HOMEWORK # 2 M301 SOLUTIONS

SECTION 3.2 #4

$$e^z = e^{x+iy} = e^x(\cos y + i\sin y)$$

$$|e^z| = e^x$$

$$\operatorname{Log} w = \ln|w| + i\operatorname{Arg}(w) \qquad -\pi < \operatorname{Arg}(w) \le \pi$$

let $w = e^z$ $\qquad \log(e^z) = \ln e^x + iy \qquad$ WITH $y = \operatorname{Arg}(z)$

$$\rightarrow \quad \log(e^z) = z, \text{ IF } \quad y \in (-\pi, \pi]$$

NOW TAKE $z = x + iy$ FOR z TO BE IN THE RANGE OF $\operatorname{Log}(e^z)$ we need

$\operatorname{IMAG}(z) \in (-\pi, \pi] \rightarrow y \in (-\pi, \pi]$.

SECTION 3.2 #5 b)

$\operatorname{Log}(w) = i\pi/2$ HAS SOLUTION $w = e^{i\pi/2}$ IF $\pi/2 \in (-\pi, \pi]$ BY PREVIOUS PROBLEM. THIS IS SATISFIED AND HENCE

$$z^2 - 1 = e^{i\pi/2} = i$$

$$z^2 = 1 + i = \sqrt{2}\, e^{i\pi/4}$$

$$z = \pm 2^{1/4} e^{i\pi/8}.$$

SECTION 3.2 # 11)

TAKE $\quad \log(z^2 + 2z + 3)$

$$\log w = \ln|w| + i[\operatorname{Arg}(w) + 2k\pi] \qquad k = 0, \pm 1, \pm 2, \ldots$$

let $w = z^2 + 2z + 3$ AND TRY $\operatorname{Log}(w)$, WITH $k = 0$.

$\rightarrow$ ANALYTIC IN $\mathbb{C} \setminus \{(-\infty, 0)\}$ IN W-PLANE.

the LINE $w = -x \quad x > 0$ GOES TO

$$z^2 + (2z) + 3 + x = 0$$

$$z = \frac{-2 \pm \sqrt{4 - 4(3+x)}}{2} \qquad z = \frac{-2 \pm \sqrt{-8 - 4x}}{2} = -1 \pm \sqrt{-2-x}$$

NOTICE THAT THERE IS NO $x < 0$ FOR WHICH $z = -1$. HENCE $z = -1$ IS NOT AN IMAGE

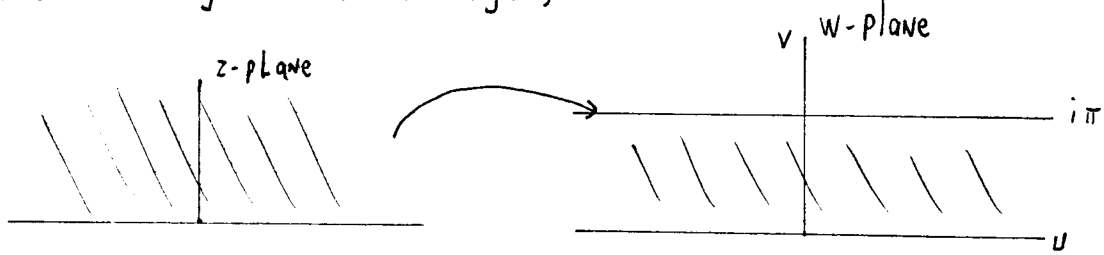
POINT ON $Re(W) \leq 0$.

$\longrightarrow$ $\log(z^2+2z+3)$ is ANALytic at $z=-1$.

$$\frac{d}{dz}\log(z^2+2z+3)\Big|_{z=-1} = 0$$

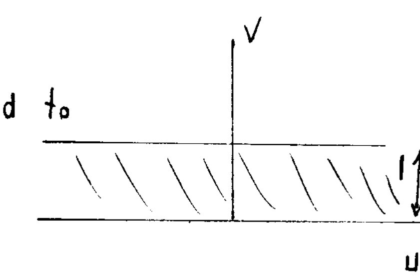
SECTION 3.2 # 15

TAKE $w = \log z = \ln|z| + i\,Arg(z)$



clearly $Arg(z) \in (0,\pi)$, $\ln|z| \in (-\infty,\infty)$.

NOW TAKE $W = \frac{1}{\pi}\log(z)$ THEN z-plane is MAPPED to



SECTION 3.3 # 1 d)

let $z = (1+i)^{1-i} = e^{(1-i)\log(1+i)}$

$$\log(1+i) = \log\sqrt{2} + i\pi/4 + 2K\pi i$$

$$z = e^{2K\pi i(1-i)}\, e^{\frac{1}{2}(1-i)\log 2 + i\pi/4(1-i)} = e^{\frac{1}{2}(1-i)\log 2 + i\pi/4 + \pi/4}\; e^{2K\pi i(1-i)}$$

OR

$$z = e^{\log(1+i)}\, e^{-i\log(1+i)}$$

$$z = (1+i)\, e^{-i[\log\sqrt{2} + i(\pi/4 + 2K\pi)]}$$

$$z = (1+i)\, e^{\pi/4 + 2K\pi - i\log\sqrt{2}} \qquad K = 0, \pm 1, \pm 2, \ldots$$

SECTION 3.3 # 3c

$z = (1+i)^{1+i} = e^{(1+i)\log(1+i)}$ FOR THE PRINCIPAL VALUE.

NOW $\log(1+i) = \log(\sqrt{2}) + i\pi/4$.

$$\longrightarrow \quad z = e^{\log(1+i) + i\log(1+i)}$$

$$z = (1+i)\, e^{i[\frac{1}{2}\log 2 + i\pi/4]}$$

$$z = (1+i)\, e^{-\pi/4 + i/2 \log 2}.$$

SECTION 3.2 # 8

$\sin z = 2$

$$z = \sin^{-1} 2 = -i \log\left[iz + (1-z^2)^{1/2} \right]\Big|_{z=2}$$

$$z = -i \log\left(i2 + (-3)^{1/2} \right) \qquad (-3)^{1/2} = \pm\sqrt{3}i$$

$+$ sign $\longrightarrow$ $z_+ = -i \log\left((2+\sqrt{3})i \right) = -i\left(\log(2+\sqrt{3}) + i\pi/2 + 2K\pi i \right)$

$$z_+ = \pi/2 + 2K\pi - i\log(2+\sqrt{3}) \qquad K = 0, \pm 1, \pm 2, \ldots$$

$-$ sign $\longrightarrow$ $z_- = -i \log\left((2-\sqrt{3})i \right) = -i\left(\log(2-\sqrt{3}) + i\pi/2 + 2K\pi \right)$

$$z_- = \pi/2 + 2K\pi - i\log(2-\sqrt{3}).$$

SECTION 3.2 # 9

(given in the notes)

SECTION 3.2 # 10

$\cos(z) = 2i$

$$z = -i\log\left(w + (w^2-1)^{1/2} \right) \qquad w = 2i$$

$$z = -i\log\left(2i + (-5)^{1/2} \right) \qquad (-5)^{1/2} = \pm i\,5^{1/2}.$$

$$\longrightarrow \quad z = -i\log\left((2 \pm \sqrt{5})i \right)$$

$$z_+ = -i\left[\log(2+\sqrt{5}) + i(\pi/2 + 2K\pi) \right] \qquad K = 0, \pm 1, \pm 2, \ldots$$

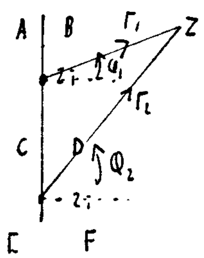
$$z_- = -i\left[\log(2-\sqrt{5}) + i(\pi/2 + 2K\pi) \right]$$

$$z_+ = \pi/2 + 2K\pi - i\log(2+\sqrt{5}) \qquad K = 0, \pm 1, \pm 2, \ldots$$

$$z_- = \pi/2 + 2K\pi - i\log(2-\sqrt{5}) \qquad K = 0, \pm 1, \pm 2, \ldots$$

SECTION 3.2 # 15b

$(4+z^2)^{1/2}$



$$z - 2i = r_1 e^{iQ_1}$$

$$z + 2i = r_2 e^{iQ_2}$$

$$\boxed{(4+z^2)^{1/2} = (r_1 r_2)^{1/2} e^{i(Q_1+Q_2)/2} \qquad Q_1 \in (-\pi/2, 3\pi/2) \quad Q_2 \in (-\pi/2, 3\pi/2)}$$

	Q_1	Q_2	$e^{i(Q_1+Q_2)/2}$
A	$\pi/2$	$\pi/2$	i
B	$\pi/2$	$\pi/2$	i
C	$3\pi/2$	$\pi/2$	-1
D	$-\pi/2$	$\pi/2$	1
E	$3\pi/2$	$3\pi/2$	$-i$
F	$-\pi/2$	$-\pi/2$	$-i$

SECTION 3.2 15 d)

let $w = z^3 - 1 = z^3(1 - 1/z^3)$

THEN $\zeta \equiv w^{1/3} = z\,(1 - 1/z^3)^{1/3}$

TAKE $(1 - 1/z^3)^{1/3} = e^{\frac{1}{3}\log(1 - 1/z^3)}$

this has a branch on $1 - 1/z^3 = X$ WITH $X < 0$

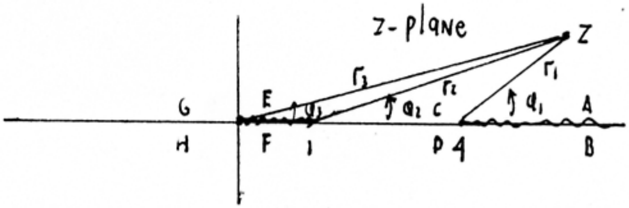
$$\rightarrow \quad z = \left(\frac{1}{1-X}\right)^{1/3} \quad \rightarrow \quad |z| \leq 1$$

THUS BRANCH CUT FOR $\log(1 - 1/z^3)$ which lies along negative real axis gets mapped to a curve inside $|z| = 1$.

HENCE $\zeta = z(1 - 1/z^3)^{1/3}$ is ANALYTIC INSIDE $|z| = 1$.

EXAMPLE

$w = \sqrt{z(z-1)(z-4)}$ WANT BRANCH BETWEEN $0 \leq z \leq 1$, $z \geq 4$.



$w = (r_1 r_2 r_3)^{1/2}\, e^{i(\varphi_1 + \varphi_2 + \varphi_3)/2}$

try $\varphi_1 \in (0, 2\pi)$

$\varphi_2 \in (0, 2\pi)$

$\varphi_3 \in (0, 2\pi)$

	φ_1	φ_2	φ_3	$e^{i(\varphi_1+\cdots}$
A	0	0	0	1
B	2π	2π	2π	$e^{3\pi i/2}$
C	π	0	0	$e^{i\pi/2}$
D	π	2π	2π	$e^{5i\pi/2}$
E	π	π	π	$e^{3\pi i/2}$
F	π	π	2π	$e^{4\pi i/2}$
G	π	π	π	$e^{3\pi i/2}$
H	π	π	π	$e^{3\pi i/2}$

FROM THE TABLE WE SEE THAT THE FUNCTION HAS JUMPS ACROSS A-B, E-F BUT IS CONTINUOUS ACROSS CD AND GH.