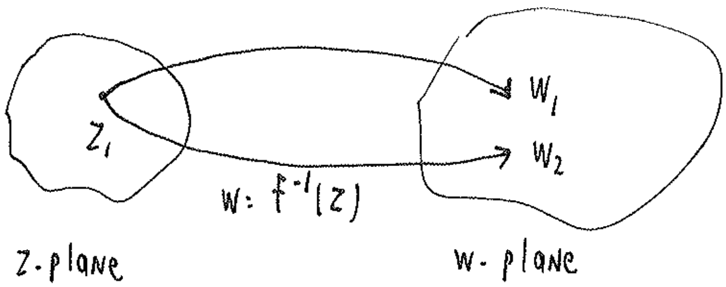
## MULTI-VALUED FUNCTIONS

THERE ARE MANY FUNCTIONS WHOSE INVERSE FUNCTION IS MULTI-VALUED FOR INSTANCE

$$z = e^{w}, \quad z = w^{2}, \quad z = \cos w, \quad z = \sin w.$$

FOR EACH OF THESE FUNCTIONS, A GIVEN VALUE OF $z$ CORRESPONDS TO MORE THAN ONE VALUE OF $w$




- $w = f^{-1}(z)$ is multi-valued
- $z = f(w)$ is single valued. GIVEN A $w$, THERE IS A UNIQUE VALUE OF $z$

**GOALS**

(i) DETERMINE ALL POSSIBLE VALUES OF THE INVERSE FUNCTION $w$.

(ii) CONSTRUCT AN INVERSE FUNCTION THAT IS SINGLE VALUED IN SOME REGION OF COMPLEX PLANE.

## LOGARITHM FUNCTION

DEFINE THE INVERSE FUNCTION FOR $z = e^{w}$.

FOR A GIVEN $z = re^{i\varphi}$ WITH $\varphi = \mathrm{ARG}(z)$ WE WRITE $w = u + iv$ WHERE $u, v$ TO BE FOUND.

SO $$re^{i\varphi} = e^{u+iv}$$

TAKING THE MODULUS WE GET $|re^{i\varphi}| = |e^{u+iv}| \rightarrow r = e^{u}$ SO $u = \ln r$.

THEN $v = \varphi + 2k\pi \qquad k = 0, \pm 1, \pm 2, \ldots$

HENCE $$w = \ln r + i[\mathrm{Arg}(z) + 2k\pi] \qquad k = 0, \pm 1, \pm 2.$$

THIS IS $$w = \ln|z| + i[\mathrm{Arg}(z) + 2k\pi] \qquad -\pi < \mathrm{ARG}(z) \leq \pi, \quad k = 0, \pm 1, \pm 2, \ldots$$

WE DEFINE THE <u>MULTI-VALUED</u> $\log z$ BY

$$w = \log z \equiv \ln|z| + i\left(\mathrm{ARG}(z) + 2k\pi\right) \qquad k = 0, \pm 1, \pm 2 \ldots$$

IT GIVES ALL THE SOLUTIONS $w$ TO $z = e^{w}$.

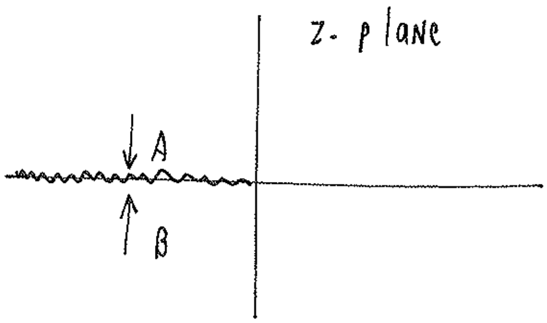
EQUIVALENTLY, WE CAN DEFINE IT AS

$$w = \log z = \ln|z| + i\left(\arg z\right) \qquad \text{since } \arg z \text{ is multi-valued.}$$

WE DEFINE THE <u>PRINCIPAL VALUE</u> OF $\log z$ BY

$$w \;\; \mathrm{LOG}\, z \equiv \ln|z| + i\,\mathrm{Arg}(z).$$

SINCE $-\pi < \mathrm{ARG}(z) \leq \pi$ WE NOTICE THAT $w$ IS NOT CONTINUOUS AT ANY POINT ON THE NEGATIVE REAL AXIS.




A: AS $z = x + iy$ WITH $y \to 0^{+}$ WITH $x < 0$ THEN

$$\mathrm{LOG}\, z \to \ln|x| + i\pi.$$

B: AS $z = x + iy$ WITH $y \to 0^{-}$ WITH $x < 0$ THEN

$$\mathrm{LOG}\, z \to \ln|x| - i\pi.$$

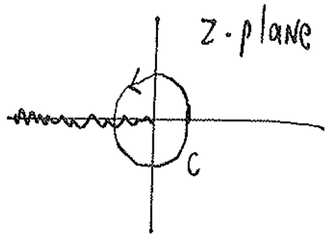
THUS, $\mathrm{LOG}\, z$ IS DISCONTINUOUS ON NEGATIVE REAL AXIS.

<u>REMARK</u> (i) $\mathrm{LOG}\, z$ IS CALLED A "BRANCH" OF THE MULTI-VALUED FUNCTION $\log z$

(ii) $\mathrm{LOG}\, z$ IS CONTINUOUS IN THE CUT PLANE $\mathbb{C} \setminus (-\infty, 0]$.

(iii) THE POINT $z = 0$ IS CALLED A "BRANCH POINT" OF $\mathrm{LOG}\, z$, SINCE IF WE ENCIRCLE $z = 0$ BY

A CLOSED CONTOUR THEN LOG Z CHANGES BY AN AMOUNT PROPORTIONAL TO $2\pi i$




CHANGE IN LOG Z AROUND PATH C IS $2\pi i$.

- IN THE CUT PLANE LOG Z IS ANALYTIC AND $\frac{d}{dz} \text{LOG} z = \frac{1}{z}$.

PROOF $\quad f(z) = \text{LOG}\, z = \frac{1}{2} \ln(x^2+y^2) + i \tan^{-1}(y/x)$

WITH $-\pi < \tan^{-1}(y/x) \leq \pi$.

THEN $\quad u = \frac{1}{2} \ln(x^2+y^2) \qquad v = \tan^{-1}(y/x)$

$$u_x = \frac{x}{x^2+y^2} \qquad v_x = \frac{-y/x^2}{1+y^2/x^2} = \frac{-y}{x^2+y^2}.$$

$$u_y = \frac{y}{x^2+y^2} \qquad v_y = \frac{1/x}{1+y^2/x^2} = \frac{x}{x^2+y^2}.$$

SO $u_x = v_y$, $u_y = -v_x$ PROVIDED $(x,y) \neq (0,0)$.

AND $\quad F'(z) = u_x + i v_x = \frac{x}{x^2+y^2} - \frac{iy}{(x^2+y^2)} = \frac{\bar{z}}{|z|^2} = \frac{\bar{z}}{z\bar{z}} = \frac{1}{z}$.

SO $F'(z) = 1/z$

EXAMPLE CALCULATE THE FOLLOWING:

(i) $\log(2i)$ $\qquad$ (ii) $\text{LOG}(1+i\sqrt{3})$ $\qquad$ (iii) $\text{LOG}(-i)$

SOLUTION

(i) $\log(2i)$: Let $z = 2i$. THEN $\text{Arg}(z) = \pi/2$.

SO $\log(2i) = \ln 2 + i\left[\pi/2 + 2k\pi\right] \quad k = 0, \pm 1, \pm 2, \ldots$

M4

(ii) $1+i\sqrt{3} = 2e^{\pi i/3}$. $\quad \text{ARG}(1+i\sqrt{3}) = \pi/3$

SO $\text{LOG}(1+i\sqrt{3}) = \ln 2 + i\,\pi/3$

(iii) $\text{LOG}(-i)$. LET $z=-i$. $|z|=1$, $\text{Arg}\, z = -\pi/2$.

SO $\text{LOG}(-i) = \ln 1 + i(-\pi/2)$. SO $\text{LOG}(-i) = -i\pi/2$.

ONE MUST BE CAREFUL WITH IDENTITIES INVOLVING $\text{LOG}\, z$, $\log z$.

THE FOLLOWING RESULTS, AS SHOWN IN HW, HOLD

(i) $\text{LOG}(z_1 z_2) \neq \text{LOG}(z_1) + \text{LOG}(z_2)$

(ii) $\log(z_1 z_2) = \log z_1 + \log z_2$
$\log(z_1/z_2) = \log z_1 - \log z_2 \quad (z_2 \neq 0)$
THE COUNTABLY INFINITE set OF values OF the left-hand-side AND right-hand-side are same

(iii) $\log(e^z) \neq z$

(iv) $\text{LOG}(e^z) = z$ IF AND ONLY IF $-\pi < \text{IM}(z) \leq \pi$.

(v) $\log z = -\log(1/z)$

(vi) $z = e^{\log z}$

(vii) $\log(z^{1/n}) = \frac{1}{n}\log z \quad$ $n$: positive integer

(viii) $\log(z^n) \neq n\log z$ IN general ($n$: positive integer).

WE NOW GIVE A PROOF FOR A FEW OF THESE. YOU ARE ASKED TO PROVE THE OTHERS IN THE HOMEWORK.

PROOF (vi) $\log z = \ln|z| + i[\operatorname{Arg} z + 2k\pi]$. (M5)

SO $e^{\log z} = e^{\ln|z| + i[\operatorname{Arg} z + 2k\pi]} = |z| e^{i \operatorname{Arg}(z)} = z$.

PROOF (viii) WE WILL SHOW THAT THE SETS OF VALUES OF

$\log(z^n)$ AND $n \log z$ DO NOT COINCIDE

LET $z = re^{i\varphi}$ WITH $\varphi = \operatorname{Arg} z$. WE GET $z^n = r^n e^{in\varphi}$

THEN $\log(z^n) = \ln(r^n) + i[n\varphi + 2k\pi]$, $k = 0, \pm 1, \pm 2, ..$

• $\log(z^n) = n \ln r + i(n\varphi + 2k\pi)$

BUT • $n \log z = n[\ln r + i(\varphi + 2m\pi)] = n \ln r + i[n\varphi + 2mn\pi]$.

COMPARING THESE TWO SETS IS EQUIVALENT TO COMPARING

$\{2k\pi\}$ AND $\{2mn\pi\}$

$k = 0, \pm 1, \pm 2..$ $\quad m = 0, \pm 1, \pm 2, ..$ ($n > 0$ integer FIXED)

THESE ARE NOT IN GENERAL THE SAME. IN PARTICULAR IF $n = 2$ THEN

$\{2k\pi\} = \{0, \pm 2\pi, \pm 4\pi, ...\}$

$\{2mn\pi\} = \{4m\pi\} = \{0, \pm 4\pi, \pm 8\pi, ...\}$

NOT SAME.

PROOF (iii) SHOW $\log(e^z) \neq z$. NOTICE LEFT-HAND-SIDE IS MULTIVALUED, BUT RHS IS SINGLE VALUED.

PUT $z = x + iy$. $\quad e^z = e^x e^{iy}$ $\quad \operatorname{Arg}(e^z) = \operatorname{Arg}(e^{iy}) = y$. IF $-\pi < y \leq \pi$

SO $\log(e^z) = \ln|e^z| + i[y + 2k\pi i] = \ln e^x + i(y + 2k\pi i)$

IF $-\pi < y \leq \pi$.

THUS $\log(e^z) = z + i(2k\pi + y)$, IF $-\pi < \operatorname{Im} z \leq \pi$.

PROOF (vii) show that the sets $\log(z^{1/n})$ AND $\frac{1}{n}\log z$ ARE THE SAME WHERE $n$ = positive integer.

write $z = re^{i\varphi}$ WITH $\varphi = \mathrm{ARG}(z)$. THEN $z^{1/n} = r^{1/n} e^{i(\varphi + 2k\pi)/n}$ $k = 0, \ldots, n-1$.

THUS $$\log(z^{1/n}) = \frac{1}{n}\ln r + i\left[\frac{\varphi + 2k\pi}{n} + 2p\pi\right] \quad k = 0, 1, \ldots, n-1;\ p = 0, \pm 1, \pm 2, \ldots$$

NOW $$\frac{1}{n}\log z = \frac{1}{n}\ln r + i\left[\frac{\varphi}{n} + \frac{2q\pi}{n}\right] \quad q = 0, \pm 1, \pm 2, \ldots$$

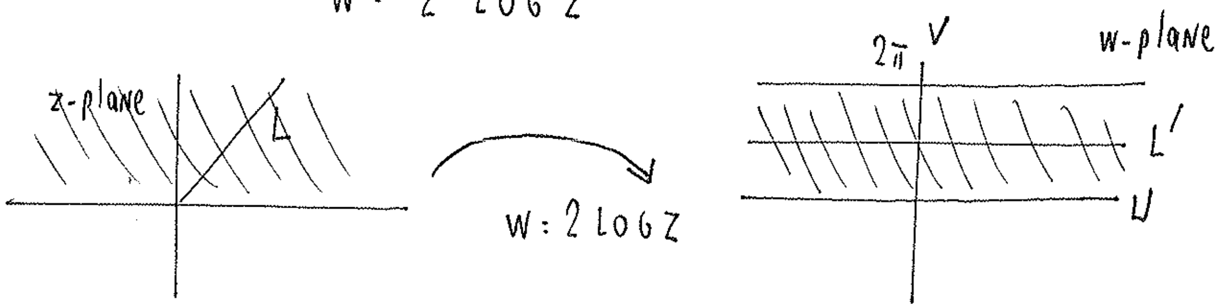
THE SET OF VALUES OF $\log(z^{1/n})$ AND $\frac{1}{n}\log z$ ARE THE SAME IF THE TWO SETS $\{k + pn\}$ $k = 0, \ldots, n-1$; $p = 0, \pm 1, \pm 2$ coincide with the set $\{q\}$ $q = 0, \pm 1, \pm 2, \ldots$

THIS IS TRUE
- FOR ANY $k$ AND $p$ WE GET A $q$
- DIVIDING $q$ BY $n$ WE GET AN INTEGER AND A REMAINDER $k$ IN $\{0, \ldots, n-1\}$.

## EXAMPLE OF MAPPING INVOLVING LOG Z

EX: FIND THE IMAGE OF $S = \{z \mid \mathrm{IM}\, z \geq 0\}$ UNDER THE MAPPING

$$W = 2\,\mathrm{LOG}\, Z$$




TO PARAMETRIZE Z-PLANE LET $z = re^{i\varphi}$. THEN

$$W = 2[\ln r + i\varphi] \quad \text{WITH } 0 \leq \varphi \leq \pi. \quad \text{WRITE } W = U + iV.$$

HENCE $U = 2\ln r$ $\quad V = 2\varphi$

- FIX A RAY WITH $\varphi$ FIXED (line L in Z-PLANE above). THEN SINCE $0 < r < \infty$ WE GET $U$ IN $(-\infty, \infty)$ AND $V$ = FIXED. THE IMAGE LINE $L'$ IS SHOWN IN W-PLANE ABOVE
- SINCE $0 \leq \varphi \leq \pi$ WE GET $U$ IN $(-\infty, \infty)$ AND $V$ IN $(0, 2\pi)$

HENCE $$S' = \{w \mid 0 \leq \mathrm{IM}\, W \leq 2\pi\}.$$

## EXPONENTS

IF $\alpha$ IS A COMPLEX NUMBER AND $z \neq 0$ THEN WE DEFINE $z^\alpha = e^{\alpha \log z}$ (multi-valued).

THUS, $z^\alpha = e^{\alpha[\ln|z| + i(\operatorname{ARG} z + 2k\pi)]}$ $k = 0, \pm 1, \pm 2, \ldots$

THIS YIELDS THAT

$$z^\alpha = |z|^\alpha e^{i[\alpha \operatorname{ARG}(z) + 2k\pi\alpha]}, \quad k = 0, \pm 1, \pm 2, \ldots$$

- THERE WILL BE A FINITE NUMBER OF VALUES OF $z^\alpha$ ONLY IF $\alpha$ IS THE RATIO OF TWO INTEGERS (I.e. IS RATIONAL). IN SUCH A CASE $\alpha k$ = integer FOR some $k$.

THE <u>PRINCIPAL VALUE</u> OF $z^\alpha$ IS DEFINED BY

$$z^\alpha = e^{\alpha \operatorname{LOG}(z)} = e^{\alpha[\ln|z| + i \operatorname{ARG}(z)]}$$

SINCE $\operatorname{LOG}(z)$ IS ANALYTIC IN THE SLIT DOMAIN $\mathbb{C} \setminus (-\infty, 0)$ AND $e^w$ IS ANALYTIC, THEN $z^\alpha$ IS ANALYTIC IN $\mathbb{C} \setminus (-\infty, 0]$

AND $\dfrac{d}{dz} z^\alpha = \alpha z^\alpha \dfrac{1}{z} = \alpha z^{\alpha - 1}$ IN $\mathbb{C} \setminus (-\infty, 0]$.

<u>EXAMPLE</u> FIND ALL SOLUTIONS TO $z^{1+i} = 4$. WE WRITE $z^{1+i} = e^{(1+i)\log z} = e^{\ln 4}$. THUS $(1+i)\log z = \ln 4 + 2\pi n i$, $n = 0, \pm 1, \pm 2$

THUS $2 \log z = (1-i)[\ln 4 + 2\pi n i] \Rightarrow \log z = (1-i)[\ln 2 + \pi n i]$.

HENCE $\log z = \ln 2 + \pi n + i(\pi n - \ln 2)$. NOW EXPONENTIATING

GIVES $z = 2e^{\pi n + i[n\pi - \ln 2]} = 2e^{\pi n}(-1)^n[\cos(\ln 2) - i \sin(\ln 2)]$

SINCE $e^{in\pi} = (-1)^n$.

DEFINITION

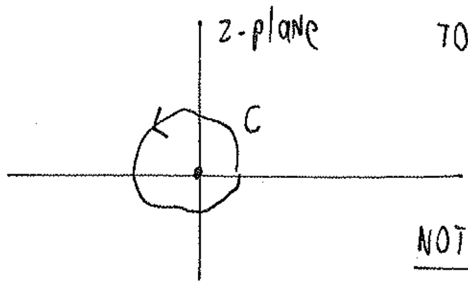
$\hat{f}(z)$ IS A <u>BRANCH</u> OF THE MULTI-VALUED FUNCTION $F(z)$ IN A DOMAIN $D$ IF $\hat{f}(z)$ IS SINGLE-VALUED AND CONTINUOUS IN $D$ AND HAS THE PROPERTY THAT FOR EACH $z$ IN $D$ THE VALUE $\hat{f}(z)$ IS ONE OF THE VALUES OF $F(z)$.

TO CONSTRUCT $\hat{f}(z)$ WE INTRODUCE A CURVE EMANATING FROM A POINT (CALLED THE BRANCH POINT) TO ENSURE THAT $\hat{f}(z)$ IS SINGLE-VALUED IN THE CUT PLANE. A <u>BRANCH POINT</u> IS A POINT FOR WHICH IF WE ENCIRCLE IT WITH AN ARBITRARY SUFFICIENTLY SMALL CURVE THE FUNCTION $F(z)$ CHANGES DISCONTINUOUSLY.

ALTHOUGH A DEEPER UNDERSTANDING OF THESE ISSUES REQUIRES MORE ADVANCED TOPICS (i.e. RIEMANN SURFACES), WE CAN STILL ILLUSTRATE THE IDEAS WITH SOME EXAMPLES.

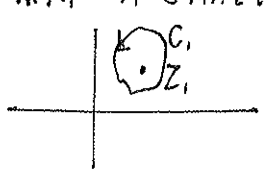
<u>EXAMPLE 1</u> LET $F(z) = \log z$. (multi-valued).

THE POINT $z=0$ IS A BRANCH POINT SINCE IF WE TAKE A PATH $C$ AS SHOWN BELOW, THEN $\log z$ DOES NOT RETURN TO ITS ORIGINAL VALUE. THE CHANGE $[\log z]_C$ IS




$$[\log z]_C = 2\pi i.$$

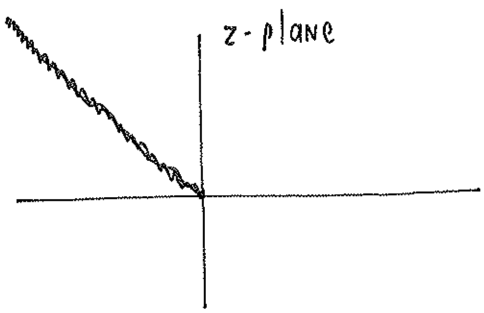
<u>NOTE</u>: IF WE ENCIRCLE ANY OTHER POINT $z_1 \neq 0$ WITH A SMALL CLOSED CURVE $C_1$ (AS SHOWN)




$[\log z]_{C_1} = 0$ THUS $z_1 \neq 0$ IS NOT A BRANCH POINT.

WE MUST INSERT A CURVE, CALLED THE BRANCH CUT, TO PREVENT COMPLETE CIRCUITS ABOUT THE BRANCH POINT, THUS RENDERING THE FUNCTION SINGLE-VALUED. THESE CUTS CAN BE LINES, CURVES ETC..

WE THEN CHOOSE A RANGE OF ARGUMENT TO UNAMBIGUOUSLY DEFINE THE FUNCTION AT EACH POINT IN THE CUT PLANE.

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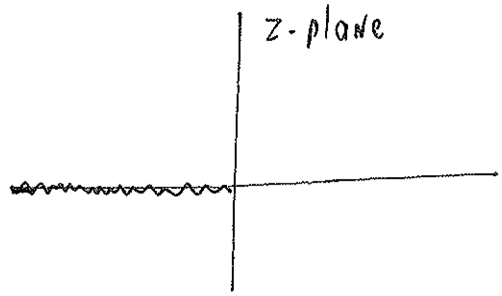
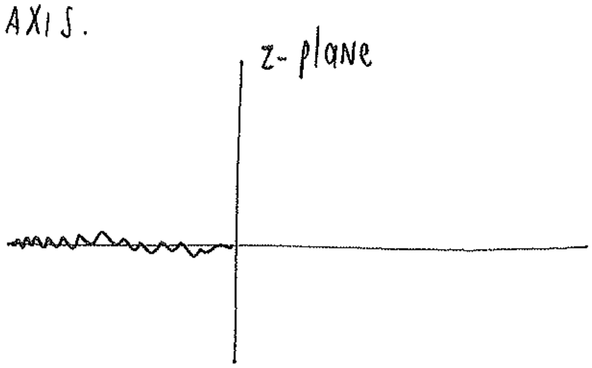



CONSTRUCT A BRANCH OF $f = \log z$ THAT IS ANALYTIC EXCEPT ON THE NEGATIVE REAL AXIS AND IS REAL-VALUED ON POSITIVE REAL AXIS.




THIS IS

$$\tilde{f}(z) = \mathrm{LOG}\, z.$$

FOR WHEN $-\pi < \mathrm{ARG}\, z \leq \pi$ AND SO FOR $z = x$ WITH $x > 0$ REAL,

$$\mathrm{IM}[\tilde{f}(z)] = 0.$$

<u>EXAMPLE</u> CONSIDER THE SINGLE-VALUED FUNCTION

$$f(z) = \mathrm{LOG}(1 - z^2)$$

WHERE IS THIS FUNCTION DISCONTINUOUS?

<u>SOLUTION</u> SINCE $\mathrm{LOG}(\zeta)$ IS ANALYTIC EXCEPT ON $\mathrm{IM}(\zeta) = 0$ AND $\mathrm{RE}(\zeta) \leq 0$, WE HAVE THAT $\mathrm{LOG}(1 - z^2)$ IS DISCONTINUOUS ONLY WHEN $\mathrm{IM}(1 - z^2) = 0$ AND $\mathrm{RE}(1 - z^2) < 0$.

M10

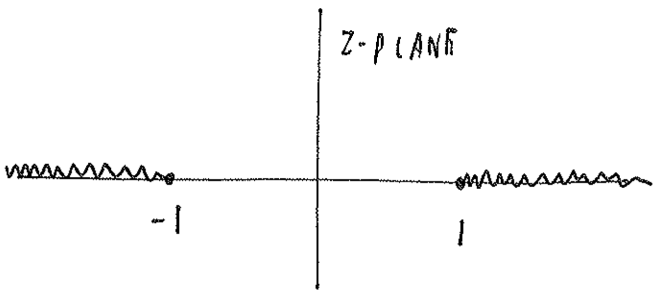
LET $z = x + iy$,

set $\text{IM}(1-z^2) = -2xy = 0$

$\text{RE}(1-z^2) = 1 - x^2 + y^2 < 0$

HENCE EITHER $x=0$ OR $y=0$. BUT IF $x=0$ THEN $1+y^2<0$ IS IMPOSSIBLE. HENCE $y=0$ AND $1-x^2+y^2 = 1-x^2 \leq 0$ IMPLIES $|x| \geq 1$.

THEREFORE THE BRANCH CUTS ARE AS SHOWN




$f(z) \equiv \text{LOG}(1-z^2)$ IS ANALYTIC IN THE CUT PLANE AS SHOWN.

EXAMPLE LET $f(z) = \text{LOG}\left(\dfrac{z-1}{z-2}\right)$, WHERE LOG DENOTES THE PRINCIPAL BRANCH OF MULTI-VALUED $\log$ FUNCTION.

WHERE IS $f(z)$ ANALYTIC?

SOLUTION THE ONLY POSSIBLE PLACES WHERE $f$ IS NOT ANALYTIC IS WHEN

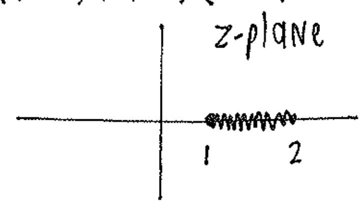
$$\text{IM}\left(\frac{z-1}{z-2}\right) = 0 \quad \text{AND} \quad \text{RE}\left(\frac{z-1}{z-2}\right) \leq 0.$$

HENCE $\dfrac{(z-1)}{(z-2)}\dfrac{\overline{(z-2)}}{\overline{(z-2)}} = \dfrac{1}{|z-2|^2}\left[z\bar{z} - \bar{z} - 2z + 2\right] = \dfrac{1}{|z-2|}\left[x^2+y^2-(x-iy)-2(x+iy)+2\right]$

SO $\text{IM}\left(\dfrac{z-1}{z-2}\right) = 0 \longrightarrow y = 0$

$\text{RE}\left(\dfrac{z-1}{z-2}\right) < 0$ WHEN $y=0$ YIELDS $x^2 - 3x + 2 = (x-2)(x-1) \leq 0$.

THUS $f(z)$ IS NOT ANALYTIC ON BRANCH CUTS AS SHOWN

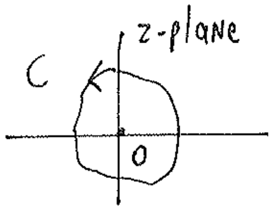


## MULTI-VALUED FUNCTIONS

CONSIDER THE FUNCTION $w = f(z) = z^{1/2}$. (multi-valued)

THE POINT $z=0$ IS A <u>BRANCH POINT</u> SINCE IF WE ENCIRCLE $z=0$ BY A SIMPLE CLOSED CURVE $C$, THE CHANGE IN $z^{1/2}$, DENOTED BY $[z^{1/2}]\big|_C$ IS, FOR $C$ COUNTER-CLOCKWISE

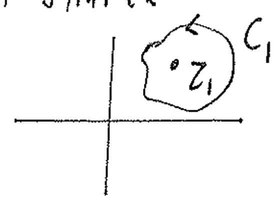
$$[z^{1/2}]\big|_C = [|z|e^{i\varphi/2}]_C = |z|e^{i\pi} \neq 0 \qquad (\varphi \mapsto \varphi + 2\pi)$$




FOR ANY OTHER POINT $z_1$, WE HAVE

$$[z^{1/2}]_{C_1} = 0$$

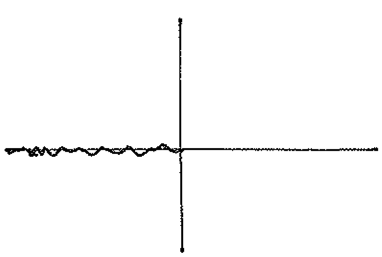
WHERE $C_1$ IS A SIMPLE CLOSED CURVE SURROUNDING $z_1$ AND NOT THE ORIGIN

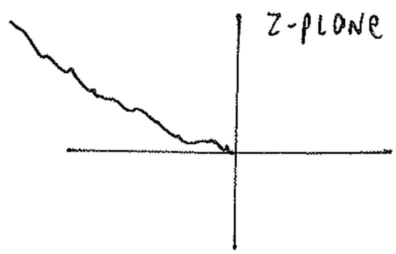


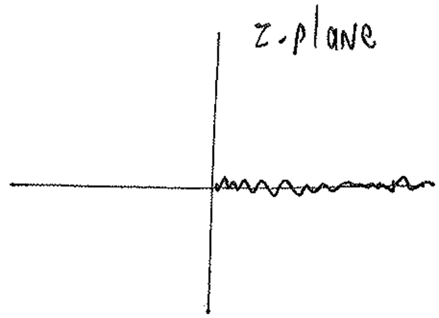

THUS $z_1 \neq 0$ IS NOT A BRANCH POINT.

WE MUST INTRODUCE A BRANCH-CUT EMANATING FROM $z=0$ AND EXTENDING TO $\infty$ TO PREVENT ENCIRCLING THE ORIGIN, AND HENCE RENDERING $z^{1/2}$ ANALYTIC IN THE CUT PLANE. THEN, WE CHOOSE A RANGE OF VALUES FOR THE ARGUMENT OF $z$ TO MAKE IT UNIQUELY DEFINED IN CUT PLANE.

POSSIBLE BRANCH CUTS FOR $z^{1/2}$

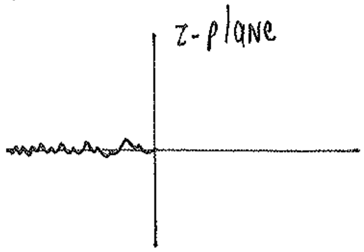




<u>EXAMPLE</u> CONSTRUCT A BRANCH OF $f(z) = z^{1/2}$ FOR WHICH $z^{1/2}$ IS ANALYTIC IN THE CUT PLANE $\mathbb{C} \setminus (-\infty, 0)$ AND FOR WHICH $RE(\sqrt{z}) \geq 0$.

<u>SOLUTION</u> WE MUST HAVE THE CUT AS SHOWN




HENCE $z = |z| e^{i\varphi}$

AND EITHER $-\pi < \varphi \leq \pi$

OR $\pi < \varphi \leq 3\pi$.

WHICH RANGE OF ANGLES WORKS?

WE CALCULATE $z^{1/2} = |z|^{1/2} e^{i\varphi/2}$

THEN $RE(z^{1/2}) = |z|^{1/2} \cos(\varphi/2)$

HENCE, IF $-\pi < \varphi \leq \pi$, THEN $\cos(\varphi/2) \geq 0 \longrightarrow RE(\sqrt{z}) \geq 0$.

WE THEN WRITE (*) $z^{1/2} = |z|^{1/2} e^{i\varphi/2}$ WITH $-\pi < \varphi \leq \pi$.

<u>REMARK</u> (i) (*) IS THE <u>PRINCIPAL BRANCH</u> OF $\sqrt{z}$.

IT COINCIDES PRECISELY WITH THE CHOICE OF BRANCH

$$z^{1/2} = e^{\frac{1}{2} \operatorname{LOG}(z)} = e^{\frac{1}{2}[\ln|z| + i \operatorname{ARG}(z)]} \qquad -\pi < \operatorname{ARG} z \leq \pi.$$

(ii) CALCULATE THE PRINCIPAL VALUE OF $(1+i)^{1/2}$.

<u>SOLUTION</u> $\operatorname{ARG} z = \pi/4$ $|z| = \sqrt{2}$. SO

$$(1+i)^{1/2} = (\sqrt{2})^{1/2} e^{i\pi/8}.$$

(iii) CONSTRUCT A BRANCH OF $z^{1/2}$ THAT IS ANALYTIC IN $\mathbb{C} \setminus (-\infty, 0)$ BUT HAS $RE(z^{1/2}) < 0$.

BY REPEATING ANALYSIS ABOVE, $z^{1/2} = |z|^{1/2} e^{i\varphi/2}$, $\pi < \varphi \leq 3\pi$

<u>EXAMPLE</u> SUPPOSE THAT $z^{1/2}$ DENOTES THE PRINCIPAL VALUE OF THE SQUARE ROOT. FIND ALL SOLUTIONS TO

$$(*) \qquad z^{1/2} + 2 - i = 0.$$

<u>SOLUTION</u> THE PRINCIPAL VALUE OF $z^{1/2}$ IS SUCH THAT IT IS ANALYTIC IN $\mathbb{C} \setminus (-\infty, 0)$ AND HAS $\mathrm{Re}(z^{1/2}) \geq 0$.

BY TAKING $\mathrm{Re}(\quad)$ OF BOTH SIDES IN $(*)$ WE OBTAIN THAT

$$\mathrm{Re}(z^{1/2}) + 2 = 0 \longrightarrow \mathrm{Re}(z^{1/2}) = -2 < 0 \longrightarrow \text{CONTRADICTION}.$$

THUS WITH THE PRINCIPAL VALUE OF $z^{1/2}$, $(*)$ HAS <u>NO SOLUTIONS</u>.

<u>REMARK</u> IT IS TEMPTING BUT <u>WRONG</u> TO CALCULATE AS

$$z^{1/2} = i - 2$$

$$\longrightarrow (z^{1/2})^2 = (i-2)^2 = 4 - 4i - 1 = 3 - 4i$$

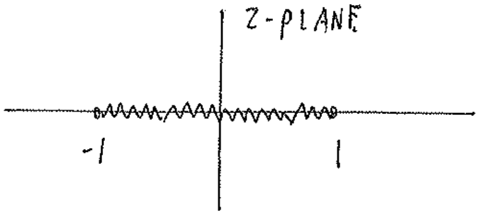
$$\text{SO} \quad z = 3 - 4i$$

<u>EXAMPLE</u> CONSTRUCT A BRANCH OF $f(z) = (z^2-1)^{1/2}$ THAT IS ANALYTIC IN $|z| > 1$ AND TAKES THE VALUE $f(2) = \sqrt{3}$.

<u>SOLUTION</u> WE USE $z_1^{1/2} z_2^{1/2} = (z_1 z_2)^{1/2}$ (multi-valued sets same) TO WRITE

$$f(z) = (z-1)^{1/2}(z+1)^{1/2}.$$

THE ONLY POINTS IN FINITE COMPLEX PLANE THAT ARE BRANCH POINTS ARE $z = -1$ AND $z = 1$. WE MUST HAVE NO BRANCH CUTS OUTSIDE $|z| > 1$, SO THE EASIEST CONSTRUCTION IS TO HAVE






METHOD 1 (RANGE OF ANGLES)

WE WRITE $(z^2-1)^{1/2} = (z-1)^{1/2}(z+1)^{1/2} = (r_1 e^{i\varphi_1})^{1/2}(r_2 e^{i\varphi_2})^{1/2}$.

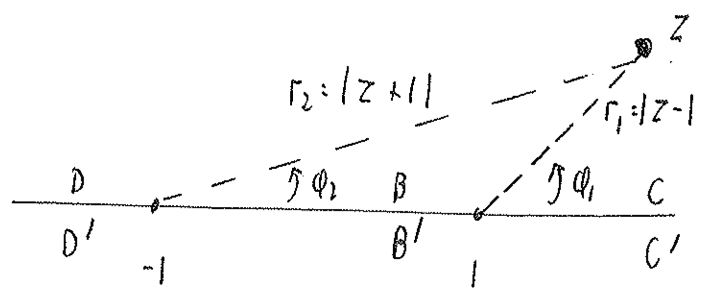
HENCE $f(z) = (z^2-1)^{1/2} = (r_1 r_2)^{1/2} e^{i(\varphi_1+\varphi_2)/2}$ $(*)$

SPECIFYING A BRANCH IS EQUIVALENT TO CHOOSING A RANGE OF ANGLES.

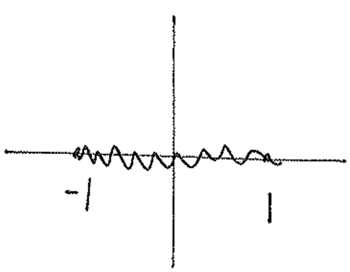
TRY $-\pi < \varphi_1 \le \pi, \quad -\pi < \varphi_2 \le \pi$

- MUST CHECK THAT DISCONTINUITY IN $f$ OCCURS BETWEEN $-1 \le X \le 1$
- MUST CHECK THAT $(*)$ GIVES $f(2) = \sqrt{3}$.

| POINT | $\varphi_1$ | $\varphi_2$ | $e^{i(\varphi_1+\varphi_2)/2}$ |
|---|---|---|---|
| C | 0 | 0 | $e^{i0} = 1$ |
| C′ | 0 | 0 | $e^{i0} = 1$ ∴ CONTINUOUS |
| B | $\pi$ | 0 | $e^{i\pi/2} = i$ |
| B′ | $-\pi$ | 0 | $e^{-i\pi/2} = -i$ ∴ DISCONTINUOUS |
| D | $\pi$ | $\pi$ | $e^{i\pi} = -1$ |
| D′ | $-\pi$ | $-\pi$ | $e^{-i\pi} = -1$ ∴ CONTINUOUS |




THUS THE CHOICE $f(z) = (r_1 r_2)^{1/2} e^{i(\varphi_1+\varphi_2)/2}$ WITH $-\pi < \varphi_1 \le \pi$ AND $-\pi < \varphi_2 \le \pi$ HAS A BRANCH CUT FROM $-1 < X < 1$ AS DESIRED



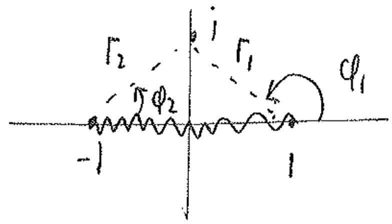

- NOW CALCULATE $f(2)$:

FOR $z = 2$ THEN $r_1 = |z-1| = 1$, $r_2 = |z+1| = 3$

AND $\varphi_1 = \varphi_2 = 0$ HENCE

$f(2) = \sqrt{1 \cdot 3}\, e^{i0} = \sqrt{3}$ AS DESIRED.

• TO CALCULATE $f(i)$ WE DRAW




AND CALCULATE $r_1 = r_2 = \sqrt{2}$.

$\varphi_1 = 3\pi/4$, $\varphi_2 = \pi/4$ SO $f(i) = (\sqrt{2}\sqrt{2})^{1/2} e^{i(3\pi/4 + \pi/4)/2}$

$\rightarrow f(i) = \sqrt{2}\, i$.

METHOD 2 ( CHOOSING A BRANCH OF $\log$ )

THIS METHOD IS LESS INTUITIVE AS IT IS NOT CLEAR APRIORI WHICH BRANCH OF $\log$ TO TAKE.

FOR INSTANCE, CONSIDER SEVERAL POSSIBLE CHOICES :

(A) $(z^2-1)^{1/2} = e^{\frac{1}{2}\log(z^2-1)} \longrightarrow f(z) = e^{1/2\, \mathrm{LOG}(z^2-1)}$

(B) $(z^2-1)^{1/2} = [-(1-z^2)]^{1/2} \Longrightarrow f(z) = \pm i\, e^{1/2\, \mathrm{LOG}(1-z^2)}$

(C) $(z^2-1)^{1/2} = [z^2(1-1/z^2)]^{1/2} \Rightarrow f(z) = \pm z\, e^{1/2\, \mathrm{LOG}(1-1/z^2)}$

(D) $(z^2-1)^{1/2} = [-z^2(-1+1/z^2)]^{1/2} \Rightarrow f(z) = \pm iz\, e^{1/2\, \mathrm{LOG}(-1+1/z^2)}$

WHICH ONE WILL GIVE US $f(z)$ ANALYTIC IN $|z|>1$ WITH $f(2) = \sqrt{3}$?

THIS IS NOT CLEAR WITHOUT CONSIDERABLE EXTRA EFFORT.

• CONSIDER THE OBVIOUS CHOICE (A): $f(z) = e^{1/2\, \mathrm{LOG}(z^2-1)}$ TO SEE IF IT WORKS.

THEN $f(z)$ IS ANALYTIC EXCEPT WHEN

$\mathrm{IM}(z^2-1) = 0$ AND $\mathrm{RE}(z^2-1) \leq 0$.

LET $z = x + iy$

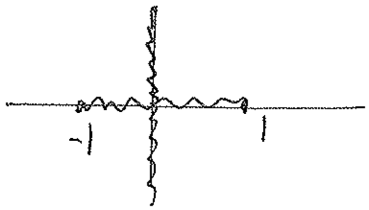
$$\mathrm{IM}(z^2-1) = 0 \longrightarrow xy = 0 \longrightarrow \underline{\text{either}}\ x=0\ \underline{\text{OR}}\ y=0.$$

$$\mathrm{RE}(z^2-1) = \mathrm{RE}\left[x^2-y^2+2ixy-1\right] = x^2-y^2-1 \leq 0.$$

$$\text{IF } x=0 \longrightarrow \mathrm{RE}(z^2-1) = -y^2-1 \leq 0 \quad \text{FOR ALL } y$$

$$\text{IF } y=0 \longrightarrow \mathrm{RE}(z^2-1) = x^2-1 \leq 0 \longrightarrow |x| \leq 1.$$

THUS, THIS CHOICE HAS BRANCH CUTS



WHICH IS $\underline{\text{NOT}}$ WHAT WE WANT.

HENCE CHOICE A FAILS.

- $\underline{\text{CONSIDER CHOICE (C)}}$ TRY $f(z) = \pm z e^{\frac{1}{2}\log(1-1/z^2)}$.

THIS IS ANALYTIC $\underline{\text{EXCEPT}}$ WHEN

$$\mathrm{IM}\left(1-\frac{1}{z^2}\right) = 0$$

$$\mathrm{RE}\left(1-\frac{1}{z^2}\right) \leq 0.$$

THUS

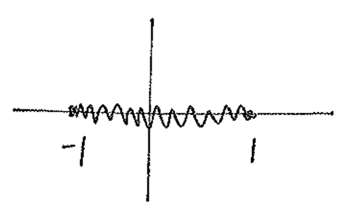
$$\mathrm{IM}\left(\frac{1}{z^2}\right) = \mathrm{IM}\left(\frac{\bar{z}^2}{|z|^4}\right) = \mathrm{IM}\left(\frac{(x-iy)^2}{|z|^4}\right) = 0 \longrightarrow xy = 0$$

$$\mathrm{RE}\left(1-\frac{1}{z^2}\right) = 1 - \mathrm{RE}\left(\frac{(x-iy)^2}{|z|^4}\right) \leq 0.$$

- SET $y=0 \longrightarrow \mathrm{RE}\left(1-\frac{1}{z^2}\right) \leq 0 \longrightarrow 1-\frac{x^2}{x^4} \leq 0 \rightarrow |x| \leq 1.$
- SET $x=0 \longrightarrow \mathrm{RE}\left(1-\frac{1}{z^2}\right) \leq 0 \rightarrow \mathrm{RE}\left(1-\frac{1}{(iy)^2}\right) = 1+\frac{1}{y^2} \leq 0$

IMPOSSIBLE.

THUS $f(z) = \pm z e^{\frac{1}{2}\log(1-1/z^2)}$ HAS DESIRED BRANCH CUTS



NOW WE MUST TAKE $\pm$ SIGN CONSISTENT WITH $f(2) = \sqrt{3}$.

TRY $+$ SIGN $\longrightarrow$ $f(2) = 2\, e^{\frac{1}{2}\operatorname{LOG}(1 - 1/4)}$

$$= 2\, e^{\frac{1}{2}\left[\ln(3/4) + i\operatorname{ARG}(3/4)\right]}, \quad \operatorname{ARG}(3/4) = 0$$

$$= 2\left(e^{\ln\left(\sqrt{3/4}\right)}\right) = 2\left(\sqrt{3/4}\right) = \sqrt{3}\ \checkmark.$$

THUS, THE DESIRED BRANCH IS

$$f(z) = z\, e^{\frac{1}{2}\operatorname{LOG}(1 - 1/z^2)}$$

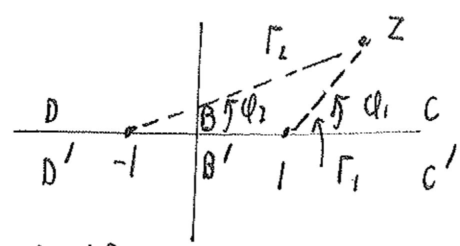
THIS WAS NOT TERRIBLY CLEAR IN ADVANCE THAT THIS CHOICE WOULD WORK. NOTICE $f(i) = i\, e^{\frac{1}{2}\operatorname{LOG}(2)} = i\, e^{\frac{1}{2}(\ln 2)} = \sqrt{2}\, i$.

<u>EXAMPLE</u> CONSTRUCT A BRANCH OF $f(z) = (z^2 - 1)^{1/2}$ THAT IS ANALYTIC IN $|z| < 1$ AND THAT TAKES THE VALUE $f(0) = i$

<u>SOLUTION</u>

<u>METHOD I</u> (RANGE OF ANGLE METHOD)

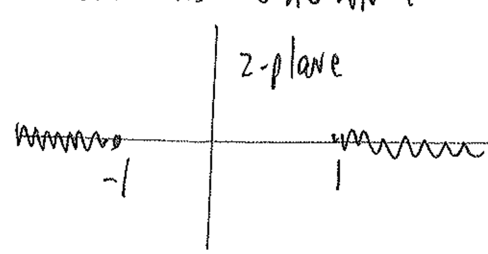
WE WRITE $f(z) = |r_1 r_2|^{1/2} e^{i(\theta_1 + \theta_2)/2}$ $(*)$




WE TRY NOW THE RANGE $0 \le \theta_1 < 2\pi$, $-\pi < \theta_2 \le \pi$.

| POINTS | $\theta_1$ | $\theta_2$ | $e^{i(\theta_1+\theta_2)/2}$ | |
|---|---|---|---|---|
| C | 0 | 0 | $e^{i0} = 1$ | $\neq$ DISCONTINUOUS |
| C' | $2\pi$ | 0 | $e^{i\pi} = -1$ | |
| B | $\pi$ | 0 | $e^{i\pi/2} = i$ | $=$ CONTINUOUS |
| B' | $\pi$ | 0 | $e^{i\pi/2} = i$ | |
| D | $\pi$ | $\pi$ | $e^{i\pi} = -1$ | $\neq$ DISCONTINUOUS |
| D' | $\pi$ | $-\pi$ | $e^{i0} = 1$ | |

$\Longrightarrow$ THIS YIELDS THE BRANCH CUTS AS SHOWN (WHAT WE WANT)



HENCE, (K) IS ANALYTIC IN $|z| < 1$.

NOW AT $z = 0$ WE CALCULATE $\theta_1 = \pi$, $\theta_2 = 0$, $r_1 = r_2 = 1$ SO THAT

$$f(0) = (z^2-1)^{1/2}\Big|_{z=0} = (r_1 r_2)^{1/2} e^{i(\theta_1+\theta_2)/2} = (1\cdot 1)^{1/2} e^{i\pi/2} = i, \text{ AS REQUIRED.}$$

METHOD 2

WE WRITE $(z^2-1)^{1/2} = [-(1-z^2)]^{1/2} = \pm i\,(1-z^2)^{1/2}$.

WE NOW CHOOSE THE PRINCIPAL VALUE $(1-z^2)^{1/2} = e^{\frac{1}{2}\operatorname{Log}(1-z^2)}$.

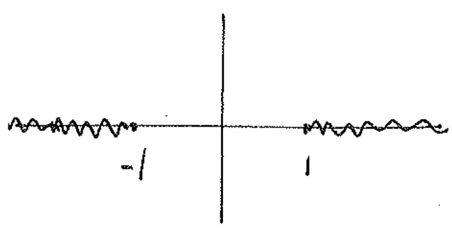
THEN, $(z^2-1)^{1/2} = (\pm i)\, e^{\frac{1}{2}\operatorname{Log}(1-z^2)}$ THIS IS CHOICE B ON PAGE (M14.5)

CHOOSE $+i$ SINCE AT $z=0$ $\operatorname{Log}(1) = \ln 1 + i0 = 0$.

THEN $f(0) = i$ AS REQUIRED.

WE OBTAIN $(z^2-1)^{1/2} = i\, e^{\frac{1}{2}\operatorname{Log}(1-z^2)}$.

BY THE EXAMPLE AT BOTTOM OF PAGE (M9) $\operatorname{Log}(1-z^2)$ HAS BRANCH CUTS AS SHOWN



HENCE $(z^2-1)^{1/2} = i\, e^{\frac{1}{2}\operatorname{Log}(1-z^2)}$

SATISFIES THE REQUIREMENTS.

- IN PARTICULAR IF $f(z) = (z^2-1)^{1/2} = i\, e^{\frac{1}{2}\operatorname{Log}(1-z^2)}$

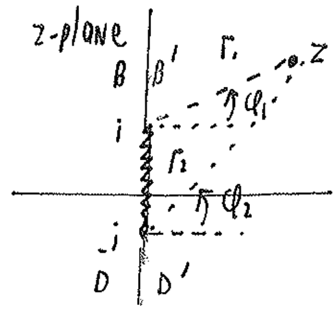
THEN $f(i) = i\, e^{\frac{1}{2}\operatorname{Log}(1-i^2)} = i\, e^{\frac{1}{2}\operatorname{Log}(2)} = i\, e^{\frac{1}{2}\ln 2} = \sqrt{2}\, i$.

- RECALL $\operatorname{Log}(1-z^2)$ IS ANALYTIC EXCEPT FOR POINTS $z$ WHERE

$\operatorname{Im}(1-z^2) = 0 \longrightarrow xy = 0 \Longrightarrow$ EITHER $x=0$ OR $y=0$

$\operatorname{Re}(1-z^2) \le 0 \longrightarrow 1-(x^2-y^2) \le 0 \Longrightarrow$ IF $x=0 \longrightarrow 1+y^2 \le 0$ IMPOSSIBLE
IF $y=0 \longrightarrow 1-x^2 \le 0 \longrightarrow |x| \ge 1$.

**EXAMPLE** CONSTRUCT A BRANCH OF $f(z) = (z^2+1)^{1/2}$ THAT IS ANALYTIC IN $|z|>1$ AND FOR WHICH $f(2i) = \sqrt{3}\, i$




**SOLUTION** WE DO THIS QUICKLY.

**METHOD 1** $f(z) = (z+i)^{1/2}(z-i)^{1/2} = (r_1 r_2)^{1/2} e^{i(\varphi_1+\varphi_2)/2}$

WE WANT A BRANCH CUT BETWEEN $-i$ AND $i$ AS SHOWN.

IF WE CHOOSE $-\pi/2 < \varphi_1 \le 3\pi/2$, $-\pi/2 < \varphi_2 \le 3\pi/2$ THEN WE GET THE DESIRED BRANCH CUT. FOR THEN WE HAVE CONTINUITY AT $B|B'$ AND $D|D'$

FOR $z = 2i$ WE GET $\varphi_1 = \varphi_2 = \pi/2$, $r_1 = 1$, $r_2 = 3$. HENCE

$$f(2i) = (1\cdot 3)^{1/2} e^{i(\pi/2+\pi/2)/2} = i\sqrt{3}.$$

**METHOD 2** THE CHOICE $(z^2+1)^{1/2} = e^{\frac{1}{2}\operatorname{LOG}(z^2+1)}$ CLEARLY DOES NOT WORK SINCE $\operatorname{LOG}(z^2+1)$ IS NOT ANALYTIC ON $z = iy$ WITH $|y|>1$.

INSTEAD WRITE $(z^2+1)^{1/2} = \left(z^2\left[1+\frac{1}{z^2}\right]\right)^{1/2} = z\,\left(1+1/z^2\right)^{1/2}$.

THEN CHOOSE $(z^2+1)^{1/2} = z\, e^{\frac{1}{2}\operatorname{LOG}(1+1/z^2)}$

NOW $z\, e^{\frac{1}{2}\operatorname{LOG}(1+1/z^2)}$ IS ANALYTIC EXCEPT ON THE SEGMENT FOR WHICH

$$\operatorname{IM}\left(1+\frac{1}{z^2}\right) = 0 \quad \text{AND} \quad \operatorname{RE}\left(1+\frac{1}{z^2}\right) \le 0.$$

IF WE PUT $z = x+iy$ THEN $\operatorname{IM}\left(1+\frac{1}{z^2}\right) = \operatorname{IM}\left(\frac{\bar{z}^2}{|z|^4}\right) = \frac{1}{|z|^4}(-2xy) = 0$

HENCE EITHER $x=0$ OR $y=0$. BUT $\operatorname{RE}\left(1+\frac{1}{z^2}\right) = \operatorname{RE}\left(\frac{\bar{z}^2}{|z|^4}\right)+1 = \frac{x^2-y^2}{(x^2+y^2)^2}+1 < 0.$

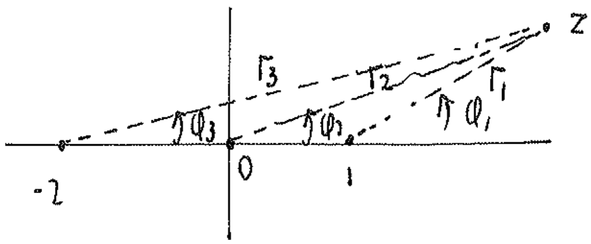
CLEARLY $y=0$ IS IMPOSSIBLE. SO $x=0$ YIELDS $\frac{-y^2}{y^4}+1 < 0 \rightarrow |y|<1$.

WE CONCLUDE THAT $(z^2+1)^{1/2} = z\, e^{\frac{1}{2}\operatorname{LOG}(1+1/z^2)}$ IS ANALYTIC IN $|z|>1$.

WE CALCULATE $f(2i) = 2i\, e^{\frac{1}{2}\operatorname{LOG}(1-1/4)} = 2i\, e^{\frac{1}{2}\ln(3/4)} = 2i\sqrt{3}/2 = i\sqrt{3}$.

EXAMPLE CONSTRUCT A BRANCH OF $f(z) = (z^3 + z^2 - 2z)^{1/2}$ THAT HAS A BRANCH CUT FROM $(0,1)$ AND FROM $(-\infty,-2)$ ALONG REAL AXIS AND FOR WHICH $f(2) = \sqrt{8}$.

SOLUTION $f(z) = \sqrt{z(z+2)(z-1)} = (r_1 r_2 r_3)^{1/2} e^{i(\varphi_1+\varphi_2+\varphi_3)/2}$



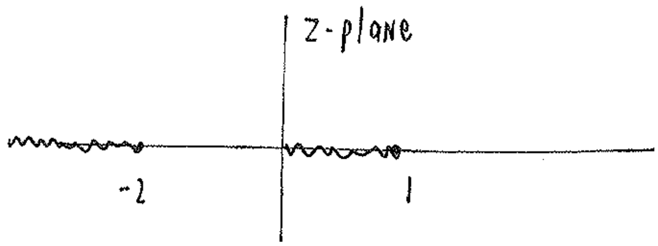

IF WE THEN CHOOSE

$-\pi < \varphi_1 \leq \pi$

$-\pi < \varphi_2 \leq \pi$

$-\pi < \varphi_3 \leq \pi$,

WE WILL OBTAIN THE BRANCH CUT STRUCTURE




TO CONVINCE YOURSELF DRAW A LITTLE TABLE OF VALUES.

WHEN $z = 2$ THEN $\varphi_1 = \varphi_2 = \varphi_3 = 0$, $r_1 = 1$, $r_2 = 2$, $r_3 = 4$.

HENCE $f(2) = (4 \cdot 2 \cdot 1)^{1/2} e^{i0} = \sqrt{8}$.

FINALLY, WE MAKE A FEW ADDITIONAL MISCELLANEOUS COMMENTS.

REMARK I NOT EVERYTHING WITH $\sqrt{z}$ HAS A BRANCH POINT AT $z = 0$. FOR WHICH OF THE FOLLOWING IS $z = 0$ A BRANCH POINT?

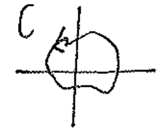
(i) $f(z) = \sin(\sqrt{z})$ (ii) $f(z) = \sqrt{z}\,\sin(\sqrt{z})$

(iii) $f(z) = \cos(\sqrt{z})$.

IN (ii) WE USE SAME CHOICE OF BRANCH OF $\sqrt{z}$.

<u>SOLUTION</u> ONLY $\sin(\sqrt{z})$ HAS A BRANCH POINT AT $z=0$.

LET'S CHECK. IN EACH CASE WE ENCIRCLE $z=0$ BY A SIMPLE CLOSED COUNTERCLOCKWISE CURVE AND WE CALCULATE $[f(z)]_C$ (THE CHANGE IN $f$ AROUND THE CURVE).



(i) $$\left[\sin(\sqrt{z})\right]\Big|_C = \left[\sin\left(r^{1/2}e^{i\varphi/2}\right)\right]\Big|_C$$
$$= \sin\left(r^{1/2}e^{2\pi i/2}\right) - \sin\left(r^{1/2}e^{i0}\right)$$
$$= \sin\left(-r^{1/2}\right) - \sin\left(r^{1/2}\right) = -2\sin\left(r^{1/2}\right) \neq 0.$$

(ii) $$\left[\sqrt{z}\sin(\sqrt{z})\right]_C = \left[r^{1/2}e^{i\varphi/2}\sin\left(r^{1/2}e^{i\varphi/2}\right)\right]\Big|_C$$
$$= +r^{1/2}e^{2\pi i/2}\sin\left(r^{1/2}e^{2\pi i/2}\right) - r^{1/2}e^{i0}\sin\left(r^{1/2}e^{i0}\right)$$
$$= -r^{1/2}\sin\left(-r^{1/2}\right) - r^{1/2}\sin\left(r^{1/2}\right) = 0.$$

(iii) $$\left[\cos(\sqrt{z})\right]\Big|_C = \left[\cos\left(r^{1/2}e^{i\varphi/2}\right)\right]\Big|_C$$
$$= \cos\left(r^{1/2}e^{2\pi i/2}\right) - \cos\left(r^{1/2}\right) = \cos\left(-r^{1/2}\right) - \cos\left(r^{1/2}\right)$$
$$= 0 \text{ SINCE } \cos(w) = \cos(-w).$$

<u>REMARK 2</u> TO CLASSIFY WHETHER $z=\infty$ IS A BRANCH POINT OF $f(z)$ WE MUST TAKE A VERY LARGE CIRCLE $|z|=R$ $R>>1$ AND SEE IF $f(z)$ RETURNS TO ITS ORIGINAL VALUE AS WE TRAVERSE THE CIRCLE.

EQUIVALENTLY, $z=\infty$ IS A BRANCH POINT OF $f(z)$ IFF $\zeta=0$ IS A BRANCH POINT OF $f(1/\zeta)$ (I.E. LET $z=1/\zeta$).

<u>EXAMPLE</u> IS $z=\infty$ A BRANCH POINT FOR

(i) $f(z)=\sqrt{(z+1)(z+2)(z-3)}$

(ii) $f(z)=\log\left(\dfrac{z+1}{z-1}\right)$

(iii) $f(z)=(z^3-z)^{1/3}$

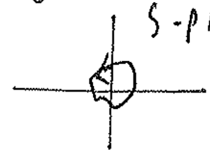
<u>SOLUTION</u>

(i) LET $\zeta=1/z$ SO $f(1/\zeta)=\sqrt{(1+1/\zeta)(2+1/\zeta)(-3+1/\zeta)}=\zeta^{-3/2}\sqrt{(1+\zeta)(1+2\zeta)(1-3\zeta)}$

FOR $|\zeta|\ll 1$, $f(1/\zeta)\cong\zeta^{-3/2}$ SO THAT $[f(1/\zeta)]\big|_C\neq 0$ WHERE

C IS THE SMALL CIRCLE $|\zeta|=\delta$ $\delta\ll 1$.

$\zeta$-plane



SO $z=\infty$ is A BP FOR $f(z)$

(ii) $f(z)=\log(1+z)-\log(z-1)$

LET $z=1/\zeta$, THEN $f(1/\zeta)=\log(1+1/\zeta)-\log(1/\zeta-1)$

$$=\log\left(\frac{1+\zeta}{\zeta}\right)-\log\left(\frac{1-\zeta}{\zeta}\right)$$

$$=\log(1+\zeta)-\log(1-\zeta).$$

LET $C:|\zeta|=\delta$, WITH $\delta\ll 1$. THEN $[f(1/\zeta)]_C=0$.

$\zeta=0$ IS NOT A BP OF $f(1/\zeta)\longrightarrow z=\infty$ IS NOT A BP OF $f(z)$.

(iii) LET $z=1/\zeta$, $f(1/\zeta)=\left(1/\zeta^3-\zeta\right)^{1/3}=\dfrac{(1-\zeta^2)^{1/3}}{(\zeta^3)^{1/3}}=\dfrac{(1-\zeta^2)^{1/3}}{\zeta}$.

NOW LET $C:|\zeta|=\delta$ WITH $\delta\ll 1$,

THEN $[f(1/\zeta)]\big|_C=0$. HENCE $\zeta=0$ IS NOT A BP OF $f(1/\zeta)$

$\longrightarrow z=\infty$ IS NOT A BP OF $f(z)$.

EXAMPLE FIND ALL POSSIBLE VALUES OF

(i) $\cos W = 2i$. (ii) $\sin W = i$

SOLUTION

(i) LET $Z = \cos W = \dfrac{e^{iW} + e^{-iW}}{2}$ SO $\dfrac{e^{iW} + e^{-iW}}{2} = 2i$.

HENCE $e^{iW} + e^{-iW} = 4i \longrightarrow e^{2iW} - 4ie^{iW} + 1 = 0$.

LET $\Lambda = e^{iW} \longrightarrow \Lambda^2 - 4i\Lambda + 1 = 0$.

THUS $\Lambda = \dfrac{4i \pm \sqrt{-16 - 4}}{2} = 2i \pm i\sqrt{5}$

NOW $e^{iW} = (2 \pm \sqrt{5})i$

+ SIGN $e^{iW} = (2 + \sqrt{5})i$

$iW = \log((2+\sqrt{5})i) = \ln(2+\sqrt{5}) + i(\pi/2 + 2K\pi)$ $K = 0, \pm 1, \pm 2, ..$

THUS $W = -i\ln(2+\sqrt{5}) + \pi/2 + 2K\pi$, $K = 0, \pm 1, \pm 2 \ldots$

- SIGN $e^{iW} = (2 - \sqrt{5})i \longrightarrow iW = \log((2-\sqrt{5})i) = \ln(\sqrt{5}-2) + i\left(-\frac{\pi}{2} + 2K\pi\right)$

THUS $W = -i\ln(\sqrt{5}-2) - \dfrac{\pi}{2} + 2K\pi$ $K = 0, \pm 1, \pm 2, ..$

SINCE $\ln(\sqrt{5}-2) = \ln\left((\sqrt{5}-2)(\sqrt{5}+2)/(\sqrt{5}+2)\right) = +\ln\left(1/\sqrt{5}+2\right) = -\ln(\sqrt{5}+2)$

THEN WE CAN WRITE +SIGN AND -SIGN TOGETHER AS

$$(*) \quad W = \pm i\ln(\sqrt{5}+2) \pm \frac{\pi}{2} + 2K\pi \qquad K = 0, \pm 1, ..$$

THE SYMMETRY IN (*) FOLLOWS FROM IDENTITY THAT $\cos W_0 = \cos(-W_0)$.

M22

(ii) FOR THE $\sin w = i$ WE PUT

$$\frac{e^{iw}-e^{-iw}}{2i} = i \longrightarrow e^{iw}-e^{-iw} = -2.$$

THUS $e^{2iw} + 2e^{iw} - 1 = 0 \rightarrow \Lambda^2 + 2\Lambda - 1 = 0$ WITH $\Lambda = e^{iw}$.

SO $\Lambda = \dfrac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}$.

+ SIGN $\quad e^{iw} = -1+\sqrt{2} \longrightarrow iw = \log(\sqrt{2}-1) = \ln|\sqrt{2}-1) + 2K\pi i.$

SO $w = -i\ln|\sqrt{2}-1) + 2K\pi$, $K = 0, \pm1, \pm2, \dots$

- SIGN $\quad e^{iw} = -1-\sqrt{2} \longrightarrow iw = \log(-1-\sqrt{2}) = \ln|\sqrt{2}+1) + i(-\pi + 2K\pi)$

SO $w = -i\ln|\sqrt{2}+1) + [-\pi + 2K\pi]$ $\quad K = 0, \pm1, \pm2, \dots$

SINCE $\ln|\sqrt{2}+1) = -\ln|\sqrt{2}-1)$ IT IS CLEARLY THAT THE SYMMETRY IN THE TWO RESULTS FOLLOWS FROM THE FACT THAT IF $W = W_0$ IS A ROOT OF $\sin(W) = Z$ THEN SO IS $W = \pi - W_0$