Sharp estimates for fully bubbling solutions of a SU(3) Toda system

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Abstract

In this paper, we obtain sharp estimates of fully bubbling solutions of SU(3) Toda system in a compact Riemann surface. In geometry, the SU(n+1) Toda system is related to holomorphic curves, harmonic maps or harmonic sequences of the Riemann surface to $\mathbb{CP}^n$. In order to compute the Leray-Schcuder degree for the Toda system, we have to obtain accurate approximations of the bubbling solutions. Our main goals in this paper are (i) to obtain a sharp convergence rate, (ii) to completely determine the locations, and (iii) to derive the ∂_z^2 condition, a unexpected and important geometric constraint.

1 Introduction

Let (M, g) be a compact Riemann surface. Consider the following system of equations:

$$\begin{cases} \Delta u_1 + 2e^{v_1} - e^{v_2} = 4\pi \sum_{j=1}^{m} \gamma_{j1}\delta_{q_j} \\ \Delta u_2 + 2e^{v_2} - e^{v_1} = 4\pi \sum_{j=1}^{m} \gamma_{j2}\delta_{q_j}, \end{cases} \tag{1}$$

where $\Delta = \Delta_g$ stands for the Laplace-Beltrami operator, γ_{jk} are nonnegative integers, q_j are distinct points in M and δ_{q_j} are the Dirac measure at q_j. The system (1) is known as the SU$(n+1)$ Toda system when $n = 2$. This system of equations arises from many different research areas in geometry and physics. In physics, it is related to the relativistic version of non-abelian Chern-Simons

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models, see [9], [28], [36], [37] and references therein. In geometry, the SU$(n+1)$ Toda system is closely related to holomorphic curves (or harmonic sequence) of M into $\mathbb{CP}^n$, see [3], [7], [11], [12]. When $M = S^2$, it was proved that the solution space of the $SU(n+1)$ Toda system is identical to the space of holomorphic curves of S^2 into $\mathbb{CP}^n$. In particular, q_j are the ramificated points of the corresponding curve and γ_{ji} represents the total ramificated index at q_j, $j = 1, 2, \ldots, m$. See [22]. However, when M is not S^2, the identity of the solution space of PDE and holomorphic curves might not hold in general. Therefore it is an interesting issue to clarify their relationship for Riemann surfaces with higher genus. This is our initial motivation to study the Toda system in a compact Riemann surface.

Integrating (1), we have

$$\begin{array}{l} \rho_1 \doteqdot \int_M e^{u_1} = \frac{8\pi m_1 + 4\pi m_2}{3} \\ \rho_2 \doteqdot \int_M e^{u_2} = \frac{4\pi m_1 + 8\pi m_2}{3}, \end{array} \tag{2}$$

where $m_1 = \sum_{j=1}^{m} \gamma_{j1}$ and $m_2 = \sum_{j=1}^{m} \gamma_{j2}$. Let $|M|$ be the area of M as usual. By introducing the Green function:

$$\begin{cases} \Delta G(x,p) = -\delta_p + \frac{1}{|M|} \\ \int_M G(x,p)dx = 0, \end{cases} \tag{3}$$

and rewriting v_i by

$$u_i = v_i + 4\pi \sum_{j=1}^{m} \gamma_{ji} G(x, q_j), \qquad i = 1, 2,$$

we have u_i satisfies the following system of equations:

$$\begin{cases} \Delta u_1 + 2\rho_1\left(\frac{h_1 e^{u_1}}{\int h_1 e^{u_1}} - \frac{1}{|M|}\right) - \rho_2\left(\frac{h_2 e^{u_2}}{\int h_2 e^{u_2}} - \frac{1}{|M|}\right) = 0, \\ \Delta u_2 + 2\rho_2\left(\frac{h_2 e^{u_2}}{\int h_2 e^{u_2}} - \frac{1}{|M|}\right) - \rho_1\left(\frac{h_1 e^{u_1}}{\int h_1 e^{u_1}} - \frac{1}{|M|}\right) = 0, \end{cases} \tag{4}$$

where

$$h_i(x) = \exp\left(-\sum_{j=1}^{m} 4\pi\gamma_{ji} G(x, q_j)\right), \quad i = 1, 2.$$

We see that $h_i(x) > 0$ in $M \setminus \{q_1, \cdots, q_m\}$. It is easy to see that if $u = (u_1, \cdots, u_n)$ is a solution of (4), $u + c = (u_1 + c_1, \cdots, u_n + c_n)$ is still a solution. Without loss of generality, we may assume each component $u_i \in \mathring{H}(M)$, where $\mathring{H}(M) = \{u_i \in H(M) | \int_M u_i = 0\}$. Obviously, the equation (4) is the Euler-Lagrange equation of the nonlinear functional Φ_ρ:

$$\Phi_\rho(u) = \frac{1}{2} \int_M \sum_{i,j=1}^{2} a^{ij} \nabla u_i \cdot \nabla u_j - \sum_{i=1}^{2} \rho_i \log \int_M h_i e^{u_i},$$

where (a^{ij}) is the inverse matrix of $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. If $h_1 = h_2$, $\rho_1 = \rho_2$ and $u_1 = u_2$, then equation (4) is reduced to the mean field equation:

$$\Delta u + \rho(\frac{he^u}{\int he^u} - \frac{1}{|M|}) = 0 \qquad \text{in } M. \tag{5}$$

Equation (5) arises also in geometry and physics. In conformal geometry, it is related to the problem of prescribing Gaussian curvature with smooth metrics or metrics with conic singularity. For the past twenty years, the equation (5) has been extensively studied because it is closely related to the Abelian Chern-Simons theory. See [1], [2], [4], [5], [10], [15], [16], [19], [23], [27], [29], [30], [31], [33] and references therein.

For equation (4), the first main issue is to determine the set of critical parameters, i.e, those $\rho = (\rho_1, \rho_2)$ such that the a-priori bounds for solutions of (4) fail. In [13], Jost-Lin-Wang proved the following a-priori estimates for equation (4). (We use $\mathbb{N}^*$ to denote the set of all positive integers.)

Theorem A. *Suppose h_i are positive smooth solutions, and $\rho_i \notin 4\pi\mathbb{N}^*$, $i = 1, 2$. Then there exists a positive constant c such that for any solution u of equation* (4)*, there holds:*

$$|u_i(x)| \le c \qquad \forall\, x \in M, \quad i = 1, 2.$$

To prove Theorem A, the authors [13] considered a sequence of bubbling solutions $u_k = (u_{1k}, u_{2k})$ to the equation: for the simplicity, let $|M| = 1$ and u_k be a solution of

$$\Delta u_{ik} + \sum_{j=1}^{2} a_{ij}\rho_{jk}(\frac{h_{jk}e^{u_{jk}}}{\int_M h_{jk}e^{u_{jk}}} - 1) = 0 \quad \text{in } M,\ i = 1, 2 \tag{6}$$

where $\rho_{jk} \to \rho_j$, $h_{jk} \to h_j$ in $C^{2+\alpha}(M)$ for some $\alpha > 0$ as $k \to +\infty$, and $S = \{p_1, \ldots, p_m\}$ is the blowup set of u_k. At each p_j, the local mass of u_k is assigned by the quantity σ:

$$\sigma_i(p_j) = \lim_{r\to 0}\lim_{k\to\infty} \frac{\int_{B_r(p_j)} \rho_{ik}h_{ik}e^{u_{ik}}}{\int_M h_{ik}e^{u_{ik}}}, \quad j = 1, 2, \ldots, m, \tag{7}$$

where $B_r(p_j)$ is the ball with center p_j and radius r. Jost-Lin-Wang [13] proved that for each p_j, there are only four possibility for (σ_1, σ_2), i.e., (σ_1, σ_2) could be one of $(4\pi, 0)$, $(0, 4\pi)$, $(8\pi, 4\pi)$, $(4\pi, 8\pi)$ and $(8\pi, 8\pi)$. It is easy to check any one of the couples could occur for global solutions in $\mathbb{R}^2$ with constant coefficients. Thus, it is a natural question to ask whether each of the couples could exist for a sequence of bubbling solutions in a compact Riemann surface M.

Obviously, $(8\pi, 8\pi)$ is the most interesting case among them. Suppose a sequence of bubbling solution u_k has the local masses $(8\pi, 8\pi)$ at p. The sequence of solutions u_k is called **fully bubbling** at P, if after a suitable scaling, the sequence of solutions will converge to (v_1, v_2) in $C^2_{loc}(\mathbb{R}^2)$ satisfying:

$$\begin{cases} \Delta v_1 + 2e^{v_1} - e^{v_2} = 0 & \text{in } \mathbb{R}^2 \\ \Delta v_2 + 2e^{v_2} - e^{v_1} = 0 & \text{in } \mathbb{R}^2 \\ \int e^{v_1} < +\infty, \ \int e^{v_2} < +\infty \end{cases} \tag{8}$$

More precisely, u_k is said to **fully blow up** at p if and only if u_k satisfies

$$\left| u_{1k}(p_{k,1}) - \ln \int_M h_{1k} e^{u_{1k}} - u_{2k}(p_{k,2}) + \ln \int_M h_{2k} e^{u_{2k}} \right| \leq c \tag{9}$$

for some constant c, where $p_{k,i}$ are the local maxima of u_{ik} in $B_r(p)$. We note that if $v = (v_1, v_2)$ is an entire solution of (8), then

$$\int_{\mathbb{R}^2} e^{v_1} = \int_{\mathbb{R}^2} e^{v_2} = 8\pi$$

This quantization result was proved by Jost-Wang [14].

In [13], Jost-Lin-Wang proved that any full bubble is *simple*, i.e., there exists a sequence of entire solutions $v = (v_1, v_2)$ to (5) such that

$$|u_{jk}(\epsilon_k y) + 2\log \epsilon_k - v_j(y)| \leq c, \quad \text{for } |y| \leq \delta_0 \epsilon_k^{-1},$$

where c, δ_0 are positive constant and

$$\varepsilon_k = \max\left\{ u_{1k}(p_{k,1}) - \ln \int_M h_{1k} e^{u_{1k}}, u_{2k}(p_{k,2}) - \ln \int_M h_{2k} e^{u_{2k}} \right\}$$

In this paper, we want to study the global behavior for a sequence of bubbling solution u_k to equation (6) and obtain some important information for this sequence of bubbling solutions. Those information will have very important applications when we come to construct bubbling solutions, to count the Morse index for each bubbling solutions and finally to compute the topological degree for solutions of equation (4). Throughout the paper, we assume that

(H) *u_k fully blows up at each p_j.*

Under the assumption (H), it is proved (see [21]) that $\rho_{jk} \to 8m\pi$ ($m \in \mathbb{N}^*$), and $u_{1k}(x) \to \sum_{i=1}^m 8\pi G(x, p_i)$ and $u_{2k}(x) \to \sum_{i=1}^m 8\pi G(x, p_i)$ in $C^2(M \setminus \{p_1, \ldots, p_m\})$ as $k \to +\infty$. Choose small $r_0 > 0$ such that $B(p_i, 2r_0) \cap B(p_j, 2r_0) = \emptyset$ for $i \neq j$, denote by $p_{k,j}$ the local maxima of u_{1k} in $B(p_j, r_0)$, $1 \leq j \leq m$, and let $\varepsilon_{k,j}$, ε_k be defined by (53) and (54). Our main result is the following sharper estimates of u_k.

Theorem 1.1. *Let $u_k \in \mathring{H}(M)$ be a sequence of blowing up solutions to* (6), *such that (H) holds. Then it holds that* *(i) Convergence rate:*

$$\rho_{1k} - 8m\pi = \sum C_{1k,j}[\Delta \ln h_{1k}(p_{k,j}) + 8m\pi - 2K(p_{k,j})]\varepsilon_{k,j}^2 |\ln \varepsilon_{k,j}| + O(\varepsilon_k^2), \tag{10}$$

$$\rho_{2k} - 8m\pi = \sum C_{2k,j}[\Delta \ln h_{2k}(p_{k,j}) + 8m\pi - 2K(p_{k,j})]\varepsilon_{k,j}^2 |\ln \varepsilon_{k,j}| + O(\varepsilon_k^2), \tag{11}$$

where C_{ik} $(i = 1, 2)$ are constants satisfying $0 < C_1 < C_{ik,j} < C_2 < \infty$ and K denotes the Gauss curvature of M. Furthermore we have

(ii) Locations of p_j:

$$8\pi \nabla_x H(p_{k,j}, p_{k,j}) + 8\pi \sum_{l \neq j} \nabla_x G(p_{k,j}, p_{k,l}) + \nabla \ln h_{1k}(p_{k,j}) = O(\varepsilon_k) \tag{12}$$

$$8\pi \nabla_x H(p_{k,j}, p_{k,j}) + 8\pi \sum_{l \neq j} \nabla_x G(p_{k,j}, p_{k,l}) + \nabla \ln h_{2k}(p_{k,j}) = O(\varepsilon_k) \tag{13}$$

where $H(x, p)$ is the regular part of $G(x, p)$.

(iii) The ∂_z^2 condition:

$$\begin{aligned} &6\pi(\partial_{11} - \partial_{22})[\ln h_{2k}(p_{k,j}) - \ln h_{1k}(p_{k,j})] \\ &+\tfrac{T_{1k,1}^j}{4}[\Delta \ln h_{1k}(p_{k,j}) + 8m\pi - 2K(p_{k,j})] \\ &+\tfrac{T_{2k,1}^j}{4}[\Delta \ln h_{2k}(p_{k,j}) + 8m\pi - 2K(p_{k,j})] = O(\varepsilon_k^\beta) \end{aligned} \tag{14}$$

$$\begin{aligned} &12\pi\partial_{12}[\ln h_{2k}(p_{k,j}) - \ln h_{1k}(p_{k,j})] \\ &+\tfrac{T_{1k,2}^j}{4}[\Delta \ln h_{1k}(p_{k,j}) + 8m\pi - 2K(p_{k,j})] \\ &+\tfrac{T_{2k,2}^j}{4}[\Delta \ln h_{2k}(p_{k,j}) + 8m\pi - 2K(p_{k,j})] = O(\varepsilon_k^\beta), \end{aligned} \tag{15}$$

where $T_{1k,1}^j$, $T_{2k,1}^j$, $T_{1k,2}^j$ and $T_{2k,2}^j$ are four constants defined in Proposition 7.1.

We note that for the mean field equation (5), an analogue theorem was proved by Chen and the first author [4]. However, this type of theorems is much harder for Toda system than for the mean field equation. In the case of scalar mean field equation, the local Pohozaev identity is a very powerful tool in the bubbling analysis since the number of Pohozaev identities equals to the number of free parameters (both are three) for the Liouville equation. For Toda system, the local Pohozaev identity only gives three equations, but there are eight free parameters in the solutions space of Toda system. See (30) and (31) in section 2. We remark here that (10)-(15) are 8 scalar conditions. Thus, the Pohozaev identity is much less powerful for equation (6). The key technical part we use for Toda system is the non-degeneracy of the entire solutions of the $SU(n)$ Toda system. This has been proved recently by Wei-Zhao-Zhou [35] for $n = 3$, and by Lin-Wei-Ye [22] for general n.

The conclusion of Theorem 1.1 is surprising when comparing with other type of Liouville system. Suppose $u_k = (u_{1k}, \ldots, u_{nk})$ is a sequence of fully blowing up solutions to the following system:

$$\Delta u_{ik} + \sum_{j=1}^{n} a_{ij}\rho_{jk}\left(\frac{h_j e^{u_{jk}}}{h_j e^{u_{jk}}} - \frac{1}{|M|}\right) = 0 \quad \text{in } M, \tag{16}$$

for $1 \le i \le n$, where h_j are positive smooth functions in M, and the matrix $A = (a_{ij})$ is a symmetric, irreducible, nonnegative matrix and $\det A \neq 0$. This system of equations has been studied by Chanillo-Kiessling [6], Chipot-Shafrir-Wolansky [8] and recently by Lin-Zhang [24], [25] and [26]. In [26], Lin-Zhang proved sharper estimates for u_k. Suppose u_k has only one blowup point p, and $\rho_{ik} \to \rho_i$. Then they proved:

(i) location of the blow-up point p:

$$\sum_{i=1}^{n} \rho_i \nabla(\log h_i(x) + 2\pi H(x,p))|_{x=p} = 0; \tag{17}$$

(ii) the convergence rate:

$$8\pi \sum_{i=1}^{n} \rho_{ik} - \sum_{i,j=1}^{n} a_{ij}\rho_{ik}\rho_{jk} = \sum_{i=1}^{n} c_i(\Delta \log h_i(p_k) - 2K(p_k) + 8\pi)\varepsilon_k^2 |\log \varepsilon_k|, \tag{18}$$

where c_i are positive constants.

From (17) and (18), we see the obvious difference between (16) and Toda system.

The conditions (10)-(15) of Theorem 1.1 already contains a lot of informations related to the geometry of the flat torus M. To explain it, let us consider the simplest case of (1),

$$\begin{cases} \Delta v_1 + 2e^{v_1} - e^{v_2} = \rho\delta_{q_1} \\ \Delta v_2 + 2e^{v_2} - e^{v_1} = \rho\delta_{q_2} \end{cases} \quad \text{in } M, \tag{19}$$

where M is a flat torus, $q_1 \neq q_2$. By (2), we see $\rho = 8\pi$ is the first ρ where the fully blowing up may occur. In this case, there is only one blowup point p. If $p \in \{q_1, q_2\}$, i.e., if blow up occures at one of the vortex points, then we can use the quantization result in [22] and show

$$\int_M e^{v_1} dx = \int_M e^{v_2} dx = 16\pi,$$

a contradiction to $\rho = 8\pi$. Therefore we conclude $p \notin \{q_1, q_2\}$.

By applying Theorem 1.1, conditions (12) and (13) imply

$$\nabla_x G(p, q_1) = \nabla_x G(p, q_2) = 0. \tag{20}$$

Without loss of generality, one may assume $p = 0$ (by translation). Let $G(x)$ denote the Green function with singularity at 0. Then (20) implies $\nabla G(q_1) = \nabla G(q_2) = 0$. Applying a result due to Lin-Wang [17], $G(x)$ has either three critical points or five critical points. We claim:

the Green function G has five critical points and $q_1 = -q_2$.

Suppose G has three critical points only. Then these three critical points are all half periods. Hence both q_1 and q_2 are half periods. Let $q_1 = \frac{\omega_i}{2}$ and $q_2 = \frac{\omega_j}{2}$ for some $i \neq j$, where ω_i, ω_j are periods of M. We can compute the second derivatives of G at $\frac{\omega_i}{2}$ and $\frac{\omega_j}{2}$ by using the Weierstrass $\mathcal{P}$ function:

$$\begin{aligned}
2\pi G_{xx}(\frac{\omega_i}{2}) &= Re(\mathcal{P}(e_i) + \eta_1) \\
2\pi G_{yy}(\frac{\omega_i}{2}) &= -Re(\mathcal{P}(e_i) + \eta_1) + \frac{2\pi}{b} \\
2\pi G_{xy}(\frac{\omega_i}{2}) &= -Im(\mathcal{P}(e_i) + \eta_1),
\end{aligned}$$

where η_1 is one of quasi-period of $\xi(z) = -\int \mathcal{P}$.

The ∂_z^2 condition implies

$$G_{xx}(\frac{\omega_i}{2}) = G_{xx}(\frac{\omega_j}{2}), \quad G_{xy}(\frac{\omega_i}{2}) = G_{xy}(\frac{\omega_j}{2}) \text{ and } G_{yy}(\frac{\omega_i}{2}) = G_{yy}(\frac{\omega_j}{2}).$$

By using the above formulas, we have $\mathcal{P}(\frac{\omega_i}{2}) = \mathcal{P}(\frac{\omega_j}{2})$, which implies $q_1 = q_2$, a contradiction to our assumption. Therefore, the claim is proved, furthermore, by the same computation, we can prove that q_i are not half periods and $q_1 = -q_2$.

As we know, either the Louville equation or the Toda system are closely related to holomorphic curves of M into $\mathbb{CP}^n$, and are completely integrable systems. The integrability of Liouville equations allow us to define *the developing map f* defined in M, and one of striking results in [17] is that if f is a developing map for a solution u of $\Delta u + e^u = 8\pi\delta_0$, then λf is also a developing for another solution u_λ, for any $\lambda > 0$. Thus, once a solution exists, there is a family of solutions u_λ and u_λ blows up at a non-half period critical point of G as $\lambda \to +\infty$. Based on this phenomenon and the calculation above, we propose the following conjecture.

Conjecture: *Suppose $\rho = 8\pi$, M is a flat torus and $q_i \in M$. Then equation (19) has one solution if and only if the Green function $G(x)$ has five critical points.*

We are also interested in studying equation (1) in a bounded domain Ω in $\mathbb{R}^2$:

$$\begin{aligned}
\Delta u_{1k} + 2\rho_{1k}\frac{h_{1k}e^{u_{1k}}}{\int_\Omega h_{1k}e^{u_{1k}}} - \rho_{2k}\frac{h_{2k}e^{u_{2k}}}{\int_\Omega h_{2k}e^{u_{2k}}} &= 0 \\
\Delta u_{2k} + 2\rho_{2k}\frac{h_{2k}e^{u_{2k}}}{\int_\Omega h_{2k}e^{u_{2k}}} - \rho_{1k}\frac{h_{1k}e^{u_{1k}}}{\int_\Omega h_{1k}e^{u_{1k}}} &= 0
\end{aligned} \qquad \text{in } \Omega. \tag{21}$$

For the Dirichlet problem, it was proved that u_k can not blow up on the boundary of Ω, see [21] and related subjects in [18], [19], [20], [29], [32], [34]. Thus, we have the sharper estimates for u_k similar to Theorem 1.1.

Theorem 1.2. *Suppose h_{ik} converges to positive functions h_i in $C^2(\overline{\Omega})$, and u_k is a sequence of blowup solutions to (21) with homogeneous Dirichlet boundary conditions and $S = \{p_1, \ldots, p_m\}$ is the blowup set. Assume u_k fully blows at p_j, $j = 1, 2, \ldots, m$. Then it holds that*

$$\rho_{1k} - 8m\pi = \sum_{j=1}^{m} C_{1k,j} \Delta \ln h_{1k}(p_{k,j}) \varepsilon_{k,j}^2 |\ln \varepsilon_{k,j}| + O(\varepsilon_k^2), \tag{22}$$

$$\rho_{2k} - 8m\pi = \sum_{j=1}^{m} C_{2k,j} \Delta \ln h_{2k}(p_{k,j}) \varepsilon_{k,j}^2 |\ln \varepsilon_{k,j}| + O(\varepsilon_k^2), \tag{23}$$

where $C_{ik,j}$ $(i = 1, 2)$ are constants satisfying $0 < C_1 < C_{ik,j} < C_2 < \infty$. Furthermore, we have

$$8\pi \nabla_x H(p_{k,j}, p_{k,j}) + 8\pi \sum_{l \neq j} \nabla_x G(p_{k,j}, p_{k,l}) + \nabla \ln h_{1k}(p_{k,j}) = O(\varepsilon_k), \tag{24}$$

$$8\pi \nabla_x H(p_{k,j}, p_{k,j}) + 8\pi \sum_{l \neq j} \nabla_x G(p_{k,j}, p_{k,l}) + \nabla \ln h_{2k}(p_{k,j}) = O(\varepsilon_k), \tag{25}$$

and

$$\begin{aligned} 6\pi(\partial_{11} - \partial_{22})[\ln h_{2k}(p_{k,j}) - \ln h_{1k}(p_{k,j})] \quad & + \tfrac{T^j_{1k,1}}{4} \Delta \ln h_{1k}(p_{k,j}) \\ & + \tfrac{T^j_{2k,1}}{4} \Delta \ln h_{2k}(p_{k,j}) = O(\varepsilon_k^{\beta}), \end{aligned} \tag{26}$$

$$\begin{aligned} 12\pi\partial_{12}[\ln h_{2k}(p_{k,j}) - \ln h_{1k}(p_{k,j})] \quad & + \tfrac{T^j_{1k,2}}{4} \Delta \ln h_{1k}(p_{k,j}) \\ & + \tfrac{T^j_{2k,2}}{4} \Delta \ln h_{2k}(p_{k,j}) = O(\varepsilon_k^{\beta}), \end{aligned} \tag{27}$$

where $T^j_{1k,1}$, $T^j_{2k,1}$, $T^j_{1k,2}$ and $T^j_{2k,2}$ are four constants defined in Proposition 7.1.

The paper is organized as follows. In section 2, we state and prove two important properties of entire solutions. For the simplicity of presentation, we will first prove Theorem 1.2. Hence, we will consider a sequence of blowing up solutions of (21) from section 3 to section 8. In section 3, we present two preliminary estimates of the blowing up solutions. Then we approximate the bubbles using the parameterized entire solutions and obtain inner estimates in Section 4. Here we need the non-degeneracy of entire solutions. Section 5 to Section 7 contain the computations of the bubbling rate and bubbling locations. Here we use the eight kernels to test the system locally. We combine the estimates to prove the main Theorem 1.2 in Section 8. In the final section, we give a brief account for the proof of Theorem 1.1. Finally the proof of Lemma 4.1 is presented at the appendix.

2 Properties of Entire Solutions

In this section, we collect several useful properties of the entire solution (v_1, v_2) to (8). It is more convenient to consider the change of variables

$$(w_1, w_2) = (2v_1 + v_2, v_1 + 2v_2) \tag{28}$$

which satisfies

$$\begin{cases} \Delta w_1 + 3e^{\frac{2w_1 - w_2}{3}} = 0 & \text{in } \mathbb{R}^2, \\ \Delta w_2 + 3e^{\frac{2w_2 - w_1}{3}} = 0 & \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^{\frac{2w_1 - w_2}{3}} < \infty, \ \int_{\mathbb{R}^2} e^{\frac{2w_2 - w_1}{3}} < \infty. \end{cases} \tag{29}$$

Explicit formula for (w_1, w_2) is (see [14] and [35])

$$w_1(y) = \ln \frac{256 a_1^2 a_2^2}{\left(a_1^2 + a_2^2 |y + b|^2 + |y^2 + cy + d|^2\right)^3}, \tag{30}$$

$$w_2(y) = \ln \frac{1024 a_1^4 a_2^4}{\left(a_1^2 a_2^2 + a_1^2 |2y + c|^2 + a_2^2 |y^2 + 2by + bc - d|^2\right)^3}. \tag{31}$$

Observe that (w_1, w_2) depends on **eight** parameters $(a_1, a_2, b, c, d) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{C}^3$.

We first recall some results about the non-degeneracy of the solutions (w_1, w_2) to the Toda system (29). The following theorem classifies the kernels of the linearized operator of (29) at (w_1, w_2). Let $\tau \in (0, 1)$ be any given number.

Theorem 2.1. *([35]) If ϕ, ψ satisfy*

$$\begin{cases} \Delta\phi + e^{v_1}(2\phi - \psi) = 0 & \textit{in } \mathbb{R}^2, \\ \Delta\psi + e^{v_2}(2\psi - \phi) = 0, & \textit{in } \mathbb{R}^2, \\ |\phi| \le C(1 + |y|^\tau), \ |\psi| \le C(1 + |y|^\tau), \end{cases}$$

then $\binom{\phi}{\psi}$ belongs to the linear space

$$\operatorname{span}\left\{ \begin{pmatrix} \partial_{a_1} w_1 \\ \partial_{a_1} w_2 \end{pmatrix}, \begin{pmatrix} \partial_{a_2} w_1 \\ \partial_{a_2} w_2 \end{pmatrix}, \begin{pmatrix} \partial_{b_1} w_1 \\ \partial_{b_1} w_2 \end{pmatrix}, \begin{pmatrix} \partial_{b_2} w_1 \\ \partial_{b_2} w_2 \end{pmatrix}, \begin{pmatrix} \partial_{c_1} w_1 \\ \partial_{c_1} w_2 \end{pmatrix}, \begin{pmatrix} \partial_{c_2} w_1 \\ \partial_{c_2} w_2 \end{pmatrix}, \begin{pmatrix} \partial_{d_1} w_1 \\ \partial_{d_1} w_2 \end{pmatrix}, \begin{pmatrix} \partial_{d_2} w_1 \\ \partial_{d_2} w_2 \end{pmatrix} \right\}.$$

Our next lemma states that the eight parameters (a_1, a_2, b, c, d) are uniquely determined by the initial values $(w_1(0), w_2(0), \partial_z w_1(0), \partial_z w_2(0), \partial_{zz} w_1(0))$.

Theorem 2.2. *Let $(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5) \in \mathbb{R} \times \mathbb{R} \times \mathbb{C}^3$ be given. Then there is a unique $(a_1, a_2, b, c, d) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{C}^3$ such that*

$$(w_1(0), w_2(0), \partial_z w_1(0), \partial_z w_2(0), \partial_{zz} w_1(0)) = (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5). \tag{32}$$

Proof. A direct computation shows that system (32) is equivalent to

$$a_1^2 + a_2^2|b|^2 + |d|^2 = \mathcal{A}_1 a_1^{\frac{2}{3}} a_2^{\frac{2}{3}}, \tag{33}$$

$$a_1^2 a_2^2 + a_1^2|c|^2 + a_2^2|bc-d|^2 = \mathcal{A}_2 a_1^{\frac{4}{3}} a_2^{\frac{4}{3}}, \tag{34}$$

$$\frac{a_2^2 b + \bar{c}d}{a_1^2 + a_2^2|b|^2 + |d|^2} = \mathcal{A}_3, \tag{35}$$

$$\frac{a_1^2 c + a_2^2|b|^2 c - a_2^2 \bar{b} d}{a_1^2 a_2^2 + a_1^2|c|^2 + a_2^2|bc-d|^2} = \mathcal{A}_4, \tag{36}$$

$$\frac{(a_2^2 b + \bar{c}d)^2 - 2d(a_1^2 + a_2^2|b|^2 + |d|^2)}{(a_1^2 + a_2^2|b|^2 + |d|^2)^2} = \mathcal{A}_5, \tag{37}$$

where

$$\mathcal{A}_1 = e^{\frac{\gamma_1}{3}} 4^{\frac{4}{3}},\ \mathcal{A}_1 = e^{\frac{\gamma_2}{3}} 16,\ \mathcal{A}_3 = -\frac{\gamma_3}{6}, \mathcal{A}_4 = -\frac{\gamma_4}{12},\ \mathcal{A}_5 = -\frac{\gamma_5}{12}. \tag{38}$$

We **claim** the existence of a_1, a_2, b, c, d with a_1, a_2 uniformly bounded above and below from zero and b, c, d bounded. First, by the equations (33)-(37) it is not difficult to find out that

$$b = \mathcal{A}_1 \mathcal{A}_3 a_1^{\frac{2}{3}} a_2^{-\frac{4}{3}} - \left[\frac{1}{4}|\mathcal{A}_3^2 - \mathcal{A}_5|^2 \mathcal{A}_1^2 \mathcal{A}_3 + \frac{1}{2}(\mathcal{A}_3^2 - \mathcal{A}_5)\mathcal{A}_2 \overline{\mathcal{A}}_4\right] a_1^{\frac{4}{3}} a_2^{-\frac{2}{3}}, \tag{39}$$

$$c = \left[\frac{1}{2}\mathcal{A}_1(\mathcal{A}_3^2 - \mathcal{A}_5)\overline{\mathcal{A}}_3 + \frac{\mathcal{A}_2\mathcal{A}_4}{\mathcal{A}_1}\right] a_1^{\frac{2}{3}} a_2^{\frac{2}{3}}, \tag{40}$$

$$d = \frac{1}{2}\mathcal{A}_1(\mathcal{A}_3^2 - \mathcal{A}_5) a_1^{\frac{2}{3}} a_2^{\frac{2}{3}}. \tag{41}$$

It remains to determine a_1 and a_2. To this end, we let $t = a_1^{\frac{2}{3}} a_2^{\frac{2}{3}}$ and we solve in t first. From (33) and (35) we have

$$a_2^2 b + \bar{c}d = \mathcal{A}_1 \mathcal{A}_3 t, \tag{42}$$

which implies that

$$a_2^4|b|^2 + |c|^2|d|^2 + a_2^2 \bar{b}\bar{c}d + a_2^2 bc\bar{d} = \mathcal{A}_1^2|\mathcal{A}_3|^2 t^2. \tag{43}$$

Multiplying (42) with $c\bar{d}$, we have $a_2^2 bc\bar{d} + |c|^2|d|^2 = \mathcal{A}_1\mathcal{A}_3 tc\bar{d}$. Adding it to its conjugate, we get $a_2^2\bar{b}\bar{c}d + a_2^2 bc\bar{d} = -2|c|^2|d|^2 + \mathcal{A}_1 t(\mathcal{A}_3 c\bar{d} + \overline{\mathcal{A}}_3\bar{c}d)$. Then (43) may be rewritten as

$$a_2^4|b|^2 = |c|^2|d|^2 + \mathcal{A}_1^2|\mathcal{A}_3|^2 t^2 - \mathcal{A}_1 t(\mathcal{A}_3 c\bar{d} + \overline{\mathcal{A}}_3 \bar{c}d). \tag{44}$$

Expansion in (34) gives

$$a_1^2 a_2^2 + a_1^2|c|^2 + a_2^2(|b|^2|c|^2 + |d|^2 - \bar{b}\bar{c}d - bc\bar{d}) = \mathcal{A}_2 t^2. \tag{45}$$

Adding (43) and (45) yields

$$(a_2^2 + |c|^2)(a_1^2 + a_2^2|b|^2 + |d|^2) = (\mathcal{A}_2 + \mathcal{A}_1^2|\mathcal{A}_3|^2)t^2 \tag{46}$$

and hence by (65) we have

$$a_2^2 + |c|^2 = (\frac{\mathcal{A}_2}{\mathcal{A}_1} + \mathcal{A}_1|\mathcal{A}_3|^2)t. \tag{47}$$

Multiplying (65) by a_2^2 we have

$$t^3 + a_2^4|b|^2 + a_2^2|d|^2 = \mathcal{A}_1 t a_2^2. \tag{48}$$

Substituting (44) into (48), we obtain

$$t^3 + |d|^2(a_2^2 + |c|^2) + \mathcal{A}_1^2|\mathcal{A}_3|^2 t^2 - \mathcal{A}_1 t(\mathcal{A}_3 c\bar{d} + \overline{\mathcal{A}}_3 \bar{c} d) = \mathcal{A}_1 t a_2^2. \tag{49}$$

Substituting (47) into (49), we get

$$t^2 + (\frac{\mathcal{A}_2}{\mathcal{A}_1}|d|^2 + \mathcal{A}_1|d\overline{\mathcal{A}}_3 - c|^2) = \mathcal{A}_2 t, \tag{50}$$

from which we can solve

$$t = a_1^{\frac{2}{3}} a_2^{\frac{2}{3}} = \frac{4\mathcal{A}_1\mathcal{A}_2}{4\mathcal{A}_1 + \mathcal{A}_1^2\mathcal{A}_2|\mathcal{A}_3^2 - \mathcal{A}_5|^2 + 4\mathcal{A}_2^2|\mathcal{A}_4|^2}. \tag{51}$$

Obviously t is uniformly bounded above and also below from zero. Therefore b, c, d are all $O(1)$'s. Then by (47) a_1, a_2 can also be solved uniquely

$$a_2^2 = (\frac{\mathcal{A}_2}{\mathcal{A}_1} + \mathcal{A}_1|\mathcal{A}_3|^2)t - |c|^2, \qquad a_1^2 = \frac{t^3}{a_2^2}. \tag{52}$$

It is also easy to see that $a_2^2 > 0$ and we can choose a_2 to be uniformly bounded below from zero. In fact,

$$\begin{aligned} a_2^2 = \Bigg[&(\frac{\mathcal{A}_2}{\mathcal{A}_1} + \mathcal{A}_1|\mathcal{A}_3|^2)(4\mathcal{A}_1 + \mathcal{A}_1^2\mathcal{A}_2|\mathcal{A}_3^2 - \mathcal{A}_5|^2 + 4\mathcal{A}_2^2|\mathcal{A}_4|^2) \\ &- 4\mathcal{A}_1\mathcal{A}_2 \left|\frac{1}{2}\mathcal{A}_1(\mathcal{A}_3^2 - \mathcal{A}_5)\overline{\mathcal{A}}_3 + \frac{\mathcal{A}_2\mathcal{A}_4}{\mathcal{A}_1}\right|^2 \Bigg] \frac{t^2}{4\mathcal{A}_1\mathcal{A}_2} \\ \geq{}& 4\mathcal{A}_2 \geq C > 0, \end{aligned}$$

where we should note that

$$|\mathcal{A}_3^2 - \mathcal{A}_5|^2 + 4|\mathcal{A}_3|^2|\mathcal{A}_4|^2 \geq 2\overline{\mathcal{A}}_3\overline{\mathcal{A}}_4(\mathcal{A}_3^2 - \mathcal{A}_5) + 2\mathcal{A}_3\mathcal{A}_4(\overline{\mathcal{A}}_3^2 - \overline{\mathcal{A}}_5).$$

Finally we have that $a_1 = t^{\frac{3}{2}}/a_2$. Therefore the claim holds. □

3 Preliminary Estimates on Blow-ups

In this section we derive two estimates of blowing-up solutions, one is near blow-up points and the other is far away from them. For simplicity of presentation, we will first prove Theorem 1.2. From now to section 8, we let (u_{1k}, u_{2k})

be a blowing up sequence of solutions to (21) satisfying (H). For $j = 1, ..., m$, let $p_{k,j}$ be the local maxima of u_{1k} near p_j, i.e., $u_{1k}(p_{k,j}) = \max_{B_\delta(p_j)} u_{1k}(x)$ where δ is sufficiently small. ($p_{k,j}$ may not be unique.)
Define

$$-2\ln\varepsilon_{k,j} \overset{\text{def}}{=} u_{1k}(p_{k,j}) - \ln\int_\Omega h_{1k}e^{u_{1k}} + \ln(\rho_{1k}h_{1k}(p_{k,j})), \tag{53}$$

$$\varepsilon_k = \max_{1\le j\le m}\varepsilon_{k,j}. \tag{54}$$

Set also

$$e^{\alpha_{1k}} = \int_\Omega h_{1k}e^{u_{1k}}, \qquad e^{\alpha_{2k}} = \int_\Omega h_{2k}e^{u_{2k}},$$
$$\tilde{u}_{1k}(x) = u_{1k}(x) - \alpha_{1k}, \qquad \tilde{u}_{2k}(x) = u_{2k}(x) - \alpha_{2k}.$$

Thus $\tilde{u}_{1k}$ and $\tilde{u}_{2k}$ satisfy

$$\begin{cases} \Delta\tilde{u}_{1k} + 2\rho_{1k}h_{1k}e^{\tilde{u}_{1k}} - \rho_{2k}h_{2k}e^{\tilde{u}_{2k}} = 0 & \text{in } \Omega, \\ \Delta\tilde{u}_{2k} - \rho_{1k}h_{1k}e^{\tilde{u}_{1k}} + 2\rho_{2k}h_{2k}e^{\tilde{u}_{2k}} = 0 & \text{in } \Omega. \end{cases}$$

The following sup + inf estimate plays an important role in later proofs. This follows from [13, Theorem 1.3].

Lemma 3.1. *Under the assumptions of Theorem 1.1 or Theorem 1.2, there exists a small $\delta > 0$ independent of k such that*

$$\tilde{u}_{ik} + 2\ln\varepsilon_{k,j} - \left[v_i\left(\frac{x-p_{k,j}}{\varepsilon_{k,j}}\right) - \ln(\rho_{ik}h_{ik}(p_{k,j}))\right] = O(1) \qquad \text{in } B_\delta(p_j) \tag{55}$$

for $i = 1, 2$ and $j = 1, \ldots, m$.

Remark: As in [13], the entire solutions v_i are chosen so that they equal to zero at the origin.

Proof. Letting

$$\bar{u}_{1k}(x) = \tilde{u}_{1k}(x) + \ln(\rho_{1k}h_{1k}(p_{k,j})), \qquad \bar{u}_{2k}(x) = \tilde{u}_{2k}(x) + \ln(\rho_{2k}h_{2k}(p_{k,j})),$$

we have

$$\begin{cases} \Delta\bar{u}_{1k} + \dfrac{2h_{1k}(x)}{h_{1k}(p_{k,j})}e^{\bar{u}_{1k}} - \dfrac{h_{2k}(x)}{h_{2k}(p_{k,j})}e^{\bar{u}_{2k}} = 0 & \text{in } B_\delta(p_j), \\ \Delta\bar{u}_{2k} - \dfrac{h_{1k}(x)}{h_{1k}(p_{k,j})}e^{\bar{u}_{1k}} + \dfrac{2h_{2k}(x)}{h_{2k}(p_{k,j})}e^{\bar{u}_{2k}} = 0 & \text{in } B_\delta(p_j). \end{cases}$$

Then $\bar{u}_{1k}$ and $\bar{u}_{2k}$ satisfy the conditions of [13, Theorem 1.3]. So we conclude that there exist two constants $\delta > 0$ and $C > 0$ independent of k such that

$$\left|\bar{u}_{ik}(x) + 2\ln\varepsilon_{k,j} - v_i\left(\frac{x-p_{k,j}}{\varepsilon_{k,j}}\right)\right| \le C \qquad \text{in } B_\delta(p_j) \quad \text{for } i = 1, 2,$$

which is equivalent to (55). ∎

Remark 3.2. *By considering $\bar{u}_{ik}(\varepsilon_{k,j}y+p_{k,j})+2\ln\varepsilon_{k,j}$ $(i=1,2)$, we have that the following holds:*

$$\varepsilon_{k,j}|\nabla\tilde{u}_{1k}(p_{k,j})|\le C, \qquad \varepsilon_{k,j}^2|\nabla^2\tilde{u}_{1k}(p_{k,j})|\le C, \tag{56}$$
$$\varepsilon_{k,j}|\nabla\tilde{u}_{2k}(p_{k,j})|\le C, \qquad \varepsilon_{k,j}^2|\nabla^2\tilde{u}_{2k}(p_{k,j})|\le C. \tag{57}$$

From Lemma 3.1 we have the following important corollary.

Corollary 3.3. *For any $i=1,2$ and $j=1,\dots,m$, it holds that*

$$\alpha_{ik}+2\ln\varepsilon_{k,j}=O(1), \tag{58}$$
$$C^{-1}\varepsilon_{k,\ell}\le\varepsilon_{k,j}\le C\varepsilon_{k,\ell} \qquad \text{for any } \ell\ne j. \tag{59}$$

Proof. Noting that $v_i(\frac{x-p_{k,j}}{\varepsilon_{k,j}})\sim 4\ln\varepsilon_{k,j}$ on $\partial B_\delta(p_j)$, we get (58). (59) follows directly from (58). □

For a fixed small $\delta>0$, we set the *local mass* $\rho_{ik,j}$ to be

$$\rho_{ik,j}=\rho_{ik}\int_{B_\delta(p_j)}h_{ik}e^{\tilde{u}_{ik}} \qquad \text{for } i=1,2.$$

By Lemma 3.1 we have

$$\rho_{ik,j}=\rho_{ik}\int_{B_\delta(p_{k,j})}h_{ik}e^{\tilde{u}_{ik}}+O(\varepsilon_{k,j}^2).$$

Observe that $\rho_{ik}=\rho_{ik}\int_\Omega h_{ik}e^{\tilde{u}_{ik}}$ and it is easy to see that

$$\rho_{ik}=\sum_{j=1}^m\rho_{ik,j}+O(\varepsilon_k^2),$$

where ε_k is defined in (54). Define again

$$w_{1k}(x)=2u_{1k}+u_{2k}-3\sum_{j=1}^m\rho_{1k,j}G(x,p_{k,j}),$$
$$w_{2k}(x)=u_{1k}+2u_{2k}-3\sum_{j=1}^m\rho_{2k,j}G(x,p_{k,j}).$$

Lemma 3.4. *It holds that, for $i=1,2$,*

$$|w_{ik}|+|\nabla w_{ik}|=O(\varepsilon_k) \qquad \text{for } x\in\Omega\setminus\bigcup_{j=1}^m B_\delta(p_j).$$

Proof. It is easy to see that

$$\begin{cases}-\Delta(2u_{1k}+u_{2k})=3\rho_{1k}h_{1k}e^{\tilde{u}_{1k}} & \text{in } \Omega,\\ -\Delta(u_{1k}+2u_{2k})=3\rho_{2k}h_{2k}e^{\tilde{u}_{2k}} & \text{in } \Omega.\end{cases}$$

This lemma follows from the Green's representation formula. In fact, for $\ell = 0, 1$ and $x \in \Omega \setminus \bigcup_{j=1}^{m} B_\delta(p_j)$

$$
\begin{aligned}
&\partial^\ell(2u_{1k}(x) + u_{2k}(x)) = 3\int_\Omega \partial^\ell G(x,z)\rho_{1k}h_{1k}(z)e^{\tilde{u}_{1k}(z)}\mathrm{d}z \\
&= 3\sum_{j=1}^{m}\int_{B_{\frac{\delta}{2}}(p_{k,j})} \partial^\ell G(x,z)\rho_{1k}h_{1k}(z)e^{\tilde{u}_{1k}(z)}\mathrm{d}z + O(\varepsilon_k^2) \\
&= 3\sum_{j=1}^{m}\int_{B_{\frac{\delta}{2}}(p_{k,j})} \partial^\ell[G(x,z) - G(x,p_{k,j})]\rho_{1k}h_{1k}(z)e^{\tilde{u}_{1k}(z)}\mathrm{d}z \\
&\qquad + 3\sum_{j=1}^{m}\int_{B_{\frac{\delta}{2}}(p_{k,j})} \partial^\ell G(x,p_{k,j})\rho_{1k}h_{1k}(z)e^{\tilde{u}_{1k}(z)}\mathrm{d}z + O(\varepsilon_k^2) \\
&= 3\int_{B_{\frac{\delta}{2}}(p_{k,j})} O(|z-p_k|)\rho_{1k}h_{1k}(z)e^{\tilde{u}_{1k}(z)}\mathrm{d}z + 3\sum_{j=1}^{m}\rho_{1k,j}\partial^\ell G(x,p_{k,j}) + O(\varepsilon_k^2) \\
&= 3\sum_{j=1}^{m}\rho_{1k,j}\partial^\ell G(x,p_{k,j}) + O(\varepsilon_k).
\end{aligned}
$$

The proof of other estimates is similar and thus omitted. □

4 Sharp approximation of the bubbles

In this section, we give a sharp description of the bubbling behavior of $2\tilde{u}_{1k} + \tilde{u}_{2k}$ and $\tilde{u}_{1k} + 2\tilde{u}_{2k}$ in the ball $B_\delta(p_{k,j})$, using the entire solutions of (29).

For simplicity, we set

$$
\begin{aligned}
\widetilde{G}_{1k,j}(x) &= \rho_{1k,j}H(x,p_{k,j}) + \sum_{\ell\neq j}\rho_{1k,\ell}G(x,p_{k,\ell}), \\
\widetilde{G}_{2k,j}(x) &= \rho_{2k,j}H(x,p_{k,j}) + \sum_{\ell\neq j}\rho_{2k,\ell}G(x,p_{k,\ell}).
\end{aligned}
$$

Set also

$$
V_{1k,j}(y) = \ln\frac{4\left(a_{1k,j}^2a_{2k,j}^2 + a_{1k,j}^2\left|2y + c_{k,j}\right|^2 + a_{2k,j}^2\left|y^2 + 2b_{k,j}y + b_{k,j}c_{k,j} - d_{k,j}\right|^2\right)}{\left(a_{1k,j}^2 + a_{2k,j}^2|y + b_{k,j}|^2 + |y^2 + c_{k,j}y + d_{k,j}|^2\right)^2\rho_{1k}h_{1k}(p_{k,j})},
$$

$$
V_{2k,j}(y) = \ln\frac{16a_{1k,j}^2a_{2k,j}^2\left(a_{1k,j}^2 + a_{2k,j}^2|y + b_{k,j}|^2 + \left|y^2 + c_{k,j}y + d_{k,j}\right|^2\right)}{\left(a_{1k,j}^2a_{2k,j}^2 + a_{1k,j}^2\left|2y + c_{k,j}\right|^2 + a_{2k,j}^2\left|y^2 + 2b_{k,j}y + b_{k,j}c_{k,j} - d_{k,j}\right|^2\right)^2} - \ln\rho_{2k}h_{2k}(p_{k,j}),
$$

and

$$
U_{1k,j}(x) = V_{1k,j}\left(\frac{x - p_{k,j}}{\varepsilon_{k,j}}\right), \qquad U_{2k,j}(x) = V_{2k,j}\left(\frac{x - p_{k,j}}{\varepsilon_{k,j}}\right),
$$

where $a_{1k,j}$, $a_{2k,j}$, $b_{k,j}$, $c_{k,j}$ and $d_{k,j}$ are chosen such that

$$(2U_{1k,j}+U_{2k,j})(p_{k,j}) = (2\tilde{u}_{1k}+\tilde{u}_{2k})(p_{k,j})+6\ln\varepsilon_{k,j}, \tag{60}$$
$$(U_{1k,j}+2U_{2k,j})(p_{k,j}) = (\tilde{u}_{1k}+2\tilde{u}_{2k})(p_{k,j})+6\ln\varepsilon_{k,j}, \tag{61}$$
$$\nabla_{\bar{x}}(2U_{1k,j}+U_{2k,j})(p_{k,j}) = \nabla_{\bar{x}}(2\tilde{u}_{1k}+\tilde{u}_{2k})(p_{k,j})-3\nabla_{\bar{x}}\widetilde{G}_{1k,j}(p_{k,j}), \tag{62}$$
$$\nabla_{\bar{x}}(U_{1k,j}+2U_{2k,j})(p_{k,j}) = \nabla_{\bar{x}}(\tilde{u}_{1k}+2\tilde{u}_{2k})(p_{k,j})-3\nabla_{\bar{x}}\widetilde{G}_{2k,j}(p_{k,j}), \tag{63}$$
$$\nabla_{\bar{x}\bar{x}}(2U_{1k,j}+U_{2k,j})(p_{k,j}) = \nabla_{\bar{x}\bar{x}}(2\tilde{u}_{1k}+\tilde{u}_{2k})(p_{k,j})-3\nabla_{\bar{x}\bar{x}}\widetilde{G}_{1k,j}(p_{k,j}). \tag{64}$$

We remark that (60)-(64) can be solved in the coefficients $a_{1k,j}$, $a_{2k,j}$, $b_{k,j}$, $c_{k,j}$ and $d_{k,j}$. In fact, a direct computation shows that

$$2V_{1k,j}+V_{2k,j} = \ln\frac{1}{\left(a_{1k,j}^2+a_{2k,j}^2|y+b_{k,j}|^2+|y^2+c_{k,j}y+d_{k,j}|^2\right)^3} + \ln\frac{256a_{1k,j}^2a_{2k,j}^2}{\rho_{1k}^2h_{1k}^2(p_{k,j})\rho_{2k}h_{2k}(p_{k,j})},$$

$$V_{1k,j}+2V_{2k,j} = \ln\frac{1}{\left(a_{1k,j}^2a_{2k,j}^2+a_{1k,j}^2\left|2y+c_{k,j}\right|^2+a_{2k,j}^2\left|y^2+2b_{k,j}y+b_{k,j}c_{k,j}-d_{k,j}\right|^2\right)^3} + \ln\frac{1024a_{1k,j}^4a_{2k,j}^4}{\rho_{1k}h_{1k}(p_{k,j})\rho_{2k}^2h_{2k}^2(p_{k,j})}.$$

We omit the subscript j for simplicity. System (60)–(64) is then rewritten as

$$a_{1k}^2+a_{2k}^2|b_k|^2+|d_k|^2 = \mathcal{A}_{1k}a_{1k}^{\frac{2}{3}}a_{2k}^{\frac{2}{3}}, \tag{65}$$
$$a_{1k}^2a_{2k}^2+a_{1k}^2|c_k|^2+a_{2k}^2|b_kc_k-d_k|^2 = \mathcal{A}_{2k}a_{1k}^{\frac{4}{3}}a_{2k}^{\frac{4}{3}}, \tag{66}$$
$$\frac{a_{2k}^2b_k+\bar{c}_kd_k}{a_{1k}^2+a_{2k}^2|b_k|^2+|d_k|^2} = \mathcal{A}_{3k}, \tag{67}$$
$$\frac{a_{1k}^2c_k+a_{2k}^2|b_k|^2c_k-a_{2k}^2\bar{b}_kd_k}{a_{1k}^2a_{2k}^2+a_{1k}^2|c_k|^2+a_{2k}^2|b_kc_k-d_k|^2} = \mathcal{A}_{4k}, \tag{68}$$
$$\frac{(a_{2k}^2b_k+\bar{c}_kd_k)^2-2d_k(a_{1k}^2+a_{2k}^2|b_k|^2+|d_k|^2)}{(a_{1k}^2+a_{2k}^2|b_k|^2+|d_k|^2)^2} = \mathcal{A}_{5k}, \tag{69}$$

where $0 < C < \mathcal{A}_{1k} \in \mathbb{R}$, $0 < C < \mathcal{A}_{2k} \in \mathbb{R}$, $\mathcal{A}_{3k}, \mathcal{A}_{4k}, \mathcal{A}_{5k} \in \mathbb{C}$ are uniquely decided by the terms on the right hand side of (60)–(64). Because of the definition of $\varepsilon_{k,j}$, the assumption in the main theorem and (56), (57), all of $\mathcal{A}_{ik}$ $(i = 1, \ldots, 5)$ are uniformly of order $O(1)$.

By the same proof as in Theorem 2.2, we obtain the existences of a_{1k}, a_{2k}, b_k, c_k, d_k with a_{1k}, a_{2k} uniformly bounded away from zero and b_k, c_k, d_k bounded.

In what follows, we define for $x \in B_\delta(p_{k,j})$

$$\eta_{1k,j}(x) = 2\tilde{u}_{1k} + \tilde{u}_{2k} + 6\ln\varepsilon_{k,j} - 2U_{1k,j} - U_{2k,j} - 3\widetilde{G}_{1k,j}(x) + 3\widetilde{G}_{1k,j}(p_{k,j}),$$
$$\eta_{2k,j}(x) = \tilde{u}_{1k} + 2\tilde{u}_{2k} + 6\ln\varepsilon_{k,j} - U_{1k,j} - 2U_{2k,j} - 3\widetilde{G}_{2k,j}(x) + 3\widetilde{G}_{2k,j}(p_{k,j}).$$

In $B_\delta(p_{k,j}) \setminus B_{\frac{\delta}{2}}(p_{k,j})$, by a Taylor expansion, it is easy to see that

$$\begin{aligned}
&2U_{1k,j} + U_{2k,j} - 6\ln\varepsilon_{k,j} \\
=& -3\ln\Big[a_{1k,j}^2\varepsilon_{k,j}^4 + a_{2k,j}^2\varepsilon_{k,j}^2|(x-p_{k,j}) + \varepsilon_{k,j}b_{k,j}|^2 \\
&\qquad + |(x-p_{k,j})^2 + \varepsilon_{k,j}c_{k,j}(x-p_{k,j}) + \varepsilon_{k,j}^2 d_{k,j}|^2\Big] \\
&\qquad\qquad + \ln\frac{256a_{1k,j}^2a_{2k,j}^2\varepsilon_{k,j}^6}{\rho_{1k}^2h_{1k}^2(p_{k,j})\rho_{2k}h_{2k}(p_{k,j})}, \\
=& -12\ln|x-p_{k,j}| + 6\ln\varepsilon_{k,j} + \ln\frac{256a_{1k,j}^2a_{2k,j}^2}{\rho_{1k}^2h_{1k}^2(p_{k,j})\rho_{2k}h_{2k}(p_{k,j})} + O(\varepsilon_{k,j}),
\end{aligned}$$

and

$$\begin{aligned}
&U_{1k,j} + 2U_{2k,j} - 6\ln\varepsilon_{k,j} \\
=& -3\ln\Big[a_{1k,j}^2a_{2k,j}^2\varepsilon_{k,j}^4 + a_{1k,j}^2\varepsilon_{k,j}^2|2(x-p_{k,j}) + \varepsilon_{k,j}c_{k,j}|^2 \\
&\qquad + a_{2k,j}^2|(x-p_{k,j})^2 + 2\varepsilon_{k,j}b_{k,j}(x-p_{k,j}) + \varepsilon_{k,j}^2(b_{k,j}c_{k,j} - d_{k,j})|^2\Big] \\
&\qquad\qquad + \ln\frac{1024a_{1k,j}^4a_{2k,j}^4\varepsilon_{k,j}^6}{\rho_{1k}h_{1k}(p_{k,j})\rho_{2k}^2h_{2k}^2(p_{k,j})} \\
=& -12\ln|x-p_{k,j}| + 6\ln\varepsilon_{k,j} + \ln\frac{1024a_{1k,j}^4a_{2k,j}^{-2}}{\rho_{1k}h_{1k}(p_{k,j})\rho_{2k}^2h_{2k}^2(p_{k,j})} + O(\varepsilon_{k,j}).
\end{aligned}$$

Thus we have, in $B_\delta(p_{k,j}) \setminus B_{\frac{\delta}{2}}(p_{k,j})$, that

$$\begin{aligned}
\eta_{1k,j} =& w_{1k} + 3\rho_{1k,j}G(x,p_{k,j}) - 2\alpha_{1k} - \alpha_{2k} - 3\rho_{1k,j}H(x,p_{k,j}) + 3\widetilde{G}_{1k,j}(p_{k,j}) \\
&+ 12\ln|x-p_{k,j}| - 6\ln\varepsilon_{k,j} - \ln\frac{256a_{1k,j}^2a_{2k,j}^2}{\rho_{1k}^2h_{1k}^2(p_{k,j})\rho_{2k}h_{2k}(p_{k,j})} + O(\varepsilon_k) \\
=& \frac{3}{2\pi}(8\pi - \rho_{1k,j})\ln|x-p_{k,j}| + A_{1k,j} + O(\varepsilon_k),
\end{aligned} \tag{70}$$

where $A_{1k,j}$ is a constant given by

$$A_{1k,j} = -2\alpha_{1k} - \alpha_{2k} - 6\ln\varepsilon_{k,j} + 3\widetilde{G}_{1k,j}(p_{k,j}) - \ln\frac{256a_{1k,j}^2a_{2k,j}^2}{\rho_{1k}^2h_{1k}^2(p_{k,j})\rho_{2k}h_{2k}(p_{k,j})}.$$

From Corollary 3.3 we derive that $A_{1k} = O(1)$. Moreover Lemma 3.4 also indicates that (70) holds for $\nabla\eta_{1k,j}$. Analogously, in $B_\delta(p_k) \setminus B_{\frac{\delta}{2}}(p_k)$,

$$\eta_{2k,j} = w_{2k} + 3\rho_{2k,j}G(x,p_{k,j}) - \alpha_{1k} - 2\alpha_{2k} - 3\rho_{2k,j}H(x,p_{k,j}) + 3\widetilde{G}_{2k,j}(p_{k,j})$$

$$+12\ln|x-p_{k,j}|-6\ln\varepsilon_{k,j}-\ln\frac{1024a_{1k,j}^4a_{2k,j}^{-2}}{\rho_{1k}^2h_{1k}^2(p_{k,j})\rho_{2k}h_{2k}(p_{k,j})}+O(\varepsilon_k)$$
$$=\frac{3}{2\pi}(8\pi-\rho_{2k,j})\ln|x-p_{k,j}|+A_{2k,j}+O(\varepsilon_k), \tag{71}$$

where $A_{2k,j}=O(1)$ and

$$A_{2k,j}=-\alpha_{1k}-2\alpha_{2k}-6\ln\varepsilon_{k,j}+3\widetilde{G}_{2k,j}(p_{k,j})-\ln\frac{1024a_{1k,j}^4a_{2k,j}^{-2}}{\rho_{1k}^2h_{1k}^2(p_{k,j})\rho_{2k}h_{2k}(p_{k,j})}.$$

Also we have that (71) holds for $\nabla\eta_{2k,j}$.

In order to estimate $\eta_{ik,j}$ in the whole $B_\delta(p_{k,j})$, let us define, for $|y|\le\frac{\delta}{\varepsilon_{k,j}}$,

$$\tilde{\eta}_{1k,j}(y)=\eta_{1k,j}(p_{k,j}+\varepsilon_{k,j}y),\qquad \tilde{\eta}_{2k,j}(y)=\eta_{2k,j}(p_{k,j}+\varepsilon_{k,j}y). \tag{72}$$

By the definition of $\tilde{\eta}_{1k,j}$ and $\tilde{\eta}_{2k,j}$, it is easy to see that they satisfy

$$\begin{cases}-\Delta\tilde{\eta}_{1k,j}=3\rho_{1k}h_{1k}(p_{k,j})e^{V_{1k,j}}D_{1k,j}(y) & \text{in } B_{\frac{\delta}{\varepsilon_{k,j}}},\\ -\Delta\tilde{\eta}_{2k,j}=3\rho_{2k}h_{2k}(p_{k,j})e^{V_{2k,j}}D_{2k,j}(y) & \text{in } B_{\frac{\delta}{\varepsilon_{k,j}}},\\ \tilde{\eta}_{1k,j}=O(1),\quad \tilde{\eta}_{2k,j}=O(1) & \text{on } \partial B_{\frac{\delta}{\varepsilon_{k,j}}},\end{cases}$$

where

$$D_{1k,j}(y)=\exp\left[\frac{1}{3}(2\tilde{\eta}_{1k,j}-\tilde{\eta}_{2k,j})+Q_{1k,j}(\varepsilon_{k,j}y+p_{k,j})-Q_{1k,j}(p_{k,j})\right]-1, \tag{73}$$

$$D_{2k,j}(y)=\exp\left[\frac{1}{3}(2\tilde{\eta}_{2k,j}-\tilde{\eta}_{1k,j})+Q_{2k,j}(\varepsilon_{k,j}y+p_{k,j})-Q_{2k,j}(p_{k,j})\right]-1. \tag{74}$$

In (73) and (74), $Q_{1k,j}(x)$ and $Q_{2k,j}(x)$ denote

$$Q_{1k,j}=2\widetilde{G}_{1k,j}-\widetilde{G}_{2k,j}+\ln h_{1k}, \tag{75}$$
$$Q_{2k,j}=2\widetilde{G}_{2k,j}-\widetilde{G}_{1k,j}+\ln h_{2k}. \tag{76}$$

Since $Q_{ik,j}(\varepsilon_{k,j}y+p_{k,j})-Q_{ik,j}(p_{k,j})=\nabla Q_{ik,j}(p_{k,j})\varepsilon_{k,j}y+O(\varepsilon_k^2|y|^2)$, we have in $B_{\frac{\delta}{\varepsilon_{k,j}}}$ that

$$\begin{cases}-\Delta\tilde{\eta}_{1k,j}=\rho_{1k}h_{1k}(p_{k,j})e^{V_{1k,j}}3\frac{e^{\frac{1}{3}(2\tilde{\eta}_{1k,j}-\tilde{\eta}_{2k,j})}-1}{2\tilde{\eta}_{1k,j}-\tilde{\eta}_{2k,j}}(2\tilde{\eta}_{1k,j}-\tilde{\eta}_{2k,j})\\ \qquad\qquad +O\left(\frac{|\nabla Q_{1k,j}(p_{k,j})|\varepsilon_k}{1+|y|^3}\right)+O(\frac{\varepsilon_k^2}{1+|y|^2}),\\ -\Delta\tilde{\eta}_{2k,j}=\rho_{2k}h_{2k}(p_{k,j})e^{V_{2k,j}}3\frac{e^{\frac{1}{3}(2\tilde{\eta}_{2k,j}-\tilde{\eta}_{1k,j})}-1}{2\tilde{\eta}_{2k,j}-\tilde{\eta}_{1k,j}}(2\tilde{\eta}_{2k,j}-\tilde{\eta}_{1k,j})\\ \qquad\qquad +O\left(\frac{|\nabla Q_{2k,j}(p_{k,j})|\varepsilon_k}{1+|y|^3}\right)+O(\frac{\varepsilon_k^2}{1+|y|^2}),\\ \tilde{\eta}_{1k}=O(1),\tilde{\eta}_{2k}=0(1)\qquad \text{on } \partial B_{\frac{\delta}{\varepsilon_{k,j}}}.\end{cases} \tag{77}$$

The following lemma plays an important role in all the subsequent estimates. The proof is lengthy and we delay it to the appendix.

Lemma 4.1. *Suppose $|\nabla Q_{ik,j}(p_{k,j})| = O(\varepsilon_{k,j}^{\sigma_0})$ for some $0 \le \sigma_0 \le 1$. Then for any $\tau \in (0,1)$ and $\tau \le \tau_0 = \frac{1+\sigma_0}{2}$, in $B_{\frac{\delta}{\varepsilon_{k,j}}}$ there holds that*

$$|\tilde{\eta}_{1k,j}| \le C_\tau(1+|y|^\tau)\left(\varepsilon_{k,j}^{2\tau} + \varepsilon_{k,j}^{\tau} \sup_{B_{\frac{\delta}{2\varepsilon_{k,j}}} \le |y| \le B_{\frac{\delta}{\varepsilon_{k,j}}}} |\tilde{\eta}_{1k,j}|\right),$$

$$|\tilde{\eta}_{2k,j}| \le C_\tau(1+|y|^\tau)\left(\varepsilon_{k,j}^{2\tau} + \varepsilon_{k,j}^{\tau} \sup_{B_{\frac{\delta}{2\varepsilon_{k.j}}} \le |y| \le B_{\frac{\delta}{\varepsilon_{k,j}}}} |\tilde{\eta}_{2k,j}|\right).$$

Remark 4.2. *We will prove $\sigma_0 = 1$ later. Hence Lemma 4.1 holds for any $\tau \in (0,1)$.*

Lemma 4.3. *For any $\frac{1}{2} \le \tau \le \tau_0$, we have*

$$A_{1k,j} = O\left(\varepsilon_k + \varepsilon_{k,j}^{\tau} \sup_{B_{\frac{\delta}{2\varepsilon_{k,j}}} \le |y| \le B_{\frac{\delta}{\varepsilon_{k,j}}}} (|\tilde{\eta}_{1k,j}| + |\tilde{\eta}_{2k,j}|)\right),$$

$$A_{2k,j} = O\left(\varepsilon_k + \varepsilon_{k,j}^{\tau} \sup_{B_{\frac{\delta}{2\varepsilon_{k,j}}} \le |y| \le B_{\frac{\delta}{\varepsilon_{k,j}}}} (|\tilde{\eta}_{1k,j}| + |\tilde{\eta}_{2k,j}|)\right).$$

Proof. By Green's formula,

$$\begin{aligned}
&2u_{1k}(p_{k,j}) + u_{2k}(p_{k,j})\\
&= 3\sum_{\ell=1}^{m}\int_{B_\delta(p_{k,\ell})} \rho_{1k}h_{1k}(x)e^{\tilde{u}_{1k}(x)}G(p_{k,j},x)\,\mathrm{d}x + O(\varepsilon_k^2)\\
&= 3\sum_{\ell=1}^{m}\int_{B_\delta(p_{k,\ell})} \rho_{1k}h_{1k}(x)e^{\tilde{u}_{1k}(x)}\left[\frac{1}{2\pi}\ln\frac{1}{|p_{k,j}-x|} + H(p_{k,j},x)\right]\mathrm{d}x + O(\varepsilon_k^2)\\
&= 3\widetilde{G}_{1k,j}(p_{k,j}) + \frac{3}{2\pi}\int_{B_\delta(p_{k,j})} \rho_{1k}h_{1k}(x)e^{\tilde{u}_{1k}(x)}\ln\frac{1}{|p_{k,j}-x|} + O(\varepsilon_k). \qquad (78)
\end{aligned}$$

Note that

$$\rho_{1k}h_{1k}e^{\tilde{u}_{1k}} = \rho_{1k}h_{1k}(p_{k,j})\varepsilon_{k,j}^{-2}e^{V_{1k,j}(\frac{x-p_{k,j}}{\varepsilon_{k,j}})}\left[1 + D_{1k,j}(\frac{x-p_{k,j}}{\varepsilon})\right]$$

where $D_{1k,j}(y)$ is given by (73). Thus we obtain, recalling (72) and using the fact that $\frac{1}{2} \le \tau \le \tau_0$,

$$-\frac{3}{2\pi}\int_{B_\delta(p_{k,j})} \rho_{1k}h_{1k}e^{\tilde{u}_{1k}}\ln|p_{k,j}-x|\,\mathrm{d}x$$

$$
\begin{aligned}
&= -\frac{3}{2\pi}\ln\varepsilon_{k,j}\int_{B_{\frac{\delta}{\varepsilon_{k,j}}}}\rho_{1k}h_{1k}(p_{k,j})e^{V_{1k,j}}(1+D_{1k,j})\\
&\qquad -\frac{3}{2\pi}\int_{B_{\frac{\delta}{\varepsilon_{k,j}}}}\rho_{1k}h_{1k}(p_{k,j})e^{V_{1k,j}}(1+D_{1k,j})\ln|y|\\
&= -12\ln\varepsilon_{k,j}+\ln\frac{1}{(a_{1k,j}^2+a_{2k,j}^2|b_{k,j}|^2+|d_{k,j}|^2)^3}\\
&\qquad +O\left(\varepsilon_{k,j}+\varepsilon_{k,j}^{\tau}\sup_{B_{\frac{\delta}{2\varepsilon_{k,j}}}\le|y|\le B_{\frac{\delta}{\varepsilon_{k,j}}}}(|\tilde\eta_{1k,j}|+|\tilde\eta_{2k,j}|)\right). \quad (79)
\end{aligned}
$$

In fact, a direct computation shows that, for $\frac{1}{2}\le\tau\le\tau_0$,

$$
\begin{aligned}
&\int_{B_{\frac{\delta}{\varepsilon_{k,j}}}}\rho_{1k}h_{1k}(p_{k,j})e^{V_{1k,j}}=8\pi+O(\varepsilon_{k,j}^2),\\
&\int_{B_{\frac{\delta}{\varepsilon_{k,j}}}}\rho_{1k}h_{1k}(p_{k,j})e^{V_{1k,j}}D_{1k,j}\\
&=\int_{B_{\frac{\delta}{\varepsilon_{k,j}}}}\rho_{1k}h_{1k}(p_{k,j})e^{V_{1k,j}}[O(|\tilde\eta_{1k,j}|+|\tilde\eta_{2k,j}|)+O(\varepsilon_{k,j}|y|)]\\
&=O\left(\varepsilon_{k,j}+\varepsilon_{k,j}^{\tau}\sup_{B_{\frac{\delta}{2\varepsilon_{k,j}}}\le|y|\le B_{\frac{\delta}{\varepsilon_{k,j}}}}(|\tilde\eta_{1k,j}|+|\tilde\eta_{2k,j}|)\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&-\frac{3}{2\pi}\int_{B_{\frac{\delta}{\varepsilon_{k,j}}}}\rho_{1k}h_{1k}(p_{k,j})e^{V_{1k,j}}\ln|y|=\frac{1}{2\pi}\int_{B_{\frac{\delta}{\varepsilon_{k,j}}}}\Delta(2V_{1k,j}+V_{2k,j})\ln|y|\\
&=2V_{1k,j}(0)+V_{2k,j}(0)\\
&\qquad+\frac{1}{2\pi}\int_{\partial B_{\frac{\delta}{\varepsilon_{k,j}}}}\left[\ln|y|\frac{\partial(2V_{1k,j}+V_{2k,j})}{\partial\nu}-(2V_{1k,j}+V_{2k,j})\frac{\partial\ln|y|}{\partial\nu}\right]\\
&=\frac{1}{2\pi}\int_{\partial B_{\frac{\delta}{\varepsilon_{k,j}}}}\left[-\ln|y|\frac{12}{|y|}-\left(\ln\frac{256a_{1k,j}^2a_{2k,j}^2}{\rho_{1k}^2h_{1k}^2(p_{k,j})\rho_{2k}h_{2k}(p_{k,j})}-12\ln|y|\right)\frac{1}{|y|}\right]\\
&\qquad+\ln\frac{256a_{1k,j}^2a_{2k,j}^2}{(a_{1k,j}^2+a_{2k,j}^2|b_{k,j}|^2+|d_{k,j}|^2)^3\rho_{1k}^2h_{1k}^2(p_{k,j})\rho_{2k}h_{2k}(p_{k,j})}+O(\varepsilon_{k,j})\\
&=\ln\frac{1}{(a_{1k,j}^2+a_{2k,j}^2|b_{k,j}|^2+|d_{k,j}|^2)^3}+O(\varepsilon_{k,j}).
\end{aligned}
$$

In the above estimate, we used the fact that on $\partial B_{\frac{\delta}{\varepsilon_{k,j}}}$

$$2V_{1k,j} + V_{2k,j} = \ln \frac{256 a_{1k,j}^2 a_{2k,j}^2}{\rho_{1k}^2 h_{1k}^2(p_{k,j})\rho_{2k} h_{2k}(p_{k,j})} - 12 \ln |y| + O(\frac{1}{|y|}),$$

$$\frac{\partial(2V_{1k,j} + V_{2k,j})}{\partial \nu} = -\frac{12}{|y|} + \frac{3(c_{k,j}\bar{y} + \bar{c}_{k,j} y)}{|y|^3} + O(\frac{1}{|y|^3}).$$

On the other hand, by our choice we have

$$\begin{aligned} & 2u_{1k}(p_{k,j}) + u_{2k}(p_{k,j}) = 2\tilde{u}_{1k}(p_{k,j}) + \tilde{u}_{2k}(p_{k,j}) + 2\alpha_{1k} + \alpha_{2k} \\ = & 2\alpha_{1k} + \alpha_{2k} - 6 \ln \varepsilon_{k,j} + \ln \frac{256 a_{1k,j}^2 a_{2k,j}^2}{\rho_{1k}^2 h_{1k}^2(p_{k,j})\rho_{2k} h_{2k}(p_{k,j})} \\ & + \ln \frac{1}{(a_{1k,j}^2 + a_{2k,j}^2 |b_{k,j}|^2 + |d_{k,j}|^2)^3}. \end{aligned} \tag{80}$$

Combining (78), (79) and (80), we get the estimate of $A_{1k,j}$. $A_{2k,j}$ can be dealt with similarly. The proof is complete. □

Using Lemma 4.3, we have from (70) and (71) that in $B_\delta(p_{k,j}) \setminus B_{\frac{\delta}{2}}(p_{k,j})$,

$$\eta_{1k,j} = \frac{3}{2\pi}(8\pi - \rho_{1k,j}) \ln |x - p_{k,j}| + O\left(\varepsilon_k + \varepsilon_{k,j}^\tau \sup_{B_{\frac{\delta}{2\varepsilon_{k,j}}} \le |y| \le B_{\frac{\delta}{\varepsilon_{k,j}}}} (|\tilde{\eta}_{1k,j}| + |\tilde{\eta}_{2k,j}|)\right),$$

$$\eta_{2k,j} = \frac{3}{2\pi}(8\pi - \rho_{2k,j}) \ln |x - p_{k,j}| + O\left(\varepsilon_k + \varepsilon_{k,j}^\tau \sup_{B_{\frac{\delta}{2\varepsilon_{k,j}}} \le |y| \le B_{\frac{\delta}{\varepsilon_{k,j}}}} (|\tilde{\eta}_{1k,j}| + |\tilde{\eta}_{2k,j}|)\right).$$

We then have

$$\sup_{B_{\frac{\delta}{2\varepsilon_{k,j}}} \le |y| \le B_{\frac{\delta}{\varepsilon_{k,j}}}} (|\tilde{\eta}_{1k,j}| + |\tilde{\eta}_{2k,j}|) \le C\left(|\rho_{1k,j} - 8\pi| + |\rho_{2k,j} - 8\pi|\right) + O(\varepsilon_k).$$

Hence Lemma 4.1 can be refined as follows.

Proposition 4.4. *It holds in $B_{\frac{\delta}{\varepsilon_{k,j}}}$ that*

$$|\tilde{\eta}_{1k,j}| + |\tilde{\eta}_{2k,j}| \le C(1 + |y|)^\tau \left[\varepsilon_k^{2\tau} + \varepsilon_k^\tau \left(|\rho_{1k,j} - 8\pi| + |\rho_{2k,j} - 8\pi|\right)\right],$$

where $\frac{1}{2} \le \tau \le \tau_0$.

5 Estimates of $\nabla Q_{1k,j}$ and $\nabla Q_{2k,j}$

In this section we estimate the gradients of the functions $Q_{1k,j}$ and $Q_{2k,j}$ defined in (75) and (76).

Proposition 5.1. *For $\frac{1}{2} \le \tau \le \tau_0$ and any $j = 1, \dots, m$, we have*

$$\begin{aligned}\nabla Q_{1k,j} &= O(\varepsilon_k|\ln\varepsilon_k|)(|\rho_{1k,j} - 8\pi| + |\rho_{2k,j} - 8\pi|)\\ &\qquad + O(\varepsilon_k^{2\tau-1})(|\rho_{1k,j} - 8\pi|^2 + |\rho_{2k,j} - 8\pi|^2) + O(\varepsilon_k),\\ \nabla Q_{2k,j} &= O(\varepsilon_k|\ln\varepsilon_k|)(|\rho_{1k,j} - 8\pi| + |\rho_{2k,j} - 8\pi|)\\ &\qquad + O(\varepsilon_k^{2\tau-1})(|\rho_{1k,j} - 8\pi|^2 + |\rho_{2k,j} - 8\pi|^2) + O(\varepsilon_k).\end{aligned}$$

Since the problem is considered locally, for simplicity of notations we omit the subscript j if there is no confusion. Similarly we use ρ_{1k}^0, ρ_{2k}^0 to denote $\rho_{1k,j}$ and $\rho_{2k,j}$.

Proof. We set

$$\begin{aligned}\psi_{1k,1}^b(y) &= (\partial_{b_k} + \partial_{\bar{b}_k})V_{1k}(y), & \psi_{2k,1}^b(y) &= (\partial_{b_k} + \partial_{\bar{b}_k})V_{2k}(y),\\ \psi_{1k,2}^b(y) &= i(\partial_{b_k} - \partial_{\bar{b}_k})V_{1k}(y), & \psi_{2k,2}^b(y) &= i(\partial_{b_k} - \partial_{\bar{b}_k})V_{2k}(y),\\ \psi_{1k,1}^c(y) &= (\partial_{c_k} + \partial_{\bar{c}_k})V_{1k}(y), & \psi_{2k,1}^c(y) &= (\partial_{c_k} + \partial_{\bar{c}_k})V_{2k}(y),\\ \psi_{1k,2}^c(y) &= i(\partial_{c_k} - \partial_{\bar{c}_k})V_{1k}(y), & \psi_{2k,2}^c(y) &= i(\partial_{c_k} - \partial_{\bar{c}_k})V_{2k}(y).\end{aligned}$$

It is easy to check that, on $\partial B_{\frac{\delta}{\varepsilon_k}}$,

$$\begin{aligned}\psi_{1k,1}^b &= \frac{4y_1}{|y|^2} + O(\frac{1}{|y|^2}), & \frac{\partial\psi_{1k,1}^b}{\partial\nu} &= -\frac{4y_1}{|y|^3} + O(\frac{1}{|y|^3}),\\ \psi_{2k,1}^b &= -\frac{8y_1}{|y|^2} + O(\frac{1}{|y|^2}), & \frac{\partial\psi_{2k,1}^b}{\partial\nu} &= \frac{8y_1}{|y|^3} + O(\frac{1}{|y|^3}),\\ \psi_{1k,2}^b &= \frac{4y_2}{|y|^2} + O(\frac{1}{|y|^2}), & \frac{\partial\psi_{1k,2}^b}{\partial\nu} &= -\frac{4y_2}{|y|^3} + O(\frac{1}{|y|^3}),\\ \psi_{2k,2}^b &= -\frac{8y_2}{|y|^2} + O(\frac{1}{|y|^2}), & \frac{\partial\psi_{2k,2}^b}{\partial\nu} &= \frac{8y_2}{|y|^3} + O(\frac{1}{|y|^3}),\\ \psi_{1k,1}^c &= -\frac{4y_1}{|y|^2} + O(\frac{1}{|y|^2}), & \frac{\partial\psi_{1k,1}^c}{\partial\nu} &= \frac{4y_1}{|y|^3} + O(\frac{1}{|y|^3}),\\ \psi_{2k,1}^c &= \frac{2y_1}{|y|^2} + O(\frac{1}{|y|^2}), & \frac{\partial\psi_{2k,1}^c}{\partial\nu} &= -\frac{2y_1}{|y|^3} + O(\frac{1}{|y|^3}),\\ \psi_{1k,2}^c &= -\frac{4y_2}{|y|^2} + O(\frac{1}{|y|^2}), & \frac{\partial\psi_{1k,2}^c}{\partial\nu} &= \frac{4y_2}{|y|^3} + O(\frac{1}{|y|^3}),\\ \psi_{2k,2}^c &= \frac{2y_2}{|y|^2} + O(\frac{1}{|y|^2}), & \frac{\partial\psi_{2k,2}^c}{\partial\nu} &= -\frac{2y_2}{|y|^3} + O(\frac{1}{|y|^3}).\end{aligned}$$

Integrating by parts, we have

$$\int_{B_{\frac{\delta}{\varepsilon_k}}} (-\Delta\tilde{\eta}_{1k})\psi_{1k,1}^c + (-\Delta\tilde{\eta}_{2k})\psi_{2k,1}^c$$

$$= \int_{B_{\frac{\delta}{\varepsilon_k}}} (-\Delta\psi^c_{1k,1})\tilde{\eta}_{1k} + (-\Delta\psi^c_{2k,1})\tilde{\eta}_{2k} + \int_{\partial B_{\frac{\delta}{\varepsilon_k}}} \left(\tilde{\eta}_{1k}\frac{\partial\psi^c_{1k,1}}{\partial\nu} - \psi^c_{1k,1}\frac{\partial\tilde{\eta}_{1k}}{\partial\nu}\right)$$
$$+ \int_{\partial B_{\frac{\delta}{\varepsilon_k}}} \left(\tilde{\eta}_{2k}\frac{\partial\psi^c_{2k,1}}{\partial\nu} - \psi^c_{2k,1}\frac{\partial\tilde{\eta}_{2k}}{\partial\nu}\right). \qquad (81)$$

By the previous estimate on $\partial B_{\frac{\delta}{\varepsilon_k}}$, (70) and (71), a straightforward calculation shows that

$$\int_{\partial B_{\frac{\delta}{\varepsilon_k}}} \left(\frac{\partial\psi^c_{ik,1}}{\partial\nu}\tilde{\eta}_{ik} - \frac{\partial\tilde{\eta}_{ik}}{\partial\nu}\psi^c_{ik,1}\right)$$
$$= O(\varepsilon_k^2|\ln\varepsilon_k|)(|\rho^0_{1k} - 8\pi| + |\rho^0_{2k} - 8\pi|) + O(\varepsilon_k^2), \qquad i = 1, 2. \quad (82)$$

where cancelation occurs due to the radial symmetry of $\ln|y|$. On the other hand, we note that, by (73) and (74),

$$3D_{1k}(y) = (2\tilde{\eta}_{1k} - \tilde{\eta}_{2k}) + 3\nabla Q_{1k}(p_k)\varepsilon_k y + O(\tilde{\eta}^2_{1k} + \tilde{\eta}^2_{2k}) + O(\varepsilon_k^2|y|^2),$$
$$3D_{2k}(y) = (2\tilde{\eta}_{2k} - \tilde{\eta}_{1k}) + 3\nabla Q_{2k}(p_k)\varepsilon_k y + O(\tilde{\eta}^2_{1k} + \tilde{\eta}^2_{2k}) + O(\varepsilon_k^2|y|^2).$$

Then

$$\int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{1k}h_{1k}(p_k)e^{V_{1k}}D_{1k}(y)\psi^c_{1k,1} + \int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{2k}h_{2k}(p_k)e^{V_{2k}}D_{2k}(y)\psi^c_{2k,1}$$
$$= \int_{B_{\frac{\delta}{\varepsilon_k}}} \rho_{1k}h_{1k}(p_k)e^{V_{1k}}(2\tilde{\eta}_{1k} - \tilde{\eta}_{2k})\psi^c_{1k,1} + \int_{B_{\frac{\delta}{\varepsilon_k}}} \rho_{2k}h_{2k}(p_k)e^{V_{2k}}(2\tilde{\eta}_{2k} - \tilde{\eta}_{1k})\psi^c_{2k,1}$$
$$+ 3\nabla Q_{1k}(p_k)\varepsilon_k \int_{B_{\frac{\delta}{\varepsilon_k}}} \rho_{1k}h_{1k}(p_k)e^{V_{1k}}y\psi^c_{1k,1}$$
$$+ 3\nabla Q_{2k}(p_k)\varepsilon_k \int_{B_{\frac{\delta}{\varepsilon_k}}} \rho_{2k}h_{2k}(p_k)e^{V_{2k}}y\psi^c_{2k,1}$$
$$+ O(\varepsilon_k^{2\tau})(|\rho^0_{1k} - 8\pi|^2 + |\rho^0_{2k} - 8\pi|^2) + O(\varepsilon_k^2), \qquad (83)$$

where Proposition 4.4 is used to estimate $\tilde{\eta}^2_{1k} + \tilde{\eta}^2_{2k}$. The equations of $\tilde{\eta}_{1k}$ and $\tilde{\eta}_{2k}$, (81), (82) and (83) give us that

$$3\nabla Q_{1k}(p_k)\varepsilon_k \int_{B_{\frac{\delta}{\varepsilon_k}}} \rho_{1k}h_{1k}(p_k)e^{V_{1k}}y\psi^c_{1k,1}$$
$$+ 3\nabla Q_{2k}(p_k)\varepsilon_k \int_{B_{\frac{\delta}{\varepsilon_k}}} \rho_{2k}h_{2k}(p_k)e^{V_{2k}}y\psi^c_{2k,1}$$
$$= O(\varepsilon_k^2|\ln\varepsilon_k|)(|\rho^0_{1k} - 8\pi| + |\rho^0_{2k} - 8\pi|)$$
$$+ O(\varepsilon_k^{2\tau})(|\rho^0_{1k} - 8\pi|^2 + |\rho^0_{2k} - 8\pi|^2) + O(\varepsilon_k^2). \quad (84)$$

Similarly, the above procedure can also be applied to $\binom{\psi^c_{1k,2}}{\psi^c_{2k,2}}$, $\binom{\psi^b_{1k,1}}{\psi^b_{2k,1}}$ and $\binom{\psi^b_{1k,2}}{\psi^b_{2k,2}}$ and then other three equalities can be gotten, which have the same form as (84) just by replacing $\binom{\psi^c_{1k,1}}{\psi^1_{2k,2}}$ by $\binom{\psi^c_{1k,2}}{\psi^c_{2k,2}}$, etc.

Now we are in position to finish the proof of Proposition 5.1. We need to show that the corresponding coefficient matrix is non-degenerate, from which the proposition follows. Since $\Delta(2\psi^c_{1k,1}+\psi^c_{2k,1})+3\rho_{1k}h_{1k}(p_k)e^{V_{1k}}\psi^c_{1k,1}=0$, one has

$$\begin{aligned}
&3\int_{B_{\frac{\delta}{\varepsilon_k}}}\rho_{1k}h_{1k}(p_k)e^{V_{1k}}y_1\psi^c_{1k,1}=-\int_{B_{\frac{\delta}{\varepsilon_k}}}\Delta(2\psi^c_{1k,1}+\psi^c_{2k,1})y_1\\
=&\int_{\partial B_{\frac{\delta}{\varepsilon_k}}}\frac{\partial y_1}{\partial\nu}(2\psi^c_{1k,1}+\psi^c_{2k,1})-\frac{\partial(2\psi^c_{1k,1}+\psi^c_{2k,1})}{\partial\nu}y_1\\
=&-12\int_{\partial B_{\frac{\delta}{\varepsilon_k}}}\frac{y_1^2}{|y|^3}+O(\varepsilon_k)=-12\pi+O(\varepsilon_k),
\end{aligned}$$

and

$$\begin{aligned}
&3\int_{B_{\frac{\delta}{\varepsilon_k}}}\rho_{1k}h_{1k}(p_k)e^{V_{1k}}y_2\psi^c_{1k,1}=-\int_{B_{\frac{\delta}{\varepsilon_k}}}\Delta(2\psi^c_{1k,1}+\psi^c_{2k,1})y_2\\
=&\int_{\partial B_{\frac{\delta}{\varepsilon_k}}}\frac{\partial y_2}{\partial\nu}(2\psi^c_{1k,1}+\psi^c_{2k,1})-\frac{\partial(2\psi^c_{1k,1}+\psi^c_{2k,1})}{\partial\nu}y_2\\
=&O(\varepsilon_k).
\end{aligned}$$

Similarly we can prove that

$$\begin{aligned}
&\int_{B_{\frac{\delta}{\varepsilon_k}}}\rho_{2k}h_{2k}(p_k)e^{V_{2k}}y_1\psi^c_{2k,1}=O(\varepsilon_k),\quad\int_{B_{\frac{\delta}{\varepsilon_k}}}\rho_{2k}h_{2k}(p_k)e^{V_{2k}}y_2\psi^c_{2k,1}=O(\varepsilon_k),\\
&\int_{B_{\frac{\delta}{\varepsilon_k}}}\rho_{1k}h_{1k}(p_k)e^{V_{1k}}y_1\psi^c_{1k,2}=O(\varepsilon_k),\\
&\int_{B_{\frac{\delta}{\varepsilon_k}}}\rho_{1k}h_{1k}(p_k)e^{V_{1k}}y_2\psi^c_{1k,2}=-12\int_{\partial B_{\frac{\delta}{\varepsilon_k}}}\frac{y_2^2}{|y|^3}+O(\varepsilon_k)=-12\pi+O(\varepsilon_k),\\
&\int_{B_{\frac{\delta}{\varepsilon_k}}}\rho_{2k}h_{2k}(p_k)e^{V_{2k}}y_1\psi^c_{2k,2}=O(\varepsilon_k),\quad\int_{B_{\frac{\delta}{\varepsilon_k}}}\rho_{2k}h_{2k}(p_k)e^{V_{2k}}y_2\psi^c_{2k,2}=O(\varepsilon_k),\\
&\int_{B_{\frac{\delta}{\varepsilon_k}}}\rho_{1k}h_{1k}(p_k)e^{V_{1k}}y_1\psi^b_{1k,1}=O(\varepsilon_k),\quad\int_{B_{\frac{\delta}{\varepsilon_k}}}\rho_{1k}h_{1k}(p_k)e^{V_{1k}}y_2\psi^b_{1k,1}=O(\varepsilon_k),\\
&\int_{B_{\frac{\delta}{\varepsilon_k}}}\rho_{2k}h_{2k}(p_k)e^{V_{2k}}y_1\psi^b_{2k,1}=-24\int_{\partial B_{\frac{\delta}{\varepsilon_k}}}\frac{y_1^2}{|y|^3}+O(\varepsilon_k)=-24\pi+O(\varepsilon_k),
\end{aligned}$$

$$\int_{B_{\frac{\delta}{\varepsilon_k}}} \rho_{2k} h_{2k}(p_k) e^{V_{2k}} y_2 \psi^b_{2k,1} = O(\varepsilon_k),$$

$$\int_{B_{\frac{\delta}{\varepsilon_k}}} \rho_{1k} h_{1k}(p_k) e^{V_{1k}} y_1 \psi^b_{1k,2} = O(\varepsilon_k), \quad \int_{B_{\frac{\delta}{\varepsilon_k}}} \rho_{1k} h_{1k}(p_k) e^{V_{1k}} y_2 \psi^b_{1k,2} = O(\varepsilon_k),$$

$$\int_{B_{\frac{\delta}{\varepsilon_k}}} \rho_{2k} h_{2k}(p_k) e^{V_{2k}} y_1 \psi^b_{2k,2} = O(\varepsilon_k),$$

$$\int_{B_{\frac{\delta}{\varepsilon_k}}} \rho_{2k} h_{2k}(p_k) e^{V_{2k}} y_2 \psi^b_{2k,2} = -24 \int_{\partial B_{\frac{\delta}{\varepsilon_k}}} \frac{y_2^2}{|y|^3} + O(\varepsilon_k) = -24\pi + O(\varepsilon_k).$$

The above computation obviously implies the non-degeneracy of the coefficient matrix. The proof is thus completed. □

6 Estimates of $\rho_{ik,j} - 8\pi$

In this section we estimate the convergence rate of $\rho_{1k,j} \to 8\pi$ and $\rho_{2k,j} \to 8\pi$ in terms of the blow-up values.

Lemma 6.1. *There holds*

$$\rho_{1k,j} - 8\pi = -\frac{1}{3} \int_{\partial B_\delta(p_{k,j})} \frac{\partial \eta_{1k,j}}{\partial \nu} + O(\varepsilon_k^2), \tag{85}$$

$$\rho_{2k,j} - 8\pi = -\frac{1}{3} \int_{\partial B_\delta(p_{k,j})} \frac{\partial \eta_{2k,j}}{\partial \nu} + O(\varepsilon_k^2). \tag{86}$$

Proof. By the definition of $\rho_{1k,j}$ and $\eta_{1k,j}$, we have

$$\begin{aligned} 3\rho_{1k,j} &= \int_{B_\delta(p_{k,j})} 3\rho_{1k} h_{1k} e^{\tilde{u}_{1k}} = -\int_{B_\delta(p_k)} \Delta(2\tilde{u}_{1k} + \tilde{u}_{2k}) \\ &= -\int_{B_\delta(p_{k,j})} \Delta \eta_{1k,j} - \int_{B_\delta(p_{k,j})} \Delta \left(2U_{1k,j} + U_{2k,j}\right) \\ &= -\int_{\partial B_\delta(p_{k,j})} \frac{\partial \eta_{1k,j}}{\partial \nu} + \int_{B_{\frac{\delta}{\varepsilon_{k,j}}}} 3\rho_{1k} h_{1k}(p_k) e^{V_{1k,j}} \\ &= 24\pi - \int_{\partial B_\delta(p_{k,j})} \frac{\partial \eta_{1k,j}}{\partial \nu} + O(\varepsilon_k^2). \end{aligned}$$

(86) can be proved similarly. □

Proposition 6.2. *We have*

$$\rho_{1k,j} - 8\pi = C_{1k,j} \Delta_x Q_{1k,j}(p_{k,j}) \varepsilon_{k,j}^2 |\ln \varepsilon_{k,j}| + O(\varepsilon_k^2),$$
$$\rho_{2k,j} - 8\pi = C_{2k,j} \Delta_x Q_{2k,j}(p_{k,j}) \varepsilon_{k,j}^2 |\ln \varepsilon_{k,j}| + O(\varepsilon_k^2),$$

where $C_{1k,j}$ and $C_{2k,j}$ are positive constants uniformly bounded below from 0 *and above from* ∞.

Proof. We omit the subscript j as in the previous section. Define

$$\psi_{1k}^{a_1}(y) = \partial_{a_{1k}} V_{1k}(y), \qquad \psi_{2k}^{a_1}(y) = \partial_{a_{1k}} V_{2k}(y), \tag{87}$$
$$\psi_{1k}^{a_2}(y) = \partial_{a_{2k}} V_{1k}(y), \qquad \psi_{2k}^{a_2}(y) = \partial_{a_{2k}} V_{2k}(y). \tag{88}$$

A Taylor expansion gives us that, for $|y| \to \infty$,

$$\begin{aligned}
\psi_{1k}^{a_1} &= O(\frac{1}{|y|^2}), & \frac{\partial \psi_{1k}^{a_1}}{\partial \nu} &= O(\frac{1}{|y|^3}),\\
\psi_{2k}^{a_1} &= \frac{2}{a_{1k}} + O(\frac{1}{|y|^2}), & \frac{\partial \psi_{2k}^{a_1}}{\partial \nu} &= O(\frac{1}{|y|^3}),\\
\psi_{1k}^{a_2} &= \frac{2}{a_{2k}} + O(\frac{1}{|y|^2}), & \frac{\partial \psi_{1k}^{a_2}}{\partial \nu} &= O(\frac{1}{|y|^3}),\\
\psi_{2k}^{a_2} &= -\frac{2}{a_{2k}} + O(\frac{1}{|y|^2}), & \frac{\partial \psi_{2k}^{a_2}}{\partial \nu} &= O(\frac{1}{|y|^3}).
\end{aligned}$$

It is then easy to check that

$$\begin{aligned}
&\int_{\partial B_{\frac{\delta}{\varepsilon_k}}} \left(\frac{\partial \psi_{1k}^{a_1}}{\partial \nu} \tilde{\eta}_{1k} - \frac{\partial \tilde{\eta}_{1k}}{\partial \nu} \psi_{1k}^{a_1} \right)\\
&\qquad = O(\varepsilon_k^2 |\ln \varepsilon_k|)(|\rho_{1k}^0 - 8\pi| + |\rho_{2k}^0 - 8\pi|) + O(\varepsilon_k^2),
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\partial B_{\frac{\delta}{\varepsilon_k}}} \left(\frac{\partial \psi_{2k}^{a_1}}{\partial \nu} \tilde{\eta}_{2k} - \frac{\partial \tilde{\eta}_{2k}}{\partial \nu} \psi_{2k}^{a_1} \right)\\
&= -\frac{2}{a_{1k}} \int_{\partial B_{\frac{\delta}{\varepsilon_k}}} \frac{\partial \tilde{\eta}_{2k}}{\partial \nu} + O(\varepsilon_k^2 |\ln \varepsilon_k|)(|\rho_{1k}^0 - 8\pi| + |\rho_{2k}^0 - 8\pi|) + O(\varepsilon_k^2).
\end{aligned}$$

So we have

$$\begin{aligned}
&\int_{B_{\frac{\delta}{\varepsilon_k}}} (-\Delta \tilde{\eta}_{1k}) \psi_{1k}^{a_1} + (-\Delta \tilde{\eta}_{2k}) \psi_{2k}^{a_1}\\
&= \int_{B_{\frac{\delta}{\varepsilon_k}}} (-\Delta \psi_{1k}^{a_1}) \tilde{\eta}_{1k} + (-\Delta \psi_{2k}^{a_1}) \tilde{\eta}_{2k}\\
&\quad + \int_{\partial B_{\frac{\delta}{\varepsilon_k}}} \left(\frac{\partial \psi_{1k}^{a_1}}{\partial \nu} \tilde{\eta}_{1k} - \frac{\partial \tilde{\eta}_{1k}}{\partial \nu} \psi_{1k}^{a_1} \right) + \int_{\partial B_{\frac{\delta}{\varepsilon_k}}} \left(\frac{\partial \psi_{2k}^{a_1}}{\partial \nu} \tilde{\eta}_{2k} - \frac{\partial \tilde{\eta}_{2k}}{\partial \nu} \psi_{2k}^{a_1} \right)\\
&= \int_{B_{\frac{\delta}{\varepsilon_k}}} (-\Delta \psi_{1k}^{a_1}) \tilde{\eta}_{1k} + (-\Delta \psi_{2k}^{a_1}) \tilde{\eta}_{2k} - \frac{2}{a_{1k}} \int_{\partial B_{\frac{\delta}{\varepsilon_k}}} \frac{\partial \tilde{\eta}_{2k}}{\partial \nu} + O(\varepsilon_k^2)\\
&\qquad + O(\varepsilon_k^2 |\ln \varepsilon_k|)(|\rho_{1k}^0 - 8\pi| + |\rho_{2k}^0 - 8\pi|).
\end{aligned} \tag{89}$$

On the other hand, using Proposition 5.1, we obtain that

$$3D_{1k}(y) = (2\tilde{\eta}_{1k} - \tilde{\eta}_{2k}) + \frac{3}{2}\nabla^2 Q_{1k}(p_k)\varepsilon_k^2 y^2 + \tilde{g}_1(y), \tag{90}$$

$$3D_{2k}(y) = (2\tilde{\eta}_{2k} - \tilde{\eta}_{1k}) + \frac{3}{2}\nabla^2 Q_{2k}(p_k)\varepsilon_k^2 y^2 + \tilde{g}_2(y), \tag{91}$$

where

$$\begin{aligned}
&\tilde{g}_i(y) = O\left(|\nabla Q_{ik}(p_{k,j})|\varepsilon_k|y| + \tilde{\eta}_{1k}^2 + \tilde{\eta}_{2k}^2 + \varepsilon_k^{2+\beta}|y|^{2+\beta}\right)\\
&= O(\varepsilon_k^{1+\tau}|y|)(|\rho_{1k}^0 - 8\pi| + |\rho_{2k}^0 - 8\pi|) + O(\varepsilon_k^{2\tau}|y|)(|\rho_{1k}^0 - 8\pi|^2 + |\rho_{2k}^0 - 8\pi|^2)\\
&\qquad + O\left(\varepsilon_k^2|y| + \tilde{\eta}_{1k}^2 + \tilde{\eta}_{2k}^2 + \varepsilon_k^{2+\beta}|y|^{2+\beta}\right)\\
&= O(\varepsilon_k^2)(1+|y|) + O(\varepsilon_k^{2+\beta}|y|^{2+\beta}) + O(\varepsilon_k^{2\tau})(1+|y|)^{2\tau}(|\rho_{1k}^0 - 8\pi| + |\rho_{2k}^0 - 8\pi|).
\end{aligned}$$

It is easy to see that

$$\int_{B_{\frac{\delta}{\varepsilon_k}}} e^{V_{ik}}\psi_{ik}^{a_1}\tilde{g}_i = O(\varepsilon_k^{2\tau})(|\rho_{1k}^0 - 8\pi| + |\rho_{2k}^0 - 8\pi|) + O(\varepsilon_k^2).$$

By the equations of $\tilde{\eta}_{1k}$ and $\tilde{\eta}_{2k}$, it holds that

$$\begin{aligned}
&\int_{B_{\frac{\delta}{\varepsilon_k}}} (-\Delta\tilde{\eta}_{1k})\psi_{1k}^{a_1} + (-\Delta\tilde{\eta}_{2k})\psi_{2k}^{a_1}\\
&\qquad = \int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{1k}h_{1k}(p_k)e^{V_{1k}}D_{1k}\psi_{1k}^{a_1} + 3\rho_{2k}h_{2k}(p_k)e^{V_{2k}}D_{2k}\psi_{2k}^{a_1}.
\end{aligned} \tag{92}$$

From (90) we have that, since $\psi_{1k}^{a_1} = O(\frac{1}{|y|^2})$ as $|y| \to \infty$,

$$\begin{aligned}
\int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{1k}h_{1k}(p_k)e^{V_{1k}}D_{1k}\psi_{1k}^{a_1} &= \int_{B_{\frac{\delta}{\varepsilon_k}}} \rho_{1k}h_{1k}(p_k)e^{V_{1k}}\psi_{1k}^{a_1}(2\tilde{\eta}_{1k} - \tilde{\eta}_{2k}) + O(\varepsilon_k^2)\\
&\qquad + O(\varepsilon_k^{2\tau})(|\rho_{1k}^0 - 8\pi| + |\rho_{2k}^0 - 8\pi|).
\end{aligned} \tag{93}$$

From (91) we know that

$$\begin{aligned}
&\int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{2k}h_{2k}(p_k)e^{V_{2k}}D_{2k}\psi_{2k}^{a_1}\\
&= \int_{B_{\frac{\delta}{\varepsilon_k}}} \rho_{2k}h_{2k}(p_k)e^{V_{2k}}\psi_{2k}^{a_1}(2\tilde{\eta}_{2k} - \tilde{\eta}_{1k}) + O(\varepsilon_k^{2\tau})(|\rho_{1k}^0 - 8\pi| + |\rho_{2k}^0 - 8\pi|)\\
&\qquad + \frac{\varepsilon_k^2}{2}\int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{2k}h_{2k}(p_k)e^{V_{2k}}\psi_{2k}^{a_1}\nabla_x^2 Q_{2k}(p_k)y^2 + O(\varepsilon_k^2).
\end{aligned} \tag{94}$$

We next **claim** that

$$\int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{2k}h_{2k}(p_k)e^{V_{2k}}\psi_{2k}^{a_1}\nabla_x^2 Q_{2k}(p_k)y^2$$

$$= \frac{\Delta_x Q_{2k}(p_k)}{2}\int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{2k}h_{2k}(p_k)e^{V_{2k}}\psi_{2k}^{a_1}|y|^2 + O(1). \quad (95)$$

Indeed, direct calculations show that

$$\int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{2k}h_{2k}(p_k)e^{V_{2k}}\psi_{2k}^{a_1}(y_1^2 - y_2^2) = -\int_{B_{\frac{\delta}{\varepsilon_k}}} \Delta(\psi_{1k}^{a_1} + 2\psi_{2k}^{a_1})(y_1^2 - y_2^2)$$

$$= \int_{\partial B_{\frac{\delta}{\varepsilon_k}}} \frac{\partial(y_1^2 - y_2^2)}{\partial\nu}(\psi_{1k}^{a_1} + 2\psi_{2k}^{a_1}) - \int_{\partial B_{\frac{\delta}{\varepsilon_k}}} \frac{\partial(\psi_{1k}^{a_1} + 2\psi_{2k}^{a_1})}{\partial\nu}(y_1^2 - y_2^2)$$

$$= O(1) \quad (96)$$

and

$$\int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{2k}h_{2k}(p_k)e^{V_{2k}}\psi_{2k}^{a_1}y_1y_2 = -\int_{B_{\frac{\delta}{\varepsilon_k}}} \Delta(\psi_{1k}^{a_1} + 2\psi_{2k}^{a_1})y_1y_2$$

$$= \int_{\partial B_{\frac{\delta}{\varepsilon_k}}} \frac{\partial(y_1y_2)}{\partial\nu}(\psi_{1k}^{a_1} + 2\psi_{2k}^{a_1}) - \int_{\partial B_{\frac{\delta}{\varepsilon_k}}} \frac{\partial(\psi_{1k}^{a_1} + 2\psi_{2k}^{a_1})}{\partial\nu}y_1y_2$$

$$= O(1). \quad (97)$$

The claim (95) follows from (96) and (97). Therefore (94) implies that

$$\int_{B_{\frac{\delta}{\varepsilon_k}}} \rho_{2k}h_{2k}(p_k)e^{V_{2k}}D_{2k}\psi_{2k}^{a_1}$$

$$= \int_{B_{\frac{\delta}{\varepsilon_k}}} \rho_{2k}h_{2k}(p_k)e^{V_{2k}}\psi_{2k}^{a_1}(2\tilde{\eta}_{2k} - \tilde{\eta}_{1k}) + O(\varepsilon_k^{2\tau})(|\rho_{1k}^0 - 8\pi| + |\rho_{2k}^0 - 8\pi|)$$

$$+ \frac{\varepsilon_k^2\Delta_x Q_{2k}(p_k)}{4}\int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{2k}h_{2k}(p_k)e^{V_{2k}}\psi_{2k}^{a_1}|y|^2 + O(\varepsilon_k^2). \quad (98)$$

Finally combing (89), (92), (93) and (98), we obtain that

$$-\frac{2}{a_{1k}}\int_{\partial B_{\frac{\delta}{\varepsilon_k}}} \frac{\partial\tilde{\eta}_{2k}}{\partial\nu} = \frac{\varepsilon_k^2\Delta_x Q_{2k}(p_k)}{4}\int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{2k}h_{2k}(p_k)e^{V_{2k}}\psi_{2k}^{a_1}|y|^2$$

$$+ O(\varepsilon_k^{2\tau})(|\rho_{1k}^0 - 8\pi| + |\rho_{2k}^0 - 8\pi|) + O(\varepsilon_k^2). \quad (99)$$

Furthermore we perform a similar procedure by replacing $(\psi_{1k}^{a_1}, \psi_{2k}^{a_1})$ by $(\psi_{1k}^{a_2}, \psi_{2k}^{a_2})$. It is easy to see that

$$\int_{B_{\frac{\delta}{\varepsilon_k}}} (-\Delta\tilde{\eta}_{1k})\psi_{1k}^{a_2} + (-\Delta\tilde{\eta}_{2k})\psi_{2k}^{a_2} = \int_{B_{\frac{\delta}{\varepsilon_k}}} (-\Delta\psi_{1k}^{a_2})\tilde{\eta}_{1k} + (-\Delta\psi_{2k}^{a_2})\tilde{\eta}_{2k}$$
$$- \frac{2}{a_{2k}} \int_{\partial B_{\frac{\delta}{\varepsilon_k}}} \frac{\partial\tilde{\eta}_{1k}}{\partial\nu} + \frac{2}{a_{2k}} \int_{\partial B_{\frac{\delta}{\varepsilon_k}}} \frac{\partial\tilde{\eta}_{2k}}{\partial\nu} + O(\varepsilon_k^2),$$

$$\int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{1k}h_{1k}(p_k)e^{V_{1k}}D_{1k}\psi_{1k}^{a_2}$$
$$= \int_{B_{\frac{\delta}{\varepsilon_k}}} \rho_{1k}h_{1k}(p_k)e^{V_{1k}}\psi_{1k}^{a_2}(2\tilde{\eta}_{1k} - \tilde{\eta}_{2k}) + O(\varepsilon_k^{2\tau})(|\rho_{1k}^0 - 8\pi| + |\rho_{2k}^0 - 8\pi|)$$
$$+ \frac{\varepsilon_k^2\Delta_x Q_{1k}(p_k)}{4} \int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{1k}h_{1k}(p_k)e^{V_{1k}}\psi_{1k}^{a_2}|y|^2 + O(\varepsilon_k^2)$$

and

$$\int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{2k}h_{2k}(p_k)e^{V_{2k}}D_{2k}\psi_{2k}^{a_2}$$
$$= \int_{B_{\frac{\delta}{\varepsilon_k}}} \rho_{2k}h_{2k}(p_k)e^{V_{2k}}\psi_{2k}^{a_2}(2\tilde{\eta}_{2k} - \tilde{\eta}_{1k}) + O(\varepsilon_k^{2\tau})(|\rho_{1k}^0 - 8\pi| + |\rho_{2k}^0 - 8\pi|)$$
$$+ \frac{\varepsilon_k^2\Delta_x Q_{2k}(p_k)}{4} \int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{2k}h_{2k}(p_k)e^{V_{2k}}\psi_{2k}^{a_2}|y|^2 + O(\varepsilon_k^2).$$

So we get that

$$- \frac{2}{a_{2k}} \int_{\partial B_{\frac{\delta}{\varepsilon_k}}} \frac{\partial\tilde{\eta}_{1k}}{\partial\nu} + \frac{2}{a_{2k}} \int_{\partial B_{\frac{\delta}{\varepsilon_k}}} \frac{\partial\tilde{\eta}_{2k}}{\partial\nu}$$
$$= \frac{\varepsilon_k^2\Delta_x Q_{1k}(p_k)}{4} \int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{1k}h_{1k}(p_k)e^{V_{1k}}\psi_{1k}^{a_2}|y|^2$$
$$+ \frac{\varepsilon_k^2\Delta_x Q_{2k}(p_k)}{4} \int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{2k}h_{2k}(p_k)e^{V_{2k}}\psi_{2k}^{a_2}|y|^2$$
$$+ O(\varepsilon_k^{2\tau})(|\rho_{1k}^0 - 8\pi| + |\rho_{2k}^0 - 8\pi|) + O(\varepsilon_k^2). \qquad (100)$$

It follows from (99) and (100) that

$$-\frac{2}{a_{2k}} \int_{\partial B_{\frac{\delta}{\varepsilon_k}}} \frac{\partial\tilde{\eta}_{1k}}{\partial\nu} = \frac{\varepsilon_k^2\Delta_x Q_{1k}(p_k)}{4} \int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{1k}h_{1k}(p_k)e^{V_{1k}}\psi_{1k}^{a_2}|y|^2$$
$$+ \frac{\varepsilon_k^2\Delta_x Q_{2k}(p_k)}{4} \int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{2k}h_{2k}(p_k)e^{V_{2k}}\left(\frac{a_{1k}}{a_{2k}}\psi_{2k}^{a_1} + \psi_{2k}^{a_2}\right)|y|^2$$

$$
\begin{aligned}
&+ O(\varepsilon_k^{2\tau})(|\rho_{1k}^0 - 8\pi| + |\rho_{2k}^0 - 8\pi|) + O(\varepsilon_k^2) \\
= &\frac{\varepsilon_k^2 \Delta_x Q_{1k}(p_k)}{4} \int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{1k} h_{1k}(p_k) e^{V_{1k}} \psi_{1k}^{a_2} |y|^2 \\
&+ O(\varepsilon_k^{2\tau})(|\rho_{1k}^0 - 8\pi| + |\rho_{2k}^0 - 8\pi|) + O(\varepsilon_k^2), \qquad (101)
\end{aligned}
$$

where $\frac{a_{1k}}{a_{2k}}\psi_{2k}^{a_1} + \psi_{2k}^{a_2} = O(|y|^{-2})$ as $|y| \to \infty$ is used in the last equality.

Since $e^{V_{1k}}$, $e^{V_{2k}} \sim |y|^{-4}$ for $|y|$ large, Lemma 6.1, (99) and (101) give that

$$
\begin{aligned}
\rho_{1k}^0 - 8\pi &= C_{1k} \Delta_x Q_{1k}(p_k) \varepsilon_k^2 |\ln \varepsilon_k| + O(\varepsilon_k^{2\tau})(|\rho_{1k}^0 - 8\pi| + |\rho_{2k}^0 - 8\pi|) + O(\varepsilon_k^2), \\
\rho_{2k}^0 - 8\pi &= C_{2k} \Delta_x Q_{2k}(p_k) \varepsilon_k^2 |\ln \varepsilon_k| + O(\varepsilon_k^{2\tau})(|\rho_{1k}^0 - 8\pi| + |\rho_{2k}^0 - 8\pi|) + O(\varepsilon_k^2),
\end{aligned}
$$

where C_{ik} is a constant such that $0 < C_1 < C_{ik} < C_2 < \infty$ $(i = 1, 2)$. Obviously we also have that

$$
\begin{aligned}
\rho_{1k}^0 - 8\pi &= C_{1k} \Delta_x Q_{1k}(p_k) \varepsilon_k^2 |\ln \varepsilon_k| + O(\varepsilon_k^2), \\
\rho_{2k}^0 - 8\pi &= C_{2k} \Delta_x Q_{2k}(p_k) \varepsilon_k^2 |\ln \varepsilon_k| + O(\varepsilon_k^2).
\end{aligned}
$$

The proof is complete. □

By Proposition 6.2, we now have a sharper estimate for $\nabla Q_{ik,j}(p_{k,j})$:

$$
|\nabla Q_{1k,j}(p_{k,j})| + |\nabla Q_{2k,j}(p_{k,j})| = O(\varepsilon_k).
$$

Hence Lemma 4.1, Lemma 4.3 and Proposition 4.4 hold for any $\tau \in (0, 1)$.

7 Estimates for $\nabla^2 Q_{1k,j}$ and $\nabla^2 Q_{2k,j}$

In this section we make use of the remaining two kernel elements to obtain the estimates on the second derivatives of the $Q_{ik,j}$'s. For convenience we still omit the subscript j.

Proposition 7.1. *It holds that*

$$
\begin{aligned}
6\pi(\partial_{11} - \partial_{22})[Q_{2k}(p_k) - Q_{1k}(p_k)] + \frac{T_{1k,1}}{4}\Delta Q_{1k}(p_k) + \frac{T_{2k,1}}{4}\Delta Q_{2k}(p_k) = O(\varepsilon_k^{\beta}), \\
12\pi\partial_{12}[Q_{2k}(p_k) - Q_{1k}(p_k)] + \frac{T_{1k,2}}{4}\Delta Q_{1k}(p_k) + \frac{T_{2k,2}}{4}\Delta Q_{2k}(p_k) = O(\varepsilon_k^{\beta}),
\end{aligned}
$$

where $T_{1k,1}$, $T_{2k,1}$, $T_{1k,2}$ and $T_{2k,2}$ are four constants defined by

$$
\begin{aligned}
T_{1k,1} &= \int_{\mathbb{R}^2} 3\rho_{1k} h_{1k}(p_k) e^{V_{1k}} \psi_{1k,1}^d |y|^2, \\
T_{2k,1} &= \int_{\mathbb{R}^2} 3\rho_{2k} h_{2k}(p_k) e^{V_{2k}} \psi_{2k,1}^d |y|^2, \\
T_{1k,2} &= \int_{\mathbb{R}^2} 3\rho_{1k} h_{1k}(p_k) e^{V_{1k}} \psi_{1k,2}^d |y|^2,
\end{aligned}
$$

$$T_{2k,2} = \int_{\mathbb{R}^2} 3\rho_{2k} h_{2k}(p_k) e^{V_{2k}} \psi^d_{2k,2} |y|^2.$$

and

$$\psi^d_{1k,1}(y) = (\partial_{d_k} + \partial_{\bar{d}_k}) V_{1k}(y), \qquad \psi^d_{2k,1}(y) = (\partial_{d_k} + \partial_{\bar{d}_k}) V_{2k}(y),$$
$$\psi^d_{1k,2}(y) = i(\partial_{d_k} - \partial_{\bar{d}_k}) V_{1k}(y), \qquad \psi^d_{2k,2}(y) = i(\partial_{d_k} - \partial_{\bar{d}_k}) V_{2k}(y).$$

Remark 7.2. *These constants $T_{ik,j}, i, j = 1, 2$ may be nonzero. See the remark at the end of this section.*

Proof. As $|y| \to \infty$, a Taylor expansion gives that

$$\psi^d_{1k,1} = -\frac{6(y_1^2 - y_2^2)}{|y|^4} + O(\frac{1}{|y|^3}), \qquad \frac{\partial \psi^d_{1k,1}}{\partial \nu} = \frac{12(y_1^2 - y_2^2)}{|y|^5} + O(\frac{1}{|y|^4}),$$
$$\psi^d_{2k,1} = \frac{6(y_1^2 - y_2^2)}{|y|^4} + O(\frac{1}{|y|^3}), \qquad \frac{\partial \psi^d_{2k,1}}{\partial \nu} = -\frac{12(y_1^2 - y_2^2)}{|y|^5} + O(\frac{1}{|y|^4}),$$
$$\psi^d_{1k,2} = -\frac{12 y_1 y_2}{|y|^4} + O(\frac{1}{|y|^3}), \qquad \frac{\partial \psi^d_{1k,2}}{\partial \nu} = \frac{24 y_1 y_2}{|y|^5} + O(\frac{1}{|y|^4}),$$
$$\psi^d_{2k,2} = \frac{12 y_1 y_2}{|y|^4} + O(\frac{1}{|y|^3}), \qquad \frac{\partial \psi^d_{2k,2}}{\partial \nu} = -\frac{24 y_1 y_2}{|y|^5} + O(\frac{1}{|y|^4}).$$

Then using the estimate (70) and (71), we have

$$\int_{B_{\frac{\delta}{\varepsilon_k}}} (-\Delta \tilde{\eta}_{1k}) \psi^d_{1k,1} + (-\Delta \tilde{\eta}_{2k}) \psi^d_{2k,1}$$
$$= \int_{B_{\frac{\delta}{\varepsilon_k}}} (-\Delta \psi^d_{1k,1}) \tilde{\eta}_{1k} + (-\Delta \psi^d_{2k,1}) \tilde{\eta}_{2k} + O(\varepsilon_k^3). \quad (102)$$

Since h_{1k} and h_{2k} are of $C^{2,\beta}(\overline{\Omega})$, it holds that

$$3D_{1k} = (2\tilde{\eta}_{1k} - \tilde{\eta}_{2k}) + 3\nabla Q_{1k}(p_k) \varepsilon_k y + \frac{3\nabla^2 Q_{1k}(p_k)}{2} \varepsilon_k^2 y^2 + O(\varepsilon_k^{2+\beta})(1 + |y|)^{2+\beta},$$
$$3D_{2k} = (2\tilde{\eta}_{2k} - \tilde{\eta}_{1k}) + 3\nabla Q_{2k}(p_k) \varepsilon_k y + \frac{3\nabla^2 Q_{2k}(p_k)}{2} \varepsilon_k^2 y^2 + O(\varepsilon_k^{2+\beta})(1 + |y|)^{2+\beta}.$$

Since

$$\int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{1k} h_{1k}(p_k) e^{V_{1k}} \psi^d_{1k,1} y = -\int_{B_{\frac{\delta}{\varepsilon_k}}} \Delta(2\psi^d_{1k,1} + \psi^d_{2k,1}) y$$
$$= \int_{\partial B_{\frac{\delta}{\varepsilon_k}}} \frac{\partial y}{\partial \nu} (2\psi^d_{1k,1} + \psi^d_{2k,1}) - \frac{\partial (2\psi^d_{1k,1} + \psi^d_{2k,1})}{\partial \nu} y = O(\varepsilon_k),$$

and clearly

$$\int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{2k}h_{2k}(p_k)e^{V_{2k}}\psi_{2k,1}^d y = O(\varepsilon_k),$$

a direct computation shows that

$$\begin{aligned}
&\int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{1k}h_{1k}(p_k)e^{V_{1k}}D_{1k}(y)\psi_{1k,1}^d \\
= &\int_{B_{\frac{\delta}{\varepsilon_k}}} \rho_{1k}h_{1k}(p_k)e^{V_{1k}}(2\tilde{\eta}_{1k}-\tilde{\eta}_{2k})\psi_{1k,1}^d \\
&\qquad + \frac{\nabla^2 Q_{1k}(p_k)}{2}\varepsilon_k^2\int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{1k}h_{1k}(p_k)e^{V_{1k}}y^2\psi_{1k,1}^d + O(\varepsilon_k^{2+\beta}),
\end{aligned}$$

and

$$\begin{aligned}
&\int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{2k}h_{2k}(p_k)e^{V_{2k}}D_{2k}(y)\psi_{2k,1}^d \\
= &\int_{B_{\frac{\delta}{\varepsilon_k}}} \rho_{2k}h_{2k}(p_k)e^{V_{2k}}(2\tilde{\eta}_{2k}-\tilde{\eta}_{1k})\psi_{2k,1}^d \\
&\qquad + \frac{\nabla^2 Q_{2k}(p_k)}{2}\varepsilon_k^2\int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{2k}h_{2k}(p_k)e^{V_{2k}}y^2\psi_{2k,1}^d + O(\varepsilon_k^{2+\beta}).
\end{aligned}$$

It is obviously that

$$\begin{aligned}
\nabla^2 Q_{1k}(p_k)y^2 &= \frac{1}{2}(\partial_{11}-\partial_{22})Q_{1k}(p_k)(y_1^2-y_2^2)+\frac{1}{2}\Delta Q_{1k}(p_k)|y|^2+2\partial_{12}Q_{1k}(p_k)y_1y_2,\\
\nabla^2 Q_{2k}(p_k)y^2 &= \frac{1}{2}(\partial_{11}-\partial_{22})Q_{2k}(p_k)(y_1^2-y_2^2)+\frac{1}{2}\Delta Q_{2k}(p_k)|y|^2+2\partial_{12}Q_{2k}(p_k)y_1y_2.
\end{aligned}$$

We can further check that

$$\begin{aligned}
&\int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{1k}h_{1k}(p_k)e^{V_{1k}}\psi_{1k,1}^d(y_1^2-y_2^2) = -\int_{B_{\frac{\delta}{\varepsilon_k}}} \Delta(2\psi_{1k,1}^d+\psi_{2k,1}^d)(y_1^2-y_2^2) \\
= &\int_{\partial B_{\frac{\delta}{\varepsilon_k}}} \frac{\partial(y_1^2-y_2^2)}{\partial\nu}(2\psi_{1k,1}^d+\psi_{2k,1}^d)-\frac{\partial(2\psi_{1k,1}^d+\psi_{2k,1}^d)}{\partial\nu}(y_1^2-y_2^2) \\
= &-24\int_{\partial B_{\frac{\delta}{\varepsilon_k}}} \frac{(y_1^2-y_2^2)^2}{|y|^5}+O(\varepsilon_k) = -24\pi+O(\varepsilon_k) \qquad (103)
\end{aligned}$$

and similarly

$$\int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{1k}h_{1k}(p_k)e^{V_{1k}}\psi_{1k,1}^d y_1y_2 = -\int_{B_{\frac{\delta}{\varepsilon_k}}} \Delta(2\psi_{1k,1}^d+\psi_{2k,1}^d)y_1y_2$$

$$= \int_{\partial B_{\frac{\delta}{\varepsilon_k}}} \frac{2y_1y_2}{|y|}(2\psi^d_{1k,1} + \psi^d_{2k,1}) - \frac{\partial(2\psi^d_{1k,1} + \psi^d_{2k,1})}{\partial\nu} y_1y_2$$

$$= O(\varepsilon_k). \tag{104}$$

Furthermore, $T_{1k,1}$ is such that

$$\int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{1k}h_{1k}(p_k)e^{V_{1k}}\psi^d_{1k,1}|y|^2 = T_{1k,1} + O(\varepsilon_k^2). \tag{105}$$

Thus, (103), (105) and (104) imply that, by symmetry,

$$\frac{\nabla^2 Q_{1k}(p_k)}{2}\varepsilon_k^2 \int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{1k}h_{1k}(p_k)e^{V_{1k}}y^2\psi^d_{1k,1}$$

$$= -6\pi(\partial_{11} - \partial_{22})Q_{1k}(p_k)\varepsilon_k^2 + \frac{1}{4}\Delta Q_{1k}(p_k)T_{1k,1}\varepsilon_k^2 + O(\varepsilon_k^3).$$

Analogously, we have

$$\frac{\nabla^2 Q_{2k}(p_k)}{2}\varepsilon_k^2 \int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{2k}h_{2k}(p_k)e^{V_{2k}}y^2\psi^d_{2k,1}$$

$$= 6\pi(\partial_{11} - \partial_{22})Q_{2k}(p_k)\varepsilon_k^2 + \frac{1}{4}\Delta Q_{2k}(p_k)T_{2k,1}\varepsilon_k^2 + O(\varepsilon_k^3),$$

where the constant $T_{2k,1}$ satisfies that

$$\int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{2k}h_{2k}(p_k)e^{V_{2k}}\psi^d_{2k,1}|y|^2 = T_{2k,1} + O(\varepsilon_k^2).$$

We conclude that

$$\int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{1k}h_{1k}(p_k)e^{V_{1,k}}D_{1k}(y)\psi^d_{1k,1} + \int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{2k}h_{2k}(p_k)e^{V_{2,k}}D_{2k}(y)\psi^d_{2k,1}$$

$$= \int_{B_{\frac{\delta}{\varepsilon_k}}} \rho_{1k}h_{1k}(p_k)e^{V_{1k}}(2\tilde\eta_{1k} - \tilde\eta_{2k})\psi^d_{1k,1} + \int_{B_{\frac{\delta}{\varepsilon_k}}} \rho_{2k}h_{2k}(p_k)e^{V_{2k}}(2\tilde\eta_{2k} - \tilde\eta_{1k})\psi^d_{2k,1}$$

$$+ 6\pi(\partial_{11} - \partial_{22})[Q_{2k}(p_k) - Q_{1k}(p_k)]\varepsilon_k^2 + \left[\frac{T_{1k}}{4}\Delta Q_{1k}(p_k) + \frac{T_{2k}}{4}\Delta Q_{2k}(p_k)\right]\varepsilon_k^2$$

$$+ O(\varepsilon_k^{2+\beta}). \tag{106}$$

Using (102) and (106), we get that

$$6\pi(\partial_{11} - \partial_{22})[Q_{2k}(p_k) - Q_{1k}(p_k)] + \frac{T_{1k,1}}{4}\Delta Q_{1k}(p_k) + \frac{T_{2k,1}}{4}\Delta Q_{2k}(p_k) = O(\varepsilon_k^{\beta}).$$

Repeating the above procedure and using

$$\int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{1k}h_{1k}(p_k)e^{V_{1k}}\psi^d_{1k,2}(y_1^2-y_2^2) = O(\varepsilon_k),$$

$$\int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{1k}h_{1k}(p_k)e^{V_{1k}}\psi^d_{1k,2}|y|^2 = T_{1k,2}+O(\varepsilon_k^2),$$

$$\int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{1k}h_{1k}(p_k)e^{V_{1k}}\psi^d_{1k,2}y_1y_2 = -12\pi+O(\varepsilon_k),$$

$$\int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{2k}h_{2k}(p_k)e^{V_{2k}}\psi^d_{2k,2}(y_1^2-y_2^2) = O(\varepsilon_k),$$

$$\int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{2k}h_{2k}(p_k)e^{V_{2k}}\psi^d_{2k,2}|y|^2 = T_{2k,2}+O(\varepsilon_k^2),$$

$$\int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{2k}h_{2k}(p_k)e^{V_{2k}}\psi^d_{2k,2}y_1y_2 = 12\pi+O(\varepsilon_k),$$

we obtain that

$$\int_{B_{\frac{\delta}{\varepsilon_k}}} (-\Delta\tilde{\eta}_{1k})\psi^d_{1k,2}+(-\Delta\tilde{\eta}_{2k})\psi^d_{2k,2}$$
$$= \int_{B_{\frac{\delta}{\varepsilon_k}}} (-\Delta\psi^d_{1k,2})\tilde{\eta}_{1k}+(-\Delta\psi^d_{2k,2})\tilde{\eta}_{2k}+O(\varepsilon_k^3),$$

and

$$\begin{aligned}
&\int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{1k}h_{1k}(p_k)e^{V_{1k}}D_{1k}(y)\psi^d_{1k,2}+\int_{B_{\frac{\delta}{\varepsilon_k}}} 3\rho_{2k}h_{2k}(p_k)e^{V_{2k}}D_{2k}(y)\psi^d_{2k,2}\\
=&\int_{B_{\frac{\delta}{\varepsilon_k}}} \rho_{1k}h_{1k}(p_k)e^{V_{1k}}(2\tilde{\eta}_{1k}-\tilde{\eta}_{2k})\psi^d_{1k,2}+\int_{B_{\frac{\delta}{\varepsilon_k}}} \rho_{2k}h_{2k}(p_k)e^{V_{2k}}(2\tilde{\eta}_{2k}-\tilde{\eta}_{1k})\psi^d_{2k,2}\\
&+12\pi\partial_{12}[Q_{2k}(p_k)-Q_{1k}(p_k)]\varepsilon_k^2+\left[\frac{T_{1k,2}}{4}\Delta Q_{1k}(p_k)+\frac{T_{2k,2}}{4}\Delta Q_{2k}(p_k)\right]\varepsilon_k^2\\
&+O(\varepsilon_k^{2+\beta}).
\end{aligned}$$

Therefore we have

$$12\pi\partial_{12}[Q_{2k}(p_k)-Q_{1k}(p_k)]+\frac{T_{1k,2}}{4}\Delta Q_{1k}(p_k)+\frac{T_{2k,2}}{4}\Delta Q_{2k}(p_k)=O(\varepsilon_k^{\beta}).$$

The proof is concluded. □

Remark 7.3. *The coefficients $T_{ik,j}$ may be nonzero. Assuming $b = c = 0$ and $d \in \mathbb{R}$, we compute $T_{1k,1}$ for $|d|$ small. We drop the dependence on k an d. Then*

$$2\psi_{1,1} + \psi_{2,1} = \frac{y_1^2 - y_2^2 + d}{a_1^2 + a_2^2|y|^2 + |y^2 + d|^2}.$$

Then it is easy to see that

$$T_{1,1} \sim \int_{R^2} e^{V_1}\psi_{1,1}|y|^2 = \lim_{R\to+\infty}\int_{B_R(0)} (-\Delta(2\psi_{1,1} + \psi_{2,1}))|y|^2$$

$$= -4\lim_{R\to+\infty}\int_{B_R(0)} (2\psi_{1,1} + \psi_{2,1}),$$

where

$$\int_{B_R(0)} (2\psi_{1,1}+\psi_{2,1}) = \int_{B_R(0)} \frac{y_1^2 - y_2^2}{a_1^2 + a_2^2|y|^2 + |y^2 + d|^2} + d\int_{B_R(0)} \frac{1}{a_1^2 + a_2^2|y|^2 + |y|^4} + O(d^2)$$

$$= \int_{B_R(0)} \left(\frac{y_1^2 - y_2^2}{a_1^2 + a_2^2|y|^2 + |y^2 + d|^2} - \frac{y_1^2 - y_2^2}{a_1^2 + a_2^2|y|^2 + |y|^4}\right) + d\int_{B_R(0)} \frac{1}{a_1^2 + a_2^2|y|^2 + |y|^4} + O(d^2)$$

$$= \int_{B_R(0)} \left(\frac{-2d(y_1^2 - y_2^2)^2}{(a_1^2 + a_2^2|y|^2 + |y|^4)^2}\right. + d\int_{B_R(0)} \frac{1}{a_1^2 + a_2^2|y|^2 + |y|^4} + O(d^2)$$

$$= d\int_{B_R(0)} \frac{a_1^2 + a_2^2|y|^2}{(a_1^2 + a_2^2|y|^2 + |y^2|^2)^2} + O(d^2).$$

Thus $\frac{T_{1,1}}{d}$ approaches a nonzero constant as $d \to 0$.

8 Proof of Theorem 1.2

Applying Proposition 6.2 to Proposition 5.1 leads to

$$\nabla Q_{1k,j}(p_{k,j}) = O(\varepsilon_k), \qquad \nabla Q_{2k,j}(p_{k,j}) = O(\varepsilon_k),$$

which implies that

$$8\pi\nabla_x H(p_{k,j}, p_{k,j}) + 8\pi\sum_{j=1}^{m} \nabla_x G(p_{k,j}, p_{k,\ell}) + \nabla \ln h_{1k}(p_{k,j}) = O(\varepsilon_k),$$

$$8\pi\nabla_x H(p_{k,j}, p_{k,j}) + 8\pi\sum_{j=1}^{m} \nabla_x G(p_{k,j}, p_{k,\ell}) + \nabla \ln h_{2k}(p_{k,j}) = O(\varepsilon_k).$$

This proves (24)-(25).

Similarly from Proposition 7.1 we have

$$6\pi(\partial_{11} - \partial_{22})\left[\ln h_{2k}(p_{k,j}) - \ln h_{1k}(p_{k,j})\right]$$

$$+\frac{T^j_{1k,1}}{4}\Delta\ln h_{1k}(p_{k,j})+\frac{T^j_{2k,1}}{4}\Delta\ln h_{2k}(p_{k,j})=O(\varepsilon_k^\beta),$$

$$12\pi\partial_{12}\left[\ln h_{2k}(p_{k,j})-\ln h_{1k}(p_{k,j})\right]$$
$$+\frac{T^j_{1k,2}}{4}\Delta\ln h_{1k}(p_{k,j})+\frac{T^j_{2k,2}}{4}\Delta\ln h_{2k}(p_{k,j})=O(\varepsilon_k^\beta),$$

which proves (26)-(27).

It remains to estimate $\rho_{1k}-8m\pi$ and $\rho_{2k}-8m\pi$. Recall that

$$\rho_{ik}=\sum_{j=1}^m\rho_{ik,j}+O(\varepsilon_k^2)\qquad\text{for } i=1,2.$$

Noting that $\Delta_x Q_{ik,j}(p_{k,j})=\Delta\ln h_{ik}(p_{k,j})$ $(i=1,2)$ and using Proposition 6.2, we easily have that

$$\rho_{1k}-8m\pi=\sum_{j=1}^m C_{1k,j}\Delta\ln h_{1k}(p_{k,j})\varepsilon_{k,j}^2|\ln\varepsilon_{k,j}|+O(\varepsilon_k^2),$$
$$\rho_{2k}-8m\pi=\sum_{j=1}^m C_{2k,j}\Delta\ln h_{2k}(p_{k,j})\varepsilon_{k,j}^2|\ln\varepsilon_{k,j}|+O(\varepsilon_k^2).$$

Hence (22)-(23) are established.

This completes the proof of Theorem 1.1.

9 The Case on a Surface: Proof of Theorem 1.1

In this section, we consider the following Toda system of SU(3)

$$\begin{cases}-\Delta_g u_{1k}=2\rho_{1k}\left(\dfrac{h_{1k}e^{u_{1k}}}{\int_M h_{1k}e^{u_{1k}}}-1\right)-\rho_{2k}\left(\dfrac{h_{2k}e^{u_{2k}}}{\int_M h_{2k}e^{u_{2k}}}-1\right) & \text{on } M,\\ -\Delta_g u_{2k}=2\rho_{2k}\left(\dfrac{h_{2k}e^{u_{2k}}}{\int_M h_{2k}e^{u_{2k}}}-1\right)-\rho_{1k}\left(\dfrac{h_{1k}e^{u_{1k}}}{\int_M h_{1k}e^{u_{1k}}}-1\right) & \text{on } M,\end{cases}\tag{107}$$

where (M,g) is a closed Riemann surface, Δ_g is the Laplace-Beltrami operator. Here we normalize the volume as $|M|=1$. In this system, ρ_{1k} and ρ_{2k} are two constants, $h_{1k}(x)$ and $h_{2k}(x)$ are two positive functions converging to $h_1(x)$ and $h_2(x)$ respectively in $C^{2,\beta}(M)$ as $k\to\infty$.

Let $\tilde u_{1k}$, $\tilde u_{2k}$, $\varepsilon_{k,j}$ and other symbols be defined as before. Set

$$w_{1k}=2u_{1k}+u_{2k}-3\sum_{j=1}^m\rho_{1k,\ell}G(x,p_{k,j})-2\bar u_{1k}-\bar u_{2k},$$
$$w_{2k}=u_{1k}+2u_{2k}-3\sum_{j=1}^m\rho_{2k,\ell}G(x,p_{k,j})-\bar u_{1k}-2\bar u_{2k},$$

where $\bar{u}_{1k}$, $\bar{u}_{2k}$ are the averages of u_{1k}, u_{2k}, and $G(x,p)$ is the Green function of Δ_g on M with singularity at p. Then by the same method, we have the similar estimates as in Lemma 3.4.

Lemma 9.1. *It holds that, for $i = 1, 2$,*

$$|w_{ik}| + |\nabla w_{ik}| = O(\varepsilon_k) \qquad \text{for } x \in M \setminus \bigcup_{j=1}^{m} B_\delta(p_j).$$

Recall that

$$\begin{cases} \Delta_g \tilde{u}_{1k} + 2\rho_{1k}\left(h_{1k}e^{\tilde{u}_{1k}} - 1\right) - \rho_{2k}\left(h_{2k}e^{\tilde{u}_{2k}} - 1\right) = 0 & \text{on } M, \\ \Delta_g \tilde{u}_{2k} - \rho_{1k}\left(h_{1k}e^{\tilde{u}_{1k}} - 1\right) + 2\rho_{2k}\left(h_{2k}e^{\tilde{u}_{2k}} - 1\right) = 0 & \text{on } M. \end{cases} \tag{108}$$

Since the computation from Section 4 to Section 8 is local. We introduce some notation for local computation. Note that isothermal coordinates always exist on Riemann surfaces. When (108) is considered locally, it is convenient to introduce a local coordinate x (still denoted by x) such that $p_{k,j}$ has the coordinate 0 and the metric $g_{ij} = e^{\phi}\delta_{ij}$ with $\phi(0) = 0$ and $\nabla\phi(0) = 0$. In this case, (108) is reduced to

$$\begin{cases} \Delta \tilde{u}_{1k} + 2\rho_{1k}e^{\phi}\left(h_{1k}e^{\tilde{u}_{1k}} - 1\right) - \rho_{2k}e^{\phi}\left(h_{2k}e^{\tilde{u}_{2k}} - 1\right) = 0 & \text{in } B_\delta(0), \\ \Delta \tilde{u}_{2k} - \rho_{1k}e^{\phi}\left(h_{1k}e^{\tilde{u}_{1k}} - 1\right) + 2\rho_{2k}e^{\phi}\left(h_{2k}e^{\tilde{u}_{2k}} - 1\right) = 0 & \text{in } B_\delta(0), \end{cases}$$

where Δ stands for the Laplacian in $\mathbb{R}^2$.

Furthermore, we set

$$\hat{u}_{1k}(x) = \tilde{u}_{1k} - (2\rho_{1k} - \rho_{2k})f_k(x) \qquad \text{and} \qquad \hat{u}_{2k}(x) = \tilde{u}_{2k} - (2\rho_{2k} - \rho_{1k})f_k(x),$$

where the function f_k is defined by

$$\begin{cases} \Delta f_k = e^{\phi} & \text{for } |x| \le \delta_0, \\ f_k(0) = 0, \quad \nabla f_k(0) = 0. \end{cases}$$

Clearly $(\hat{u}_{1k}, \hat{u}_{2k})$ satisfies that

$$\begin{cases} \Delta \hat{u}_{1k} + 2\rho_{1k}\hat{h}_{1k}e^{\hat{u}_{1k}} - \rho_{2k}\hat{h}_{2k}e^{\hat{u}_{2k}} = 0 & \text{for } |x| \le \delta_0, \\ \Delta \hat{u}_{2k} - \rho_{1k}\hat{h}_{1k}e^{\hat{u}_{1k}} + 2\rho_{2k}\hat{h}_{2k}e^{\hat{u}_{2k}} = 0 & \text{for } |x| \le \delta_0, \end{cases} \tag{109}$$

where

$$\hat{h}_{1k}(x) = e^{\phi}h_{1k}e^{(2\rho_{1k} - \rho_{2k})f_k}, \qquad \hat{h}_{2k}(x) = e^{\phi}h_{2k}e^{(2\rho_{2k} - \rho_{1k})f_k}.$$

Thus the similar proceeding from Section 4 to Section 8 can be carried out to (109). Note that now

$$Q_{1k,j}(x) = 2\widetilde{G}_{1k,j}(x) - \widetilde{G}_{2k,j}(x) + \ln \hat{h}_{1k}$$

$$= 2\widetilde{G}_{1k,j}(x) - \widetilde{G}_{2k,j}(x) + \ln h_{1k} + \phi + (2\rho_{1k} - \rho_{2k}) f_k,$$
$$Q_{2k,j}(x) = 2\widetilde{G}_{2k,j}(x) - \widetilde{G}_{1k,j}(x) + \ln h_{2k} + \phi + (2\rho_{2k} - \rho_{1k}) f_k.$$

Using

$$\begin{aligned}
\nabla \phi(p_{k,j}) &= \nabla f_k(p_{k,j}) = 0,\\
\Delta \phi(p_{k,j}) &= -2K(p_{k,j}) \quad \text{where } K \text{ is the Gauss curvature},\\
(2\rho_{1k} - \rho_{2k}) \Delta f_k(p_{k,j}) &= 8m\pi + 2(\rho_{1k} - 8m\pi) - (\rho_{2k} - 8m\pi),\\
(2\rho_{2k} - \rho_{1k}) \Delta f_k(p_{k,j}) &= 8m\pi + 2(\rho_{2k} - 8m\pi) - (\rho_{1k} - 8m\pi),\\
(\partial_{11} - \partial_{22}) \left[(2\rho_{1k} - \rho_{2k}) f_k - (2\rho_{2k} - \rho_{1k}) f_k\right] &= O(|\rho_{1k} - 8m\pi| + |\rho_{2k} - 8m\pi|),\\
\partial_{12} \left[(2\rho_{1k} - \rho_{2k}) f_k - (2\rho_{2k} - \rho_{1k}) f_k\right] &= O(|\rho_{1k} - 8m\pi| + |\rho_{2k} - 8m\pi|),
\end{aligned}$$

we therefore obtain Theorem 1.1.

Appendix: Proof of Lemma 4.1

Here we give the proof of Lemma 4.1. We shall follow the proof in [4]. Several key ingredients needed already exist: first of all, we have the non-degeneracy of entire solution; secondly, each $\tilde{\eta}_{ik,j}$ satisfies a linear equation with potential decaying like $|z|^{-4}$:

$$e^{V_{ik,j}(z)} = \frac{c_i}{|z|^4} + O(\frac{1}{|z|^5}) \qquad i = 1, 2. \tag{110}$$

Lastly, we have two bounded (non decaying) kernels $(\psi_{1k}^{a_i}, \psi_{2k}^{a_i})$ (as defined at (87)-(88)).

For $0 < \tau \le \tau_0$, let

$$R = \frac{\delta}{\varepsilon_{k,j}}, \qquad \alpha = \varepsilon_{k,j}^{2\tau} + \varepsilon_{k,j}^{\tau} \sup_{\frac{R}{2} \le |z| \le R} (|\tilde{\eta}_{1k,j}| + |\tilde{\eta}_{2k,j}|)$$

and

$$N_k^i = \sup_{|z| \le R} \frac{|\tilde{\eta}_{ik,j}|}{\alpha(1 + |z|)^{\tau}}, \qquad i = 1, 2.$$

We **claim** that

$$N_k^i \le C \tag{111}$$

for some constant C. To prove (111), we follow the proof in [4] and divide it into several steps. We prove it by contradiction. Without loss of generality, we may assume that $N_k := N_k^1 \ge N_k^2$. Assume that

$$N_k \to +\infty \qquad \text{as } k \to +\infty. \tag{112}$$

Step 1 $|\tilde\eta_{1k,j}(y)|+|\tilde\eta_{2k,j}(y)|=o(\alpha N_k)$ in any compact set.
Let

$$\bar\eta_{1k,j}(y)=\frac{\tilde\eta_{1k,j}(y)}{\|(1+|y|)^{-\tau}(|\tilde\eta_{1k,j}|+|\tilde\eta_{2k,j}|)\|_{L^\infty(B_{\frac{\delta}{\varepsilon_{k,j}}})}},$$

$$\bar\eta_{2k,j}(y)=\frac{\tilde\eta_{2k,j}(y)}{\|(1+|y|)^{-\tau}(|\tilde\eta_{1k,j}|+|\tilde\eta_{2k,j}|)\|_{L^\infty(B_{\frac{\delta}{\varepsilon_{k,j}}})}}.$$

Clearly $\bar\eta_{1k,j}$ and $\bar\eta_{2k,j}$ are locally bounded. Recall that $a_{1k,j}$, $a_{2k,j}$, $b_{k,j}$, $c_{k,j}$ and $d_{k,j}$ are determined by (60)–(64) such that $0<C_1<a_{1k,j},\ a_{2k,j}<C_2<\infty$ and the other coefficients are of order $O(1)$ in k. Recall the system (77) for $\tilde\eta_{ik,j}$ and we find that the inhomogeneous terms in the equations of $\bar\eta_{ik,j}$ are $O\left(\frac{\varepsilon_k^{2\tau_0}}{\alpha N_k(1+|y|^2)}\right)$. Obviously, $\frac{\varepsilon_k^{2\tau_0}}{\alpha N_k}\to 0$ as k goes to infinity, so the inhomogeneous terms tend to zero. Standard elliptic regularity then implies that there exist $\bar\eta_{i\infty}$ such that $\bar\eta_{ik,j}\to\bar\eta_{i\infty}$ in $C^2_{\rm loc}(\mathbb{R}^2)$ and

$$-\Delta\bar\eta_{1\infty}=e^{v_1}(2\bar\eta_{1\infty}-\bar\eta_{2\infty}),\qquad -\Delta\bar\eta_{2\infty}=e^{v_2}(2\bar\eta_{2\infty}-\bar\eta_{1\infty}),\tag{113}$$

where (v_1,v_2) is the entire solution with the parameters determined by the convergence. Since $\bar\eta_{1\infty}=O(1+|y|^\tau)$, $\bar\eta_{2\infty}=O(1+|y|^\tau)$, by Theorem 2.1 we deduce that $\binom{\bar\eta_{1\infty}}{\bar\eta_{2\infty}}=\sum_{\ell=1}^{8}\gamma_\ell Z_\ell$ where $Z_1=\binom{\partial_{a_1}w_1}{\partial_{a_1}w_2}$, $Z_2=\binom{\partial_{a_2}w_1}{\partial_{a_2}w_2}$, ... Since the choices of $a_{1k,j}$, $a_{2k,j}$, $b_{k,j}$, $c_{k,j}$ and $d_{k,j}$ imply that

$$\nabla^\ell\bar\eta_{1\infty}(0)=\nabla^\ell\bar\eta_{2\infty}(0)=0\qquad\text{for }\ell=0,1,$$
$$\partial_{11}\bar\eta_{1\infty}(0)=\partial_{22}\bar\eta_{1\infty}(0),\qquad \partial_{12}\bar\eta_{1\infty}(0)=0,$$

we deduce that $\gamma_\ell=0$ and hence $\bar\eta_{1\infty}=\bar\eta_{2\infty}\equiv 0$. So $\bar\eta_{ik,j}\to 0$ as $k\to\infty$ for $i=1,2$. This concludes Step 1.

Step 2 There exists C_1 such that

$$|y||\nabla\tilde\eta_{ik,j}(y)|\le C_1\left(\sup_{\frac{|y|}{2}\le|z|\le 2|y|}(|\tilde\eta_{1k,j}(z)|+|\tilde\eta_{2k,j}(z)|)+\varepsilon_k^{2\tau_0}\right).$$

This is the standard gradient estimates. We omit the proof (see [4]).

Step 3 It holds that

$$N_{ik}^*:=\sup_{|z|\le R}\sup_{|z|=|z'|}\frac{|\tilde\eta_{ik,j}(z)-\tilde\eta_{ik,j}(z')|}{\alpha(1+|z|)^\tau}=o(N_k),\qquad i=1,2.\tag{114}$$

We prove it by contradiction. Assume that $N_{ik}^*\ge c_0N_k$ for some $c_0>0$. Without loss of generality, we might assume $N_{1k}^*=\max(N_{1k}^*,N_{2k}^*)$. Let z_k' and z_k'' be such that $|z_k'|=|z_k''|$ and

$$N_{1k}^*=\frac{|\tilde\eta_{1k,j}(z_k')-\tilde\eta_{1k,j}(z_k'')|}{\alpha(1+|z_k|)^\tau}.$$

As in [4], we can prove $|z_k'| = |z_k''| < \frac{R}{2}$ and the angle between $\overrightarrow{Oz_k'}$ and $\overrightarrow{Oz_k''}$ $\theta_k \geq \theta_0 > 0$. This follows from the gradient estimate of Step 2 and $\tau \leq \tau_0$. Without loss of generality, we may assume z_k' and z_k'' are symmetric with respect to z_1-axis and $\tilde{\eta}_{1k,j}(z_k') > \tilde{\eta}_{1k,j}(z_k'')$.

Set

$$\omega_k^*(z) = \tilde{\eta}_{1k,j}(z) - \tilde{\eta}_{1k,j}(z^-) \qquad \text{for } z_2 > 0,$$

where $z = (z_1, z_2)$ and $z^- = (z_1, -z_2)$, and set

$$\bar{\omega}_k(z) = \frac{\omega_k^*(z)}{\alpha(1 + z_{k,2})^\tau}.$$

Let z_k^* be the maximum of $|\bar{\omega}_k(z)|$ in $B_R^+ = \{|z| \leq R, z_2 > 0\}$ and denote the maximum of $|\bar{\omega}_k(z)|$ by N_k^{**}. Then by the assumption, $N_k^{**} \geq c_1 N_{1k}^* \geq c_2 N_k$ for some positive constants c_1 and c_2, due to $\theta_k \geq \theta_0 > 0$. In particular, $N_k^{**} \to +\infty$ as $k \to +\infty$. By (70), (71) and Step 1, $1 << |z_k^*| \leq \frac{R}{2}$.

Straightforward computations show that $\bar{\omega}_k(z)$ satisfies, for $|z| > 1$,

$$\begin{aligned}\Delta\bar{\omega}_k(z) &+ 2\tau\nabla\log(1 + z_2)\nabla\bar{\omega}_k - \frac{\tau(1-\tau)}{(1 + z_2)^2}\bar{\omega}_k \\ &= O\left(\frac{1}{\alpha|z|^4(1 + z_2)^\tau}(|\tilde{\eta}_{1k,j}| + |\tilde{\eta}_{2k,j}|)\right) + O\left(\frac{\varepsilon_{k,j}^{2\tau_0}}{\alpha|z|^2(1 + z_2)^\tau}\right),\end{aligned}$$

At $z = z_k^*$, we have $\nabla\bar{\omega}_k = 0, \Delta\bar{\omega}_k \leq 0$ and thus we obtain

$$\begin{aligned}c_2 N_k \leq N_k^{**} = \bar{\omega}_k(z_k^*) &\leq C\frac{(1 + z_2)^2}{\alpha|z|^4(1 + z_2)^\tau}(|\tilde{\eta}_{1k,j}| + |\tilde{\eta}_{2k,j}|)) + C \\ &\leq CN_k|z_k^*|^{-2} + C,\end{aligned}$$

which clearly gives a contradiction. This proves Step 3.

Step 4 Set the radial average of $\tilde{\eta}_{ik,j}$ as

$$\varphi_{ik,j}(r) = \frac{1}{2\pi}\int_0^{2\pi} \tilde{\eta}_{ik,j}(re^{i\theta})\mathrm{d}\theta, \qquad i = 1, 2.$$

By Step 3, $\max_{i=1,2} \sup_{|z|\leq R} \frac{|\varphi_{ik,j}|}{\alpha(1+|z|)^\tau} \geq c_3 N_k$ for some positive constant c_3. Let us assume that $\frac{\varphi_{1k,j}(s_k)}{(1+s_k)^\tau} = \sup_{|z|\leq R} \frac{|\varphi_{1k,j}|}{(1+|z|)^\tau} \geq C\alpha N_k$. By Step 3, we have $1 << s_k < \frac{R}{2}$. Then multiplying the equations for $\tilde{\eta}_{ik,j}$ by the kernel functions $(\psi_{1k}^{a_i}, \psi_{2k}^{a_i})$ (as defined at (87)-(88)) respectively, we obtain, as in the proof of Estimate C of [4] and also as in the proof of (99), that

$$\begin{aligned}\int_{|z|=r} &\left[\frac{\partial}{\partial\nu}(\tilde{\eta}_{1k,j}(z))\psi_{1k}^{a_1} - \frac{\partial}{\partial\nu}(\psi_{1k}^{a_1})\tilde{\eta}_{1k,j}(z)\right] \\ &+ \int_{|z|=r}\left[\frac{\partial}{\partial\nu}(\tilde{\eta}_{2k,j}(z))\psi_{2k}^{a_1} - \frac{\partial}{\partial\nu}(\psi_{2k}^{a_1})\tilde{\eta}_{2k,j}(z)\right]\end{aligned}$$

$$= \int_{|z|\le r} \left[\rho_{1k} h_{1k}(p_{k,j}) e^{V_{1k,j}} \psi_{1k}^{a_1} O(|\tilde\eta_{1k,j}|^2 + |\tilde\eta_{2k,j}|^2)\right]$$
$$+ \int_{|z|\le r} \left[\rho_{2k} h_{2k}(p_{k,j}) e^{V_{2k,j}} \psi_{2k}^{a_1} O(|\tilde\eta_{1k,j}|^2 + |\tilde\eta_{2k,j}|^2)\right] + O(\varepsilon_k^{2\tau}).$$

Using the asymptotic expansion of $(\psi_{1k}^{a_1}, \psi_{2k}^{a_1})$ (see (88))

$$\psi_{1k}^{a_1} = O(\frac{1}{|y|^2}), \qquad \frac{\partial \psi_{1k}^{a_1}}{\partial \nu} = O(\frac{1}{|y|^3}),$$
$$\psi_{2k}^{a_1} = \frac{2}{a_{1k}} + O(\frac{1}{|y|^2}), \qquad \frac{\partial \psi_{2k}^{a_1}}{\partial \nu} = O(\frac{1}{|y|^3}),$$

we obtain, similar to [4], that

$$|\varphi_{2k}'(r)| \le c\left(\frac{\alpha N_k}{r^{2-\tau}} + \varepsilon_k^{2\tau}\frac{\ln(r+2)}{r} + (\alpha N_k)^2 r^{-1}\right).$$

Similarly, using the kernel $(\psi_{1k}^{a_2}, \psi_{2k}^{a_2})$, we may further obtain

$$|\varphi_{1k}'(r)| \le c\left(\frac{\alpha N_k}{r^{2-\tau}} + \varepsilon_k^{2\tau}\frac{\ln(r+2)}{r} + (\alpha N_k)^2 r^{-1}\right).$$

Note that

$$\alpha N_k = \sup_{|z|\le R} \frac{|\tilde\eta_{1k,j}(z)|}{(1+|z|)^2} \to 0 \qquad \text{as } k \to \infty,$$

because $\tilde\eta_{1k,j}$ is uniformly bounded and $\tilde\eta_{1k,j}(z) \to 0$ uniformly in any compact set. Hence

$$\alpha N_k(1+s_k)^\tau \le |\varphi_{1k}(s_k)| \le C\int_{r_0}^{s_k} |\varphi_{1k}'(r)| \mathrm{d}r \le C(\ln s_k)^2(\varepsilon^{2\tau} + \alpha N_k) + r_0^{-1}\alpha N_k,$$

where r_0 is a fixed but large positive number. Since $s_k \to \infty$, we get that

$$N_k \le \frac{C(\ln s_k)^2}{(1+s_k)^\tau}\frac{\varepsilon_k^{2\tau}}{\alpha} = o(1) \qquad \text{as } k \to \infty,$$

which yields a contradiction. This completes the proof of Lemma 4.1.

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