CLUSTERED BUBBLES FOR AN ELLIPTIC PROBLEM WITH CRITICAL GROWTH

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Abstract. We consider the following nonlinear Schrödinger equation in $\mathbb{R}^3$

$$\begin{cases} -\Delta u + \lambda V(|y|)u = u^5, \ u > 0 & \text{in } B_R \\ u = 0, & \text{on } \partial B_R \end{cases}$$

where λ is large. Ambrosetti, Malchiodi and Ni [2] established the existence of one layered solution if $M(r) = r^2 V(r)$ has a nondegenerate critical point $r = r_0 > 0$. Brezis and Peletier [12] asked if there are more and more (radial or nonradial) solutions as λ increases. We partially solve this question by constructing solutions with clustered bubbles at the origin and a layer at $r = r_0$, provided that λ is large and is away from certain *resonant numbers.*

1. Introduction

In this paper, we consider the existence of positive solutions with clustered bubbles for the following problem:

$$\begin{cases} -\Delta u + \lambda V(|y|)u = u^5, \ u > 0 & \text{in } B_R \\ u = 0, & \text{on } \partial B_R, \end{cases} \tag{1.1}$$

where B_R is the ball centered at the origin with radius R in $\mathbb{R}^3$, $V(|y|)$ is smooth in B_R and $V(|y|) \geq V_0 > 0$ in B_R.

Problem (1.1) involves critical growth. By using Pohozaev identity, one can show that if $(r^2V(r))' \geq 0$ in $(0, R)$, (1.1) does not have any solution. Concerning the existence result for (1.1), Ambrosetti, Malchiodi and Ni ([2], [3]) showed that if $r^2V(r)$ has a non-degenerate critical point $r_0 \in (0, R)$, then for large $\lambda > 0$, (1.1) has a radial solution u_λ concentrating as $\lambda \to +\infty$ at r_0. (See also [8] and [10].) More precisely,

$$u_\lambda(r) \sim \big(V(r_0)\lambda\big)^{1/4} w\Big(\sqrt{V(r_0)\lambda}(r - r_0)\Big),$$

where w is the unique solution of

$$w'' - w + w^5 = 0\,, \ u > 0, \quad w'(0) = 0, \quad w(\pm\infty) = 0.$$

One of the open problems raised by Brezis and Peletier in [12] is that if $r^2V(r)$ has a critical point $r_0 > 0$, then the number of the solutions for (1.1) is unbounded as $\lambda \to +\infty$. See *Open Question 8.5* of [12]. In this paper, we will partially prove this claim.

Problem (1.1) is related to the following elliptic problem on the spherical caps:

$$\begin{cases} -\Delta_{S^3} u = u^5 + \bar\lambda u, \ u > 0, & \text{in } B', \\ u = 0, & \text{on } \partial B' \end{cases} \tag{1.2}$$

where Δ_{S^3} is the Laplace-Beltrami operator in the unit sphere S^3, and B' is a spherical caps in S^3, that is, B' is the set of points in S^3, such that its geodesic distance to the north pole in S^3 is less than a constant θ^*. By using the stereographic projection with vertex at the south pole of S^3, (1.2) can be transformed into

$$\begin{cases} -\Delta u + \lambda \frac{1}{(1+|y|^2)^2} u = u^5, \ v > 0, & \text{in } B_R \\ u = 0, & \text{on } \partial B_R, \end{cases} \tag{1.3}$$

where $R = \tan\frac{\theta^*}{2}$, $\lambda = -4\bar\lambda - 3$. Note that $R > 1$ if $\theta^* \in (\frac{\pi}{2}, \pi)$, and $\lambda \to +\infty$ as $\bar\lambda \to -\infty$.

Note that $\frac{r^2}{(1+r^2)^2}$ has a nondegenerate critical point $r_0 = 1$, and $\left(\frac{r^2}{(1+r^2)^2}\right)' > 0$ in $(0, 1)$. So, by Pohozaev identity, (1.3) has no solution if $R < 1$. On the other hand, it follows from the result in [2] that (1.3) has a radial solution for large λ if $R > 1$. Another open problem raised in [12] is that (1.3) has non-radial solution if $R > 1$. See *Remark 1.2* and *Remark 1.3* in [12]. In this paper, we will also partially give a positive result for this open problem. For the study of problem (1.2), we refer to [5], [6], [7], [11], [12], [14], [34] and the references therein.

The aim of this paper is to show that the existence of a layer solution near a non-degenerate critical point $r_0 \in (0, R)$ of $r^2 V(r)$ will create new type of solutions for (1.1) with multiple bubbles clustering near the origin. Since we aim at constructing solutions with bubbles near the origin, we first define an approximate solution for (1.1) in the following way.

Let $PU_{\bar x,\bar\mu}$ be the solution of

$$\begin{cases} -\Delta PU_{\bar x,\bar\mu} + \lambda V(y) PU_{\bar x,\bar\mu} = U^5_{\bar x,\bar\mu}, & \text{in } B_R, \\ PU_{\bar x,\bar\mu} = 0, & \text{on } \partial B_R, \end{cases}$$

where, for any $\bar x \in \mathbb{R}^3$, $\bar\mu > 0$,

$$U_{\bar x,\bar\mu}(y) = \frac{c_0 \bar\mu^{1/2}}{(1 + \bar\mu^2 |y - \bar x|^2)^{1/2}}, \quad c_0 = 3^{1/4}. \tag{1.4}$$

Note that $U_{\bar x,\bar\mu}$ satisfies $-\Delta U_{\bar x,\bar\mu} = U^5_{\bar x,\bar\mu}$ in $\mathbb{R}^3$. In this paper, we will use the following notation: $U = U_{0,1}$.

For any $u, v \in H^1_0(B_R)$, we define

$$\langle u, v\rangle = \int_{B_R} \big(\nabla u \nabla v + \lambda V(|y|) uv\big)\, dy, \quad \|u\| = \langle u, u\rangle^{1/2}.$$

For any integer $k \geq 2$, let H_s be the subspace of $H^1_0(B_R)$, consisting of functions u satisfying

$$u(y_1,y_2,-y_3)=u(y_1,y_2,y_3),\quad u\big(r\cos(\tilde\theta+\frac{2\pi}{k}),r\sin(\tilde\theta+\frac{2\pi}{k}),y_3\big)=u(r\cos\tilde\theta,r\sin\tilde\theta,y_3),\tag{1.5}$$

where $y_1=r\cos\tilde\theta, y_2=r\sin\tilde\theta$.

Let $\bar\mu_m$, $m=0,1,\cdots$, be all the eigenvalues of $-\Delta_{S^2}$ in the symmetric subspace of $L^2(S^2)$, consisting of functions $\Theta(y)$, $y\in S^2$, satisfying (1.5).

The main result of this paper is the following:

Theorem 1.1. *Suppose that $M(r)=r^2V(r)$ has a non-degenerate critical point at $r_0\in(0,R)$ and*

$$M''(r_0)\neq-\bar\mu_mV(r_0)\quad for\ m=1,\cdots.\tag{1.6}$$

Let u_λ be the solution of (1.1) with a layer near r_0. Then, for any integer $k\geq2$, there exists $\lambda_0>0$, such that for all $\lambda>\lambda_0$ satisfying the gap condition

$$\big|8-\frac{\bar\mu_m}{\lambda r_0V(r_0)}\big|\geq\frac{c}{\sqrt\lambda},\ \forall\, m=0,1,2\cdots,\tag{1.7}$$

where $c>0$ is any small fixed constant, (1.1) has a solution $u\in H_s$, which has the form

$$u=\sum_{j=1}^kPU_{x_{\lambda,j},\bar\mu_\lambda}+u_\lambda+\omega_{\lambda,k},$$

satisfying that as $\lambda\to+\infty$,

(i) $x_{\lambda,j}=\big(r_\lambda\cos\frac{2(j-1)\pi}{k},r_\lambda\sin\frac{2(j-1)\pi}{k},0\big),\quad r_\lambda\sqrt\lambda\geq a>0,\ \ |r_\lambda|\to0,\ \ j=1,\cdots,k;$

(ii) $\bar\mu_\lambda\to+\infty$;

(iii) $\|\omega_{\lambda,k}\|=O\big(\frac{1}{\bar\mu_\lambda^{\frac12+\tau}}\big)$ *for some constant $\tau>0$.*

Let us make a few remarks on Theorem 1.1.

Remark 1.2. *By working in the symmetric space H_s, the set of eigenvalues $\bar\mu_m$ is a subset of all the eigenvalues of $-\Delta_{S^2}$ in the whole space. By simple computations, it is easy to see*

$$\bar\mu_m=2m(2m+1),m=0,1,....\tag{1.8}$$

(See Section 10.3 and 10.6 of [33]*.) In particular, $\bar\mu_m\geq6$ for $m\geq1$. This is important when we apply the result in Theorem 1.1 to the elliptic problem in a spherical cap. See the discussion before Corollary 1.6.*

Remark 1.3. *By Lemma 3.1 of* [37]*, we have*

$$u_\lambda(t)=e^{-(\int_t^{r_0}\sqrt{V(\tau)}\,d\tau+\gamma_\lambda(t))\sqrt\lambda},\tag{1.9}$$

where $\gamma_\lambda(t)\to0$ uniformly for $t\leq r_0-\theta$, as $\lambda\to\infty$, for any $\theta>0$. In [37]*, we prove that for all large $\lambda>0$, (1.1) has a radial solution with a layer near r_0 and a bubble exactly in the origin. The main reason that we can not prove Theorem 1.1 for all large λ is that the linear operator of the layer solution u_λ has many small eigenvalues in the space H_s. More* *precisely, condition (1.7) represents a kind of* resonance *phenomena for layered solutions, which has appeared in many other problems. See e.g.* [22] *and* [26] *and the references therein.*

Remark 1.4. *Theorem 1.1 also holds in the whole space, i.e., for the following nonlinear Schrödinger equation in* $\mathbb{R}^3$*:*

$$\begin{cases} -\Delta u + \lambda V(|y|)u = u^5, \ u > 0 \quad in\ \mathbb{R}^3 \\ u \in H^1(\mathbb{R}^3). \end{cases} \tag{1.10}$$

We remark that for Schrödinger equation with critical exponent,

$$\begin{cases} -\Delta u + \lambda V(x)u = u^p, \ u > 0 \quad in\ \mathbb{R}^n \\ u \in H^1(\mathbb{R}^n). \end{cases} \tag{1.11}$$

where $p = \frac{n+2}{n-2}$*, there are very few results. Benci and Cerami in* [9] *proved the existence of one solution* $\lambda\|V\|_{L^{\frac{n}{2}}}$ *is small. On the other hand, it is proved by Cingonali and Pistoia in* [15] *that there are no single bubble solutions when* $n \geq 5$*, as* $\lambda \to +\infty$*. Results in the nearly critical case are contained in* [30, 28]*: setting* $p = \frac{n+2}{n-2} + \delta$*, they find multiple solutions concentrating as* $\delta \to 0^+$*, at a critical point of* V *with negative value for* $n \geq 7$*. There* $\lambda\|V\|_{L^{\frac{n}{2}}}$ *is also required to be globally small, so that in particular the maximum principle holds.*

On the other hand, for Schrödinger equation with subcritical growth, many results on existence of concentrating solutions have been proved, under various assumptions on the potential or the nonlinearity, with the aid of perturbation or variational methods, lifting non-degeneracy and also allowing the potential to vanish in some region or even be negative somewhere, see for instance [1, 4, 13, 16, 17, 18, 19, 20, 23, 24, 25, 35, 36] *and the references therein. In particular, it is proved that there is no solution with clustering peaks near a minimum point of* V*. See* [25]*. In Theorem 1.1, no condition is imposed on* V *at* $y = 0$*. In the present situation, it is the layer solution that creates new solutions with clustering bubbles near the origin.*

Suppose that $M(r) = r^2V(r)$ has a non-degenerate critical point $r_0 \in (0, R)$. If $M(r)$ is a local maximum point of $M(r)$, then, (1.1) has (radially symmetric) solutions with multiple layers clustering near r_0, [27]. Thus, (1.1) has more and more solution as $\lambda \to +\infty$. On the other hand, if r_0 is a local minimum point of $M(r)$, then $M''(r_0) > 0$, which gives $M''(r_0) \neq -\bar{\mu}_m V(r_0)$ for any nonnegative integer m. By Theorem 1.1, (1.1) has solutions with multiple bubbles clustering near the origin for some large λ away from certain resonant intervals. So we have the following result which partially answers *Open Question 8.5* raised by Brezis and Peletier in [12].

Corollary 1.5. *Suppose that* $M(r) = r^2V(r)$ *has a non-degenerate critical point* $r_0 \in (0, R)$*. Then, for any integer* $k \geq 1$*, there exists* $\lambda_0 > 0$ *such that* (1.1) *has at least* k *solutions, provided that* $\lambda > \lambda_0$ *satisfies (1.7).*

For the elliptic problem (1.2) on the spherical cap, $V(r) = \frac{1}{(1+r^2)^2}$. Direct calculations show that $M(r) = \frac{r^2}{(1+r^2)^2}$ has a critical point at $r_0 = 1$ with $M''(1) = -\frac{1}{2}$. By Remark 1.2, $\bar\mu_m \geq 6$, $m = 1, 2, \cdots$. So, we have $M''(1) \neq -\bar\mu_m V(r_0)$, $m = 0, 1, 2, \cdots$. An application of Theorem 1.1 is the following result that partially answers another question put forward by Brezis and Peletier [12]. (See *Remark 1.2* and *Remark 1.3* of [12].)

Corollary 1.6. *If $\theta^* \in (\pi/2, \pi)$, then* (1.2) *has non-symmetric solutions for $\lambda > \lambda_0$ satisfying (1.7).*

Before we close this section, we outline the proof of Theorem 1.1. The proof uses the so-called "*localized energy method*", which combines varitional techniques and Liapunov-Schmidt reduction method. Firstly, we carry out the reduction argument in the symmetric space H_s. To achieve this goal, we need to arrange the location of the peaks x_i in a symmetric way. Thus, we let

$$x = (x_1, \cdots, x_k), \quad x_j = (r\cos\frac{2(j-1)\pi}{k}, r\sin\frac{2(j-1)\pi}{k}, 0), \;\; j = 1, \cdots, k,$$

and

$$\mu = (\bar\mu, \cdots, \bar\mu).$$

Define

$$M_{r,\bar\mu} = \Big\{(r, \bar\mu) : \; d_1\lambda^{-1/2} \leq r \leq d_\lambda, \;\; \frac{b_0 g_\lambda^2(r)\lambda}{u_\lambda^2(r)} \leq \bar\mu \leq \frac{b_1 g_\lambda^2(r)\lambda}{u_\lambda^2(r)}\Big\},$$

where $b_1 > b_0$, and d_1 are some constants, and $d_\lambda \to 0$ as $\lambda \to +\infty$, and $g_\lambda(r)$ is the function defined in (3.4).

Let

$$I(u) = \frac{1}{2}\int_{B_R}\big(|\nabla u|^2 + \lambda V(|y|)u^2\big)\,dy - \frac{1}{6}\int_{B_R}|u|^6\,dy, \quad u \in H_0^1(B_R),$$

and

$$J_\lambda(r, \bar\mu, \omega) = I\Big(\sum_{j=1}^k PU_{x_j,\bar\mu} + u_\lambda + \omega\Big), \quad (r, \bar\mu) \in M_{r,\bar\mu}, \; \omega \in E_{r,\bar\mu},$$

where

$$E_{r,\bar\mu} = \Big\{\omega : \; \omega \in H_s, \;\; \big\langle\omega, \frac{\partial(PU_{x_1,\bar\mu})}{\partial\bar\mu}\big\rangle = \big\langle\omega, \frac{\partial(PU_{x_1,\bar\mu})}{\partial x_h}\big\rangle = 0, \; h = 1, 2, 3.\Big\}$$

In section 2, we will prove that there exists a large $\lambda_0 > 0$, such that for any $\lambda > \lambda_0$ satisfying (1.7), there is a C^1 map $\omega_{r,\bar\mu}$ from $M_{r,\bar\mu}$ to H_s, satisfying $\omega_{r,\bar\mu} \in E_{r,\bar\mu}$, and

$$\frac{\partial J_\lambda(r, \bar\mu, \omega_{r,\bar\mu})}{\partial\omega} = A\frac{\partial(PU_{x_1,\bar\mu})}{\partial\bar\mu} + \sum_{h=1}^3 B_h\frac{\partial(PU_{x_1,\bar\mu})}{\partial x_h},$$

for some constants A and B_h. Next, we will choose $(r,\bar\mu)\in M_{r,\bar\mu}$, such that corresponding to this $(r,\bar\mu)$, all the constants A and B_h are zero. It is well known that if $(r,\bar\mu)$ is a critical point of the function $K(r,\bar\mu)$ defined as

$$K(r,\bar\mu)=J_\lambda(r,\bar\mu,\omega_{r,\bar\mu}),$$

then, all the constants A and B_h are zero. See for example [21], [31], [32] and [38].

In section 3, we will prove the existence of a critical point of $K(r,\bar\mu)$ in $M_{r,\bar\mu}$ by using a min-max procedure, and thus prove Theorem 1.1.

We put all the calculations for the energy expansion in Appendix A. In Appendix B, we analyze the spectrum for the linear operator of the layer solution, which lays a foundation for the reduction argument.

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2. The reduction

In the section, we will reduce the problem of finding solution for (1.1) to a finite dimension problem.

Proposition 2.1. *Let $\lambda>0$ be large and satisfy (1.7). Then there exists a C^1 map $\omega_{r,\bar\mu}$ from $M_{r,\bar\mu}$ to H_s, satisfying $\omega_{r,\bar\mu}\in E_{r,\bar\mu}$ and*

$$\frac{\partial J_\lambda(r,\bar\mu,\omega_{r,\bar\mu})}{\partial\omega}=A\frac{\partial(PU_{x_1,\bar\mu})}{\partial\bar\mu}+\sum_{h=1}^{3}B_h\frac{\partial(PU_{x_1,\bar\mu})}{\partial x_h},\tag{2.1}$$

for some constants A and B_h. Moreover,

$$\|\omega_{r,\bar\mu}\|\le\frac{C}{\bar\mu^{1/2+\tau}},$$

for some positive constant τ.

Proof. We expand $J_\lambda(r,\bar\mu,\omega)$ at $\omega=0$ as follows:

$$J_\lambda(r,\bar\mu,\omega)=J_\lambda(r,\bar\mu,0)+\langle l_\lambda,\omega\rangle+\frac12\langle Q_\lambda\omega,\omega\rangle+R_\lambda(\omega),$$

where $l_\lambda\in E_{r,\bar\mu}$ satisfying

$$\begin{aligned}\langle l_\lambda,\omega\rangle=&\int_{B_R}\nabla\Big(\sum_{j=1}^{k}PU_{x_j,\bar\mu}+u_\lambda\Big)D\omega\,dy+\lambda\int_{B_R}V(|y|)\Big(\sum_{j=1}^{k}PU_{x_j,\bar\mu}+u_\lambda\Big)\omega\,dy\\&-\int_{B_R}\Big(\sum_{j=1}^{k}PU_{x_j,\bar\mu}+u_\lambda\Big)^5\omega\,dy,\quad\forall\omega\in E_{r,\bar\mu},\end{aligned}\tag{2.2}$$

and Q_λ is a bounded linear map from $E_{r,\bar\mu}$ to $E_{r,\bar\mu}$, satisfying

$$\begin{aligned}\langle Q_\lambda\omega,\eta\rangle=&\int_{B_R}\nabla\omega\nabla\eta\,dy+\lambda\int_{B_R}V(|y|)\omega\eta\,dy\\&-5\int_{B_R}(\sum_{j=1}^k PU_{x_j,\bar\mu}+u_\lambda)^4\omega\eta\,dy,\quad \omega,\ \eta\in E_{r,\bar\mu},\end{aligned}\tag{2.3}$$

and $R_\lambda(\omega)$ collects all the other terms, satisfying

$$R_\lambda^{(j)}(\omega)=O(\|\omega\|^{3-j}),\quad j=0,1,2.$$

Thus, to find a critical point for $J_\lambda(r,\bar\mu,\omega)$ in $E_{r,\bar\mu}$ is equivalent to solving

$$l_\lambda+Q_\lambda\omega+R'_\lambda(\omega)=0.\tag{2.4}$$

By Lemma 2.3, for large λ satisfying (1.7), Q_λ is invertible in $E_{r,\bar\mu}$, and there is a constant $C>0$, such that $\|Q_\lambda^{-1}\|\le C\lambda$. It follows from the implicit function theory that for this λ, there is a C^1 map $\omega_{r,\bar\mu}$ from $M_{r,\bar\mu}$ to H_s, satisfying $\omega_{r,\bar\mu}\in E_{r,\bar\mu}$, and

$$\|\omega_{r,\bar\mu}\|\le C\lambda\|l_\lambda\|.$$

Thus, the result follows from Lemma 2.2.

□

Lemma 2.2. *There is a constant $\tau>0$, such that for any $(r,\bar\mu)\in M_{r,\bar\mu}$,*

$$\|l_\lambda\|=O\Big(\frac{1}{\bar\mu^{1/2+2\tau}}\Big).$$

Proof. We have

$$\begin{aligned}\langle l_\lambda,\omega\rangle=&-\int_{B_R}\sum_{j=1}^k\big((PU_{x_j,\bar\mu})^5-U^5_{x_j,\bar\mu}\big)\omega\\&-\int_{B_R}\Big((\sum_{j=1}^k PU_{x_j,\bar\mu})^5-\sum_{j=1}^k(PU_{x_j,\bar\mu})^5\Big)\omega\\&-\int_{B_R}\Big((\sum_{j=1}^k PU_{x_j,\bar\mu}+u_\lambda)^5-(\sum_{j=1}^k PU_{x_j,\bar\mu})^5-u_\lambda^5\Big)\omega\\=:&\,l_1+l_2+l_3.\end{aligned}$$

It is easy to check that

$$|l_2|\le C\sum_{i\ne j}\varepsilon_{ij}\|\omega\|,$$

where $\varepsilon_{ij}=\frac{1}{\bar\mu|x_i-x_j|}$, and

$$
\begin{aligned}
|l_3| \le & \int_{B_R} \Big(\big(\sum_{j=1}^k PU_{x_j,\bar\mu}\big)^4 u_\lambda + u_\lambda^4 \sum_{j=1}^k PU_{x_j,\bar\mu}\Big)|\omega| \\
\le & \Big(\int_{B_R} \Big(\big(\sum_{j=1}^k PU_{x_j,\bar\mu}\big)^4 u_\lambda + u_\lambda^4 \sum_{j=1}^k PU_{x_j,\bar\mu}\Big)^{6/5}\Big)^{5/6} \|\omega\| \\
\le & C\sum_{j=1}^k \Big(u_\lambda(r)\mu_j^{-1/2} + u_\lambda^4(r)\bar\mu^{-1/2} + \lambda\bar\mu^{-5/2}\Big)\|\omega\|.
\end{aligned}
$$

Finally, by Lemma A.1

$$
\begin{aligned}
|l_1| \le & C\sum_{j=1}^k \int_{B_R} U_{x_j,\bar\mu}^4 \varphi_{x_j,\bar\mu}|\omega| \le \frac{C\sqrt\lambda}{\bar\mu^{1/2}} \sum_{j=1}^k \int_{B_R} U_{x_j,\bar\mu}^4|\omega| \\
\le & \frac{C\sqrt\lambda}{\bar\mu^{1/2}} \sum_{j=1}^k \Big(\int_{B_R} U_{x_j,\bar\mu}^{24/5}\Big)^{5/6}\|\omega\| \le \frac{C\sqrt\lambda}{\bar\mu}\|\omega\|.
\end{aligned}
\tag{2.5}
$$

Thus, the result follows.

□

Lemma 2.3. *For λ satisfying (1.7), and for any $(r,\bar\mu)\in M_{r,\bar\mu}$, it holds*

$$
\|Q_\lambda \omega\| \ge c_0\lambda^{-1}\|\omega\|, \quad \forall\, \omega\in E_{r,\bar\mu},
$$

where $c_0>0$ is a constant, independent of λ.

Proof. The proof of this lemma is quite standard. We just sketch it.

Let λ satisfy (1.7). So Corollary B.5 holds. We argue by contradiction. Suppose that there are $\lambda_n\to+\infty$ satisfying (1.7), $(r_n,\bar\mu_n)\in M_{r_n,\bar\mu_n}$, $\omega_n\in E_{r_n,\bar\mu_n}$, with

$$
\|Q_{\lambda_n}\omega_n\| = o(\lambda_n^{-1})\|\omega_n\|. \tag{2.6}
$$

We may assume $\|\omega_n\|=1$. Let

$$
\bar\omega_n(y) = \bar\mu_n^{-1/2}\omega(\bar\mu_n^{-1}y + x_{1,n}).
$$

We define

$$
\big\langle u,v\big\rangle_{n,*} = \int_{B_{R,n}} \big(\nabla u\nabla v + \lambda_n\bar\mu_n^{-2}V(\bar\mu_n^{-1}y+x_{1,n})uv\big), \quad \|u\|_{n,*} = \big\langle u,u\big\rangle_{n,*}^{1/2},
$$

where $B_{R,n}=\big\{y:\bar\mu_n^{-1}y+x_{1,n}\in B_R\big\}$. Then, $\|\bar\omega_n\|_{n,*}=1$, and

$$
\begin{aligned}
&\int_{B_{R,n}} \nabla \bar\omega_n \nabla \eta \, dy + \lambda_n \bar\mu_n^{-2} \int_{B_{R,n}} V(\bar\mu_n^{-1} y + x_{1,n}) \bar\omega_n \eta \, dy \\
&- 5 \int_{B_{R,n}} \Big(\sum_{j=1}^{k} \bar U_{j,n} + u_\lambda(\bar\mu_n^{-1} y + x_{1,n}) \Big)^4 \bar\omega_n \eta \, dy = o(\lambda_n^{-1}) \|\eta\|, \quad \forall\, \eta \in \bar E_n,
\end{aligned} \tag{2.7}
$$

where $\bar U_{j,n} = \bar\mu_n^{-1/2} P U_{x_{j,n},\bar\mu_n}(\bar\mu_n^{-1} y + x_{1,n})$, and

$$
\bar E_n = \Big\{ \eta : \eta \in H_0^1(B_{R,n}) : \ \big\langle \eta, \frac{\partial \bar U_{1,n}}{\partial x_h} \big\rangle_{n,*} = \big\langle \eta, \frac{\partial \bar U_{1,n}}{\partial \bar\mu} \big\rangle_{n,*} = 0, \ h = 1, 2, 3. \Big\}.
$$

From (2.7), we can deduce that there are A,and B_h, $h = 1, 2, 3$, such that

$$
\begin{aligned}
&- \Delta \bar\omega_n + \lambda_n \bar\mu_n^{-2} V(\bar\mu_n^{-1} y + x_{1,n}) \bar\omega_n - 5 \Big(\sum_{j=1}^{k} \bar U_{j,n} + u_{\lambda_n}(\bar\mu_n^{-1} y + x_{1,n}) \Big)^4 \bar\omega_n \\
=& A \frac{\partial \bar U_{1,n}}{\partial \bar\mu} + \sum_{h=1}^{3} B_h \frac{\partial \bar U_{1,n}}{\partial x_h} + o(\lambda_n^{-1}).
\end{aligned} \tag{2.8}
$$

Let ξ be a smooth function, such that $\xi = 1$ in $B_{\bar\mu_n^{7/8}}(0)$, $\xi = 0$ in $R^3 \setminus B_{2\bar\mu_n^{7/8}}(0)$, $|D\xi| \le C \bar\mu_n^{-7/8}$, and $|D^2 \xi| \le C \bar\mu_n^{-7/4}$. Then, noting that $\bar\mu_n \ge e^{c\lambda_n}$ for some $c > 0$, we find

$$
\begin{aligned}
&- \Delta(\xi \bar\omega_n) + \lambda_n \bar\mu_n^{-2} V(x_{1,n}) \xi \bar\omega_n - 5 \bar U_{1,n}^4 \xi \bar\omega_n \\
=& \xi \Big(A \frac{\partial \bar U_{1,n}}{\partial \bar\mu} + \sum_{h=1}^{3} B_h \frac{\partial \bar U_{1,n}}{\partial x_{1h}} \Big) + o(\lambda_n^{-1}).
\end{aligned} \tag{2.9}
$$

Let P_n is the projection of $H_0^1(B_{R,n})$ to $\bar E_n$. We obtain from (2.9) that

$$
P_n \Big(-\Delta(\xi \bar\omega_n) + \lambda_n \bar\mu_n^{-2} V(x_{1,n}) \xi \bar\omega_n - 5 \bar U_{1,n}^4 \xi \bar\omega_n \Big) = o(\lambda_n^{-1}). \tag{2.10}
$$

Choose $\tilde A$ and $\tilde B_h$, such that

$$
\tilde\omega_n = \xi \bar\omega_n - \tilde A \frac{\partial \bar U_{1,n}}{\partial \bar\mu} - \sum_{h=1}^{3} \tilde B_h \frac{\partial \bar U_{1,n}}{\partial x_h} \in \bar E_n.
$$

Then, from $\bar\omega_n \in \bar E_n$, we can check $\tilde A$, $\tilde B_n = O(e^{-c' \lambda_n^\theta})$, where $c' > 0$ is a constant. So, from (2.10), we find

$$
P_n \Big(-\Delta \tilde\omega_n + \lambda_n \mu_n^{-2} V(x_{1,n}) \tilde\omega_n - 5 \bar U_{1,n}^4 \tilde\omega_n \Big) = o(\lambda_n^{-1}). \tag{2.11}
$$

On the other hand, it is easy to check that

$$
\|\tilde L_n \omega\|_{n,*} \ge c' \|\omega\|_{n,*}, \quad \forall\, \omega \in \bar E_n,
$$

where $\tilde{L}_n$ is the linear operator, defined by

$$\tilde{L}_n\omega = P_n\Big(-\Delta\omega + \lambda_n\bar{\mu}_n^{-2}V(x_{1,n})\omega - 5\bar{U}_{1,n}^4\omega\Big).$$

So, we obtain from (2.11) that

$$\|\tilde{\omega}_n\|_{n,*} = o(\lambda_n^{-1}),$$

which gives

$$\int_{B_{\bar{\mu}_n^{-1/8}}(x_{1,n})} \omega_n^2 = o(\lambda_n^{-1}).$$

By the symmetry, we find

$$\int_{B_{\bar{\mu}_n^{-1/8}}(x_{i,n})} \omega_n^2 = o(\lambda_n^{-1}), \quad i = 1, \cdots, k. \tag{2.12}$$

For $|y - x_j| \ge \bar{\mu}^{-1/8}$, we have $U_{x_j,\bar{\mu}} \le \frac{1}{\bar{\mu}^{1/4}}$. So, using (2.12), we obtain

$$Q_{\lambda_n}\omega_n = L_{\lambda_n}\omega_n + o(\lambda_n^{-1}),$$

which, together with (2.6), gives

$$\|L_{\lambda_n}\omega_n\| = o(\lambda_n^{-1}) = o(\lambda_n^{-1})\|\omega_n\|.$$

This is a contradiction to Corollary B.5.

□

3. Proof of the main result

Since λ satisfies (1.7), Proposition 2.1 holds. Let

$$K(r,\bar{\mu}) = J_\lambda(r,\bar{\mu},\omega_{r,\bar{\mu}}), \quad (r,\bar{\mu}) \in M_{r,\bar{\mu}},$$

where $\omega_{r,\bar{\mu}}$ is the map obtained in Proposition 2.1. Then, we obtain from Propositions 2.1 and A.6 that

$$\begin{aligned} K(r,\bar{\mu}) =& J_\lambda(r,\bar{\mu},0) + O(\|l_\lambda\|\|\omega_{r,\bar{\mu}}\| + \|\omega_{r,\bar{\mu}}\|^2) = J_\lambda(r,\bar{\mu},0) + O\big(\frac{1}{\bar{\mu}^{1+\tau}}\big) \\ =& I(u_\lambda) + kA + \frac{kc_0B_1\sqrt{V(0)}\sqrt{\lambda}}{2\bar{\mu}} - \frac{1}{2}c_0B_1\sum_{i\neq j} e^{-\sqrt{V(0)\lambda}|x_i-x_j|}\varepsilon_{ij} \\ & - B_1k\frac{u_\lambda(r)}{\bar{\mu}^{1/2}} + O\Big(\frac{1}{\bar{\mu}\sqrt{\lambda}} + \frac{\sqrt{\lambda}r^2}{\bar{\mu}}\Big), \end{aligned} \tag{3.1}$$

where $\varepsilon_{ij} = \frac{1}{\bar{\mu}|x_i-x_j|}$.

Note that $|x_j - x_1| = \alpha_j|x_1|$, where $\alpha_j = \sqrt{2 - 2\cos\frac{2(j-1)\pi}{k}}$. So (3.1) becomes

$$
\begin{aligned}
K(r,\bar\mu) =& I(u_\lambda)+kA+\frac{kc_0B_1\sqrt{V(0)}\sqrt{\lambda}}{2\bar\mu}-\frac12 kc_0B_1\sum_{j=2}^{k}e^{-\sqrt{V(0)}\alpha_j r\sqrt{\lambda}}\frac{1}{\alpha_j\bar\mu r}\\
&-B_1k\frac{u_\lambda(r)}{\bar\mu_j^{1/2}}+O\Big(\frac{1}{\bar\mu\sqrt{\lambda}}+\frac{\sqrt{\lambda}r^2}{\bar\mu}\Big).
\end{aligned}\tag{3.2}
$$

We need the following expansions of the derivative of $K(r,\bar\mu)$.

Proposition 3.1. *Assume* $(r,\bar\mu)\in M_{r,\bar\mu}$. *Then*

$$
\begin{aligned}
\frac{\partial K(r,\bar\mu)}{\partial\bar\mu}=&-\frac12 kc_0B_1\frac{\sqrt{V(0)\lambda}}{\bar\mu^2}+\frac12 k\sum_{j=2}^{k}\frac{c_0B_1}{\alpha_j\bar\mu^2 r}e^{-\sqrt{\lambda V(0)}\alpha_j r}+\frac{B_1k}{2}\frac{u_\lambda(r)}{\bar\mu^{3/2}}\\
&+O\Big(\frac{1}{\bar\mu^2\sqrt{\lambda}}+\frac{\sqrt{\lambda}r^2}{\bar\mu^2}\Big).
\end{aligned}\tag{3.3}
$$

We will put the proof of Proposition 3.1 to the end of this section. Now, we prove the main result of this paper.

Let

$$
g_\lambda(r)=\frac{c_0B_1\sqrt{V(0)}}{2}-\frac12 c_0B_1\sum_{j=2}^{k}e^{-\sqrt{V(0)}\alpha_j r\sqrt{\lambda}}\frac{1}{\alpha_j r\sqrt{\lambda}}.\tag{3.4}
$$

Let $r_0>0$ be the largest number, such that $g_\lambda(r_0\lambda^{-1/2})=0$.

Define

$$
S=\Big\{(r,\bar\mu): r\in[d_1\lambda^{-1/2},d_\lambda],\ \bar\mu\in\Big[(1-r^\theta)\frac{4g_\lambda^2(r)\lambda}{B_1^2u_\lambda^2(r)},(1+r^\theta)\frac{4g_\lambda^2(r)\lambda}{B_1^2u_\lambda^2(r)}\Big]\Big\},
$$

where $d_1=r_0+e^{-\theta\sqrt{\lambda}}$, $\theta>0$ is a small constant, and $d_\lambda\to0$ as $\lambda\to+\infty$, satisfying $d_\lambda\sqrt{\lambda}\to+\infty$, $d_\lambda^{-1}\gamma_\lambda(t)\to0$ uniformly for $t\le\frac12 r_0$. Here $\gamma_\lambda(t)$ is the function in (1.9).

Let

$$
c_2=I(u_\lambda)+kA+\eta,
$$

and

$$
c_{1,\lambda}=I(u_\lambda)+kA-L\frac{u_\lambda^2(\bar r\lambda^{-1/2})}{\sqrt{\lambda}},
$$

where $\eta>0$ is small fixed small constant, and $\bar r>0$ and $L>0$ are fixed large constants. For any c, let $K^c=\{(r,\bar\mu):K(r,\bar\mu)<c\}$.

Consider the following flow:

$$\begin{cases} \frac{dr(t)}{dt} = -\nabla_r K(r(t), \bar\mu(t)), & t > 0; \\ \frac{d\bar\mu(t)}{dt} = -\nabla_{\bar\mu} K(r(t), \bar\mu(t)), & t > 0; \\ (r(0), \bar\mu(0)) = (r_0, \bar\mu_0) \in K^{c_2}. \end{cases} \tag{3.5}$$

Then

Proposition 3.2. *$(r(t), \bar\mu(t))$ will not leave S before it reaches $K^{c_{1,\lambda}}$.*

Proof. By the definition of $g(r)$, we find that if $\bar\mu = (1 - r^\theta)\frac{4g_\lambda^2(r)\lambda}{B_1^2 u_\lambda^2(r)}$, then

$$\frac{\partial K(r, \bar\mu)}{\partial \bar\mu} = -\frac{B_1^4 u_\lambda^4(r)}{16\lambda^{3/2} g_\lambda^3(r)}\frac{1}{2} r^\theta + O\Big(\frac{1}{\bar\mu^2\sqrt{\lambda}} + \frac{\sqrt{\lambda} r^2}{\bar\mu^2}\Big) < 0,$$

and if $\bar\mu = (1 + r^\theta)\frac{4g_\lambda^2(r)\lambda}{B_1^2 u_\lambda^2(r)}$, then

$$\frac{\partial K(r, \bar\mu)}{\partial \bar\mu} = \frac{B_1^4 u_\lambda^4(r)}{16\lambda^{3/2} g_\lambda^3(r)}\frac{1}{2} r^\theta + O\Big(\frac{1}{\bar\mu^2\sqrt{\lambda}} + \frac{\sqrt{\lambda} r^2}{\bar\mu^2}\Big) > 0.$$

On the other hand, if $r = d_1\lambda^{-1/2}$, then,

$$\begin{aligned} K(r, \bar\mu) \le & I(u_\lambda) + kA - c' \frac{k^2 B_1^2 u_\lambda^2(d_1\lambda^{-1/2})}{2\sqrt{\lambda} g_\lambda(d_1\lambda^{-1/2})} \\ = & I(u_\lambda) + kA - c' \frac{k^2 B_1^2 u_\lambda^2(\bar r\lambda^{-1/2})}{2\sqrt{\lambda}} \frac{u_\lambda^2(d_1\lambda^{-1/2})}{u_\lambda^2(\bar r\lambda^{-1/2}) g_\lambda(d_1\lambda^{-1/2})} < c_{1,\lambda}, \end{aligned}$$

since $g_\lambda(d_1\lambda^{-1/2}) \sim e^{-\theta\sqrt{\lambda}}$, and by (1.9)

$$\frac{u_\lambda^2(d_1\lambda^{-1/2})}{u_\lambda^2(\bar r\lambda^{-1/2}) g_\lambda(d_1\lambda^{-1/2})} \sim e^{(\theta + o(1))\sqrt{\lambda}}.$$

Suppose that $r = d_\lambda$. Then

$$K(r, \bar\mu) \le I(u_\lambda) + kA - c' \frac{k^2 B_1^2 u_\lambda^2(d_\lambda)}{2\sqrt{\lambda} g_\lambda(d_\lambda)} \le I(u_\lambda) + kA - c'' \frac{u_\lambda^2(d_\lambda)}{\sqrt{\lambda}} < c_{1,\lambda},$$

since by (1.9)

$$\frac{u_\lambda(d_\lambda)}{u_\lambda(\bar r\lambda^{-1/2})} \ge e^{\left(\int_{\bar r\lambda^{-1/2}}^{d_\lambda} \sqrt{V(t)}\,dt - |\gamma_\lambda(d_\lambda)| - |\gamma_\lambda(\bar r\lambda^{-1/2})|\right)\sqrt{\lambda}} \ge e^{\frac{1}{2}\int_0^{d_\lambda} \sqrt{V(t)}\,dt\sqrt{\lambda}}$$

and $d_\lambda\sqrt{\lambda} \to +\infty$. □

Proof of Theorem 1.1. We will prove that $K(r, \bar\mu)$ has a critical point in $K^{c_2} \setminus K^{c_{1,\lambda}}$.

Let Λ be the set of maps $h(r, \bar\mu)$ from S to S, satisfying

$$h_1(r, \bar\mu) = r, \quad \text{if } r = d_1\lambda^{-1/2}, \text{ or } r = d_\lambda,$$

where $h(r, \bar\mu) = (h_1(r, \bar\mu), h_2(r, \bar\mu))$, $h_1 \in [d_1\lambda^{-1/2}, d_\lambda]$, and h_2 is the $\bar\mu$ component.

Define

$$c_\lambda = \inf_{h\in\Lambda} \sup_{(r,\bar\mu)\in S} K(h(r,\bar\mu)).$$

We will show that c_λ is a critical value of $\bar K(r,\bar\mu)$. To prove this claim, we need to prove

(i) $c_{1,\lambda} < c_\lambda < c_2$;

(ii) if $r = d_1\lambda^{-1/2}$, or $r = d_\lambda$, then $K(h(r,\bar\mu)) < c_{1,\lambda}$, $\forall\ h\in\Lambda$.

To prove (ii), let $h\in\Lambda$. Then, for any $(r,\bar\mu)\in S$ with $r = d_1\lambda^{-1/2}$, or $r = d_\lambda$, we have $h(r,\bar\mu) = (r,\hat\mu)$ for some $\hat\mu$. By Proposition 3.2, we obtain

$$K(r,\hat\mu) < c_{1,\lambda}.$$

Now, we prove (i). It is easy to see $c_\lambda < c_2$.

Let $\tilde\mu(r) = \frac{4g_\lambda^2(r)}{B_1^2 u_\lambda^2(r)}$. For any $h\in\Lambda$, $\bar h(r) =: h_1(r,\tilde\mu(r))$ is a map from $[d_1\lambda^{-1/2}, d_\lambda]$ to $[d_1\lambda^{-1/2}, d_\lambda]$, satisfying

$$\bar h(r) = r, \quad \text{if } r = d_1\lambda^{-1/2}, \ \text{ or } \ r = d_\lambda.$$

Therefore, there is a $r\in[d_1\lambda^{-1/2}, d_\lambda]$, such that $\bar h(r) = \bar r\lambda^{-1/2}$. Let $\bar\mu = h_2(r,\tilde\mu(r))$. We have

$$\sup_{(r,\bar\mu)\in S} K(h(r,\bar\mu)) \ge K(\bar r\lambda^{-1/2},\bar\mu).$$

But

$$K(\bar r\lambda^{-1/2},\bar\mu) = I(u_\lambda) + kA - O\Big(\frac{u_\lambda^2(\bar r\lambda^{-1/2})}{\sqrt{\lambda} g_\lambda(\bar r\lambda^{-1/2})}\Big) > c_{1,\lambda}$$

if $L > 0$ is large enough.

□

In the rest of this section, we prove Proposition 3.1.

We use ∂ to denote either $\frac{\partial}{\partial\bar\mu}$ or $\frac{\partial}{\partial x_h}$. Using Proposition 2.1, we find

$$\begin{aligned}
\partial K(r,\bar\mu) &= \partial J_\lambda(r,\bar\mu,\omega_{r,\bar\mu}) + \Big\langle\frac{\partial J_\lambda(r,\bar\mu,\omega_{r,\bar\mu})}{\partial\omega}, \partial\omega_{r,\bar\mu}\Big\rangle \\
&= \partial J_\lambda(r,\bar\mu,\omega_{r,\bar\mu}) + A\Big\langle\frac{\partial PU_{x_1,\bar\mu}}{\partial\bar\mu}, \partial\omega_{r,\bar\mu}\Big\rangle + \sum_{h=1}^{3} B_h\Big\langle\frac{\partial PU_{x_1,\bar\mu}}{\partial x_h}, \partial\omega_{r,\bar\mu}\Big\rangle \\
&= \partial J_\lambda(r,\bar\mu,\omega_{r,\bar\mu}) - A\Big\langle\partial\frac{\partial PU_{x_1,\bar\mu}}{\partial\bar\mu}, \omega_{r,\bar\mu}\Big\rangle - \sum_{h=1}^{3} B_h\Big\langle\partial\frac{\partial PU_{x_1,\bar\mu}}{\partial x_h}, \omega_{r,\bar\mu}\Big\rangle.
\end{aligned} \tag{3.6}$$

Thus, to estimate $\partial K(r,\bar\mu)$, we need to estimate $\partial J_\lambda(r,\bar\mu,\omega_{r,\bar\mu})$, A and B_h.

First, we estimate $\partial J_\lambda(r,\bar\mu,\omega_{r,\bar\mu})$.

Lemma 3.3. *Let $\omega_{r,\bar\mu}$ be the function obtained in Proposition 2.1 and $(r,\bar\mu)\in M_{r,\bar\mu}$. Then*

$$\begin{aligned}\frac{\partial J_\lambda(r,\bar\mu,\omega_{r,\bar\mu})}{\partial\bar\mu}=&-\frac{1}{2}kc_0B_1\frac{\sqrt{V(0)\lambda}}{\bar\mu^2}+\frac12\sum_{j=2}^k\frac{c_0B_1}{\alpha_j\bar\mu^2 r}e^{-\sqrt{\lambda V(0)}\alpha_j r}+\frac{kB_1}{2}\frac{u_\lambda(r)}{\bar\mu^{3/2}}\\&+O\Big(\frac{1}{\bar\mu^2\sqrt\lambda}+\frac{\sqrt\lambda r^2}{\bar\mu^2}\Big).\end{aligned}\tag{3.7}$$

and

$$\frac{\partial J_\lambda(r,\bar\mu,\omega_{r,\bar\mu})}{\partial x_h}=O\big(\sqrt\lambda\big).\tag{3.8}$$

Proof. We have

$$\begin{aligned}&\partial J_\lambda(r,\bar\mu,\omega_{r,\bar\mu})\\=&\partial J_\lambda(r,\bar\mu,0)\\&-\int_{B_R}\Big(\big(\sum_{j=1}^k PU_{x_j,\bar\mu}+u_\lambda+\omega_{r,\bar\mu}\big)^5-\big(\sum_{j=1}^k PU_{x_j,\bar\mu}+u_\lambda\big)^5\Big)\partial\sum_{i=1}^k PU_{x_i,\bar\mu}.\end{aligned}$$

So, using the estimate for $\|\omega_{x,\mu}\|$, similar to the proof of Lemma 2.2, we find

$$\frac{\partial J_\lambda(r,\bar\mu,\omega_{r,\bar\mu})}{\partial\bar\mu}=\frac{\partial J_\lambda(r,\bar\mu,0)}{\partial\bar\mu}+O\Big(\frac{1}{\bar\mu^{2+\tau}}\Big),$$

and

$$\frac{\partial J_\lambda(r,\bar\mu,\omega_{r,\bar\mu})}{\partial x_h}=\frac{\partial J_\lambda(r,\bar\mu,0)}{\partial x_h}+O\Big(\frac{1}{\bar\mu^{\tau}}\Big).$$

Thus, the result follows from Proposition A.7. □

Next, we estimate A and B_h.

Lemma 3.4. *Let A and B_h be the constants obtained in Proposition 2.1. Then, we have*

$$A=O\big(\sqrt\lambda\big),\quad B_h=O\big(\frac{\sqrt\lambda}{\bar\mu^2}\big).$$

Proof. From Proposition 2.1 and Lemma 3.3, we know that A_i and B_{ih} satisfy

$$\begin{aligned}&\Big\langle\frac{\partial PU_{x_1,\bar\mu}}{\partial\bar\mu},\frac{\partial PU_{x_1,\bar\mu}}{\partial\bar\mu}\Big\rangle A+\sum_{h=1}^3\Big\langle\frac{\partial PU_{x_1,\bar\mu}}{\partial x_h},\frac{\partial PU_{x_1,\bar\mu}}{\partial\bar\mu}\Big\rangle B_h\\=&\Big\langle\frac{\partial J_\lambda}{\partial\omega},\frac{\partial PU_{x_1,\bar\mu}}{\partial\bar\mu}\Big\rangle=\frac{\partial J_\lambda(r,\bar\mu,\omega_{r,\bar\mu})}{\partial\bar\mu}=O\big(\frac{\sqrt\lambda}{\bar\mu^2}\big);\end{aligned}\tag{3.9}$$

$$
\begin{aligned}
&\langle\frac{\partial PU_{x_1,\bar\mu}}{\partial\bar\mu},\frac{\partial PU_{x_1,\bar\mu}}{\partial x_{\tilde h}}\rangle A+\sum_{h=1}^{3}\langle\frac{\partial PU_{x_1,\bar\mu}}{\partial x_h},\frac{\partial PU_{x_1,\bar\mu}}{\partial x_{\tilde h}}\rangle B_h\\
=&\langle\frac{\partial J_\lambda}{\partial\omega},\frac{\partial PU_{x_1,\bar\mu}}{\partial x_{\tilde h}}\rangle=\frac{\partial J_\lambda(r,\bar\mu,\omega_{r,\bar\mu})}{\partial x_{\tilde h}}=O\big(\sqrt{\lambda}\big)
\end{aligned}
\tag{3.10}
$$

Thus, we can solve (3.9) and (3.10) to obtain the result. □

We are now ready to prove Proposition 3.1.

Proof of Proposition 3.1. It follows directly from (3.6), Lemmas 3.3 and 3.4. □

Appendix A. Energy expansion

In this section, we will expand $I(PU_{x_j,\mu_j})$ and its derivatives. Throughout this section, we assume that $d_1\lambda^{-1/2}\le|x_j|$ and $|x_i-x_j|\ge c'\lambda^{-1/2}$, $i\ne j$, $\mu_j\ge e^{c'\sqrt{\lambda}}$ for some $c'>0$. Let us emphasize here that in the section, we do not assume that x_j, $j=1,\cdots,k$, are arranged in a symmetric way. Note that we have

$$
\frac{1}{\mu_j^\tau|x_i-x_j|^T}=O\big(\frac{1}{\mu_j^{\tau/2}}\big).
$$

Let

$$
\varphi_{x_j,\mu_j}=U_{x_j,\mu_j}-PU_{x_j,\mu_j}.
$$

We first estimate φ_{x_j,μ_j} and and its derivatives. By definition, φ_{x_j,μ_j} satisfies

$$
\begin{cases}-\Delta\varphi_{x_j,\mu_j}+\lambda V(y)\varphi_{x_j,\mu_j}=\lambda V(y)U_{x_j,\mu_j}, & \text{in } B_R,\\ \varphi_{x_j,\mu_j}=U_{x_j,\mu_j}, & \text{on } \partial B_R.\end{cases}
\tag{A.1}
$$

It is easy to see

$$
\Big|\frac{\partial\varphi_{x_j,\mu_j}}{\partial\mu_j}\Big|\le\frac{C}{\mu_j}\varphi_{x_j,\mu_j},\qquad\Big|\frac{\partial\varphi_{x_j,\mu_j}}{\partial x_{jh}}\Big|\le C\mu_j\varphi_{x_j,\mu_j}.
$$

Thus, we only need to estimate φ_{x_j,μ_j}.

Lemma A.1. *We have*

$$
\varphi_{x_j,\mu_j}(y)=\frac{c_0}{\mu_j^{1/2}}\frac{1}{|y-x_j|}\big(1-e^{-\sqrt{\lambda V(0)}|y-x_j|}\big)+O\big(\frac{1}{\mu_j^{1/2}\lambda^{1/2}}+\frac{\lambda^{1/2}|x_j|^2}{\mu_j^{1/2}}\big),\quad y\in B_R.
$$

Proof. Let $\xi_{x_j,\mu_j}=\frac{c_0}{\mu_j^{1/2}}\frac{1}{|y-x_j|}\big(1-e^{-\sqrt{\lambda V(0)}|y-x_j|}\big)$. Then ξ satisfies

$$-\Delta\xi_{x_j,\mu_j}+\lambda V(0)\xi_{x_j,\mu_j}=\frac{\lambda V(0)}{\mu_j^{1/2}}\frac{c_0}{|y-x_j|}.$$

Write

$$\varphi_{x_j,\mu_j}(y)=\xi_{x_j,\mu_j}+\psi_{x_j,\mu_j}. \tag{A.2}$$

Then,

$$\begin{aligned}&-\Delta\psi_{x_j,\mu_j}+\lambda V(y)\psi_{x_j,\mu_j}\\ =&\lambda V(y)\big(U_{x_j,\mu_j}-\frac{c_0}{\mu_j^{1/2}|y-x_j|}\big)+\lambda\big(V(y)-V(0)\big)\frac{c_0}{\mu_j^{1/2}|y-x_x|}e^{-\sqrt{\lambda}|y-x_j|}.\end{aligned}$$

Decompose $\psi_{x_j,\mu_j}=\psi_1+\psi_2+\psi_3+\psi_4$, where

$$\begin{cases}-\Delta\psi_1+\lambda V(y)\psi_1=\lambda V(y)\big(U_{x_j,\mu_j}-\frac{c_0}{\mu_j^{1/2}|y-x_j|}\big), & \text{in } B_R,\\ \psi_1=0, & \text{on } \partial B_R,\end{cases}$$

$$\begin{cases}-\Delta\psi_2+\lambda V(y)\psi_2=\lambda\big(V(y)-V(0)\big)\frac{c_0}{\mu_j^{1/2}|y-x_j|}e^{-\sqrt{\lambda}|y-x_j|}, & \text{in } B_R,\\ \bar\psi_1=0, & \text{on } \partial B_R,\end{cases}$$

$$\begin{cases}-\Delta\psi_3+\lambda V(y)\psi_3=0, & \text{in } B_R\\ \psi_3=U_{x_j,\mu_j}-\frac{c_0}{\mu_j^{1/2}}\frac{1}{|y-x_j|}, & \text{on } \partial B_R,\end{cases}$$

and

$$\begin{cases}-\Delta\psi_4+\lambda V(y)\psi_4=0, & \text{in } B_R\\ \psi_4=\frac{c_0}{\mu_j^{1/2}}\frac{1}{|y-x_j|}e^{-\sqrt{\lambda V(0)}|y-x_j|}, & \text{on } \partial B_R.\end{cases}$$

Then there is a $\kappa>0$,

$$\psi_4=O\big(\frac{e^{-\kappa\sqrt{\lambda}}}{\mu_j^{1/2}}\big).$$

On the other hand, for any $y\in\partial B_R$,

$$U_{x_j,\mu_j}-\frac{c_0}{\mu_j^{1/2}}\frac{1}{|y-x_j|}=O\big(\frac{1}{\mu_j^{5/2}}\big).$$

So

$$|\psi_3|=O\big(\frac{1}{\mu_j^{5/2}}\big).$$

Let $\bar\psi_1(z)=\mu_j^{-1/2}\psi_1(\mu_j^{-1}z+x_j)$. Then

$$\begin{cases} -\Delta\bar{\psi}_1 + \lambda\mu_j^{-2}V(\mu_j^{-1}z + x_j)\bar{\psi}_1 = \lambda\mu_j^{-2}V(\mu_j^{-1}z + x_j)\big(U - \frac{c_0}{|z|}\big), & \text{in } (B_R)_{x_j,\mu_j} \\ \bar{\psi}_1 = 0, & \text{on } \partial(B_R)_{x_j,\mu_j}. \end{cases}$$

So,

$$\bar{\psi}_1(z) = O\big(\frac{\lambda}{\mu_j^2}\big).$$

As a result,

$$\psi_1(y) = O\big(\frac{\lambda}{\mu_j^{3/2}}\big).$$

Lastly, let $\bar{\psi}_2(z) = \psi_1(\lambda^{-1/2}z + x_j)$. Then

$$\begin{cases} -\Delta\bar{\psi}_2 + V(\lambda^{-1/2}z + x_j)\bar{\psi}_2 = \big(V(\lambda^{-1/2}z + x_j) - V(0)\big)\frac{c_0\sqrt{\lambda}}{\mu_j^{1/2}|x|}e^{-|z|}, & \text{in } (B_R)_{x_j,\sqrt{\lambda}} \\ \bar{\psi}_2 = 0, & \text{on } \partial(B_R)_{x_j,\sqrt{\lambda}}, \end{cases}$$

Since

$$|V(\lambda^{-1/2}z + x_j) - V(0)| \leq C|\lambda^{-1/2}z + x_j|^2 = C\lambda^{-1}|z|^2 + C|x_j|^2,$$

by comparison theorem, we obtain

$$|\bar{\psi}_2(z)| = O\big(\frac{1}{\mu_j^{1/2}\lambda^{1/2}} + \frac{\lambda^{1/2}|x_j|^2}{\mu_j^{1/2}}\big).$$

□

Remark A.2. Using the above techniques, we can prove

$$\frac{\partial\varphi_{x_j,\mu_j}(y)}{\partial\mu_j} = -\frac{c_0}{2\mu_j^{3/2}}\frac{1}{|y - x_j|}\big(1 - e^{-\sqrt{\lambda V(0)}|y-x_j|}\big) + O\big(\frac{1}{\mu_j^{3/2}\lambda^{1/2}} + \frac{\lambda^{1/2}|x_j|^2}{\mu_j^{3/2}}\big), \quad y \in B_R,$$

Let

$$A = \frac{1}{3}\int_{R^N} U^6.$$

Proposition A.3. *We have the following estimate:*

$$I(PU_{x_j,\mu_j}) = A + \frac{c_0B_1\sqrt{V(0)}\sqrt{\lambda}}{2\mu_j} + O\big(\frac{1}{\mu_j\sqrt{\lambda}} + \frac{\sqrt{\lambda}|x_j|^2}{\mu_j}\big), \tag{A.3}$$

where $B_1 = \int_{R^3} U^5$.

Proof. We have

$$
\begin{aligned}
&I(PU_{x_j,\mu_j})\\
=&\frac{1}{2}\int_{B_R} U^5_{x_j,\mu_j} PU_{x_j,\mu_j} - \frac{1}{6}\int_{B_R} (PU_{x_j,\mu_j})^6\\
=&A + \frac{1}{2}\int_{B_R} U^{2^*-1}_{x_j,\mu_j}\varphi_{x_j,\mu_j} + O\Big(\int_{B_R} U^{2^*-2}_{x_j,\mu_j}\varphi^2_{x_j,\mu_j} + \int_{B_R}\varphi^6_{x_j,\mu_j}\Big)
\end{aligned}
\tag{A.4}
$$

It follows from Lemma A.1 that

$$
\begin{aligned}
&\int_{B_R} U^5_{x_j,\mu_j}\varphi_{x_j,\mu_j}\\
=&\int_{B_R} U^5_{x_j,\mu_j}\frac{c_0}{\mu_j^{1/2}}\frac{1}{|y-x_j|}\big(1-e^{-\sqrt{\lambda V(0)}|y-x_j|}\big) + O\big(\frac{1}{\mu_j\sqrt{\lambda}} + \frac{\sqrt{\lambda}|x_j|^2}{\mu_j}\big)\\
=&c_0\int_{B_R} U^5_{x_j,\mu_j}\frac{1}{\mu_j^{1/2}|y-x_j|}\sqrt{\lambda V(0)}|y-x_j| + O\big(\frac{1}{\mu_j\sqrt{\lambda}} + \frac{\sqrt{\lambda}|x_j|^2}{\mu_j}\big)\\
=&\frac{c_0\sqrt{V(0)\lambda}}{\mu_j}\int_{R^3} U^5 + O\big(\frac{1}{\mu_j\sqrt{\lambda}} + \frac{\sqrt{\lambda}|x_j|^2}{\mu_j}\big),
\end{aligned}
$$

$$
\begin{aligned}
&\int_{B_R} U^4_{x_j,\mu_j}\varphi^2_{x_j,\mu_j} \le C\int_{B_R} U^4_{x_j,\mu_j}\frac{1}{\mu_j|y-x_j|^2}\big(1-e^{-\sqrt{\lambda V(x_j)}|y-x_j|}\big)^2 + O\big(\frac{1}{\mu_j^2}\big)\\
\le& C\int_0^{2R}\frac{\mu_j}{(1+\mu_j^2r^2)^2}\big(1-e^{-\sqrt{\lambda V(x_j)}r}\big)^2\,dr + O\big(\frac{1}{\mu_j^2}\big) = O\big(\frac{\sqrt{\lambda}}{\mu_j^2}\big),
\end{aligned}
$$

and

$$
\int_{B_R}\varphi^6_{x_j,\mu_j} \le \frac{C}{\mu_j^3}\int_0^{2R}\frac{1}{r^4}\big(1-e^{-\sqrt{\lambda V(x_j)}r}\big)^6 + O\big(\frac{1}{\mu_j^3}\big) = O\big(\frac{\lambda^{3/2}}{\mu_j^3}\big).
$$

□

For $i \ne j$, we define

$$
\varepsilon_{ij} = \frac{1}{\mu_i^{1/2}\mu_j^{1/2}|x_i - x_j|}.
$$

Proposition A.4. *We have the following estimate:*

$$
\begin{aligned}
I\big(\sum_{j=1}^{k} PU_{x_j,\mu_j}\big) =&kA+\sum_{j=1}^{k}\frac{c_0B_1\sqrt{V(0)}\sqrt{\lambda}}{2\mu_j}-\frac{1}{2}c_0B_1\sum_{i\neq j}e^{-\sqrt{V(0)\lambda}|x_i-x_j|}\varepsilon_{ij}\\
&+O\Big(\sum_{j=1}^{k}\big(\frac{1}{\mu_j\sqrt{\lambda}}+\frac{\sqrt{\lambda}|x_j|^2}{\mu_j}\big)\Big)
\end{aligned}
\tag{A.5}
$$

Proof. We have

$$
\begin{aligned}
&I\big(\sum_{j=1}^{k} PU_{x_j,\mu_j}\big)\\
=&\sum_{j=1}^{k} I\big(PU_{x_j,\mu_j}\big)-\frac{1}{2}\sum_{i\neq j}\int_{B_R}U^5_{x_i,\mu_i}PU_{x_j,\mu_j}\\
&+O\Big(\sum_{i\neq j}\int_{B_R}\big(U^4_{x_i,\mu_i}\varphi_{x_i,\mu_i}+\varphi^5_{x_i,\mu_i}\big)U_{x_j,\mu_j}\Big)+O\Big(\sum_{i\neq j}\int_{B_R}U^4_{x_i,\mu_i}U^2_{x_j,\mu_j}\Big).
\end{aligned}
\tag{A.6}
$$

For any $y\notin B_{\mu_j^{-1/2}|x_i-x_j|}(x_j)$, it follows from Lemma A.1 that

$$
\begin{aligned}
&PU_{x_j,\mu_j}=U_{x_j,\mu_j}-\varphi_{x_j,\mu_j}\\
=&\frac{c_0}{\mu_j^{1/2}}\frac{1}{|y-x_j|}e^{-\sqrt{\lambda V(0)}|y-x_j|}+O\Big(\frac{1}{\mu_j^{5/2}|x_i-x_j|^3}+\frac{1}{\mu_j^{1/2}\sqrt{\lambda}}+\frac{\sqrt{\lambda}|x_j|^2}{\mu_j^{1/2}}\Big),
\end{aligned}
$$

from which, we deduce

$$
\begin{aligned}
&\int_{B_R}U^5_{x_i,\mu_i}PU_{x_j,\mu_j}\\
=&\Big(\int_{B_{\mu_i^{-1/2}|x_i-x_j|}(x_i)}+\int_{B_{\mu_j^{-1/2}|x_i-x_j|}(x_j)}\Big)U^5_{x_i,\mu_i}PU_{x_j,\mu_j}\\
&+\int_{B_R\setminus(B_{\mu_i^{-1/2}|x_i-x_j|}(x_i)\cup B_{\mu_j^{-1/2}|x_i-x_j|}(x_j))}U^5_{x_i,\mu_i}PU_{x_j,\mu_j}\\
=&\int_{B_{\mu_i^{-1/2}|x_i-x_j|}(x_i)}U^5_{x_i,\mu_i}PU_{x_j,\mu_j}+O\big(\frac{1}{\mu_j\sqrt{\lambda}}+\frac{\sqrt{\lambda}|x_j|^2}{\mu_j^{1/2}}\big)\\
=&c_0B_1\varepsilon_{ij}e^{-\sqrt{\lambda V(0)}|x_i-x_j|}+O\big(\frac{1}{\mu_j\sqrt{\lambda}}+\frac{\sqrt{\lambda}|x_j|^2}{\mu_j^{1/2}}\big).
\end{aligned}
\tag{A.7}
$$

On the other hand, there is a constant $\tau>0$, such that

$$\begin{aligned}
&\int_{B_R} U^4_{x_i,\mu_i}\varphi_{x_i,\mu_i}U_{x_j,\mu_j} = \int_{B_{\mu_i^{-1/2}|x_i-x_j|}(x_i)} U^4_{x_i,\mu_i}\varphi_{x_i,\mu_i}U_{x_j,\mu_j} + O\big(\frac{1}{\mu_j^{1+\tau}}\big)\\
=&O\Big(\frac{1}{\mu_j^{1/2}|x_i-x_j|}\int_{B_{\mu_i^{-1/2}|x_i-x_j|}(x_i)} U^4_{x_i,\mu_i}\varphi_{x_i,\mu_i} + \frac{1}{\mu_j^{1+\tau}}\Big)\\
=&O\Big(\varepsilon_{ij}\int_{B_{\mu_i^{1/2}|x_i-x_j|}(0)} U^4\frac{1}{|z|}(1-e^{-\sqrt{\lambda V(x_i)}|z|/\mu_i}) + \frac{1}{\mu_j^{1+\tau}}\Big)\\
=&\varepsilon_{ij}O\Big(\frac{\sqrt{\lambda}}{\mu_i} + \frac{1}{\mu_j^{1+\tau}}\Big) = O\big(\frac{1}{\mu_j^{1+\tau}}\big),
\end{aligned}\tag{A.8}$$

and

$$\begin{aligned}
&\int_{B_R} \varphi^5_{x_i,\mu_i}U_{x_j,\mu_j}\\
=&O\Big(\varepsilon_{ij}\int_{B_{\mu_i^{-1/2}|x_i-x_j|\mu_i}(0)} \frac{1}{|z|^5}(1-e^{-\sqrt{\lambda V(x_i)}|z|/\mu_i})^5 + \frac{1}{\mu_j^{1+\tau}}\Big) = O\big(\frac{1}{\mu_j^{1+\tau}}\big).
\end{aligned}\tag{A.9}$$

Finally,

$$\sum_{i\neq j}\int_{B_R} U^4_{x_i,\mu_i}U^2_{x_j,\mu_j} = \sum_{i\neq j}\varepsilon_{ij}^{1+2\sigma} = O\big(\frac{1}{\mu_j^{1+\tau}}\big). \tag{A.10}$$

Combining (A.6), (A.7), (A.8), (A.9), and (A.10), we obtain the desired estimate. □

Next, we estimate the derivatives of $I\big(\sum_{j=1}^k PU_{x_j,\mu_j}\big)$. Intuitively, these estimates can be obtained by differentiating (A.5).

Proposition A.5. *We have the following estimates:*

$$\begin{aligned}
\frac{\partial}{\partial\mu_i}I\big(\sum_{j=1}^k PU_{x_j,\mu_j}\big) =& -\frac{1}{2}c_0B_1\frac{\sqrt{V(0)\lambda}}{\mu_i^2} + \frac{1}{2}\sum_{j\neq i}\frac{c_0B_1}{\mu_i^{3/2}\mu_j|x_i-x_j|}e^{-\sqrt{\lambda V(0)}|x_i-x_j|}\\
&+O\Big(\sum_{j=1}^k\big(\frac{1}{\mu_j^2\sqrt{\lambda}} + \frac{\sqrt{\lambda}|x_j|^2}{\mu_j^2}\big)\Big),
\end{aligned}\tag{A.11}$$

and

$$\frac{\partial}{\partial x_{ih}}I\big(\sum_{j=1}^k PU_{x_j,\mu_j}\big) = O\big(\sqrt{\lambda}\big). \tag{A.12}$$

Proof. We use ∂_i to denote either $\frac{\partial}{\partial\mu_i}$, or $\frac{\partial}{\partial x_{ih}}$. We have

$$\begin{aligned}
&\partial_i I\Big(\sum_{j=1}^{k} PU_{x_j,\mu_j}\Big)\\
=&\sum_{j=1}^{k}\int_{B_R}\big(U^5_{x_j,\mu_j}-(PU_{x_j,\mu_j})^5\big)\partial_i(PU_{x_i,\mu_i})\\
&-\int_{B_R}\Big(\big(\sum_{j=1}^{k} PU_{x_j,\mu_j}\big)^5-\sum_{j=1}^{k}(PU_{x_j,\mu_j})^5\Big)\partial_i(PU_{x_i,\mu_i})\\
=&\sum_{j=1}^{k}\int_{B_R}\big(U^5_{x_j,\mu_j}-(PU_{x_j,\mu_j})^5\big)\partial_i(PU_{x_i,\mu_i})-\int_{B_R}\sum_{j\neq i}5(PU_{x_i,\mu_i})^4\partial_i(PU_{x_i,\mu_i})PU_{x_j,\mu_j}\\
&+O\Big(\sum_{j\neq i}\int_{B_R}\big((PU_{x_i,\mu_i})^3(PU_{x_j,\mu_j})^2+PU_{x_i,\mu_i}(PU_{x_j,\mu_j})^4\big)\big|\partial_i(PU_{x_i,\mu_i})\big|\Big).
\end{aligned}$$

We prove (A.11) first.
Step 1. The estimate of $\int_{B_R}\big(U^5_{x_j,\mu_j}-(PU_{x_j,\mu_j})^5\big)\frac{\partial(PU_{x_i,\mu_i})}{\partial\mu_i}$.
If $j=i$, then

$$\begin{aligned}
&\int_{B_R}\big(U^5_{x_i,\mu_i}-(PU_{x_i,\mu_i})^5\big)\frac{\partial(PU_{x_i,\mu_i})}{\partial\mu_i}\\
=&5\int_{B_R}U^4_{x_i,\mu_i}\frac{\partial U_{x_i,\mu_i}}{\partial\mu_i}\varphi_{x_i,\mu_i}+O\Big(\sum_{j=1}^{k}\big(\frac{1}{\mu_j^2\sqrt{\lambda}}+\frac{\sqrt{\lambda}|x_j|^2}{\mu_j^2}\big)\Big)\\
=&5\int_{B_R}U^4_{x_i,\mu_i}\frac{\partial U_{x_i,\mu_i}}{\partial\mu_i}\frac{c_0}{\mu_i^{1/2}}\frac{1}{|y-x_i|}\big(1-e^{-\sqrt{\lambda V(0)}|y-x_i|}\big)+O\Big(\sum_{j=1}^{k}\big(\frac{1}{\mu_j^2\sqrt{\lambda}}+\frac{\sqrt{\lambda}|x_j|^2}{\mu_j^2}\big)\Big)\\
=&5\int_{B_R}U^4_{x_i,\mu_i}\frac{\partial U_{x_i,\mu_i}}{\partial\mu_i}\frac{c_0}{\mu_i^{1/2}}\sqrt{\lambda V(0)}+O\Big(\sum_{j=1}^{k}\big(\frac{1}{\mu_j^2\sqrt{\lambda}}+\frac{\sqrt{\lambda}|x_j|^2}{\mu_j^2}\big)\Big)\\
=&\frac{c_0}{\mu_i^{1/2}}\sqrt{\lambda V(0)}\frac{\partial}{\partial\mu_j}\int_{R^3}U^5_{x_i,\mu_i}+O\Big(\sum_{j=1}^{k}\big(\frac{1}{\mu_j^2\sqrt{\lambda}}+\frac{\sqrt{\lambda}|x_j|^2}{\mu_j^2}\big)\Big)\\
=&-\frac{1}{2}c_0B_1\frac{\sqrt{V(0)\lambda}}{\mu_i^2}+O\Big(\sum_{j=1}^{k}\big(\frac{1}{\mu_j^2\sqrt{\lambda}}+\frac{\sqrt{\lambda}|x_j|^2}{\mu_j^2}\big)\Big).
\end{aligned}\tag{A.13}$$

On the other hand, for $i\neq j$,

$$
\begin{aligned}
&\int_{B_R}\big(U_{x_j,\mu_j}^5-(PU_{x_j,\mu_j})^5\big)\frac{\partial(PU_{x_i,\mu_i})}{\partial\mu_i}\\
\le&\frac{C}{\mu_i}\int_{B_R}\big|U_{x_j,\mu_j}^5-(PU_{x_j,\mu_j})^5\big|PU_{x_i,\mu_i}=O\big(\frac{1}{\mu_i^2}\big).
\end{aligned}
\tag{A.14}
$$

Step 2. The estimate of $\int_{B_R}5(PU_{x_i,\mu_i})^4\frac{\partial(PU_{x_i,\mu_i})}{\partial\mu_i}PU_{x_j,\mu_j}$, $j\ne i$.
By Remark A.2, we find

$$
\begin{aligned}
&\int_{B_R}5(PU_{x_i,\mu_i})^4\frac{\partial(PU_{x_i,\mu_i})}{\partial\mu_i}PU_{x_j,\mu_j}\\
=&5\int_{B_R}U_{x_i,\mu_i}^4\frac{\partial U_{x_i,\mu_i}}{\partial\mu_i}PU_{x_j,\mu_j}+O\big(\frac{1}{\mu_i^2}\big)\\
=&5\int_{B_{\frac12|x_i-x_j|}(x_i)}U_{x_i,\mu_i}^4\frac{\partial U_{x_i,\mu_i}}{\partial\mu_i}\frac{c_0}{\mu_j^{1/2}}\frac{1}{|x_i-x_j|}e^{-\sqrt{\lambda V(x_j)}|x_i-x_j|}+O\big(\frac{1}{\mu_i^2}\big)\\
=&-\frac12\frac{c_0B_1}{\mu_i^{3/2}\mu_j^{1/2}|x_i-x_j|}e^{-\sqrt{\lambda V(x_j)}|x_i-x_j|}+O\Big(\sum_{j=1}^k\frac{1}{\mu_j^2}\Big).
\end{aligned}
$$

Step 3. The estimate of $\int_{B_R}(PU_{x_i,\mu_i})^{5-m}(PU_{x_j,\mu_j})^m\big|\frac{\partial(PU_{x_i,\mu_i})}{\partial\mu_i}\big|$, $m=2,3,4$, $j\ne i$.
Using Lemma A.1, we can deduce

$$
\int_{B_R}(PU_{x_i,\mu_i})^{5-m}(PU_{x_j,\mu_j})^m\big|\frac{\partial(PU_{x_i,\mu_i})}{\partial\mu_i}\big|=O\big(\frac{1}{\mu_i}\varepsilon_{ij}^{1+\tau}\big)=O\big(\frac{1}{\mu_i^2}\big).
$$

The proof (A.12) is similar.

□

Let

$$
\tilde J_\lambda(x,\mu)=I\Big(\sum_{j=1}^k PU_{x_j,\mu_j}+u_\lambda\Big).
$$

Now, we expand $\tilde J_\lambda(x,\mu)$ and its derivatives.

Proposition A.6. *We have*

$$
\begin{aligned}
\tilde J_\lambda\big(x,\mu\big)=&I(u_\lambda)+kA+\sum_{j=1}^k\frac{c_0B_1\sqrt{V(0)}\sqrt\lambda}{2\mu_j}-\frac12c_0B_1\sum_{i\ne j}e^{-\sqrt{V(0)\lambda}|x_i-x_j|}\varepsilon_{ij}\\
&-B_1\sum_{j=1}^k\frac{u_\lambda(x_j)}{\mu_j^{1/2}}+\sum_{j=1}^kO\Big(\big(\frac{u_\lambda(x_j)}{\mu_j^{1/2}}\big)^{1+\tau}+\frac{1}{\mu_j\sqrt\lambda}+\frac{\sqrt\lambda|x_j|^2}{\mu_j}\Big),
\end{aligned}
\tag{A.15}
$$

where $\tau>0$ is a constant, and $B_1=\int_{R^3}U^{2^*-1}$.

Proof. Let $H=\sum_{j=1}^k PU_{x_j,\mu_j}$. We have

$$\begin{aligned}&J(x,\mu,0)=I(H+u_\lambda)\\ =&I(H)+I(u_\lambda)-\int_{B_R}H^{2^*-1}u_\lambda-I_1,\end{aligned}\tag{A.16}$$

where

$$I_1=\frac{1}{6}\int_{B_R}\Big((H+u_\lambda)^6-H^6-u_\lambda^6-6H^5u_\lambda-6Hu_\lambda^5\Big).$$

It follows from Lemma A.1 and Remark A.2 that

$$\begin{aligned}I_1\le&C\int_{B_R}\big(H^4u_\lambda^2+H^2u_\lambda^4\big)\\ =&C\int_{B_\delta}\big(H^4u_\lambda^2+H^2u_\lambda^4\big)+C\int_{B_R\setminus B_\delta}\big(H^4u_\lambda^2+H^2u_\lambda^4\big)\\ =&\sum_{j=1}^k O\Big(\big(\frac{u_\lambda(x_j)}{\mu_j^{1/2}}\big)^{1+\tau}\Big),\end{aligned}$$

for some constant $\tau>0$. But

$$\begin{aligned}&\int_{B_R}H^5u_\lambda=\sum_{j=1}^k\int_{B_R}(PU_{x_j,\mu_j})^5u_\lambda+O\Big(\sum_{i\neq j}\int_{B_R}U_{x_i,\mu_i}^4U_{x_j,\mu_j}u_\lambda\Big)\\ =&\sum_{j=1}^k\int_{B_R}U_{x_j,\mu_j}^5u_\lambda+\sum_{j=1}^k O\Big(\int_{B_R}U_{x_j,\mu_j}^4\varphi_{x_j,\mu_j}u_\lambda+\big(\frac{u_\lambda(x_j)}{\mu_j^{1/2}}\big)^{1+\tau}\Big).\end{aligned}\tag{A.17}$$

On the other hand,

$$\begin{aligned}&\int_{B_R}U_{x_j,\mu_j}^4\varphi_{x_j,\mu_j}u_\lambda=\int_{B_{\mu_j^{-1/4}}(x_j)}U_{x_j,\mu_j}^4\varphi_{x_j,\mu_j}u_\lambda+\int_{B_R\setminus B_{\mu_j^{-1/4}}(x_j)}U_{x_j,\mu_j}^4\varphi_{x_j,\mu_j}u_\lambda\\ =&O\Big(u_\lambda(x_j)\int_{B_{\mu_j^{-1/4}}(x_j)}U_{x_j,\mu_j}^4\varphi_{x_j,\mu_j}\Big)+O\Big(\frac{1}{\mu_j^{1+\tau}}\Big)\\ =&O\Big(\frac{u_\lambda(x_j)}{\mu_j^{1+\tau}}+\frac{1}{\mu_j^{1+\tau}}\Big).\end{aligned}$$

Moreover

$$
\begin{aligned}
&\int_{B_R} U_{x_j,\mu_j}^{2^*-1} u_\lambda = \int_{B_{\mu_j^{-1/4}(x_j)}} U_{x_j,\mu_j}^{2^*-1} u_\lambda + O\big(\frac{1}{\mu_j^{1+\tau}}\big)\\
=&\frac{B_1 u_\lambda(0)}{\mu_j^{1/2}} + O\big(\frac{1}{\mu_j^{1+\tau}}\big).
\end{aligned}
$$

Thus, the result follows.

□

Next, we expand the derivatives of $\tilde{J}_\lambda(x,\mu)$ with respect to x and μ. Intuitively, we can differentiate (A.15) and obtain the desired results.

Proposition A.7. *For any $i = 1, \cdots, k$, we have*

$$
\begin{aligned}
\frac{\partial \tilde{J}_\lambda(x,\mu)}{\partial \mu_i} =& -\frac{1}{2} c_0 B_1 \frac{\sqrt{V(0)\lambda}}{\mu_i^2} + \frac{1}{2}\sum_{j\neq i} \frac{c_0 B_1}{\mu_i^{3/2}\mu_j |x_i - x_j|} e^{-\sqrt{\lambda V(0)}|x_i - x_j|} + \frac{B_1}{2}\frac{u_\lambda(x_i)}{\mu_i^{3/2}}\\
&+ \sum_{j=1}^k \frac{1}{\mu_j} O\Big(\big(\frac{u_\lambda(x_j)}{\mu_j^{1/2}}\big)^{1+\tau} + \frac{1}{\mu_j\sqrt{\lambda}} + \frac{\sqrt{\lambda}|x_j|^2}{\mu_j}\Big),
\end{aligned}
$$

and

$$
\frac{\partial \tilde{J}(x,\mu)}{\partial x_{ih}} = O\big(\sqrt{\lambda}\big).
$$

Proof. We use ∂_i to denote either $\frac{\partial}{\partial \mu_i}$, or $\frac{\partial}{\partial x_{ih}}$. Then

$$
\begin{aligned}
&\partial_i I\big(\sum_{j=1}^k PU_{x_j,\mu_j} + u_\lambda\big)\\
=&\partial_i I\big(\sum_{j=1}^k PU_{x_j,\mu_j}\big) - 5\int_{B_R} H^4 \partial_i (PU_{x_i,\mu_i}) u_\lambda\\
&-\int_{B_R}\Big(\big(\sum_{j=1}^k PU_{x_j,\mu_j} + u_\lambda\big)^5 - \big(\sum_{j=1}^k PU_{x_j,\mu_j}\big)^5 - u_\lambda^5 - 5\big(\sum_{j=1}^k PU_{x_j,\mu_j}\big)^4 u_\lambda\Big)\partial_i(PU_{x_i,\mu_i}).
\end{aligned}
$$

Using

$$
u_\lambda(y) = u_\lambda(x_i) + u_\lambda'(x_i)\langle \frac{x_i}{|x_i|}, y - x_i\rangle + O\big(u_\lambda(x_i)\lambda|y - x_i|^2\big),
$$

we can obtain the estimates as we did in Proposition A.5. □

Appendix B. Spectrum Estimates of Layered Solutions

Consider

$$\begin{cases} -\Delta u + \lambda V(|y|)u = u^p, \ u > 0 \quad \text{in } B_R \\ u \in H_0^1(B_R), \end{cases} \tag{B.1}$$

where $p > 1$, $n \geq 2$, B_R is the ball in $\mathbb{R}^n$, with radius R, centered at the origin, and V is a smooth function with

$$\inf_{y \in \mathbb{R}^n} V(y) \ > \ 0 \ .$$

Let

$$M(r) = r^{n-1} V^{\sigma}(r) \tag{B.2}$$

where

$$\sigma = \frac{p+1}{p-1} - \frac{1}{2}. \tag{B.3}$$

Ambrosetti, Malchiodi and Ni [2] showed that if $M(r)$ has a non-degenerate critical point $r_0 \in (0, R)$, then (B.1) has a radial solutions $u_\lambda(|y|)$ exhibiting concentration on a sphere $|y| = r_0$ in the form

$$u_\lambda(r) \sim \big(\lambda V(r_0)\big)^{\frac{1}{p-1}} w\big((\lambda V(r_0))^{\frac{1}{2}}(r - r_0)\big) \ ,$$

where w is the solution of

$$w^{''} - w + w^p = 0 \text{ in } \mathbb{R}^1, w(0) = \max_{\mathbb{R}^1} w(y), w(y) \to 0 \text{ as } |y| \to +\infty. \tag{B.4}$$

Let $f(t) = t^p$. The aim of this section is to prove that the linear operator L_λ in $H_0^1(B_R)$ defined by

$$\big\langle L_\lambda \psi, \eta \big\rangle = \int_{B_R} \big(\nabla\psi\nabla\eta + \lambda V(|y|)\psi\eta - f'(u_\lambda)\psi\eta\big) \tag{B.5}$$

satisfies

$$\|L_\lambda \psi\| \geq c_0 \lambda^{-1} \|\psi\|,$$

for some constant $c_0 > 0$. For this aim, we need to study the spectrum properties of this layered solution u_λ.

Let $\varepsilon = \frac{1}{\sqrt{\lambda}}$. By scaling, (B.1) can be transformed into

$$-\varepsilon^2 \Delta u + V(y)u = u^p, \ u > 0, \quad u \in H_0^1(B_R), \tag{B.6}$$

Let u_ε be the layered solution of (B.6) constructed in [2]. A standard argument (similar to the proof of Claim 1 of Theorem 2.1 of [29]) shows that u_ε has a unique maximum point r_ε where $r_\varepsilon \to r_0$ with $M^{'}(r_0) = 0$.

It is well-known that the following eigenvalue problem

$$\phi^{''} - \phi + f^{'}(w) = \nu\phi \text{ in } \mathbb{R}^1, \phi \in H^1(\mathbb{R}^1) \tag{B.7}$$

admits the following eigenvalues

$$\nu_1 > 0,\ \nu_2 = 0,\ \nu_3 < 0 \tag{B.8}$$

where the eigenfunction corresponding to ν_1 can be made positive and even. We denote the first eigenfunction as $\Psi_0(y)$ with $\int_{\mathbb{R}^1} \Psi_0^2 = 1$. In fact,

$$\nu_1 = \frac{1}{4}(p-1)(p+3), \Psi_0 = \frac{1}{\int_{\mathbb{R}} w^{p+1}} w^{\frac{p+1}{2}}. \tag{B.9}$$

Consider the following eigenvalue problem

$$\varepsilon^2 \Delta\psi - V(r)\psi + f^{'}(u_\varepsilon)\psi = \nu_\varepsilon f'(u_\varepsilon)\psi, \psi \in H_0^1(B_R). \tag{B.10}$$

The main result in this section is the following theorem.

Theorem B.1. *Assume that there exists $c > 0$ such that the following gap condition holds*

$$\Big|\nu_1 - \frac{\varepsilon^2 m(m+n-2)}{r_0 V(r_0)}\Big| \geq c\varepsilon,\ \forall\ m = 0, 1, \cdots, \tag{B.11}$$

and the following non-degenerate condition holds

$$M^{''}(r_0) \neq -m(m+n-2) r_0^{n-3} V^{\sigma}(r_0), \quad \forall\ m = 0, 1, \cdots. \tag{B.12}$$

If $(\bar{\psi}_\varepsilon, \bar{\nu}_\varepsilon)$, $\bar{\psi}_\varepsilon \neq 0$, is a solution of (B.10), *then we have*

$$|\bar{\nu}_\varepsilon| \geq C\varepsilon^2. \tag{B.13}$$

To prove Theorem B.1, we first need some asymptotic behavior of the layered solution u_ε. By a scaling argument, we may assume that $V(r_0) = 1$. Let $x = r_\varepsilon + \varepsilon y$ and

$$\tilde{u}_\varepsilon(y) = V(r_\varepsilon)^{-\frac{1}{p-1}} u_\varepsilon\big(\varepsilon_1 y + r_\varepsilon\big), \quad \varepsilon_1 = \frac{\varepsilon}{\sqrt{V(r_\varepsilon)}},$$

and

$$\Delta^{'}_{\varepsilon_1} u = u^{''} + \frac{\varepsilon_1(n-1)}{r_\varepsilon + \varepsilon_1 y} u^{'}, \text{ where } y \in I_\varepsilon = \Big[-\frac{r_\varepsilon}{\varepsilon_1}, \frac{R}{\varepsilon_1}\Big). \tag{B.14}$$

In [37], we proved

Lemma B.2. *(Lemma 2.2 of* [37]*.) It holds*

(1) $r_\varepsilon = r_0 + o(\varepsilon)$;

(2) $\tilde{u}_\varepsilon(y) = w(y) + \varepsilon_1\phi_1(y) + \varepsilon_1^2\phi_2(y) + o(\varepsilon^2)$, *where $\phi_1(y)$ is the unique (odd) solution of*

$$\phi_1^{''} - \phi_1 + f^{'}(w)\phi_1 + \frac{n-1}{r_0} w^{'} - V^{'}(r_0) y w(y) = 0, \phi_1^{'}(0) = 0, \quad in\ \mathbb{R}^1, \tag{B.15}$$

and $\phi_2(y)$ is the unique (even) solution of

$$\begin{aligned} &\phi_2^{''} - \phi_2 + f^{'}(w)\phi_2 \\ &- \frac{n-1}{r_0^2} w^{'} y - \frac{1}{2} V^{''}(r_0) y^2 w(y) + \frac{n-1}{r_0}\phi_1^{'} - V^{'}(r_0) y\phi_1 + \frac{1}{2} f^{''}(w)\phi_1^2 = 0. \end{aligned} \tag{B.16}$$

Using Lemma B.2, we are now ready to prove Theorem B.1.

Proof of Theorem B.1: Let (ψ, ν) satisfy (B.10). The symmetry of $V(y)$ allows us to expand ψ into spherical harmonics. We write ψ as

$$\psi(x) = \sum_{m=0}^{\infty} \psi_m(r)\Theta_m(\theta), \quad r \in \big[0, \frac{R}{\varepsilon}\big), \theta \in S^{n-1} \tag{B.17}$$

where Θ_m, $m \geq 0$ are the eigenfunctions of the Laplace-Beltrami operator $-\Delta_{S^{n-1}}$ on the sphere S^{n-1}, normalized so that they constitute an orthonormal system in $L^2(S^{n-1})$. We take Θ_0 to be a positive constant, associated to the eigenvalue 0 and Θ_i, $1 \leq i \leq n$ is an appropriate multiple of $\frac{x_i}{|x|}$ which has eigenvalue $\lambda_i = n-1$, $1 \leq i \leq n$. In general, $\mu_m = m(m+n-2)$ denotes the eigenvalue associated to Θ_m, we repeat eigenvalues according to their multiplicity and we arrange them in an non-decreasing sequence.

The components ψ_m then satisfy the differential equations

$$\varepsilon^2\Delta\psi_m - V(r)\psi_m - \frac{\epsilon^2\mu_m}{r^2}\psi_m + f'(u_\varepsilon)\psi_m = \nu f'(u_\varepsilon)\psi_m, \psi_m = \psi_m(r) \in H_0^1(B_R). \tag{B.18}$$

Since $\psi \not\equiv 0$, there exists a m such that $\psi_m \not\equiv 0$. From now on, we consider (B.18) instead. For simplicity of notation, we denote ψ_m as ψ. Let us assume that there exists $(\bar{\psi}_\varepsilon, \bar{\nu}_\varepsilon)$ satisfying (B.18) such that $\bar{\nu}_\varepsilon = o(\varepsilon^2)$. We shall derive a contradiction.

Since $f'(u_\varepsilon) \leq C$, it is easy to see that

$$\varepsilon^2\mu_m \leq C. \tag{B.19}$$

Let

$$L_\varepsilon(\phi) = \Delta'_{\varepsilon_1}\phi - V_\varepsilon(\varepsilon_1 y + r_\varepsilon)\phi - \frac{\varepsilon_1^2\mu_m}{(r_\varepsilon + \varepsilon_1 y)^2}\phi + f'(\tilde{u}_\varepsilon)\phi$$

and

$$\psi_\varepsilon(y) = \bar{\psi}_\varepsilon\big(\varepsilon_1 y + r_\varepsilon\big), \quad \varepsilon_1 = \frac{\varepsilon}{\sqrt{V(r_\varepsilon)}}, \; \nu_\varepsilon = \frac{1}{V(r_\varepsilon)}\bar{\nu}_\varepsilon$$

Then

$$\Delta'_{\varepsilon_1}\psi_\varepsilon - V_\varepsilon(\varepsilon_1 y + r_\varepsilon)\psi_\varepsilon - \frac{\varepsilon_1^2\mu_m}{(r_\varepsilon + \varepsilon_1 y)^2}\psi_\varepsilon + f'(\tilde{u}_\varepsilon)\psi_\varepsilon = \nu_\varepsilon f'(\tilde{u}_\varepsilon)\psi_\varepsilon,$$

and $\nu_\varepsilon = o(\varepsilon^2)$.

Now we let ψ_1 be the unique even function of

$$\psi_1'' - \psi_1 + f'(w)\psi_1 = -\frac{(n-1)}{r_0}w'' + V'(r_0)yw' - f''(w)\phi_1 w' \quad \text{in } \mathbb{R}^1. \tag{B.20}$$

Set

$$\Psi_1(y) = w'(y) + \varepsilon_1\psi_1(y). \tag{B.21}$$

By direct calculation, we have

$$\begin{aligned} L_\varepsilon(\Psi_1) =& \varepsilon_1^2[-\frac{n-1}{r_0^2}yw'' - \frac{1}{2}V''(r_0)y^2w'] + \varepsilon_1^2[\frac{n-1}{r_0}\psi_1' - V'(r_0)y\psi_1] \\ &+ \varepsilon_1^2[f''(w)\psi_1\phi_1 + \frac{1}{2}f'''(w)\phi_1^2w' + f''(w)\phi_2 w'] - \frac{\varepsilon_1^2\mu_m}{(r_\varepsilon + \varepsilon_1 y)^2}\Psi_1 + o(\varepsilon^2). \end{aligned} \tag{B.22}$$

See (2.16) in [37] for details.

On the other hand, it is easy to see

$$L_\varepsilon(\Psi_0)=\frac{\varepsilon_1(n-1)}{r_\varepsilon+\varepsilon y}\Psi_0^{'}+(\nu_1-\frac{\varepsilon_1^2\mu_m}{(r_\varepsilon+\varepsilon_1 y)^2})\Psi_0+\varepsilon_1 f^{''}(w)\phi_1\Psi_0-\varepsilon_1 V'(r_\varepsilon)y\Psi_0+O(\varepsilon^2). \quad \text{(B.23)}$$

We decompose

$$\psi_\varepsilon=d_\varepsilon\Psi_0(y)+c_\varepsilon\Psi_1(y)+\psi_\varepsilon^\perp, \quad \int_{I_\varepsilon}\Psi_1(y)\psi_\varepsilon^\perp=0, \int_{I_\varepsilon}\Psi_0(y)\psi_\varepsilon^\perp=0, \quad \text{(B.24)}$$

where $\psi_\varepsilon^\perp$ satisfies

$$-L_\varepsilon(\psi_\varepsilon^\perp)=c_\varepsilon L_\varepsilon\Psi_1+d_\varepsilon L_\varepsilon(\Psi_0)-\nu_\varepsilon f'(\tilde{u}_\varepsilon)\psi_\epsilon \text{ in } I_\varepsilon, \quad \text{(B.25)}$$

and

$$\int_{I_\varepsilon}\Psi_1(y)\psi_\varepsilon^\perp=0, \int_{I_\varepsilon}\Psi_0(y)\psi_\varepsilon^\perp=0.$$

Then by the same argument as in of [2], we have that

$$|\psi_\varepsilon^\perp|\le C\Bigg((\varepsilon^2+\varepsilon^2\mu_m)|c_\varepsilon|+(\varepsilon+|\nu_1-\frac{\varepsilon^2\mu_m}{r_\varepsilon^2}|)|d_\varepsilon|\Bigg). \quad \text{(B.26)}$$

Now we multiply (B.18) by $w^{'}$, integrate over I_ε and use (B.26) to obtain

$$\begin{aligned}&c_\varepsilon\int_{I_\varepsilon}L_\varepsilon(\Psi_1)w^{'}+d_\varepsilon\int_{I_\varepsilon}L_\varepsilon(\Psi_0)w^{'}\\&=\nu_\varepsilon\big[c_\varepsilon\int_{I_\varepsilon}f'(\tilde{u}_\varepsilon)\Psi_1 w^{'}+O(\varepsilon d_\varepsilon)\big]+O\Big(\varepsilon(\varepsilon^2+\varepsilon^2\mu_m)|c_\varepsilon|+O(\varepsilon(\varepsilon+|\nu_1-\frac{\varepsilon_1^2\mu_m}{r_\varepsilon}|)|d_\varepsilon|\Big).\end{aligned} \quad \text{(B.27)}$$

Similarly, we multiply (B.18) by Ψ_0, integrate over I_ε to obtain

$$\begin{aligned}&c_\varepsilon\int_{I_\varepsilon}L_\varepsilon(\Psi_1)\Psi_0+d_\varepsilon\int_{I_\varepsilon}L_\varepsilon(\Psi_0)\Psi_0\\&=\nu_\varepsilon\big[d_\varepsilon\int_{I_\varepsilon}f'(\tilde{u}_\varepsilon)\Psi_0^2+O(\varepsilon|d_\varepsilon|)\big]+O\Big(\varepsilon(\varepsilon^2+\varepsilon^2\mu_m)|c_\varepsilon|+O(\varepsilon(\varepsilon+|\nu_1-\frac{\varepsilon_1^2\mu_m}{r_\varepsilon}|)|d_\varepsilon|\Big).\end{aligned} \quad \text{(B.28)}$$

We now analyze both sides of (B.27). The right hand side is relatively easy to understand since

$$\int_{I_\varepsilon}f'(\tilde{u}_\varepsilon)\Psi_1 w^{'}=\int_{\mathbb{R}}f'(w)(w^{'})^2+O(\varepsilon). \quad \text{(B.29)}$$

Using (B.22), we can deduce

$$\int_{I_\varepsilon}(L_\varepsilon(\Psi_1))w^{'}=B_0M^{''}(r_0)\varepsilon_1^2-\frac{\varepsilon_1^2\mu_m}{r_\varepsilon^2}\int_{\mathbb{R}}(w^{'})^2+o(\varepsilon^2)+O(\varepsilon^3\mu_m), \quad \text{(B.30)}$$

where

$$B_0 = -r_0^{1-n} \int_{\mathbb{R}} (w')^2,$$

See the calculations of (2.22)–(2.27) in [37] for details.

Using (B.22) again, we can also deduce

$$\int_{I_\varepsilon} (L_\varepsilon(\Psi_1))\Psi_0 = O(\varepsilon^3 \mu_m + \varepsilon^2). \tag{B.31}$$

Next, by (B.23), we find

$$\int_{I_\varepsilon} (L_\varepsilon(\Psi_0))w' = O(\varepsilon^3 \mu_m + \varepsilon), \tag{B.32}$$

and

$$\int_{I_\varepsilon} (L_\varepsilon(\Psi_0))\Psi_0 = O(\varepsilon^2) + (\nu_1 - \frac{\varepsilon_1^2 \mu_m}{r_\varepsilon^2}) \int_{\mathbb{R}} \Psi_0^2. \tag{B.33}$$

Substituting (B.30)-(B.33) into (B.27) and (B.28), we obtain

$$c_\varepsilon \Big(B_0 M''(r_0)\varepsilon_1^2 - \varepsilon_1^2 \frac{\mu_m}{r_\varepsilon^2} + O(\varepsilon^3 \mu_m) + o(\varepsilon_1^2) \Big) + O(\varepsilon d_\varepsilon) = 0, \tag{B.34}$$

and

$$c_\varepsilon O(\varepsilon_1^3 \frac{\mu_m}{r_\varepsilon^2}) + d_\varepsilon \big(\nu_1 - \frac{\varepsilon_1^2 \mu_m}{r_\varepsilon^2} + O(\varepsilon^2) \big) = 0. \tag{B.35}$$

From (B.35) and the gap condition (1.7), we have that

$$d_\varepsilon = O(\frac{\varepsilon_1^3 \mu_m}{|\nu_1 - \frac{\varepsilon_1^2 \mu_m}{r_\varepsilon^2}|}) c_\varepsilon. \tag{B.36}$$

Substituting (B.36) into (B.34), we obtain that

$$c_\varepsilon \left[B_0 M''(r_0)\varepsilon_1^2 - \varepsilon_1^2 \frac{\mu_m}{r_\varepsilon^2} + o(\varepsilon_1^2) + O(\frac{\varepsilon_1^4 \mu_m}{|\nu_1 - \frac{\varepsilon_1^2 \mu_m}{r_\varepsilon^2}|}) \right] = 0,$$

which, together with conditions (B.11) and (B.12), implies that

$$c_\varepsilon = 0, \quad d_\varepsilon = 0. \tag{B.37}$$

This forces $\psi_\varepsilon^\perp \equiv 0$, by (B.26). Thus, $\psi_\varepsilon \equiv 0$. A contradiction.

This completes the proof of Theorem B.1.

□

If we work in a sub-space H_s of $H_0^1(B_R(0))$, then we have the following result:

Theorem B.3. *Assume that the following gap condition holds*

$$\big|\nu_1 - \frac{\varepsilon^2 \bar{\mu}_m}{r_0 V(r_0)}\big| \geq c\varepsilon, \ \forall \ m \in \mathcal{N} \tag{B.38}$$

and the following non-degenerate condition holds

$$M''(r_0) \neq -\bar{\mu}_m r_0^{n-3} V^\sigma(r_0), \quad \forall m \in \mathcal{N}, \tag{B.39}$$

where $\bar{\mu}_m$, $m = 1, \cdots$, *are all the eigenvalues of* $-\Delta_{S^{n-1}}$ *on* S^{n-1}, *with the corresponding eigenfunctions* $\Theta_m(\theta)$ *satisfying* $\psi(r)\Theta_m(\theta) \in H_s$ *for any* $\psi(r)$.

If $(\bar{\psi}_\varepsilon, \bar{\nu}_\varepsilon)$, $\bar{\psi}_\varepsilon \in H_s$, $\bar{\psi}_\varepsilon \neq 0$, *is a solution of* (B.10), *then we have*

$$|\bar{\nu}_\varepsilon| \geq C\varepsilon^2. \tag{B.40}$$

Remark B.4. If $n = 3$ and H_s is the space defined in (1.5), then $\bar{\mu}_m \geq 6$, $m = 1, 2, \cdots$.

From Theorem B.1, or Theorem B.3, we can deduce the following corollary:

Corollary B.5. *Suppose that the conditions in Theorem B.1 (or Theorem B.3) hold. There is* $\lambda_0 > 0$ *such that for* $\lambda > \lambda_0$ *satisfying (B.11) (or (B.38)), the linearized operator* L_λ *from* $H_0^1(B_R)$ *(or* H_s*) into itself defined in* (B.5) *is invertible operator and satisfies*

$$\|L_\lambda \psi\| \geq c_0 \lambda^{-1} \|\psi\| \tag{B.41}$$

where $c_0 > 0$ *is a constant, independent of* λ.

Proof. Let $\varepsilon = \frac{1}{\sqrt{\lambda}}$. From Theorem B.1, we have that the following eigenvalue problem:

$$\varepsilon^2 \Delta\psi - V(|y|)\psi + f'(u_\varepsilon)\psi = \lambda_\varepsilon\big(\varepsilon^2\Delta\psi - V(|y|)\psi\big), \quad \psi \in H_0^1(B_R) \ \ (\text{or} H_s), \tag{B.42}$$

has a spectrum gap $|\lambda_\varepsilon| \geq c_0\varepsilon^2 = c_0\lambda^{-1}$. So, the result follows. □

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