MATH 300 FUNDAMENTALS

DEFINE $i = \sqrt{-1}$. A COMPLEX NUMBER Z HAS THE FORM

$$z = a + ib$$

WHERE a AND b ARE REAL.

(i) TWO COMPLEX NUMBERS $z_1 = a_1 + ib_1$ AND $z_2 = a_2 + ib_2$ ARE EQUAL IFF $a_1 = a_2$ AND $b_1 = b_2$.

(ii) $RE(z) = a$, $IM(z) = b$.

AXIOMS

- $z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2)$
- $z_1 z_2 = x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1)$
- $z_1/z_2 = \dfrac{x_1 + iy_1}{x_2 + iy_2} = \dfrac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \dfrac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\dfrac{[x_2 y_1 - x_1 y_2]}{x_2^2 + y_2^2}$.

ALGEBRAIC PROPERTIES THE FOLLOWING ARE EASILY PROVED BY THE AXIOMS

(i) COMMUTATIVE : $z_1 + z_2 = z_2 + z_1$, $z_1 z_2 = z_2 z_1$

(ii) ASSOCIATIVE : $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$, $(z_1 z_2) z_3 = z_1 (z_2 z_3)$

(iii) DISTRIBUTIVE : $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$.

REMARKS AND SIMPLE CALCULATIONS

(i) $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^{4K} = 1$, $i^{4K+1} = i$, $i^{4K+2} = -1$ $K = 0, 1, 2, \ldots$

(ii) $z_1 z_2 = 0$ MEANS EITHER $z_1 = 0$ OR $z_2 = 0$

PROOF $z_1 z_2 = x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1) = 0$

THUS $x_2 x_1 = y_1 y_2$, $x_2 y_1 + x_1 y_2 = 0$.

SUPPOSE $z_1 \neq 0$ THEN $x_1 \neq 0$ AND $y_1 \neq 0$ SIMULTANEOUSLY.

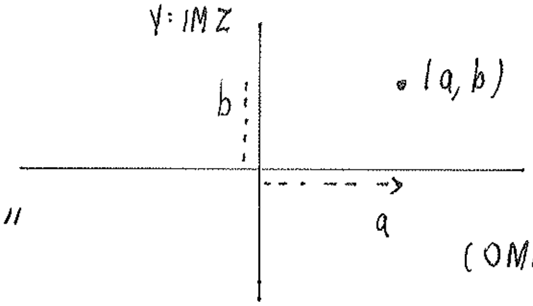
WE WRITE IN MATRIX FORM $\begin{pmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $M\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

NOW $\det M = x_2^2 + y_2^2 = 0$ SINCE $(x_1, y_1) \neq (0,0)$. THUS BOTH $x_2 = y_2 = 0$.

POLAR REPRESENTATION

$z = a + ib.$

WE CAN THINK OF $z = a+ib$ AS A "VECTOR"



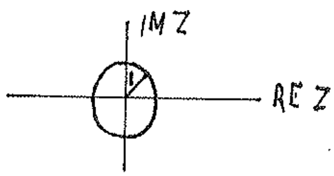
COMPLEX PLANE

DEFINITION THE MODULUS OF z (i.e. LENGTH OF "VECTOR" z) IS DEFINED AS

$$|z| = (a^2 + b^2)^{1/2} \geq 0.$$

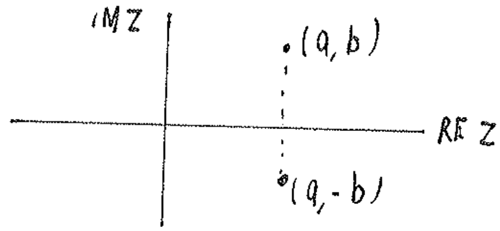
REMARKS (i) IT MAKES NO SENSE TO WRITE $z_1 < z_2$ (NO ORDERING PRINCIPLE FOR "VECTORS"). BUT $|z_1| < |z_2|$ HAS MEANING.

(ii) GEOMETRICALLY $|z| = 1$ REPRESENTS ALL POINTS z AT A DISTANCE 1 FROM THE ORIGIN IN COMPLEX PLANE



i.e. IT IS UNIT CIRCLE IN COMPLEX PLANE.

DEFINITION THE CONJUGATE OF $z = a+ib$, DENOTED BY $\bar{z}$ IS DEFINED BY $\bar{z} = a - ib.$



i.e. REFLECTION IN HORIZONTAL AXIS.

THE BASIC PROPERTIES ARE

(i) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$

(ii) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$. PROOF $\overline{(a_1 + ib_1)(a_2 + ib_2)} = \overline{a_1 a_2 - b_1 b_2 + i(b_1 a_2 + a_1 b_2)}$

SO LHS $= a_1 a_2 - b_1 b_2 - i(b_1 a_2 + a_1 b_2)$

THE RHS IS $(a_1 - ib_1)(a_2 - ib_2) = a_1 a_2 - b_1 b_2 - i(b_1 a_2 + a_1 b_2) =$ LHS ✓.

(iii) $\mathrm{RE}(z) = \dfrac{z + \bar{z}}{2}$, $\mathrm{IM}\, z = \dfrac{z - \bar{z}}{2i}$. HENCE IF $z = \bar{z}$ THEN $\mathrm{IM}(z) = 0$.

(iv) $\overline{(\bar{z})} = z$.

(v) $\quad \text{RE } z \leq |x| \leq \sqrt{x^2+y^2} = |z| \longrightarrow \text{RE}(z) \leq |z|$ IF $z = x+iy$

$\text{IM } z \leq |y| \leq \sqrt{x^2+y^2} = |z| \longrightarrow \text{IM}(z) \leq |z|$

(vi) $\quad |z|^2 = (a^2+b^2) = z\bar{z} \longrightarrow |z| = \sqrt{z\bar{z}}$.

(vii) $\quad |\bar{z}| = |z|$

(viii) $\quad |z_1 z_2| = |z_1||z_2|$.

PROOF $\quad |z_1 z_2|^2 = z_1 z_2 \overline{z_1 z_2} = z_1 z_2 \bar{z}_1 \bar{z}_2 = (z_1\bar{z}_1)(z_2\bar{z}_2) = |z_1|^2|z_2|^2$

TAKE $\sqrt{} \longrightarrow |z_1 z_2| = |z_1||z_2|$.

(ix) $\quad |z_1/z_2| = |z_1|/|z_2|$

(x) $\quad z_1/z_2 = z_1\bar{z}_2/z_2\bar{z}_2 = z_1\bar{z}_2/|z_2|^2$.

INEQUALITIES

- TRIANGLE INEQUALITY $\quad$ FOR ANY z_1, z_2 WE HAVE $|z_1+z_2| \leq |z_1|+|z_2|$.

PROOF $\quad |z_1+z_2|^2 = (z_1+z_2)(\overline{z_1+z_2}) = z_1\bar{z}_1 + z_2\bar{z}_2 + z_1\bar{z}_2 + z_2\bar{z}_1$.

BUT $\text{RE}(w) = \dfrac{w+\bar{w}}{2}$, SO IF $w = z_1\bar{z}_2$ THEN $2\,\text{RE}(z_1\bar{z}_2) = z_1\bar{z}_2 + \bar{z}_1 z_2$.

THIS GIVES $\quad |z_1+z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\text{RE}(z_1\bar{z}_2)$.

NOW RECALL $\text{RE}(w) \leq |w|$. HENCE $\quad |z_1+z_2|^2 \leq |z_1|^2 + |z_2|^2 + 2|z_1\bar{z}_2|$

$\leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$

THIS GIVES $\quad |z_1+z_2|^2 \leq (|z_1|+|z_2|)^2 \longrightarrow |z_1+z_2| \leq |z_1|+|z_2|$ $\square$

- BY INDUCTION WE HAVE

$$|z_1 + \dots + z_N| \leq |z_1| + \dots + |z_N|.$$

- ANOTHER FUNDAMENTAL INEQUALITY IS

$$||z_1| - |z_2|| \le |z_1 + z_2|.$$

PROOF $\quad |z_1| = |(z_1 + z_2) + (-z_2)| \le |z_1 + z_2| + |-z_2|$ BY Δ INEQUALITY

$$|z_1| \le |z_1 + z_2| + |z_2|$$

THUS $\quad |z_1| - |z_2| \le |z_1 + z_2|.$

NOW INTERCHANGE $z_1, z_2 \longrightarrow |z_2| - |z_1| \le |z_1 + z_2|.$

COMBINING THESE $\longrightarrow ||z_1| - |z_2|| \le |z_1 + z_2|.$

REMARKS

(i) $\text{RE}(iz) = -\text{IM}(z)$.

(ii) $\text{RE}(1/z) > 0$ IFF $\text{RE}\, z > 0$.

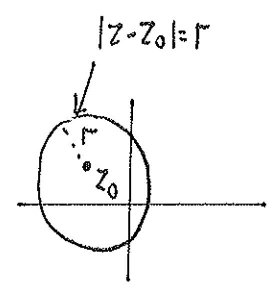
PROOF $\quad \text{RE}\left(\frac{1}{z}\right) = \text{RE}\left(\frac{\bar{z}}{z\bar{z}}\right) = \frac{1}{|z|^2}\,\text{RE}(\bar{z}) = \frac{x}{|z|^2} = \frac{\text{RE}(z)}{|z|^2}$

SO $\quad \text{RE}\left(\frac{1}{z}\right) = \frac{\text{RE}(z)}{|z|^2} \longrightarrow \text{RE}(1/z) > 0$ IFF $\text{RE}(z) > 0$.

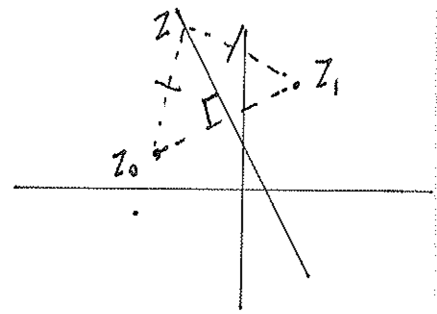
(iii) $|z_1| - |z_2| \le |z_1 - z_2|.$

PROOF IS BY Δ INEQUALITY.

PLANAR SETS + GEOMETRY



(i) $|z - z_0| = r$ IS A CIRCLE OF RADIUS r IN z-plane centered at z_0



(ii) THE SET OF z FOR WHICH $|z - z_0| = |z - z_1|$ is the perpendicular bisector of the line passing through z_0 AND z_1

(iii) FIND SET OF Z FOR WHICH

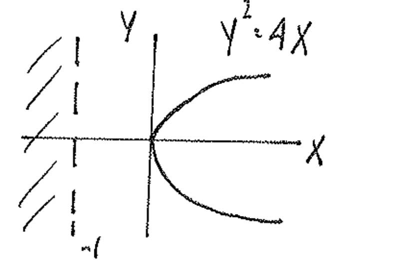
$|z-1| = RE(z)+1$. NOTE: SINCE $|z-1| \geq 0$ WE REQUIRE $RE(z) \geq -1$. (shaded Region)
WE MUST CHECK THIS LATER

LET $Z = X+iy$. THEN $|X-1+iy|^2 = (X+1)^2$

$$(X-1)^2 + y^2 = (X+1)^2$$

$$\longrightarrow \quad -2X + y^2 = 2X$$

SO $y^2 = 4X \rightarrow$ PARABOLA



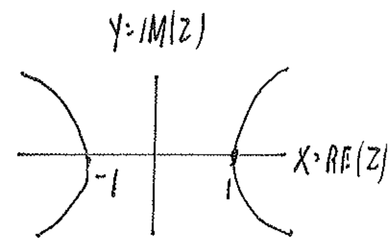
NOTE: SINCE INDEED $RE(z) \geq -1$ FOR THIS PARABOLA WE ARE OKAY ✓

(iv) FIND SET OF Z FOR WHICH

$$z^2 + \bar{z}^2 = 2.$$

$$(x+iy)(x+iy) + (x-iy)(x-iy) = 2$$

$$x^2 - y^2 + 2ixy + x^2 - y^2 - 2ixy = 2 \longrightarrow x^2 - y^2 = 1 \rightarrow \text{hyperbola}$$



(V) LET $|z| = 3|z-1|$. DESCRIBE THE SET OF Z FOR WHICH THIS HOLDS

$$|z|^2 = 9|z-1|^2$$

$$\longrightarrow x^2+y^2 = 9[(x-1)^2+y^2] \rightarrow \frac{x^2+y^2}{9} = x^2 - 2x + 1 + y^2.$$

$$\frac{8}{9}(x^2+y^2) - 2X + 1 = 0 \rightarrow x^2+y^2 - \frac{9}{4}x + \frac{9}{8} = 0$$

$$\rightarrow \left(x^2 - \frac{9}{4}x + \frac{81}{64}\right) + y^2 = -\frac{9}{8} + \frac{81}{64} = \frac{9}{64}$$

SO $(X - 9/8)^2 + Y^2 = (3/8)^2$. A CIRCLE OF RADIUS $3/8$ CENTERED AT $(9/8, 0)$.

(vi) FIND SET OF Z FOR WHICH

$$|z+3| + |z-3| = 10.$$

IF ONE STARTS BY WRITING $Z = X+iy$ THE CALCULATION IS BAD. INSTEAD THINK GEOMETRICALLY!

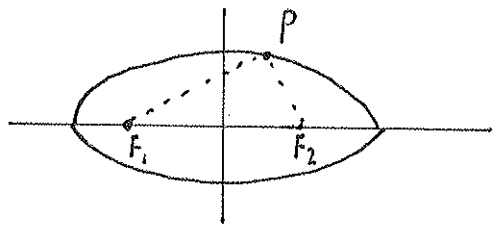
RECALL THAT FOR AN ELLIPSE WITH $x^2/a^2 + y^2/b^2 = 1$ AND $a > b$

THEN $e = (a^2 - b^2)^{1/2}/a$ IS ECCENTRICITY.

THE FOCI ARE AT $F_1 = (-ae, 0)$, $F_2 = (ae, 0)$

AND AN ELLIPSE IS CHARACTERIZED BY FINDING ALL POINTS P FOR

WHICH $\text{dist}(P, F_1) + \text{dist}(P, F_2) = 2a$.



THUS $|z+3| + |z-3| = 10$ IS AN ELLIPSE WITH $a = 5$

AND FOCI AT $F_\pm = (\pm 3, 0) \rightarrow ae = 3 \rightarrow e = 3/5$.

SO $(a^2 - b^2)^{1/2} = 3$ WITH $a = 5 \rightarrow b = 4$.

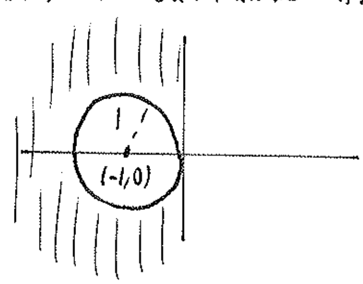
IT IS THE ELLIPSE $x^2/25 + y^2/16 = 1$.

(vii) FIND SET OF Z FOR WHICH $RE(z) < 0$ AND $|z+1| \geq 1$.

NOTE: $|z+1| \geq 1$ IS A CIRCLE OF RADIUS 1 CENTERED AT $(-1, 0)$.

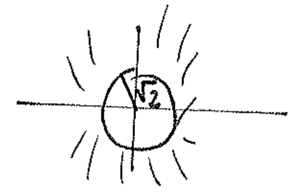
AND $RE(z) < 0$ IS LEFT HALF PLANE.

THE REGION IS SHADED AS $\longrightarrow$



(viii) FIND REGION IN Z-PLANE WITH $z\bar{z} - 2 \geq 0$.

z-plane



NOW $z\bar{z} \geq 2$.

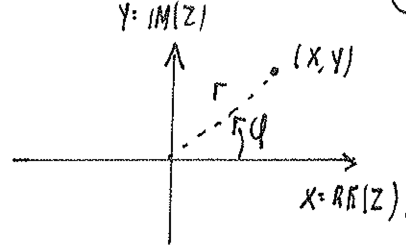
$\longrightarrow |z|^2 \geq 2 \longrightarrow |z| \geq \sqrt{2}$.

THIS IS OUTSIDE a circle of radius $\sqrt{2}$ centered at ORIGIN

POLAR REPRESENTATION

LET $z = x + iy$ AND LET $x = r\cos\varphi$, $y = r\sin\varphi$.

THEN $r = |z|$ AND $\cos\varphi = x/|z|$, $\sin\varphi = y/|z|$



REMARK (i) (*) $\cos\varphi = x/|z|$, $\sin\varphi = y/|z|$ DETERMINES φ UNIQUELY IN THE INTERVAL $(-\pi, \pi]$.

NOW IF φ_0 SATISFIES (*) IN $(-\pi, \pi]$ THEN THE OTHER SOLUTIONS TO (*) ARE $\varphi = \varphi_0 \pm 2k\pi$ $k = 0, 1, 2, 3, \ldots$

(ii) WE DEFINE $\operatorname{Arg}(z) = \varphi_0$ "PRINCIPAL VALUE OF ARGUMENT OF z"

$\operatorname{Arg}(z) \in (-\pi, \pi]$. ← Single-valued.

$\arg(z) = \varphi_0 \pm 2k\pi$ ← Multivalued Function.

EXAMPLE (i) LET $z = 1 + \sqrt{3}\, i$. THEN $|z| = \sqrt{1^2 + (\sqrt{3})^2} = 2$.

SO $\cos\varphi = 1/2$, $\sin\varphi = \sqrt{3}/2$

$\longrightarrow$ $\operatorname{Arg}(z) = \pi/3$ IN $(-\pi, \pi]$

$\arg(z) = \pi/3 \pm 2k\pi$ $k = 0, 1, 2, \ldots$

(ii) LET $z = 1 - i$. THEN $|z| = \sqrt{1+1} = \sqrt{2}$.

$\cos\varphi = 1/\sqrt{2}$, $\sin\varphi = -1/\sqrt{2}$.

$\operatorname{ARG}(z) = -\pi/4$ IN $(-\pi, \pi]$ (NOT $\operatorname{ARG}(z) = 7\pi/4$!!)

$\arg(z) = -\pi/4 + 2k\pi$, $k = 0, \pm 1, \pm 2, \ldots$

NOW WE WRITE $z = r\cos\varphi + i r\sin\varphi$.

AS SUCH IT IS CONVENIENT TO DEFINE

$$(*) \quad e^{i\varphi} \equiv \cos\varphi + i\sin\varphi \quad \text{WITH } \varphi \text{ real.}$$

THE PROPERTIES IMPLIED BY (*) ARE:

(i) $e^{-i\varphi} = 1/e^{i\varphi} = \overline{e^{i\varphi}}$

PROOF $e^{-i\varphi} = \cos(-\varphi) + i\sin(-\varphi) = \cos\varphi - i\sin\varphi = \overline{e^{i\varphi}} = \dfrac{1}{\cos\varphi + i\sin\varphi} = \dfrac{1}{e^{i\varphi}}$

(ii) $|e^{i\varphi}| = (\cos^2\varphi + \sin^2\varphi)^{1/2} = 1.$

(iii) $e^{i\varphi_1} e^{i\varphi_2} = e^{i(\varphi_1+\varphi_2)}$

PROOF $e^{i\varphi_1} e^{i\varphi_2} = (\cos\varphi_1 + i\sin\varphi_1)(\cos\varphi_2 + i\sin\varphi_2)$

$$= (\cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2) + i[\cos\varphi_1\sin\varphi_2 + \cos\varphi_2\sin\varphi_1]$$

$$= \cos(\varphi_1+\varphi_2) + i\sin(\varphi_1+\varphi_2) = e^{i(\varphi_1+\varphi_2)} \quad \square$$

(iv) $e^{i\varphi_1}/e^{i\varphi_2} = e^{i(\varphi_1-\varphi_2)}$

PROOF $1/e^{i\varphi_2} = e^{-i\varphi_2}$ BY (i).

SO $e^{i\varphi_1}/e^{i\varphi_2} = e^{i\varphi_1}e^{-i\varphi_2} = e^{i(\varphi_1-\varphi_2)}$ BY (iii).

(v) $e^{i(\varphi_1+2k\pi)} = e^{i\varphi_1}$. $\quad k = 0, 1, 2, 3, \dots$

PROOF $e^{i(\varphi_1+2k\pi)} = \cos(\varphi_1+2k\pi) + i\sin(\varphi_1+2k\pi)$

$= \cos(\varphi_1) + i\sin(\varphi_1)$ BY PERIODICITY

$= e^{i\varphi_1}.$

CLEARLY IF (iii) HOLDS THEN WE HAVE

$$e^{i(\varphi_1+\varphi_2+\dots+\varphi_N)} = e^{i\varphi_1} e^{i\varphi_2} e^{i\varphi_3} \dots e^{i\varphi_N}$$

OR $\underline{e^{iN\varphi} = (e^{i\varphi})^N}$. N = integer.

THEREFORE, IF $Z = r(\cos\varphi + i\sin\varphi) = re^{i\varphi}$

THEN $Z^N = (re^{i\varphi})^N = r^N e^{iN\varphi}$.

IF WE SET $r=1$, THEN WE HAVE <u>DE'MOIVRE'S IDENTITY</u>

$$(\cos\varphi + i\sin\varphi)^N = [\cos N\varphi + i\sin N\varphi].$$

OR EQUIVALENTLY, $\cos N\varphi = \mathrm{RE}\left[(\cos\varphi + i\sin\varphi)^N\right]$

$\sin N\varphi = \mathrm{IM}\left[(\cos\varphi + i\sin\varphi)^N\right]$.

<u>EXAMPLES</u>

(i) EXPRESS $\sin^3\varphi$ AS A LINEAR COMBINATION OF $\sin\varphi$ AND $\sin 3\varphi$.

LET $N=3$. THEN $\sin 3\varphi = \mathrm{IM}\left[(\cos\varphi + i\sin\varphi)^3\right]$.

LET $c = \cos\varphi$, $s = \sin\varphi$. $\sin 3\varphi = \mathrm{IM}\left[(c+is)(c+is)(c+is)\right]$

$$= \mathrm{IM}\left[(c^2 - s^2 + 2ics)(c+is)\right)$$

$$= 2c^2 s + c^2 s - s^3 = 3sc^2 - s^3$$

$$= 3s(1-s^2) - s^3 = 3s - 4s^3.$$

SO $\sin 3\varphi = 3\sin\varphi - 4\sin^3\varphi \longrightarrow \sin^3\varphi = \frac{3}{4}\sin\varphi - \frac{1}{4}\sin 3\varphi$.

(ii) $e^{i\varphi} = \cos\varphi + i\sin\varphi$, $e^{-i\varphi} = \cos\varphi - i\sin\varphi$.

SO $\cos\varphi = \mathrm{RE}(e^{i\varphi}) = (e^{i\varphi} + e^{-i\varphi})/2$

$\sin\varphi = \mathrm{IM}(e^{i\varphi}) = (e^{i\varphi} - e^{-i\varphi})/2i$.

(iii) CALCULATE $\frac{1+i}{\sqrt{3}-i}$ IN POLAR FORM.

$$1+i = \sqrt{2}\, e^{\pi i/4}, \qquad \sqrt{3}-i = 2 e^{-\pi i/6}.$$

SO
$$\frac{1+i}{\sqrt{3}-i} = \frac{\sqrt{2}\, e^{\pi i/4}}{2 e^{-i\pi/6}} = \frac{1}{\sqrt{2}}\, e^{5\pi i/12}.$$

(iii) PROVE THAT
$$\sum_{K=0}^{N} \cos(K\varphi) = \frac{1}{2} + \frac{1}{2}\, \frac{\sin[(N+1/2)\varphi]}{\sin(\varphi/2)}.$$

<u>PROOF</u> WE RECALL

$$1 + Z + Z^2 + \dots + Z^N = \frac{Z^{N+1}-1}{Z-1} \quad \text{FINITE GEOMETRIC SERIES.}$$

(THE PROOF OF THIS IS TO WRITE $(1+Z+\dots+Z^N)(1-Z) = 1+Z+Z^2+\dots+Z^N - Z - Z^2 \dots - Z^{N+1} = 1 - Z^{N+1}$)

NOW LET $Z = e^{i\varphi}$

$$\mathrm{RE}\left(1 + e^{i\varphi} + \dots + e^{iN\varphi}\right) = \mathrm{RE}\left(\frac{e^{i(N+1)\varphi}-1}{e^{i\varphi}-1}\right)$$

$$\longrightarrow \quad \sum_{K=0}^{N} \cos(K\varphi) = \mathrm{RE}\left(\frac{e^{i(N+1)\varphi}-1}{e^{i\varphi}-1}\right) = \mathrm{RE}\left(\frac{(e^{i(N+1)\varphi}-1)e^{-i\varphi/2}}{e^{i\varphi/2}-e^{-i\varphi/2}}\right)$$

$$= \mathrm{RE}\left(\frac{e^{i(N+1/2)\varphi}-e^{-i\varphi/2}}{2i\sin(\varphi/2)}\right)$$

$$= \frac{1}{2\sin(\varphi/2)}\, \mathrm{RE}\left[-i\left(e^{i(N+1/2)\varphi}-e^{-i\varphi/2}\right)\right]$$

BUT $\mathrm{RE}(iZ) = -\mathrm{IM}(Z)$ $\longrightarrow$

$$= \frac{1}{2\sin(\varphi/2)}\, \mathrm{RE}\left[i\left(e^{-i\varphi/2}-e^{i(N+1/2)\varphi}\right)\right]$$

$$= -\frac{1}{2\sin(\varphi/2)}\, \mathrm{IM}\left[e^{-i\varphi/2}-e^{i(N+1/2)\varphi}\right]$$

$$= -\frac{1}{2\sin(\varphi/2)}\left[+\sin(-\varphi/2) - \sin((N+1/2)\varphi)\right]$$

$$\longrightarrow \quad \sum_{K=0}^{N} \cos(K\varphi) = 1/2 + \sin[(N+1/2)\varphi]/2\sin(\varphi/2)$$

PROPERTIES OF $\arg(z)$ AND $\mathrm{ARG}(z)$

LET $z = re^{i\varphi}$ $\mathrm{ARG}(z) = \varphi_0$ WITH $\varphi_0 \in (-\pi, \pi]$ ← single valued.

$\arg(z) = \varphi_0 \pm 2k\pi$, $k = 0, 1, 2, ..$ ← multi-valued

LET $z_1 = r_1 e^{i\varphi_1}$, $z_2 = r_2 e^{i\varphi_2}$.

THEN (i) $\arg(\bar{z}) = -\arg(z)$

(ii) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

this is interpreted as saying that if any particular values of $\arg(z_1)$ and $\arg(z_2)$ are chosen, then one can find a value for $\arg(z_1 z_2)$ that satisfies this equation. i.e. the <u>set</u> of values for $\{\arg(z_1 z_2)\}$ AND $\{\arg(z_1)\} + \{\arg(z_2)\}$ are the <u>same</u>.

(iii) $\arg(1/z) = -\arg z$

(iv) $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2)$ same interpretation as in (ii)

(v) NOTE <u>HOWEVER</u> THAT $\mathrm{ARG}(z_1 z_2) \neq \mathrm{ARG}(z_1) + \mathrm{ARG}(z_2)$ IN GENERAL. TO ILLUSTRATE THIS LET

$$z_1 = e^{3\pi i/4}, \quad z_2 = e^{3\pi i/4} \longrightarrow \mathrm{ARG}(z_1) = \mathrm{ARG}(z_2) = 3\pi/4.$$

NOW $z_1 z_2 = e^{3\pi i/2} \longrightarrow \mathrm{ARG}(z_1 z_2) = -\pi/2$ SINCE MUST GET RESULT IN $(-\pi, \pi]$.

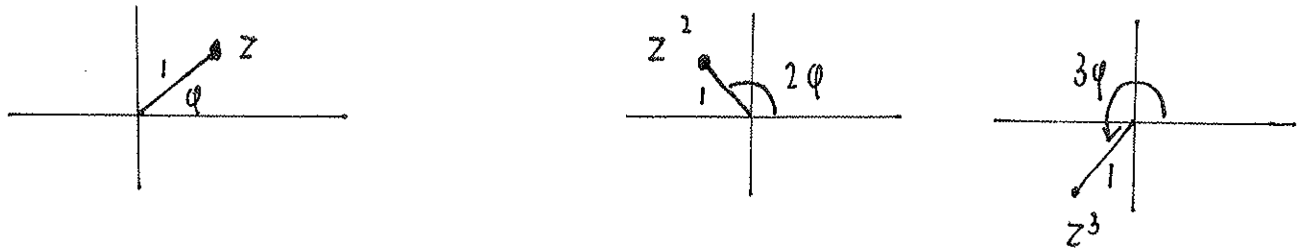
THUS FOR THIS EXAMPLE $\mathrm{ARG}(z_1 z_2) \neq \mathrm{ARG}\, z_1 + \mathrm{ARG}\, z_2$.

POWERS AND ROOTS

LET $z = e^{i\varphi}$. THEN

$z^2 = e^{2i\varphi}$, $z^3 = e^{3i\varphi}$ SO MULTIPLICATION BY z

CORRESPONDS TO A ROTATION WITH ANGLE φ



ROOTS OF UNITY

SUPPOSE THAT WE WANT TO FIND THE VALUES OF z FOR WHICH $z^n = 1$ WHERE n IS A POSITIVE INTEGER.

(i) THIS IS EQUIVALENT TO FINDING ROOTS OF $p(z) = 0$ WHERE $p(z) \equiv z^n - 1$. $\longrightarrow$ n SUCH ROOTS.

(ii) THE ROOTS z ARE CALLED "THE ROOTS OF UNITY".

CALCULATION LET $z = re^{i\varphi}$. THEN $z^n = 1$ YIELDS

$r^n e^{in\varphi} = 1$. TAKE $|\;|$ OF BOTH SIDES $\longrightarrow$ $r^n = 1$

SO THAT $r = 1$ AND

$$e^{in\varphi} = 1$$

BUT $1 = e^{2\pi i k}$, $k = 0, 1, 2, \dots$. HENCE $e^{in\varphi} = e^{2\pi i k}$, WHICH GIVES

$\varphi = 2\pi k/n$ $\quad k = 0, 1, .., n-1$. (NOTE: MAX$|k| = n-1$ OTHERWISE $\varphi \geq 2\pi$)

THUS THE ROOTS OF UNITY ARE $z_k = e^{2\pi i k/n}$, $k = 0, .., n-1$.
THEY SATISFY $z_k^n = 1$.

IF WE DEFINE w_n BY

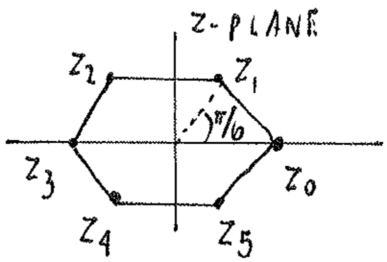
$$w_n = e^{2\pi i/n}$$

THEN THE ROOTS OF UNITY ARE THE SET

$$\{1, z_1, z_2, .., z_{n-1}\} = \{1, w_n, w_n^2, w_n^3, .., w_n^{n-1}\}.$$

GEOMETRICALLY THE ROOTS OF UNITY ARE AT VERTICES OF A REGULAR GON IN COMPLEX PLANE. FOR EXAMPLE, IF $n = 6$, THE ROOTS ARE AT

$$\{1, z_1, .., z_5\} = \{1, w_n, .., w_n^5\}$$



IDENTITIES PROVE THAT

$$\sum_{k=1}^{n-1} \cos(2\pi k/n) = -1 , \qquad \sum_{k=1}^{n-1} \sin(2\pi k/n) = 0.$$

PROOF LET $w_n = e^{2\pi i/n}$.

THEN $w_n^n = e^{2\pi i} = 1$.

NOW $1 + w + w^2 + .. + w^{n-1} = \dfrac{1 - w^n}{1 - w}$. LET $w = w_n$.

THEN $$1 + w_n + w_n^2 + .. + w_n^{n-1} = \frac{1 - w_n^n}{1 - w_n} = 0$$

SO $$w_n + w_n^2 + .. + w_n^{n-1} = e^{2\pi i/n} + e^{4\pi i/n} + .. + e^{2\pi i(n-1)/n} = -1.$$

TAKING THE REAL AND IMAGINARY PARTS THEN GIVES THE RESULT

$$\Longrightarrow \cos(2\pi/n) + \cos(4\pi/n) + .. + \cos\left(\frac{2\pi(n-1)}{n}\right) = -1, \quad \sum_{k=1}^{n-1} \sin(2\pi k/n) = 0.$$

MORE GENERALLY, LET ζ BE A GIVEN COMPLEX NUMBER. FIND THE n VALUES OF z FOR WHICH

$$z^n = \zeta \qquad n: \text{positive integer.}$$

WE FIRST WRITE ζ IN POLAR FORM WITH

$$\zeta = \rho e^{i\phi} \qquad \rho = |\zeta| > 0 \qquad \rho, \phi \text{ TO BE CALCULATED}$$

THEN $z^n = \rho e^{i\phi}$

WE SET $z = re^{i\varphi}$ WITH $r = |z| > 0$, SO THAT

$$r^n e^{in\varphi} = \rho e^{i\phi}$$

- TAKING MODULUS OF BOTH SIDES $\longrightarrow$ $r^n = \rho$ SO $r = \rho^{1/n}$.
- THEN $e^{in\varphi} = e^{i\phi} \longrightarrow in\varphi = i\phi + 2k\pi i \quad k = 0, .., n-1.$

  HENCE $\varphi = \dfrac{\phi}{n} + \dfrac{2k\pi}{n}, \quad k = 0, .., n-1.$

IN SUMMARY, THE ROOTS z OF $z^n = \zeta$ ARE

$$(*) \left\{ z_k = \rho^{1/n} e^{i(\phi/n + 2k\pi/n)} \quad k = 0, .., n-1. \right. \quad \text{WITH } \zeta = \rho e^{i\phi}$$

<u>EXAMPLE</u>

(i) SOLVE $z^5 = 1 + i$.

<u>SOLUTION</u> LET $1 + i = \sqrt{2}\, e^{\pi i/4} \longrightarrow z^5 = \sqrt{2}\, e^{\pi i/4}$

THUS $\rho = \sqrt{2}$ AND $\phi = \pi/4 \longrightarrow z_k = (\sqrt{2})^{1/5} e^{i(\pi/4 + 2k\pi)/5}$

$k = 0, 1, .., 4.$

(ii) IF $z^5 = -3$ THEN WRITE $z^5 = 3e^{i\pi}$ SO $\rho = 3$, $\phi = \pi$ ETC...

EXAMPLE

$$z^n = (z+1)^n$$

SUPPOSE $z \neq -1$, THEN $w^n = 1$ WITH $w = \dfrac{z}{z+1} \longrightarrow zw + w = z$

SO $z(w-1) = -w$ OR $z = \dfrac{w}{1-w}$

NOW $w = e^{2\pi i k/n}$ $k = 0, .., n-1$. MUST EXCLUDE $k=0$ SINCE $w \neq 1$.

THEN $z = \dfrac{e^{2\pi i k/n}}{1 - e^{2\pi i k/n}} = \dfrac{e^{\pi i k/n}}{-(e^{\pi i k/n} - e^{-\pi i k/n})}$

THIS YIELDS $z = \dfrac{e^{\pi i k/n}}{-2i \sin(\pi k/n)}$

$$(*) \quad z = \frac{i}{2\sin(k\pi/n)} e^{\pi i k/n} = \frac{i}{2\sin(k\pi/n)} \left(\cos(k\pi/n) + i\sin(k\pi/n)\right)$$

NOW IF $z = x + iy$ THEN $(*)$ IMPLIES THAT

$$x = -1/2 \qquad y = \cot(k\pi/n)$$

SO $z = -1/2 + i\cot(k\pi/n)$ $\quad k = 1, .., n-1$.

EXAMPLE

(i) $e^z = 1$ iff $z = 2k\pi i$

(ii) $e^{z_1} = e^{z_2}$ iff $z_1 = z_2 + 2k\pi i$

(iii) ALSO, $f(z) = f(z + 2\pi i)$ is periodic where $f(z) = e^z$.

THE COMPLEX EXPONENTIAL

LET $z = x + iy$, THEN WE DEFINE

$$e^{z} \equiv e^{x}(\cos y + i\sin y) = e^{x}e^{iy} = e^{x+iy}.$$

NOW SINCE $e^{i\varphi} = \cos\varphi + i\sin\varphi$ HAS PROPERTIES AS WRITTEN ON PAGE F8 THEN IT FOLLOWS THAT IF z_1, z_2 ARE COMPLEX NUMBERS

(i) $e^{z_1+z_2} = e^{z_1}e^{z_2}$

(ii) $e^{z_1-z_2} = e^{z_1}/e^{z_2}$

(iii) $e^{z+2\pi i} = e^{z}$, $e^{z+\pi i} = e^{z}[\cos(\pi) + i\sin(\pi)] = -e^{z}$.

WE NOW SHOW TWO FURTHER PROPERTIES:

(i) PROVE THAT $|e^{z}| \leq 1$ IF $\mathrm{Re}\, z \leq 0$.

<u>PROOF</u> $|e^{z}| = |e^{x+iy}| = |e^{x}\cos y + ie^{x}\sin y|$

$$= e^{x}|\cos y + i\sin y| = e^{x}[\cos^2 y + \sin^2 y]^{1/2} = e^{x}.$$

THUS $|e^{z}| = e^{x} \leq 1$ IF $x = \mathrm{RE}(z) < 0$.

(ii) PROVE THAT $\overline{e^{z}} = e^{\bar{z}}$.

$$\overline{e^{z}} = \overline{e^{x}(\cos y + i\sin y)} = e^{x}(\cos y - i\sin y) = e^{x}[\cos(-y) + i\sin(-y)]$$
$$= e^{x-iy} = e^{\bar{z}}.$$

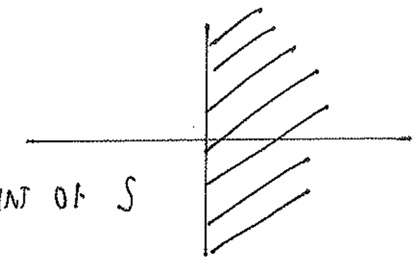
(iii) NOTICE THAT FOR x FIXED THEN AS A FUNCTION OF y WE HAVE THAT $\mathrm{RE}[e^{z}]$ IS AN OSCILLATORY FUNCTION OF y.

REGIONS IN THE COMPLEX PLANE

LET S BE A SET IN THE COMPLEX Z-PLANE.

(i) z_0 IS AN <u>INTERIOR POINT</u> OF S IF $\exists$ A NEIGHBORHOOD OF z_0 WHICH IS COMPLETELY CONTAINED WITHIN S. BY A NEIGHBORHOOD WE MEAN A SMALL DISK OF RADIUS ρ CENTRED AT z_0.

<u>EXAMPLE</u> LET $S = \{ z \mid Re(z) \geq 0 \}$



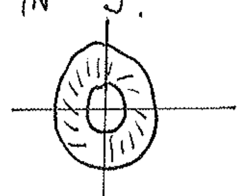
- THEN IF $Re(z_0) > 0 \longrightarrow z_0$ IS AN INTERIOR POINT OF S
- IF $z_0 = i$, THEN THIS POINT IS <u>NOT</u> AN INTERIOR POINT OF S.

(ii) IF EVERY POINT OF A SET S IS AN INTERIOR POINT OF S, THEN WE SAY THAT S IS AN <u>OPEN SET</u>

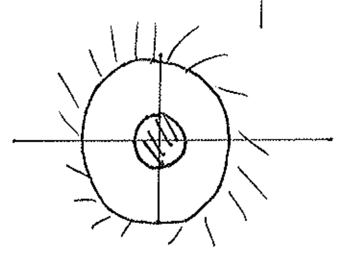
- $S = \{ z \mid Re(z) \leq 0 \}$ IS NOT AN OPEN SET SINCE $z_0 = i$ IS IN S, BUT IS NOT AN INTERIOR POINT.
- $S = \{ z \mid Re(z) < 0 \}$ AND $S = \{ z \mid |z - 1| < 2 \}$ ARE BOTH OPEN SETS.

(iii) AN OPEN SET IS SAID TO BE <u>CONNECTED</u> IF EVERY PAIR OF POINTS z_1 AND z_2 IN S CAN BE JOINED BY A FINITE NUMBER OF LINE SEGMENTS LYING ENTIRELY IN S.

<u>EXAMPLE</u>
- $S = \{ z \mid 1 < |z| < 2 \}$ IS CONNECTED



- $S = \{ z \mid |z| < 1 \text{ OR } |z| > 3 \}$ IS NOT CONNECTED



ROUGHLY SPEAKING, CONNECTED SETS DO NOT HAVE DISJOINT PIECES.

(iv) AN OPEN CONNECTED SET IS CALLED A DOMAIN.

(v) A POINT z_0 IS CALLED A BOUNDARY POINT OF S IF EVERY NEIGHBORHOOD OF z_0 CONTAINS AT LEAST ONE POINT IN S AND AT LEAST ONE POINT NOT IN S

- $S = \{ z \mid \mathrm{Re}(z) \le 0 \} \rightarrow$ BOUNDARY POINTS SATISFY $\mathrm{Re}(z) = 0$.
- $S = \{ z \mid 1 \le |z| \le 2 \} \rightarrow$ BOUNDARY POINTS SATISFY $|z| = 1$ AND $|z| = 2$.
- $S = \{ z \mid |\mathrm{Im}(z)| > 1 \} \rightarrow$ BOUNDARY POINTS SATISFY $\mathrm{Im}(z) = \pm 1$.

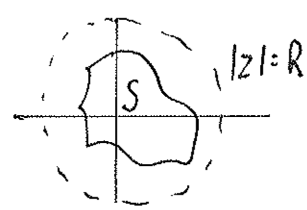
THE BOUNDARY OF S IS THE SET OF ALL OF ITS BOUNDARY POINTS.

(vi) A SET S IS CLOSED IF IT CONTAINS ALL OF ITS BOUNDARY POINTS.

- $S = \{ z \mid 1 \le |z| \le 2 \} \rightarrow$ IS CLOSED
- $S = \{ z \mid 1 \le |z| < 2 \} \rightarrow$ NOT CLOSED SINCE IT DOES NOT CONTAIN ITS BOUNDARY POINTS ON $|z| = 2$.

(vii) A SET S IS CALLED BOUNDED IF THERE IS A POSITIVE REAL NUMBER R FOR WHICH ALL POINTS IN S CAN BE ENCLOSED IN THE BALL $|z| < R$.

IT IS CALLED UNBOUNDED IF SUCH A BALL CANNOT BE FOUND.

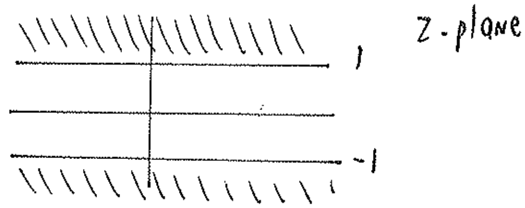


KEY WORDS interior point, open set, connected, domain, boundary point, boundary, closed, bounded, unbounded.

EXAMPLE CLASSIFY THE FOLLOWING SETS:

(i) $S = \{ z \mid |\mathrm{Im}(z)| > 1 \}$

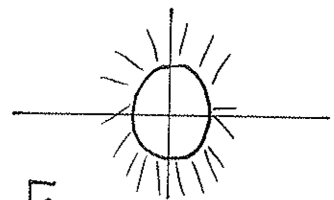
(ii) $S = \{ z \mid z\bar{z} \geq 2 \}$

SOLUTION (i) WE PLOT 

S IS NOT connected. THE BOUNDARY OF S IS $\mathrm{IM}\,z = \pm 1$.

IT IS AN OPEN SET. NOT A DOMAIN. IT IS UNBOUNDED.

(ii) $z\bar{z} = |z|^2 \geq 2 \rightarrow |z| \geq \sqrt{2}$.



S is a connected, unbounded, closed set. THE BOUNDARY IS $|z| = \sqrt{2}$.

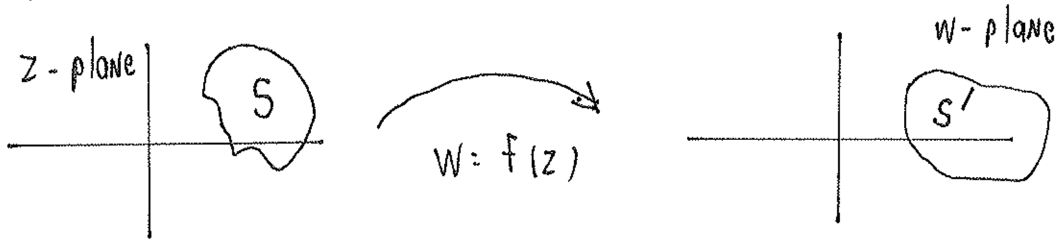
MATH 300 FUNCTIONS OF A COMPLEX VARIABLE

IN THIS SECTION WE CONSIDER FUNCTIONS OF A COMPLEX VARIABLE WRITTEN AS

$$w = f(z) \quad \text{WITH} \quad z \in S$$

WHERE S IS SOME REGION IN COMPLEX Z-PLANE.

LET $S' = \{ w \mid w = f(z),\ z \in S \}$ BE THE IMAGE, OR RANGE, OF S UNDER THE MAPPING. WE PLOT SYMBOLICALLY



SOME EXAMPLES OF SIMPLE MAPPINGS ARE:

(i) $w = z + \beta$ $\longrightarrow$ translation

(ii) $w = e^{i\varphi_0} z$ $\longrightarrow$ Rotation by angle φ_0, preservation of lengths

(iii) $w = az$ a REAL WITH $a > 0$ $\rightarrow$ stretching OR contraction depending on whether $a > 1$ OR $a < 1$

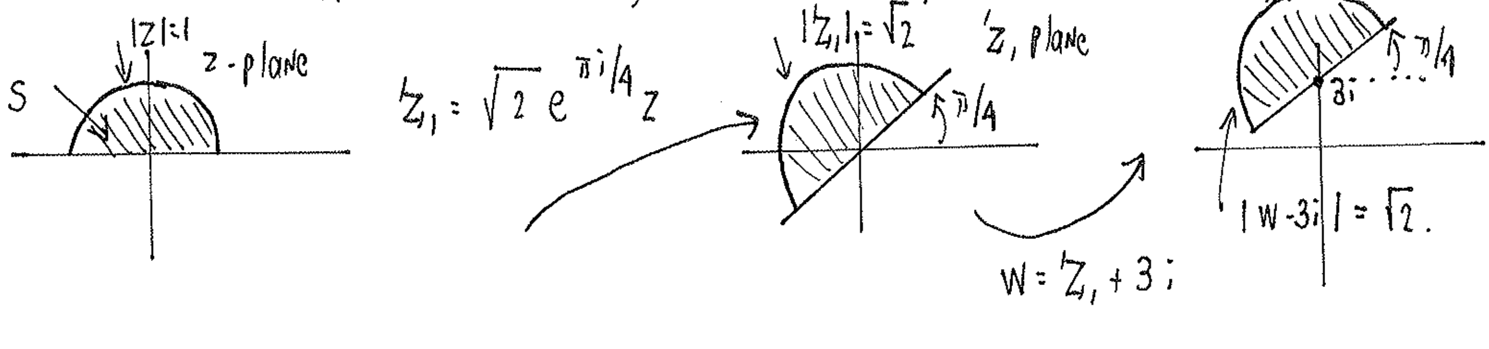
(iv) $w = 1/z$ INVERSION $\longrightarrow$ property described below.

<u>EXAMPLE I</u> LET $w = (1+i)z + 3i \equiv f(z)$. FIND THE IMAGE OF THE SET $S = \{ z \mid |z| \le 1,\ \text{IM}(z) \ge 0 \}$ UNDER $w = f(z)$.

WE PLOT A SERIES OF PICTURES.

$$w = \sqrt{2}\, e^{\pi i/4} z + 3i.$$

WE DEFINE $Z_1 = \sqrt{2}\, e^{\pi i/4} z$, AND $w = Z_1 + 3i$ (translation)



THEREFORE $S' = \{ w \mid |w-3i| \le \sqrt{2},\ \pi/4 \le \arg(w-3i) \le 5\pi/4 \}$

IS THE IMAGE SET.

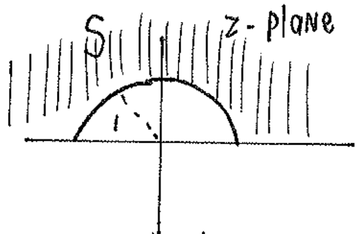
EXAMPLE 2 LET $w = f(z) = 2/z$ AND $S = \{ z \mid |z| \ge 1, \operatorname{Im}(z) \ge 0 \}$.

FIND THE IMAGE OF S UNDER $w = f(z)$.

TO FIND IMAGE SET LET

$$w = \frac{2\bar{z}}{z\bar{z}} = \frac{2\bar{z}}{|z|^2}.$$

DEFINE $Z_1 = z/|z|^2$ THEN $w = 2\bar{Z_1}$.

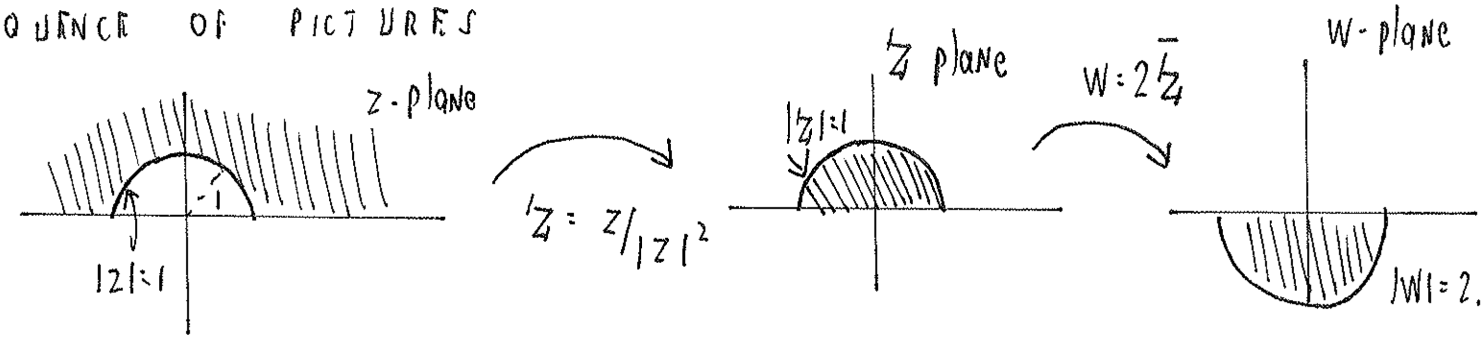


S is shaded region written as $z = re^{i\varphi}$, $r \ge 1$ WITH $0 \le \varphi \le \pi$.

THEN $$Z_1 = \frac{re^{i\varphi}}{r^2|e^{2i\varphi}|} = \frac{1}{r}e^{i\varphi},\quad 0 \le \varphi \le \pi,\ \text{FOR } r \ge 1$$

HENCE $\operatorname{Im}(Z_1) \ge 0$ AND $|Z_1| \le 1$. THIS YIELDS THE

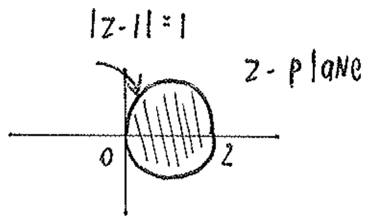
SEQUENCE OF PICTURES



THUS $S' = \{ w \mid |w| \le 2, \operatorname{Im}(w) \le 0 \}$.

EXAMPLE 3 FIND THE IMAGE OF $S = \{ z \mid |z-1| \le 1 \}$ UNDER THE MAPPING $w = f(z) = 1/z$.

IN Z-PLANE WE HAVE



<u>NOTE:</u> $z = 0$ IS A BOUNDARY POINT OF S THAT IS MAPPED TO $w = \infty$.

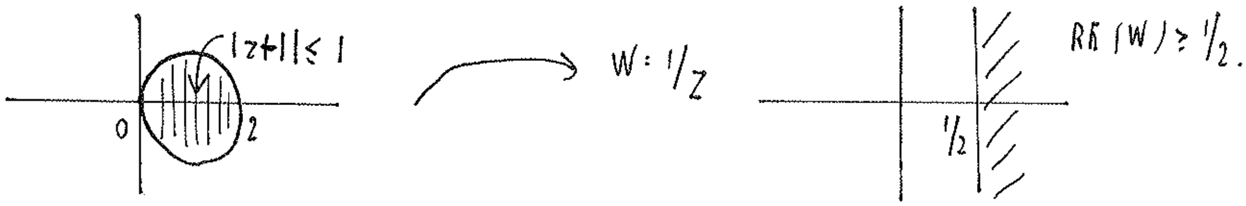
NOW LET $w = 1/z$ OR $z = 1/w$ IN $|z-1| \le 1$.

THUS $|1/w - 1| \le 1 \longrightarrow |w-1| \le |w|$. THIS YIELDS WITH $w = u + iv$

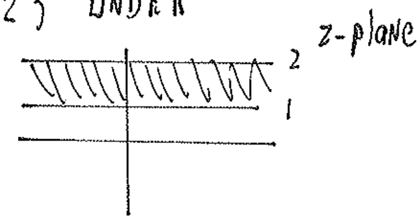
$$|w-1|^2 \le |w|^2 \longrightarrow (u-1)^2 + v^2 \le u^2 + v^2$$

$$\longrightarrow -2u + 1 \le 0 \quad \text{OR} \quad u \ge 1/2.$$

HENCE, $S' = \{ w \mid \mathrm{RE}(w) \ge 1/2 \}$.



EXAMPLE 4 FIND THE IMAGE OF $S = \{ z \mid 1 \le \mathrm{IM}(z) \le 2 \}$ UNDER THE MAPPING $w = \dfrac{z+1}{z-1}$. S IS AS SHOWN



TO SOLVE THIS WE WRITE $w = \dfrac{(z-1)+2}{(z-1)} = 1 + \dfrac{2}{z-1}$.

THUS $\dfrac{w-1}{2} = \dfrac{1}{z-1}$ OR $z - 1 = \dfrac{2}{w-1} \longrightarrow z = 1 + \dfrac{2}{w-1}$.

NOW $1 \le \mathrm{IM}(z) \le 2$ BECOMES $1 \le \mathrm{IM}\left(1 + \dfrac{2}{w-1}\right) \le 2$

SO $1 \le 2\,\mathrm{IM}\left(\dfrac{\bar{w}-1}{|w-1|^2}\right) \le 2 \longrightarrow \dfrac{1}{2} \le \dfrac{1}{|w-1|^2}\,\mathrm{IM}(\bar{w}) \le 1$.

PUT $w = u + iv$. WE GET $\dfrac{1}{2} \le \dfrac{1}{(u-1)^2 + v^2}\,\mathrm{IM}(u - iv) \le 1$.

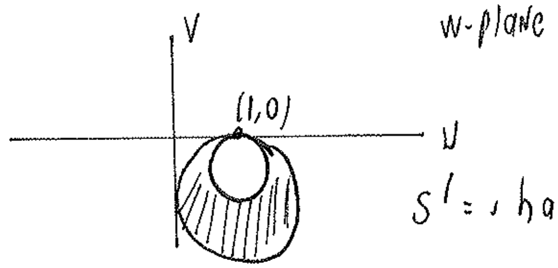
HENCE, $1/2 \le -v/[(u-1)^2 + v^2] \le 1$.

THIS YIELDS $(u-1)^2 + v^2 \ge -v \longrightarrow (u-1)^2 + (v + 1/2)^2 \ge (1/2)^2$

AND $(u-1)^2 + v^2 \le -2v \longrightarrow (u-1)^2 + (v+1)^2 \le 1$.

THUS $S' = \{ u + iv \mid (u-1)^2 + (v+1/2)^2 \geq (1/2)^2$ AND $(u-1)^2 + (v+1)^2 \leq 1 \}$.

GRAPHICALLY WE HAVE

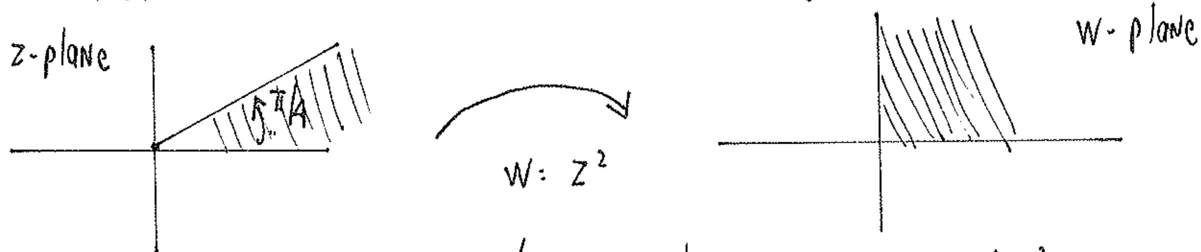


S' = shaded region.

EXAMPLE 5 FIND IMAGE OF $S = \{ z \mid 0 \leq \arg(z) \leq \pi/4 \}$ UNDER $w = z^2$.

WRITE $z = re^{i\varphi}$ WITH $0 \leq \varphi \leq \pi/4 \rightarrow w = r^2 e^{2i\varphi}$.

THUS $|w| = r^2$ WITH $0 < r < \infty$ AND $0 \leq \arg(w) \leq \pi/2$.

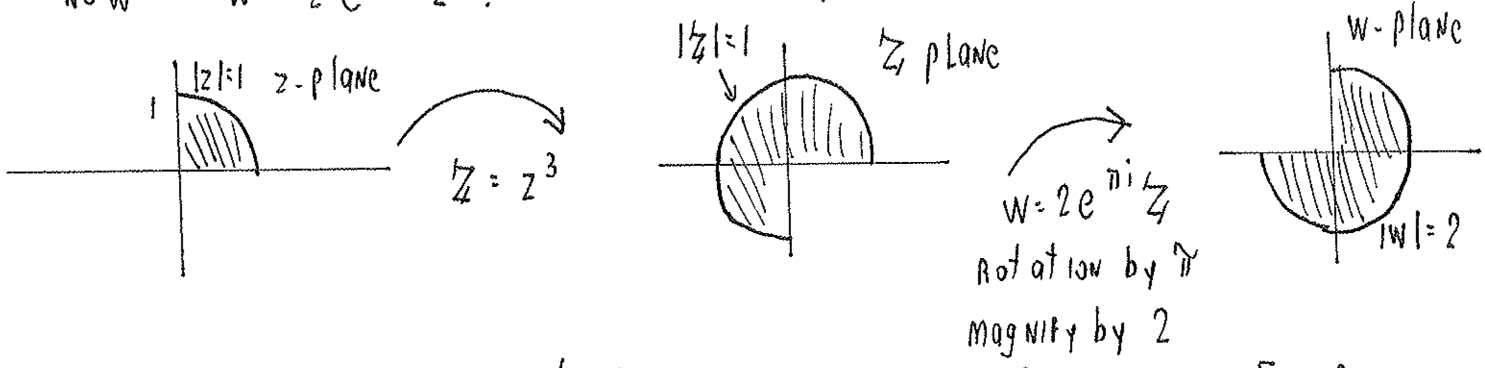


$S' = \{ w \mid 0 \leq \arg(w) \leq \pi/2 \}$.

EXAMPLE 6 FIND IMAGE OF $S = \{ z \mid |z| \leq 1, \ 0 \leq \arg(z) \leq \pi/2 \}$

UNDER $w = -2z^3$.

NOW $w = 2e^{\pi i} z^3$. WE WRITE $Z_1 = z^3$ AND $w = 2e^{\pi i} Z_1$.



NOW $S' = \{ w \mid |w| \leq 2$ AND $\pi < \arg(w) < 5\pi/2 \}$.

EXAMPLE 7 THERE IS A SPECIAL PROPERTY OF THE INVERSION MAP $w = 1/z$ THAT IS NOTABLE.

CLAIM: IN z-plane the class OF circles OR lines will get mapped via $w = 1/z$ to the class of circles and lines.

PROOF IN z-plane WE CONSIDER

$$a(x^2+y^2)+bx+cy+d=0 \quad a,b,c,d \text{ real}$$

IF $a \neq 0$ THIS IS A CIRCLE. IF $a=0$ IT IS A LINE.

NOW LET $x=(z+\bar{z})/2$, $y=(z-\bar{z})/2i$.

WE CALCULATE,

$$a\left[\frac{1}{4}(z+\bar{z})^2-\frac{1}{4}(z-\bar{z})^2\right]+\frac{b}{2}(z+\bar{z})+\frac{c}{2i}(z-\bar{z})=d.$$

SO $\quad a z\bar{z}+zB+\bar{z}\bar{B}+d=0$ WITH $B=b/2+c/2i$

NOW LET $w=1/z$ OR $z=1/w$. THEN

$$\frac{a}{w\bar{w}}+\frac{B}{w}+\frac{\bar{B}}{\bar{w}}+d=0$$

MULTIPLY BY $w\bar{w} \longrightarrow \quad dw\bar{w}+a+B\bar{w}+\bar{B}w=0$

THIS HAS PRECISELY THE FORM OF A CIRCLE (IF $d \neq 0$)

OR LINE IF $d=0$ IN w-plane □

REMARK (i) IF A BOUNDARY POINT OF S IS MAPPED TO $w=\infty$ UNDER $w=1/z$ WHEN S IS A CIRCLE, THEN THE IMAGE S' MUST BE UNBOUNDED $\rightarrow$ i.e. IT MUST BE A LINE IN w-plane. THIS IS MECHANISM AT WORK IN EXAMPLE 3.