REVIEW PROBLEMS

PROBLEM 1 CALCULATE THE FOLLOWING INTEGRALS USING COMPLEX CONTOUR INTEGRATION:

(i) $I = \displaystyle\int_0^\infty \frac{x^\alpha}{x^2+\sqrt{2}\,x+1}\,dx \qquad -1<\alpha<1.$ (ii) $I = \displaystyle\int_0^\infty \frac{\ln x}{x^3+1}\,dx$

PROBLEM 2 CONSIDER THE MULTI-VALUED FUNCTION

$$w = f(z) = \left(\frac{z+1}{z-6}\right)^{2/3}$$

(i) DEFINE A BRANCH THAT IS CONTINUOUS AT $z = \pm 7$ WITH $f(7) = 4$ AND EVALUATE $f(-7)$.

(ii) DEFINE A DIFFERENT BRANCH WITH $f(7) = 4$ THAT IS CONTINUOUS AT $z = \pm 7$ AND AT THE ORIGIN.

PROBLEM 3 FIND ALL ROOTS OF

(i) $\cos z = 2i$ (ii) $2i = e^{3z}$

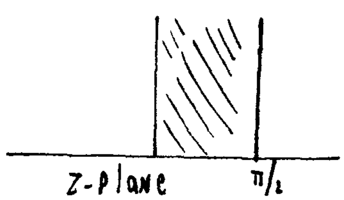
PROBLEM 4 FIND ALL BRANCH POINTS OF

$$f(z) = \log\left(z+\frac{1}{z}\right).$$

DEFINE A BRANCH THAT IS CONTINUOUS ON THE CIRCLE $|z-i| = \pi/2$ WITH $f(2i) = \ln\frac{3}{2} + i\pi/2$

PROBLEM 5 CALCULATE THE FOLLOWING INTEGRAL USING COMPLEX INTEGRATION $\displaystyle\int_0^3 \sqrt{x(3-x)}\,dx.$

PROBLEM 6 FIND THE IMAGE OF THE REGION SHOWN UNDER THE MAP $w = \sin z$

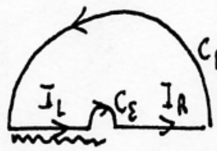


REVIEW SOLUTIONS

PROBLEM 1(i) LET $I = \int_0^\infty \frac{x^\alpha}{x^2+\sqrt{2}x+1}\,dx$

CONSIDER THE CONTOUR INTEGRAL $\int_C \frac{z^\alpha}{z^2+\sqrt{2}z+1}\,dz$ PRINCIPAL BRANCH OF z^α

$$f(z) = \frac{z^\alpha}{z^2+\sqrt{2}z+1}$$



ON $C_\varepsilon \to$ $\left|\int_{C_\varepsilon} \frac{z^\alpha}{z^2+\sqrt{2}z+1}\,dz\right| \le \max_{\text{ON } C_\varepsilon} \left|\frac{z^\alpha}{z^2+\sqrt{2}z+1}\right| \pi\varepsilon$

$$\le d_0\,\varepsilon^\alpha(\pi\varepsilon) = O(\varepsilon^{\alpha+1})$$

$\longrightarrow 0$ AS $\varepsilon \to 0$ SINCE $\alpha >$

NOW ON C_R: $\left|\int_{C_R} \frac{z^\alpha}{z^2+\sqrt{2}z+1}\,dz\right| \le \max_{C_R} \left|\frac{z^\alpha}{z^2+\sqrt{2}z+1}\right| \pi R \le d_0 \frac{R^\alpha}{R^2}\pi R = O(R^{\alpha-1})$

$\longrightarrow 0$ AS $R \to \infty$ SINCE $\alpha <$

THE POLES ARE AT $0 = z^2+\sqrt{2}z+1 = z^2+\sqrt{2}z+\frac{1}{2}+\frac{1}{2} \Rightarrow \left(z+\frac{\sqrt{2}}{2}\right)^2 = \pm\frac{i}{\sqrt{2}}$

HENCE $z_\pm = -\frac{\sqrt{2}}{2} \pm \frac{i\sqrt{2}}{2}$ $\qquad z_+ = e^{3\pi i/4}$ INSIDE CONTOUR.

HENCE $I_L + I_R = 2\pi i \,\mathrm{RES}\left[f(z); e^{3\pi i/4}\right] = 2\pi i\left[\frac{e^{(3\pi i/4)\alpha}}{2z+\sqrt{2}}\Big|_{z_+}\right]$

THIS GIVES $(*)$ $I_L + I_R = \frac{2\pi i\, e^{3\pi i\alpha/4}}{i\sqrt{2}} = \sqrt{2}\,\pi\, e^{(3\pi i/4)\alpha}$.

NOW $I_R = I$.

FOR I_L: $z = xe^{i\pi}$ $\quad dz = -dx$ $\qquad I_L = \int_\infty^0 \frac{x^\alpha e^{i\pi\alpha}}{x^2-\sqrt{2}x+1}(-dx)$

THIS GIVES $I_L = e^{i\pi\alpha}\int_0^\infty \frac{x^\alpha\,dx}{x^2-\sqrt{2}x+1}$. LET $J \equiv \int_0^\infty \frac{x^\alpha\,dx}{x^2-\sqrt{2}x+1}$

THEN $(*)$ GIVES $\qquad e^{i\pi\alpha}J + I = \sqrt{2}\,e^{(3\pi i/4)\alpha}\cdot\pi$

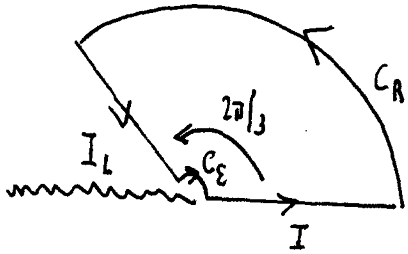
TO EXTRACT I MULTIPLY BY $e^{-i\pi\alpha}$ AND TAKE IMAGINARY PARTS.

$$J + Ie^{-i\pi\alpha} = \sqrt{2}\,\pi\, e^{-i\pi\alpha/4}$$

SO $\mathrm{IM}\left(Ie^{-i\pi\alpha}\right) = \mathrm{IM}\left(\sqrt{2}\,\pi\,e^{-i\pi\alpha/4}\right) \Rightarrow$ $\qquad I = \sqrt{2}\,\pi\,\frac{\sin(\pi\alpha/4)}{\sin(\pi\alpha)}$.

(ii) $I = \int_0^\infty \frac{\ln x}{x^3+1}\, dx.$

CONSIDER THE CONTOUR SHOWN $\int_C f(z)\,dz = \int_C \frac{\operatorname{Log} z}{z^3+1}\,dz$



NOW $\left|\int_{C_\varepsilon} \frac{\operatorname{Log} z}{z^3+1}\,dz\right| \le \max_{C_\varepsilon}\left|\frac{\operatorname{Log} z}{z^3+1}\right| \frac{2\pi}{3}\varepsilon$

$\le d_0\, \varepsilon \ln \varepsilon \to 0$ AS $\varepsilon \to 0$

ALSO $\left|\int_{C_R} \frac{\operatorname{Log} z}{z^3+1}\,dz\right| \le \max_{C_R}\left|\frac{\operatorname{Log} z}{z^3+1}\right| \frac{2\pi}{3} R$

$\le d_0\left(\frac{\ln R}{R^3}\right) R \to 0$ AS $R \to \infty$

NOW POLE AT $z = e^{\pi i/3}$.

HENCE $(*)$ $I_L + I_R = 2\pi i \operatorname{RES}\left[\frac{\operatorname{Log} z}{z^3+1} ; e^{\pi i/3}\right] = 2\pi i\left[\frac{\operatorname{Log}(e^{\pi i/3})}{3e^{2\pi i/3}}\right]$

$$= \frac{2\pi i}{3} e^{-2\pi i/3}\left(\frac{i\pi}{3}\right) = -\frac{2\pi^2}{9} e^{-2\pi i/3}.$$

NOW $I_R = I$.

ON I_L: $z = x e^{2\pi i/3}$, $dz = e^{2\pi i/3}\,dx$ SO $\int_{I_L} \longrightarrow I_L = \int_\infty^0 \frac{\operatorname{Log}(x e^{2\pi i/3})}{x^3+1} e^{2\pi i/3}\,dx$

NOW $I_L = e^{\frac{2\pi i}{3}} \int_\infty^0 \left(\frac{\ln x}{x^3+1} + \frac{2\pi i/3}{x^3+1}\right) dx = -e^{2\pi i/3} I - \frac{2\pi i}{3} e^{2\pi i/3} \int_0^\infty \frac{dx}{x^3+1}$

LET $J = \int_0^\infty \frac{1}{x^3+1}\,dx.$

SO SUBSTITUTING IN $(*)$ $-e^{2\pi i/3} I - \frac{2\pi i}{3} e^{2\pi i/3} J + I = -\frac{2\pi^2}{9} e^{-2\pi i/3}$

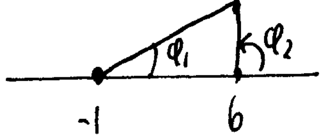
WE CAN WRITE THIS AS $I\left(1 - e^{2\pi i/3}\right) - \frac{2\pi i}{3} e^{2\pi i/3} J = -\frac{2\pi^2}{9} e^{-2\pi i/3}.$

MULTIPLY BOTH SIDES BY $i e^{-2\pi i/3}$ $\rightarrow$ $i I\left(e^{-2\pi i/3} - 1\right) + \frac{2\pi}{3} J = -\frac{2\pi^2}{9} i e^{-4\pi i/3}$

TAKE IMAGINARY PARTS $I \operatorname{IM}\left[i\left(e^{-2\pi i/3} - 1\right)\right] = -\frac{2\pi^2}{9} \operatorname{IM}\left(e^{-\pi i/3}\right) = \frac{2\pi^2}{9} \sin(\pi/3)$

SO $I\left(\cos(2\pi/3) - 1\right) = \frac{2\pi^2}{9} \sin(\pi/3)$ $\rightarrow$ $I\left(-3/2\right) = \frac{2\pi^2}{9}\frac{\sqrt{3}}{2}$ $\rightarrow$ $I = -\frac{2\pi^2}{9\sqrt{3}}.$

PROBLEM 2 $\quad w = f(z) = \left(\dfrac{z+1}{z-6}\right)^{2/3}$



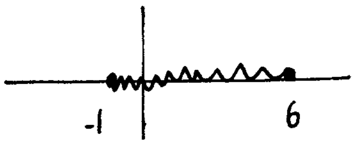
LET $\quad z = -1 + r_1 e^{i\varphi_1} \qquad z = 6 + r_2 e^{i\varphi_2}$

THEN
$$w = \left(\frac{r_1}{r_2}\right)^{2/3} e^{2i(\varphi_1 - \varphi_2)/3}.$$

POINTS $z=-1$ AND $z=6$ ARE BRANCH POINTS SINCE IF WE CIRCLE EITHER OF $z=-1$ OR $z=6$ BY CLOSED LOOP THEN EITHER φ_1 OR φ_2 INCREASES BY 2π $\longrightarrow$ w DOES NOT RETURN TO SAME VALUE.

(i) WANT CONTINUITY AT $z = \pm 7$ WITH $f(7) = 4$.

TRY $0 \le \varphi_1 < 2\pi$, $0 \le \varphi_2 < 2\pi$ THIS GIVES A CUT AS SHOWN BELOW



NOW CALCULATE $f(7)$. FOR $z=7$ $\quad r_1 = 8$, $r_2 = 1$, $\varphi_1 = \varphi_2 = 0$

SO $\quad w = \left(\dfrac{8}{1}\right)^{2/3} e^{2i(\varphi_1 - \varphi_2)/3} = 4.$ ✓✓ $f(7) = 4$.

NOW EVALUATE $f(-7)$. AT $z=-7$ WE HAVE $\varphi_1 = \varphi_2 = \pi$

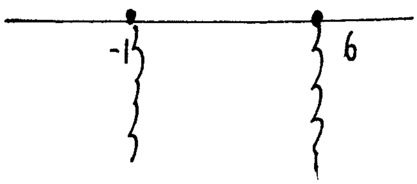
$r_1 = 6$, $r_2 = 13$

HENCE $\quad f(-7) = \left(\dfrac{6}{13}\right)^{2/3} e^{2i(\pi - \pi)/3} = \left(\dfrac{6}{13}\right)^{2/3}$

(ii) WE WANT CONTINUITY AT $z = \pm 7$ AND AT ORIGIN WITH AGAIN $f(7) = 4$.

THIS SUGGESTS THAT WE SHOULD NOT HAVE BRANCH CUT ON REAL AXIS. SO INSTEAD TRY

$$w = \left(\frac{r_1}{r_2}\right)^{2/3} e^{2i(\varphi_1 - \varphi_2)/3} \quad \text{WITH} \quad \frac{-\pi}{2} < \varphi_1 < \frac{3\pi}{2}, \quad \frac{-\pi}{2} < \varphi_2 < \frac{3\pi}{2}$$



PROBLEM 3

(i) $\cos z = 2i$

$$\frac{e^{z}+e^{-z}}{2} = 2i \quad \text{so} \quad e^{z}+e^{-z} = 4i \Rightarrow w + \frac{1}{w} = 4i \qquad w = e^{z}.$$

so $$w^2 - 4iw + 1 = 0 \qquad w_{\pm} = \frac{4i \pm \sqrt{-16-4}}{2} = 2i \pm i\sqrt{5}$$

TAKE $w_+ = (2+\sqrt{5})i \rightarrow z_+ = \log((2+\sqrt{5})i)$

$$\rightarrow z_{k_+} = \ln(2+\sqrt{5}) + i\left(\frac{\pi}{2} + 2k\pi\right)$$

$k = 0, \pm 1, \ldots$

TAKE $w_- = (2-\sqrt{5})i \rightarrow z_- = \log((2-\sqrt{5})i)$

$$\rightarrow z_{k_-} = \ln(\sqrt{5}-2) + i\left(-\frac{\pi}{2} + 2k\pi\right)$$

$k = 0, \pm 1, \ldots$

NOTICE THAT $\ln(\sqrt{5}-2) = -\ln\left(\frac{1}{\sqrt{5}-2}\,\frac{\sqrt{5}+2}{\sqrt{5}+2}\right) = -\ln(\sqrt{5}+2).$

HENCE
$$\left\{\begin{array}{l} z_{k_+} = \ln(2+\sqrt{5}) + i\left(\frac{\pi}{2}+2k\pi\right) \\ z_{k_-} = -\ln(2+\sqrt{5}) - i\left(\frac{\pi}{2}+2k\pi\right) \end{array}\right.$$
ARE ROOTS OF $\cos z = 2i$

(ii) $2i = e^{3z}$ so $3z = \log(2i) = \ln 2 + i\left(\frac{\pi}{2} + 2k\pi\right)$

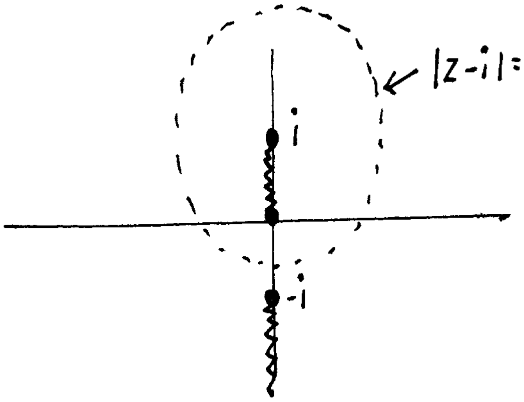
$$z = \frac{1}{3}\ln 2 + \frac{i}{3}\left(\frac{\pi}{2} + 2k\pi\right) \qquad k = 0, \pm 1, \pm 2, \ldots$$

PROBLEM 4

WE REWRITE $f(z) = \log\left(z + \frac{1}{z}\right) = \log\left(\frac{z^2+1}{z}\right) = \log(z^2+1) - \log z$

SO $f(z) = \log(z+i) + \log(z-i) - \log z$. BRANCH POINTS AT $z = i, -i, 0$.

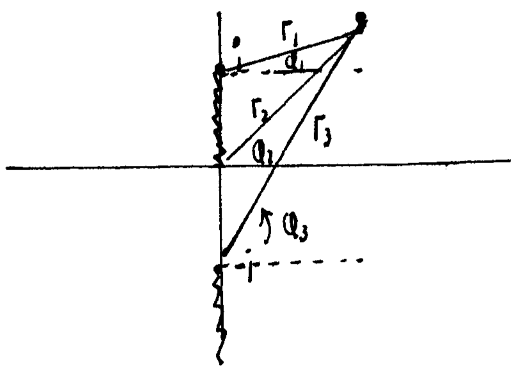
WE WANT $f(z)$ ANALYTIC ON $|z-i| = \frac{\pi}{2}$.



$\leftarrow |z-i| = \pi/2$. (WANT ANALYTIC ON THIS)

THIS SUGGESTS TO TAKE BRANCH CUTS AS SHOWN ABOVE

SO DRAW ANGLES



SO $f(z) = \ln(r_1 r_3 / r_2) + i(\theta_1 + \theta_3 - \theta_2)$

CHOOSE $-\frac{\pi}{2} < \theta_1 < \frac{3\pi}{2}$, $-\frac{\pi}{2} < \theta_2 < \frac{3\pi}{2}$,

$-\frac{\pi}{2} < \theta_3 < \frac{3\pi}{2}$

THIS GIVES CUTS AS SHOWN.

NOW WHEN $z = 2i$ $r_1 = 1$, $r_2 = 2$, $r_3 = 3$

$\theta_1 = \theta_2 = \theta_3 = \pi/2$

SO $f(2i) = \ln\left(\frac{3}{2}\right) + i\left(\frac{\pi}{2} + \frac{\pi}{2} - \frac{\pi}{2}\right)$

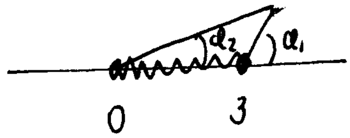
SO $f(2i) = \ln(3/2) + i\pi/2$.

PROBLEM 5 $\qquad I = \int_0^3 \sqrt{x}\sqrt{3-x}\; dx.$

LET $\quad f(z) = \sqrt{z(z-3)} = (r_1 r_2)^{1/2} e^{i(\theta_1+\theta_2)/2}$

WANT A BRANCH CUT BETWEEN $0<z<3$, z REAL $\longrightarrow$ θ_1 AND θ_2

MUST SATISFY $\qquad 0 \le \theta_1 \le 2\pi, \quad 0 \le \theta_2 \le 2\pi.$



NOW FOR z - LARGE

$$f(z) = \sqrt{z(z-3)} = \sqrt{z\,z\left(1-\frac{3}{z}\right)} = \pm z\left(1-\frac{3}{z}\right)^{1/2}$$

SO FOR $|z|>3$ WE HAVE $\qquad \left(1-\frac{3}{z}\right)^{1/2} = 1 - \frac{3}{2z} - \frac{1}{4}\left(\frac{3}{z}\right)^2 + \dots$

RECALL $f(h) = (1+h)^{1/2} \sim 1 + \frac{h}{2} - \frac{h^2}{8} + \dots$ AS $h \to 0$.

$f(h) = f(0) + f'(0)h + \frac{f''(0)h^2}{2} + \dots$

$f'(0) = \frac{1}{2}$, $f''(0)$

HENCE $\qquad f(z) \sim \pm z\left(1 - \frac{3}{2z} - \frac{9}{8z^2} + \dots\right)$

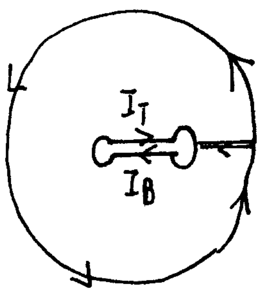
CLEARLY THE + SIGN IS CONSISTENT WITH BRANCH CUT AS CHOSEN

(SEE THE CLASS NOTES) SO

$$f(z) \sim z - \frac{3}{2} - \frac{9}{8z} + \dots \qquad \text{AS} \quad |z| \to \infty.$$

THIS IMPLIES $\displaystyle \lim_{R\to\infty} \oint_{C_R} f(z)\,dz = -\frac{9}{8}\,2\pi i = -\frac{9}{4}\pi i.$

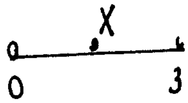
NOW TO CALCULATE INTEGRAL CONSIDER $\displaystyle \int_C f(z)\,dz = \int_C \sqrt{z(z-3)}\,dz.$



$$I_T + I_B + \lim_{R\to\infty}\oint_{C_R} f(z)\,dz = 0 \qquad \text{BY RESIDUE THEOREM}$$

SO $\quad I_T + I_B = \frac{9}{4}\pi i.$

ON I_T

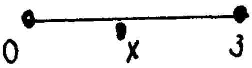


$\theta_1 = \pi$, $\theta_2 = 0$, $r_1 = 3-x$, $r_2 = x$

$f(z) = \left(x(3-x)\right)^{1/2} e^{i\pi/2}.$

SO $I_T = \int_0^3 \sqrt{x(3-x)}\, e^{i\pi/2}\, dx = i\, I \qquad I = \int_0^3 \sqrt{x(3-x)}\, dx.$

NOW ON $\underline{I_B}$:



$r_1 = 3 - x \qquad \theta_1 = \pi$

$r_2 = x \qquad \theta_2 = 2\pi$

$$f(z) = \sqrt{x(3-x)}\, e^{3\pi i/2} = -i\sqrt{x(3-x)}$$

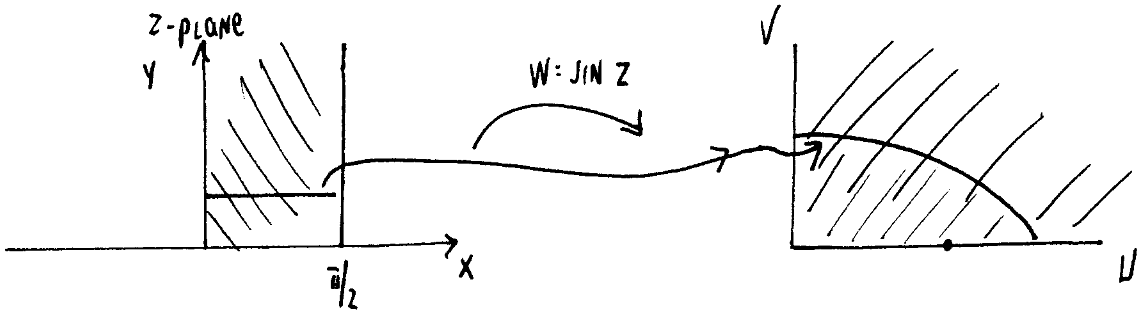
SO $I_B = -i\int_3^0 \sqrt{x(3-x)}\, dx = i\, I.$

HENCE $I_T + I_B = \frac{9\pi}{4} i$

$$\rightarrow \quad 2i\, I = \frac{9\pi}{4} i$$

$$I = \frac{9\pi}{8} = \int_0^3 \sqrt{x(3-x)}\, dx.$$

PROBLEM 6



LET $W = U + iV$, $Z = X + iY$

SO $U + iV = \sin(X+iY) = \sin X \cosh Y + i \cos X \sinh Y$

SO $U = \sin X \cosh Y$

$V = \cos X \sinh Y$

RECALL THAT $\sin^2 X + \cos^2 X = 1$

SO A LINE $Y = Y_0 \geq 0$ CONSTANT GETS MAPPED TO

$$\frac{U^2}{\cosh^2 Y_0} + \frac{V^2}{\sinh^2 Y_0} = 1 \quad \text{BUT} \quad 0 \leq X \leq \pi/2$$

$\rightarrow$ QUARTER ELLIPSE

NOTICE $\cosh Y_0$, $\sinh Y_0$ MONOTONICALLY INCREASING IN Y_0

$\longrightarrow$ THIS IMPLIES THAT $\mathrm{IM}(W) \geq 0$, $\mathrm{RE}(W) \geq 0$ IS IMAGE