LUMP TYPE SOLUTIONS: BACKLUND TRANSFORMATION AND SPECTRAL PROPERTIES

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ABSTRACT. There are various different ways to obtain traveling waves of lump type for the KP equation. We propose a general and simple approach to derive them via a Backlund transformation. This enables us to establish an explicit connection between those low energy solutions and high energy ones. Based on this construction, spectral analysis of the degree 6 solutions is then carried out in details. The analysis of higher energy ones can be done in an inductive way.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

KP equation is a two dimensional analogy of the classical KdV equation. It naturally appears in the theory of shallow water waves. As an important $2+1$ dimensional integrable system, it has been extensively studied for more than forty years, and many other integrable systems can be regarded as its suitable reduction. However, there are still some interesting questions remained to be answered for this equation. As a matter of fact, even the properties of its traveling wave solutions are not fully understood yet. For traveling waves, the KP equation reduces to the so called Boussinesq equation:

$$\partial_x^2 \left(\partial_x^2 u - u + 3u^2 \right) - \partial_y^2 u = 0. \tag{1}$$

In principle, solutions to this equation should play important role in the long time dynamics of the KP equation.

The Boussinesq equation in the form of (1) is of elliptic type and closely related to other PDEs such as GP equation. While there already exist a lot of works concerning elliptic equations of second order, the study of fourth order equations with both mathematical and physical significance is relatively few. In this paper, we would like to study the spectral properties of "lump type" solutions to (1). By lump type solutions, we mean solutions of (1) which decay to zero at infinity. This is a natural class of physically meaningful solutions. The name "lump type" comes from the fact that the following "classical" lump solution solves (1):

$$U(x, y) = 4 \frac{y^2 - x^2 + 3}{\left(x^2 + y^2 + 3\right)^2}.$$

The analysis of U has a long history. It is first obtained in [16, 22] by parameter degeneration. The spectral property of U is now well understood. Indeed, we have proved in [14] using Backlund transformation that U is nondegenerated, in the sense that the linearized operator at U does not have any nontrivial kernels. A direct consequence of this property is that the lump is orbitally stable under the

KP-I flow, which is globally well posed [11, 17, 18]. The asymptotical stability of U remains to be an unsolved open problem in this direction.

If we introduce the tau function τ by $u = 2\partial_x^2 \ln \tau$, then the equation (1) turns into the following bilinear equation:

$$\left(D_x^4 - D_x^2 - D_y^2\right)\tau \cdot \tau = 0. \tag{2}$$

Here D is the bilinear derivative operator introduced by Hirota [10]. One easily checks that the lump solution U corresponds to $\tau = x^2 + y^2 + 3$.

Our recent result [15] shows that real valued solution(with a mild decaying assumption) of (1) has to be rational, and the corresponding tau function, which solves (2), will be a polynomial of degree $k(k+1)$ with $k \in \mathbb{N}$. In the case of $k = 1$, it is not difficult to show that up to translation in the x, y variables and multiplication by a constant, real valued solution to (2) has to be $x^2 + y^2 + 3$.

According to our classification result mentioned above, the next family of tau functions of (2) are polynomials of degree 6, corresponding to $k = 2$. In Section 2, we show, using Backlund transformation, that the following family of polynomials $h_{A,B}$ solves (2), where

$$\begin{aligned} h_{A,B}(x,y) = {} & x^6 + 3x^4y^2 + 3x^2y^4 + y^6 + 25x^4 + 90x^2y^2 + 17y^4 \\ & + Bx^3 + 3Ax^2y - 3Bxy^2 - Ay^3 - 125x^2 + 475y^2 \\ & - Bx + 5Ay + 1875 + \frac{A^2}{4} + \frac{B^2}{4}. \end{aligned}$$

Here $A, B \in \mathbb{R}$ are parameters. The solutions $2\partial_x^2 \ln h_{A,B}$ of (1) will be denoted by $u_{A,B}$. Presumably, any degree 6 solution should belong to this family. Note that this family of solutions are first obtained in [9], using a limiting procedure on the involved parameters.

As we know, solutions of most elliptic equations do not have explicit expression. Therefore the existence of such a family of non-radially-symmetric, rational, solutions $u_{A,B}$ to the Boussinesq equation is by itself an interesting phenomenon and provides an important scenario for us to analyze various properties of the solutions relevant in the subject of elliptic equations.

Our first result is the following

Theorem 1. *For any A, B, the solution $u = u_{A,B}$ is nondegenerated in the following sense: If ϕ is a solution of the linearized operator*

$$\partial_x^2\left(\partial_x^2\phi - \phi + 6u\phi\right) - \partial_y^2\phi = 0,$$

with

$$\phi(x,y) \to 0, \text{ as } x^2 + y^2 \to +\infty,$$

then there exist constants $c_1, ..., c_4$, such that

$$\phi = c_1\partial_x u + c_2\partial_y u + c_3\partial_A u + c_4\partial_B u.$$

Moreover, the Morse index of u is equal to 3.

The Morse index of u is, by definition, the number of negative eigenvalues(counted by multiplicity) of the operator

$$\eta \to \partial_x^2 \phi - \phi + 6u\phi - \partial_x^{-2}\partial_y^2 \phi.$$

In the special case that $A = B = 0$, the solution $u_{0,0}$ will be even in both variables. This is precisely the solution obtained for the first time in [19]. If we consider kernel which is also even in both variables, then from Theorem 1, we know that it has to be 0. As a direct application of this fact, one can then construct new subsonic traveling wave solutions to the GP equation from $u_{0,0}$, using the same perturbation method developed in [13]. More precisely, we have

Corollary 2. *For each $\varepsilon > 0$ sufficiently small, there exists a traveling wave solution*

$$\Phi(t,x,y) = \phi\left(x - \left(\sqrt{2} - \varepsilon^2\right) t, y\right),$$

to the following GP equation

$$i\partial_t \Phi + \Delta\Phi + \left(1 - |\Phi|^2\right)\Phi = 0,$$

which has the asymptotic expansion

$$\phi(x,y) = 1 + i\varepsilon\partial_x^{-1} u^* + O\left(\varepsilon^2\right),$$

where

$$u^*(x,y) = -\frac{3}{2} u_{0,0}\left(2^{\frac{3}{4}} x, 2^{\frac{1}{2}} y\right).$$

To the best of our knowledge, this is the first result in the construction of high energy traveling wave solution for the GP equation in the subsonic regime, although there are quite a few existence results on the least energy solutions.

From the even solution $u_{0,0}$ and our nondegeneracy result, one can actually also construct new nontrivial solutions to following generalized KP-I equation:

$$\partial_x^2 u - u + |u|^\alpha - \partial_x^{-2}\partial_y^2 u = 0,$$

provided that the exponent α is sufficiently close to 2. Note that for this type of generalized KP-I equation, one can use variational method (See [4, 5, 12]) to construct the ground state solution, whose Morse index is presumably equal to 1. On the other hand, the new solutions close to $u_{0,0}$ will have Morse index 3, and it seems to be hard to construct them by variational method. It is also interesting to see whether this type of solutions exist for α not close to 2.

The method developed in this paper to prove Theorem 1 can actually be used to construct and analyze the spectral properties of higher energy lump type solutions. We emphasize that general lump type solutions of KP equation with degree $k(k+1)$, and k free parameters have already been found in [8, 19, 20, 21], using the Wronskian representation. It is also worth mentioning that there are also other methods to construct these solutions, see [1, 2, 6, 7, 23, 24]. Our method has the advantage that it establishes explicit connection between low energy solutions and high energy ones. The complete classification of lump type solutions remains open.

The paper is organized in the following way. In Section 2, we explain how to use the Backlund transformation to create higher energy solutions from the low energy solutions. We point out that these solutions are in general complex valued. In Section 3, we analyze the precise asymptotic behavior of the eigenfunctions of the linearized operator and show that the Morse index of $h_{A,B}$ is equal to 3. This is based on a "reverse" Lyapunov-Schmidt reduction type argument. The very delicate point here is that the reduced problem is actually degenerated.

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2. BACKLUND TRANSFORMATION FROM LOW ENERGY SOLUTIONS TO HIGH ENERGY ONES

In this section, we propose a general scheme to create high energy lump type solutions from low energy ones. Although there are other methods to construct these solutions, our method has the advantage that it establishes explicit link between solutions with different energy, which in turn enables us to study their spectral properties using an inductive argument.

The Boussinesq equation has the following Backlund transformation(See [14]):

$$\begin{cases} \left(D_x^2 + \mu D_x + \frac{i}{\sqrt{3}} D_y - \lambda\right) f \cdot g = 0, \\ \\ \left(\left(3\lambda - 1\right) D_x - \sqrt{3} i \mu D_y + D_x^3 - \sqrt{3} i D_x D_y + v\right) f \cdot g = 0. \end{cases} \tag{3}$$

Here μ, λ, v are arbitrary parameters, and throughout the paper, i will be the imaginary unit. The Backlund transformation (3) has the following property: If f satisfies the bilinear equation (2), and f, g satisfy the system (3), then g will automatically be a solution of (2). In this way, to get a new solution, we only need to solve a linear problem involving third order derivatives, instead of the original nonlinear problem with fourth order derivatives.

If f, g are polynomials, then in view of the highest degree terms, necessarily $\lambda = v = 0$. Then inspecting the highest degree terms, we find that $\mu = \pm \frac{1}{\sqrt{3}}$. We are thus lead to consider the following

$$\begin{cases} \left(D_x^2 + \mu D_x + \frac{i}{\sqrt{3}} D_y\right) f \cdot g = 0, \\ \\ \left(-D_x - \sqrt{3} i \mu D_y + D_x^3 - \sqrt{3} i D_x D_y\right) f \cdot g = 0. \end{cases} \tag{4}$$

Let f be a real valued polynomial solution of (2) of degree $2n$. The classification result in [15] tells us that

$$n = \frac{k(k+1)}{2}, \text{ for some } k \in \mathbb{N}.$$

Those degree j terms in f will be denoted by f_j. Our first key observation is that it will be more convenient to consider the problem in the z-$\bar{z}$ coordinate, rather than the usual x-y coordinate, where

$$z = x + yi \text{ and } \bar{z} = x - yi.$$

It follows from Lemma 13 of [15] that $f_{2n} = z^n \bar{z}^n$, and by suitable translation in the x and y variables, we can assume that $f_{2n-1} = 0$. We also observe that

$$D_x + iD_y = 2D_{\bar{z}} \text{ and } D_x - iD_y = 2D_z.$$

This formula, although very simple, will be frequently used in this section.

The following lemma tells us that if f, g are connected through Backlund transformation, then the degree of g will be a square number and essentially determined by that of f, and more importantly, g will be complex valued.

Lemma 3. *Let f be a real valued polynomial solution of (2) of degree $2n$ with $f_{2n-1} = 0$. Suppose f, g satisfies (4). Then the highest degree term g_m of g has the form*

$$g_m = z^j \bar{z}^n,$$

where j satisfies

$$n(n-1) - 2nj + j(j-1) = 0. \tag{5}$$

In particular, if $n = \frac{k(k+1)}{2}$ for some integer k, then the degree $j + n$ of g is equal to

$$k^2 \text{ or } (k+1)^2.$$

Moreover, $g_{m-1} = \sqrt{3} n z^j \bar{z}^{n-1} + c z^{j-1} \bar{z}^n$, where

$$c = -\frac{(j-n)(j+n-1)}{2\sqrt{3}(n-j+1)},$$

and g_{m-2} solves

$$2D_{\bar{z}} f_{2n} \cdot g_{m-2} + 2D_{\bar{z}} f_{2n-2} \cdot g_m + \sqrt{3} D_x^2 f_{2n} \cdot g_{m-1} = 0.$$

Proof. Balancing highest degree terms in (4) requires

$$D_{\bar{z}} f_{2n} \cdot g_m = 0.$$

This readily implies

$$g_m = c z^j \bar{z}^n,$$

for some constant c and non-negative integer j. The constant c can be normalized to be 1.

Since $f_{2n-1} = 0$, the degree $m-1$ terms in g should satisfy

$$\begin{cases} 2D_{\bar{z}} f_{2n} \cdot g_{m-1} + \sqrt{3} D_x^2 f_{2n} \cdot g_m = 0, \\ \\ 2D_{\bar{z}} f_{2n} \cdot g_{m-1} + \sqrt{3} i D_x D_y f_{2n} \cdot g_m = 0. \end{cases} \tag{6}$$

We compute

$$\begin{aligned} D_x^2 (z^n \bar{z}^n) \cdot \left(z^j \bar{z}^k\right) = & \left[n(n-1) - 2nj + j(j-1)\right] z^{j+n-2} \bar{z}^{n+k} \\ & + \left[2n^2 - 2nk - 2nj + 2jk\right] z^{j+n-1} \bar{z}^{n+k-1} \\ & + \left[n(n-1) - 2nk + k(k-1)\right] z^{j+n} \bar{z}^{n+k-2}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} D_x D_y (z^n \bar{z}^n) \cdot \left(z^j \bar{z}^k\right) = & \left[n(n-1) - 2nj + j(j-1)\right] i z^{j+n-2} \bar{z}^{k+n} \\ & + \left[-n(n-1) + 2nk - k(k-1)\right] i z^{j+n} \bar{z}^{k+n-2}. \end{aligned}$$

From the system (6), we deduce

$$D_x^2 f_{2n} \cdot g_m = i D_x D_y f_{2n} \cdot g_m,$$

which implies that when $k = n$, there holds

$$\begin{aligned} & [n(n-1) - 2nj + j(j-1)] z^{j+n-2} \bar{z}^{n+k} + [n(n-1) - 2nk + k(k-1)] z^{j+n} \bar{z}^{n+k-2} \\ & = -[n(n-1) - 2nj + j(j-1)] z^{j+n-2} \bar{z}^{k+n} \\ & \quad - [-n(n-1) + 2nk - k(k-1)] z^{j+n} \bar{z}^{k+n-2}. \end{aligned}$$

From this we get the relation (5) between j and n.

Now g_{m-1} satisfies

$$D_{\bar{z}} f_{2n} \cdot g_{m-1} = -\frac{\sqrt{3}}{2} D_x^2 f_{2n} \cdot g_m = \sqrt{3} n z^{j+n} \bar{z}^{2n-2}.$$

Therefore we obtain

$$g_{m-1} = \sqrt{3} n z^j \bar{z}^{n-1} + c z^{j-1} \bar{z}^n,$$

where c is a constant to be determined.

To find c, we consider the equations to be satisfied by g_{m-2} :

$$\begin{cases} 2D_{\bar{z}} f_{2n} \cdot g_{m-2} + 2D_{\bar{z}} f_{2n-2} \cdot g_m + \sqrt{3} D_x^2 f_{2n} \cdot g_{m-1} = 0, \\ \\ 2D_{\bar{z}} f_{2n} \cdot g_{m-2} + 2D_{\bar{z}} f_{2n-2} \cdot g_m + \sqrt{3} i D_x D_y f_{2n} \cdot g_{m-1} - D_x^3 f_{2n} \cdot g_m = 0. \end{cases}$$

We compute

$$\begin{aligned} & D_x^3 (z^n \bar{z}^n) \cdot \left(z^j \bar{z}^k \right) \\ & = [n(n-1)(n-2) - 3kn(n-1) + 3nk(k-1) - k(k-1)(k-2)] z^{j+n} \bar{z}^{k+n-3} \\ & \quad + \left[3n^2 (n-1) - 3jn(n-1) - 6n^2 k + 6njk + 3nk(k-1) - 3jk(k-1) \right] z^{j+n-1} \bar{z}^{k+n-2} \\ & \quad + \left[3n^2 (n-1) - 6n^2 j - 3nk(n-1) + 6njk + 3nj(j-1) - 3j(j-1)k \right] z^{j+n-2} \bar{z}^{k+n-1} \\ & \quad + [n(n-1)(n-2) - 3nj(n-1) + 3nj(j-1) - j(j-1)(j-2)] z^{j+n-3} \bar{z}^{k+n}. \end{aligned}$$

We find that if j and n satisfy (5), then

$$D_x^3 (z^n \bar{z}^n) \cdot \left(z^j \bar{z}^n \right) = \left(6jn - 6n^2 \right) z^{j+n-1} \bar{z}^{2n-2} + 2(j-n)(j+n-1) z^{j+n-3} \bar{z}^{2n},$$

and moreover,

$$\begin{aligned} & \sqrt{3} D_x^2 f_{2n} \cdot g_{m-1} - \sqrt{3} i D_x D_y f_{2n} \cdot g_{m-1} + D_x^3 f_{2n} \cdot g_m \\ & = 2\sqrt{3} D_x D_z (z^n \bar{z}^n) \left(\sqrt{3} n z^j \bar{z}^{n-1} + c z^{j-1} \bar{z}^n \right) + D_x^3 (z^n \bar{z}^n) \left(z^j \bar{z}^n \right) \\ & = 2\sqrt{3} c (2n - 2j + 2) z^{j+n-3} \bar{z}^{2n} + 2(j-n)(j+n-1) z^{j+n-3} \bar{z}^{k+n}. \end{aligned}$$

Compatibility of the two equations in (4) requires the right hand side to be 0. It follows that

$$c = -\frac{(j-n)(j+n-1)}{2\sqrt{3}(n-j+1)}.$$

This proves the assertion. $\square$

This lemma tells us that if $n = 1$, then the degree of g has to be 1 or 4. We now proceed to construct explicit Backlund transformations from

$$f = x^2 + y^2 + 3, \tag{7}$$

the first nontrivial solution, to a family of degree 4 polynomial. We will see that there will be a free complex parameter appearing in the process.

Proposition 4. *Let f be given by* (7) *and*

$$g = z^3\bar{z} + \sqrt{3}z^3 + \sqrt{3}z^2\bar{z} + 12z^2 - 3\bar{z}^2 + 3z\bar{z} + 9\sqrt{3}z + \alpha\bar{z} - 36 + \sqrt{3}\alpha,$$

where $\alpha \in \mathbb{C}$ is a parameter. Then f, g satisfies

$$\begin{cases} \left(D_x^2 + \frac{1}{\sqrt{3}}D_x + \frac{i}{\sqrt{3}}D_y\right) f \cdot g = 0, \\ \left(D_x^3 - \sqrt{3}iD_xD_y - D_x - iD_y\right) f \cdot g = 0. \end{cases}$$

As a consequence, g is a solution to the bilinear equation (2).

Proof. Note that $f_2 = z\bar{z}$ and $f_0 = 3$. Applying Lemma 3, we find $g_4 = z^3\bar{z}$ and $g_3 = \sqrt{3}z^3 + \sqrt{3}z^2\bar{z}$.

To get g_2, we observe that it satisfies

$$2D_{\bar{z}}f_2 \cdot g_2 + 2D_{\bar{z}}f_0 \cdot g_4 + \sqrt{3}D_x^2 f_2 \cdot g_3 = 0.$$

Solving this equation gives

$$g_2 = 12z^2 - 3\bar{z}^2 + az\bar{z},$$

where a is a parameter still to be determined in the next step.

To find a and g_1, we use the fact that g_1 satisfies

$$\begin{cases} 2D_{\bar{z}}f_2 \cdot g_1 + 2D_{\bar{z}}f_0 \cdot g_3 + \sqrt{3}D_x^2 f_2 \cdot g_2 + \sqrt{3}D_x^2 f_0 \cdot g_4 = 0, \\ 2D_{\bar{z}}f_2 \cdot g_1 + 2D_{\bar{z}}f_0 \cdot g_3 + \sqrt{3}iD_xD_y f_2 \cdot g_2 + \sqrt{3}iD_xD_y f_0 \cdot g_4 - D_x^3 f_2 \cdot g_3 = 0. \end{cases}$$

Compatibility of these two equations implies that $a = 3$ and

$$g_2 = 12z^2 - 3\bar{z}^2 + 3z\bar{z}.$$

Now solving the following equation for g_1 :

$$2D_{\bar{z}}f_2 \cdot g_1 + 2D_{\bar{z}}f_0 \cdot g_3 + \sqrt{3}D_x^2 f_2 \cdot g_2 + \sqrt{3}D_x^2 f_0 \cdot g_4 = 0,$$

we get

$$g_1 = 9\sqrt{3}z + \alpha\bar{z},$$

where α is a parameter.

To see whether or not α can be arbitrary, we consider the equations satisfied by g_0 :

$$\begin{cases} 2D_{\bar{z}}f_2 \cdot g_0 + 2D_{\bar{z}}f_0 \cdot g_2 + \sqrt{3}D_x^2 f_2 \cdot g_1 + \sqrt{3}D_x^2 f_0 \cdot g_3 = 0, \\ 2D_{\bar{z}}f_2 \cdot g_0 + 2D_{\bar{z}}f_0 \cdot g_2 + \sqrt{3}iD_xD_y f_2 \cdot g_1 + \sqrt{3}iD_xD_y f_0 \cdot g_3 - D_x^3 f_2 \cdot g_2 - D_x^3 f_0 \cdot g_4 = 0. \end{cases}$$

Direct computation tells us that this system is compatible for any α, and we are lead to

$$2D_{\bar{z}} f_2 \cdot g_0 + \left(72 - 2\sqrt{3}\alpha\right) z = 0,$$

which implies that $g_0 = -36 + \sqrt{3}\alpha$. One then checks that the function g obtained in this way indeed solves (4) . This finishes the proof. □

2.1. **A family of degree 6 tau functions and their Backlund transformation.** We have obtained a family of degree 4 polynomials:

$$g = z^3\bar{z} + \sqrt{3}z^3 + \sqrt{3}z^2\bar{z} + 12z^2 - 3\bar{z}^2 + 3z\bar{z} + 9\sqrt{3}z + \alpha\bar{z} - 36 + \sqrt{3}\alpha.$$

We would like to find all the degree 6 polynomial h such that g, h are connected through the Backlund transformation. They are supposed to satisfy the following system

$$\begin{cases} \left(D_x - iD_y - \sqrt{3}D_x^2\right) g \cdot h = 0, \\ \\ \left(D_x - iD_y + \sqrt{3}iD_xD_y - D_x^3\right) g \cdot h = 0. \end{cases}$$

In the z-$\bar{z}$ coordinate, it takes the form:

$$\begin{cases} \left(2D_z - \sqrt{3}D_x^2\right) g \cdot h = 0, \\ \\ \left(2D_z + \sqrt{3}iD_xD_y - D_x^3\right) g \cdot h = 0. \end{cases} \tag{8}$$

The following result provides the explicit formula of a family of degree 6 polynomial solutions.

Proposition 5. *Let α, β be parameters and*

$$\begin{aligned} h = {} & z^3\bar{z}^3 - 2\sqrt{3}z^2\bar{z}^3 + 2\sqrt{3}z^3\bar{z}^2 - 3z^4 + 15z^2\bar{z}^2 + 6z\bar{z}^3 - 3\bar{z}^4 + 6z^3\bar{z} \\ & + \beta z^3 + 24\sqrt{3}z^2\bar{z} - 24\sqrt{3}z\bar{z}^2 + \left(3\sqrt{3} + \alpha\right)\bar{z}^3 \\ & - \left(90 + 2\sqrt{3}\right) z^2 + 63z\bar{z} - \left(72 - 2\sqrt{3}\alpha\right)\bar{z}^2 \\ & + \left(189\sqrt{3} - 3\alpha + 6\beta\right) z + \left(-180\sqrt{3} + 6\alpha - 3\beta\right)\bar{z} \\ & + 1161 - 6\sqrt{3}\alpha + 9\sqrt{3}\beta + \alpha\beta. \end{aligned}$$

Then g, h satisfy (8) .

Proof. The highest degree terms of h has to be $h_6 = z^3\bar{z}^3$. The h_5 term can be obtained by solving the equation:

$$2D_z g_4 \cdot h_5 = \sqrt{3}D_x^2 g_4 \cdot h_6 - 2D_z g_3 \cdot h_6,$$

which gives

$$h_5 = -2\sqrt{3}z^2\bar{z}^3 + cz^3\bar{z}^2,$$

where c is a parameter to be determined using information of h_4, which satisfies

$$2D_z g_4 \cdot h_4 + 2D_z g_3 \cdot h_5 + 2D_z g_2 \cdot h_6 - \sqrt{3}D_x^2 g_4 \cdot h_5 - \sqrt{3}D_x^2 g_3 \cdot h_6 = 0,$$

and

$$2D_z g_4 \cdot h_4 + 2D_z g_3 \cdot h_5 + 2D_z g_2 \cdot h_6 + \sqrt{3} i D_x D_y g_4 \cdot h_5$$
$$+ \sqrt{3} i D_x D_y g_3 \cdot h_6 - D_x^3 g_4 \cdot h_6 = 0.$$

The compatibility of these two equations gives $c = 2\sqrt{3}$ and hence

$$h_5 = -2\sqrt{3} z^2 \bar{z}^3 + 2\sqrt{3} z^3 \bar{z}^2.$$

With h_5 at hand, h_4 can be found by solving the equation

$$2D_z g_4 \cdot h_4 + 2D_z g_3 \cdot h_5 + 2D_z g_2 \cdot h_6 - \sqrt{3} D_x^2 g_4 \cdot h_5 - \sqrt{3} D_x^2 g_3 \cdot h_6 = 0.$$

This gives

$$h_4 = -3z^4 + 15z^2\bar{z}^2 + 6z\bar{z}^3 - 3\bar{z}^4 + cz^3\bar{z}.$$

Again, c is a parameter to be determined, using the equation of h_3.

Now h_3 satisfies

$$2D_z g_4 \cdot h_3 + 2D_z g_3 \cdot h_4 + 2D_z g_2 \cdot h_5 + 2D_z g_1 \cdot h_6$$
$$- \sqrt{3} D_x^2 g_4 \cdot h_4 - \sqrt{3} D_x^2 g_3 \cdot h_5 - \sqrt{3} D_x^2 g_2 \cdot h_6 = 0,$$

and

$$2D_z g_4 \cdot h_3 + 2D_z g_3 \cdot h_4 + 2D_z g_2 \cdot h_5 + 2D_z g_1 \cdot h_6 + \sqrt{3} i D_x D_y g_4 \cdot h_4$$
$$+ \sqrt{3} i D_x D_y g_3 \cdot h_5 + \sqrt{3} i D_x D_y g_2 \cdot h_6 - D_x^3 g_4 \cdot h_5 - D_x^3 g_3 \cdot h_6 = 0.$$

The compatibility implies that $c = 6$, and we deduce that

$$h_4 = -3z^4 + 15z^2\bar{z}^2 + 6z\bar{z}^3 - 3\bar{z}^4 + 6z^3\bar{z}.$$

Having obtained h_4, we then proceed to solve the equation for h_3 and find that for some parameter β,

$$h_3 = 24\sqrt{3} z^2 \bar{z} - 24\sqrt{3} z \bar{z}^2 + \left(3 + \frac{\alpha}{\sqrt{3}}\right) \sqrt{3} \bar{z}^3 + \beta z^3.$$

The rest of terms h_2, h_1, h_0 follow from routine computation, and it turns out that β is a free parameter. We omit the details. □

Observe that the function h given by Proposition 5 is not real valued. But we are mainly interested in real valued solutions. In view of this, we replace y by $y - \frac{2\sqrt{3}i}{3}$, and choose the complex parameters α, β such that

$$\alpha - \beta = Ai - \frac{211}{3\sqrt{3}},$$
$$\alpha + \beta = B - 3\sqrt{3},$$

where A, B are real numbers. Then the function h becomes

$$h_{A,B}(x, y) := x^6 + 3x^4y^2 + 3x^2y^4 + y^6 + 25x^4 + 90x^2y^2 + 17y^4$$
$$+ Bx^3 + 3Ax^2y - 3Bxy^2 - Ay^3 - 125x^2 + 475y^2$$
$$- Bx + 5Ay + 1875 + \frac{A^2}{4} + \frac{B^2}{4}.$$

This is a family of real valued solution. Note that translation along the x or y direction still yields a solution(the rotation will not), hence there are all together 4 free real parameters in the whole family of solutions. Now if

$$A = 0 \text{ and } B = 0,$$

then h will be an even solution, which equals

$$\begin{aligned} &x^6 + 3x^4y^2 + 3x^2y^4 + y^6 + 25x^4 + 90x^2y^2 + 17y^4 \\ &- 125x^2 + 475y^2 + 1875. \end{aligned}$$

Note that this function is not radially symmetry. For general parameters A, B, the solution does not have any symmetry.

We should emphasize that although $h_{A,B}$ is real valued, it is connected by Backlund transformation, via a degree 4 polynomial, to the following degree 2 polynomial:

$$x^2 + \left(y - \frac{2\sqrt{3}i}{3}\right)^2 + 3,$$

which is not real valued.

2.2. **The general case.** Our construction can then be iterated to created higher energy solutions(Note that energy is completely determined by the degree, [9]). One can indeed directly write down an algorithm to do this computation. At each stage, one find the solutions from their highest degree terms to lower degree terms. More precisely, suppose we have already found a polynomial solution f to the bilinear equation, with highest degree term equals $z^n\bar{z}^n$, and we want to find its Backlund transformation g. Then once we have found g_j for $j > m$, then from the Backlund transformation, we see that g_m will satisfy a system of equations of the form

$$\begin{aligned} D_{\bar{z}}\left(z^n\bar{z}^n\right) \cdot g_m &= \mathrm{RHS}_1, \\ D_{\bar{z}}\left(z^n\bar{z}^n\right) \cdot g_m &= \mathrm{RHS}_2, \end{aligned}$$

where RHS_1 and RHS_2 contain terms explicitly polynomial terms from g_j with $j > m$.

An important question is, in which form, should a free parameter appear. Assume that the term g_{m+1} has a parameter term $\sigma z^j \bar{z}^n$ to be determined, where $j + n = m + 1$. The compatibility of these two equations is $\mathrm{RHS}_1 - \mathrm{RHS}_2 = 0$. In this equation, the parameter σ appears as

$$\sigma \left(D_x^2 - iD_xD_y\right)\left(z^n\bar{z}^n\right) \cdot \left(z^j\bar{z}^n\right).$$

Direct computation tells us that this equals

$$\sigma \left[n\left(n-1\right) - 2nj + j\left(j-1\right)\right] z^{j+n-2}\bar{z}^{2n}.$$

Hence we conclude that free parameter(that is, no restriction on σ) can occur only if j satisfies

$$n\left(n-1\right) - 2nj + j\left(j-1\right) = 0.$$

In the case of $n = 3$, the free parameter term is $\sigma z\bar{z}^3$.

Next, let us assume that the polynomial solution g, with highest degree term $z^p \bar{z}^n$ has been found. We would like to find function h by another Backlund transformation. Similar as above, once we have found h_j for $j > m$, then h_m will satisfy a system of equations of the form

$$D_z \left(z^p \bar{z}^n \right) \cdot h_m = \text{RHS}_1,$$
$$D_z \left(z^p \bar{z}^n \right) \cdot h_m = \text{RHS}_2.$$

Assume that h_{m+1} has a parameter term $\alpha z^p \bar{z}^j$ to be determined. Then α enters into the compatibility condition as

$$\alpha \left(D_x^2 + i D_x D_y \right) \left(z^p \bar{z}^n \right) \cdot \left(z^p \bar{z}^j \right).$$

This equals

$$\alpha \left[n (n-1) - 2nj + j (j-1) \right] z^{2p} \bar{z}^{j+n-2}.$$

Therefore, again, free parameter can occur only if

$$n (n-1) - 2nj + j (j-1) = 0.$$

In the case of $n = 3$, the free parameter term is $\alpha z^6 \bar{z}$.

Remark 6. *Our algorithm tells us that, at least formally, if we consider those solutions(complex valued) whose leading term is $z^n \bar{z}^n$, where $n = \frac{1}{2} k (k+1)$, then the space of these solutions should have complex dimension $2k$. Moreover, the space of real valued solutions with degree $k (k+1)$ is expected to have real dimension $2k$.*

3. NONDEGENERACY AND MORSE INDEX OF DEGREE 6 SOLUTIONS

In this section, we will show that the family of real valued solutions $u_{A,B}$ to the Boussinesq equation corresponding to $h_{A,B}$ have Morse index 3.

Our starting point in the computation of Morse index is to analyze the asymptotic behavior of $u = u_{0,B}$ for B large.

In view of the fact that

$$u = 2\partial_x^2 \ln h_{0,B} = 2 \frac{h_{0,B} \partial_x^2 h_{0,B} - \left(\partial_x h_{0,B} \right)^2}{h_{0,B}^2},$$

for B large, the maximum of u should take place around the points (x, y) which solve the system of algebraic equations:

$$\begin{cases} \phi = 0, \\ \partial_x \phi = 0, \end{cases} \tag{9}$$

where ϕ designates the main order term of $h_{0,B}$(away from the maximum of u) and is defined by

$$\phi (x, y) = x^6 + 3x^4 y^2 + 3x^2 y^4 + y^6 + Bx^3 - 3Bxy^2 + \frac{B^2}{4}.$$

Solving (9) and setting $\gamma = \left(\frac{B}{2} \right)^{\frac{1}{3}}$, we obtain the following three points P_j on the (x, y) plane:

$$P_1 = (-\gamma, 0), \; P_2 = \frac{1}{2} \left(\gamma, -\sqrt{3}\gamma \right), \; P_3 = \frac{1}{2} \left(\gamma, \sqrt{3}\gamma \right). \tag{10}$$

These three points are the vertices of an equilateral triangle. This is in agreement with the formal computation for the dynamics of peaks of KP-I equation, carried out in [9]. The reason that P_j are in this position will be clear later on.

Let us set

$$L(x,y) = x^2 + y^2 + 3.$$

Recall that we use U to denote the classical lump solution. That is,

$$U(x,y) = 2\partial_x^2 \ln L = 4\frac{y^2 - x^2 + 3}{(x^2+y^2+3)^2}. \tag{11}$$

Then we define

$$L_j(\cdot) = L\left(\cdot - P_j\right), \text{ and } U_j = 2\partial_x^2 L_j, \;\; j = 1,2,3.$$

The following result describes the asymptotic behavior of u as B(or γ) tends to $+\infty$.

Lemma 7. *The error between u and $U_1 + U_2 + U_3$ satisfies*

$$\|u - (U_1 + U_2 + U_3)\|_{L^\infty(\mathbb{R}^2)} \to 0, \text{ as } B \to +\infty.$$

Proof. We have

$$\begin{aligned} u - (U_1+U_2+U_3) &= 2\partial_x^2 \left(\ln h - \ln L_1 - \ln L_2 - \ln L_3\right) \\ &= -2\partial_x^2\left[\ln\left(1 - \frac{h - L_1L_2L_3}{h}\right)\right]. \end{aligned}$$

Direct computation tells us that

$$\begin{aligned} \eta &:= h - L_1L_2L_3 \\ &= 16x^4 + 72x^2y^2 + 8y^4 - \left(152 + 9\gamma^2\right)x^2 \\ &\quad + \left(448 - 9\gamma^2\right)y^2 - Bx + 1848 - 27\gamma^2 - 9\gamma^4. \end{aligned}$$

Observe that at P_j, the main contribution to h comes from those degree 4 terms. Therefore, to estimate the error around P_j, we need to have a better control of η at P_j. It turns out that $\eta\left(P_j\right)$ is of the order $O\left(\gamma^2\right)$. More precisely,

$$\eta(P_1) = -179\gamma^2 + 1848, \quad \eta(P_2) = \eta(P_3) = 271\gamma^2 + 1848.$$

From these, we then conclude with little work that

$$\left\|\frac{h - L_1L_2L_3}{h}\right\|_{L^\infty(\mathbb{R}^2)} \to 0,$$

which readily implies

$$\|u - (U_1 + U_2 + U_3)\|_{L^\infty(\mathbb{R}^2)} \to 0, \text{ as } B \to +\infty.$$

This finishes the proof. □

Lemma 7 provides a rough picture of the solution u. However, to analyze its Morse index, we need more precise expansion of u. To obtain the required expansion, it turns out that the explicit form of the function h does not help too much. Therefore we use the mapping property of the linearized Boussinesq operator. This will be done in sequel.

For a function q, we use $\mathcal{L}_q$ to denote the linearized Boussinesq operator at q, with the following form:

$$\mathcal{L}_q \eta = \partial_x^2 \eta - \eta + 6q\eta - \partial_x^{-2} \partial_y^2 \eta.$$

Here $\partial_x^{-1} = \int_{-\infty}^{x}$. One of the reason that we integrate twice in the original form of the Boussinesq equation is that the operator $\mathcal{L}_q$ is self adjoint. We emphasize that in view of the definition of ∂_x^{-1}, one should be very careful about the integrability of the function.

We let $\left(x_j^*, y_j^*\right)$ be the point close to $P_j = (\tilde{x}_j, \tilde{y}_j)$, which is introduced in (10). Setting

$$U_j^* (x, y) := U\left(x - x_j^*, y - y_j^*\right)$$

and writing

$$u = U^* + \xi,$$

where U^* is the "approximate" solution defined by

$$U^* = U_1^* + U_2^* + U_3^*,$$

and ξ is a perturbation term satisfying the following orthogonality condition: For $k = 1, 2, 3,$

$$\int_{\mathbb{R}^2} \xi \partial_x U_k^* dxdy = 0 \quad \text{and} \quad \int_{\mathbb{R}^2} \xi \partial_y U_k^* dxdy = 0.$$

This can always be achieved by perturbing $(\tilde{x}_k, \tilde{y}_k)$ into $\left(x_k^*, y_k^*\right)$. Note that $\partial_x U, \partial_y U$ are kernels of the operator $\mathcal{L}_U$. Hence $\partial_x U_k^*, \partial_y U_k^*$ are "approximate" kernels of $\mathcal{L}_{U^*}$.

By Lemma 7, if the distance between $\left(x_j^*, y_j^*\right)$ and P_j is close enough, then $\|\xi\|_{L^\infty(\mathbb{R}^2)}$ will be small, provided that B is large. Since u satisfies the Boussinesq equation, the perturbation ξ should satisfy the following nonlinear equation

$$\mathcal{L}_{U^*} \xi = -E(U^*) - 3\xi^2, \tag{12}$$

where $E(U^*)$ is the "error" of the approximate solution U^* :

$$\begin{aligned} E(U^*) &= \partial_x^2 U^* - U^* + 3U^{*2} - \partial_x^{-2} \partial_y^2 U^* \\ &= 6\left(U_1^* U_2^* + U_2^* U_3^* + U_1^* U_3^*\right). \end{aligned} \tag{13}$$

We see that essentially $E(U^*)$ gives the interaction between U_j^*, and therefore the presence of ξ is due to this interaction. $E(U^*)$ is of the order $O\left(\gamma^{-2}\right)$.

We introduce the complex numbers $z_j^* = x_j^* + iy_j^*, j = 1, 2, 3,$ and define

$$d^* = 24 \int_{\mathbb{R}^2} U^2 dxdy. \tag{14}$$

An important ingredient of the analysis is to understand the projection of the error $E = E(U^*)$ onto the kernels. This is the content of the following

Lemma 8. *There holds*

$$\int_{\mathbb{R}^2} E \partial_x U_j^* dxdy = -d^* \sum_{k \neq j} \operatorname{Re} \frac{1}{\left(z_j^* - z_k^*\right)^3} + O\left(\gamma^{-4}\right), \quad j = 1, 2, 3, \tag{15}$$

and

$$\int_{\mathbb{R}^2} E\partial_y U_j^* dxdy = d^* \sum_{k\neq j} \operatorname{Im} \frac{1}{\left(z_j^* - z_k^*\right)^3} + O\left(\gamma^{-4}\right), \quad j = 1,2,3. \tag{16}$$

Proof. We shall prove (15) and (16) for the case $j = 1$, the other ones can be treated similarly. For the former one, using (13) we get

$$\int_{\mathbb{R}^2} E\partial_x U_1^* dxdy = 6\int_{\mathbb{R}^2} \left(U_1^* U_2^* + U_1^* U_3^* + U_2^* U_3^*\right) \partial_x U_1^* dxdy.$$

Let us compute each integral appeared in the right hand side. Integrating by parts, we get

$$\int_{\mathbb{R}^2} U_2^* U_1^* \partial_x U_1^* dxdy = -\frac{1}{2}\int_{\mathbb{R}^2} U_1^{*2} \partial_x U_2^* dxdy. \tag{17}$$

To estimate this integral, we need the identity

$$\frac{y^2 - x^2}{\left(x^2+y^2\right)^2} = -\frac{1}{2}\left(\frac{1}{z^2} + \frac{1}{\bar{z}^2}\right), \tag{18}$$

which implies

$$\partial_x \left(\frac{y^2 - x^2}{\left(x^2+y^2\right)^2}\right) = 2\operatorname{Re}\frac{1}{z^3}, \quad \partial_y \left(\frac{y^2 - x^2}{\left(x^2+y^2\right)^2}\right) = -2\operatorname{Im}\frac{1}{z^3}, \tag{19}$$

$$\partial_x^2 \left(\frac{y^2 - x^2}{\left(x^2+y^2\right)^2}\right) = -6\operatorname{Re}\frac{1}{z^4}, \quad \partial_y^2 \left(\frac{y^2 - x^2}{\left(x^2+y^2\right)^2}\right) = 6\operatorname{Re}\frac{1}{z^4},$$

and

$$\partial_x\partial_y \left(\frac{y^2 - x^2}{\left(x^2+y^2\right)^2}\right) = 6\operatorname{Im}\frac{1}{z^4}.$$

We then use the explicit form of the lump solution U to deduce that within a ball of radius $\frac{\gamma}{2}$ centered at P_1,

$$\left|\partial_x U_2^* - 8\operatorname{Re}\frac{1}{z^3}\right| = O\left(\gamma^{-4}\right).$$

Inserting this estimate into (17) , we then get

$$6\int_{\mathbb{R}^2} U_2^* U_1^* \partial_x U_1^* dxdy = -d^* \operatorname{Re}\frac{1}{\left(z_1^* - z_2^*\right)^3} + O\left(\gamma^{-4}\right),$$

where d^* is defined in (14) .

Similarly,

$$6\int_{\mathbb{R}^2} U_3^* U_1^* \partial_x U_1^* dxdy = -d^* \operatorname{Re}\frac{1}{\left(z_1^* - z_3^*\right)^3} + O\left(\gamma^{-4}\right).$$

Moreover, we use the decay of U_2^*, U_3^* to conclude directly that

$$\int_{\mathbb{R}^2} U_2^* U_3^* \partial_x U_1^* dxdy = O\left(\gamma^{-4}\right).$$

Combining all these estimates, we then get

$$\int_{\mathbb{R}^2} E\partial_x U_1^* dxdy = -d^* \operatorname{Re}\frac{1}{\left(z_1^* - z_2^*\right)^3} - d^* \operatorname{Re}\frac{1}{\left(z_1^* - z_3^*\right)^3} + O\left(\gamma^{-4}\right).$$

The equation (16) can be obtained in a very similar way, using the second identity of (19) . □

In view of Lemma 8, the position z_j^* of the single lumps should approximately satisfy the following balancing condition: For each fixed j,

$$\sum_{k\neq j} \frac{1}{\left(z_j^* - z_k^*\right)^3} = 0. \tag{20}$$

As we have mentioned above, this condition has already been observed in Section 3 of [9] for the KP-I equation, from a more physically inspired point of view. On the other hand, the space M of points z_j^* satisfying these balancing equations (20) has been investigated in [3], where rational solutions of the KdV equation has been studied.

Taking into account of the previous computation, we shall define the map

$$\mathcal{F} : (z_1, z_2, z_3)^T \to (F_1, F_2, F_3)^T, \ z_j \in \mathbb{C},$$

where

$$F_j = \sum_{k\neq j} \frac{1}{(z_j - z_k)^3}, \ j = 1, 2, 3.$$

Note that for $\tilde{z}_1 := -1, \tilde{z}_2 := \frac{1+\sqrt{3}i}{2}, \tilde{z}_3 := \frac{1-\sqrt{3}i}{2}$, we have

$$\mathcal{F}(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3) = 0.$$

The linearization of $\mathcal{F}$ will play important role in our later analysis. The derivative $D\mathcal{F}$ of $\mathcal{F}$ at (z_1, z_2, z_3) is a matrix of the form $\left[F_{j,k}\right]$, where $F_{j,k} = \partial_{z_k} F_j$.

The next lemma follows from direct computation of eigenvectors.

Lemma 9. *The kernels of $M := D\mathcal{F}|_{(\tilde{z}_1,\tilde{z}_2,\tilde{z}_3)}$ are given by*

$$c_1\mathbf{b}_1 + c_2\mathbf{b}_2,$$

where c_1, c_2 are complex numbers and

$$\mathbf{b}_1 = (1, 1, 1)^T, \ \mathbf{b}_2 = (\tilde{z}_1, \tilde{z}_2, \tilde{z}_3)^T.$$

We remark that the vector $\mathbf{b}_1$ reflects the translation invariance of the system, and $\mathbf{b}_2$ is corresponding to scaling and rotation(multiplication by a complex number c).

To proceed, we use $d_k(x, y)$ to denote the distance between (x, y) and $\left(x_k^*, y_k^*\right)$. Let

$$\theta_\alpha(x, y) = \left(\sum_{k=1}^{3} (1 + d_k)^{-1}\right)^{-\alpha}.$$

We need some apriori estimates for the linearized operator.

Lemma 10. *Let $\varepsilon > 0$ be a fixed small constant. Suppose η satisfies $\|\eta\theta_\varepsilon\|_{L^\infty(\mathbb{R}^2)} < +\infty$ and*

$$\mathcal{L}_{U^*}\eta = f, \tag{21}$$

where $\|f\theta_2\|_{L^\infty(\mathbb{R}^2)} < +\infty$. Assume for $k = 1, 2, 3$,

$$\int_{\mathbb{R}^2} \eta \partial_x U_k^* dxdy = 0, \text{ and } \int_{\mathbb{R}^2} \eta \partial_y U_k^* dxdy = 0.$$

Then for any $\sigma \in (0, 2-\varepsilon)$, there holds

$$\|\eta \theta_\sigma\|_{L^\infty(\mathbb{R}^2)} \leq C \|f\theta_2\|_{L^\infty(\mathbb{R}^2)},$$

where C is independent of σ and f.

Proof. Consider the cone $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ with vertex at the origin containing those points whose angle coordinate are in the range $\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$, $\left(\frac{2\pi}{3}, 2\pi\right)$, $\left(0, \frac{2\pi}{3}\right)$, respectively. Then $P_k \in \mathcal{C}_k$. Let ρ_1, ρ_2, ρ_3 be a partition of unity such that ρ_k equals 1 in most part of $\mathcal{C}_k$ and $\nabla \rho_k$ is supported in a radius 1 tubular neighborhood of $\partial \mathcal{C}_k$.

We rewrite the equation (21) into the form

$$\mathcal{L}_0 \eta := \partial_x^2 \eta - \eta - \partial_x^{-2} \partial_y^2 \eta = f - 6U^* \eta.$$

Hence $\eta = \eta_1 + \eta_2 + \eta_3$, where $f_k = \rho_k f$ and η_k is determined by the equation

$$\mathcal{L}_0 \eta_k = f_k - 6U_k^* \eta.$$

Observe that $\mathcal{L}_0$ is a hypo-elliptic operator with constant coefficient, and the decay properties of its Green function K have been established in [5]. In particular,

$$\left\| r^2 K \right\|_{L^\infty(\mathbb{R}^2)} \leq C.$$

Using this decay estimate, we obtain

$$\left\| (1+d_k)^\sigma \eta_k \right\|_{L^\infty(\mathbb{R}^2)} \leq C \left\| (1+d_k)^2 f_k \right\|_{L^\infty(\mathbb{R}^2)} + C \left\| (1+d_k)^\sigma U_k^* \eta \right\|_{L^\infty(\mathbb{R}^2)}.$$

It follows that for some r_0 sufficiently large,

$$\left\| (1+d_k)^\sigma \eta_k \right\|_{L^\infty(\mathbb{R}^2)} \leq C \left\| (1+d_k)^2 f_k \right\|_{L^\infty(\mathbb{R}^2)} + C \|\eta\|_{L^\infty\left(B_{r_0}(P_k)\right)}.$$

Now we claim

$$\|\eta\|_{L^\infty\left(B_{r_0}(P_k)\right)} \leq C \left\| (1+d_k)^2 f_k \right\|_{L^\infty(\mathbb{R}^2)}.$$

Otherwise, there would exist a sequence of $\eta\left(x - x_k^*, y - y_k^*\right)$ which converges to solution of the equation

$$\mathcal{L}_U \eta_\infty = 0.$$

However, this contradicts with the nondegeneracy of lump solution [14] and the assumption that η is orthogonal to the $\partial_x U_k^*, \partial_y U_k^*$. □

The next lemma deals with the explicit expression of the function related to the main order correction of the approximate solution.

Lemma 11. *Let U be the classical lump solution defined by* (11) *and*

$$\omega = \partial_x \left[\frac{24x\left(y^2 - 3\right)}{\left(x^2 + y^2 + 3\right)^2} \right].$$

Then

$$\mathcal{L}_U \omega = -6U, \text{ and } \mathcal{L}_U \left[\partial_x \omega\right] = -6\partial_x U - 6\partial_x U \omega. \tag{22}$$

Proof. This follows from direct computation. Note that ω decays at the rate $O\left(r^{-2}\right)$ at infinity. □

With Lemma 11 being understood, we introduce the notation

$$\mathbf{p}_k = -2 \sum_{j \neq k} \operatorname{Re} \frac{1}{\left(z_k^* - z_j^*\right)^2}, \quad \text{for} \quad k = 1,2,3.$$

From the explicit formula of U and (18), we deduce

$$U_2^*\left(x_1^*, y_1^*\right) + U_3^*\left(x_1^*, y_1^*\right) = \mathbf{p}_1 + O\left(\gamma^{-4}\right),$$

and $|\mathbf{p}_k| = O\left(\gamma^{-2}\right)$. We then define, for $k = 1,2,3$,

$$\omega_k\left(x, y\right) := \mathbf{p}_k \omega\left(x - x_k^*, y - y_k^*\right).$$

With the help of ω_k, we will prove the following result, which gives us a more precise description of the solution u.

Proposition 12. *There exists $\varepsilon > 0$ such that the function $\xi = u - U^*$ has the following expansion:*

$$\left\| \left(\xi - \sum_{k=1}^{3} \omega_k \right) \theta_\varepsilon \right\|_{L^\infty(\mathbb{R}^2)} \leq C\gamma^{-3}.$$

Proof. We write $\xi = \sum_{k=1}^{3} \omega_k + \eta$, then η satisfies

$$\mathcal{L}_{U^*}\eta = -\sum_{k=1}^{3} \mathcal{L}_{U^*}\omega_k - E\left(U^*\right) - 3\left(\sum_{k=1}^{3} \omega_k + \eta\right)^2.$$

We have

$$\left\| \left(-\sum_{k=1}^{3} \mathcal{L}_{U^*}\omega_k - E\left(U^*\right) \right) \theta_2 \right\|_{L^\infty(\mathbb{R}^2)} \leq C\gamma^{-3}.$$

Applying Lemma 10, we conclude that for some $\varepsilon > 0$,

$$\left\| \eta \theta_\varepsilon \right\|_{L^\infty(\mathbb{R}^2)} \leq C\gamma^{-3}.$$

This finishes the proof. □

In the next result, we need to use the following constants:

$$a^* = \int_{\mathbb{R}^2} \left(3\omega^2 + 6\omega\right) \partial_x^2 U dxdy,$$
$$b^* = \int_{\mathbb{R}^2} \left(3\omega^2 + 6\omega\right) \partial_x \partial_y U dxdy,$$
$$c^* = \int_{\mathbb{R}^2} \left(3\omega^2 + 6\omega\right) \partial_y^2 U dxdy.$$

Lemma 13. *For any index $j \in \{1,2,3\}$ it holds that*

$$\int_{\mathbb{R}^2} \partial_x U_j^* \mathcal{L}_u \left[\partial_x U_j^*\right] dxdy = -3d^* \sum_{k \neq j} \operatorname{Re}\left[\left(z_j^* - z_k^*\right)^{-4}\right] + a^* \mathbf{p}_j^2 + O\left(\gamma^{-5}\right),$$

$$\int_{\mathbb{R}^2} \partial_x U_j^* \mathcal{L}_u \left[\partial_y U_j^*\right] dxdy = 3d^* \sum_{k \neq j} \operatorname{Im}\left[\left(z_j^* - z_k^*\right)^{-4}\right] + b^* \mathbf{p}_j^2 + O\left(\gamma^{-5}\right), \tag{23}$$

$$\int_{\mathbb{R}^2} \partial_y U_j^* \mathcal{L}_u \left[\partial_y U_j^*\right] dxdy = 3d^* \sum_{k \neq j} \operatorname{Re}\left[\left(z_j^* - z_k^*\right)^{-4}\right] + c^* \mathbf{p}_j^2 + O\left(\gamma^{-5}\right).$$

While for different indices $j, k \in \{1,2,3\}$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \partial_x U_j^* \mathcal{L}_u \left[\partial_x U_k^*\right] dxdy = 3d^* \operatorname{Re}\left[\left(z_j^* - z_k^*\right)^{-4}\right] + O\left(\gamma^{-5}\right), \\
&\int_{\mathbb{R}^2} \partial_x U_j^* \mathcal{L}_u \left[\partial_y U_k^*\right] dxdy = -3d^* \operatorname{Im}\left[\left(z_j^* - z_k^*\right)^{-4}\right] + O\left(\gamma^{-5}\right), \\
&\int_{\mathbb{R}^2} \partial_y U_j^* \mathcal{L}_u \left[\partial_y U_k^*\right] dxdy = -3d^* \operatorname{Re}\left[\left(z_j^* - z_k^*\right)^{-4}\right] + O\left(\gamma^{-5}\right).
\end{aligned} \tag{24}
$$

Proof. We shall firstly verify the three equations in (23). Without loss of generality we may assume that $j = 1$. Since $\partial_x U_1^*$ is a kernel of the operator $\mathcal{L}_{U_1^*}$, we have

$$
\mathcal{L}_u \left[\partial_x U_1^*\right] = \mathcal{L}_{U_1^*} \left[\partial_x U_1^*\right] + 6\left(u - U_1^*\right) \partial_x U_1^* = 6\left(U_2^* + U_3^* + \xi\right) \partial_x U_1^*. \tag{25}
$$

As a direct consequence,

$$
\int_{\mathbb{R}^2} \partial_x U_1^* \mathcal{L}_u \left[\partial_x U_1^*\right] dxdy = 6 \int_{\mathbb{R}^2} \left(\partial_x U_1^*\right)^2 \left(U_2^* + U_3^* + \xi\right) dxdy. \tag{26}
$$

To estimate the right hand side, we first differentiate the equation

$$
\mathcal{L}_{U^*} \xi = -E\left(U^*\right) - 3\xi^2 := J
$$

with respect to x. This yields

$$
\mathcal{L}_{U^*} \left[\partial_x \xi\right] + 6 \partial_x U^* \xi = \partial_x J.
$$

Multiplying both sides with $\partial_x U_1^*$ and integrating by parts, we obtain

$$
\int_{\mathbb{R}^2} \left(-\partial_x \left(\mathcal{L}_{U^*} \left[\partial_x U_1^*\right]\right) + 6 \partial_x U_1^* \partial_x U^*\right) \xi dxdy = \int_{\mathbb{R}^2} \partial_x J \partial_x U_1^* dxdy.
$$

Reorganizing terms, we have

$$
\begin{aligned}
6 \int_{\mathbb{R}^2} \left(\partial_x U_1^*\right)^2 \xi dxdy &= \int_{\mathbb{R}^2} \partial_x J \partial_x U_1^* dxdy - 6 \int_{\mathbb{R}^2} \left(\partial_x U_2^* + \partial_x U_3^*\right) \partial_x U_1^* \xi dxdy \\
&\quad + 6 \int_{\mathbb{R}^2} \partial_x \left[\left(U_2^* + U_3^*\right) \partial_x U_1^*\right] \xi dxdy \\
&= \int_{\mathbb{R}^2} \partial_x J \partial_x U_1^* dxdy + 6 \int_{\mathbb{R}^2} \left(U_2^* + U_3^*\right) \partial_x^2 U_1^* \xi dxdy.
\end{aligned}
$$

Inserting this into (26), we get

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \partial_x U_1^* \mathcal{L}_u \left[\partial_x U_1^*\right] dxdy \\
&= 6 \int_{\mathbb{R}^2} \left(\partial_x U_1^*\right)^2 \left(U_2^* + U_3^*\right) dxdy + \int_{\mathbb{R}^2} \partial_x J \partial_x U_1^* dxdy + 6 \int_{\mathbb{R}^2} \left(U_2^* + U_3^*\right) \partial_x^2 U_1^* \xi dxdy \\
&= 3 \int_{\mathbb{R}^2} U_1^{*2} \partial_x^2 \left(U_2^* + U_3^*\right) dxdy - 3 \int_{\mathbb{R}^2} \partial_x \left(\xi^2\right) \partial_x U_1^* dxdy \\
&\quad + 6 \int_{\mathbb{R}^2} \left(U_2^* + U_3^*\right) \partial_x^2 U_1^* \xi dxdy + O\left(\gamma^{-5}\right).
\end{aligned}
$$

Note that

$$
\int_{\mathbb{R}^2} U_1^{*2} \partial_x^2 \left(U_2^* + U_3^*\right) dxdy = -d^* \operatorname{Re}\left[\left(z_1^* - z_2^*\right)^{-4} + \left(z_1^* - z_3^*\right)^{-4}\right] + O\left(\gamma^{-5}\right).
$$

We then get

$$\int_{\mathbb{R}^2} \partial_x U_1^* \mathcal{L}_u \left[\partial_x U_1^*\right] dxdy = -3d^* \operatorname{Re}\left[\left(z_1^* - z_2^*\right)^{-4} + \left(z_1^* - z_3^*\right)^{-4}\right] + \mathbf{p}_1^2 \int_{\mathbb{R}^2} \left(3\omega_1^2 + 6\omega_1\right) \partial_x^2 U_1^* dxdy + O\left(\gamma^{-5}\right).$$

This is the required identity.

Next we compute

$$\begin{aligned}
&\int_{\mathbb{R}^2} \partial_x U_1^* \mathcal{L}_u \left[\partial_y U_1^*\right] dxdy \\
&= \int_{\mathbb{R}^2} \partial_x U_1^* \mathcal{L}_{U_1^*} \left[\partial_y U_1^*\right] dxdy + 6 \int_{\mathbb{R}^2} \partial_x U_1^* \left(u - U_1^*\right) \partial_y U_1^* dxdy \\
&= 6 \int_{\mathbb{R}^2} \partial_x U_1^* \partial_y U_1^* \left(U_2^* + U_3^* + \xi\right) dxdy.
\end{aligned}$$

On the other hand,

$$\mathcal{L}_{U^*}\left[\partial_y \xi\right] + 6\partial_y U^* \xi = \partial_y J,$$

which implies

$$\int_{\mathbb{R}^2} \left(-\partial_y \left(\mathcal{L}_{U^*}\left[\partial_x U_1^*\right]\right) + 6\partial_x U_1^* \partial_y U^*\right) \xi dxdy = \int_{\mathbb{R}^2} \partial_y J \partial_x U_1^* dxdy.$$

From this identity, we get, using similar computation as before,

$$\begin{aligned}
&\int_{\mathbb{R}^2} \partial_x U_1^* \mathcal{L}_u \left[\partial_y U_1^*\right] dxdy \\
&= 6 \int_{\mathbb{R}^2} \partial_x U_1^* \partial_y U_1^* \left(U_2^* + U_3^*\right) dxdy + \int_{\mathbb{R}^2} \partial_y J \partial_x U_1^* dxdy + 6 \int_{\mathbb{R}^2} \left(U_2^* + U_3^*\right) \partial_x \partial_y U_1^* \xi dxdy \\
&= 3 \int_{\mathbb{R}^2} \partial_x \partial_y \left(U_2^* + U_3^*\right) U_1^{*2} dxdy - 3 \int_{\mathbb{R}^2} \partial_y \left(\xi^2\right) \partial_x U_1^* dxdy \\
&\quad + 6 \int_{\mathbb{R}^2} \left(U_2^* + U_3^*\right) \partial_x \partial_y U_1^* \xi dxdy + O\left(\gamma^{-5}\right).
\end{aligned}$$

In view of the estimate

$$\int_{\mathbb{R}^2} U_1^{*2} \partial_x \partial_y \left(U_2^* + U_3^*\right) dxdy = -3d^* \operatorname{Im}\left[\left(z_1^* - z_2^*\right)^{-4} + \left(z_1^* - z_3^*\right)^{-4}\right] + O\left(\gamma^{-5}\right),$$

we then arrive at

$$\int_{\mathbb{R}^2} \partial_x U_1^* \mathcal{L}_u \left[\partial_y U_1^*\right] dxdy = 3d^* \operatorname{Im}\left[\left(z_1^* - z_2^*\right)^{-4} + \left(z_1^* - z_3^*\right)^{-4}\right] + \mathbf{p}_1^2 \int_{\mathbb{R}^2} \left(3\omega_1^2 + 6\omega_1\right) \partial_x \partial_y U_1^* dxdy + O\left(\gamma^{-5}\right).$$

Regarding the last equation in (23), as what we have done for the first one we get that

$$\begin{aligned}
&\int_{\mathbb{R}^2} \partial_y U_1^* \mathcal{L}_u \left[\partial_y U_1^*\right] dxdy \\
&= 6 \int_{\mathbb{R}^2} \left(\partial_y U_1^*\right)^2 \left(U_2^* + U_3^*\right) dxdy + \int_{\mathbb{R}^2} \partial_y J \partial_y U_1^* dxdy + 6 \int_{\mathbb{R}^2} \left(U_2^* + U_3^*\right) \partial_y^2 U_1^* \xi dxdy \\
&= 3 \int_{\mathbb{R}^2} U_1^{*2} \partial_y^2 \left(U_2^* + U_3^*\right) dxdy - 3 \int_{\mathbb{R}^2} \partial_y \left(\xi^2\right) \partial_y U_1^* dxdy \\
&\quad + 6 \int_{\mathbb{R}^2} \left(U_2^* + U_3^*\right) \partial_y^2 U_1^* \xi dxdy + O\left(\gamma^{-5}\right).
\end{aligned}$$

Using the fact that

$$\int_{\mathbb{R}^2} U_1^{*2} \partial_y^2 \left(U_2^* + U_3^*\right) dxdy = d^* \operatorname{Re}\left[\left(z_1^* - z_2^*\right)^{-4} + \left(z_1^* - z_3^*\right)^{-4}\right] + O\left(\gamma^{-5}\right).$$

We obtain that

$$\begin{aligned}\int_{\mathbb{R}^2} \partial_y U_1^* \mathcal{L}_u \left[\partial_y U_1^*\right] dxdy = 3d^* \operatorname{Re}\left[\left(z_1^* - z_2^*\right)^{-4} + \left(z_1^* - z_3^*\right)^{-4}\right] \\ + \mathbf{p}_1^2 \int_{\mathbb{R}^2} \left(3\omega_1^2 + 6\omega_1\right) \partial_y^2 U_1^* dxdy + O\left(\gamma^{-5}\right).\end{aligned}$$

Now we verify the equations in (24).

$$\begin{aligned}&\int_{\mathbb{R}^2} \partial_x U_j^* \mathcal{L}_u \left[\partial_x U_k^*\right] dxdy \\ &= \int_{\mathbb{R}^2} \partial_x U_j^* \mathcal{L}_{U_k^*} \left[\partial_x U_k^*\right] dxdy + 6 \int_{\mathbb{R}^2} \partial_x U_j^* \left(u - U_k^*\right) \partial_x U_k^* dxdy \\ &= 6 \int_{\mathbb{R}^2} \partial_x U_j^* \partial_x U_k^* \left(\sum_{\ell \neq k} U_\ell^* + \xi\right) dxdy \\ &= 6 \int_{\mathbb{R}^2} \partial_x U_j^* \partial_x U_k^* U_j^* dxdy + O\left(\gamma^{-5}\right) \\ &= 3d^* \operatorname{Re}\left[\left(z_j^* - z_k^*\right)^{-4}\right] + O\left(\gamma^{-5}\right).\end{aligned}$$

Since the computation of other integrals like $\int_{\mathbb{R}^2} \partial_x U_j^* \mathcal{L}_u \left[\partial_y U_k^*\right] dxdy$ is quite similar, the details of these computation will be omitted. □

The key result of this section is the following

Proposition 14. *The Morse index of u is equal to 3, provided that B is sufficiently large.*

Proof. Let λ_B be a negative eigenvalue of the linearized operator, with ϕ_B being an eigenfunction normalized such that $\|\phi_B\|_{L^\infty(\mathbb{R}^2)} = 1$. Then

$$\partial_x^2 \phi_B - \phi_B + 6u\phi_B - \partial_x^{-2} \partial_y^2 \phi_B = \lambda_B \phi_B.$$

Taking x-derivative twice, we get

$$\partial_x^2 \left(\partial_x^2 \phi_B - \phi_B + 6u\phi_B\right) - \partial_y^2 \phi_B = \lambda_B \partial_x^2 \phi_B.$$

Our first step is to show that there exists c_0 independent of B such that

$$\lambda_B \leq c_0 < 0. \tag{27}$$

Assume to the contrary that (27) was not true. Then there was a sequence $\lambda_j \to 0$ with corresponding normalized eigenfunctions ϕ_j.

Consider the translated sequence $\tilde{\phi}_j(x, y) := \phi_j\left(x - x_1^*, y - y_1^*\right)$. Since we have assumed that $\|\phi_j\|_{L^\infty(\mathbb{R}^2)} = 1$, the new functions $\tilde{\phi}_j$ will converge to a function Φ, solution of the equation

$$\partial_x^2 \left(\partial_x^2 \Phi - \Phi + 6U\Phi\right) - \partial_y^2 \Phi = 0.$$

By the nondegeneracy of lump, there exist constants c_1, c_2(could be zero) such that

$$\Phi = c_1 \partial_x U + c_2 \partial_y U.$$

To simplify the notation, we omit the subscript j, if no confusion will arise. Then we can write ϕ as $\phi^* + \zeta$, where

$$\phi^* = \sum_{k=1}^{3} \left(\sigma_k \partial_x U_k^* + \tau_k \partial_y U_k^* \right),$$

and ζ is small and satisfies the orthogonality condition: For $k = 1, 2, 3$,

$$\int_{\mathbb{R}^2} \zeta \partial_x U_k^* dxdy = 0, \text{ and } \int_{\mathbb{R}^2} \zeta \partial_y U_k^* dxdy = 0.$$

We would like to estimate ζ, using the equation

$$\mathcal{L}_u \zeta = -\mathcal{L}_u \phi^* + \lambda \phi^* + \lambda \zeta.$$

To do this, we first observe that $\mathcal{L}_u \phi^*$ is of the order $O\left(\gamma^{-2}\right)$. Hence to obtain a precise expansion for ζ, we write

$$\zeta = \sum_{k=1}^{3} \left(\sigma_k \partial_x \omega_k + \tau_k \partial_y \omega_k \right) + \zeta^*.$$

Using $(22), (25)$, for any $k = 1, 2, 3$, we can estimate

$$\mathcal{L}_u \left[\partial_x U_k^* \right] + \mathcal{L}_u \left[\partial_x \omega_k \right] = O\left(\gamma^{-3} \right), \tag{28}$$

$$\mathcal{L}_u \left[\partial_y U_k^* \right] + \mathcal{L}_u \left[\partial_y \omega_k \right] = O\left(\gamma^{-3} \right). \tag{29}$$

It follows that

$$\begin{aligned} \mathcal{L}_u \zeta^* - \lambda \xi^* = & - \sum_{k=1}^{3} \left(\sigma_k \mathcal{L}_u \left[\partial_x U_k^* + \partial_x \omega_k \right] + \tau_k \mathcal{L}_u \left[\partial_y U_k^* + \partial_y \omega_k \right] \right) \\ & + \lambda \left(\phi^* + \sum_{k=1}^{3} \left(\sigma_k \partial_x \omega_k + \tau_k \partial_y \omega_k \right) \right). \end{aligned}$$

Let us denote the right hand side by Q.

For a function η, we define

$$\eta_{k,x}^{\|} = \int_{\mathbb{R}^2} \eta \partial_x U_k^* dxdy, \quad \eta_{k,y}^{\|} = \int_{\mathbb{R}^2} \eta \partial_y U_k^* dxdy,$$

and

$$\eta^{\|} = \sum_{k=1}^{3} \left(\eta_{k,x}^{\|} \partial_x U_k^* + \eta_{k,y}^{\|} \partial_y U_k^* \right),$$

and $\eta^{\perp} = \eta - \eta^{\|}$. The function $\eta^{\perp}$ can be understood as the projection orthogonal to the kernels of the linearized operator. Then

$$\mathcal{L}_u \zeta^* - \lambda \xi^* = Q^{\perp} + Q^{\|}. \tag{30}$$

Estimates (28) and (29) imply that $Q^{\perp} = O\left(\gamma^{-3}\right)$. On the other hand, multiplying (30) by $\partial_x U_k^*, \partial_y U_k^*$ and integrating tell us that $Q^{\|} = o\left(\zeta^*\right)$. Following the proof of Lemma 10, we find that there exists $\varepsilon > 0$, such that

$$\left\| \zeta^* \theta_\varepsilon \right\|_{L^\infty(\mathbb{R}^2)} \leq C \gamma^{-3}.$$

With the estimate of ζ at hand, now we project the equation $\mathcal{L}_u\phi = \lambda\phi$ onto the kernels $\partial_x U_k^*$ and $\partial_y U_k^*$. More precisely, we consider the following two identities

$$\int_{\mathbb{R}^2} \partial_x U_k^* \left(\mathcal{L}_u\phi - \lambda\phi\right) dxdy = 0, \quad \int_{\mathbb{R}^2} \partial_y U_k^* \left(\mathcal{L}_u\phi - \lambda\phi\right) dxdy = 0.$$

Now we compute

$$\begin{aligned}\int_{\mathbb{R}^2} \partial_x U_j^* \mathcal{L}_u \left[\partial_x \omega_j\right] dxdy &= 6\int_{\mathbb{R}^2} \partial_x U_j^* \left(\sum_{k\neq j} U_k^* + \xi\right) \partial_x \omega_j dxdy \\ &= 6\int_{\mathbb{R}^2} \left(-\omega_j \mathbf{p}_j - \frac{1}{2}\omega_j^2\right) \partial_x^2 U_j^* dxdy + O\left(\gamma^{-5}\right) \\ &= -a^* \mathbf{p}_j^2 + O\left(\gamma^{-5}\right).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\int_{\mathbb{R}^2} \partial_x U_j^* \mathcal{L}_u \left[\partial_y \omega_j\right] dxdy &= 6\int_{\mathbb{R}^2} \partial_x U_j^* \left(\sum_{k\neq j} U_k^* + \xi\right) \partial_y \omega_j dxdy \\ &= 6\int_{\mathbb{R}^2} \left(-\omega_j \mathbf{p}_j - \frac{1}{2}\omega_j^2\right) \partial_x \partial_y U_j^* dxdy \\ &= -b^* \mathbf{p}_j^2 + O\left(\gamma^{-5}\right),\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^2} \partial_y U_j^* \mathcal{L}_u \left[\partial_y \omega_j\right] dxdy &= 6\int_{\mathbb{R}^2} \partial_y U_j^* \left(\sum_{k\neq j} U_k^* + \xi\right) \partial_y \omega_j dxdy \\ &= 6\int_{\mathbb{R}^2} \left(-\omega_j \mathbf{p}_j - \frac{1}{2}\omega_j^2\right) \partial_y^2 U_j^* dxdy \\ &= -c^* \mathbf{p}_j^2 + O\left(\gamma^{-5}\right).\end{aligned}$$

While for $j \neq k$, we have

$$\int_{\mathbb{R}^2} \partial_x U_j^* \mathcal{L}_u \left[\partial_x \omega_k\right] dxdy = O\left(\gamma^{-5}\right), \quad \int_{\mathbb{R}^2} \partial_x U_j^* \mathcal{L}_u \left[\partial_y \omega_k\right] dxdy = O\left(\gamma^{-5}\right),$$

and

$$\int_{\mathbb{R}^2} \partial_y U_j^* \mathcal{L}_u \left[\partial_y \omega_k\right] dxdy = O\left(\gamma^{-5}\right).$$

From the above estimates and Lemma 13, we deduce that for $k = 1, 2, 3$,

$$\int_{\mathbb{R}^2} \left(\partial_x U_k^* - i\partial_y U_k^*\right) \left(\mathcal{L}_u\phi - \lambda\phi\right) dxdy = \sum_{j=1}^{3} \left(F_{k,j}\mathbf{e}_j\right) + O\left(\gamma^{-5}\right),$$

where $\mathbf{e}_j = \sigma_j + i\tau_j$ is complex number. Moreover, there exists universal positive constants δ_1, δ_2, such that

$$\delta_1 \leq \sum_j \left|\mathbf{e}_j\right|^2 \leq \delta_2.$$

Hence, after a scaling, in terms of the matrix M defined in Lemma 9, we obtain

$$M\left(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\right)^T = O\left(\gamma^{-1}\right) + O\left(|\lambda|\right).$$

We then deduce that for some constants c_1, c_2, with $c_1^2 + c_2^2$ uniformly bounded away from zero, such that

$$(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = c_1 \mathbf{b}_1 + c_2 \mathbf{b}_2 + O\left(\gamma^{-1}\right) + O\left(|\lambda|\right).$$

On the other hand, using the explicit four-parameter family of solutions $u_{A,B}$ for the Boussinesq equation, we deduce that there exists a kernel φ of $\mathcal{L}_u$ whose projection on $\partial_x U_k^*, \partial_y U_k^*$ is close to $c_1 \mathbf{b}_1 + c_2 \mathbf{b}_2$. We then conclude that ϕ_j will not be L^2-orthogonal to the φ for j large enough. This contradicts with the fact that ϕ_j, as an eigenfunction with respect to a negative eigenvalue, has to be L^2-orthogonal to φ.

Now we have proved that negative eigenvalue λ_j will be uniformly bounded away from 0. Using this information, we then deduce that λ_j has to converge to the unique negative eigenvalue of the operator $\mathcal{L}_U$. By result in [14], the Morse index of the standard lump U is equal to 1, we then find that the Morse index of u is at most 3. On the other hand, by constructing explicit test functions using the eigenfunctions(of negative eigenvalue) of $\mathcal{L}_U$, we know that the Morse index of u is at least 3. Hence we conclude that the Morse index of u is equal to 3. This finishes the proof. □

Having analyzed the solutions $u_{0,B}$ for B large, we proceed to show that all the solutions $u_{A,B} = 2\partial_x^2 \ln h_{A,B}$ has Morse index 3.

Proof of Theorem 1. Straightforward application of the results in the previous section tells us that there is a Backlund transformation from the translated degree 2 polynomial

$$f(x,y) = x^2 + \left(y - \frac{2\sqrt{3}i}{3}\right)^2 + 3$$

to the degree 4 polynomial g, which is defined explicitly by

$$\begin{aligned} g(x,y) = {} & x^4 + 2ix^3y + 2ixy^3 - y^4 + \frac{10x^3}{\sqrt{3}} + 4\sqrt{3}ix^2y + 2\sqrt{3}xy^2 + \frac{\sqrt{3}iy^3}{8} \\ & + 20x^2 + 30ixy + 2y^2 + \left(\frac{50}{\sqrt{3}} + \frac{Ai+B}{2}\right)x \\ & + \left(\frac{80i}{\sqrt{3}} + \frac{A-Bi}{2}\right)y - 25 + \frac{Ai+B}{2\sqrt{3}}. \end{aligned}$$

Then the function g is Backlund transformed to $h_{A,B}$. Observe that both f and g have finitely many simple zeros. Using this fact, the nondegeneracy of the linearized operator $\mathcal{L}_{u_{A,B}}$ then follows directly from the same argument as that of [14], that is, by analyzing the associated linearized Backlund transformations.

To show that the Morse index of $u_{A,B}$ is equal to 3, let m be sufficiently large and consider the family of solutions $u_{At,Bt+(1-t)m}$. When $t = 1$, it is $u_{A,B}$, and for $t = 0$, the solution is $u_{0,m}$. We now know that for any $t \in [0,1]$, the solution is nondegenerated in the sense that the linearized operator has no nontrivial kernels. This together with the continuous dependence of the negative eigenvalues with respect to t, implies that as t decreases from 1 to 0, a negative eigenvalue can not diminish to the zero eigenvalue. We then see that the Morse index of $u_{A,B}$ should be the same as that of $u_{0,m}$. By Proposition 14, $u_{0,m}$ has Morse index 3, provided that m is large. We then conclude that for any A, B, the Morse index of $u_{A,B}$ equals 3. This completes the proof. □

As a final remark, we point out that for solutions with higher degrees, the above arguments also work. The only delicate part is, we need to show, as one of the parameter tends to infinity, the solution splits into a number of classical lumps which are far away from each other. This is to ensure that procedure of reverse Lyapunov-Schmidt reduction can be started. For polynomial tau functions of degree $k(k+1)$, the corresponding solution should have Morse index $k(k+1)/2$. Once this is proved, it will yield the existence of infinitely many solutions for the GP and generalized KP equation. Rigorous justification of this fact would require a complete classification of the moduli space of lump type solutions. This is an ongoing project.

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