M305 REVIEW QUESTIONS (SOME NOTES)

Q1 (i) $I = \int_C \frac{\sin z}{z^2(z-4)}\,dz$: $C: |z| = 2$ C.C.

ONLY $z=0$ IS INSIDE C; NOTE $z=0$ IS A SIMPLE POLE.

L-SERIES NEAR $z=0$;

$$\frac{1}{z^2(z-4)}\sin z \sim \frac{(z - z^3/6 + \dots)}{-z^2 4(1-z/4)} \sim -\frac{1}{4z}(1 - z^2/6)(1+z/4) \sim -\frac{1}{4z} + \dots$$

THUS $a_{-1} = -1/4 \quad \to \quad I = 2\pi i\, a_{-1} = -i\pi/2$.

(ii) $I = \int_C \frac{e^{zt}}{z^2(z+1)}\,dz$: $C: |z| = 2$, C.C.

NOTICE $z=0$ AND $z=-1$ ARE INSIDE C.

$\cdot$ $z=0$ is double pole, $z=-1$ simple pole.

$$I = 2\pi i \,\mathrm{RES}\left[\frac{e^{zt}}{z^2(z+1)};0\right] + 2\pi i\,\mathrm{RES}\left[\frac{e^{zt}}{z^2(z+1)};-1\right]$$

$$= 2\pi i \lim_{z\to 0}\frac{d}{dz}\left[(z+1)^{-1}e^{zt}\right] + 2\pi i\, e^{-t}$$

$$= 2\pi i\left[\left(-(z+1)^{-2}e^{zt} + t(z+1)^{-1}e^{zt}\right)\Big|_{z=0}\right] + 2\pi i\, e^{-t}$$

$$I = 2\pi i\left[-e^0 + t\,e^0\right] + 2\pi i\,e^{-t} = 2\pi i\left(e^{-t} + t - 1\right).$$

(iii) $I = \int_C \frac{z}{z^3-9}\,dz \quad C: |z| = 4.$

$\cdot$ simple poles are at $|z| = 9^{1/3} < 4$, AND SO ARE ALL INSIDE C.

$\cdot$ DEFORM TO C_R WITH $|z| = R > 4$ AND LET $R \to \infty$.

$$I = \lim_{R\to\infty}\int_{C_R}\frac{z}{z^3-9}\,dz = 0 \quad \text{SINCE} \quad \left|\int_{C_R}\frac{z}{z^3-9}\,dz\right| \le \frac{R}{R^3+9}(2\pi R) \to 0$$

AS $R\to\infty$.

(iv) $\quad I = \int_C \frac{e^z}{(z-\pi)\tan z}\,dz. \qquad C: |z| = 2 \quad$ c.c.

- $z = k\pi$, $k = \pm 1, \pm 2$ are poles (outside C)
- $z = 0$ is a simple pole inside C. $\qquad p = e^z/(z-\pi) \quad Q = \tan z$

Thus $\quad I = 2\pi i \operatorname{Res}\left(\frac{e^z/(z-\pi)}{\tan z}; 0\right) = 2\pi i \left(\frac{e^0/(0-\pi)}{\sec^2(0)} \right) = +2\pi i\left(-\frac{1}{\pi}\right)$

so $I = -2i$.

(v) $\quad I = \int_C z^{-7}[1-\cos(z)]^2\,dz \qquad C: |z| = 1.$

$z = 0$ is a pole.

Near $z = 0$

$$\frac{1}{z^7}[1-\cos z]^2 = \frac{1}{z^7}\left[1-\left(1-z^2/2! + z^4/4! + \dots\right)\right]^2$$

$$= \frac{1}{z^7}\left[\left(z^2/2! - z^4/4! + \dots\right)^2\right]$$

so $\quad \frac{1}{z^7}[1-\cos z]^2 \sim \frac{1}{z^7}\left[z^4/2^2 - 2z^2z^4/2!\,4! + \dots\right] = \frac{1}{2^2z^3} - \frac{1}{4!\,z} + \dots$

Thus $z = 0$ is a pole of order 3 and $a_{-1} = -1/4!$.

Thus $\quad I = 2\pi i\left(-1/4!\right) = -\pi i/12.$

(vi) $\quad I = \int_C e^{1/z}\sin(1/z)\,dz \qquad C: |z| = 1$

$z = 0$ is an essential singularity.

The L-series valid for $|z| > 0$ is

$$e^{1/z}\sin(1/z) = \left(1 + \frac{1}{z} + \frac{1}{z^2} + \dots\right)\left(\frac{1}{z} - \frac{1}{3!\,z^3} + \dots\right) = \frac{1}{z} + \dots$$

Thus $a_{-1} = 1 \longrightarrow I = 2\pi i\, a_{-1} = 2\pi i.$

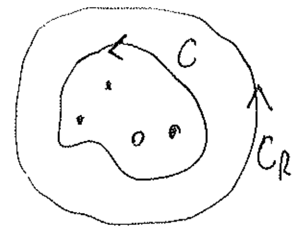
Q2 LET $f(z) = \frac{P(z)}{Q(z)}$ $\deg Q = N \geq 2$
$\deg P \leq N-2.$

THUS $\frac{|P(z)|}{|Q(z)|} \leq \frac{d}{R^2}$ ON $|z| = R >> 1$ WITH d = CONSTANT.

NOW LET $I = \int_C \frac{P(z)}{Q(z)}\, dz.$

• SUPPOSE C IS CHOSEN WHICH ENCLOSES ALL ZEROS OF $Q(z)$.

WE DEFORM C TO $C_R : |z| = R$ AS SHOWN



AND SINCE $f(z)$ IS ANALYTIC BETWEEN C AND C_R

WE OBTAIN BY CAUCHY-INTEGRAL THEOREM,

$$I = \int_C \frac{P}{Q}\, dz = \int_{C_R} \frac{P}{Q}\, dz.$$

NOW LET $R \to \infty$ AND USE $\left| \int_{C_R} \frac{P}{Q}\, dz \right| \leq \max_{z \text{ on } C_R} \left| \frac{P}{Q} \right| 2\pi R = \frac{d}{R^2}(2\pi R) \to 0$
AS $R \to \infty$.

THUS $I = 0.$ $(*)$

• ON THE OTHER HAND BY THE RESIDUE THEOREM, SINCE ALL ZEROS OF $Q(z) = 0$ ARE INSIDE C (AND labelled by $z_1, \ldots, z_N$)

THEN $$I = \int_C \frac{P}{Q}\, dz = 2\pi i \sum_{j=1}^{N} \operatorname{RES}\left(\frac{P}{Q}; z_j\right) \quad (**)$$

EQUATING $(*)$ AND $(**)$ gives $\sum_{j=1}^{N} \operatorname{RES}\left(P/Q ; z_j \right) = 0.$

<u>Q3</u> SUPPOSE $z_1, .., z_N$ ARE DISTINCT ROOTS OF $P(z) = 0$.

THEN BY RESIDUE THEOREM

$$I = \frac{1}{2\pi i} \int_C \frac{P'(z)}{P(z)}\, dz = 2\pi i \sum_{\substack{j \\ z_j \text{ inside } C}} \text{RES}\left(\frac{1}{2\pi i} \frac{P'(z)}{P(z)} ; z_j \right)$$

SINCE z_j IS A SIMPLE POLE,

$$I = 2\pi i \sum_{\substack{j \\ z_j \text{ inside } C}} \frac{1}{2\pi i} \frac{P'(z_j)}{P(z_j)} = \sum_{\substack{j \\ z_j \text{ inside } C}} (1) = M = \text{\# ZEROES OF } P(z) \text{ inside } C.$$

- NOW IF $I = 1$ THEN $M = 1$. i.e. $\exists$ EXACTLY ONE ZERO OF $P(z) = 0$ inside C. TO calculate this ROOT, we write

$$J = \frac{1}{2\pi i} \int_C z \frac{P'(z)}{P(z)}\, dz = 2\pi i \text{ RES}\left(\frac{z P'(z)}{P(z) 2\pi i} ; z^* \right)$$

$$= z^* \frac{P'(z^*)}{P'(z^*)} = z^*.$$

THUS J DETERMINES THIS ROOT !!

<u>Q4</u> $$F(z) = \sum_{k=0}^{\infty} \frac{k^4 z^k}{2^{2k}}. \qquad a_k = \frac{k^4 z^k}{2^{2k}}$$

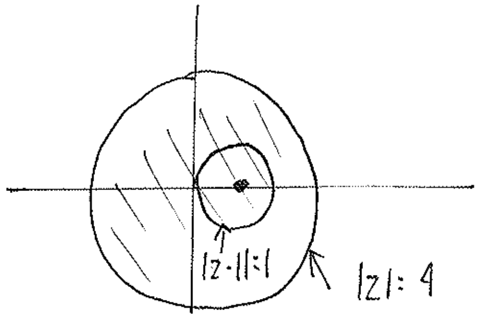
BY RATIO TEST $$\left| \frac{a_{k+1}}{a_k} \right| = \left| \frac{(k+1)^4 z^{k+1}}{2^{2(k+1)}} \right| \left| \frac{2^{2k}}{k^4 z^k} \right| =$$

need $$\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| < 1 \rightarrow \frac{|z|}{4} < 1 \longrightarrow |z| < 4.$$

THUS THE SERIES CONVERGES IN $|z| < 4$, AND DIVERGES FOR $|z| > 4$.

IN THE REGION $|z| < 4$, $F(z)$ DEFINES AN ANALYTIC FUNCTION.

- $I_1 = \int_C \cos(iz)\, f(z)\, dz \qquad C: |z-1| = 1$



f analytic in shaded region

SINCE C IS <u>INSIDE</u> $|z| \leq 4 \rightarrow I_1 = 0$ BY CAUCHY integral theorem.

- $I_2 = \int_C \frac{1}{z^3} f(z)\, dz \qquad C; |z| = \pi.$

NOW $f(z)$ IS ANALYTIC IN $|z| \leq \pi$ SINCE $\pi < 4$.

ONLY SINGULARITY IS AT $z=0$ WHICH IS A POLE OF order 3.

we calculate Residue by L-SERIES

$$\frac{1}{z^3}\left[f(0) + z f'(0) + z^2 \frac{f''(0)}{2} + \ldots \right] \Rightarrow \frac{f''(0)}{2} = a_{-1}$$

NOW $f(z) \sim \frac{z}{4} + \frac{z^2 (2)^4}{2^4} + \ldots \Rightarrow f''(0) = 2.$

SO $a_{-1} = 1.$

THUS $I_2 = 2\pi i.$

<u>Q5</u> BY MAX-MODULUS PRINCIPLE

$$\max_{|z|\leq 4} |e^{iz^2}| = \max_{|z|=4} |e^{iz^2}| = \max_{0\leq\varphi\leq 2\pi} \left| e^{16i[\cos(2\varphi) + i\sin(2\varphi)]} \right|$$

$$= \max_{0\leq\varphi\leq 2\pi} e^{-16\sin(2\varphi)} = e^{16}$$

ATTAINED WHEN $\sin(2\varphi) = -1.$

Q6 : $z^5 = -4 = 4e^{i\pi}$.

NOW $z = \rho e^{i\pi} \longrightarrow \rho^5 = 4, \quad \rho = 4^{1/5}$.

$5\phi = \pi i + 2k\pi i \qquad k = 0, 1, 2, 3, 4.$

so $\quad z_k = 4^{1/5}\, e^{(\pi + 2k\pi)i/5}, \quad k = 0, 1, 2, 3, 4.$

Q7 $\quad |z| + |z-2| = 4$

see page F5-F6 of notes on "FUNDAMENTALS".

THIS is geometric characterization of an ellipse.

Q8 $\quad 1 + w + \dots + w^n = \dfrac{w^{n+1} - 1}{w - 1}$.

Let $w = e^{i\varphi}$ AND TAKE REAL PARTS

$$1 + e^{i\varphi} + \dots + e^{in\varphi} = \frac{\left(e^{(n+1)i\varphi} - 1\right)}{\left(e^{i\varphi} - 1\right)} \frac{\left(e^{-i\varphi/2}\right)}{\left(e^{-i\varphi/2}\right)} = \frac{e^{i(n+1/2)\varphi} - e^{-i\varphi/2}}{e^{i\varphi/2} - e^{-i\varphi/2}}$$

$$1 + e^{i\varphi} + \dots + e^{in\varphi} = \frac{e^{i(n+1/2)\varphi} - e^{-i\varphi/2})}{2i\sin(\varphi/2)}$$

take REAL PARTS

$$1 + \cos\varphi + \dots + \cos(n\varphi) = \mathrm{RE}\left[\frac{-i}{2\sin(\varphi/2)}\left(e^{i(n+1/2)\varphi} - e^{-i\varphi/2}\right)\right]$$

BUT $\mathrm{RE}(-i\zeta) = \mathrm{IM}(\zeta)$ so $\quad 1 + \cos\varphi + \dots + \cos(n\varphi) = \mathrm{IM}\left[\dfrac{e^{i(n+1/2)\varphi} - e^{-i\varphi/2}}{2\sin(\varphi/2)}\right]$

THIS GIVES $\quad 1 + \cos\varphi + \dots + \cos(n\varphi) = \dfrac{1}{2} + \dfrac{\sin((n+1/2)\varphi)}{2\sin(\varphi/2)}$ ..

Q9 $\quad f(z) = \dfrac{1}{z(1-z)}$

(i) L-SERIES IN $0<|z|<1$. USE $\dfrac{1}{1-z} = 1 + z + z^2 + \dots$ IN $|z|<1$

$$f(z) = \frac{1}{z}\left[1 + z + z^2 + \dots\right] = \sum_{j=0}^{\infty} \frac{z^j}{z} = \sum_{j=0}^{\infty} z^{j-1}.$$

(ii) IN $|z|>1$:

$$f(z) = \frac{1}{-z\left[z(1-1/z)\right]} = -\frac{1}{z^2}\sum_{j=0}^{\infty} 1/z^j \quad \text{IN } |z|>1.$$

SO $\quad f(z) = -\displaystyle\sum_{j=0}^{\infty} \frac{1}{z^{j+2}} \quad$ IN $|z|>1$.

NOW $\displaystyle\int_C f(z)\,dz = 0 \quad$ WITH $\quad C: |z| = 4 > 1$

SINCE $\quad f(z) \sim \dfrac{1}{z^2} + O\left(1/z^3\right) + \dots \quad$ (NO $1/z$ term).

Q10 $\quad I = \displaystyle\int_0^{2\pi} (\cos(\varphi))^m \cos(n\varphi)\,d\varphi. \qquad \cos(n\varphi) = \mathrm{RE}\left(e^{in\varphi}\right).$

WE WRITE $\quad I = \mathrm{RE}\left[\displaystyle\int_0^{2\pi} e^{in\varphi} (\cos\varphi)^m \, d\varphi\right]$

NOW LET $\cos\varphi = (z + 1/z)/2$. THUS, $\dfrac{dz}{iz} = d\varphi$.

$$I = \mathrm{RE}\left[\int_C \frac{z^n}{2^m}\left(z + 1/z\right)^m \frac{dz}{iz}\right] \qquad C: |z|=1 \text{ C.C.}$$

$$I = \mathrm{RE}\left[\frac{-i}{2^m}\int_C \frac{z^n}{z}\left(z + 1/z\right)^m dz\right] \qquad \text{NOTE: } \mathrm{RE}(-i\zeta) = \mathrm{IM}(\zeta)$$

$$I = \mathrm{IM}\left[\frac{1}{2^m}\int_C z^{n-1}\left(z + 1/z\right)^m dz\right]$$

THUS $\quad I = \dfrac{1}{2^{2m}} \, \mathrm{IM}\left[2\pi i \; \mathrm{RES}\left(z^{n-1}\left(z + 1/z\right)^m ; 0 \right)\right]$

REMARK (i) IF $n-1 > m$ THEN $a_{-1} = \mathrm{RES}\left[z^{n-1}\left(z+1/z\right)^m ; 0\right] = 0$

SINCE NO TERM OF FORM $(1/z)\,a_{-1}$ IN L-SERIES. $\rightarrow I = 0$ IF $n > m+1$.

(ii) THUS WE REQUIRE THAT $0 \le n \le m+1$.

NOW $(a+b)^m = \sum_{j=0}^{m} a^{m-j} b^j \binom{m}{j}$ BY BINOMIAL SERIES

let $a = z$, $b = 1/z$ $\longrightarrow$ $(z+1/z)^m = \sum_{j=0}^{m} z^{m-2j}\binom{m}{j}$

NOW $z^{n-1}(z+1/z)^m = z^{n-1}\sum_{j=0}^{m} z^{m-2j}\binom{m}{j}$. $\qquad \binom{j}{k} \equiv \dfrac{j!}{k!(j-k)!}$

RESIDUE TERM IS WHEN $n-1+m-2j = -1 \rightarrow n+m = 2j$.

THUS IF $n+m = \text{EVEN} \longrightarrow j = \dfrac{n+m}{2}$ AND $a_{-1} = \binom{m}{(m+n)/2}$

HENCE $I = \begin{cases} \dfrac{1}{2^{2m}}\, 2\pi \dbinom{m}{(m+n)/2} & \text{WHEN } 0 \le n \le m+1 \text{ AND } n+m = \text{EVEN}. \\ 0 & \text{OTHERWISE}. \end{cases}$

THIS PROBLEM IS interesting, BUT TOO HARD FOR THE FINAL.