A COROLLARY OF THE ARGUMENT PRINCIPLE: ROUCHÉ THEOREM

♠ Let $F(z)$ and $G(z)$ be holomorphic on and inside a closed contour C.

If $|F(z)| > |G(z) - F(z)|$ on C,

then $F(z)$ and $G(z)$ have the same number of zeros inside C.

Let $w = \dfrac{G}{F}$; consider $\dfrac{1}{2\pi i} \oint_C \dfrac{w'(z)}{w(z)}\, dz$.

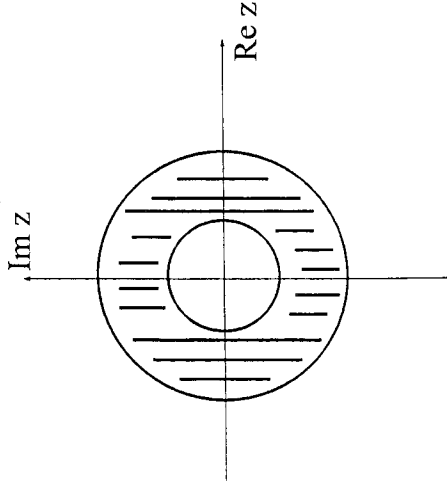$$|w(z) - 1| = \frac{|G - F|}{|F|} < 1 \text{ on } C .$$

Therefore the image of C lies inside $|w - 1| < 1$

$$\implies \Delta_C \arg w = 0 \implies N = P \text{ for } w(z) .$$

Thus the number of zeros of F (P) equals the number of zeros of G (N).

EXAMPLE

◇ Show that the polynomial $P(z) = z^5 + 14z + 2$ has 4 roots in the annulus $3/2 < |z| < 2$.

• Consider C_2 circle $|z| = 2$. Take $G = P(z)$, $F(z) = z^5$.

$|G - F| < |F|$ on $C_2 \implies P(z)$ has as many zeros inside C_2 as $F(z)$, which is 5.

• Next consider C_1 circle $|z| = 3/2$. Take $G = P(z)$, $F(z) = 14z$.

$|G - F| < |F|$ on $C_1 \implies P(z)$ has as many zeros inside C_1 as $F(z)$, which is 1.

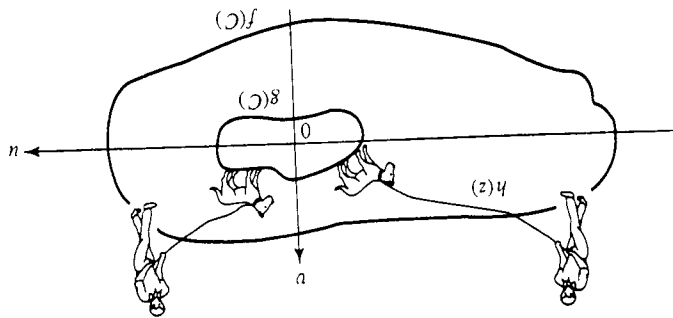Thus $5 - 1 = 4$ zeros of $P(z)$ lie between C_2 and C_1.

Figure 6.25 M. Rouché and Truffles.

Theorem 4 (Rouché's Theorem). If f and h are each functions that are analytic inside and on a simple closed contour C and if the strict inequality

$$|h(z)| > |f(z)| \tag{7}$$

holds at each point on C, then f and $f + h$ must have the same total number of zeros (counting multiplicities) inside C.

Observe that the inequality (7) need only hold *on* C, not inside, and that (7) prevents f (as well as $g = f + h$) from being zero on C. See Prob. 15 for an extension to the case when f is meromorphic inside C.

One typically uses Rouché's theorem to deduce some information about the location of zeros of a complicated analytic function g by comparing it with an analytic function f whose zeros are known.

Example 2

Prove that all five zeros of the polynomial

$$g(z) = z^5 + 3z + 1$$

lie in the disk $|z| < 2$.

Solution. We take C as the circle $|z| = 2$, and we regard g as a perturbation of the function $f(z) = z^5$, which clearly has five zeros inside C. To test condition (7) we estimate the perturbation $h(z) = 3z + 1$ on C by

$$|h(z)| = |3z + 1| \le 3|z| + 1 = 3 \cdot 2 + 1 = 7,$$

which sure enough, is strictly less than $|f(z)| = |z^5| = 2^5 = 32$. Therefore, g also has five zeros inside $|z| < 2$. ∎