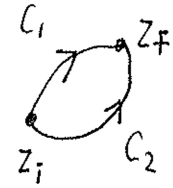
AT THIS STAGE WE HAVE THE FOLLOWING THEOREM:

THEOREM LET $f(z)$ BE CONTINUOUS ON A DOMAIN D. THE FOLLOWING STATEMENTS ARE EQUIVALENT:

(i) $f(z)$ HAS AN ANTI-DERIVATIVE $\mathcal{F}(z)$ IN D

(ii) IF C IS ANY LOOP IN D THEN $\int_C f(z)\,dz = 0$.

(iii) PATH INDEPENDENCE holds, i.e. $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$



REMARKS (1) THIS THEOREM IS RATHER USELESS IN THAT HOW CAN WE POSSIBLY CHECK WHETHER $\int_C f(z)\,dz = 0$ FOR EVERY CURVE C IF WE CANNOT CALCULATE EASILY THE ANTI-DERIVATIVE.

(2) IT IS ONLY USEFUL IF WE CAN SOMEHOW "SPOT" OR "GUESS" THE $\mathcal{F}(z)$. WE WOULD PREFER A THEOREM REFERRING TO A SPECIFIC PROPERTY OF $f(z)$ THAT IS EASILY CHECKED. (i.e. IMPRACTICAL TO USE FOR $\int_C z^5 \cos(z^2)\,dz$).

PROOF OF THEOREM

(ii) $\longleftrightarrow$ (iii) IS TRIVIAL

(i) $\longrightarrow$ (iii) IS TRIVIAL ; (i) $\longrightarrow$ (ii) IS ALSO TRIVIAL.

THE ONLY TRICKY ISSUE IS TO PROVE THAT IF (iii) HOLDS THEN (i) MUST HOLD.

PROOF THAT (iii) $\longrightarrow$ (i) SUPPOSE THAT (iii) HOLDS. THEN DEFINE

$$\mathcal{F}(z) = \int_{z_0}^{z} f(\zeta)\,d\zeta$$ WHERE WE HAVE TAKEN ANY PATH FROM z_0 TO z

THUS, BY PATH-INDEPENDENCE $\mathcal{F}(z)$ IS WELL-DEFINED. WE CALCULATE

$$\mathcal{F}(z+\Delta z) - \mathcal{F}(z) = \int_{z}^{z+\Delta z} f(\zeta)\,d\zeta$$ WHERE WE TAKE STRAIGHT LINE FROM z TO Δz.

THUS IF $\zeta(t) = z + t\Delta z$ WITH $0 \le t \le 1 \longrightarrow d\zeta/dt = \Delta z$ AND

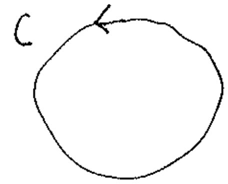
$$\mathcal{F}(z+\Delta z) - \mathcal{F}(z) = \int_0^1 f(z+t\Delta z)\,\Delta z\,dt \Longrightarrow \frac{\mathcal{F}(z+\Delta z)-\mathcal{F}(z)}{\Delta z} = \int_0^1 f(z+t\Delta z)\,dt$$

NOW TAKE $\Delta z \to 0$ AND $\mathcal{F}'(z) = \int_0^1 f(z)\,dt = f(z)$.

THEOREM (CAUCHY'S INTEGRAL THEOREM)

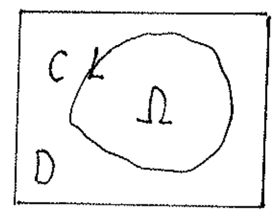
SUPPOSE THAT $f(z)$ IS ANALYTIC, WITH A CONTINUOUS DERIVATIVE f', INSIDE AND ON A SIMPLE CLOSED CURVE C. THEN,

$$(*) \qquad \int_C f(z)\,dz = 0$$



PROOF WE FIRST MUST RECALL GREEN'S THEOREM IN THE PLANE. LET P, Q, P_x, P_y, Q_x, Q_y BE CONTINUOUS IN A DOMAIN D THAT CONTAINS A SIMPLE CLOSED CURVE C ORIENTED COUNTERCLOCKWISE. THEN GREEN'S THEOREM IS

$$(+) \qquad \int_C (P\,dx + Q\,dy) = \iint_\Omega (Q_x - P_y)\,dx\,dy$$



WHERE Ω IS THE REGION INSIDE C. I WILL OUTLINE ON NEXT PAGE THE DERIVATION OF THIS FROM DIVERGENCE THEOREM.

TO PROVE $(*)$ WE LET $f = u + iv$ AND $dz = dx + i\,dy$.

THEN
$$\int_C f(z)\,dz = \int_C (u+iv)(dx + i\,dy) = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy) \,. \quad (1)$$

NOW SINCE f IS ANALYTIC $\longrightarrow$ $u_x = v_y$, $u_y = -v_x$ (CR) EQUATION

f' IS CONTINUOUS $\longrightarrow$ u_x, u_y, v_x, v_y CONTINUOUS

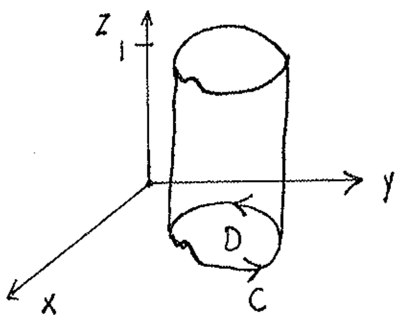
USING GREEN'S THEOREM ON EACH TERM IN (1) WE GET

$$\int_C f(z)\,dz = -\int_\Omega [v_x + u_y]\,dx\,dy + i\int_\Omega [u_x - v_y]\,dx\,dy = 0 \text{ BY CR.}$$

THUS
$$\int_C f(z)\,dz = 0 .$$

GREEN'S THEOREM DERIVATION

WE CONSIDER A 3-D REGION V WITH BASE D IN x,y PLANE THAT IS EXTENDED FROM $z=0$ TO $z=1$ TRIVIALLY.



LET C BOUND D WITH counterclockwise orientation.

LET $\underline{F}$ BE THE VECTOR FIELD

$\underline{F} = F_1 \hat{i} + F_2 \hat{j} + 0\hat{k}$ WITH NO z component.

ASSUME THAT $F_1, F_2, F_{1x}, F_{1y}, F_{2x}, F_{2y}$ ARE CONTINUOUS IN D.

THEN THE DIVERGENCE THEOREM GIVES

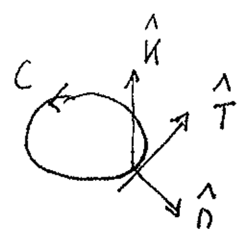
$$\iiint_V \nabla\cdot\underline{F}\, dv = \iint_S \hat{F}\cdot\hat{n}\, ds$$

WHERE S IS SURFACE AREA OF V AND $\hat{n}$ IS UNIT OUTWARD NORMAL TO S.

NOTICE THAT S CONSISTS OF THE SIDES OF THE "OIL CAN" AND THE TOP AND BOTTOM OF THE CAN. NOW THE FLUX ON THE TOP SURFACE $(z=1)$ AND BOTTOM SURFACE $z=0$ IS ZERO SINCE $\underline{F}\cdot\hat{k}=0$ (i.e. $\hat{n}=\hat{k}$ ON $z=1$)

THUS
$$\iiint_V \nabla\cdot\underline{F}\, dx\,dy\,dz = \int_0^1\Big(\iint_D (F_{1x}+F_{2y})\,dx\,dy\Big)dz = \iint_D (F_{1x}+F_{2y})\,dx\,dy$$
$$= \int_0^1\Big(\oint_C \underline{F}\cdot\hat{n}\,ds\Big)dz = \oint_C \underline{F}\cdot\hat{n}\,ds.$$

THIS GIVES
$$\iint_D (F_{1x}+F_{2y})\,dx\,dy = \oint_C \underline{F}\cdot\hat{n}\,ds$$



LET $\hat{T}$ TANGENT TO C, $\hat{k}$ IS $\perp$ TO D, SO

$\hat{n} = \hat{T}\times\hat{k}$ AND $\hat{T}\,ds = d\underline{r}$ WITH $d\underline{r} = (dx, dy) = \hat{i}\,dx + \hat{j}\,dy$.

THUS, $(*)$
$$\iint_D (F_{1x}+F_{2y})\,dx\,dy = \oint_C \underline{F}\cdot[\hat{T}\times\hat{k}]\,ds = \oint_C (\hat{k}\times\underline{F})\cdot\hat{T}\,ds = \int_C (\hat{k}\times\underline{F})\cdot d\underline{r}$$

HERE WE USED THE VECTOR IDENTITY $\underline{a}\cdot[\underline{b}\times\underline{c}] = (\underline{a}\times\underline{b})\cdot\underline{c}$. WE THEN CALCULATE

$$\hat{k}\times\bar{F} = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ F_1 & F_2 & 0 \end{vmatrix} = -F_2\hat{i} + F_1\hat{j}.$$

THEN IN $(*)$

$$\iint_D (F_{1x}+F_{2y})\,dx\,dy = \oint_C (-F_2\hat{i}+F_1\hat{j})\cdot(\hat{i}\,dx+\hat{j}\,dy)$$

GREEN'S THEOREM.
$$\iint_D (F_{1x}+F_{2y})\,dx\,dy = \oint_C (F_1\,dy - F_2\,dx)$$

REMARKS (i) WE ASSUMED

a) $f(z)$ IS ANALYTIC INSIDE Ω AND ON C (Analyticity $\rightarrow f'$ exists)

b) $f'(z)$ IS CONTINUOUS IN Ω $\longleftarrow$ would like to remove this condition

c) C is a simple closed curve $\leftarrow$ can we have figure 8? more general curves?

A MORE PRECISE VERSION OF CAUCHY'S THEOREM BY E. GOURSAT REMOVED CONDITION (ii). FOR MATHEMATICALLY INCLINED READERS SEE

- SECTION 2.3 OF THE BOOK "BASIC COMPLEX ANALYSIS" BY J. MARSDEN.
- THE REMOVAL OF CONDITION (ii) IS MENTIONED BUT NOT PROVED IN OUR BOOK BY SAFF AND SNIDER.

A REGION D IS CALLED SIMPLY-CONNECTED IF D IS CONNECTED AND EVERY CLOSED CURVE IN D CONTAINS ONLY POINTS IN D.

simply connected

not simply connected

ROUGHLY speaking, simply connected domains are connected domains with no holes.

CAUCHY-GOURSAT THEOREM LET $f(z)$ BE ANALYTIC ON A SIMPLY CONNECTED DOMAIN D AND LET C BE A SIMPLE CLOSED CURVE IN D. THEN $\int_C f(z)\,dz = 0$

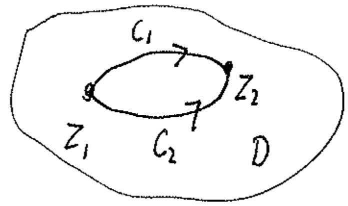
REMARKS (i) WE CAN APPLY THIS THEOREM TO FIGURE 8 TYPE LOOPS, BY CONSIDERING EACH LOOP SEPARATELY.

(ii) CAN ALSO GENERALIZE AS SHOWN IN THE NEXT DEFORMATION OF CONTOUR APPROACH.

WE NOW STATE A FEW CONSEQUENCES OF CAUCHY-GOURSAT THEOREM.

THEOREM (PATH INDEPENDENCE) SUPPOSE f IS ANALYTIC ON A SIMPLY CONNECTED REGION D. THEN FOR ANY TWO CURVES C_1 AND C_2 WITH SAME ORIENTATION JOINING THE SAME POINTS z_0 AND z_1 WE HAVE

$$\int_{C_1} f\,dz = \int_{C_2} f\,dz$$

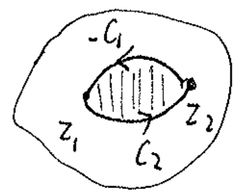


i.e. path independence.

PROOF we make a closed curve C oriented counterclockwise AND use cauchy-goursat theorem

$$\int_C f(z)\,dz = \int_{C_2} f(z)\,dz + \int_{-C_1} f(z)\,dz = 0$$

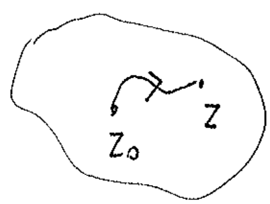
since f analytic.



THUS, $\displaystyle\int_{C_2} f(z)\,dz = -\int_{-C_1} f(z)\,dz = \int_{C_1} f(z)\,dz.$ ☒

THEOREM (ANTI-DERIVATIVE) SUPPOSE $f(z)$ IS ANALYTIC IN A SIMPLY CONNECTED DOMAIN D. THEN $f(z)$ HAS AN ANTI-DERIVATIVE $\mathcal{F}(z)$ WHERE

$$\mathcal{F}(z) = \int_{z_0}^{z} f(\zeta)\,d\zeta$$



ζ-plane.

PROOF SINCE $f(\zeta)$ IS ANALYTIC, WE HAVE THAT THE INTEGRAL FROM $\zeta = z_0$ TO $\zeta = z$ IS INDEPENDENT OF PATH AND HENCE IS WELL DEFINED. WE WANT TO SHOW THAT $\mathcal{F}'(z) = f(z)$. WE CALCULATE

$$\mathcal{F}(z+\Delta z) - \mathcal{F}(z) = \int_{z_0}^{z+\Delta z} f(\zeta)\,d\zeta - \int_{z_0}^{z} f(\zeta)\,d\zeta = \int_{z}^{z+\Delta z} f(\zeta)\,d\zeta.$$

NOW WRITING $f(z) = \dfrac{1}{\Delta z}\displaystyle\int_z^{z+\Delta z} f(z)\,d\zeta$ WE GET BY ADDING AND SUBTRACTING $f(z)$

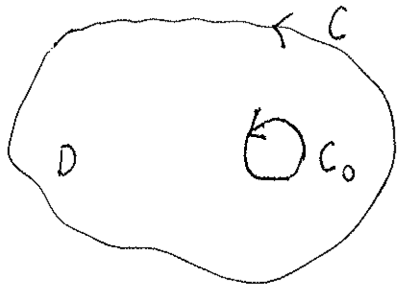
$$\frac{\mathcal{F}(z+\Delta z) - \mathcal{F}(z)}{\Delta z} = \int_z^{z+\Delta z} \frac{(f(\zeta) - f(z))}{\Delta z}\,d\zeta + f(z).$$

LET $\Delta z \to 0$ THEN

$$\mathcal{F}'(z) = \lim_{\Delta z \to 0} \frac{\mathcal{F}(z+\Delta z) - \mathcal{F}(z)}{\Delta z} \Rightarrow f(z).$$

DEFORMATION RESULT I

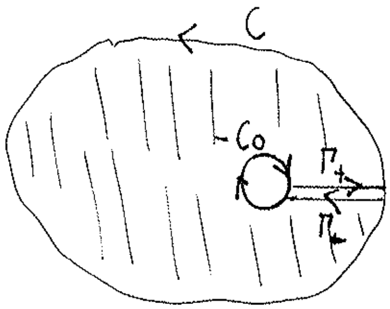
CONSIDER THE DOMAIN D AS SHOWN, THAT HAS A "HOLE".



SUPPOSE THAT $f(z)$ IS ANALYTIC BETWEEN C_0 AND C, AND IS ANALYTIC ON C AND C_0.

THEN
$$\int_C f(z)\,dz = \int_{C_0} f(z)\,dz$$
WITH BOTH CONTOURS ORIENTED COUNTERCLOCKWISE.

DERIVATION WE CONSIDER THE CONTOUR AS SHOWN. WE INTRODUCE A PATH



Γ_- FROM C TO $-C_0$, AND A PATH Γ_+ FROM $-C_0$ BACK TO C.

WE LET $\Gamma = C + \Gamma_- + (-C_0) + \Gamma_+$

INSIDE Γ (shaded region), $f(z)$ IS ANALYTIC

AND BY CAUCHY'S INTEGRAL THEOREM $\int_\Gamma f(z)\,dz = 0$

(NOTE: STRICTLY speaking Γ IS NOT a simple curve, but one can take Γ_- arbitrarily close to Γ_+).

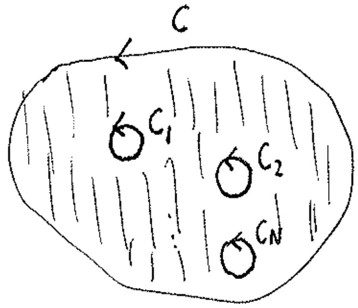
THEN
$$\int_\Gamma f(z)\,dz = \int_C f(z)\,dz + \int_{\Gamma_-} f(z)\,dz + \int_{\Gamma_+} f(z)\,dz + \int_{-C_0} f(z)\,dz = 0.$$

BUT $\int_{\Gamma_-} f(z)\,dz = -\int_{\Gamma_+} f(z)\,dz$ (same curve traversed in different directions).

THUS
$$\underset{\text{counterclockwise}}{\int_C f(z)\,dz} = -\int_{-C_0} f(z)\,dz = \underset{\text{counterclockwise}}{\int_{C_0} f(z)\,dz} \qquad \boxtimes$$

DEFORMATION RESULT II

SUPPOSE THAT WE HAVE A DOMAIN WITH N HOLES AS SHOWN.

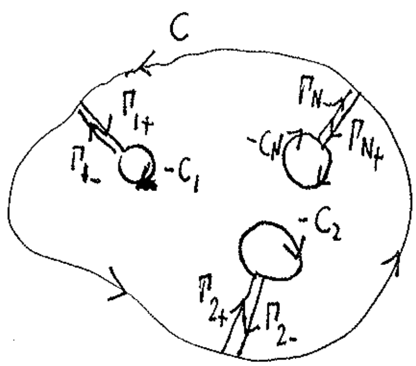


SUPPOSE THAT $f(z)$ IS ANALYTIC IN THE SHADED REGION INSIDE AND ON C, BUT OUTSIDE $C_1, \ldots, C_N$. SUPPOSE f IS ANALYTIC ON $C_1, \ldots, C_N$.

THEN $$\int_C f(z)\,dz = \sum_{j=1}^{N} \int_{C_j} f(z)\,dz$$

counterclockwise orientation.

THE DERIVATION OF THIS IS TO USE CAUCHY'S THEOREM ON THE CONTOUR AS SHOWN



THEN
$$\left(\int_C + \sum_{j=1}^{N} \left(\int_{\Gamma_{j+}} + \int_{\Gamma_{j-}} \right) + \sum_{j=1}^{N} \int_{-C_j} \right) f(z)\,dz = 0$$

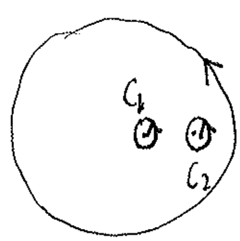
BUT $\int_{\Gamma_{j+}} = -\int_{\Gamma_{j-}}$ SO

$$\int_C f(z)\,dz = -\sum_{j=1}^{N} \int_{-C_j} f(z)\,dz = \sum_{j=1}^{N} \int_{C_j} f(z)\,dz.$$

REMARK (i) TYPICALLY HOW THIS WILL BE USED IS THAT WE WILL PUT AN ISOLATED "SINGULARITY" OF $f(z)$ INSIDE A SMALL DISK.

i.e. IF $f(z) = \dfrac{1}{(z-1)(z-2)}$ AND $C: \ |z| = 3$

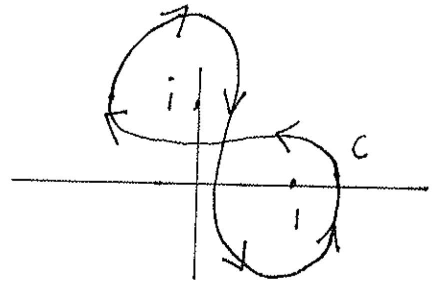
THEN WE DRAW



$|z| = 3$

WHERE C_1 CONTAINS $z = 1$ AND C_2 CONTAINS $z = 2$.

TO ILLUSTRATE THE USE OF THE DEFORMATION RESULT WE CONSIDER HERE A SIMPLE EXAMPLE. (MANY MORE EXAMPLES ARE CONSIDERED LATER)



<u>EXAMPLE 1</u> CALCULATE $I = \int_C \frac{z}{(z-i)(z-1)}\,dz$

WHERE C IS THE FIGURE 8 PATH AS SHOWN.

<u>SOLUTION</u> NOTICE THAT $f = \frac{z}{(z-i)(z-1)}$ IS <u>NOT</u> ANALYTIC AT $z=i$ AND AT $z=1$.

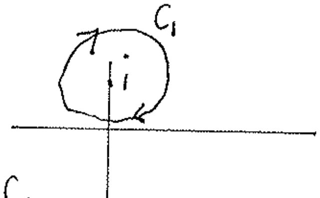
WE FIRST USE PARTIAL FRACTION DECOMPOSITION

$$\frac{z}{(z-i)(z-1)} = \frac{A}{z-i} + \frac{B}{z-1} \longrightarrow z = A(z-1) + B(z-i)$$

$$z = i \longrightarrow A = \frac{i}{i-1} = \frac{i(-i-1)}{2} = \frac{1-i}{2}$$

$$z = 1 \longrightarrow B = \frac{1}{1-i} = \frac{1+i}{2}.$$

THUS $I = \int_C \left(\frac{A}{z-i} + \frac{B}{z-1}\right) dz = \int_C \frac{A}{z-i}\,dz + \int_C \frac{B}{z-1}\,dz.$

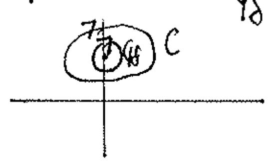


- DEFINE C_1 TO BE <u>CLOCKWISE LOOP</u> AROUND $z=i$

THEN $\int_{C_1} \frac{B}{z-1}\,dz = 0$ SINCE $1/z-1$ IS <u>ANALYTIC</u> inside C_1.

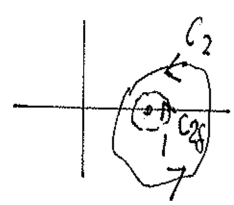
BY PATH DEFORMATION RESULT I, WE GET $\int_{C_1} \frac{A}{z-i}\,dz = \int_{C_\delta} \frac{A}{z-i}\,dz$

WHERE C_δ IS <u>CLOCKWISE</u> CONTOUR $|z-i| = \delta$.



BUT $\int_{C_\delta} \frac{A}{z-i}\,dz = -2\pi i\,(A)$.

HENCE $\int_{C_1} \frac{B}{z-1}\,dz = 0$, $\int_{C_1} \frac{A}{z-i}\,dz = -2\pi i A$.



- NOW DEFINE C_2 TO BE <u>COUNTERCLOCKWISE</u> LOOP AROUND $z=1$.

THEN $\int_{C_2} \frac{A}{z-i}\,dz = 0$ SINCE $1/z-i$ IS ANALYTIC INSIDE C_2

AND BY PATH DEFORMATION $\int_{C_2} \frac{B}{z-1}\,dz = \int_{C_{2\delta}} \frac{B}{z-1}\,dz = 2\pi i B$ $C_{2\delta}$ counterclockwise

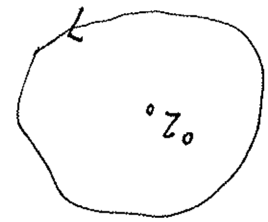
THUS, $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz = -2\pi i A + 2\pi i B$

$$= 2\pi i\,(B-A) = 2\pi i\,(i) = -2\pi.$$

THEOREM (CAUCHY-INTEGRAL FORMULA)

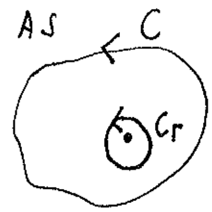
LET C BE A SIMPLE CLOSED CONTOUR ORIENTED COUNTERCLOCKWISE. IF $f(z)$ IS ANALYTIC INSIDE AND ON C. THEN IF z_0 IS ANY POINT INSIDE C, WE HAVE

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz$$



PROOF NOTICE THAT $\dfrac{f(z)}{z - z_0}$ IS ANALYTIC INSIDE AND ON C EXCEPT AT z_0.

THUS BY CAUCHY-GOURSAT WE CAN DEFORM AS



$$I \equiv \int_C \frac{f(z)}{z - z_0}\, dz = \int_{C_r} \frac{f(z)}{z - z_0}\, dz$$

WHERE $C_r = \{ z \mid |z - z_0| = r \}$.

NOW WE WRITE,
$$\int_{C_r} \frac{f(z)}{z - z_0}\, dz = \int_{C_r} \frac{f(z_0)}{z - z_0}\, dz + \int_{C_r} \left(\frac{f(z) - f(z_0)}{z - z_0} \right) dz.$$

THIS YIELDS THAT
$$I = 2\pi i\, f(z_0) + \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0}\, dz.$$

SINCE I AND $2\pi i\, f(z_0)$ ARE INDEPENDENT OF r, WE CAN TAKE THE LIMIT $r \to 0$ TO OBTAIN

$$I - 2\pi i\, f(z_0) = \lim_{r \to 0} \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0}\, dz \qquad (+)$$

BUT
$$\left| \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0}\, dz \right| \leq \max_{C_r} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| 2\pi r = \max_{C_r} |f(z) - f(z_0)|\, 2\pi. \qquad (*)$$

NOW DEFINE $M_r = \max_{C_r} |f(z) - f(z_0)|$. SINCE f IS CONTINUOUS AT z_0 IT FOLLOWS THAT $f(z) \to f(z_0)$ AS $r \to 0$, SO THAT $M_r \to 0$ AS $r \to 0$.

HENCE, $(*)$ IMPLIES $\left| \int_{C_r} \dfrac{f(z) - f(z_0)}{z - z_0}\, dz \right| \to 0$ AS $r \to 0$. THEN $(+)$ YIELDS THAT

$$I = \int_C \frac{f(z)}{z - z_0}\, dz = 2\pi i\, f(z_0) \longrightarrow f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz$$

EXAMPLE 1 EVALUATE $I = \int_C \frac{1}{z}\,dz$ WHERE C IS ELLIPSE $x^2 + 4y^2 = 1$ ORIENTED COUNTERCLOCKWISE.

SOLUTION WE DEFORM TO CONTOUR AS SHOWN SINCE $1/z$ ANALYTIC EXCEPT AT $z=0$.

THEN $\left(\int_C + \int_{C_\delta}\right)(1/z)\,dz = 0$ BY CAUCHY-GOURSAT

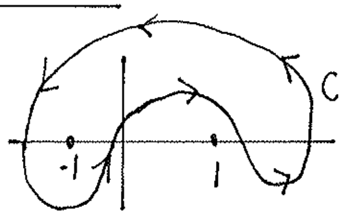
WHERE C_δ IS CIRCLE $|z| = \delta$ ORIENTED CLOCKWISE

THUS $\int_C \frac{1}{z}\,dz = -\int_{C_\delta} \frac{1}{z}\,dz = \int_{-C_\delta} \frac{1}{z}\,dz = 2\pi i$ BY PREVIOUS NOTES.

NOW $-C_\delta$ ORIENTED COUNTERCLOCKWISE

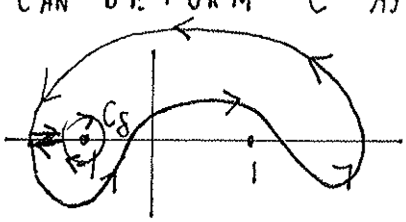
$$\int_{-C_\delta} \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{\delta e^{it}}\, i\delta e^{it}\,dt = i2\pi. \quad \text{THUS } I = 2\pi i.$$

EXAMPLE 2 CALCULATE $I = \int_C \frac{dz}{z^2-1}$ WHERE C IS PATH SHOWN BELOW.



NOTICE: $z=-1, 1$ ARE POINTS WHERE $f(z) = \frac{1}{z^2-1}$ IS NOT ANALYTIC, BUT ONLY $z=-1$ IS INSIDE C.

WE CAN DEFORM C AS SHOWN:



THUS $\left(\int_C + \int_{C_\delta}\right) \frac{1}{z^2-1}\,dz = 0$ BY CAUCHY-GOURSAT.

BUT $\frac{1}{z^2-1} = \frac{1}{2(z-1)} - \frac{1}{2(z+1)}$

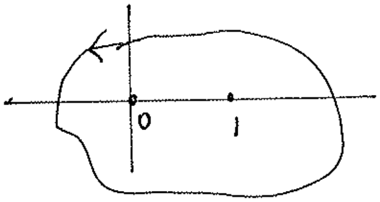
WE CONCLUDE THAT $\int_C \frac{1}{2(z-1)}\,dz = 0$ SINCE $f_1(z) = \frac{1}{2(z-1)}$ IS ANALYTIC INSIDE AND ON C. THUS $\int_C \frac{1}{z^2-1}\,dz = \int_C \frac{-1}{2(z+1)}\,dz$.

NOW BY PATH DEFORMATION $\int_C \frac{1}{2(z+1)}\,dz + \int_{C_\delta} \frac{1}{2(z+1)}\,dz = 0$. BY CAUCHY-GOURSAT.

HENCE, $\int_C \frac{-1}{2(z+1)}\,dz = +\int_{C_\delta} \frac{1}{2(z+1)}\,dz = \frac{-1}{2}\int_{-C_\delta} \frac{1}{z+1}\,dz = \frac{-1}{2}(2\pi i)$ SINCE $-C_\delta$ COUNTERCLOCKWISE

THUS, $\int_C \frac{dz}{z^2-1} = -\frac{1}{2}(2\pi i) = -\pi i.$

EXAMPLE 3 CALCULATE $I = \int_C \frac{3z-2}{z(z-1)}\,dz$ WITH C AS SHOWN.



THE INTEGRAND $\frac{3z-2}{z(z-1)}$ IS ANALYTIC EXCEPT AT $z=0$ AND $z=1$.

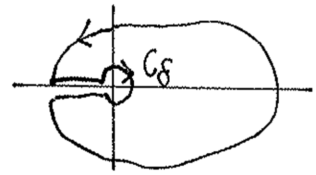
WE USE PARTIAL FRACTIONS: $\frac{3z-2}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1}$

HENCE $3z-2 = A(z-1) + Bz$. IF $z=0 \rightarrow A=2$

IF $z=1 \rightarrow B=1$.

THUS $I = \int_C \frac{2}{z}\,dz + \int_C \frac{1}{z-1}\,dz$.

- FOR $I_1 = \int_C \frac{2}{z}\,dz$ WE DEFORM AS FOLLOWS



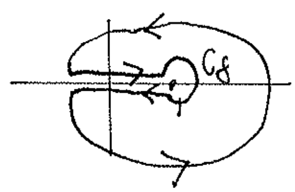
THEN BY CAUCHY-GOURSAT

$$\int_C \frac{2}{z}\,dz + \int_{C_\delta} \frac{2}{z}\,dz = 0.$$

$$I_1 = \int_C \frac{2}{z}\,dz = -\int_{C_\delta} \frac{2}{z}\,dz = \int_{-C_\delta} \frac{2}{z}\,dz = 2(2\pi i).$$

COUNTERCLOCKWISE

- FOR $I_2 = \int_C \frac{1}{z-1}$ WE DEFORM AS



SO $\int_C \frac{1}{z-1}\,dz + \int_{C_\delta} \frac{1}{z-1}\,dz = 0$ BY CAUCHY-GOURSAT.

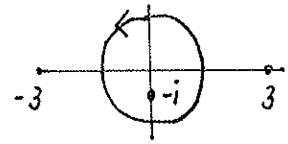
THUS $\int_C \frac{1}{z-1}\,dz = -\int_{C_\delta} \frac{1}{z-1}\,dz = \int_{-C_\delta} \frac{1}{z-1}\,dz = 2\pi i$

COUNTERCLOCKWISE

THEREFORE, $I = \int_C \frac{3z-2}{z(z-1)}\,dz = I_1 + I_2 = 4\pi i + 2\pi i = 6\pi i$.

EXAMPLE 4 CALCULATE $I = \int_C \frac{z\,dz}{(9-z^2)(z+i)}$ WHERE C IS THE DISK $|z|=2$

ORIENTED COUNTERCLOCKWISE.



NOTICE THAT $f(z) = \frac{z}{(9-z^2)(z+i)}$ IS ANALYTIC EXCEPT AT $z=-i, 3, -3$.

ONLY $z=-i$ IS INSIDE C. WE CAN USE PARTIAL FRACTIONS BUT INSTEAD USE CAUCHY INTEGRAL FORMULA

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz$$

WHEN C counterclockwise containing z_0, AND f Analytic inside and on C.

LET $z_0 = -i$ SO $\quad f(-i) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z+i}\,dz$

LET $f(z) = z/9-z^2$ THEN $\quad 2\pi i\, f(-i) = \int_C \frac{f(z)}{z+i}\,dz = \int_C \frac{z}{(9-z^2)(z+i)}\,dz$

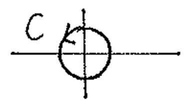
THUS $\quad \int_C \frac{z/(9-z^2)}{z+i}\,dz = 2\pi i \left(\frac{z}{9-z^2}\right)\Big|_{z=i} = 2\pi i\left(\frac{-i}{9+1}\right) = \pi/5$

SO $I = \pi/5$.

EXAMPLE 5 LET C BE THE UNIT CIRCLE $|z|=1$ ORIENTED COUNTERCLOCKWISE.

CALCULATE $\int_C \frac{e^{az}}{z}\,dz$ FOR ANY CONSTANT a, AND FROM THIS PROVE

THAT $\int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\,d\theta = \pi$.



PROOF BY CAUCHY-INTEGRAL FORMULA SINCE $f(z) = e^{az}$ IS ANALYTIC INSIDE AND ON C WE OBTAIN,

$f(0) = \frac{1}{2\pi i}\int_C \frac{e^{az}}{z-0}\,dz$. THUS $\int_C \frac{e^{az}}{z}\,dz = 2\pi i$ SINCE $f(0)=1$.

NOW LET $z = e^{i\varphi} \rightarrow dz/d\varphi = ie^{i\varphi}$ $\quad \int_C \frac{e^{az}}{z}\,dz = \int_0^{2\pi} \frac{e^{az(\varphi)}}{z(\varphi)} \frac{dz}{d\varphi}\,d\varphi$

THUS,
$$\int_C \frac{e^{az}}{z}\,dz = \int_0^{2\pi} \frac{e^{a(\cos\varphi + i\sin\varphi)}}{e^{i\varphi}}\, ie^{i\varphi}\,d\varphi = i\int_0^{2\pi} e^{a\cos\varphi}\,[\cos(a\sin\varphi) + i\sin(a\sin\varphi)]\,d\varphi$$

THUS
$$2\pi i = i\left[\int_0^{2\pi} e^{a\cos\varphi}\cos(a\sin\varphi)\,d\varphi\right] - \int_0^{2\pi} e^{a\cos\varphi}\sin(a\sin\varphi)\,d\varphi$$

TAKE IMAGINARY PARTS OF BOTH SIDES,
$$(*)\quad \int_0^{2\pi} e^{a\cos\varphi}\cos(a\sin\varphi)\,d\varphi = 2\pi.$$

SINCE INTEGRAND IN $(*)$ IS 2π PERIODIC, THEN
$$\int_{-\pi}^{\pi} e^{a\cos\varphi}\cos(a\sin\varphi)\,d\varphi = 2\pi.$$

HOWEVER, SINCE INTEGRAND IN $(*)$ IS EVEN IN φ, THEN
$$\int_0^{\pi} e^{a\cos\varphi}\cos(a\sin\varphi)\,d\varphi = \pi.$$

<u>EXAMPLE 6</u> CALCULATE $I = \int_0^{2\pi} \cos^{2n}\varphi\,d\varphi$ WITH n A POSITIVE INTEGER.

WE WRITE $\cos\varphi = \frac{1}{2}(z + 1/z)$ WITH $z = e^{i\varphi}$ AND $dz = ie^{i\varphi}d\varphi \rightarrow \frac{dz}{iz} = d\varphi$.

WE CONVERT I TO A LINE INTEGRAL OVER UNIT CIRCLE ORIENTED COUNTERCLOCKWISE.

WE WRITE
$$I = \int_0^{2\pi} \cos^{2n}\varphi\,d\varphi = \int_C \left[\frac{1}{2}(z + 1/z)^{2n}\right]\frac{dz}{iz} = \frac{1}{i2^{2n}}\int_C (z + 1/z)^{2n}\,\frac{dz}{z}.$$

NOW THE BINOMIAL THEOREM IS $(a+b)^{2n} = \sum_{k=0}^{2n} a^{2n-k}b^k \binom{2n}{k}$ AND

WE APPLY IT WITH $a = z$, $b = 1/z$. WE CALCULATE,
$$I = \frac{1}{i2^{2n}}\int_C \frac{1}{z}\sum_{k=0}^{2n} z^{2n-k}\left(\frac{1}{z}\right)^k \binom{2n}{k}\,dz = \frac{1}{i2^{2n}}\sum_{k=0}^{2n}\binom{2n}{k}\int_C z^{2n-2k-1}\,dz.$$

WE NOW DEFORM TO A CIRCULAR DISK OF RADIUS δ CENTERED AT THE ORIGIN

OR MORE SIMPLY NOTICE AND RECALL THAT
$$\int_C z^m\,dz = \begin{cases} 0 & \text{IF } m \neq -1 \\ 2\pi i & \text{IF } m = -1 \end{cases}$$

THUS, THE ONLY TERM IN THE SUM TO CONTRIBUTE IS WHEN $K = n$.

THUS,

$$I = \int_0^{2\pi} \cos^{2n}\varphi \, d\varphi = \frac{1}{i\,2^{2n}} \binom{2n}{n} 2\pi i = \frac{2\pi}{2^{2n}} \frac{(2n)!}{(n!)^2}.$$

(WE RECALL $\binom{K}{j} = \frac{K!}{j!(K-j)!}$.).

THUS,

$$I = \frac{2\pi}{2^{2n}} \frac{2n(2n-1)(2n-2)\cdots 1}{(1\cdot 2\cdot 3\cdots n)(1\cdot 2\cdot 3\cdots n)} = \frac{2\pi}{2^{2n}} \frac{[2n(2n-2)(2n-4)\cdots 2][(2n-1)(2n-3)\cdots 1]}{[1\cdot 2\cdots n][1\cdot 2\cdots n]}$$

$$I = 2\pi \frac{[2n(2n-2)(2n-4)\cdots 2][(2n-1)(2n-3)\cdots 1]}{[2\cdot 4\cdots (2n)][2\cdot 4\cdots 2n]}$$

THUS

$$I = 2\pi \frac{(1\cdot 3\cdots (2n-1))}{(2\cdot 4\cdots 2n)}.$$

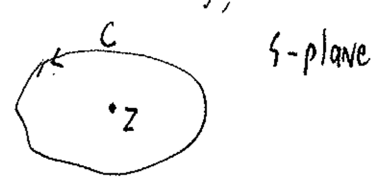
FOR INSTANCE

$$\int_0^{2\pi} \cos^{10}(\varphi)\, d\varphi = 2\pi \left(\frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8\cdot 10} \right).$$

CONSEQUENCES OF THE CAUCHY INTEGRAL FORMULA

KEY POINT 1 IF f IS ANALYTIC WITHIN AND ON A SIMPLE CLOSED CONTOUR C, THEN FOR ANY POINT z INSIDE C (ORIENTED COUNTERCLOCKWISE), WE HAVE

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta$$



AND THAT $f', f'', \ldots, f^{(n)}$ ARE ALL ANALYTIC AT z. IN OTHER WORDS, IF f IS ANALYTIC AT POINT z THEN ITS DERIVATIVES OF ALL ORDERS ARE ALSO ANALYTIC AT POINT z. MOREOVER,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta.$$

PROOF WE FIRST SHOW THAT $f'(z)$ EXISTS FOR ANY z IN C, AND THAT $f'(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta$ (i.e. we can differentiate under integral sign).

TO PROVE THIS, WE NOTE THAT SINCE z IS INSIDE C, THEN FOR $z + \Delta z$ IT MUST ALSO BE IN C FOR $|\Delta z|$ SMALL ENOUGH.

WE CALCULATE
$$J \equiv \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{1}{2\pi i}\int_C \left[\frac{1}{\zeta - z - \Delta z} - \frac{1}{\zeta - z}\right]\frac{f(\zeta)}{\Delta z}\, d\zeta \quad (*)$$

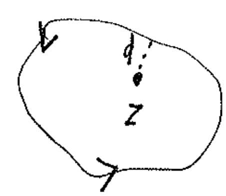
THUS
$$J = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z - \Delta z)(\zeta - z)}\, d\zeta$$
OBTAINED BY COMBINING $(*)$.

NOW ADD AND SUBTRACT
$$J - \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^2}\, dz = \frac{1}{2\pi i}\int_C \left[\frac{1}{(\zeta - z - \Delta z)(\zeta - z)} - \frac{1}{(\zeta - z)^2}\right] f(\zeta)\, d\zeta$$
$$= \frac{1}{2\pi i}\int_C \frac{1}{(\zeta - z)}\left[\frac{1}{\zeta - z - \Delta z} - \frac{1}{\zeta - z}\right] f(\zeta)\, d\zeta$$
$$J - \frac{1}{2\pi i}\int \frac{f(\zeta)}{(\zeta - z)^2}\, dz = \frac{1}{2\pi i}\Delta z \int_C \frac{f(\zeta)}{(\zeta - z - \Delta z)(\zeta - z)^2}\, d\zeta. \quad (+)$$

NOW LET $|\Delta z| \to 0$ AND DEFINE $M = \max_C |f(\zeta)|$. WE WANT TO SHOW THAT $|\text{RHS}|$ OF $(+)$ TENDS TO 0 AS $|\Delta z| \to 0$.

NOW SINCE z IS <u>INSIDE</u> C, THEN THERE IS A POSITIVE CONSTANT d SUCH THAT $|\zeta - z| \geq d > 0$ FOR ALL ζ ON CONTOUR C. LET $|\Delta z| < d/2$.

THEN BY Δ INEQUALITY



$$|\zeta - z - \Delta z| \geq \big|\, |\zeta - z| - |\Delta z| \,\big| \geq d - d/2 = d/2$$

FOR ζ ON CONTOUR C. THEREFORE,
$$\frac{1}{|\zeta - z - \Delta z|\,|\zeta - z|^2} \leq \frac{1}{(d/2)\, d^2} \quad \text{FOR } \zeta \text{ ON } C.$$

WE THEN ESTIMATE

$$\left| \frac{1}{2\pi i} \Delta z \int_C \frac{f(\zeta)}{(\zeta - z - \Delta z)(\zeta - z)^2} d\zeta \right| \leq \frac{|\Delta z|}{2\pi} \left| \int_C \frac{f(\zeta)}{(\zeta - z - \Delta z)(\zeta - z)^2} dz \right| \qquad (**)$$

BUT

$$\left| \frac{f(\zeta)}{(\zeta - z - \Delta z)(\zeta - z)^2} \right| \leq \max_{\zeta \text{ on } C} |f| \frac{1}{(d/2) d^2}$$

THUS $(**)$ BECOMES,

$$\left| \frac{1}{2\pi i} \Delta z \int_C \frac{f(\zeta)}{(\zeta - z - \Delta z)(\zeta - z)^2} d\zeta \right| \leq \frac{|\Delta z|}{2\pi} \max_{\zeta \text{ on } C} |f(\zeta)| \frac{2}{d^3} L$$

WHERE $L = \text{length}(C)$.

FINALLY IN $(+)$

$$\left| J - \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta \right| \leq \frac{|\Delta z|}{2\pi} \max_{\zeta \text{ on } C} |f(\zeta)| \frac{2}{d^3} L \to 0 \text{ AS } \Delta z \to 0.$$

SINCE $J \to f'$ AS $\Delta z \to 0$, WE HAVE

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta$$

WHICH JUSTIFIES DIFFERENTIATING UNDER THE INTEGRAL SIGN.

FURTHER DIFFERENTIATIONS YIELD THAT $f^{(n)}(z)$ EXISTS FOR ANY n

AND SO

$$f^{(n)}(z) = \frac{1}{2\pi i} n! \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta.$$

SINCE z IS ARBITRARY INSIDE C, THEN $f^{(n)}$ IS ANALYTIC AT EACH POINT INSIDE C FOR ANY $n > 0$.

<u>NOTE</u>: THIS IS VERY DIFFERENT FROM CALCULUS. I.E. LET $f(x) = x^3 \sin(1/x)$

IT IS EASY TO SEE THAT $f'(x)$ EXISTS BUT $f''(x)$ DOES NOT AT $x = 0$.

<u>KEY POINT 2</u> LET C BE A SIMPLE CLOSED CONTOUR AND LET $f(\zeta)$ BE ANALYTIC INSIDE AND ON C. THEN IF z IS INSIDE C,

ζ plane

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta. \qquad (*)$$

TAKING C_R TO BE THE DISK $|\zeta - z| = R$, AND ASSUMING THAT $|f(\zeta)| \le M$ ON C_R, WE ESTIMATE FROM $(*)$ THAT

$$|f^{(n)}(z)| \le \frac{n!}{2\pi} \frac{M}{R^{n+1}} (2\pi R) = \frac{n!\, M}{R^n}. \qquad (+)$$

CAUCHY'S INEQUALITY.

THIS ESTIMATE LEADS TO AN IMPORTANT RESULT KNOWN AS LIOUVILLE'S THEOREM:

<u>LIOUVILLE'S THEOREM</u> LET $f(\zeta)$ BE AN ENTIRE FUNCTION THAT IS BOUNDED (i.e. $|f(\zeta)| \le M$) FOR ALL ζ IN THE COMPLEX PLANE. THEN $f(\zeta)$ IS CONSTANT FOR ALL ζ.

<u>PROOF</u> PICK ANY POINT z IN COMPLEX PLANE, AND DEFINE C_R: $|\zeta - z| = R$. THEN FROM (+) WITH $n = 1$ WE HAVE

$$|f'(z)| \le \frac{M}{R}.$$

SINCE R CAN BE MADE ARBITRARILY LARGE, WHILE $f'(z)$ IS A FIXED NUMBER, IT FOLLOWS THAT $f'(z) = 0$. SINCE z IS ARBITRARY, IT FOLLOWS THAT $f'(z) = 0$ FOR ALL z, AND CONSEQUENTLY

$$f(z) = c \qquad c = \text{CONSTANT} \quad \forall z.$$

REMARKS (i) IF $f(\zeta)$ IS ENTIRE AND $|\mathrm{Re}\, f| \leq M \;\; \forall \zeta \longrightarrow f$ IS CONSTANT.

PROOF LET $g(\zeta) = e^{f(\zeta)}$. THEN $g(\zeta)$ IS ENTIRE SINCE f IS.

NOW LET $f = u + iv$. $\longrightarrow g = e^{u} e^{iv}$ AND $|g(\zeta)| = e^{u}$.

BUT $u \leq |u| \leq M$ IN ζ-plane $\Longrightarrow |g(\zeta)| \leq e^{M}$.

THUS $g(\zeta)$ IS ENTIRE BOUNDED FUNCTION. LIOUVILLE'S THEOREM YIELDS THAT $g(\zeta)$ IS A CONSTANT $\longrightarrow f(\zeta) =$ CONSTANT.

(ii) IF $f(\zeta)$ IS ENTIRE AND $|\mathrm{Im}\, f| \leq M \;\; \forall \zeta \longrightarrow f$ IS CONSTANT.

TO PROVE THIS START WITH $g(\zeta) = e^{-if(\zeta)}$ AND FOLLOW (i).

KEY POINT 3 LET $f(z)$ BE ANALYTIC INSIDE AND ON A SIMPLE CLOSED CONTOUR C THAT IS A CIRCLE (COUNTERCLOCKWISE) OF RADIUS R CENTERED AT z_0. THEN

$$(*) \qquad f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\varphi})\, d\varphi$$

= average of the values of f on the circle.

$(*)$ IS CALLED THE MEAN-VALUE PROPERTY.

PROOF

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z(\varphi))}{z(\varphi) - z_0} \frac{dz}{d\varphi}\, d\varphi.$$

NOW LET $z - z_0 = Re^{i\varphi} \longrightarrow dz/d\varphi = iRe^{i\varphi} \qquad z(\varphi) = z_0 + Re^{i\varphi}$.

THUS
$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + Re^{i\varphi})}{Re^{i\varphi}}\, iRe^{i\varphi}\, d\varphi$$

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\varphi})\, d\varphi \qquad \blacksquare$$

PROPOSITION (MAX-MODULUS PRINCIPLE - LOCAL VERSION)

LET f BE ANALYTIC IN A DOMAIN D AND ASSUME THAT $|f(z)|$ HAS A LOCAL MAXIMUM AT z_0 IN D, i.e. $|f(z)| \leq |f(z_0)|$ FOR ALL z IN SOME NEIGHBOURHOOD OF z_0. THEN $f(z)$ IS A CONSTANT IN SOME NEIGHBORHOOD OF z_0.

PROOF LET z_0 IN D, AND SUPPOSE THAT $|f(z)| \leq |f(z_0)|$ IN SOME DISK $|z-z_0| < r_0$ IN D. WE WANT TO SHOW THAT $|f(z)| = |f(z_0)|$ ON $|z-z_0| < r_0$. SUPPOSE INSTEAD THAT THERE IS A POINT z_1 IN $|z-z_0| < r_0$ WHERE $|f(z_1)| < |f(z_0)|$. SINCE f IS CONTINUOUS, AND WE CAN WRITE $z_1 = z_0 + re^{i\varphi_1}$ WITH $r < r_0$, THEN THERE IS AN $\varepsilon > 0$ AND $\delta > 0$ SUCH THAT

$$|f(z_0 + re^{i\varphi})| < |f(z_0)| - \delta \quad \text{WHEN } |\varphi - \varphi_1| < \varepsilon.$$

EQUIVALENTLY, $|f(z_0 + re^{i(\varphi_1+\phi)})| < |f(z_0)| - \delta$ when $|\phi| < \varepsilon$.

WE NOW OBTAIN A CONTRADICTION USING MEAN-VALUE PROPERTY

$$
\begin{aligned}
|f(z_0)| &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(z_0 + re^{i(\varphi_1+\phi)})\, d\phi \right| \\
&\leq \frac{1}{2\pi} \int_{-\pi}^{-\varepsilon} |f(z_0 + re^{i(\varphi_1+\phi)})|\, d\phi + \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} |f(z_0 + re^{i(\varphi_1+\phi)})|\, d\phi \\
&\quad + \frac{1}{2\pi} \int_{\varepsilon}^{\pi} |f(z_0 + re^{i(\varphi_1+\phi)})|\, d\phi \\
&< \frac{1}{2\pi} \Big(|f(z_0)| (\pi - \varepsilon) + [|f(z_0)| - \delta]\, 2\varepsilon + |f(z_0)| (\pi - \varepsilon) \Big)
\end{aligned}
$$

$$|f(z_0)| < |f(z_0)| - \frac{\varepsilon\delta}{\pi} \longrightarrow \text{CONTRADICTION SINCE } \varepsilon, \delta > 0.$$

THUS THERE IS NO POINT IN $|z-z_0| < r_0$ WITH $|f(z)| < |f(z_0)|$. IT FOLLOWS THAT $|f(z)| = |f(z_0)|$ FOR ALL z IN $|z-z_0| < r_0$.

THUS $|f(z)|$ IS A CONSTANT IN $|z-z_0|<r_0$, AND SO $f(z)$ IS CONSTANT IN $|z-z_0|<r_0$. $\longrightarrow$ $z=z_0$ CANNOT BE A STRICT LOCAL MAX OF $|f(z)|$ IN THE DISK.

THEOREM (MAX-MODULUS PRINCIPLE) GLOBAL VERSION

LET $f(z)$ BE ANALYTIC INSIDE AND ON A SIMPLE CLOSED CURVE C. THEN THE MAX OF $|f(z)|$ CANNOT OCCUR AT AN INTERIOR POINT, AND INSTEAD MUST OCCUR ON C, UNLESS $f(z)$ IS CONSTANT.

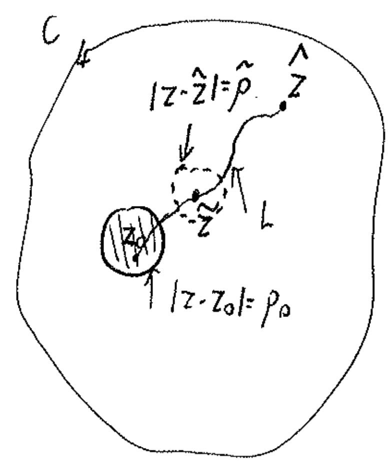
PROOF LET D DENOTE THE OPEN INTERIOR OF C. SUPPOSE THAT

$$|f(z)| \le M_0 \quad \text{FOR ALL } z \text{ IN } D$$

AND THAT THERE IS A POINT z_0 IN D SUCH THAT

$$|f(z_0)| = M_0.$$

SINCE D IS OPEN, THERE IS SOME DISK $|z-z_0|<\rho_0$ CONTAINED IN D, AND BY THE LOCAL MAXIMUM PRINCIPLE (PROPOSITION P. I32) IT FOLLOWS THAT $|f(z)| = M_0$ FOR ALL z IN $|z-z_0|<\rho_0$ (i.e. in shaded region).



NOW ASSUME THAT $|f(z)|$ IS NOT IDENTICALLY CONSTANT. THEN THERE IS A POINT $\hat{z}$ IN D FOR WHICH $|f(\hat{z})| \ne M_0$. SINCE M_0 IS MAX-VALUE, WE MUST HAVE THE STRICT INEQUALITY $|f(\hat{z})| < M_0$.

SINCE D IS A CONNECTED SET, WE CAN JOIN $\hat{z}$ TO z_0 WITH SOME CURVE, OR BROKEN LINE, LYING ENTIRELY IN D (see figure above). BY THE CONTINUITY OF $|f(z)|$ THERE MUST BE POINTS ON THIS CURVE L NEAR $\hat{z}$ AT WHICH $|f(z)| < M_0$. EVENTUALLY, WE MUST ARRIVE AT A POINT $\tilde{z}$ FOR WHICH $|f(\tilde{z})| = M_0$.

(ONE SUCH POINT WOULD OCCUR AT INTERSECTION OF $|z - z_0| = \rho_0$ WITH CURVE L).

NOW CONSIDER THE CIRCLE (DOTTED ONE IN FIGURE) $|z - \hat{z}| = \hat{\rho}$ LYING INSIDE D. BY THE LOCAL MAXIMUM PRINCIPLE SINCE $|f(\hat{z})| = M_0$, WE MUST HAVE $|f(z)| = M_0$ <u>FOR ALL</u> z IN $|z - \hat{z}| < \hat{\rho}$.

HOWEVER THERE ARE POINTS INSIDE $|z - \hat{z}| < \rho$ WHICH ALSO LIE ON CURVE L FOR WHICH $|f(z)| < M_0$. HENCE, WE HAVE A CONTRADICTION AND SO THERE CAN BE NO POINTS FOR WHICH $|f(z)| < M_0$.

i.e. $|f(z)|$ MUST BE A CONSTANT EVERYWHERE IN D, i.e.

$$|f(z)| = M_0 \text{ IN } D \longrightarrow f(z) = \text{CONSTANT IN } D.$$

NOW IF $f(z)$ IS A NONCONSTANT FUNCTION, IT FOLLOWS THAT THE MAXIMUM OF $|f(z)|$ <u>CANNOT</u> OCCUR AT AN INTERIOR POINT.

THUS THE MAXIMUM OF $|f(z)|$ MUST OCCUR ON THE BOUNDARY C.

<u>THEOREM</u> LET C BE A SIMPLE CLOSED CURVE. LET $\phi(x,y)$ BE A HARMONIC FUNCTION INSIDE C AND BOUNDED ON C. THEN $\phi(x,y)$ ATTAINS ITS MAXIMUM VALUE ON C.

<u>PROOF</u> LET $f = \phi + iV$ WHERE V IS HARMONIC CONJUGATE.

THEN $f(z)$ IS ANALYTIC AND HENCE SO IS $g(z) = e^{f(z)}$.

BY MAXIMUM-MODULUS PRINCIPLE $|g| = e^{\phi}$ ATTAINS ITS MAXIMUM VALUE ON C. SINCE e^s IS A MONOTONE INCREASING FUNCTION IN s IT FOLLOWS THAT $\phi(x,y)$ ATTAINS ITS MAXIMUM VALUE ON C.

REMARK (i) SUPPOSE THAT $f(z)$ IS ANALYTIC AND NONZERO INSIDE AND ON A CLOSED SIMPLE CONTOUR C. SHOW THAT $|f|$ HAS NO STRICT LOCAL MINIMA INSIDE C, AND HENCE $\min|f|$ OCCURS ON C.

PROOF LET $g(z) = 1/f(z)$. IF $f(z) \neq 0$ FOR ANY z inside C THEN $g(z)$ IS Analytic inside C AND ON C.

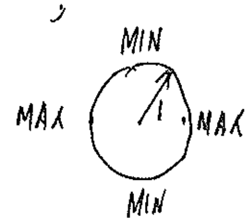
THUS BY M-M THEOREM $|g(z)| = \dfrac{1}{|f(z)|}$ ATTAINS ITS MAXIMUM VALUE ON $C \longrightarrow |f(z)|$ ATTAINS ITS MINIMUM VALUE ON C.

EXAMPLE 1 FIND MAX AND MIN OF $|e^{z^2}|$ IN $|z| \leq 1$.

SINCE $f(z) = e^{z^2}$ IS ANALYTIC, THE MAX AND MIN OF $|e^{z^2}|$ OCCURS ON $|z|=1$.

ON $|z|=1$ WE PUT $|e^{z^2}| = |e^{RE(z^2) + i IM(z^2)}| = e^{RE(z^2)} = e^{\cos 2\varphi}$,

WHERE $z = e^{i\varphi}$. THUS $|e^{z^2}| = e^{\cos 2\varphi}$ ON $|z|=1$.



$$\max |e^{z^2}| = e^{1} \quad (\text{when } \varphi = 0, \pi;\ \text{i.e. } z = \pm 1)$$

$$\min |e^{z^2}| = e^{-1} \quad (\text{when } \varphi = \pi/2, 3\pi/2;\ \text{i.e. } z = \pm i)$$

EXAMPLE 2 FIND MIN AND MAX OF $|z^3 e^z|$ ON $|z| \leq 1$.

LET $f(z) = z^3 e^z$. SINCE $f(0) = 0$ THE MIN-MODULUS PRINCIPAL DOES <u>NOT</u> APPLY AND $\displaystyle\min_{|z|\leq 1} |f(z)| = |f(0)| = 0$.

BUT, BY MAX-MODULUS PRINCIPLE, SINCE $f(z)$ IS ANALYTIC

$$\longrightarrow \max_{|z|\leq 1} |f(z)| = \max_{|z|=1} |f(z)| = \max_{|z|=1} |e^z| = \max_{0\leq\varphi\leq 2\pi} (e^{\cos\varphi}) = 1$$

EXAMPLE 3 SUPPOSE THAT $f(z)$ IS ANALYTIC INSIDE AND ON C AND THAT $|f(z)-1|<1$ FOR ALL z ON C. PROVE THAT $f(z)$ HAS NO ZEROES INSIDE C (i.e. no points z inside C where $f(z)=0$).

PROOF LET $g(z)=f(z)-1$. $g(z)$ is analytic inside and on C. THEN BY ASSUMPTION $|g(z)|<1$ ON C.

BY MAX-MODULUS PRINCIPLE, $|g(z)|<1$ FOR ALL z inside C.

SINCE $f(z_0)=0$ FOR SOME z_0 inside C implies that $|g(z_0)|=1$, THE CONDITION $|g(z)|<1$ FOR ALL z inside C leads to a CONTRADICTION.

THUS, $f(z)=0$ HAS NO ROOTS inside C.

EXAMPLE 4 PROVE THAT THE POLYNOMIAL $p(z)=\frac{1}{3}z^6+\frac{1}{4}z^4+\frac{1}{6}z^3+1=0$ HAS NO ROOTS z inside $|z|\leq 1$.

SOL WE CALCULATE $|p(z)-1|\leq \frac{1}{3}+\frac{1}{4}+\frac{1}{6}=\frac{3}{4}<1$ FOR $|z|\leq 1$ BY Δ INEQUALITY.

THUS THE PROOF OF EXAMPLE 3 $\longrightarrow$ NO ROOTS TO $p(z)=0$ IN $|z|\leq 1$.

EXAMPLE 5 LET $f(z)$ BE ANALYTIC IN $1\leq |z|\leq 2$ AND $|f(z)|\leq 3$ ON $|z|=1$ AND $|f(z)|\leq 12$ ON $|z|=2$. PROVE THAT $|f(z)|\leq 3|z|^2$ FOR $1\leq |z|\leq 2$.

PROOF DEFINE $g(z)=f(z)/3z^2$. THIS IS ANALYTIC IN $1\leq |z|\leq 2$ AND THE DATA ON $|f(z)|$ ON $|z|=1$ AND $|z|=3$ PROVE THAT $|g(z)|<1$ ON $|z|=1$ AND ON $|z|=2$. THUS, BY MAX-MODULUS PRINCIPLE

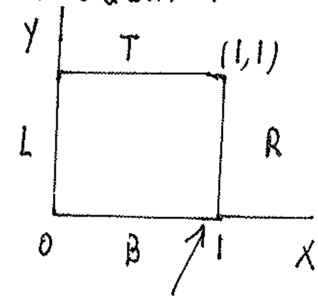
$$|g(z)|\leq 1 \text{ ON } 1\leq |z|\leq 2 \longrightarrow |f(z)|\leq 3|z|^2 \text{ ON } 1\leq |z|\leq 2.$$

EXAMPLE 6

FIND MAXIMUM OF $U = \mathrm{RE}(z^3)$ ON THE SQUARE $[0,1] \times [0,1]$/

SINCE U IS A HARMONIC FUNCTION (REAL PART OF ANALYTIC FUNCTION) THEN THE MAXIMUM OF U OCCURS ON THE BOUNDARY OF THE SQUARE.

WE CALCULATE $U = \mathrm{RE}\left((x+iy)^3\right) = x^3 - 3xy^2$.



ON L: $x = y = 0 \longrightarrow U = 0$

ON B: $y = 0 \longrightarrow U = x^3$ MAX AT $x = 1 \rightarrow U_{MAX} = 1$

ON R: $x = 1 \longrightarrow U = 1 - 3y^2$ MAX AT $y = 0 \rightarrow U_{MAX} = 1$

ON T: $y = 1 \longrightarrow U = x^3 - 3x = x^2(x-3)$ MAX AT $x = 0 \rightarrow U_{MAX} = 0$.

THUS $U(x,y) \leq 1$ IN $[0,1] \times [0,1]$

FUNDAMENTAL THEOREM OF ALGEBRA

LET $a_0, \dots a_n$ BE COMPLEX AND SUPPOSE $n > 1$ WITH $a_n \neq 0$.
DEFINE THE POLYNOMIAL $p(z) = a_0 + a_1 z + \dots + a_n z^n$.
THEN, THERE IS A POINT z_0 SUCH THAT $p(z_0) = 0$.

PROOF (BY LIOUVILLE THEOREM AND CONTRADICTION). WE PROCEED BY CONTRADICTION.
SUPPOSE $p(z) \neq 0$ FOR ANY z. THEN DEFINE $f(z) = 1/p(z)$. $f(z)$ IS AN ENTIRE FUNCTION, AND WE CAN FIND AN $R > 0$ SO THAT

$$|f(z)| = \frac{1}{|p(z)|} < \frac{C}{|a_n| R^n}$$ FOR ALL z IN $|z| \geq R$. (FOR SOME $C > 0$).

THUS $f(z)$ IS BOUNDED FOR ALL z. BY LIOUVILLE'S THEOREM, $f(z)$ AND CONSEQUENTLY $p(z)$, MUST BE A CONSTANT. BUT $p(z)$ IS A NONCONSTANT POLYNOMIAL. THIS CONTRADICTS ASSUMPTION $p(z) \neq 0$ FOR ANY z.