Math 305 Review Questions

(*Received 03 December 2010*)

Q1: Calculate the following integrals. In each case the simple closed curve C is oriented counterclockwise.

- $I = \int_C \frac{\sin z}{z^2(z-4)}\, dz$. Here C is $|z| = 2$. (**Answer:** $I = -i\pi/2$.)
- $I = \int_C \frac{e^{zt}}{z^2(z+1)}\, dz$. Here C is $|z| = 2$. (**Answer:** $I = 2\pi i(e^{-t} + t - 1)$.)
- $I = \int_C z/(z^3 - 9)\, dz$. Here C is $|z| = 4$. (**Answer:** $I = 0$.)
- $I = \int_C e^z/((z-\pi)\tan(z))\, dz$. Here C is $|z| = 2$. (**Answer:** $I = -2i$.)
- $I = \int_C z^{-7}(1 - \cos(z))^2\, dz$. Here C is $|z| = 1$. (**Answer:** $I = -\pi i/12$.)
- $I = \int_C e^{1/z}\sin(1/z)\, dz$. Here C is $|z| = 1$. (**Answer:** $I = 2\pi i$.)

Q2: Let $P(z)$ and $Q(z)$ be two polynomials in z, with $\deg(Q) = N \geq 2$. Assume that $\deg(P) \leq N - 2$, and that $Q(z)$ has simple zeroes at distinct points z_j for $j = 1, \ldots, N$. Let C be a simple closed curve oriented counterclockwise, and assume that z_j are all inside C. Prove that

$$\sum_{j=1}^{N} \text{Res}\left(\frac{P}{Q}\,;\, z_j\right) = 0$$

(Hint: define $I = \int_C P(z)/Q(z)\, dz$. Then, $I = 0$ by deforming to a large circle $|z| = R$, and letting $R \to \infty$. Then, use the residue theorem inside the curve.)

Q3: Let $P(z)$ be a polynomial of degree N with N distinct roots z_j for $j = 1, \ldots, N$. Let C be a simple closed curve oriented counterclockwise, and assume that there are no roots of $P(z) = 0$ on the curve C.

- Using the residue theorem, show that $I = (2\pi i)^{-1} \int_C P'(z)/P(z)\, dz = M$, where M is the number of zeroes of $P(z)$ inside C.
- Suppose that you can verify that for some choice of C that $I = 1$ (I defined above). Then, what does the calculation of the integral $J = (2\pi i)^{-1} \int_C zP'(z)/P(z)\, dz$ tell you? (Hint:: again use the residue theorem) **Answer:** $J = z_*$ where z_* is the only root of $P(z) = 0$ inside C. In other words, calculating J determines the root, since $I = 1$ means that there is only one root inside C. In practice we would compute I and J as a line integral using a numerical quadrature, i.e. trapezoidal rule)

Q4: Define the function $f(z)$ by the infinite series

$$f(z) = \sum_{k=0}^{\infty} k^4 z^k/(2^{2k})\,.$$

Let C be a simple closed curve oriented counterclockwise. Then, calculate the following two integrals:

- $I_1 = \int_C \cos(iz)f(z)\,dz$ with C the curve $|z-1| = 1$
- $I_2 = \int_C z^{-3}f(z)\,dz$ with C the curve $|z| = \pi$.

(**Answer for I_1:** $I_1 = 0$. Using the ratio test, the series for $f(z)$ converges in $|z| < 4$. The result must be analytic in $|z| < 4$ (theorem in class). Thus, since $|z| < 4$ contains the curve C, and the product of two analytic functions $\cos(iz)$ and $f(z)$ are analytic, we get by Cauchy-Goursat that $I_1 = 0$.) (**Answer for I_2:** $I_2 = 2\pi i$ by simply identifing the residue term at $z = 0$.)

Q5: Let $f(z) = e^{iz^2}$ calculate the maximum M of $|f(z)|$ over the closed disk $|z| \le 4$. (**Answer:** by maximum modulus principle, and by evaluating on $z = 4e^{i\theta}$, we conclude that $M = e^{16}$, which is attained at two points on the boundary $|z| = 4$.)

Q6: Find the roots of $z^5 + 4 = 0$. (**Answer:** $z = 4^{1/5}e^{i\pi(1+2k)/5}$ for $k = 0, 1, 2, 3, 4$.)

Q7: Describe the set of z for which $|z| + |z-2| = 4$. (**Answer:** the ellipse $(x-1)^2/4 + y^2/3 = 1$.)

Q8: Show that

$$1 + \cos\theta + \cos(2\theta) + \cdots \cos(n\theta) = \frac{1}{2} + \frac{\sin\left[(n+1/2)\theta\right]}{2\sin(\theta/2)}\,.$$

(Hint: start with a finite geometric series $1 + w + \cdots w^{N-1} = (1-w^N)/(1-w)$, and set $w = e^{i\theta}$. Then, take the real part of both sides of the equation.)

Q9: Expand

$$f(z) = \frac{1}{z(1-z)}$$

in a Laurent series that converges in each of the following two regions:

- $0 < |z| < 1$
- $|z| > 1$. Then, using the Laurent series, calculate $\int_C f(z)\,dz$, where C is the circle $|z| = 4$ oriented counterclockwise.

(**Answer:** this is easy; see notes for similar example).

Q10: Show how to calculate the following integral $I = \int_0^{2\pi} [\cos(\theta)]^m \cos(n\theta)\,d\theta$, by integrating

$$J = (2\pi i)^{-1} 2^{-m} \int_C z^{-n}\left(z + \frac{1}{z}\right)^m dz$$

where C is $|z| = 1$ oriented counterclockwise.

Q11: Using contour integration verify the results of the following integrals

- $\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+1}\,dx = \pi/e$.
- $I = \int_0^{\infty} \frac{\ln(x)}{1+x^2}\,dx = 0$
- $I = \int_0^{\infty} \frac{[\ln(x)]^2}{1+x^2}\,dx = \pi^3/8$.
- $I = \int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^2}\,dx = \pi/2$
- $I = \int_{-\infty}^{\infty} \frac{\cos(2x)}{x-i\pi}\,dx = i\pi e^{-2\pi}$.
- $I = \int_{-\infty}^{\infty} \frac{1}{1+x^4}\,dx = \pi/\sqrt{2}$

- $I = \int_0^\infty \frac{\ln(x)}{x^2+5x+6}\, dx = \ln(3/2)$.
- $I = \int_0^\infty \frac{x^{1/3}}{(x+1)^2}\, dx = \frac{\pi}{3\sin(\pi/3)}$.