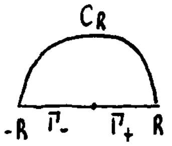
M301 HW #3 SOLUTIONS

SECTION 6.6 #1

let $I = \int_0^\infty \frac{\sqrt{x}}{x^2+1}\,dx.$

CONSIDER $\int_C \frac{z^{1/2}}{z^2+1}\,dz$



TAKE BRANCH CUT ALONG NEGATIVE Z AXIS.

$$\lim_{R\to\infty}\left(\int_{C_R} + \int_{\Gamma_+} + \int_{\Gamma_-}\right) = 2\pi i \, \mathrm{RES}\left[\frac{\sqrt{z}}{z^2+1}; i\right] = \frac{2\pi i}{2i} e^{\pi i/4} = \pi e^{\pi i/4}.$$

NOW $\lim_{R\to\infty} \int_{C_R} \frac{z^{1/2}}{z^2+1}\,dz \le K R^{-1/2} \to 0$ AS $R \to \infty$.

ON Γ_+: $z = r$ $\quad \int_{\Gamma_+} = \int_0^\infty \frac{r^{1/2}}{r^2+1}\,dr$

ON Γ_- $\quad z = re^{i\pi}$ $\quad \int_{\Gamma_-} = \int_\infty^0 \frac{r^{1/2} e^{i\pi/2}}{r^2+1}(-dr) = i\int_0^\infty \frac{r^{1/2}}{1+r^2}\,dr.$

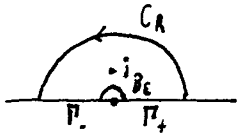
THUS $\quad (1+i)\, I = \pi e^{\pi i/4} = \pi\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$

$$\longrightarrow \quad I = \pi/\sqrt{2} \quad \text{WHERE} \quad I = \int_0^\infty \frac{r^{1/2}}{1+r^2}\,dr.$$

SECTION 6.6 #4

CONSIDER $\int_C \frac{z^\alpha}{(z^2+1)^2}\,dz$

$-1 < \alpha < 3$



double pole at $z = i$

take principal branch for z^α.

NOW ON C_R: $\left|\int_{C_R} \frac{z^\alpha}{(z^2+1)^2}\,dz\right| \le K R^{\alpha-3} \to 0$ AS $R\to\infty$ SINCE $\alpha < 3$.

NOW ON B_ε: $\left|\int_{B_\varepsilon} \frac{z^\alpha}{(z^2+1)^2}\,dz\right| \le K \varepsilon^{1+\alpha} \to 0$ AS $\varepsilon \to 0$ SINCE $\alpha > -1$.

NOW $$\int_{\Gamma_+} + \int_{\Gamma_-} = 2\pi i\, \mathrm{RES}\left[\frac{z^\alpha}{(z^2+1)^2}; i\right] = 2\pi i \lim_{z\to i}\left[\frac{d}{dz}\left(z^\alpha (z+i)^{-2}\right)\right]$$

$$\longrightarrow \int_{\Gamma_+} + \int_{\Gamma_-} = 2\pi i\left[\alpha z^{\alpha-1}(z+i)^{-2} - 2z^\alpha (z+i)^{-3}\right]\Bigg|_{z=i} = 2\pi i\left(\frac{i^\alpha}{4i} - \frac{\alpha\, i^\alpha}{4i}\right)$$

NOW ON Γ_+: $\quad z = r \quad \int_{\Gamma_+} = \int_0^\infty \frac{r^\alpha}{(r^2+1)^2}\,dr$

NOW ON Γ_- : $z = re^{i\pi}$ $\quad dz = -dr$

$$\int_{\Gamma_-} = \int_\infty^0 \frac{r^\alpha e^{i\pi\alpha}}{(r^2+1)^2}(-dr)$$

$$\to \int_0^\infty \frac{r^\alpha}{(r^2+1)^2}dr + e^{i\pi\alpha}\int_0^\infty \frac{r^\alpha}{(r^2+1)^2}dr = \frac{2\pi i}{4i}\, i^\alpha(1-\alpha) = \frac{\pi}{2}(1-\alpha)e^{\pi i\alpha/2}.$$

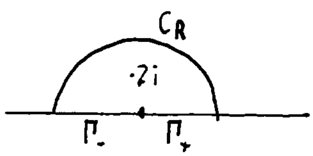
$$\to (1+e^{\pi i\alpha})\int_0^\infty \frac{r^\alpha}{(r^2+1)^2}dr = \frac{\pi}{2}(1-\alpha)e^{\pi i\alpha/2}.$$

$$\left(\frac{e^{\pi i\alpha/2}+e^{-\pi i\alpha/2}}{2}\right)\int_0^\infty \frac{r^\alpha}{(r^2+1)^2}dr = \frac{\pi}{4}(1-\alpha)$$

HENCE, $$\int_0^\infty \frac{r^\alpha}{(r^2+1)^2}dr = \frac{\pi(1-\alpha)}{4\cos(\pi\alpha/2)} \qquad -1<\alpha<3.$$

SECTION 6.6 # 8

CONSIDER $\int_C \frac{\log(z)}{z^2+4}dz$ $\qquad$ Log(z) principal branch



NOW $$\lim_{R\to\infty}\left(\int_{C_R}\frac{\log(z)}{z^2+4}dz + \int_{\Gamma_+} + \int_{\Gamma_-}\right) = 2\pi i\ \text{RES}\left[\frac{\log(z)}{z^2+4}; 2i\right]$$

$$\lim_{R\to\infty}\left|\int_{C_R}\frac{\log(z)}{z^2+4}dz\right| = K\frac{\log(R)}{R} \to 0 \text{ AS } R\to\infty.$$

NOW ON Γ_+ $\quad z = r$

$$\int_{\Gamma_+} = \int_0^\infty \frac{\log(r)}{r^2+4}dr$$

NOW ON Γ_- : $z = re^{i\pi}$

$$\int_{\Gamma_-} = \int_\infty^0 \frac{(\log(r)+i\pi)}{r^2+1}(-dr)$$

HENCE $$\int_0^\infty \frac{\log(r)}{r^2+4}dr + \int_0^\infty \frac{\log(r)}{r^2+4}dr + i\pi\int_0^\infty \frac{1}{r^2+1}dr = \frac{2\pi i}{4i}\left[\log(2)+i\pi/2\right]$$

FROM REAL PART WE GET $2\int_0^\infty \frac{\log(r)}{r^2+4}dr = \frac{\pi}{2}\log(2) \longrightarrow \int_{-\infty}^\infty \frac{\log|r|}{r^2+4}dr = \frac{\pi}{2}\log(2)$

SECTION 6.6 # 10

CONSIDER $\displaystyle\int_C \frac{\log(z)}{(z^2+1)^2}\,dz$ $\qquad z=i$ has a double pole.

NOW $\displaystyle\lim_{R\to\infty}\left(\int_{C_R}+\int_{\Gamma_+}+\int_{\Gamma_-}\right)=2\pi i\ \operatorname{RES}\left[\frac{\log(z)}{(z^2+1)^2}\ ;i\right].$

$$\lim_{R\to\infty}\left|\int_{C_R}\frac{\log(z)}{(z^2+1)^2}\,dz\right|\le K\,\frac{\log(R)}{R^3}\to 0 \text{ AS } R\to\infty.$$

NOW ON Γ_+: $\quad z=r \qquad \displaystyle\int_{\Gamma_+}=\int_0^\infty\frac{\log(r)}{(r^2+1)^2}\,dr.$

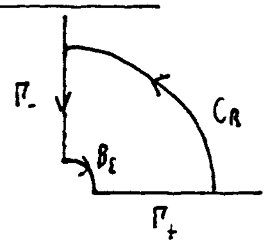
NOW ON Γ_-: $\quad z=re^{i\pi} \qquad \displaystyle\int_{\Gamma_-}=\int_\infty^0\frac{(\log(r)+i\pi)(-dr)}{(r^2+1)^2}=\int_0^\infty\frac{(\log(r)+i\pi)\,dr}{(r^2+1)^2}$

THUS $\displaystyle\int_{\Gamma_+}+\int_{\Gamma_-}=2\pi i\lim_{z\to i}\left[\frac{d}{dz}\left((z+i)^{-2}\log(z)\right)\right]=2\pi i\left[\frac{(z+i)^{-2}}{z}-\frac{2\log(z)}{(z+i)^3}\right]\Bigg|_{z=i}$

THUS $\displaystyle 2\int_0^\infty\frac{\log(r)}{(r^2+1)^2}\,dr+i\pi\int_0^\infty\frac{1}{(r^2+1)^2}\,dr=2\pi\left(-\frac{1}{4}+\frac{\pi i}{4}\right)$

equate real AND imaginary parts $\longrightarrow$ $\displaystyle\int_0^\infty\frac{\log(r)}{(r^2+1)^2}\,dr=-\pi/4\ ,\quad\int_0^\infty\frac{1}{(r^2+1)^2}\,dr=\pi/2.$

SECTION 6.6 # 11



$\displaystyle\int_C z^{\alpha-1}e^{-z}\,dz=0$ SINCE NO POLES.

NOW ON C_R: $\quad z=R(\cos\varphi+i\sin\varphi)$

$\operatorname{Re}(z)>0 \to |z^{\alpha-1}e^{-z}|\to 0$ AS $R\to\infty$.

HENCE $\displaystyle\lim_{R\to\infty}\int_{C_R}=0.$

NOW ON B_ε; $\quad z=\varepsilon e^{i\varphi} \qquad \displaystyle\left|\int_{B_\varepsilon}\right|=K\varepsilon^\alpha\to 0$ AS $\varepsilon\to 0$ IF $\alpha>0$.

THUS $\displaystyle\int_{\Gamma_-}+\int_{\Gamma_+}=0 \longrightarrow \int_{\Gamma_-}=-\int_{\Gamma_+}$

ON Γ_+ : $z = r$ $\qquad \displaystyle\int_{\Gamma_+} = \int_0^\infty r^{\alpha-1} e^{-r}\, dr = \Gamma(\alpha)$

ON Γ_- : $z = r e^{i\pi/2}$ $\qquad \displaystyle\int_{\Gamma_-} = \int_\infty^0 i\, r^{\alpha-1} e^{i(\alpha-1)\pi/2} e^{-ir}\, dr = \int_\infty^0 \frac{i}{i} r^{\alpha-1} e^{i\alpha\pi/2} e^{-ir}\, dr$

$$\longrightarrow \int_\infty^0 r^{\alpha-1} e^{i\alpha\pi/2} e^{-ir}\, dr = -\int_0^\infty r^{\alpha-1} e^{-r}\, dr$$

$$\rightarrow \int_0^\infty r^{\alpha-1} e^{-ir}\, dr = e^{-i\alpha\pi/2} \int_0^\infty r^{\alpha-1} e^{-r}\, dr.$$

NOW TAKE IMAGINARY PARTS OF BOTH SIDES

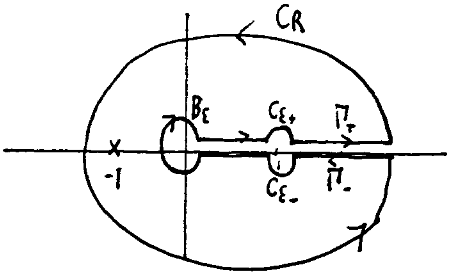
$$\longrightarrow \int_0^\infty r^{\alpha-1} \sin(-r)\, dr = \sin(-\alpha\pi/2) \int_0^\infty r^{\alpha-1} e^{-r}\, dr.$$

$$\longrightarrow \int_0^\infty r^{\alpha-1} \sin(r)\, dr = \sin(\alpha\pi/2)\, \Gamma(\alpha).$$

ECTION 6.6 # 6

$z^\alpha = r^\alpha e^{i\theta\alpha}$ $0<$

$\displaystyle\int_C \frac{z^\alpha}{z^2-1}\, dz.$ poles at $z = \pm 1$. BRANCH OF z^α ALONG positive Real AXIS.



NOW $\displaystyle\left|\int_{C_R} \frac{z^\alpha}{z^2-1}\, dz\right| \leq K R^{\alpha-1} \rightarrow 0$ AS $R \rightarrow \infty$ SINCE α

$\displaystyle\left|\int_{B_\varepsilon} \frac{z^\alpha}{z^2-1}\, dz\right| \leq K \varepsilon^{\alpha+1} \rightarrow 0$ AS $\varepsilon \rightarrow 0$ SINCE α

THUS $\displaystyle\int_{\Gamma_+} + \int_{C_{\varepsilon+}} + \int_{\Gamma_-} + \int_{C_{\varepsilon-}} = 2\pi i \operatorname{RES}\left[\frac{z^\alpha}{z^2-1}; -1\right] = \frac{2\pi i}{(-2)}\left(e^{\alpha\pi i}\right)$

NOW ON Γ_+ : $z = r$ $\qquad \displaystyle\int_{\Gamma_+} = \int_0^\infty \frac{r^\alpha}{r^2-1}\, dr.$

ON Γ_- : $z = r e^{2\pi i}$ $\qquad \displaystyle\int_{\Gamma_-} = \int_\infty^0 \frac{r^\alpha e^{2\pi i\alpha}}{r^2-1}\, dr$

THUS (1) $\displaystyle\left(1 - e^{2\pi i\alpha}\right)\int_0^\infty \frac{r^\alpha}{r^2-1}\, dr = -\lim_{\varepsilon\to 0}\int_{C_{\varepsilon+}} - \lim_{\varepsilon\to 0}\int_{C_{\varepsilon-}} - \pi i\, e^{\alpha\pi i}.$

NOW ON $C_{\varepsilon+}$: $z = 1 + \varepsilon e^{i\varphi}$. $\quad dz = \varepsilon i e^{i\varphi} d\varphi$

$$(1)^{\alpha} = e^{\alpha \log(1)} = 1.$$

HENCE

$$\lim_{\varepsilon \to 0} \int_{C_{\varepsilon+}} = \int_{\pi}^{0} \frac{i\varepsilon e^{i\varphi}\, d\varphi}{2(\varepsilon e^{i\varphi})} = \frac{-i\pi}{2}.$$

NOW ON $C_{\varepsilon-}$: $z = 1 + \varepsilon e^{i\varphi}$ $\quad dz = \varepsilon i e^{i\varphi} d\varphi$

$$(1)^{\alpha} = e^{\alpha \log 1} = e^{2\pi i \alpha}.$$

$$\lim_{\varepsilon \to 0} \int_{C_{\varepsilon-}} = \int_{2\pi}^{\pi} \frac{e^{2\pi i \alpha}\, i\varepsilon e^{i\varphi}}{2(\varepsilon e^{i\varphi})} d\varphi = -\frac{i\pi}{2} e^{2\pi i \alpha}.$$

NOW SUBSTITUTE IN (1)

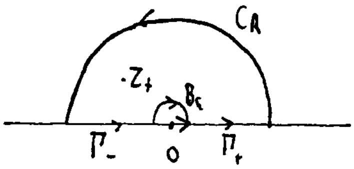
$$\left(1 - e^{2\pi i \alpha}\right) \int_0^{\infty} \frac{r^{\alpha}}{r^2 - 1} dr = -i\pi e^{\alpha\pi i} + \frac{i\pi}{2}\left(1 + e^{2\pi i \alpha}\right)$$

$$\frac{\left(e^{\pi i \alpha} - e^{-\pi i \alpha}\right)}{2i} \int_0^{\infty} \frac{r^{\alpha}}{r^2 - 1} dr = \frac{i\pi}{2i} - \frac{i\pi}{2(2i)}\left(e^{\pi i \alpha} + e^{-i\pi\alpha}\right)$$

$$\sin(\pi\alpha) \int_0^{\infty} \frac{r^{\alpha}}{r^2 - 1} dr = \frac{\pi}{2} - \frac{\pi}{2}\cos(\pi\alpha)$$

$$\int_0^{\infty} \frac{r^{\alpha}}{r^2 - 1} dr = \frac{\pi}{2}\left(\frac{1 - \cos(\pi\alpha)}{\sin(\pi\alpha)}\right).$$

SECTION 6.6 # 5



$z^2 + z + 1 = 0$ $\qquad z_{\pm} = \dfrac{-1 \pm \sqrt{-3}}{2}$

$$z_{\pm} = \frac{-1 \pm i\sqrt{3}}{2} \qquad z_+ = \frac{-1 + i\sqrt{3}}{2}.$$

$$\left| \int_{C_R} \frac{z^{\alpha-1}}{z^2 + z + 1} dz \right| \le K R^{\alpha - 2} \to 0 \text{ AS } R \to \infty \text{ SINCE } \alpha < 2.$$

$$\left| \int_{B_\varepsilon} \frac{z^{\alpha-1}}{z^2 + z + 1} dz \right| \le K \varepsilon^{\alpha} \to 0 \text{ AS } \varepsilon \to 0 \text{ SINCE } \alpha > 0.$$

NOW ON Γ_+: $z = r$

$$\int_{\Gamma_+} = \int_0^\infty \frac{r^{\alpha-1}}{r^2+r+1}\, dr.$$

NOW ON Γ_-: $z = re^{i\pi}$ $dz = -dr$.

$$\int_{\Gamma_-} = \int_\infty^0 \frac{r^{\alpha-1} e^{i\pi(\alpha-1)}}{r^2-r+1}(-dr) = -\int_0^\infty \frac{r^{\alpha-1}}{r^2-r+1}\, dr \left(e^{i\pi\alpha}\right).$$

THUS $$I_+ - e^{\pi i\alpha} I_- = 2\pi i \,\text{RES}\left[\frac{z^{\alpha-1}}{z^2+z+1}; z_+\right]. \qquad I_\pm = \int_0^\infty \frac{r^{\alpha-1}}{r^2\pm r+1}\, dr.$$

NOW $$I_+ - e^{\pi i\alpha} I_- = \frac{2\pi i}{2z_+ + 1}(z_+)^{\alpha-1} \qquad 2z_+ + 1 = i\sqrt{3}. \quad z_+ = e^{2\pi i/3}$$

$$\rightarrow \quad I_+ - e^{\pi i\alpha} I_- = \frac{2\pi}{\sqrt{3}} e^{2\pi i(\alpha-1)/3}.$$

TAKE REAL PARTS $$I_+ - \cos(\pi\alpha) I_- = \frac{2\pi}{\sqrt{3}}\cos\left[\frac{2\pi}{3}(\alpha-1)\right].$$

TAKE IMAG. PARTS $$-\sin(\pi\alpha) I_- = \frac{2\pi}{\sqrt{3}}\sin\left(2\pi(\alpha-1)/3\right).$$

HENCE $$I_+ + \cot(\pi\alpha)\sin\left(2\pi(\alpha-1)/3\right)\frac{2\pi}{\sqrt{3}} = \cos\left[\frac{2\pi}{3}(\alpha-1)\right]\frac{2\pi}{\sqrt{3}}$$

$$\rightarrow \quad I_+ = \frac{2\pi}{\sqrt{3}}\left[\cos\left(\frac{2\pi}{3}(\alpha-1)\right) - \cot(\pi\alpha)\sin\left(\frac{2\pi(\alpha-1)}{3}\right)\right].$$

$$\rightarrow \quad I_+ = \frac{2\pi}{\sqrt{3}}\left[\frac{\cos\left(\frac{2\pi}{3}(\alpha-1)\right)\sin(\pi\alpha)}{\cos(\alpha\pi)} - \sin\left(\frac{2\pi}{3}(\alpha-1)\right)\right]\cot(\alpha\pi)$$

$$I_+ = \frac{2\pi}{\sqrt{3}}\left[\cos\left(\frac{2\pi}{3}(\alpha-1)\right)\sin(\pi\alpha) - \sin\left(\frac{2\pi}{3}(\alpha-1)\right)\cos(\alpha\pi)\right]\csc(\alpha\pi)$$

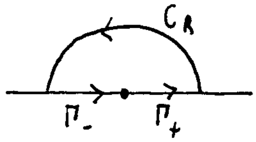
$$I_+ = \frac{2\pi}{\sqrt{3}}\sin\left[\pi\alpha - (\alpha-1)\frac{2\pi}{3}\right]\csc(\pi\alpha) = \frac{2\pi}{\sqrt{3}}\sin\left(\frac{\alpha\pi}{3}+\frac{2\pi}{3}\right)\csc(\pi\alpha)$$

BUT $\sin(z) = \cos(z \mp \pi/2)$

$$I_+ = \frac{2\pi}{\sqrt{3}}\cos\left[\frac{\alpha\pi}{3}+\frac{\pi}{6}\right]\csc(\alpha\pi) \qquad 0<\alpha<2.$$

EXTRA # 1

$$I = \int_0^\infty \frac{x^{1/2} \ln x}{x^2+1}\, dx.$$



NOW ON Γ_+: $z = r$ $\quad \int_{\Gamma_+} = \int_0^\infty \frac{r^{1/2} \ln r}{r^2+1}\, dr$

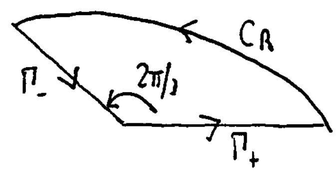
NOW ON Γ_-: $z = re^{i\pi}$ $\quad \int_{\Gamma_-} = \int_\infty^0 \frac{r^{1/2} e^{i\pi/2} [\log r + i\pi]}{r^2+1}(-dr).$

THUS, $\int_{\Gamma_+} + \int_{\Gamma_-} = 2\pi i \, \mathrm{RES}\left[\frac{z^{1/2} \log z}{z^2+1} ; i \right] = \frac{2\pi i}{2i} e^{\pi i/4}\, i\pi/2 = \frac{i\pi^2}{2} e^{\pi i/4}.$

NOW $\int_0^\infty \frac{r^{1/2} \log r}{r^2+1}\, dr + i \int_0^\infty \frac{r^{1/2} (\log r + i\pi)}{r^2+1}\, dr = \frac{i\pi^2}{2} e^{\pi i/4}$

TAKE IMAGINARY PART $\int_0^\infty \frac{r^{1/2} \log r}{r^2+1}\, dr = \frac{\pi^2}{2} \frac{\sqrt{2}}{2} = \frac{\sqrt{2}\pi}{4}$

EXTRA # 2 $\quad I = \int_0^\infty \frac{x \log x}{1+x^3}\, dx.$



pole at $z = e^{\pi i/3}.$

$\int_C \frac{z \log z}{1+z^3}\, dz$

ON Γ_+: $z = r$ $\quad \int_{\Gamma_+} = \int_0^\infty \frac{r \log r}{r^3+1}\, dr$

NOW ON Γ_-: $z = re^{2\pi i/3}$, $dz = e^{2\pi i/3}\, dr$ $\quad \int_{\Gamma_-} = \int_\infty^0 e^{2\pi i/3} \frac{re^{2\pi i/3} [\log r + 2\pi i/3]}{r^3+1}\, dr$

$$\rightarrow \int_0^\infty \frac{r \log r}{r^3+1}\, dr \left(1 - e^{4\pi i/3}\right) - \frac{2\pi i}{3} e^{4\pi i/3} \int_0^\infty \frac{r}{r^3+1}\, dr = 2\pi i \, \mathrm{RES}\left[\frac{z \log(z)}{z^3+1} ; e \right.$$

$$\rightarrow \int_0^\infty \frac{r \log r}{r^3+1}\, dr \left(1 - e^{4\pi i/3}\right) - \frac{2\pi i}{3} e^{4\pi i/3} \int_0^\infty \frac{r}{r^3+1}\, dr = \frac{2\pi i \left[e^{\pi i/3}\, \pi i/3 \right]}{3e^{2\pi i/3}} = -\frac{2\pi^2}{9} e^{-\pi i/}$$

$$\int_0^\infty \frac{r \log r}{r^3+1}\, dr \left(e^{-2\pi i/3} - e^{2\pi i/3}\right) - \frac{2\pi i}{3} e^{2\pi i/3} \int_0^\infty \frac{r}{r^3+1}\, dr = \frac{2\pi^2}{9}.$$

$$\rightarrow -2i \sin(2\pi/3) \int_0^\infty \frac{r \log r}{r^3+1}\, dr - \frac{2\pi i}{3} e^{2\pi i/3} \int_0^\infty \frac{r}{r^3+1}\, dr = \frac{2\pi^2}{9}.$$

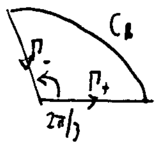
TAKE IMAGINARY PART

$$-2\sin(2\pi/3)\int_0^\infty \frac{r\log r}{r^3+1}\,dr - \frac{2\pi}{3}\cos(2\pi/3)\int_0^\infty \frac{r}{r^3+1}\,dr = 0$$

$$\int_0^\infty \frac{r\log r}{r^3+1}\,dr = -\frac{\pi}{3}\cot(2\pi/3)\int_0^\infty \frac{r}{r^3+1}\,dr \qquad \cot(2\pi/3) = \frac{-1/2}{\sqrt{3}/2} = -\frac{1}{\sqrt{3}}$$

$$(1) \qquad \int_0^\infty \frac{r\log r}{r^3+1}\,dr = \frac{\pi}{3\sqrt{3}}\int_0^\infty \frac{r}{r^3+1}\,dr.$$

NOW $\displaystyle\int_C \frac{z}{z^3+1}\,dz.$



$$\int_C \frac{z}{z^3+1}\,dz = 2\pi i \,\mathrm{Res}\left[\frac{z}{z^3+1};\, e^{\pi i/3}\right] = \frac{2\pi i}{3e^{\pi i/3}}.$$

ON Γ_+: $z = r \quad \longrightarrow \quad \displaystyle\int_{\Gamma_+} = \int_0^\infty \frac{r}{r^3+1}\,dr$

ON Γ_-: $z = re^{2\pi i/3} \quad \longrightarrow \quad \displaystyle\int_{\Gamma_-} = e^{4\pi i/3}\int_\infty^0 \frac{r}{r^3+1}\,dr.$

$$\rightarrow \left(1 - e^{4\pi i/3}\right)\int_0^\infty \frac{r}{r^3+1}\,dr = \frac{2\pi i}{3}\,e^{-\pi i/3}$$

$$\frac{\left(e^{-2\pi i/3} - e^{2\pi i/3}\right)}{2i}\int_0^\infty \frac{r}{r^3+1}\,dr = \frac{\pi}{3}\,e^{-\pi i} = -\frac{\pi}{3}$$

$$\rightarrow \qquad \int_0^\infty \frac{r}{r^3+1}\,dr = \frac{\pi}{3\sin(2\pi/3)}$$

SUBSTITUTE IN (1)

$$\int_0^\infty \frac{r\log r}{r^3+1}\,dr = \frac{\pi}{3\sqrt{3}}\,\frac{\pi}{3\sin(2\pi/3)} = \frac{\pi^2}{9\sqrt{3}\,(\sqrt{3}/2)}$$

$$\int_0^\infty \frac{r\log r}{r^3+1}\,dr = 2\pi^2/27.$$