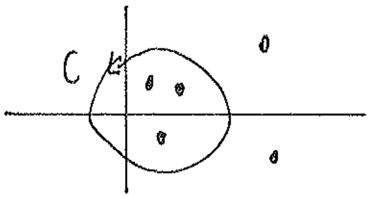
NYQUIST CRITERION AND WINDING NUMBER

SUPPOSE FOR SIMPLICITY THAT $P(z)$ IS A POLYNOMIAL OF DEGREE D WITH DISTINCT ROOTS $z_1, \ldots, z_D$. WE WOULD LIKE TO COUNT THE NUMBER N OF THESE ROOTS INSIDE C. FOR EXAMPLE IF $D=5$ AND $N=3$ WE HAVE A PICTURE AS SHOWN.



WE FIRST CLAIM THAT $N = \frac{1}{2\pi i} \int_C \frac{P'(z)}{P(z)}\, dz$ WHERE C IS CLOSED CURVE (SIMPLE) ORIENTED COUNTERCLOCKWISE, WHERE WE ASSUME THAT P HAS NO ZEROS ON C.

<u>DERIVATION</u> WE WRITE $P(z) = (z-z_1)\cdots(z-z_D)$

WE DIFFERENTIATE USING EXTENDED PRODUCT RULE:

$$P'(z) = (z-z_2)\cdots(z-z_D) + (z-z_1)(z-z_3)\cdots(z-z_D) + \cdots + (z-z_1)\cdots(z-z_{D-1})$$

NOW DIVIDE BY $P(z)$ TO GET

$$\frac{P'(z)}{P(z)} = \frac{1}{z-z_1} + \cdots + \frac{1}{z-z_D}.$$

NOW THE RESULT FOLLOWS SINCE $\frac{1}{2\pi i}\int_C \frac{1}{z-z_j}\, dz = \begin{cases} 1 & \text{if } z_j \text{ inside } C \\ 0 & \text{otherwise} \end{cases}$.

THUS

$$N = \frac{1}{2\pi i}\int_C \frac{P'(z)}{P(z)}\, dz. \qquad (1)$$

NOW WE CAN EVALUATE THIS INTEGRAL DIRECTLY TO OBTAIN THAT

<u>LEMMA</u> $\int_C \frac{P'(z)}{P(z)}\, dz = i\left[\arg P(z)\right]\Big|_C$ WHEN P HAS NO ZEROS ON C.

HERE $\left[\arg P(z)\right]\Big|_C$ DENOTES THE CHANGE IN THE ARGUMENT OF $P(z)$ AS z TRAVERSES C COUNTERCLOCKWISE.

DERIVATION

LET $I = \dfrac{1}{2\pi i} \displaystyle\int_C \dfrac{P'(z)}{P(z)}\, dz$ $\qquad$ $C: z = z(t),\ a \le t \le b.$

let C BE SMOOTH PATH

DEFINE $w = P(z)$. A FORMAL CALCULATION GIVES

$$w'(t) = P'(z(t))\, z'(t)$$

SO THAT $\qquad I = \dfrac{1}{2\pi i} \displaystyle\int_{C^*} \dfrac{w'(t)}{w(t)}\, dt$ $\qquad$ C^* IS THE IMAGE OF C UNDER $P(z)$.

NOW C^* HAS ENDPOINTS $w(a) = P[z(a)]$, $w(b) = P(z(b))$, WITH $w(a) = w(b)$

NOTICE THAT C^* IS NOW PERHAPS NO LONGER A SIMPLE CURVE. IN FACT, C^* MAY ENCIRCLE ORIGIN SEVERAL TIMES.

WE THEN BREAK UP THE INTEGRAL AS

$$I = \frac{1}{2\pi i} \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \frac{w'(t)}{w(t)}\, dt \qquad \text{WHERE} \quad t_0 = a,\ t_N = b.$$

ON EACH SEGMENT $t_k < t < t_{k+1}$ WE CAN DEFINE AN ANALYTIC ANTI-DERIVATIVE

$$\hat{f}_k(t) = \log(w(t)) \qquad \text{FOR WHICH} \qquad \hat{f}'(t) = \frac{w'(t)}{w(t)},$$

AND WE ENSURE THAT THE COMPOSITE FUNCTION IS CONTINUOUS.

$$\hat{f}(t) = \begin{cases} \hat{f}_0(t) & t_0 < t < t_1 \\ \vdots & \vdots \\ \hat{f}_{N-1}(t) & t_{N-1} < t < t_N \end{cases}$$

HENCE

$$I = \frac{1}{2\pi i} \sum_{k=0}^{N-1} \log[w(t)] \Big|_{t=t_k}^{t=t_{k+1}} = \frac{1}{2\pi i} \sum_{k=0}^{N-1} \left(\ln(|w(t_{k+1})|) - \ln|w(t_k)| \right) + \frac{i}{2\pi i} \sum_{k=0}^{N-1} \left(\arg w(t_{k+1}) - \arg w(t_k) \right)$$

THIS IS A TELESCOPING SUM. THUS

$$I = \frac{1}{2\pi i} \left[\ln|w(b)| - \ln|w(a)| \right] + \frac{1}{2\pi} \left[\arg(w(b)) - \arg w(a) \right]$$

$w(b) = P(z(b))$
$w(a) = P(z(a))$

BUT $|w(b)| = |w(a)|$ $\qquad$ SO $\qquad I = \dfrac{1}{2\pi} \left[\arg P(z) \right] \Big|_C$ = change IN argument OVER C.

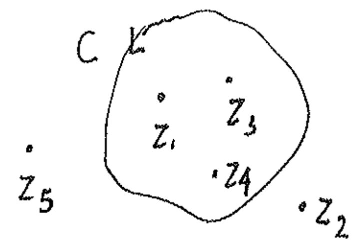
THIS GIVES THE MAJOR RESULT:

<u>THEOREM</u> LET C BE A SIMPLE CLOSED CURVE ORIENTED COUNTERCLOCKWISE AND LET $P(z)$ BE A POLYNOMIAL OF degree n WITH DISTINCT ROOTS $z_1, \ldots, z_n$. ASSUME THAT NO ROOT IS ON C.

THEN, IF N = NUMBER OF ZEROES OF $P(z) = 0$ INSIDE C,

WE HAVE

$$(*) \qquad 2\pi N = \left[\arg P(z)\right]\Big|_C$$



picture $N = 3$ AND $n = 5$.

WHERE

$[\arg P(z)]\big|_C$ DENOTES THE CHANGE IN THE ARGUMENT OF $P(z)$ OVER C. THIS MUST STILL BE CALCULATED FOR A GIVEN EXAMPLE OF $P(z)$.

<u>REMARK</u> (i) $(*)$ STILL HOLDS FOR ROOTS OF HIGHER MULTIPLICITY. IF THE ROOT HAS MULTIPLICITY 2, THEN WE COUNT IT'S MULTIPLICITY IN THE SUM $(*)$.

(ii) LATER WE WILL SHOW BY THE RESIDUE THEOREM THAT $(*)$ IN FACT HOLDS NOT JUST FOR POLYNOMIALS BUT FOR <u>ANY</u> ANALYTIC FUNCTION $P(z)$.

APPLICATION WHEN DO WE NEED TO DETERMINE WHETHER A POLYNOMIAL $p(z)=0$ HAS ROOTS IN $|z|<1$?

CONSIDER THE DIFFERENCE EQUATION

$$\underline{x}_{m+1} = A\,\underline{x}_m, \quad m=0,1,2,\ldots \quad \text{WHERE } A \text{ IS } n\times n \text{ MATRIX.}$$

TO SOLVE THIS RECURSION RELATION WE LET

$$\underline{x}_m = z^m \underline{c}. \quad \text{THEN} \quad z^{m+1}\underline{c} = A(z^m \underline{c}) = z^m A\underline{c}.$$

THEREFORE, $A\underline{c} = z\,\underline{c}$ THUS $z, \underline{c}$ ARE AN EIGENPAIR OF A.

NOW $p(z) = \det(A - zI) = 0$ IS A POLYNOMIAL OF degree n.

THUS $z_1, .., z_n$ ARE THE ROOTS OF $p(z)=0$ AND $\underline{c}_1, .., \underline{c}_n$ ARE THE CORRESPONDING EIGENVECTORS IF $z_1, .., z_n$ ARE DISTINCT.

THUS BY SUPERPOSITION, $\underline{x}_m = \sum_{k=1}^{n} B_k z_k^m \underline{c}_k$. WHEN $z_1, .., z_n$ ARE DISTINCT. NOW $\underline{x}_m \to \underline{0}$ AS $m \to \infty$ WHEN $|z_j|<1$ FOR $j=1,..,n$.

THEREFORE, IF $z_1, .., z_n$ ARE DISTINCT, THEN IF ALL THE ROOTS OF $p(z) = \det(A - \lambda I) = 0$ ARE IN $|z|<1$, THEN $\underline{x}_m \to 0$ AS $m\to\infty$.

EXAMPLE WE MUST SHOW THAT ALL ROOTS OF A POLYNOMIAL ARE IN THE UNIT DISK IN ORDER THAT THE RECURSION IS STABLE.

LET $C: |z|=1$, C.C.; $N = \#$ ZEROS OF $p(z)=0$ IN C:

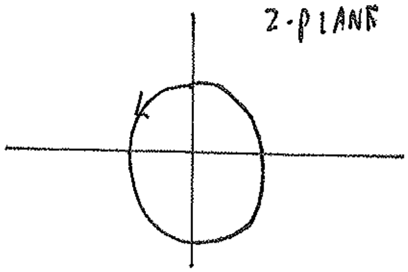
THEN $N = \frac{1}{2\pi}[\arg p]\big|_C$ $\quad p$ = POLYNOMIAL OF degree n.

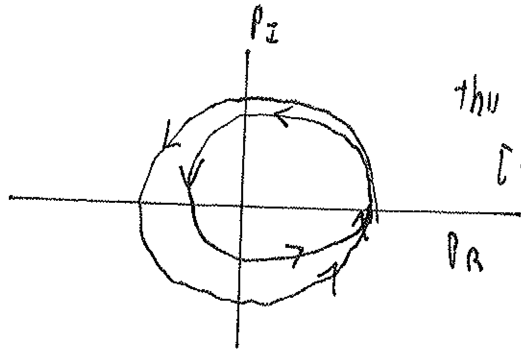
WE MUST SHOW THAT $[\arg p]\big|_C = n/2\pi$ SO THAT $N=n$

WE WRITE $p(z)$ ON C AS $z = e^{i\varphi}$ AND

$$p(e^{i\varphi}) = p_R(\varphi) + i\,p_I(\varphi)$$

THE GOAL IS TO PLOT THE PATH IN THE P_I VS P_R PLANE AS φ RANGES FROM 0 TO 2π





this would have $[\arg P]\big|_C = 4\pi$ (it encircles origin twice in counterclockwise sense.

IF $n = 3$ AND $P(0) > 0$ WITH A A REAL MATRIX

<u>EXAMPLE</u> SHOW THAT THE POLYNOMIAL $p(z) = 8z^4 + z^3 + \frac{8}{5}z^2 + 4z + 1 = 0$ HAS ALL OF ITS ROOTS IN $|z| < 1$.

<u>PROOF</u> WE EITHER USE THE WINDING NUMBER CRITERION

$$[\arg p]\big|_C = 2\pi N \qquad N = \#\text{ ZEROES INSIDE } \quad C : |z| = 1$$

OR ELSE PROCEED ANALYTICALLY. (SEE BELOW).

TO USE WINDING NUMBER PUT $z = e^{i\varphi}$ SO THAT BY DE'MOIVRES IDENTITY

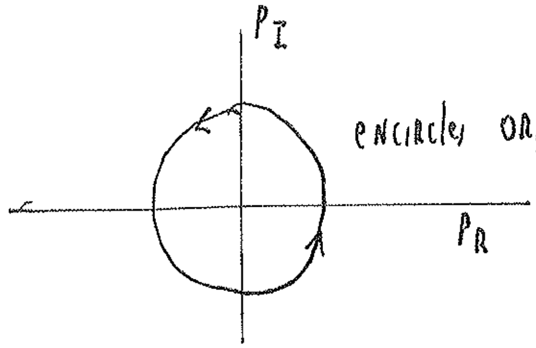
$$P(e^{i\varphi}) = 8e^{4i\varphi} + e^{3i\varphi} + \frac{8}{5}e^{2i\varphi} + 4e^{i\varphi} + 1$$
$$= \left(8\cos(4\varphi) + \cos 3\varphi + \frac{8}{5}\cos(2\varphi) + 4\cos\varphi + 1\right)$$
$$+ i\left(8\sin 4\varphi + \sin 3\varphi + \frac{8}{5}\sin 2\varphi + 4\sin\varphi\right)$$

AND $0 \leq \varphi \leq 2\pi$.

WE DEFINE

$$P_R(\varphi) = 8\cos 4\varphi + \cos 3\varphi + \frac{8}{5}\cos(2\varphi) + 4\cos\varphi + 1$$
$$P_I(\varphi) = 8\sin 4\varphi + \sin 3\varphi + \frac{8}{5}\sin 2\varphi + 4\sin\varphi$$

AND WE PLOT P_I VS P_R AS $0 \leq \varphi \leq 2\pi$ (see ATTAched plot generated by a computer)



encircles origin 4 times in counterclockwise sense (the path's overlap)

so $[\arg P]|_C = 8\pi$

$\Longrightarrow \quad 2\pi N = 8\pi$ so $N = 4$.

WE CONCLUDE THAT $[\arg P]|_C = 8\pi$ (IT ENCIRCLES ORIGIN 4 TIMES).

<u>ALTERNATE METHOD</u> LET $Z = 1/W$ AND DEFINE THE 4^{th} DEGREE POLYNOMIAL

$$g(W) = W^4 \, p(1/W) \qquad \text{WITH} \quad |W| < 1.$$

THUS IF THERE ARE NO ROOTS OF $g(W) = 0$ IN $|W| < 1$, THEN THERE ARE NO ROOTS OF $p(Z) = 0$ IN $|Z| > 1$. HENCE ALL ROOTS OF $p(Z) = 0$ MUST BE IN $|Z| \leq 1$.

WE OBTAIN $\quad g(W) = W^4\left(\dfrac{8}{W^4} + \dfrac{1}{W^3} + \dfrac{8/5}{W^2} + \dfrac{4}{W} + 1\right) = W^4 + 4W^3 + \dfrac{8}{5}W^2 + W + 8$

NOW $g(W) = 0 \rightarrow f(W) \equiv \dfrac{W^4}{8} + \dfrac{W^3}{2} + \dfrac{W^2}{5} + \dfrac{W}{8} + 1 = 0.$

RECALL A THEOREM FROM BEFORE IF $f(W)$ IS ANALYTIC INSIDE AND ON C AND IF $|f(W) - 1| < 1$ ON C, THEN f HAS NO ZEROES INSIDE OR ON C. HERE C IS A SIMPLE CLOSED CURVE.

NOW $\quad f(W) - 1 = \dfrac{W^4}{8} + \dfrac{W^3}{2} + \dfrac{W^2}{5} + \dfrac{W}{8}$ IS ANALYTIC. NOW BY Δ-INEQUALITY

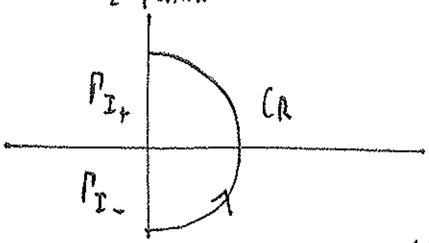
WE CALC $\quad |f(W) - 1| \leq \dfrac{|W|^4}{8} + \dfrac{|W|^3}{2} + \dfrac{|W|^2}{5} + \dfrac{|W|}{8} \leq \dfrac{1}{8} + \dfrac{1}{2} + \dfrac{1}{5} + \dfrac{1}{8} < 1$ ON $|W| = 1$

THUS $f(W) = 0$ HAS NO ZEROES IN $|W| \leq 1 \longrightarrow p(Z) = 0$ HAS NO ZEROES IN $|Z| \geq 1 \longrightarrow$ ALL ZEROES OF $p(Z) = 0$ MUST BE INSIDE $C: |Z| = 1$.

APPLICATION 2

IN CALCULATING THE SOLUTION TO DIFFERENTIAL EQUATIONS WE WOULD LIKE TO KNOW IF POLYNOMIALS $p(z) = z^n + a_0 z^{n-1} + \dots + a_n$ HAVE ROOTS IN $\operatorname{Re} z > 0$. TO DETERMINE THIS WE USE

$$C = C_R \cup \Gamma_{I_+} \cup \Gamma_{I_-}$$



AND

$$\lim_{R\to\infty} [\arg p]\Big|_{C_R} + [\arg p]\Big|_{\Gamma_{I_+}} + [\arg p]\Big|_{\Gamma_{I_-}} = 2\pi N$$

WHERE $N = \#$ ZEROES OF $p(z) = 0$ IN $\operatorname{Re} z > 0$.

NOW SINCE $\deg p = n$ WE HAVE WITH $p \sim z^n$ AS $n \to +\infty$ THAT

$$\lim_{R\to\infty} [\arg p]\Big|_{C_R} = n\pi.$$

THUS $\quad n\pi + [\arg p]\Big|_{\Gamma_{I_+}} + [\arg p]\Big|_{\Gamma_{I_-}} = 2\pi N$. ASSUME COEFFICIENTS OF p ARE REAL.

NOW ON Γ_{I_+}: $0 < y < \infty$ traversed downwards

Γ_{I_-}: $0 < y < \infty$ traversed downwards

SO

$$[\arg p]\Big|_{\Gamma_{I_+}} = \arg p(0) - \arg p(i\infty)$$

$$[\arg p]\Big|_{\Gamma_{I_-}} = \arg p(-i\infty) - \arg p(0).$$

NOW WLOG WE CAN TAKE $p(0) > 0$ REAL SO THAT $\arg p(0) = 0$.

THEN SINCE $p(-i\infty) = \overline{p(i\infty)} \longrightarrow \arg p(-i\infty) = +\arg \overline{p(i\infty)} = -\arg p(i\infty)$.

THUS $[\arg p]\Big|_{\Gamma_{I_+}} = [\arg p]\Big|_{\Gamma_{I_-}}$. WE CONCLUDE THAT

$$N = \frac{1}{2\pi}\left[n\pi + 2\,[\arg p]\Big|_{\Gamma_{I_+}} \right]$$

WHEN $p(z)$ IS A POLYNOMIAL OF DEGREE n WITH REAL COEFFICIENTS AND $p(0) > 0$.

EXAMPLE 1

PROVE THAT THERE ARE NO ZEROES OF $p(z) = z^3 + 2z^2 + z + 1 = 0$ IN $\text{RE } z \geq 0$.

SINCE $\lim_{R \to \infty} [\arg p]\big|_{C_R} = 3\pi$

THEN, $N = \frac{1}{2\pi}\left[3\pi + 2[\arg p]\big|_{\Gamma_{I_+}}\right]$ WHERE $\Gamma_{I_+} : 0 < y < \infty$ DOWNWARD,

AND $N = \#$ ZEROES INSIDE $\text{RE } z > 0$.

NOW ON Γ_{I_+}: $z = iy$ AND $p(iy) = (iy)^3 + 2(iy)^2 + iy + 1$.

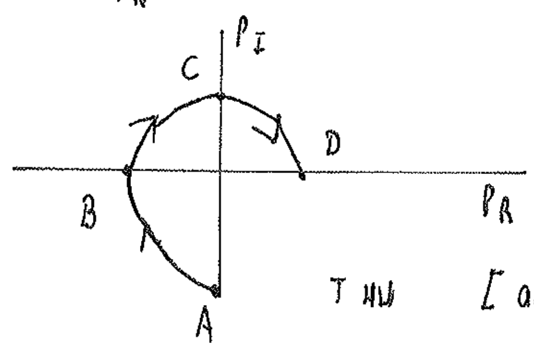
THUS $p(iy) = p_R(y) + i\,p_I(y)$ WITH $p_R(y) = 1 - 2y^2$

$p_I(y) = y - y^3$.

NOW $p_R = 0$ WHEN $y_R = \pm 1/\sqrt{2}$; $p_I = 0$ WHEN $y = 0, \pm 1$.

	Γ	p_R	p_I
A	∞	<0	<0
B	1	<0	$=0$
C	$1/\sqrt{2}$	$=0$	>0
D	0	>0	$=0$

NOW $p_I/p_R \to +\infty$ AS $y \to +\infty$



THUS $[\arg p]\big|_{\Gamma_{I_+}} = \frac{-3\pi}{2}$.

THUS $N = \frac{1}{2\pi}\left[3\pi + 2(-3\pi/2)\right] = 0$.

THUS $N = 0 \to$ NO ZEROES IN $\text{RE } z > 0$

EXAMPLE 2

FIND THE NUMBER OF ZEROES OF $p(z) = z^3 - 2z^2 + 4$ IN $\text{RE } z > 0$. AGAIN WE HAVE

$$N = \frac{1}{2\pi}\left[3\pi + 2[\arg p]\big|_{\Gamma_{I_+}}\right].$$

HERE $\Gamma_{I_+} : 0 < y < \infty$ DOWNWARDS.

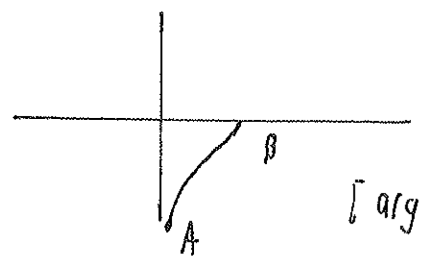
$$P(iy) = (iy)^3 - 2(iy)^2 + 4 = -iy^3 + 2y^2 + 4.$$

WE HAVE $P(iy) = P_R(y) + i P_I(y)$ $\quad P_R(y) = 2y^2 + 4, \quad P_I(y) = -y^3.$

NOW WE HAVE $P_I < 0$ ON $0 < y < \infty$, $P_R > 0$.

	Γ	P_R	P_I
A	∞	>0	<0
B	0	>0	$=0$

AND $P_I / P_R \to -\infty$ AS $y \to +\infty$



$$[\arg P]\big|_{\Gamma_I} = \pi/2.$$

THUS $$N = \frac{1}{2\pi}\left[3\pi + 2\left(\pi/2\right)\right] = 2.$$

THUS $N = 2 \longrightarrow$ TWO ZEROS IN $\operatorname{RE} z > 0$.

<u>APPLICATION</u> CONSIDER A SECOND ORDER OR HIGHER DIFFERENTIAL EQUATION OF THE FORM

$$y^{(n)} + a_0 y^{(n-1)} + \dots + a_{n-1} y = 0$$

WITH $y(0), \dots, y^{(n-1)}(0)$ GIVEN.

IF YOU CALCULATE THE LAPLACE TRANSFORM YOU GET

$$\mathcal{L}(y) = \frac{Q(s)}{P(s)} \qquad P(s) = s^n + a_0 s^{n+1} + \dots + a_{n-1}.$$

THEN $y \to 0$ AS $t \to \infty$ FOR ANY INITIAL CONDITIONS $y(0), \dots, y^{(n-1)}(0)$

IFF ALL THE ZEROS OF $P(s) = 0$ ARE IN $\operatorname{RE} s < 0$.

TO DETERMINE THE ROOTS OF $P(s) = 0$ IN $\operatorname{RE}(s) > 0$

WE CAN SIMPLY USE THE WINDING NUMBER CRITERION

$$N = \frac{1}{2\pi}\left[n\pi + 2\,[\arg P]\big|_{\Gamma_{I+}}\right]$$ WHERE Γ_{I+}, $s = iy$, TRAVERSED DOWNWARDS.