FURTHER EXAMPLES OF BRANCH CUTS

EXAMPLE 1 FIND THE BRANCH POINTS OF

$$f(z) = (z^3 - z)^{1/3} \quad \text{IN EXTENDED } z\text{-PLANE.}$$

INTRODUCE BRANCH CUTS. IF $f(2) = \sqrt[3]{6}$ WHAT IS $f(-2)$?

SOLUTION WE WRITE

$$f(z) = (z(z^2-1))^{1/3} = z^{1/3}(z-1)^{1/3}(z+1)^{1/3}.$$

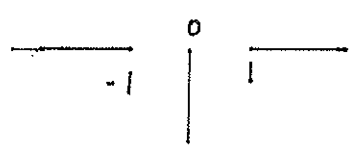
THERE ARE BRANCH POINTS AT $z = -1, 0, 1$. WE CONSIDER THE POINT AT INFINITY. WE LET $z = 1/\zeta$ AND CALCULATE

$$f(1/\zeta) = (1/\zeta)^{1/3}(1/\zeta - 1)^{1/3}(1/\zeta + 1)^{1/3} = \frac{1}{\zeta}(1-\zeta)^{1/3}(1+\zeta)^{1/3}.$$

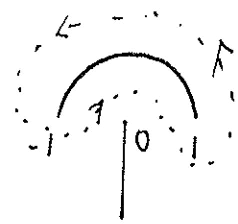
FOR $|\zeta| \ll 1$, $f(1/\zeta) \simeq 1/\zeta$ WHICH DOES NOT HAVE A BRANCH POINT AT $\zeta = 0$. $\longrightarrow$ $z = \infty$ IS NOT A BRANCH POINT FOR $f(z)$.

CONSIDER THE FOLLOWING SETS OF POSSIBLE BRANCH CUTS.

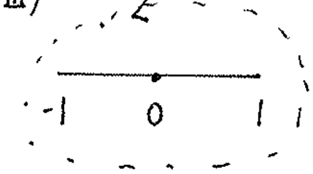
(I)



(II)



(III)



NOTICE (I) AND (III) MAKE $f(z)$ SINGLE-VALUED, WHILE (II) WILL NOT. TO SHOW THAT (II) DOES NOT MAKE f SINGLE-VALUED CONSIDER WALKING ALONG CLOSED DOTTED CURVE IN (II)

THEN IF $\varphi_1, \varphi_2, \varphi_3$ MEASURE ANGLES WRT $z=-1, z=1, z=0$, RESPECTIVELY, IT FOLLOWS THAT φ_1 CHANGES BY 2π, φ_2 CHANGES BY 2π AND φ_3 DOES NOT CHANGE IN GOING AROUND CIRCUIT.

THEN SINCE $f(z) = (r_1 r_2 r_3)^{1/3} e^{i(\theta_1 + \theta_2 + \theta_3)/3}$

IT FOLLOWS THAT THE CHANGE IN $f(z)$ AROUND DOTTED LOOP IS

$$[f(z)] = (r_1 r_2 r_3)^{1/3} [1 - e^{4\pi i/3}] \neq 0.$$

THUS (II) DOES NOT MAKE $f(z)$ SINGLE-VALUED AS IT ALLOWS US TO TAKE A PATH AROUND BOTH $z=-1$ AND $z=1$, BUT NOT $z=0$.

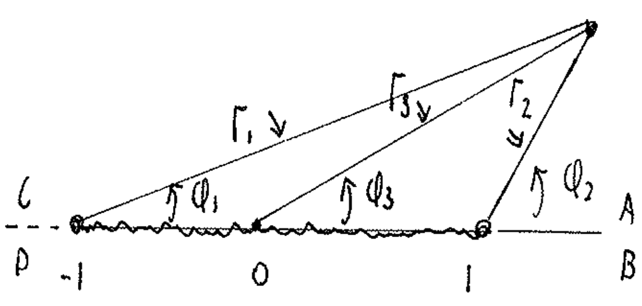
NOTE: (I) MAKES $f(z)$ SINGLE-VALUED SINCE WE CANNOT WALK AROUND ANY INDIVIDUAL BRANCH POINT.

IN (III) IF WE TAKE DOTTED PATH AROUND ALL 3 BRANCH POINTS $z=0,-1,1$, THEN $\theta_1, \theta_2, \theta_3$ ALL CHANGE BY 2π. THUS

$$[f(z)] = (r_1 r_2 r_3)^{1/3} (1 - e^{6\pi i/3}) = 0.$$

$\longrightarrow$ $f(z)$ RETURNS TO SAME VALUE AFTER TAKING THE DOTTED LOOP.

LET'S TRY THE OPTION (III)



$$f(z) = (r_1 r_2 r_3)^{1/3} e^{i(\theta_1 + \theta_2 + \theta_3)/3}$$

TRY
$$\left.\begin{aligned} -\pi < \theta_1 \leq \pi \\ -\pi < \theta_2 \leq \pi \\ -\pi < \theta_3 \leq \pi . \end{aligned}\right\} (*)$$

AT POINT A: $\theta_1 = \theta_2 = \theta_3 = 0$ AT POINT B: $\theta_1 = \theta_2 = \theta_3 = 0 \longrightarrow$ CONTINUOUS

AT POINT C: $\theta_1 = \theta_2 = \theta_3 = \pi \longrightarrow f(z) = (r_1 r_2 r_3)^{1/3} e^{3\pi i/3} = -(r_1 r_2 r_3)^{1/3}$

AT POINT D: $\theta_1 = \theta_2 = \theta_3 = -\pi \longrightarrow f(z) = (r_1 r_2 r_3)^{1/3} e^{-3\pi i/3} = -(r_1 r_2 r_3)^{1/3}$

THUS f IS CONTINUOUS FROM C TO D.

INDEED WITH THIS CHOICE $f(2) = (1 \cdot 2 \cdot 3)^{1/3} e^{i0} = \sqrt[3]{6}$.

NOW $f(-2) = (1 \cdot 2 \cdot 3)^{1/3} e^{3\pi i/3} = -\sqrt[3]{6}$.

EXAMPLE 2 CONSTRUCT A BRANCH OF

$$f(z) = (z^2+1)^{1/3}$$

SUCH THAT $f(0) = \frac{1}{2}(-1 + i\sqrt{3})$.

SOLUTION WE FACTOR

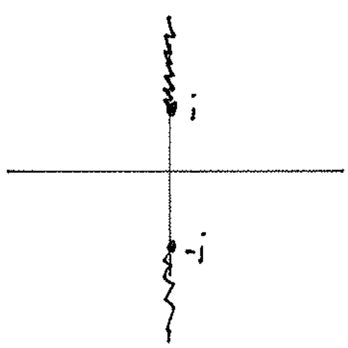
$$f(z) = (z-i)^{1/3}(z+i)^{1/3} = (r_1 r_2)^{1/3} e^{i(\theta_1+\theta_2)/3}$$

THERE ARE BRANCH POINTS AT $z = i$, $z = -i$, AND $z = \infty$.

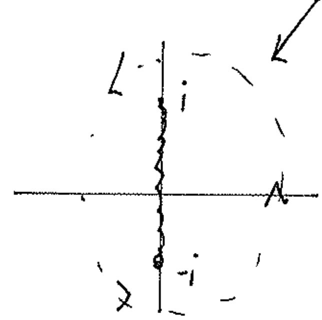
NOTE: IF WE PUT A BRANCH CUT FROM $z = -i$ TO i AS SHOWN IN (II) BELOW, THEN $f(z)$ WOULD NOT BE SINGLE VALUED AS WE COULD TAKE DOTTED PATH FOR WHICH CHANGE IN θ_1 AND θ_2 WOULD BE 2π SO $[f(z)] = (r_1 r_2)^{1/3}[1 - e^{4\pi i/3}] \neq 0$.

POSSIBLE CHOICES INCLUDE

(I)



(II)



THIS DOES NOT MAKE F SINGLE-VALUED

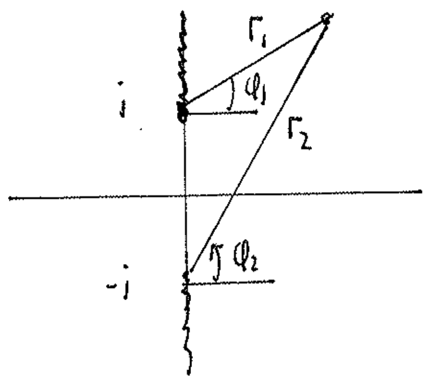
THUS WE RULE OUT (II).

IN (I) SINCE WE CANNOT WALK AROUND ANY INDIVIDUAL BRANCH POINT, THIS CHOICE WILL MAKE $f(z)$ SINGLE VALUED.

SO LET'S TRY (I) AND MAKE A CHOICE FOR THE ANGLES.

CHOICE (i) TRY $-\pi/2 < \theta_2 < 3\pi/2$, $-3\pi/2 < \theta_1 < \pi/2$

THIS WILL CLEARLY MAKE f DISCONTINUOUS ACROSS THE CUTS



$$f(z) = (r_1 r_2)^{1/3} e^{i(\varphi_1 + \varphi_2)/3}.$$

NOW WITH CHOICE (i) WE CALCULATE $f(0)$.

AT $z=0 \longrightarrow \varphi_1 = -\pi/2 \quad r_1 = 1$

$\varphi_2 = \pi/2 \quad r_2 = 1$

SO $f(0) = 1\, e^{i0} = 1.$

THIS IS <u>NOT</u> THE VALUE WE WANT.

SO ALTHOUGH THE BRANCH CUTS MAKE $f(z)$ SINGLE-VALUED, THE CHOICE (i) OF ANGLES GIVES THE <u>WRONG</u> BRANCH AND GIVES $f(0)=1$ INSTEAD OF $f(0) = \frac{1}{2}(-1 + i\sqrt{3})$, AS DESIRED.

SO LET'S TRY <u>CHOICE (ii)</u>

<u>CHOICE (ii)</u> $\quad -\pi/2 < \varphi_2 \le 3\pi/2 \longrightarrow$ SAME AS choice (i).

$\pi/2 < \varphi_1 \le 5\pi/2 \quad$ (i.e. ADD 2π TO CHOICE (i))

WITH THIS CHOICE, AT $z=0$ WE GET

$$\varphi_2 = \pi/2, \quad \varphi_1 = 3\pi/2. \qquad r_1 = r_2 = 1$$

SO $f(0) = 1\, e^{i(\varphi_1+\varphi_2)/3} = e^{i(\pi/2 + 3\pi/2)/3} = e^{2\pi i/3}$

$f(0) = e^{2\pi i/3} = (-1 + i\sqrt{3})/2$ WHICH IS WHAT WE WANTED.

<u>CONCLUSION</u> TAKE THE BRANCH CUTS AS IN (I) AND SPECIFY

$$-\pi/2 < \varphi_2 \le 3\pi/2, \quad \pi/2 < \varphi_1 \le 5\pi/2,$$

FOR THE CHOICE OF ANGLES.

<u>QUESTION FOR STUDENTS:</u> HOW CAN WE GET $f(0) = e^{4\pi i/3}$?