ANALYTICITY

(A1)

SUPPOSE THAT $f(z)$ IS DEFINED ON A DOMAIN S (OPEN CONNECTED SET) IN THE COMPLEX PLANE. IF z_0 IS A POINT IN S, THEN $f(z)$ IS CONTINUOUS AT z_0 IF

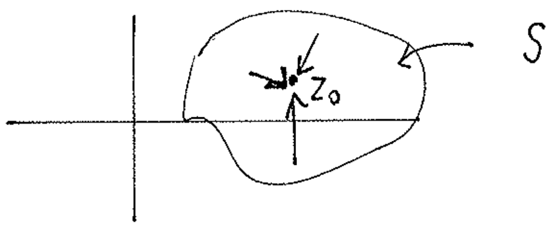
$$\lim_{z \to z_0} f(z) = f(z_0)$$

THAT IS, f IS CONTINUOUS AT z_0 IF THE VALUES OF $f(z)$ GET ARBITRARILY CLOSE TO $f(z_0)$, SO LONG AS $z \in S$ AND z IS SUFFICIENTLY CLOSE TO z_0. THE TECHNICAL DEFINITION IS

FOR ANY $\varepsilon > 0$, $\exists \delta > 0$ SUCH THAT $|f(z) - f(z_0)| < \varepsilon$ WHENEVER $0 < |z - z_0| < \delta$.

THE KEY POINT IS:

(*) FOR A FUNCTION TO BE CONTINUOUS AT z_0 WE REQUIRE THAT $f(z) \to f(z_0)$ AS $z \to z_0$ IN <u>ANY</u> DIRECTION IN THE COMPLEX PLANE



let C = complex plane

<u>EX 1</u> $f(z) = |z|^2$ IS CONTINUOUS AT EVERY POINT $z \in C$

<u>EX 2</u> $f(z) = \dfrac{1}{4-z}$ IS CONTINUOUS FOR $z \in C$ EXCEPT $z = 4$.

<u>EX 3</u> $f(z) = \dfrac{(z^4 - 1)}{(z - i)}$ IS CONTINUOUS FOR $z \in C$

PROVIDED THAT WE DEFINE $f(i) = -4i$.

<u>EX 4</u> FOR $f(z) = \dfrac{z}{\bar{z}}$ THEN $f(z)$ IS NOT CONTINUOUS AT $z = 0$.

- LET $z = x$ WITH $x \to 0^{\pm}$. THEN $\lim_{z \to 0} f(z) = 1$. PATH 1
- LET $z = iy$ WITH $y \to 0^{\pm}$. THEN $\lim_{z \to 0} f(z) = \lim_{z \to 0} \dfrac{iy}{-iy} = -1$. PATH 2

SINCE THE VALUE OF $\lim_{z\to 0} f(z)$ IS DIFFERENT ON PATH 1 THAN ON PATH 2, $f(z)$ IS NOT CONTINUOUS AT $z=0$.

EXAMPLE (HW) IDENTIFY ANY POINTS OF DISCONTINUITY OF

$$f(z)=\begin{cases} z, & \text{IF } |z|\le 1 \\ |z|^2, & \text{IF } |z|>1. \end{cases}$$

DEFINITION A FUNCTION $f(z)$ FOR z IN A DOMAIN S IS DIFFERENTIABLE AT A POINT z_0 IN S IF

$$(*) \quad \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$$

EXISTS. IF THIS LIMIT EXISTS WE LABEL IT BY $f'(z_0)$.

KEY POINT 1: FOR $f(z)$ TO BE DIFFERENTIABLE AT $z=z_0$ WE REQUIRE THAT THE LIMIT IN $(*)$ GIVES THE SAME VALUE FOR ANY PATH FOR WHICH $z\to z_0$.

EXAMPLE 1 SHOW THAT $f(z)=\bar{z}$ IS NOT DIFFERENTIABLE AT ANY POINT z_0.

PROOF WE WRITE $h=\Delta z$ COMPLEX, AND CALCULATE

$$L=\lim_{\Delta z\to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \lim_{\Delta z\to 0} \frac{\overline{(z_0+\Delta z)}-\overline{z_0}}{(\Delta z)} = \lim_{\Delta z\to 0} \frac{\overline{(\Delta z)}}{(\Delta z)}$$

- PATH 1 LET $\Delta z=\Delta x$ WITH $\Delta x\to 0$ REAL. THEN

$$L=\lim_{\Delta x\to 0} \frac{\overline{(\Delta x)}}{\Delta x} = \lim_{\Delta x\to 0} \frac{\Delta x}{\Delta x} = 1.$$

- PATH 2 LET $\Delta z=i\Delta y$ WITH $\Delta y\to 0$. THEN

$$L=\lim_{\Delta y\to 0} \frac{\overline{(i\Delta y)}}{i\Delta y} = \lim_{\Delta y\to 0} \frac{-i\Delta y}{i\Delta y} = -1.$$

DIFFERENT

THUS $f(z) = \bar{z}$ IS NOT DIFFERENTIABLE AT ANY POINT z_0.

EXAMPLE 2 LET $f(z) = |z|^2$. $f(z)$ IS CONTINUOUS FOR ALL z_0.
HOWEVER, WE NOW SHOW THAT $f(z)$ IS NOT DIFFERENTIABLE AT ANY POINT $z_0 \neq 0$, BUT IS DIFFERENTIABLE AT $z_0 = 0$.

PROOF LET z_0 BE GIVEN. WE CALCULATE

$$L = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{|z_0 + \Delta z|^2 - |z_0|^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{[(z_0 + \Delta z)(\overline{z_0 + \Delta z}) - z_0 \bar{z}_0]}{\Delta z}$$

$$L = \lim_{\Delta z \to 0} \frac{[z_0 \bar{z}_0 + \bar{z}_0 \Delta z + z_0 \overline{\Delta z} + |\Delta z|^2 - z_0 \bar{z}_0]}{\Delta z}$$

$$\text{SO } (+) \Big\{ L = \bar{z}_0 + z_0 \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

FROM (+) WE OBSERVE THAT IF $z_0 = 0 \to L = 0$ i.e. DIFFERENTIABLE
BUT, IF $z_0 \neq 0$, THEN SINCE $\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$ DEPENDS ON PATH FOR WHICH $\Delta z \to 0$ AS IN EXAMPLE 1 IT FOLLOWS THAT L IS NOT INDEPENDENT OF THE PATH FOR WHICH $\Delta z \to 0$.

THUS $f(z) = |z|^2$ IS <u>NOT</u> DIFFERENTIABLE FOR ANY $z \neq 0$.

DEFINITION A FUNCTION $f(z)$ IS ANALYTIC AT A POINT z_0 IF ITS DERIVATIVE EXISTS NOT ONLY AT z_0 BUT ~~AT~~ ANY z IN A SMALL NEIGHBORHOOD OF z_0.

DEFINITION A FUNCTION $f(z)$ IS ANALYTIC IN A DOMAIN D IF IT HAS A DERIVATIVE AT EVERY POINT IN D.

REMARKS

(i) $f(z) = |z|^2$ IS DIFFERENTIABLE AT $z=0$ BUT IS NOT ANALYTIC AT $z=0$. WHY? BECAUSE, WE CAN FIND NO SMALL NEIGHBORHOOD ABOUT $z=0$ FOR WHICH f HAS A DERIVATIVE AT EACH POINT IN THE NEIGHBORHOOD. (RECALL $f(z) = |z|^2$ IS NOT DIFFERENTIABLE FOR ANY $z \neq 0$).

DEFINITION $f(z)$ IS AN ENTIRE FUNCTION IF IT IS ANALYTIC AT EACH POINT IN THE COMPLEX PLANE.

EXAMPLES (i) POLYNOMIALS $p(z) = a_N z^N + \dots a_0$ ARE ANALYTIC FOR ALL z, i.e. ENTIRE FUNCTIONS.

(ii) $f(z) = \dfrac{z}{z^2+1}$ IS ANALYTIC FOR ALL z EXCEPT AT $z = \pm i$. SUCH POINTS ARE "SINGULARITIES".

(iii) $f(z) = 8\bar{z} + i$ IS NOT DIFFERENTIABLE AT ANY POINT z. HENCE, NOWHERE ANALYTIC.

NOTE ANALYTIC AT $z_0 \longrightarrow$ DIFFERENTIABLE AT $z_0 \longrightarrow$ CONTINUITY AT z_0.

(MEANS DIFFERENTIABLE AT z_0 AND IN ANY SMALL NEIGHBORHOOD OF z_0)

NOTE IF $f(z)$ IS DIFFERENTIABLE AT A POINT THEN "USUAL" RULES OF CALCULUS STILL HOLD AND CAN BE PROVED FROM DEFINITION.

REMARKS IF $f(z)$ AND $g(z)$ ARE DIFFERENTIABLE AT z THEN "USUAL" FORMULAE STILL HOLD:

$$(f \pm g)'(z) = f'(z) \pm g'(z)$$

$$(fg)'(z) = f'(z)g(z) + f(z)g'(z) \qquad \text{product rule}$$

IF $f(z)$ AND $g(z)$ ARE differentiable for all z, THEN

$$\frac{d}{dz} f(g(z)) = f'(g(z))\, g'(z) \qquad \text{chain rule.}$$

CAUCHY RIEMANN EQUATIONS (SECTION 2.4)

WE WRITE $f(z) = u(x,y) + i\, v(x,y)$ $\qquad u = Re(f),\ v = Im(f)$.

THEOREM 1 SUPPOSE THAT $f(z)$ IS DIFFERENTIABLE AT A POINT z_0. THEN THE CAUCHY-RIEMANN EQUATIONS

$$CR \begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, & \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

ARE SATISFIED AT $z_0 = x_0 + i y_0$.

PROOF SINCE $f(z) = u(x,y) + i v(x,y)$ IS DIFFERENTIABLE AT $z_0 = x_0 + i y_0$ THEN THE LIMIT

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

CAN BE CALCULATED BY TAKING ANY PATH FOR WHICH $\Delta z \to 0$.

PATH 1 LET $\Delta z = \Delta x \to 0$. THEN

$$f'(z_0) = \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) + i v(x_0 + \Delta x, y_0) - u(x_0, y_0) - i v(x_0, y_0)}{\Delta x}$$

$$f'(z_0) = \lim_{\Delta x \to 0} \left[\frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x}\right] + i \lim_{\Delta x \to 0} \left[\frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x}\right]$$

THUS $$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0). \quad \Big\} \quad (1)$$

PATH 2 LET $\Delta z = i\Delta y$ WITH $\Delta y \to 0$.

$$f'(z_0) = \lim_{\Delta y \to 0} \frac{u(x_0, y_0+\Delta y) + i v(x_0, y_0+\Delta y) - u(x_0, y_0) - i v(x_0, y_0)}{i\Delta y}$$

$$= -i \lim_{\Delta y \to 0}\left(\frac{u(x_0, y_0+\Delta y) - u(x_0, y_0)}{\Delta y}\right) + \lim_{\Delta y \to 0}\left(\frac{v(x_0, y_0+\Delta y) - v(x_0, y_0)}{\Delta y}\right)$$

THUS, $$f'(z_0) = -i\frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0). \quad \Big\} \quad (2)$$

SINCE (1) AND (2) MUST BE THE SAME, THEN

$$(*)\left\{\begin{array}{ll} \partial u/\partial x = \partial v/\partial y & \text{AT } (x_0, y_0) \\ \partial u/\partial y = -\partial v/\partial x & \text{AT } (x_0, y_0) \text{ MUST HOLD} \end{array}\right.$$

(*) ARE CALLED CAUCHY-RIEMANN EQUATIONS.

REMARK (i) IF THE CR EQUATIONS DO NOT HOLD AT (x_0, y_0) THEN $f(z)$ IS NOT DIFFERENTIABLE AT z_0.

(ii) KEY POINT IF $f(z)$ IS ANALYTIC ON SOME DOMAIN D THEN CR EQUATIONS MUST HOLD AT EVERY POINT IN D.

(iii) v IS CALLED THE HARMONIC CONJUGATE OF u (EXPLAINED LATER).

HOWEVER, WHAT IS MORE USEFUL IS TO DETERMINE FOR A GIVEN $U(X,Y)$ AND $V(X,Y)$ WHETHER $F(z) = U(X,Y) + iV(X,Y)$ IS AN ANALYTIC FUNCTION. FOR THIS WE NEED:

<u>THEOREM 2</u> LET $f(z) = U(X,Y) + iV(X,Y)$ BE DEFINED IN A DOMAIN S AND LET Z_0 BE A POINT IN S (i.e. $Z_0 \in S$).

THEN, IF

(i) U_X, U_Y, V_X, V_Y EXIST IN A NEIGHBORHOOD OF Z_0 AND ARE <u>CONTINUOUS</u> AT Z_0

AND (ii) IF CR EQUATIONS ARE SATISFIED AT $Z_0 = X_0 + iY_0$

i.e. $U_X = V_Y$, $V_X = -U_Y$ at (X_0, Y_0)

THEN f IS DIFFERENTIABLE AT Z_0.

THUS IF U_X, U_Y, V_X, V_Y EXIST AND ARE CONTINUOUS IN S AND CR HOLD IN S, THEN $f(z)$ IS ANALYTIC IN S. □

<u>REMARK</u> (i) THE PROOF IS TECHNICAL (SEE P 75 OF [SS]).

(ii) NOTE THAT THE CONTINUITY ASSUMPTION IN (i) IS NEEDED.

WE CAN SUMMARIZE THEOREM 1 AND 2 AS THE FOLLOWING:

(TH1) IF $f(z)$ IS DIFFERENTIABLE AT ANY $z \in S$ (i.e. ANALYTIC IN S)

$\Longrightarrow$ CR EQUATIONS ARE SATISFIED AT EACH z IN S.

THUS IF CR NOT SATISFIED AT SOME $Z_0 \in S$

$\Longrightarrow$ $f(z)$ IS NOT DIFFERENTIABLE AT Z_0

(TH 2) IF CR ARE SATISFIED AT ANY $z \in S$,
AND u_x, u_y, v_x, v_y CONTINUOUS AT ANY $z \in S$
$\longrightarrow$ $f(z)$ IS ANALYTIC IN S.

EXAMPLE 1 LET $f(z) = |z|^2$. THEN WITH $z = x + iy$

$$f(z) = x^2 + y^2 + i\,0$$

$$u = x^2 + y^2,\ v = 0 \longrightarrow \begin{array}{l} u_x = v_y \longrightarrow 2x = 0 \\ u_y = -v_x \longrightarrow 2y = 0. \end{array}$$

THUS CR SATISFIED ONLY AT $x = y = 0$. ALSO u_x, u_y, v_x, v_y ARE CONTINUOUS ALWAYS. TH 2 YIELDS $f(z)$ IS DIFFERENTIABLE ONLY AT $z = 0$. IT IS NOT ANALYTIC AT $z = 0$ SINCE $f(z)$ IS NOT DIFFERENTIABLE AT ANY POINT IN A SMALL NEIGHBORHOOD OF $z = 0$.

EXAMPLE 2 LET $f(z) = x^3 + 3xy^2 - 3x + i(y^3 + 3x^2y - 3y)$.
SHOW THAT $f(z)$ IS DIFFERENTIABLE ON COORDINATE AXES BUT IS NOWHERE ANALYTIC.

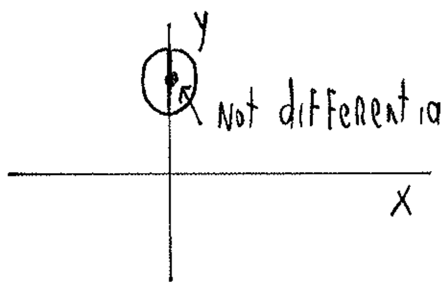
PROOF $u = x^3 + 3xy^2 - 3x \qquad v = y^3 + 3x^2y - 3y$

$$u_x = 3x^2 + 3y^2 - 3 \quad = \quad v_y = 3y^2 + 3x^2 - 3.$$

$$u_y = 6xy \qquad v_x = 6xy$$

NOTE: $u_x = v_y$ FOR ANY x, y BUT $u_y = -v_x \longrightarrow 12xy = 0$.
THUS WE NEED EITHER $x = 0$ OR $y = 0$ FOR CR TO BE SATISFIED.

SO, CR ARE SATISFIED ONLY ON $X=0$ OR $y=0$ AND U_x, U_y, V_x, V_y ARE CONTINUOUS $\longrightarrow$ $f(z)$ IS DIFFERENTIABLE ON $X=0$ AND ON $y=0$. NOTE: $f(z)$ IS NOWHERE ANALYTIC SINCE WE CAN NEVER INSERT A SMALL NEIGHBORHOOD ABOUT A POINT ON COORDINATE AXES FOR WHICH f IS DIFFERENTIABLE AT EACH POINT IN THE NEIGHBORHOOD.



Not differentiable in here, only on $X=0$.

EXAMPLE 3 THE FUNCTION $f(z) = \begin{cases} (\bar{z})^2/z, & \text{IF } z \neq 0 \\ 0, & \text{IF } z=0 \end{cases}$

IS NOT DIFFERENTIABLE AT $z=0$, BUT THE CR EQUATIONS ARE SATISFIED AT $z=0$. (SEE HW #3)

IS THIS CONTRADICTING THEOREM 2? NO, ONE CAN SHOW THAT IF WE WRITE $f(z) = u+iv$ THEN u_x, u_y, v_x, v_y ARE NOT ALL CONTINUOUS AT $x=y=0$.

EXAMPLE 4 LET $u(x,y)$ BE GIVEN, FIND THE FUNCTION $v(x,y)$ (THE HARMONIC CONJUGATE) SO THAT $f=u+iv$ IS ANALYTIC.

(i) LET $u = x^3 - 3xy^2 + y$. WE WILL FIND V BY CR EQUATIONS

NOW $\quad U_x = 3x^2 - 3y^2 = V_y$.

THUS $\quad V = 3x^2 y - y^3 + h(x)$

$$V_x = 6xy + \underline{h'(x)} = -[U_y] = -[-6xy + \underline{1}]$$

THUS, $\quad h'(x) = -1$ OR $\quad h(x) = -x$. (IGNORE CONSTANT WLOG).

THIS GIVES $\quad V = 3x^2 y - y^3 - x$

SO $\quad f(z) = x^3 - 3xy^2 + y + i[3x^2 y - y^3 - x]$.

(ii) LET $U = x^2 - y^2$. FIND HARMONIC CONJUGATE V.

NOW $\quad U_x = V_y \longrightarrow 2x = V_y \quad$ SO $\quad V = 2xy + h(x)$

$\quad U_y = -V_x \longrightarrow -2y = -[2y + h'(x)] \longrightarrow h'(x) = 0$.

TAKE $h(x) = 0$ WLOG.

SO $\quad V = 2xy \quad$ AND $\quad f(z) = x^2 - y^2 + 2ixy$ IS ANALYTIC

Notice Also $\quad f(z) = z^2$.

<u>REMARKS</u>

(i) CAUCHY RIEMANN IN "POLAR" COORDINATES $\qquad r^2 = x^2 + y^2$

LET $\quad f(z) = U(x,y) + iV(x,y) \qquad \tan\varphi = y/x$

$$\bar{U}(r,\varphi) = U(r\cos\varphi, r\sin\varphi), \quad \bar{V}(r,\varphi) = U(r\cos\varphi, r\sin\varphi)$$

NOW CALCULATE $\quad r_x = x/r = \cos\varphi, \quad r_y = \sin\varphi, \quad \varphi_x = -y/(x^2+y^2) = -\sin\varphi/r,$

$$\varphi_y = \frac{x}{x^2+y^2} = \frac{\cos\varphi}{r}.$$

$$U_x = \bar{U}_r r_x + \bar{U}_\varphi \varphi_x$$

$$\rightarrow U_x = \bar{U}_r \cos\varphi + \bar{U}_\varphi\left(-\sin\varphi/r\right)$$

$$U_y = \bar{U}_r r_y + \bar{U}_\varphi \varphi_y = \bar{U}_r \sin\varphi + \bar{U}_\varphi \cos\varphi/r.$$

SIMILARLY $\quad V_x = \bar{V}_r \cos\varphi - \frac{1}{r}\sin\varphi \bar{V}_\varphi, \quad V_y = \bar{V}_r \sin\varphi + \frac{1}{r}\cos\varphi \bar{V}_\varphi.$

NOW SET $U_x = V_y \longrightarrow \left(U_r - \frac{1}{r}V_\varphi\right)\cos\varphi - \left(V_r + U_\varphi/r\right)\sin\varphi = 0$

$U_y = -V_x \longrightarrow \left(\frac{1}{r}U_\varphi + V_r\right)\cos\varphi + \left(U_r - \frac{1}{r}V_\varphi\right)\sin\varphi = 0.$

THIS HAS THE FORM $\begin{pmatrix} a & -b \\ +b & a \end{pmatrix}\begin{pmatrix}\cos\varphi \\ \sin\varphi\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ $\quad a = U_r - \frac{1}{r}V_\varphi$, $\quad b = V_r + \frac{1}{r}U_\varphi.$

THUS TAKING THE DETERMINANT AND SETTING $=0$ TO ENSURE A NONTRIVIAL SOLUTION $\longrightarrow$ REQUIRE $a^2 + b^2 = 0$

$\longrightarrow a = 0$ AND $b = 0$

THUS $\boxed{U_r = \frac{1}{r}V_\varphi \qquad V_r = -\frac{1}{r}U_\varphi.}$ ARE CR IN POLAR FORM.

<u>EXAMPLE</u> (i) SHOW THAT $U = r^n\cos n\varphi$, $V = r^n\sin n\varphi$ $\quad n > 0$ n integer SATISFY CR, AND SINCE THEY ARE SMOOTH FUNCTIONS IT FOLLOWS THAT $f = U + iV$ IS ANALYTIC.

(ii) <u>ANALYTIC FUNCTIONS MUST BE IN TERMS OF z.</u>

LET $f(x,y) = u(x,y) + iv(x,y).$ (1)

SUPPOSE u, v ARE SMOOTH FUNCTIONS AND LET

(*) $x = (z + \bar{z})/2$, $\quad y = (z - \bar{z})/2i$

SUPPOSE THAT CR ARE SATISFIED. SHOW THAT IF WE SUBSTITUTE (*) INTO (1) THEN THERE IS <u>NO</u> $\bar{z}$-DEPENDENCE.

<u>DERIVATION</u> LET $\hat{f}(z,\bar{z}) = f\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$

WE CALCULATE :

$$\frac{\partial \hat{f}}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{1}{2} - \frac{\partial f}{\partial y}\frac{1}{2i}$$

SO $$\frac{\partial \hat{f}}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\Big((u_x + i v_x) + i(u_y + i v_y)\Big).$$

THUS $$\frac{\partial \hat{f}}{\partial \bar{z}} = \frac{1}{2}\Big((u_x - v_y) + i(v_x + u_y)\Big) = 0$$ SINCE $u_x = v_y$ by CR, $u_y = -v_x$

HENCE $$\frac{\partial \hat{f}}{\partial \bar{z}} = 0. \rightarrow \hat{f} = \hat{f}(z).$$

IF (1) IS ANALYTIC IN A DOMAIN S, THEN THERE WILL BE NO $\bar{z}$ DEPENDENCE IF WE SUBSTITUTE (*) INTO (1).

(iii) <u>THEOREM 3</u> IF $f(z)$ IS ANALYTIC IN A DOMAIN S AND IF $f'(z) = 0$ EVERYWHERE IN S THEN $f(z)$ IS A CONSTANT IN S. (<u>RECALL</u>: DOMAIN IS OPEN AND CONNECTED)

<u>PROOF</u> WE WILL GIVE THE IDEA IN CLASS (SEE p. 76 AND SECTION 1.6 P. 40 OF SAFF-SNIDER.

<u>EXAMPLE</u> SUPPOSE THAT $RE[f(z)]$ IS CONSTANT INSIDE A DOMAIN S AND $f(z)$ IS ANALYTIC IN S. PROVE THAT $f(z)$ IS CONSTANT IN S.

<u>PROOF</u> $u = RE[f(z)]$. SINCE u IS CONSTANT, THEN $u_x = u_y = 0$. BUT BY CR, WE GET $v_x = v_y = 0$.

RECALL $f'(z) = u_x + i v_x$. HENCE $f'(z) = 0$.

BY THEOREM 3, $f(z) =$ CONSTANT IN S.

(iv) JACOBIAN

SUPPOSE THAT $f(z) = u + iv$ WITH $u(x,y) = u$, $v(x,y) = v$, IS ANALYTIC IN S. SUPPOSE WE THINK OF CHANGING COORDINATES

$(x,y) \to (u,v)$ VIA

$$u = u(x,y) \qquad v = v(x,y).$$

WHAT IS THE JACOBIAN OF THE TRANSFORMATION?

$$\left.\begin{aligned} \Delta u &= u_x \Delta x + u_y \Delta y + \cdots \\ \Delta v &= v_x \Delta x + v_y \Delta y + \cdots \end{aligned}\right\} \text{FROM TAYLOR-SERIES IN 2 VARIABLES}$$

THUS AS A MATRIX $\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix}$

$J = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$ IS JACOBIAN.

BY CR EQUATIONS, $\det J = u_x v_y - v_x u_y = u_x^2 + v_x^2 = |f'(z)|^2$.

THUS $\boxed{\det J = |f'(z)|^2}$.

(v) LEVEL CURVES

IF $f(z)$ IS ANALYTIC AND WE WRITE

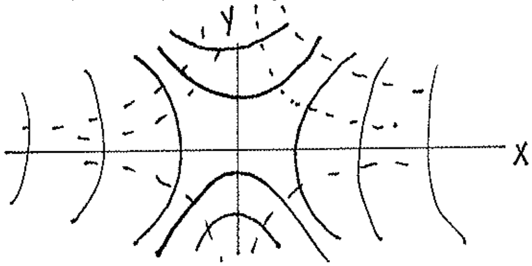
$$f(z) = u(x,y) + i\,v(x,y)$$

THEN WE CLAIM THAT THE LEVEL CURVES

$$u(x,y) = \text{CONSTANT} \qquad \text{AND} \qquad v(x,y) = \text{CONSTANT}$$

ARE ORTHOGONAL AT EVERY POINT WHERE $f'(z) = 0$.

EX: $f(z) = z^2 = (x^2 - y^2) + 2ixy$.



SOLID: level lines for $u = x^2 - y^2 = $ CONSTANT

dotted level lines FOR $v = 2xy = $ constant

PROOF THE LEVEL LINES ARE ORTHOGONAL IF

$$\nabla u \cdot \nabla v = 0 \qquad \text{(RECALL } \nabla u \text{ IS } \perp \text{ to } u = \text{constant).}$$

$$\longrightarrow \quad (u_x, u_y) \cdot (v_x, v_y) = u_x v_x + u_y v_y = -u_x u_y + u_y u_x = 0$$

BY CR EQUATIONS.

THUS $\nabla u \cdot \nabla v = 0$.

CONSEQUENTLY, IT IS EASY TO FIND LEVEL CURVES THAT ARE ORTHOGONAL. SIMPLY TAKE THE REAL AND IMAGINARY PART OF A COMPLEX FUNCTION.

EX
$$f(z) = z^2 = x^2 - y^2 + 2ixy$$
$$f(z) = e^z = e^x \cos y + i e^x \sin y$$

(vi) HARMONIC FUNCTIONS

A HARMONIC FUNCTION $H(x,y)$ IS ONE FOR WHICH H SATISFIES LAPLACE'S EQUATION

$$H_{xx} + H_{yy} = 0.$$

THUS, H CAN BE INTERPRETED AS A STEADY-STATE TEMPERATURE DISTRIBUTION. TYPICALLY SOME BOUNDARY CONDITIONS FOR H MUST BE GIVEN.

WE NOW SHOW AN IMPORTANT RESULT. IF $f(z) = u(x,y) + iv(x,y)$ IS ANALYTIC IN A DOMAIN S THEN

$$(*) \quad \begin{cases} u_{xx} + u_{yy} = 0 & \text{IN } S \\ v_{xx} + v_{yy} = 0 & \text{IN } S \end{cases}$$

i.e. BOTH u AND v SATISFY LAPLACE'S EQUATION.

REMARK (i) ONE MIGHT THINK THAT EXTRA CONDITIONS TO ENSURE THAT u_{xx}, u_{yy} ETC, EXIST NEED TO BE IMPOSED. WE DO NOT WORRY ABOUT THIS HERE. IN FACT WE SHOW LATER IN COURSE THAT IF $f(z)$ IS ANALYTIC THEN ALL HIGHER DERIVATIVES f', f'', f''', .. EXIST!

THE LITTLE PROOF OF (*) IS EASY.

WE HAVE BY ANALYTICITY THAT CR ARE SATISFIED

$$u_x = v_y$$

$$u_y = -v_x$$

THUS IF u, v SMOOTH ENOUGH (NOT AN EXTRA CONDITION BY REMARK I)

THEN $$(u_x)_x = (v_y)_x = (v_x)_y = (-u_y)_y .$$

HENCE $$u_{xx} + u_{yy} = 0$$

SIMILARLY $$v_{xx} + v_{yy} = 0 .$$

WE WILL GIVE EXAMPLES OF SOLVING LAPLACE'S EQUATION THROUGH ELEMENTARY MAPPINGS IN NEXT SECTION.