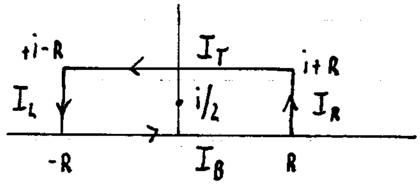
MATH 301 HOMEWORK SOLUTIONS # 1

6.3 # 9 CONSIDER $I = \oint_C \dfrac{e^{2z}}{\cosh(\pi z)}\, dz$, WHERE



NOTICE $\cosh(\pi z) = 0$ WHEN

$$z = \pm i \frac{(2n-1)\pi}{2} \quad n \text{ integer. HENCE } z = i/_2 \text{ LIES INSIDE the contour.}$$

$$\text{RES}\left[\frac{e^{2z}}{\cosh(\pi z)} ; i/_2\right] = \frac{e^{i}}{\frac{d}{dz}(\cosh(\pi z))\Big|_{z=i/_2}} = \frac{e^{i}}{\pi \sinh(i\pi/_2)} = \frac{e^{i}}{i\pi\left(\frac{e^{i\pi/_2} - e^{-i\pi/_2}}{2i}\right)} = \frac{-ie^{i}}{\pi \sin(\pi/_2)}$$

THUS
$$I = 2\pi i \ \text{RES}\left[\frac{e^{2z}}{\cosh(\pi z)} ; i/_2\right] = \frac{2\pi e^{i}}{\pi} = 2e^{i}.$$

NOW
$$|I_R| \le \int_0^1 \left|\frac{e^{2R}}{\cosh(\pi(R+is))}\right| ds \le C e^{(2-\pi)R} \to 0 \quad \text{AS} \quad R \to \infty.$$

similarly $|I_L| \to 0$ AS $R \to \infty$.

NOW
$$\int_{I_T} + \int_{I_B} = 2e^{i} \qquad \text{ON } I_T: \quad z = i + s$$

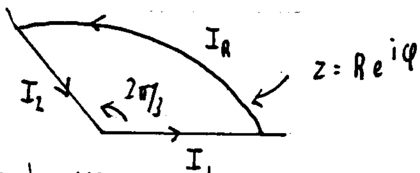
$$\lim_{R\to\infty}\left(\int_R^{-R} \frac{e^{2(i+s)}}{\cosh(\pi(i+s))}\, ds + \int_{-R}^{R} \frac{e^{2s}}{\cosh(\pi s)}\, ds\right) = 2e^{i}$$

BUT $\cosh(\pi(i+s)) = -\cosh(\pi s)$.

$$\rightarrow \quad (1 + e^{2i}) \int_{-\infty}^{\infty} \frac{e^{2s}}{\cosh(\pi s)}\, ds = 2e^{i}$$

$$\rightarrow \quad \int_{-\infty}^{\infty} \frac{e^{2s}}{\cosh(\pi s)}\, ds = \frac{2}{e^{i}+e^{-i}} = \sec 1$$

6.3 # 11 CONSIDER $I = \oint_C \dfrac{dz}{z^3+1}$



Poles are at $z^3 + 1 = 0 \rightarrow z = e^{\pi i/_3}$ which LIES inside the contour:

$$\text{ON } I_2: \quad z = re^{2\pi i/_3} \longrightarrow z^3 + 1 = r^3 + 1 \qquad dz = e^{2\pi i/_3}\, dr$$
$$\text{ON } I_1: \quad z = r$$

THUS,

$$\oint_C = \int_{I_1} + \int_{I_2} + \int_{I_R} = 2\pi i \ \mathrm{RES}\left[\frac{1}{z^3+1} \ ; \ e^{\pi i/3}\right] = \frac{2\pi i}{3\, e^{2\pi i/3}}$$

NOW $$\left| \int_{I_R} \frac{1}{z^3+1}\, dz \right| \leq \int_0^{2\pi/3} \frac{R\, d\varphi}{|R^3 e^{3i\varphi}+1|} \leq C R^{-2} \to 0 \text{ AS } R \to \infty$$

NOW ON I we get

$$I = \int_0^\infty \frac{1}{r^3+1}\, dr + \int_\infty^0 \frac{e^{2\pi i/3}}{r^3+1}\, dr = \frac{2\pi i}{3} e^{-2\pi i/3}$$

$$\left(1 - e^{2\pi i/3}\right) \int_0^\infty \frac{1}{r^3+1}\, dr = \frac{2\pi i}{3} e^{-2\pi i/3}$$

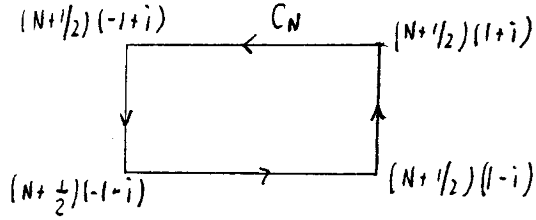
$$\to \quad \left(e^{-\pi i/3} - e^{\pi i/3}\right) \int_0^\infty \frac{1}{r^3+1}\, dr = \frac{2\pi i}{3}(-1) = -\frac{2\pi}{3} i$$

$$\left(\frac{e^{\pi i/3} - e^{-\pi i/3}}{2i}\right) \int_0^\infty \frac{1}{r^3+1}\, dr = \frac{\pi}{3}$$

$$\to \quad \int_0^\infty \frac{1}{r^3+1}\, dr = \frac{\pi}{3 \sin(\pi/3)} = \frac{\pi}{3\left(\sqrt{3}/2\right)} = \frac{2\pi}{3\sqrt{3}}$$

$$\to \quad \int_0^\infty \frac{1}{r^3+1}\, dr = \frac{2\pi\sqrt{3}}{9} .$$

.3 # 18 $\quad I = \oint_{C_N} \pi f(z) \csc(\pi z)\, dz$



NOW let $f(z)$ HAVE poles at $z = z_1, \ldots, z_J$

NOTE: $\csc(\pi z)$ HAS POLES AT $z = n$ $\quad n = -N, -N+1, \ldots, N$ which lie inside C_N.

THUS $$I = 2\pi i \sum_{j=1}^{J} \mathrm{RES}\left[\pi f(z) \csc(\pi z) ; z_j\right] + 2\pi i \sum_{j=-N}^{N} \mathrm{RES}\left[\pi f(z) \csc(\pi z) ; j\right]$$

NOW let $N \to \infty$ AND ASSUME that $I \to 0$ AS we show below. Then,

$$\sum_{j=-\infty}^{\infty} \mathrm{RES}\left[\pi f(z) \csc(\pi z) ; j\right] = -\sum_{j=1}^{J} \mathrm{RES}\left[\pi f(z) \csc(\pi z) ; z_j\right]$$

simple poles at $z = j$.

$$\mathrm{RES}\left[\pi f(z) \csc(\pi z) ; j\right] = \frac{\pi f(j)}{\pi \cos(\pi j)} = (-1)^j f(j)$$

HENCE $\rightarrow$ $$\sum_{j=-\infty}^{\infty} (-1)^j f(j) = -\sum_{j=1}^{J} \mathrm{RES}\left[\pi f(z) \csc(\pi z) ; z_j\right].$$

NOW NEED TO SHOW THAT $\int_{C_N} \to 0$ AS $N \to \infty$.

THIS GIVES $$\left| \int_{C_N} \right| \le 4\pi \max_{C_N}\left(|f(z)| \, |\csc(\pi z)| \right) N$$ since length grows like N.

NOW BY ASSUMPTION $|f(z)| \le K |z|^{-2}$ AS $|z| \to \infty$

AND NOTE $|\csc(\pi z)| \le K$ ON C_N

K is a generic constant.

HENCE $$\left| \int_{C_N} \right| \le K N^{-1} \longrightarrow 0 \quad \text{AS } N \to \infty.$$

6.3 # 19 LET $f(z) = 1/z^2$

$$f(z)\,\csc(\pi z) = \frac{\csc(\pi z)}{z^2}$$ HAS A POLE OF order 3 at $z=0$.

let's FIND the Residue

$$f(z)\csc(\pi z) \sim \frac{1}{z^2(\pi z - \pi^3 z^3/6 + \cdots)} \sim \frac{1}{\pi z^3}\left(1 + \pi^2 z^2/6 + \cdots\right) = \frac{1}{\pi z^3} + \frac{\pi}{6z} + \cdots$$

$$\text{RES}\left[f(z)\csc(\pi z); 0\right] = \pi/6.$$

HENCE
$$\lim_{N\to\infty} \oint_{C_N} \pi \frac{\csc(\pi z)}{z^2}\,dz = \frac{\pi}{6} + \sum_{\substack{K=-\infty \\ K\neq 0}}^{\infty} \frac{(-1)^K}{K^2}.$$

BUT
$$\sum_{\substack{K=-\infty \\ K\neq 0}}^{\infty} \frac{(-1)^K}{K^2} = 2\sum_{K=1}^{\infty} \frac{(-1)^K}{K^2}.$$

HENCE
$$\sum_{K=1}^{\infty} \frac{(-1)^K}{K^2} = -\pi^2/12.$$

6.3 # 17

17a)
$$\sum_{K=-\infty}^{\infty} f(K) = -\sum_{j=1}^{J} \text{RES}\left[\pi f(z)\cot(\pi z); z_j\right]$$

take $f(z) = \frac{1}{(z+a)^2}$. POLE OF ORDER (2) at $z=-a$.

$$\sum_{K=-\infty}^{\infty} \frac{1}{(z+a)^2} = -\lim_{z\to -a}\left[\frac{d}{dz}\left[\pi \frac{(z+a)^2 \cot(\pi z)}{(z+a)^2}; -a\right]\right] = -\pi^2\csc^2(-\pi a) = -\pi^2\csc^2(\pi a$$

$$\rightarrow \sum_{K=-\infty}^{\infty}\left(\frac{1}{K+a}\right)^2 = +\pi^2\csc^2(\pi a)$$

17 b) $\displaystyle\sum_{K=-\infty}^{\infty} \frac{1}{K^2+a^2}$

take $f(z) = \dfrac{1}{z^2+a^2} \rightarrow$ simple poles at $z = \pm ia$

NOW $$\sum_{K=-\infty}^{\infty} \frac{1}{K^2+a^2} = -\operatorname{RES}\left[\frac{\pi}{z^2+a^2}\cot(\pi z);\ ia\right] - \operatorname{RES}\left[\frac{\pi}{z^2+a^2}\cot(\pi z);\ -ia\right]$$

$$= -\frac{\pi}{2ia}\cot(\pi i a) - \frac{\pi}{-2ia}\cot(-\pi i a) = \frac{-\pi}{ia}\cot(\pi i a)$$

BUT $\cot(\pi i a) = -i\coth(\pi a)$.

HENCE, $$\sum_{K=-\infty}^{\infty} \frac{1}{K^2+a^2} = \frac{\pi}{a}\coth(\pi a).$$

17 c) $\displaystyle\sum_{K=-\infty}^{\infty} \frac{K^2-a^2}{(K^2+a^2)^2}$ $\qquad f(z) = \dfrac{z^2-a^2}{(z^2+a^2)^2}$ $\qquad$ double poles at $z = ia, -ia$

$(z^2+a^2)^2 = (z+ia)^2(z-ia)^2$.

NOW $$\sum_{K=-\infty}^{\infty} \frac{K^2-a^2}{(K^2+a^2)^2} = -\lim_{z\to ia}\left[\frac{d}{dz}\left[\frac{\cancel{(z-ia)^2}(z^2-a^2)}{\cancel{(z-ia)^2}(z+ia)^2}\pi\cot(\pi z)\right]\right]$$
$$-\lim_{z\to -ia}\left[\frac{d}{dz}\left[\frac{(z+ia)^2(z^2-a^2)}{(z-ia)^2(z+ia)^2}\pi\cot(\pi z)\right]\right]$$

$$= -\lim_{z\to ia}\left[\frac{d}{dz}\left[(z^2-a^2)\pi(z+ia)^{-2}\cot(\pi z)\right]\right] \qquad \text{BY symmetry}$$
$$-\lim_{z\to -ia}\left[\frac{d}{dz}\left[(z^2-a^2)\pi(z-ia)^{-2}\cot(\pi z)\right]\right]$$

$$= -\ 2z\pi(z+ia)^{-2}\cot(\pi z)\Big|_{ia} + 2(z^2-a^2)\pi(z+ia)^{-3}\cot(\pi z)\Big|_{ia}$$
$$+\pi^2(z^2-a^2)(z+ia)^{-2}\csc^2(\pi z)\Big|_{ia}$$
$$-\ 2z\pi(z-ia)^{-2}\cot(\pi z)\Big|_{-ia} + 2\pi(z^2-a^2)(z-ia)^{-3}\cot(\pi z)\Big|_{-ia} + \pi(z^2-a^2)(z-ia)^{-2}\csc^2(\pi z)\Big|$$

THE COT TERMS CANCEL TO get

$$\sum_{K=-\infty}^{\infty} \frac{K^2-a^2}{K^2+a^2} = \frac{-2\pi^2a^2}{(2ia)^2}\csc^2(\pi i a) - \frac{2\pi^2}{(-2ia)^2}a^2\csc^2(\pi i a)$$

BUT $\csc(\pi i a) = -i\operatorname{csch}(\pi a)$

HENCE $\displaystyle\sum_{K=-\infty}^{\infty} \frac{K^2-a^2}{K^2+a^2} = -\pi^2 \operatorname{csch}^2(\pi a)$

d) take $f(z) = \dfrac{1}{(z-r)^2+a^2}$

pole at $(z-r) = \pm i a$

$z_\pm = r \pm i a.$

$$\sum_{K=-\infty}^{\infty} \frac{1}{(K-r)^2+a^2} = -\lim_{z\to z_+}\left[\left(\frac{(z-z_+)\,\pi\cot(\pi z)}{(z-z_+)(z-z_-)}\right)\right] - \lim_{z\to z_-}\left[\frac{(z-z_-)\pi\cot(\pi z)}{(z-z_+)(z-z_-)}\right]$$

$$\sum_{K=-\infty}^{\infty} \frac{1}{(K-r)^2+a^2} = -\frac{\pi\cot(\pi z_+)}{2(z_+-r)} - \frac{\pi\cot(\pi z_-)}{2(z_--r)}$$

$$\sum_{K=-\infty}^{\infty} \frac{1}{(K-r)^2+a^2} = -\pi\frac{\cot(\pi r+i\pi a)}{2ia} - \pi\frac{\cot(\pi r-i\pi a)}{-2ia}$$

NOW

$$\sum_{K=-\infty}^{\infty} \frac{1}{(K-r)^2+a^2} = \frac{\pi}{2ia}\left[\frac{\cos(\pi r-i\pi a)}{\sin(\pi r-i\pi a)} - \frac{\cos(\pi r+i\pi a)}{\sin(\pi r+i\pi a)}\right]$$

$$= \frac{\pi}{2ia}\,\frac{\sin[2\pi i a]}{\sin[\pi r-i\pi a]\sin[\pi r+i\pi a]}$$

NOW $\sin[\pi r - i\pi a] = \dfrac{e^{i\pi r+\pi a} - e^{-i\pi r-\pi a}}{2i}$

$\sin[\pi r + i\pi a] = \dfrac{e^{i\pi r-\pi a} - e^{-i\pi r+\pi a}}{2i}$

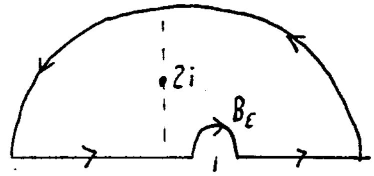
MULTIPLY TO get

$\sin[\pi r - i\pi a]\sin[\pi r+i\pi a]$
$= -i\left[\sin^2(\pi r) + \sinh^2(\pi a)\right]$

HENCE $\displaystyle\sum_{K=-\infty}^{\infty} \frac{1}{(K-r)^2+a^2} = \frac{\pi}{2a}\left(\frac{\sin(2a\pi)}{\sin^2\pi r + \sinh^2(\pi a)}\right)$

SECTION 6.5 #6

$$I = \text{p.v.} \int_{-\infty}^{\infty} \frac{\sin x}{(x^2+4)(x-1)}\, dx$$



CONSIDER $\tilde{I} = \int_C \frac{e^{iz}}{(z^2+4)(z-1)}\, dz.$

clearly, $\left| \int_{C_R} \frac{e^{iz}}{(z^2+4)(z-1)}\, dz \right| \leq M|z|^{-2} \to 0$ AS $|z| \to \infty$ IN UPPER 1/2 PLANE.

THUS $\int_{C_R} \to 0$ AS $R \to \infty$.

THUS $$\lim_{\varepsilon \to 0} \left(\int_{-\infty}^{1-\varepsilon} \frac{e^{ix}}{(x^2+4)(x-1)}\, dx + \int_{1+\varepsilon}^{\infty} \frac{e^{ix}}{(x^2+4)(x-1)}\, dx + \int_{B_\varepsilon} \frac{e^{iz}}{(z^2+4)(z-1)}\, dz \right) = 2\pi i \,\text{RES}\left[\frac{e^{iz}}{(z^2+4)(z-1)}; \right.$$

NOW $$\text{p.v.} \int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+4)(x-1)}\, dx + \lim_{\varepsilon \to 0} \int_{\pi}^{0} \frac{e^{i(1+\varepsilon e^{i\varphi})}}{5(\cancel{\varepsilon} \cancel{e}^{i\varphi})} i\cancel{\varepsilon}\cancel{e}^{i\varphi}\, d\varphi = \frac{2\pi i\, e^{-2}}{(4i)(2i-1)}$$

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+4)(x-1)}\, dx - e^{i} \frac{\pi i}{5} = \frac{\pi e^{-2}}{2} \frac{1}{2i-1} = \frac{-\pi e^{-2}}{2 \cdot 5}(1+2i)$$

NOW take imaginary parts,

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+4)(x-1)}\, dx = \frac{\pi}{5}\cos(1) - \frac{\pi e^{-2}}{5}$$

SECTION 6.5 #8 CALCULATE $I = \int_{-\infty}^{\infty} \frac{\cos(2x)}{x^3+1}\, dx.$

NOW $x^3 + 1 = 0 \to x = -1$ AND $x = e^{i\pi/3}$ IN $\text{IM}(x) \geq 0$.

THUS TAKE $I = \oint_C \frac{e^{2iz}}{z^3+1}\, dz$

NOW $\left| \int_{C_R} \right| \to 0$ AS $R \to \infty$

$$I = \lim_{\varepsilon \to 0} \left(\int_{-\infty}^{-1-\varepsilon} \frac{e^{2ix}}{x^3+1}\, dx + \int_{B_\varepsilon} \frac{e^{2iz}}{z^3+1}\, dz + \int_{-1+\varepsilon}^{\infty} \frac{e^{2ix}}{x^3+1}\, dx \right) = 2\pi i \,\text{RES}\left(\frac{e^{2iz}}{z^3+1}; e^{\pi i/3}\right.$$

NOW $\displaystyle \lim_{\varepsilon\to 0}\int_{B_\varepsilon}\frac{e^{2iz}}{z^3+1}\,dz = \lim_{\varepsilon\to 0}\int_{B_\varepsilon}\frac{e^{2iz}}{(z+1)(z^2-z+1)}\,dz = \lim_{\varepsilon\to 0}\int_{\pi}^{0}\frac{e^{-2i}\,\varepsilon i e^{i\varphi}}{\varepsilon e^{i\varphi}(3)}\,d\varphi = -\frac{\pi i e^{-2i}}{3}$

HENCE $$\text{p.v.}\int_{-\infty}^{\infty}\frac{e^{2ix}}{x^3+1}\,dx = \frac{\pi i e^{-2i}}{3} + \frac{2\pi i\, e^{2ie^{\pi i/3}}}{3e^{2\pi i/3}}.$$

NOW $$\text{p.v.}\int_{-\infty}^{\infty}\frac{e^{2ix}}{x^3+1}\,dx = \frac{\pi i e^{-2i}}{3} + \frac{2\pi i}{3}\, e^{-2\pi i/3}\, e^{2i\left[i\sin(\pi/3)+\cos(\pi/3)\right]}$$

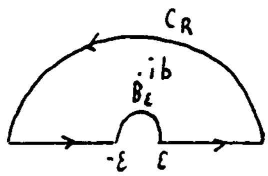
NOW TAKE REAL PARTS,

$$\text{p.v.}\,\text{RE}\left(\int_{-\infty}^{\infty}\frac{e^{2ix}}{x^3+1}\,dx\right) = \frac{\pi}{3}\sin(2) + \frac{2\pi}{3}\,\text{RE}\left[e^{-2\pi i/3}e^{-\sqrt{3}+i}\right]$$

HENCE $$\int_{-\infty}^{\infty}\frac{\cos 2x}{x^3+1}\,dx = \frac{\pi}{3}\sin(2) + \frac{2\pi}{3}\left[\text{RE}\left(e^{-\sqrt{3}}(\cos 1 + i\sin 1)\right)\left(\cos(-2\pi/3) + i\sin(\right.\right.$$

$$\int_{-\infty}^{\infty}\frac{\cos 2x}{x^3+1}\,dx = \frac{\pi\sin 2}{3} + \frac{2\pi}{3}e^{-\sqrt{3}}\left(\frac{1}{2}\sin(1) + \frac{\sqrt{3}}{2}\cos(1)\right).$$

SECTION 6.5 #12



$$I = \int_0^{\infty}\frac{\sin(ax)}{x(x^2+b^2)}\,dx \qquad a>0,\ b>0$$

let $$\tilde{I} = \oint_C \frac{e^{iaz}}{z(z^2+b^2)}\,dz.$$

NOW $$\tilde{I} = \lim_{R\to\infty}\int_{I_R} + \lim_{\varepsilon\to 0}\int_{B_\varepsilon} + \text{p.v.}\int_{-\infty}^{\infty}\frac{e^{iax}}{x(x^2+b^2)}\,dx.$$

NOW $\displaystyle \lim_{R\to\infty}\int_{I_R} \to 0$, SINCE $a>0$.

$$\lim_{\varepsilon\to 0}\int_{B_\varepsilon} = \int_{\pi}^{0}\frac{i\varepsilon e^{i\varphi}\,d\varphi}{\varepsilon e^{i\varphi}\,b^2} = -\frac{i\pi}{b^2}.$$

THUS $$\text{p.v.}\int_{-\infty}^{\infty}\frac{e^{iax}}{x(x^2+b^2)}\,dx = \left(\frac{+i\pi}{b^2}\right) + 2\pi i\ \text{RES}\left(\frac{e^{iaz}}{z(z^2+b^2)};\ ib\right) = \frac{i\pi}{b^2} + \frac{\cancel{2}\pi\cancel{i}\,e^{-ab}}{\cancel{i}b(\cancel{2}ib)}$$

NOW TAKE real parts, $$\int_{-\infty}^{\infty}\frac{\cos(ax)}{x(x^2+b^2)}\,dx = \frac{\pi}{b^2}\left(1-e^{-ab}\right).$$