SOME SIMPLE FUNCTIONS

THE COMPLEX EXPONENTIAL IS DEFINED BY

$$e^{z} = e^{x}[\cos y + i\sin y] = e^{x+iy} = e^{x}e^{iy}.$$

NOW SINCE $e^{i\varphi} = \cos\varphi + i\sin\varphi$ HAS PROPERTIES AS WRITTEN ON PAGE (F8) THEN IT FOLLOWS THAT IF z_1, z_2 ARE COMPLEX NUMBERS

(i) $e^{z_1+z_2} = e^{z_1}e^{z_2}$

(ii) $e^{z_1-z_2} = e^{z_1}/e^{z_2}$

(iii) $e^{z+2\pi i} = e^{z}$, $\quad e^{z+\pi i} = e^{z}[\cos(\pi)+i\sin(\pi)] = -e^{z}$.

(iv) $\overline{(e^{z})} = e^{\bar{z}}$

(v) $|e^{z}| \leq 1$ IF $\text{RE}(z) \leq 0$

(vi) $d/dz\ e^{z} = e^{z}$, e^{z} IS AN ENTIRE FUNCTION.

WE NOW DERIVE SOME OF THESE PROPERTIES:

(i) $e^{z_1+z_2} = e^{x_1+x_2+i(y_1+y_2)} = e^{x_1+x_2}e^{i(y_1+y_2)} = e^{x_1+x_2}e^{iy_1}e^{iy_2}$ by (iii) on Page F8.

THUS $e^{z_1+z_2} = e^{x_1+iy_1}e^{x_2+iy_2} = e^{z_1}e^{z_2}$.

(ii) IS SAME

(iii) $e^{z+2\pi i} = e^{x+iy+2\pi i} = e^{x+iy}e^{2\pi i} = e^{x+iy} = e^{z}$ SINCE $e^{2\pi i} = 1$.

(iv) $\overline{e^{z}} = \overline{e^{x}e^{iy}} = e^{x}(\overline{e^{iy}}) = e^{x}[\cos y - i\sin y] = e^{x}e^{-iy} = e^{x-iy} = e^{\bar{z}}$.

(v) $|e^{z}| = |e^{x}\cos y + ie^{x}\sin y| = \sqrt{e^{2x}\cos^2 y + e^{2x}\sin^2 y} = e^{x}$

SO $|e^{z}| = e^{x} \leq 1$ IF $x \leq 0$ I.E. RE $z \leq 0$.

(vi) WE PROCEED BY DEFINITION: $e^{z} = e^{x}\cos y + ie^{x}\sin y$. IT IS EASY TO SEE THAT $u = e^{x}\cos y$, $v = e^{x}\sin y$ SATISFY CR EQUATIONS AND u_x, u_y, v_x, v_y ARE CONTINUOUS. THUS $f(z) = e^{z}$ IS ANALYTIC everywhere, I.E. AN entire FUNCTION.

WE THEN CALCULATE

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \quad \text{where } h = \Delta z$$

$$= \lim_{\Delta z \to 0} \frac{e^{z+\Delta z} - e^{z}}{\Delta z} = e^{z} \lim_{\Delta z \to 0} \frac{e^{\Delta z} - 1}{\Delta z}$$

$$= e^{z} \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} = e^{z}$$

SINCE WE showed that f is Analytic, we can take $\Delta z = \Delta x + i0$ WLOG.

EXAMPLE 1 FIND ALL z FOR WHICH

$$e^{z} = -4i.$$

WE WRITE $e^{x+iy} = 4e^{-i\pi/2}$

TAKING $|\ |$ OF BOTH SIDES $|e^{x+iy}| = |e^{x}||e^{iy}| = e^{x} = 4|e^{-i\pi/2}| = 4.$

THUS $e^{x} = 4 \longrightarrow x = +\ln 4$

$e^{iy} = e^{-i\pi/2} \longrightarrow y = -\pi/2 + 2k\pi \quad k = 0, 1, 2, \ldots$

THUS $z = +\ln 4 + i\left(-\pi/2 + 2k\pi\right) \quad k = 0, 1, \ldots$

NOW WE VIEW $w = e^{z}$ AS A MAPPING FUNCTION

EXAMPLE 2 FIND THE IMAGE UNDER $w = ie^{z}$ OF THE REGION $S = \{ z \mid 0 \leq \text{Re}\, z \leq 1,\ 0 \leq \text{Im}\, z \leq \pi \}$.

SOLUTION WE WRITE THE MAP IN TWO STEPS:

$\tilde{W} = e^{z}$

$w = i\tilde{W}$. counterclockwise rotation by $e^{i\pi/2} \rightarrow \pi/2^{\circ}$ rotation.

WE FIRST CONSIDER

$$W = e^Z = U + iV$$

WE PUT $Z = X + iy$ WITH $0 \le X \le 1$ <u>FIXED</u> AND $0 \le y \le \pi$.

THEN $U + iV = e^X \cos y + i e^X \sin y$.

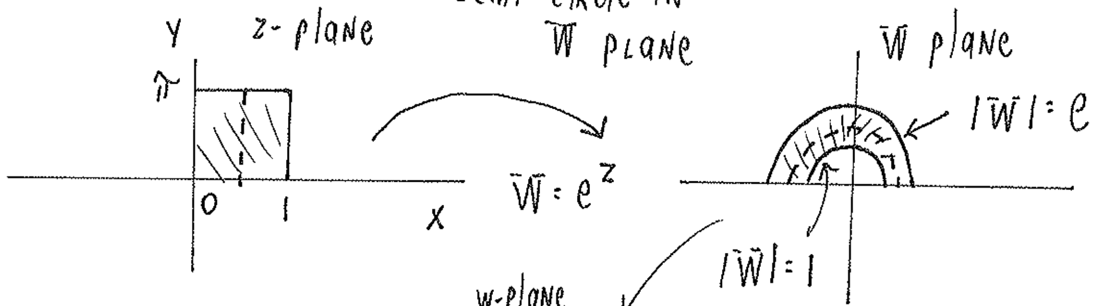
$\longrightarrow$ $U = e^X \cos y$, $V = e^X \sin y$.

FOR X $\left(U/e^X\right)^2 + \left(V/e^X\right)^2 = 1$ CIRCLE OF RADIUS e^X

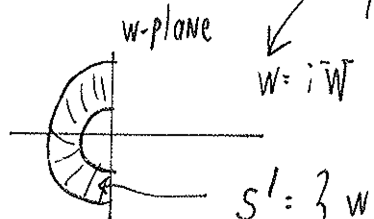
BUT $0 \le y \le \pi$ IMPLIES $V \ge 0$. i.e. $\mathrm{IM}(W) \ge 0$

<u>NOTE:</u> Dotted line in z plane maps to dotted semi-circle in W plane

WE OBTAIN



THEN $w = iW$ rotates by $\pi/2$.



$S' = \{ w \mid 1 \le |w| \le e^1, \ \mathrm{RE}\, w \le 0 \}$.

<u>EXAMPLE 3</u> FIND IMAGE OF $S = \{ z \mid -\pi < \mathrm{IM}(z) \le \pi \}$ UNDER $W = e^Z$.

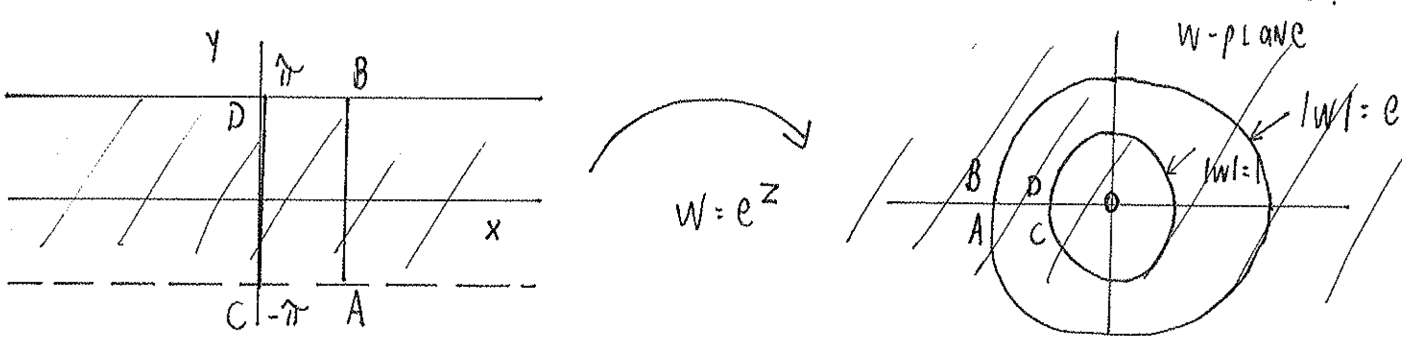
WE TAKE A VERTICAL LINE WITH $\mathrm{RE}(Z) = X$ FIXED AND $-\pi < y \le \pi$.

THEN $u + iv = e^X(\cos y + i \sin y) \longrightarrow u = e^X \cos y$, $v = e^X \sin y$.

HENCE $\left(u/e^X\right)^2 + \left(v/e^X\right)^2 = 1$ circle of radius e^X with $-\infty < X < \infty$

i.e. of radius in $(0, \infty)$

WE GET FULL CIRCLE SINCE $-\pi < y \le \pi$ SWEEPS A FULL PERIOD.



THIS GENERATES THE WHOLE OF COMPLEX W-PLANE EXCEPT $W = 0$.

THE NEXT SIMPLE FUNCTIONS ARE SIN Z AND COS Z.

WE RECALL THAT

$$e^{iX} = \cos X + i\sin X, \quad e^{-iX} = \cos(-X) + i\sin(-X) = \cos X - i\sin X$$

SO THAT
$$\sin X = \frac{e^{iX} - e^{-iX}}{2i}, \quad \cos X = \frac{e^{iX} + e^{-iX}}{2}.$$

THIS SUGGESTS THAT FOR A COMPLEX VARIABLE $Z = X + iy$

WE DEFINE

$$* \left\{ \sin Z \equiv \frac{e^{iZ} - e^{-iZ}}{2i}, \quad \cos Z \equiv \frac{e^{iZ} + e^{-iZ}}{2}. \right.$$

SINCE e^{iz}, e^{-iz} ARE ANALYTIC EVERYWHERE, i.e. ENTIRE FUNCTIONS, THE $\sin Z$, $\cos Z$ ARE ENTIRE FUNCTIONS.

FOLLOWING THE DEFINITION (*) WE HAVE MANY KEY PROPERTIES:

(i) $\frac{d}{dz} \sin Z = \cos Z, \quad \frac{d}{dz} \cos Z = -\sin Z$

(ii) $\sin(Z + 2\pi) = \sin Z, \quad \cos(Z + 2\pi) = \cos Z.$

(iii) $\sin(-Z) = -\sin Z, \quad \cos(-Z) = \cos Z$

(iv) $\cos Z = \cos X \cosh y - i \sin X \sinh y = \mathrm{RE}[\cos Z] + i\, \mathrm{IM}[\cos Z]$

$\sin Z = \cosh y \sin X + i \cos X \sinh y = \mathrm{RE}[\sin Z] + i\, \mathrm{IM}[\sin Z].$

(v) FROM (iv) $\longrightarrow$ $\cos(iy) = \cosh y, \quad \sin(iy) = i \sinh y.$ (set $X = 0$ IN (iv))

RECALL $\quad \sinh y = \frac{e^{y} - e^{-y}}{2}, \quad \cosh y = \frac{e^{y} + e^{-y}}{2}$

(vi) $\overline{\cos Z} = \cos(\bar{z})$

(vii) $|\cos Z| = (\sinh^2 y + \cos^2 X)^{1/2}$ NOTE: $|\cos z| \to \infty$ AS $|y| \to \infty$

$|\sin Z| = (\sinh^2 y + \sin^2 X)^{1/2}$ $\quad |\sin Z| \longrightarrow \infty$ AS $|y| \to \infty$.

(viii) $\cos^2 Z + \sin^2 Z = 1 \quad \forall Z.$

WE NOW PROVE THESE RESULTS FROM DEFINITION (*)

(i) $\frac{d}{dz}\sin z = \frac{d}{dz}\left(\frac{e^{iz}-e^{-iz}}{2i}\right) = \frac{1}{2i}\left(ie^{iz}+ie^{-iz}\right) = \frac{1}{2}\left(e^{iz}+e^{-iz}\right) = \cos z.$

$\frac{d}{dz}\cos z = \frac{d}{dz}\left(\frac{e^{iz}+e^{-iz}}{2}\right) = \frac{1}{2}\left(ie^{iz}-ie^{-iz}\right) = \frac{i}{2}\left(e^{iz}-e^{-iz}\right) = -\frac{1}{2i}\left(e^{iz}-e^{-iz}\right) = -\sin z$

(ii) $\sin(z+2\pi) = \frac{e^{i(z+2\pi)}-e^{-i(z+2\pi)}}{2i} = \frac{e^{iz}-e^{-iz}}{2i}$ SINCE $e^{\pm 2\pi i} = 1.$

(iii) IS CLEAR

(iv) $\cos z = \frac{e^{iz}+e^{-iz}}{2} = \frac{1}{2}\left(e^{i(x+iy)}+e^{-i(x+iy)}\right)$

$= \frac{1}{2}e^{-y}(\cos x + i\sin x) + \frac{1}{2}e^{y}(\cos x - i\sin x)$

$= \cos x\left[\frac{1}{2}(e^{y}+e^{-y})\right] - i\sin x\left(\frac{e^{y}-e^{-y}}{2}\right) = \cos x\cosh y - i\sin x\sinh y.$

similar calculation FOR $\sin z = \sin(x+iy)$ (omitted).

(v) SET $x=0 \longrightarrow \cos(iy) = \cosh y - i\sin(0)\sinh y = \cosh y$

$\sin(iy) = 0 + i\cos(0)\sinh y = i\sinh y.$

THUS $\cos(iy)$ INCREASES exponentially AS $y \to +\infty$.

(vi) $\overline{\cos z} = \frac{1}{2}\overline{\left(e^{iz}+e^{-iz}\right)} = \frac{1}{2}\left(\overline{e^{iz}}+\overline{e^{-iz}}\right) = \frac{1}{2}\left(e^{-i\bar z}+e^{i\bar z}\right) = \cos(\bar z).$

RECALL $\overline{e^{iw}} = e^{\overline{iw}} = e^{-i\bar w}.$

(vii) USE RESULTS IN (iv) TO OBTAIN USING $\cosh^2 y = 1+\sinh^2 y$

$|\cos z| = \sqrt{\cos^2 x\cosh^2 y + \sin^2 x\sinh^2 y} = \sqrt{\cos^2 x\left(1+\sinh^2 y\right) + \sin^2 x\sinh^2 y}$

$|\cos z| = \sqrt{\cos^2 x + \sinh^2 y}$ UPON USING $\sin^2 x + \cos^2 x = 1.$

SIMILAR CALCULATION FOR $|\sin z|$ (OMITTED).

(viii) $\cos^2 z + \sin^2 z = \frac{1}{4}\left(e^{iz}+e^{-iz}\right)^2 - \frac{1}{4}\left(e^{iz}-e^{-iz}\right)^2 = \frac{1}{4}\left(e^{2iz}+2+e^{-2iz}\right)$

$- \frac{1}{4}\left(e^{2iz}-2+e^{-2iz}\right)$

$\cos^2 z + \sin^2 z = \frac{1}{2}+\frac{1}{2} = 1. \longleftarrow$

EXAMPLE 1 FIND ALL THE ROOTS OF

$$\sin z = \cosh 4.$$

WE WRITE $\sin z = \sin(x+iy) = \cosh y \sin x + i \cos x \sinh y = \cosh 4.$

THUS $\cos x \sinh y = 0 \rightarrow$ either $y = 0$ OR $x = (2n+1)\pi/2 \quad n = 0, \pm 1, \pm 2, ..$

AND $\cosh y \sin x = \cosh 4 \rightarrow$ this implies $y \neq 0$ since $\cosh 4 > 1$ and $\sin x = \cosh 4$ impossible

THUS $\sin\left[\frac{(2n+1)\pi}{2}\right] = 1$ IS NEEDED, AND $\cosh y = \cosh 4$ implies $y = 4$, AND $y = -4$.

SO $n = 0, \pm 2, \pm 4, \ldots$

HENCE $z_n = (2n+1)\pi/2 \pm 4i$ FOR $n = 0, \pm 2, \pm 4, \ldots$

EXAMPLE 2 FIND ALL ROOTS OF $\cos z = 2.$

$$\cos z = \cos x \cosh y - i \sin x \sinh y = 2.$$

THUS $\cos x \cosh y = 2$

$\sin x \sinh y = 0 \rightarrow$ either $y = 0$ OR $x = n\pi, \quad n = \pm 1, \pm 2, .. 0$

NOTE $y = 0$ IMPOSSIBLE SINCE THIS GIVES $\cos x = 2.$

ALSO NEED $\cos[n\pi] = 1$ SO $n = 0, \pm 2, \pm 4, ..$

NOW $\cosh y = 2$ IMPLIES $\frac{e^y + e^{-y}}{2} = 2 \rightarrow e^y + e^{-y} - 4 = 0.$

$$\lambda^2 + 1 - 4\lambda = 0 \quad \text{WITH} \quad \lambda = e^y.$$

$$\lambda = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3} \rightarrow y = \ln[2 \pm \sqrt{3}].$$

THUS $z = 2m\pi + i \ln[2 \pm \sqrt{3}] \quad m = 0, \pm 1, \pm 2, ..$

ARE ROOTS OF $\cos z = 2.$

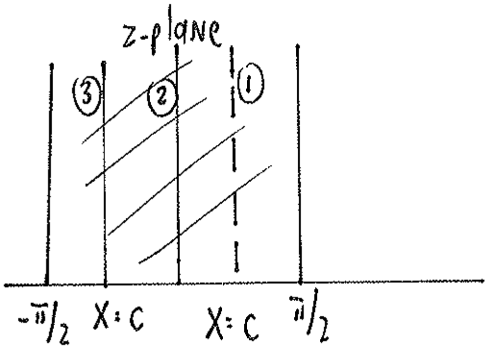
MAPPINGS

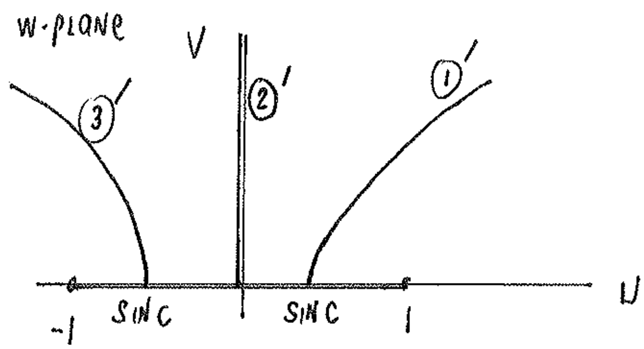
EXAMPLE 3 LET $S = \{ z \mid |RE(z)| \leq \pi/2,\ IM\, z \geq 0 \}$. FIND IMAGE OF S UNDER $W = SIN\, Z$.

SOLUTION $W = U + iV = SIN(X+iy) = COSH\, y\, SIN\, X + i\, COS\, X\, SINH\, y$

THUS $U = \cosh y \sin X, \quad V = \cos X \sinh y.$

NOTICE: SINCE $-\pi/2 \leq X \leq \pi/2$ AND $y \geq 0 \longrightarrow V \geq 0.$

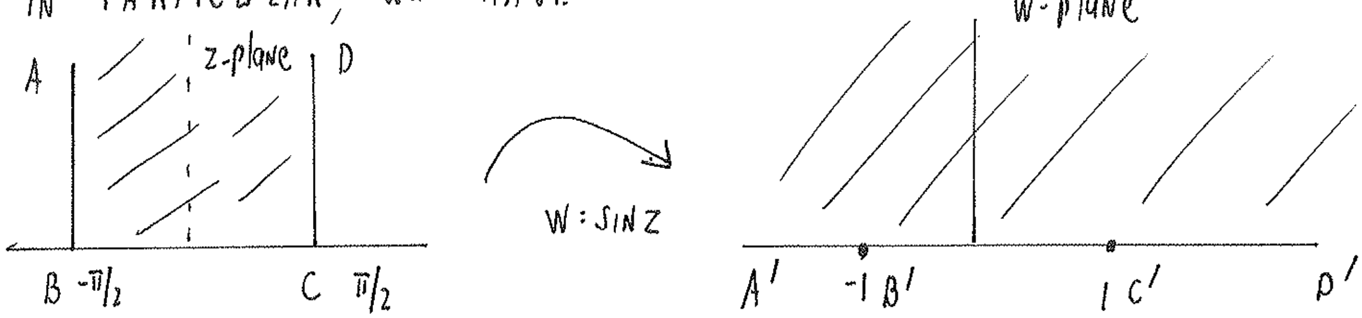




- CONSIDER IMAGE OF $X = C$ WITH $0 < C < \pi/2$ FIXED AND $0 \leq y < \infty$.
  THEN USE $\cosh^2 y - \sinh^2 y = 1$ TO GET
  $$U^2/\sin^2 C - V^2/\cos^2 C = 1 \longrightarrow \text{hyperbola, but with } V \geq 0.$$
  THIS IS LINE ①' IN W-plane
- CONSIDER IMAGE OF $X = 0$, $0 \leq y < \infty$ THEN $U = 0$ AND $0 \leq V \leq \infty$
  THIS IS LINE ②'
- IF WE TAKE $X = C$ WITH $-\pi/2 < C < 0$ THEN $U < 0$ AND $V > 0$.
  WE GET hyperbola $U^2/\sin^2 C - V^2/\cos^2 C = 1$ WITH $U = \sin C$ when $V = 0$.
  this is LINE ③'
- FINALLY IF $-\pi/2 \leq X \leq \pi/2$ AND $y = 0$ THEN $V = 0$ AND $|U| \leq 1$.
  This is a segment of Real AXIS IN W-plane.

CLAIM IMAGE SET $S' = \{ w \mid IM(W) \geq 0 \}$.

IN PARTICULAR, WE HAVE



- LINE $X = -\pi/2$, $0 \le y < \infty$ MAPS TO $U \le -1$, $V = 0$.
- LINE $X = \pi/2$, $0 \le y < \infty$ MAPS TO $U \ge 1$, $V = 0$
- LINE $y = 0$, $|X| \le \pi/2$ MAPS TO $|U| \le 1$ AND $V = 0$.

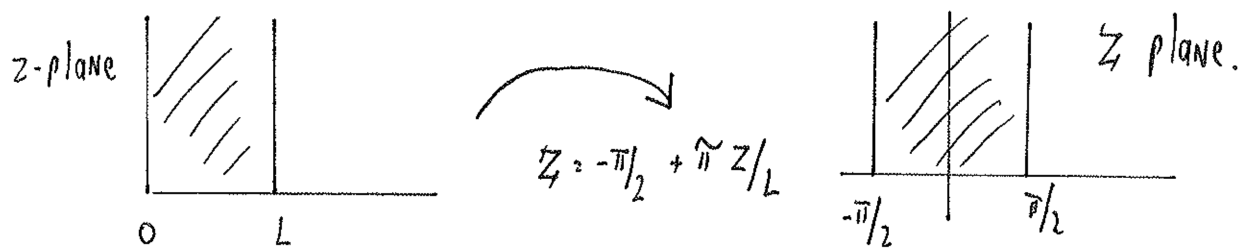
<u>EXAMPLE 4</u> CONSIDER $S = \{ z \mid 0 \le RE(z) \le L,\ 0 \le IM(z) < \infty \}$.

FIND A MAPPING THAT TAKES S TO LOWER HALF PLANE $IM(W) \le 0$.

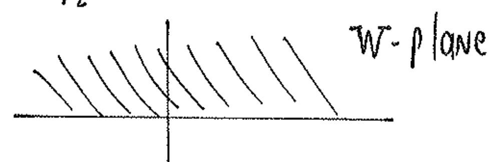
<u>SOLUTION</u> IF WE PUT $z = \frac{L}{\pi} Z_1$ THEN $0 \le RE(z) \le L$ BECOMES

$0 \le \frac{L}{\pi} RE(Z_1) \le L \longrightarrow 0 \le RE(Z_1) \le \pi$. NOW TRANSLATE BY $\pi/2$.

THUS PUT $z = \frac{L}{\pi}\left(Z + \pi/2 \right)$ OR $Z = -\pi/2 + \frac{\pi}{L} z$.

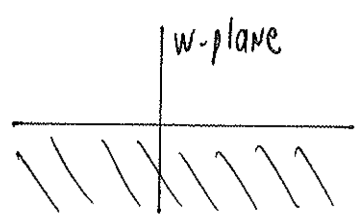


NOW LET $\mathcal{W} = \sin Z$. THIS GIVES



ROTATE BY π. SO $W = \mathcal{W} e^{\pi i}$ BUT $e^{i\pi} = -1$

HENCE $W = -\sin\left[\frac{-\pi}{2} + \frac{\pi z}{L} \right]$ GIVES $IM\, W \le 0$



COMPLEX HYPERBOLIC FUNCTIONS

WE DEFINE $\sinh z = \dfrac{e^{z} - e^{-z}}{2}$, $\cosh z = \dfrac{e^{z} + e^{-z}}{2}$.

WHEN z IS REAL, THESE AGREE WITH USUAL DEFINITION OF $\sinh x$, $\cosh x$.

SOME SIMPLE PROPERTIES ARE

(i) $\sinh z = -i \sin(iz)$, $\cosh z = \cos(iz)$.

(ii) $\sinh(z + 2\pi i) = \sinh z$, $\cosh(z + 2\pi i) = \cosh z$

(iii) $\sinh z$, $\cosh z$ are entire FUNCTIONS with $\dfrac{d}{dz} \sinh z = \cosh z$ AND $\dfrac{d}{dz} \cosh z = \sinh z$.

(iv) $\sinh(iy) = i \sin y$, $\cosh(iy) = \cos y$, $\sinh x = -i \sin(ix)$,
$\cosh x = \cos(ix)$.

(v) $\cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y$
$\sinh(x + iy) = \sinh x \cos y + i \cosh x \sin y$.

THE DERIVATION OF THESE PROPERTIES IS SIMPLE (OMITTED).

EXAMPLE 1 FIND ALL ROOTS OF $\cosh z = 1/2$. NOTICE IF $z = x$ IS REAL, THERE ARE NO ROOTS SINCE $\cosh x \geq 1$. NEED z TO BE COMPLEX.

WE WRITE $\cosh z = \cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y = \dfrac{1}{2}$.

THUS $\cosh x \cos y = 1/2$

$\sinh x \sin y = 0 \longrightarrow x = 0$ OR $y = n\pi$

WE MUST TAKE $x = 0$. THUS $\cos y = 1/2 \longrightarrow y = (\pi/3 + 2n\pi)$ $n = 0, \pm 1, \pm 2, \ldots$
$y = (-\pi/3 + 2n\pi)$

THUS $y = \pm \pi/3 + 2n\pi$ $n = 0, \pm 1, \pm 2, \ldots$ AND $z = iy$.

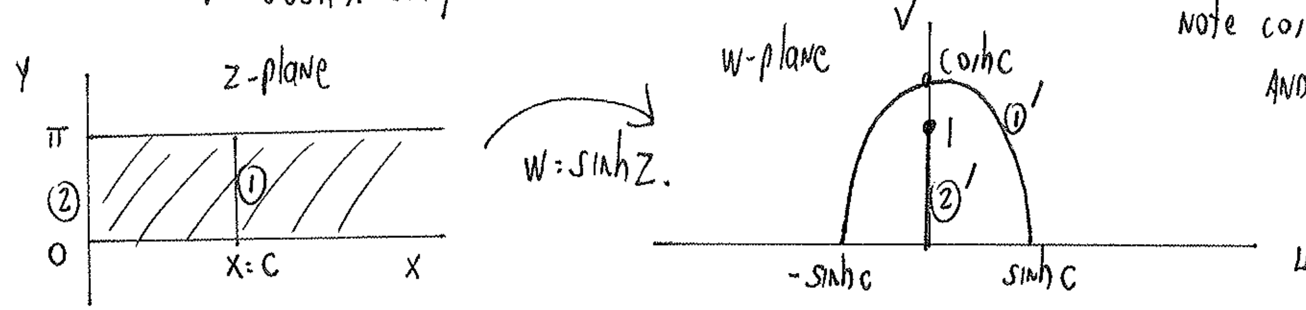
EXAMPLE 2 (MAPPING)

FIND IMAGE OF $S = \{ z \mid 0 \leq \mathrm{Im}(z) \leq \pi,\ \mathrm{Re}\, z \geq 0 \}$ UNDER $w = \sinh z$.

SOL: $u + iv = \sinh(x + iy) = \sinh x \cos y + i \cosh x \sin y$.

THUS $u = \sinh x \cos y$ NOTE IF $0 \leq y \leq \pi \longrightarrow v \geq 0$.

$v = \cosh x \sin y$



Note $\cosh c > \sinh c$ AND $\cosh c > 1$.

- FIND IMAGE OF $x = c$, $c > 0$ FIXED WITH $0 \leq y \leq \pi$.

  THEN $u^2 / \sinh^2 c + v^2 / \cosh^2 c = 1 \longrightarrow$ ELLIPSE BUT NEED $v \geq 0$.

  i.e. ① MAPS TO ①'.

- NOW IF $x = 0$, $0 \leq y \leq \pi \longrightarrow u = 0$, $0 \leq v \leq 1$. i.e. ② MAPS TO ②'.

NOW IF WE SWEEP OUT ALL c IN $0 \leq c < \infty$ WE OBTAIN UPPER 1/2 PLANE $\mathrm{Im}(w) \geq 0$