0.2) when ω is close to resonant frequencies ω_k. To each ω_k is associated a sequence of n_k-spikes solutions with $n_k \to +\infty$ as $k \to +\infty$. This can be considered as bifurcation from infinity (see Bahri [2]). More precisely we use here the so-called localized energy method (see Del Pino, Felmer and Musso [7], Rey and Wei [22], and Wei [27]) which goes through the choice of suitable approximate solutions (this is done in this section) and the use of finite-dimensional reduction (carried over in the following section).

Let $P_1 = (1,0,0,0)$ in S^3 and $k \in \mathbb{N}$, $k \ge 1$. We define the P_i's, $i = 1,\dots,k$, by $P_i = \left(e^{i\theta_i}, 0\right) \in S^3 \subset \mathbb{R}^2 \times \mathbb{R}^2$, where $\theta_i = \frac{2\pi(i-1)}{k}$. Let G_k be the maximal isometry group of (S^3, g) which leaves globally invariant the set $\{P_1,\dots,P_k\}$. Let also $\Sigma_k \subset S^3$ be the slice

$$\Sigma_k = \left\{ \left(re^{i\theta}, z\right), r > 0, z \in \mathbb{C}, r^2 + |z|^2 = 1, -\frac{\pi}{k} \le \theta \le \frac{\pi}{k} \right\} . \tag{5.1}$$

We consider the nonlinear critical equation

$$\frac{\hbar^2}{2m_0^2}\Delta_g u + \Lambda(m_0)^2 u = u^5$$

in S^3, with $u > 0$. Its solutions are all known and given, see (2.4), by

$$U_{\beta,x_0} = \frac{\sqrt{\hbar}\left(3(\beta^2-1)\right)^{1/4}}{2^{1/4}\sqrt{m_0}\sqrt{2(\beta-\cos r)}} ,$$

where $\beta > 1$ is arbitrary, $r = d_g(x_0, \cdot)$, and $x_0 \in S^3$ is also arbitrary. These solutions can be rewritten as

$$U_{\varepsilon,x_0} = \frac{3^{1/4}\sqrt{\hbar}}{8^{1/4}\sqrt{m_0}} \left(\frac{\varepsilon}{\varepsilon^2 \cos^2 \frac{r}{2} + \sin^2 \frac{r}{2}} \right)^{1/2} , \tag{5.2}$$

where $\varepsilon \in (0,1)$. There holds that $U_{\varepsilon,x_0} = U_{\beta_\varepsilon,x_0}$ by letting $\beta_\varepsilon = \frac{1+\varepsilon^2}{1-\varepsilon^2}$. Also we do have an explicit expression for the Green's function G_ω of $\frac{\hbar^2}{2m_0^2}\Delta_g + \omega^2$. Namely,

$$G_\omega(x,y) = \frac{m_0^2 \sinh\left(\mu_\omega(\pi - r)\right)}{2\pi\hbar^2 \sinh(\mu_\omega \pi) \sin r} \tag{5.3}$$

for all $x, y \in S^3$, $x \neq y$, where $r = d_g(x,y)$ and $\mu_\omega = \sqrt{\frac{2m_0^2}{\hbar^2}\omega^2 - 1}$. When $\omega^2 = \Lambda(m_0)^2$ we recover the Green's function of the conformal Laplacian. We write G_0 instead of G_ω when $\omega^2 = \Lambda(m_0)^2$ and there holds that

$$G_0(x,y) = \frac{m_0^2 \cos \frac{r}{2}}{2\pi\hbar^2 \sin r} \tag{5.4}$$

for all $x, y \in S^3$, $x \neq y$, where $r = d_g(x,y)$. At this point we let R_ω be given by $G_\omega = G_0 + R_\omega$, and we define

$$\eta_k(\omega) = R_\omega(P_1, P_1) + \sum_{i=2}^{k} G_\omega(P_1, P_i) , \tag{5.5}$$

where the second term in the right hand side of (5.5) is zero if $k = 1$. There holds

$$R_\omega(P_1, P_1) = -\frac{m_0^2}{2\pi\hbar^2}\mu_\omega \coth(\mu_\omega \pi)$$

so that $\eta_1(\omega) = 0$ if and only if $\omega = \pm\Lambda(m_0)$, while $\eta_1'\left(\Lambda(m_0)\right) < 0$. It is easily checked that $\eta_k(\omega) \to -\infty$ as $\omega \to \pm\infty$, while $\eta_k\left(\hbar/\sqrt{2}m_0\right) > 0$ for $k \geq 2$. There also holds that $\frac{d}{d\mu}(\mu \coth(\mu\pi) > 0$ while, by the maximum principle, $G_\omega \leq G_{\omega_0}$ if $\omega \geq \omega_0$. Hence there exists a unique $\omega_k \geq \Lambda(m_0)$ such that $\eta_k(\omega_k) = 0$. We define

$$\begin{aligned} \omega_k &= \inf\left\{\omega \geq \Lambda(m_0) \text{ s.t. } \eta_k(\omega) = 0\right\} , \\ &= \sup\left\{\omega \geq \Lambda(m_0) \text{ s.t. } \eta_k(\omega) = 0\right\} , \end{aligned} \tag{5.6}$$

where $\eta_k(\omega)$ is given by (5.5). Then $\eta_k(\omega) > 0$ if $\omega < \omega_k$ and $\eta_k(\omega) < 0$ if $\omega > \omega_k$. When $k = 1$, $\omega_1 = \Lambda(m_0)$. Since $\sinh(tx)/\sin(x) > t$ for $x \in (-\pi, \pi)$, there holds that $\omega_k \to +\infty$ as $k \to +\infty$. Independently, we can check

$$\frac{\hbar^2}{2m_0^2}R_{\omega,P_1} = -\frac{\mu_\omega \coth(\mu_\omega \pi)}{4\pi} + \frac{1}{8\pi}\left(\frac{2m_0^2}{\hbar^2}\omega^2 - \frac{3}{4}\right) r + O(r^2) , \tag{5.7}$$

where $r = d_g(P_1, \cdot)$ and $R_{\omega,P_1} = G_\omega(P_1, \cdot) - G_0(P_1, \cdot)$. The R_{ω,P_1}'s satisfy the equation

$$\frac{\hbar^2}{2m_0^2}\Delta_g R_{\omega,P_1} + \omega^2 R_{\omega,P_1} = \left(\Lambda(m_0)^2 - \omega^2\right) G_0 . \tag{5.8}$$

In what follows we define the projections $\mathcal{U}_{\varepsilon,P_i}$, $i = 1, \ldots, k$, by

$$\frac{\hbar^2}{2m_0^2}\Delta_g \mathcal{U}_{\varepsilon,P_i} + \omega^2 \mathcal{U}_{\varepsilon,P_i} = U_{\varepsilon,P_i}^5 \tag{5.9}$$

and we define $\varphi_{\varepsilon,P_i}$ to be given by

$$\mathcal{U}_{\varepsilon,P_i} = U_{\varepsilon,P_i} + \varphi_{\varepsilon,P_i}\ , \tag{5.10}$$

where U_{ε,P_i} is given by (5.2). There holds that

$$\frac{\hbar^2}{2m_0^2}\Delta_g\varphi_{\varepsilon,P_i} + \omega^2\varphi_{\varepsilon,P_i} = \left(\Lambda(m_0)^2 - \omega^2\right)U_{\varepsilon,P_i}\ . \tag{5.11}$$

Independently we let $\psi \in \dot{H}^1(\mathbb{R}^3)$ be the solution of

$$\frac{\hbar^2}{2m_0^2}\Delta\psi = \frac{1}{\sqrt{4+|x|^2}} - \frac{1}{|x|} \tag{5.12}$$

in $\mathbb{R}^3$. By the Green's representation formula we get that $|\psi(x)| \le C\frac{\ln(2+|x|)}{1+|x|}$ and $|\nabla\psi(x)| \le C\frac{\ln(2+|x|)}{(1+|x|)^2}$ as $|x| \to +\infty$. The first lemma we prove is the following where we obtain the asymptotic expansion of the $\varphi_{\varepsilon,P_i}$'s, and thus of the approximate solutions $\mathcal{U}_{\varepsilon,P_i}$'s.

Lemma 5.1. *There holds that*

$$\varphi_{\varepsilon,P_1} = A\sqrt{\varepsilon}R_{\omega,P_1} + B_\omega\varepsilon^{3/2}\psi\left(\frac{r}{\varepsilon}\right) + o\left(\varepsilon^{3/2}\right) \tag{5.13}$$

in Σ_k*, where*

$$A = 3^{1/4}\pi 4\sqrt{2}\left(\frac{\hbar}{\sqrt{2}m_0}\right)^{5/2} \quad \textit{and} \quad B_\omega = \frac{3^{1/4}2\sqrt{\hbar}}{8^{1/4}\sqrt{m_0}}\left(\Lambda(m_0)^2 - \omega^2\right)\ , \tag{5.14}$$

Σ_k *is as in* (5.1)*,* $\varphi_{\varepsilon,P_1}$ *is as in* (5.10)*,* $R_{\omega,P_1} = G_\omega(P_1,\cdot) - G_0(P_1,\cdot)$*,* ψ *is as in* (5.12)*, and* $r = d_g(P_1,\cdot)$*.*

Proof of Lemma 5.1. Thanks to the equations (5.8) and (5.11) satisfied by R_{ω,P_1} and $\varphi_{\varepsilon,P_1}$,

$$\frac{\hbar^2}{2m_0^2}\Delta_g\left(\varphi_{\varepsilon,P_1} - A\sqrt{\varepsilon}R_{\omega,P_1}\right) + \omega^2\left(\varphi_{\varepsilon,P_1} - A\sqrt{\varepsilon}R_{\omega,P_1}\right) = CK_\varepsilon\ ,$$

where $C = \frac{3^{1/4}\sqrt{\hbar}}{8^{1/4}\sqrt{m_0}}\left(\Lambda(m_0)^2 - \omega^2\right)$ and $K_\varepsilon = \left(\frac{\varepsilon}{\varepsilon^2\cos^2\frac{r}{2}+\sin^2\frac{r}{2}}\right)^{1/2} - \frac{\sqrt{\varepsilon}}{\sin\frac{r}{2}}$. Let $v_\varepsilon = \varphi_{\varepsilon,P_1} - A\sqrt{\varepsilon}R_{\omega,P_1}$, $\tilde{v}_\varepsilon, g_\varepsilon$ be given by

$$\tilde{v}_\varepsilon(x) = \varepsilon^{-3/2}v_\varepsilon\left(\exp_{P_1}(\varepsilon x)\right)\ ,$$

and $g_\varepsilon(x) = \left(\exp_{P_1}^\star g\right)(\varepsilon x)$, where $x \in \mathbb{R}^3$. There holds $g_\varepsilon \to \xi$ in $C^2_{loc}(\mathbb{R}^3)$, where ξ is the Euclidean metric, and we have that

$$\frac{\hbar^2}{2m_0^2}\Delta_{g_\varepsilon}\tilde{v}_\varepsilon + \varepsilon^2\omega^2\tilde{v}_\varepsilon = C\varepsilon\tilde{K}_\varepsilon\ ,$$

where $\tilde{K}_\varepsilon = \left(\varepsilon^2\cos^2\frac{\varepsilon|x|}{2} + \sin^2\frac{\varepsilon|x|}{2}\right)^{-1/2} - \left(\sin\frac{\varepsilon|x|}{2}\right)^{-1}$. The functions $\frac{1}{2}\varepsilon\tilde{K}_\varepsilon$ converge in $C^0_{loc}(\mathbb{R}^3)$ to the right hand side of (5.12), in the sense that the difference converges to 0 in C^0_{loc}, while by the Green's representation formula, $\varepsilon^{-3/2}v_\varepsilon(x_\varepsilon) \to 0$ if $\frac{1}{\varepsilon}d_g(P_1,x_\varepsilon) \to +\infty$. In particular, we do have (5.13) and this proves Lemma 5.1. $\square$

At this point we define $\mathcal{W}_\varepsilon$ to be the sum of the $\mathcal{U}_{\varepsilon,P_i}$'s. Then

$$\mathcal{W}_\varepsilon = \sum_{i=1}^k \mathcal{U}_{\varepsilon,P_i} . \tag{5.15}$$

In particular $\mathcal{W}_\varepsilon$ is G_k-invariant. As a direct consequence of Lemma 5.1 we get that the following expansion of the $\mathcal{W}_\varepsilon$'s holds true. Namely

$$\mathcal{W}_\varepsilon = U_{\varepsilon,P_1} + A\sqrt{\varepsilon}\left(R_{\omega,P_1} + \sum_{i=2}^k G_{\omega,P_i}\right) + B_\omega \varepsilon^{3/2}\psi\left(\frac{r}{\varepsilon}\right) + o\left(\varepsilon^{3/2}\right) \tag{5.16}$$

in Σ_k, where A, B_ω are as in (5.14), Σ_k is as in (5.1), U_{ε,P_1} is as in (5.2), the $G_{\omega,P_i} = G_\omega(P_i,\cdot)$'s are as in (5.3), $R_{\omega,P_1} = G_\omega(P_1,\cdot) - G_0(P_1,\cdot)$, ψ is as in (5.12), and $r = d_g(P_1,\cdot)$. We define $U_0 : \mathbb{R}^3 \to \mathbb{R}$ to be given by

$$U_0(x) = \left(\frac{1}{1+\frac{|x|^2}{4}}\right)^{1/2} , \tag{5.17}$$

and let K_0 be the constant

$$K_0 = \frac{3^{1/4}\sqrt{\hbar}}{8^{1/4}\sqrt{m_0}} . \tag{5.18}$$

Also we let $\Phi_k(\omega) : S^3 \to \mathbb{R}$ be the solution of

$$\Delta_g \Phi_k(\omega) + m_1^2 \Phi_k(\omega) = K_1\left(\sum_{i=1}^k G_{\omega,P_i}\right)^2 , \tag{5.19}$$

where $G_{\omega,P_1} = G_\omega(P_1,\cdot)$, G_ω is as in (5.3), and $K_1 = 128\sqrt{3}\pi^3\left(\frac{\hbar}{\sqrt{2m_0}}\right)^5$. The right hand side in (5.19) is in L^p for all $p < \frac{3}{2}$. Thus, (5.19) makes sense. Now, thanks to (5.16), we get that the following asymptotics for the energy hold true.

Lemma 5.2. *There holds that*

$$\begin{aligned} I_6\left(\mathcal{W}_\varepsilon\right) = {} & A_{0,k} + A_{1,k}\varepsilon\eta_k(\omega) + A_{2,k}(\omega)q^2\varepsilon^2 \\ & + A_{3,k}(\omega)\varepsilon^2 + O\left(\eta_k(\omega)^2\varepsilon^2\right) + o\left(\varepsilon^2\right) , \end{aligned} \tag{5.20}$$

where

$$\begin{aligned} A_{0,k} &= \frac{kK_0^6}{3}\int_{\mathbb{R}^3} U_0^6 dx , \quad A_{1,k} = -\frac{kK_0^5 A}{2}\int_{\mathbb{R}^3} U_0^5 dx , \\ A_{2,k}(\omega) &= \frac{1}{16\pi}\int_{S^3}\left(|\nabla\Phi_k(\omega)|^2 + m_1^2\Phi_k(\omega)^2\right) dv_g , \\ A_{3,k}(\omega) &= -\frac{128\pi 6^{1/4} kK_0^5}{3}\left(\frac{m_0}{\hbar}\right)^{3/2}\left(\omega^2 - \Lambda(m_0)^2\right)\int_0^{+\infty}\frac{dr}{4+r^2} , \end{aligned} \tag{5.21}$$

I_6 is the functional given by (2.2)*, A is as in* (5.14)*, $\mathcal{W}_\varepsilon$ is as in* (5.15)*, U_0 is as in* (5.17)*, K_0 is as in* (5.18)*, and Φ is as in* (5.19)*.*

Proof of Lemma 5.2. There holds that

$$\frac{\hbar^2}{2m_0^2}\int_{S^3}|\nabla\mathcal{W}_\varepsilon|^2 dv_g + \omega^2\int_{S^3}\mathcal{W}_\varepsilon^2 dv_g = k\int_{\Sigma_k} U_{\varepsilon,P_1}^5 \mathcal{W}_\varepsilon dv_g + O\left(\varepsilon^{5/2}\right) .$$

Then, by (5.16), and thus by Lemma 5.1,

$$\begin{aligned}&\frac{\hbar^2}{2m_0^2}\int_{S^3}|\nabla\mathcal{W}_\varepsilon|^2dv_g+\omega^2\int_{S^3}\mathcal{W}_\varepsilon^2dv_g\\&=k\int_{\Sigma_k}U_{\varepsilon,P_1}^5\left(U_{\varepsilon,P_1}+A\sqrt{\varepsilon}\Psi+B_\omega\varepsilon^{3/2}\psi\left(\frac{r}{\varepsilon}\right)+o\left(\varepsilon^{3/2}\right)\right)dv_g\\&\quad+o\left(\varepsilon^2\right)\ ,\end{aligned}\tag{5.22}$$

where

$$\Psi=R_{\omega,P_1}+\sum_{i=2}^kG_{\omega,P_i}\ .\tag{5.23}$$

Let $\delta_\varepsilon>0$ be such that $\delta_\varepsilon\to0$ as $\varepsilon\to0$ and $\delta_\varepsilon^{-6}\varepsilon^3=o\left(\varepsilon^2\right)$. There holds

$$\int_{\Sigma_k}U_{\varepsilon,P_1}^6dv_g=\int_{B_{P_1}(\delta_\varepsilon)}U_{\varepsilon,P_1}^5+o\left(\varepsilon^2\right)\ .$$

We have

$$\frac{\varepsilon}{\varepsilon^2\cos^2\frac{r}{2}+\sin^2\frac{r}{2}}=\frac{\varepsilon}{\varepsilon^2+\frac{r^2}{4}}\left(1+\frac{r^2}{12}+\frac{\varepsilon^2r^2}{6(\varepsilon^2+\frac{r^2}{4})}+o(r^2)\right)$$

in $B_{P_1}(\delta_\varepsilon)$, while $dv_g=\left(\frac{\sin r}{r}\right)^2dx$ in geodesic normal coordinates. It follows that

$$\int_{\Sigma_k}U_{\varepsilon,P_1}^6dv_g=K_0^6\int_{\mathbb{R}^3}U_0^6dx+o(\varepsilon^2)\ ,\tag{5.24}$$

where K_0 is as in (5.18). By (5.7),

$$\Psi=\eta_k(\omega)+C_\omega r+C^ix_i+O(r^2)\ ,$$

where Ψ is as in (5.23), C_ω is given by $C_\omega=\frac{m_0^4}{2\pi\hbar^4}\left(\omega^2-\Lambda(m_0)^2\right)$, and the sum C^ix_i is given by $C^ix_i=\sum_{j=2}^k\sum_{i=1}^3\frac{\partial G_{\omega,P_j}}{\partial x_i}(P_1)x_i$. Splitting Σ_k into $B_{P_1}(\delta_\varepsilon)$ and $\Sigma_k\backslash B_{P_1}(\delta_\varepsilon)$, we obtain that

$$\int_{\Sigma_k}U_{\varepsilon,P_1}^5\Psi dv_g=K_0^5\sqrt{\varepsilon}\eta_k(\omega)\int_{\mathbb{R}^3}U_0^5dx+K_0^5C_\omega\varepsilon^{3/2}\int_{\mathbb{R}^3}U_0^5rdx+o\left(\varepsilon^{3/2}\right)\ ,\tag{5.25}$$

where $r=|x|$. In a similar way, thanks to the bounds at infinity we have on ψ, there holds that

$$\begin{aligned}\int_{\Sigma_k}U_{\varepsilon,P_1}^5\psi\left(\frac{r}{\varepsilon}\right)dv_g&=\int_{B_{P_1}(\delta_\varepsilon)}U_{\varepsilon,P_1}^5\psi\left(\frac{r}{\varepsilon}\right)dv_g+o\left(\varepsilon^{3/2}\right)\\&=K_0^5\sqrt{\varepsilon}\int_{\mathbb{R}^3}U_0^5\psi(r)dx+o\left(\varepsilon^{3/2}\right)\ .\end{aligned}\tag{5.26}$$

Plugging (5.24)–(5.26) into (5.22) we get that

$$\begin{aligned}&\frac{\hbar^2}{2m_0^2}\int_{S^3}|\nabla\mathcal{W}_\varepsilon|^2dv_g+\omega^2\int_{S^3}\mathcal{W}_\varepsilon^2dv_g\\&=kK_0^6\int_{\mathbb{R}^3}U_0^6dx+kK_0^5A\varepsilon\eta_k(\omega)\int_{\mathbb{R}^3}U_0^5dx\\&\quad+kK_0^5AC_\omega\varepsilon^2\int_{\mathbb{R}^3}U_0^5rdx+kK_0^5B_\omega\varepsilon^2\int_{\mathbb{R}^3}U_0^5\psi dx+o\left(\varepsilon^2\right)\ .\end{aligned}\tag{5.27}$$

Independently, still by (5.16), there holds that

$$\int_{S^3} \mathcal{W}_\varepsilon^6 dv_g = k \int_{\Sigma_k} U_{\varepsilon,P_1}^6 dv_g + 6k \int_{\Sigma_k} U_{\varepsilon,P_1}^5 \left(A\sqrt{\varepsilon}\Psi + B_\omega \varepsilon^{3/2} \psi\left(\frac{r}{\varepsilon}\right)\right) + 15A^2 k\varepsilon \int_{\Sigma_k} U_{\varepsilon,P_1}^4 \Psi^2 dv_g + o\left(\varepsilon^2\right) \; .$$

Noting that

$$\begin{aligned}\int_{\Sigma_k} U_{\varepsilon,P_1}^4 \Psi^2 dv_g &= \int_{B_{P_1}(\delta_\varepsilon)} U_{\varepsilon,P_1}^4 \Psi^2 dv_g + o(\varepsilon) \\ &= \eta_k(\omega)^2 K_0^4 \varepsilon \int_{\mathbb{R}^3} U_0^4 dx + o(\varepsilon)\end{aligned}$$

we get as above that

$$\begin{aligned}\int_{S^3} \mathcal{W}_\varepsilon^6 dv_g &= kK_0^6 \int_{\mathbb{R}^3} U_0^6 dx + 6kK_0^5 A\varepsilon\eta_k(\omega) \int_{\mathbb{R}^3} U_0^5 dx \\ &+ 6kK_0^5 AC_\omega \varepsilon^2 \int_{\mathbb{R}^3} U_0^5 r dx + 6kK_0^5 B_\omega \varepsilon^2 \int_{\mathbb{R}^3} U_0^5 \psi dx \\ &+ O\left(\varepsilon^2 \eta_k(\omega)^2\right) + o\left(\varepsilon^2\right) \; .\end{aligned} \tag{5.28}$$

At last, by noting that $\frac{1}{\varepsilon}\Phi(\mathcal{W}_\varepsilon) \to q\Phi_k(\omega)$ in H^1, where $\Phi_k(\omega)$ is as in (5.19), we get that

$$\begin{aligned}q\int_{S^3} \Phi(\mathcal{W}_\varepsilon)\mathcal{W}_\varepsilon^2 dv_g &= \frac{1}{4\pi}\int_{S^3} \left(|\nabla\Phi(\mathcal{W}_\varepsilon)|^2 + m_1^2 \Phi(\mathcal{W}_\varepsilon)^2\right) dv_g \\ &= \frac{\varepsilon^2 q^2}{4\pi}\int_{S^3} \left(|\nabla\Phi_k(\omega)|^2 + m_1^2\Phi_k(\omega)^2\right) dv_g + o(\varepsilon^2)\end{aligned} \tag{5.29}$$

Combining (5.27), (5.28) and (5.29) we get (5.20) where $A_{3,k}(\omega)$ is given by

$$\begin{aligned}A_{3,k}(\omega) &= -\frac{kK_0^5}{2}\left(AC_\omega \int_{\mathbb{R}^3} U_0^5 r dx + B_\omega \int_{\mathbb{R}^3} U_0^5 \psi dx\right) \\ &= -\frac{6^{1/4} k K_0^5 m_0^{3/2}}{2\hbar^{3/2}}\left(\omega^2 - \Lambda(m_0)^2\right)\int_{\mathbb{R}^3} U_0^5\left(r - \psi_0(x)\right) dx \; ,\end{aligned}$$

and ψ_0 solves

$$\frac{1}{2}\Delta\psi_0 = \frac{1}{\sqrt{4+|x|^2}} - \frac{1}{|x|}$$

in $\mathbb{R}^3$. Noting that $\Delta U_0 = \frac{3}{4}U_0^5$, integrating by parts, we get the expression of $A_{3,k}(\omega)$ stated in the lemma. This ends the proof of Lemma 5.2. □

An additional result we prove is the following.

Lemma 5.3. *There holds that*

$$\lim_{k\to+\infty}\left|\frac{A_{3,k}(\omega_k)}{A_{2,k}(\omega_k)}\right| = +\infty \; , \tag{5.30}$$

where the ω_k's are as in (5.6), and $A_{2,k}, A_{3,k}(\omega)$ are as in (5.21).

Proof of Lemma 5.3. By the uniqueness of $\Phi_k(\omega)$ in (5.19) it is G_k-invariant. There holds that $\Phi_k(\omega) \geq 0$, and we can thus write by Hölder's inequalities that

$$\begin{aligned}
&\int_{S^3} \left(|\nabla \Phi_k(\omega)|^2 + m_1^2 \Phi_k(\omega)^2\right) dv_g \\
&\leq C \sum_{i=1}^k \int_{S^3} G_{\omega,P_i}^2 \Phi_k(\omega) dv_g \\
&\leq Ck \int_{S^3} G_{\omega,P_1}^2 \Phi_k(\omega) dv_g \\
&\leq Ck \|G_{\omega,P_1}\|_{L^{12/5}} \|\Phi_k(\omega)\|_{L^6} \ .
\end{aligned}$$

By the maximum principle, $G_{\omega,P_1} \leq G_{\omega_0,P_1}$ for all $\omega \geq \omega_0$. Since $\omega_k \to +\infty$, it follows that $A_{2,k}(\omega_k) \leq Ck^2$, where $C > 0$ is independent of k. On the other hand, by the definition of ω_k, $\mu_k \coth(\mu_k \pi) \geq CG_{\omega_k,P_1}(P_2)$, where $\mu_k \leq C\omega_k$, and we thus get that $\omega_k \geq Ck$, where $C > 0$ is independent of k. Then $|A_{3,k}(\omega_k)| \geq CkA_{2,k}(\omega_k)$, where $C > 0$ is independent of k. This proves the lemma. ☐

6. Equivariant finite-dimensional reduction

We develop in this section the finite-dimensional reduction argument we need in order to prove the second part of Theorem 0.2, following in large parts previous arguments by Rey and Wei [22] and Del Pino, Felmer and Musso [7], and we prove the second part of Theorem 0.2 (for the finite dimensional reduction method in the subcritical case, we refer to the book by Ambrosetti and Malchiodi [1] and the survey paper by Wei [27]). We let Θ_k be given by

$$\Theta_k = q^2 + \frac{A_{3,k}(\omega_k)}{A_{2,k}(\omega_k)} \ . \tag{6.1}$$

Then we let $\varepsilon = \Lambda\tilde{\varepsilon}$, where $\frac{1}{C} \leq \Lambda \leq C$ for $C \gg 1$, and we define $\tilde{\varepsilon} = \eta_k(\omega)$ for $\omega \in (\omega_k - \delta, \omega_k)$ with $\delta > 0$ small in case $\Theta_k > 0$, and $\tilde{\varepsilon} = -\eta_k(\omega)$ for $\omega \in (\omega_k, \omega_k + \delta)$ with $\delta > 0$ small in case $\Theta_k < 0$. Since $A_{3,k}(\omega_1) = 0$ we have that $\Theta_1 > 0$. On the other hand, by Lemma 5.3, there holds that $\Theta_k < 0$ for $k \gg 1$. In the above constructions, $\tilde{\varepsilon} > 0$ and $\tilde{\varepsilon} \to 0$ as $\omega \to \omega_k$. We let

$$f_{\tilde{\varepsilon}} : \frac{1}{\tilde{\varepsilon}} S^3 \to S^3$$

be the map given by $f_{\tilde{\varepsilon}}(x) = \tilde{\varepsilon}x$. If $g_{\tilde{\varepsilon}}$ is the standard metric on $\frac{1}{\tilde{\varepsilon}}S^3$, induced from the Euclidean metric, then $f_{\tilde{\varepsilon}}^\star g = \tilde{\varepsilon}^2 g_{\tilde{\varepsilon}}$. Given $u : S^3 \to \mathbb{R}$, we define the $\sim$-procedure which, to u, associate $\tilde{u} : \frac{1}{\tilde{\varepsilon}}S^3 \to \mathbb{R}$, where

$$\tilde{u} = \sqrt{\tilde{\varepsilon}} u \circ f_{\tilde{\varepsilon}} \ .$$

We let $\tilde{Y} = \frac{\partial \tilde{\mathcal{W}}_\varepsilon}{\partial \Lambda}$, where $\tilde{\mathcal{W}}_\varepsilon$ is obtained from $\mathcal{W}_\varepsilon$ in (5.15) by the $\sim$-procedure, and we define

$$\tilde{Z} = \frac{\hbar^2}{2m_0^2} \Delta_{g_{\tilde{\varepsilon}}} \tilde{Y} + \tilde{\varepsilon}^2 \omega^2 \tilde{Y} \ . \tag{6.2}$$

There holds that $\langle \tilde{Y}, \tilde{Z} \rangle = \gamma_0 + o(1)$, where $\gamma_0 > 0$ and $\langle \cdot, \cdot \rangle$ is the L^2-scalar product with respect to $g_{\tilde{\varepsilon}}$. We say in what follows that a function $\tilde{u}$ in $\frac{1}{\tilde{\varepsilon}}S^3$ is

G_k-invariant if u is G_k-invariant in S^3. In particular $\tilde{Y}$ and $\tilde{Z}$ are G_k-invariant. By the $\sim$-procedure, the equation

$$\frac{\hbar^2}{2m_0^2}\Delta_g u + \omega^2 u + q\Phi(u)u = u^5$$

in S^3 is equivalent to

$$\frac{\hbar^2}{2m_0^2}\Delta_{g_{\tilde{\varepsilon}}}\tilde{u} + \tilde{\varepsilon}^2\omega^2\tilde{u} + q\tilde{\varepsilon}^2\overline{\Phi(u)}\tilde{u} = \tilde{u}^5$$

in $\frac{1}{\tilde{\varepsilon}}S^3$, where $\overline{\Phi(u)} = \Phi(u)\circ f_{\tilde{\varepsilon}}$. Now we define the norms $\|\cdot\|_{\star,\sigma}$ and $\|\cdot\|_{\star\star,\sigma}$ by

$$\begin{aligned} \|u\|_{\star,\sigma} &= \sup_{x\in\frac{1}{\tilde{\varepsilon}}S^3}\left(\min_{i=1,\dots,k}\left(1+d_{g_{\tilde{\varepsilon}}}(\tilde{P}_i,x)\right)^{\sigma}\right)|u(x)| \ , \\ \|u\|_{\star\star,\sigma} &= \sup_{x\in\frac{1}{\tilde{\varepsilon}}S^3}\left(\min_{i=1,\dots,k}\left(1+d_{g_{\tilde{\varepsilon}}}(\tilde{P}_i,x)\right)^{2+\sigma}\right)|u(x)| \end{aligned} \tag{6.3}$$

for $u\in L^\infty\left(\frac{1}{\tilde{\varepsilon}}S^3\right)$, where $0<\sigma<1$ and $f_{\tilde{\varepsilon}}(\tilde{P}_i) = P_i$, $i=1,\dots,k$. Given a function $h\in L^\infty\left(\frac{1}{\tilde{\varepsilon}}S^3\right)$ we consider the problem

$$\begin{cases} \frac{\hbar^2}{2m_0^2}\Delta_{g_{\tilde{\varepsilon}}}\phi + \tilde{\varepsilon}^2\omega^2\phi - 5\tilde{W}_{\varepsilon}^4\phi = h + c_0\tilde{Z} \\ \int_{\frac{1}{\tilde{\varepsilon}}S^3}\tilde{Z}\phi dv_{g_{\tilde{\varepsilon}}} = 0 \ , \end{cases} \tag{6.4}$$

where $c_0\in\mathbb{R}$, and $\tilde{Z}$ is as in (6.2). A key point in the equivariant finite-dimensional reduction argument we develop here is given by the following lemma.

Lemma 6.1. *Let $(h_{\tilde{\varepsilon}})_{\tilde{\varepsilon}}$ be a family in $L^\infty\left(\frac{1}{\tilde{\varepsilon}}S^3\right)$ of G_k-invariant functions such that $\|h_{\tilde{\varepsilon}}\|_{\star\star,\sigma}\to 0$ as $\tilde{\varepsilon}\to 0$, and $(\phi_{\tilde{\varepsilon}})_{\tilde{\varepsilon}}$ be a family of G_k-invariant solutions of* (6.4) *with $h = h_{\tilde{\varepsilon}}$. There holds $\|\phi_{\tilde{\varepsilon}}\|_{\star,\sigma}\to 0$ as $\tilde{\varepsilon}\to 0$.*

Proof of lemma 6.1. Let $\sigma'<\sigma$. We prove by contradiction that $\|\phi_{\tilde{\varepsilon}}\|_{\star,\sigma'}\to 0$ as $\tilde{\varepsilon}\to 0$. We can assume that $\|\phi_{\tilde{\varepsilon}}\|_{\star,\sigma'} = 1$. In what follows we let $G_{\tilde{\varepsilon}}$ be the Green's function of $\Delta_{g_{\tilde{\varepsilon}}} + \tilde{\varepsilon}^2\omega^2$. Then

$$G_{\tilde{\varepsilon}}(\tilde{x},\tilde{y}) \le \tilde{\varepsilon}G(x,y) \le \frac{C}{d_{g_{\tilde{\varepsilon}}}(\tilde{x},\tilde{y})} \ , \tag{6.5}$$

where $f_{\tilde{\varepsilon}}(\tilde{x}) = x$, $f_{\tilde{\varepsilon}}(\tilde{y}) = y$, and G is the Green's function of $\Delta_g+\omega^2$. Thanks to the G_k-symmetries, using the Green's representation formula and (6.5), we get that $\|\phi_{\tilde{\varepsilon}}\|_{L^\infty}\le C$ and that

$$|\phi_{\tilde{\varepsilon}}(\tilde{x})| \le C\left(\frac{1}{\min_i d_{g_{\tilde{\varepsilon}}}(\tilde{P}_i,\tilde{x})}\right)^{\sigma} \tag{6.6}$$

for all $\tilde{x}\ne\tilde{P}_i$, $i=1,\dots,k$. There also holds that

$$\begin{aligned} c_0 &= O\left(\|h_{\tilde{\varepsilon}}\|_{\star\star,\sigma}\right) + o\left(\|\phi_{\tilde{\varepsilon}}\|_{\star,\sigma'}\right) \\ &= o(1) \ . \end{aligned} \tag{6.7}$$

Let $\hat{\phi}_{\tilde{\varepsilon},i} = \phi_{\tilde{\varepsilon}}\circ\exp_{\tilde{P}_i}$, $i=1,\dots,k$. Then $\exp^\star_{\tilde{P}_i} g_{\tilde{\varepsilon}}\to\xi$ in $C^2_{loc}(\mathbb{R}^3)$ as $\tilde{\varepsilon}\to 0$, where ξ is the Euclidean metric, and by standard elliptic theory, $\hat{\phi}_{\tilde{\varepsilon},i}\to\hat{\phi}$ in $C^2_{loc}(\mathbb{R}^3)$ as $\tilde{\varepsilon}\to 0$, where

$$\frac{\hbar^2}{2m_0^2}\Delta\hat{\phi}_i = 5U_{\Lambda,0}^4\hat{\phi}_i \ ,$$

$|\hat{\phi}_i| \le C$, $|x|^\sigma |\hat{\phi}_i| \le C$ for all x, and

$$U_{\Lambda,0} = K_0 \Lambda^{1/2}\left(\Lambda^2 + |x|^2/4\right)^{-1/2} \ .$$

By Bianchi-Egnell [3] this implies that $\hat{\phi}_i = \alpha_i \frac{\partial U_{\Lambda,0}}{\partial \Lambda}$ since by the G_k-invariance, $\hat{\phi}_i$ is even. Still by the G_k-invariance, $\alpha_1 = \cdots = \alpha_k$. Let α be the common value to the α_i's. By (6.6) and since $\|\phi_{\tilde{\varepsilon}}\|_{\star,\sigma'} = 1$, there exist $R > 0$ and $\delta > 0$ such that

$$\|\phi_{\tilde{\varepsilon}}\|_{L^\infty(B_0(R))} \ge \delta \ . \tag{6.8}$$

There holds $\int_{\frac{1}{\tilde{\varepsilon}}S^3} \tilde{Z}\phi_{\tilde{\varepsilon}} dv_{g_{\tilde{\varepsilon}}} = 0$ and we have that

$$\begin{aligned}\int_{\frac{1}{\tilde{\varepsilon}}S^3} \tilde{Z}\phi_{\tilde{\varepsilon}} dv_{g_{\tilde{\varepsilon}}} &\to k\alpha \int_{\mathbb{R}^3} \left(\Delta \frac{\partial U_{\Lambda,0}}{\partial \Lambda}\right) \frac{\partial U_{\Lambda,0}}{\partial \Lambda} dx \\ &\ge \delta\alpha \ ,\end{aligned}$$

where $\delta > 0$. Hence $\alpha = 0$ and we get a contradiction with (6.8). This proves that $\|\phi_{\tilde{\varepsilon}}\|_{\star,\sigma'} \to 0$ as $\tilde{\varepsilon} \to 0$. Noting that

$$\|\phi_{\tilde{\varepsilon}}\|_{\star,\sigma} \le C\left(\|\phi_{\tilde{\varepsilon}}\|_{\star,\sigma'} + \|h_{\tilde{\varepsilon}}\|_{\star\star,\sigma} + |c_0|\right) \ ,$$

we then get with (6.7) that $\|\phi_{\tilde{\varepsilon}}\|_{\star,\sigma} \to 0$ as $\tilde{\varepsilon} \to 0$. This ends the proof of the lemma. □

At this point we define $R_{1,\tilde{\varepsilon}}$, $R_{2,\tilde{\varepsilon}}$, and $R_{\tilde{\varepsilon}}$ by

$$\begin{aligned} R_{1,\tilde{\varepsilon}} &= \tilde{\mathcal{W}}_\varepsilon^5 - \frac{\hbar^2}{2m_0^2}\Delta_{g_{\tilde{\varepsilon}}}\tilde{\mathcal{W}}_\varepsilon - \omega^2\tilde{\varepsilon}^2\tilde{\mathcal{W}}_\varepsilon \ , \\ R_{2,\tilde{\varepsilon}} &= -q\tilde{\varepsilon}^2\overline{\Phi(\mathcal{W}_\varepsilon)}\tilde{\mathcal{W}}_\varepsilon \ , \text{ and } R_{\tilde{\varepsilon}} = R_{1,\tilde{\varepsilon}} + R_{2,\tilde{\varepsilon}} \ . \end{aligned} \tag{6.9}$$

Thanks to the asymptotic expansion in Lemma 5.1, noting that $|\Phi(\mathcal{W}_\varepsilon)| = O\left(\varepsilon^\sigma\right)$ for any $0 < \sigma < 1$, we get that $\|R_{i,\tilde{\varepsilon}}\|_{\star\star,\sigma} \le C\tilde{\varepsilon}$ and $\|D_\Lambda R_{i,\tilde{\varepsilon}}\|_{\star\star,\sigma} \le C\tilde{\varepsilon}$ for all $i = 1, 2$. Following almost word by word the arguments in Rey and Wei [22], see also Del Pino, Felmer and Musso [7], we get with Lemma 6.1 that there exist $\tilde{\varepsilon}_0 > 0$ and $C > 0$ such that

(R1) for any $\tilde{\varepsilon} \in (0, \tilde{\varepsilon}_0)$ and any G_k-invariant function $h \in L^\infty\left(\frac{1}{\tilde{\varepsilon}}S^3\right)$, (6.4) has a unique G_k-invariant solution $\phi = \mathcal{L}_{\tilde{\varepsilon}}(h)$ with $\|\phi\|_{\star,\sigma} \le C\|h\|_{\star\star,\sigma}$. Moreover, the map $\mathcal{L}_{\tilde{\varepsilon}}$ is C^1 w.r.t. Λ and $\|D_\Lambda \mathcal{L}_{\tilde{\varepsilon}}(h)\|_{\star,\sigma} \le C\|h\|_{\star\star,\sigma}$.

(R2) for any $\tilde{\varepsilon} \in (0, \tilde{\varepsilon}_0)$, (6.10) has a unique G_k-invariant solution $\tilde{\phi} = \tilde{\phi}_{\tilde{\varepsilon}}$ with $\|\tilde{\phi}_{\tilde{\varepsilon}}\|_{\star,\sigma} \le C\tilde{\varepsilon}$ and $\|D_\Lambda\tilde{\phi}_{\tilde{\varepsilon}}\|_{\star,\sigma} \le C\tilde{\varepsilon}$, where (6.10) is the problem

$$\begin{cases} \frac{\hbar^2}{2m_0^2}\Delta_{g_{\tilde{\varepsilon}}}(\hat{W}_\varepsilon + \tilde{\phi}) + \tilde{\varepsilon}^2\omega^2(\hat{W}_\varepsilon + \tilde{\phi}) \\ \quad +q\tilde{\varepsilon}^2\overline{\Phi(\mathcal{W}_\varepsilon + K_\varepsilon + \phi)}(\hat{W}_\varepsilon + \tilde{\phi}) = (\hat{W}_\varepsilon + \tilde{\phi})^5 + c_0\tilde{Z} \\ \int_{\frac{1}{\tilde{\varepsilon}}S^3} \tilde{Z}\tilde{\phi} dv_{g_{\tilde{\varepsilon}}} = 0 \ , \end{cases} \tag{6.10}$$

$\hat{W}_\varepsilon = \tilde{\mathcal{W}}_\varepsilon + \mathcal{L}_{\tilde{\varepsilon}}(R_{\tilde{\varepsilon}})$, $c_0 \in \mathbb{R}$,

$$\overline{\Phi(\mathcal{W}_\varepsilon + K_\varepsilon + \phi)} = \Phi(\mathcal{W}_\varepsilon + K_\varepsilon + \phi) \circ f_{\tilde{\varepsilon}} \ ,$$

and $\tilde{K}_\varepsilon = \mathcal{L}_{\tilde{\varepsilon}}(R_{\tilde{\varepsilon}})$.

We get (R1) by an application of the Fredholm theorem, and (R2) by an application of the fixed point theorem (and we assume σ is not to small). Now we let

$$\hat{\mathcal{U}}_\varepsilon = \tilde{\mathcal{W}}_\varepsilon + \mathcal{L}_{\tilde{\varepsilon}}(R_{\tilde{\varepsilon}}) + \tilde{\phi}_{\tilde{\varepsilon}} \ . \tag{6.11}$$

There holds by (R1) that $\|\mathcal{L}_{\tilde{\varepsilon}}(R_{\tilde{\varepsilon}})\|_{\star,\sigma} \le C\tilde{\varepsilon}$. Thus $\hat{\mathcal{U}}_\varepsilon > 0$. We define $\rho : \mathbb{R}^+ \to \mathbb{R}$ by

$$\begin{aligned}\rho(\Lambda) = \frac{\hbar^2}{4m_0^2}\int_{\frac{1}{\tilde{\varepsilon}}S^3} |\nabla\hat{\mathcal{U}}_\varepsilon|^2 dv_{g_{\tilde{\varepsilon}}} + \frac{\omega^2\tilde{\varepsilon}^2}{2}\int_{\frac{1}{\tilde{\varepsilon}}S^3} \hat{\mathcal{U}}_\varepsilon^2 dv_{g_{\tilde{\varepsilon}}} \\ + \frac{q\tilde{\varepsilon}^2}{4}\int_{\frac{1}{\tilde{\varepsilon}}S^3} \overline{\Phi(\mathcal{U}_\varepsilon)}\hat{\mathcal{U}}_\varepsilon^2 dv_{g_{\tilde{\varepsilon}}} - \frac{1}{6}\int_{\frac{1}{\tilde{\varepsilon}}S^3} \hat{\mathcal{U}}_\varepsilon^6 dv_{g_{\tilde{\varepsilon}}} \ ,\end{aligned} \tag{6.12}$$

where $\mathcal{U}_\varepsilon$ is such that $\tilde{\mathcal{U}}_\varepsilon = \hat{\mathcal{U}}_\varepsilon$, namely such that $\hat{\mathcal{U}}_\varepsilon$ is obtained from $\mathcal{U}_\varepsilon$ by the $\sim$-procedure. The following proposition holds true.

Proposition 6.1. *The function* $\hat{\mathcal{U}}_\varepsilon > 0$ *is a solution of*

$$\frac{\hbar^2}{2m_0^2}\Delta_{g_{\tilde{\varepsilon}}}\tilde{\mathcal{U}} + \tilde{\varepsilon}^2\omega^2\tilde{\mathcal{U}} + q\tilde{\varepsilon}^2\overline{\Phi(\mathcal{U})}\tilde{\mathcal{U}} = \tilde{\mathcal{U}}^5 \tag{6.13}$$

in $\frac{1}{\tilde{\varepsilon}}S^3$ *if and only if* Λ *is a critical point of* ρ.

Proof of Proposition 6.1. We define $I_{\tilde{\varepsilon}}$ by

$$\begin{aligned}I_{\tilde{\varepsilon}}(\tilde{U}) = \int_{\frac{1}{\tilde{\varepsilon}}S^3}\left(\frac{\hbar^2}{4m_0^2}|\nabla\tilde{U}|^2 + \frac{\omega^2\tilde{\varepsilon}^2}{2}\tilde{U}^2\right)dv_{g_{\tilde{\varepsilon}}} + \frac{q\tilde{\varepsilon}^2}{4}\int_{\frac{1}{\tilde{\varepsilon}}S^3}\overline{\Phi(U)}\tilde{U}^2 dv_{g_{\tilde{\varepsilon}}} \\ - \frac{1}{6}\int_{\frac{1}{\tilde{\varepsilon}}S^3}(\tilde{U}^+)^6 dv_{g_{\tilde{\varepsilon}}} \ .\end{aligned}$$

Then $I_{\tilde{\varepsilon}}(\tilde{U}) = I_6(U)$ and there holds that $\hat{\mathcal{U}}_\varepsilon$ is a solution of (6.13) if and only if $\mathcal{U}_\varepsilon = \mathcal{W}_\varepsilon + K_\varepsilon + \phi_{\tilde{\varepsilon}}$ is a solution of

$$\frac{\hbar^2}{2m_0^2}\Delta_g U + \omega^2 U + q\Phi(U)U = U^5 \ .$$

This is in turn equivalent to $c_0 = 0$, where c_0 is as in (6.10), which is again equivalent to $I'_{\tilde{\varepsilon}}(\hat{\mathcal{U}}_\varepsilon).(\tilde{Y}) = 0$ since $I'_{\tilde{\varepsilon}}(\hat{\mathcal{U}}_\varepsilon).(\tilde{Y}) = c_0\langle\tilde{Y},\tilde{Z}\rangle$ and $\langle\tilde{Y},\tilde{Z}\rangle = \gamma_0 + o(1)$, where $\gamma_0 > 0$. Independently, there holds that $\rho'(\Lambda) = 0$ if and only if

$$I'_{\tilde{\varepsilon}}\left(\hat{\mathcal{U}}_\varepsilon\right).\left(\tilde{Y} + \frac{\partial\Psi_\varepsilon}{\partial\Lambda}\right) = 0 \ ,$$

where $\Psi_\varepsilon = \tilde{K}_\varepsilon + \tilde{\phi}_{\tilde{\varepsilon}}$, while if we let $y_0 = \frac{\partial\Psi_\varepsilon}{\partial\Lambda}$, then $\|y_0\|_{\star,\sigma} \le C\varepsilon$. We write that $y_0 = y_0' + a\tilde{Y}$, where $(y_0',\tilde{Y})_{\tilde{\varepsilon}} = 0$ and $(\cdot,\cdot)_{\tilde{\varepsilon}}$ is the scalar product associated to $\frac{\hbar^2}{2m_0^2}\Delta_{g_{\tilde{\varepsilon}}} + \tilde{\varepsilon}^2\omega^2$. Then $\rho'(\Lambda) = 0$ if and only if

$$(1+a)I'_{\tilde{\varepsilon}}(\hat{\mathcal{U}}_\varepsilon).(\tilde{Y}) = 0$$

since $\langle y_0',\tilde{Z}\rangle = (y_0',\tilde{Y})_{\tilde{\varepsilon}}$. There holds that $(y_0,\tilde{Y})_{\tilde{\varepsilon}} = o(1)$ and this implies that $a = o(1)$. This ends the proof of the proposition. □

Now, thanks to Proposition 6.1, we are in position to prove point (ii) in Theorem 0.2. This is the subject of what follows.

Proof of the second part of Theorem 0.2. Given $\sigma \in (0,1)$ sufficiently close to 1, we compute

$$\begin{aligned}
\int_{\frac{1}{\tilde\varepsilon}S^3} \overline{\Phi(\mathcal{U}_\varepsilon)}\tilde{\mathcal{U}}_\varepsilon^2 dv_{g_{\tilde\varepsilon}} &= \int_{\frac{1}{\tilde\varepsilon}S^3} \overline{\Phi(\mathcal{W}_\varepsilon)}\tilde{\mathcal{W}}_\varepsilon^2 dv_{g_{\tilde\varepsilon}} + o(1)\ ,\\
\int_{\frac{1}{\tilde\varepsilon}S^3} \tilde{\mathcal{U}}_\varepsilon^6 dv_{g_{\tilde\varepsilon}} &= \int_{\frac{1}{\tilde\varepsilon}S^3} \tilde{\mathcal{W}}_\varepsilon^6 dv_{g_{\tilde\varepsilon}} + 6\int_{\frac{1}{\tilde\varepsilon}S^3} \tilde{\mathcal{W}}_\varepsilon^5\tilde\psi_{\tilde\varepsilon} dv_{g_{\tilde\varepsilon}}\\
&\quad +15\int_{\frac{1}{\tilde\varepsilon}S^3} \tilde{\mathcal{W}}_\varepsilon^4\tilde\psi_{\tilde\varepsilon}^2 dv_{g_{\tilde\varepsilon}} + o\left(\tilde\varepsilon^2\right)\ ,\\
\int_{\frac{1}{\tilde\varepsilon}S^3} \tilde{\mathcal{U}}_\varepsilon^5\tilde\psi_{\tilde\varepsilon} dv_{g_{\tilde\varepsilon}} &= \int_{\frac{1}{\tilde\varepsilon}S^3} \tilde{\mathcal{W}}_\varepsilon^5\tilde\psi_{\tilde\varepsilon} dv_{g_{\tilde\varepsilon}} + 5\int_{\frac{1}{\tilde\varepsilon}S^3} \tilde{\mathcal{W}}_\varepsilon^4\tilde\psi_{\tilde\varepsilon}^2 dv_{g_{\tilde\varepsilon}} + o\left(\tilde\varepsilon^2\right)\ ,\\
\int_{\frac{1}{\tilde\varepsilon}S^3} \overline{\Phi(\mathcal{U}_\varepsilon)}\tilde{\mathcal{U}}_\varepsilon\tilde\psi_{\tilde\varepsilon} dv_{g_{\tilde\varepsilon}} &= \int_{\frac{1}{\tilde\varepsilon}S^3} \overline{\Phi(\mathcal{W}_\varepsilon)}\tilde{\mathcal{U}}_\varepsilon\tilde\psi_{\tilde\varepsilon} dv_{g_{\tilde\varepsilon}} + o(1)\ ,
\end{aligned}$$

where $\mathcal{U}_\varepsilon = \mathcal{W}_\varepsilon + \psi_{\tilde\varepsilon}$ and $\psi_{\tilde\varepsilon} = K_\varepsilon + \phi_{\tilde\varepsilon}$. Then

$$\rho(\Lambda) = I_6(\mathcal{W}_\varepsilon) - \frac{1}{2}\int_{\frac{1}{\tilde\varepsilon}S^3} R_{1,\tilde\varepsilon}\tilde\psi_{\tilde\varepsilon} dv_{g_{\tilde\varepsilon}} + \frac{1}{2}\int_{\frac{1}{\tilde\varepsilon}S^3} R_{2,\tilde\varepsilon}\tilde\psi_{\tilde\varepsilon} dv_{g_{\tilde\varepsilon}} + o\left(\tilde\varepsilon^2\right)\ ,$$

where $R_{1,\tilde\varepsilon}$ and $R_{2,\tilde\varepsilon}$ are as in (6.9). By our choices of ε, $\tilde\varepsilon$, and since $\frac{1}{C} \le \Lambda \le C$ for $C > 1$ fixed, we then get by direct computations that $\rho(\Lambda) = I_6(\mathcal{W}_\varepsilon) + o\left(\varepsilon^2\right)$. Assume now that $\Theta_k > 0$, where Θ_k is as in (6.1). Then, by Lemma 5.2,

$$\begin{aligned}
\rho(\Lambda) &= A_{0,k} + A_{1,k}\tilde\varepsilon^2\Lambda + A_{2,k}(\omega)q^2\tilde\varepsilon^2\Lambda^2 + A_{3,k}(\omega)\tilde\varepsilon^2\Lambda^2 + o\left(\tilde\varepsilon^2\right)\Lambda^2\\
&= A_{0,k} + A_{1,k}\tilde\varepsilon^2\Lambda + A_{2,k}(\omega_k)\Theta_k\tilde\varepsilon^2\Lambda^2 + o\left(\tilde\varepsilon^2\right)\Lambda^2
\end{aligned}$$

and since $A_{1,k} < 0$ and $\Theta_k > 0$, ρ has an absolute minimum Λ_ω in $\left(\frac{1}{C}, C\right)$ for $C \gg 1$ when $\omega \in (\omega_k - \delta, \omega_k)$ and $0 < \delta \ll 1$. Pick any sequence $(\omega_\alpha)_\alpha$ of phases in $(\omega_k - \delta, \omega_k)$ such that $\omega_\alpha \to \omega_k$ as $\alpha \to +\infty$. By Proposition 6.1 we then get that there is an associated sequence $(\mathcal{U}_\alpha, \Phi(\mathcal{U}_\alpha))$ of solutions of (0.2) with $\omega = \omega_\alpha$, where $\mathcal{U}_\alpha = \mathcal{U}_{\varepsilon_\alpha}$ and $\varepsilon_\alpha = \Lambda_{\omega_\alpha}\eta_k(\omega_\alpha)$, such that $(\mathcal{U}_\alpha)_\alpha$ is a k-spikes type solution of the first equation in (0.2). In particular, $\|\mathcal{U}_\alpha\|_{L^\infty} \to +\infty$ as $\alpha \to +\infty$. Similarly, if we assume that $\Theta_k < 0$, then by Lemma 5.2,

$$\begin{aligned}
\rho(\Lambda) &= A_{0,k} - A_{1,k}\tilde\varepsilon^2\Lambda + A_{2,k}(\omega)q^2\tilde\varepsilon^2\Lambda^2 + A_{3,k}(\omega)\tilde\varepsilon^2\Lambda^2 + o\left(\tilde\varepsilon^2\right)\Lambda^2\\
&= A_{0,k} - A_{1,k}\tilde\varepsilon^2\Lambda + A_{2,k}(\omega_k)\Theta_k\tilde\varepsilon^2\Lambda^2 + o\left(\tilde\varepsilon^2\right)\Lambda^2
\end{aligned}$$

and ρ has an absolute maximum in $\left(\frac{1}{C}, C\right)$ for $C \gg 1$ when $\omega \in (\omega_k, \omega_k + \delta)$ and $0 < \delta \ll 1$. Pick any sequence $(\omega_\alpha)_\alpha$ of phases in $(\omega_k, \omega_k + \delta)$ such that $\omega_\alpha \to \omega_k$ as $\alpha \to +\infty$. By Proposition 6.1 we then get that there is an associated sequence $(\mathcal{U}_\alpha, \Phi(\mathcal{U}_\alpha))$ of solutions of (0.2) with $\omega = \omega_\alpha$, where $\mathcal{U}_\alpha = \mathcal{U}_{\varepsilon_\alpha}$ and $\varepsilon_\alpha = -\Lambda_{\omega_\alpha}\eta_k(\omega_\alpha)$, such that $(\mathcal{U}_\alpha)_\alpha$ is a k-spikes type solution of the first equation in (0.2). In particular, $\|\mathcal{U}_\alpha\|_{L^\infty} \to +\infty$ as $\alpha \to +\infty$. We know that $\Theta_k > 0$ for $k = 1$ and, by Lemma 5.3, that $\Theta_k < 0$ for $k \gg 1$. This ends the proof of the second part of Theorem 0.2. □

As a remark it can be noted that we obtain the existence of solutions to (0.2) for ω sufficiently close to the ω_k's with $\omega < \omega_k$ if $\Theta_k > 0$ and $\omega > \omega_k$ if $\Theta_k > 0$.

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