CLASSIFICATION OF SINGULARITIES

SUPPOSE THAT $f(z)$ IS ANALYTIC IN THE REGION $r_1 < |z - z_0| < r_2$ WITH $r_1 \geq 0$, $r_2 > r_1$ AND POSSIBLY $r_2 = \infty$. SUPPOSE THAT $f(z)$ HAS AN ISOLATED SINGULARITY AT $z = z_0$. THEN $f(z)$ HAS A LAURENT SERIES IN $r_1 < |z - z_0| < r_2$ OF THE FORM

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}$$

$$(*) \quad f(z) = \left(a_0 + a_1 (z - z_0) + \cdots + a_n (z - z_0)^n + \cdots \right) + \left(\frac{a_{-1}}{(z - z_0)} + \frac{a_{-2}}{(z - z_0)^2} + \cdots + \frac{a_{-n}}{(z - z_0)^n} + \cdots \right)$$

THE SERIES CONVERGES ON $r_1 < |z - z_0| < r_2$.

WE CLASSIFY THE SINGULARITY AT $z = z_0$ USING THE LAURENT SERIES $(*)$

(i) $f(z)$ HAS A <u>POLE OF ORDER $m > 0$</u> AT $z = z_0$ IF $a_{-m} \neq 0$ BUT $a_{-m-1} = a_{-m-2} = \cdots = 0$. THE L-SERIES HAS THE FORM

terminates here

$$f(z) = \left(a_0 + a_1 (z - z_0) + \cdots \right) + \left(\frac{a_{-1}}{z - z_0} + \cdots + \frac{a_{-m}}{(z - z_0)^m} \right)$$

(ii) $f(z)$ HAS AN <u>ESSENTIAL SINGULARITY</u> AT $z = z_0$ IF THE L-SERIES HAS AN <u>INFINITE NUMBER</u> OF TERMS IN POSITIVE POWERS OF $1/z - z_0$

(iii) $f(z)$ HAS A <u>REMOVABLE SINGULARITY</u> AT $z = z_0$ IF $f(z)$ IS NOT DEFINED AT $z = z_0$ BUT THAT $\lim_{z \to z_0} f(z)$ EXISTS.

IN THIS CASE $a_{-1} = a_{-2} = \cdots = 0$.

<u>REMARK</u> (i) THE PROOF OF THE CONVERGENCE PROPERTY OF THE L-SERIES

(ii) A POLE OF ORDER 1 IS CALLED A SIMPLE POLE.

(iii) <u>IMPORTANT</u> $a_{-1} = \text{RESIDUE}\left[f(z) ; z_0 \right]$

IS A CRITICAL TERM TO CALCULATE.

CLASSIFICATION OF SINGULARITIES: GENERAL REMARKS

REMARK 1 SUPPOSE THAT $f(z) = \dfrac{P(z)}{Q(z)}$

WHERE $P(z)$, $Q(z)$ ANALYTIC AT $z = z_0$ WITH

$$P(z_0) \neq 0, \quad Q(z_0) = 0, \quad Q'(z_0) \neq 0,$$

SO THAT z_0 IS A SIMPLE ZERO OF $Q(z) = 0$. THEN $f(z)$ HAS A SIMPLE POLE AT $z = z_0$ AND AS $z \to z_0$

$$f(z) \approx \frac{P(z_0) + (z - z_0) P'(z_0) + \cdots}{(z - z_0) Q'(z_0) + \cdots} = \frac{P(z_0)/Q'(z_0)}{z - z_0} + O(1)$$

THUS $a_{-1} = \text{RES}[f; z_0] = P(z_0)/Q'(z_0)$.

THIS IS THE EASIEST METHOD TO CALCULATE THE RESIDUE AT A SIMPLE POLE.

REMARK 2 SUPPOSE THAT $f(z) = P(z)/Q(z)$

WITH $P(z)$, $Q(z)$ ANALYTIC AT $z = z_0$ WITH

$$P(z_0) \neq 0$$

$$Q(z_0) = 0, \quad Q'(z_0) = 0, \quad \ldots \quad Q^{(m)}(z_0) = 0, \quad Q^{(m+1)}(z_0) \neq 0$$

THEN $Q(z) \approx (z - z_0)^{m+1} \dfrac{Q^{(m+1)}(z_0)}{(m+1)!}$ AS $z \to z_0$.

THUS AS $z \to z_0$,

$$f(z) \approx \frac{\left(P(z_0)/Q^{(m)}(z_0) \right)(m+1)!}{(z - z_0)^{m+1}}$$

$\longrightarrow$ pole at $z = z_0$ OF ORDER $m+1$.

REMARK 3 CALCULATION OF RESIDUE FOR A POLE OF ORDER m AT $z = z_0$.

THE LAURENT SERIES OF $f(z)$, WHICH CONVERGES IN $0 < |z - z_0| < r_0$ FOR SOME r_0, IS

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \dots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

TO CALCULATE a_{-1} MULTIPLY BY $(z-z_0)^m$:

$$(z-z_0)^m f(z) = a_{-m} + \dots + a_{-1}(z-z_0)^{m-1} + a_0 (z-z_0)^m + \dots$$

NOW DIFFERENTIATE $m-1$ TIMES:

$$\frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z) = a_{-1}(m-1)! + [m(m-1)\cdots 2]\, a_0 (z-z_0) + \dots$$

NOW LET $z \to z_0$, THEN

$$a_{-1} = \frac{1}{(m-1)!} \lim_{z \to z_0} \left[\frac{d^{m-1}}{dz^{m-1}} \left((z-z_0)^m f(z) \right) \right]. \quad (*)$$

EXAMPLE CALCULATE THE RESIDUE OF $f(z) = \dfrac{z^2 - 2z}{[z+1]^2 (z^2+4)}$ AT $z = -1$.

DO THIS IN TWO DIFFERENT WAYS

METHOD 1 USE THE FORMULA $(*)$. NOTICE THAT $z = -1$ IS A POLE OF ORDER 2. SO

$$a_{-1} = \lim_{z \to -1} \frac{d}{dz} \left[(z+1)^2 f(z) \right] = \lim_{z \to -1} \frac{d}{dz} \left[\frac{z^2 - 2z}{z^2+4} \right]$$

THIS GIVES

$$a_{-1} = \lim_{z \to -1} \left[\frac{(z^2+4)(2z-2) - (z^2-2z)2z}{(z^2+4)^2} \right] = \frac{5(-4) + (+1+2)2}{25} = -\frac{14}{25}.$$

METHOD 2 WRITE $f(z) = \dfrac{P(z)}{Q(z)}$ WITH $Q(z) = (z+1)^2$, $P(z) = (z^2 - 2z)/(z^2+4)$.

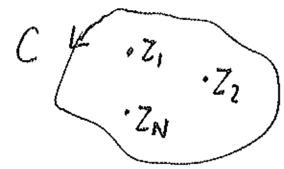
NOW AS $z \to -1$ $\quad P(z) \sim P(-1) + P'(-1)(z-1) + \dots$

THUS $f(z) \sim \dfrac{P(-1)}{(z+1)^2} + \dfrac{P'(-1)}{(z+1)} + \dots$ AS $z \to -1$. SO $a_{-1} = P'(-1) = -14/25$.

REMARK 4 IF $f(z)$ HAS AN ESSENTIAL SINGULARITY AT $z = z_0$, THE ONLY WAY TO CALCULATE THE RESIDUE IS TO CALCULATE THE LAURENT SERIES OF $f(z)$ AS $z \to z_0$ TO IDENTIFY THE COEFFICIENT $a_{-1}/(z-z_0)$.

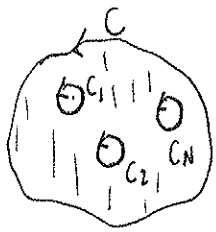
RESIDUE THEOREM LET C BE A SIMPLE CLOSED CURVE, ORIENTED COUNTERCLOCKWISE, AND SUPPOSE THAT $f(z)$ IS ANALYTIC INSIDE AND ON C EXCEPT AT THE ISOLATED SINGULARITIES $z_1, \ldots, z_N$, WITH z_j inside C FOR $j=1,\ldots,N$. THEN

$$\int_C f(z)\,dz = 2\pi i \sum_{j=1}^{N} \mathrm{RES}(f; z_j).$$



DERIVATION BY DEFORMING THE CONTOUR AS ON PAGE (I20) WE OBTAIN

$$\int_C f(z)\,dz = \sum_{j=1}^{N} \int_{C_j} f(z)\,dz \quad (*), \qquad C_j : |z-z_j| = \delta \text{ COUNTERCLOCKWISE.}$$



NOW SINCE δ IS SMALL, $f(z)$ HAS A LAURENT SERIES THAT CONVERGES IN $0 < |z-z_j| < \delta$ OF THE FORM

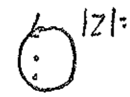
$$f(z) = \sum_{m=0}^{\infty} a_{m_j} (z-z_j)^m + \sum_{m=1}^{\infty} a_{-m_j} (z-z_j)^{-m}. \quad \text{FOR SOME } a_{m_j} \text{ AND } a_{-m_j}.$$

THEN $\displaystyle\int_{C_j} f(z)\,dz = 2\pi i\, a_{-1j}$ SINCE $\displaystyle\int_{C_j} (z-z_j)^n\,dz = \begin{cases} 0, & n \neq -1 \\ 2\pi i, & n = -1 \end{cases}.$

FROM $(*)$ WE GET $\displaystyle\int_C f(z)\,dz = 2\pi i \sum_{j=1}^{N} a_{-1j} = 2\pi i \sum_{j=1}^{N} \mathrm{RES}[f; z_j].$

REMARKS TO CALCULATE $\mathrm{RES}[f; z_j]$ WE USE REMARKS 2, 3, 4 ON PAGE S11-S12.

EXAMPLE 0 CALCULATE $I = \displaystyle\int_C \frac{dz}{z^2+z+1}$ $\quad C: |z| = 2$ counter-clockwise



METHOD 1 THE SIMPLE POLES ARE AT $z_{\pm} = \dfrac{-1 \pm \sqrt{3}\, i}{2}$ SO $|z_{\pm}| = 1$. THESE ARE INSIDE C. SO BY RESIDUE THEOREM AND REMARK 1 WE GET

$$I = 2\pi i \left[\mathrm{RES}(f; z_+) + \mathrm{RES}(f; z_-) \right] = 2\pi i \left[\frac{1}{2z_+ + 1} + \frac{1}{2z_- + 1} \right] = 2\pi i \left(\frac{2(z_- + z_+) + 2}{(2z_+ + 1)(2z_- + 1)} \right) = 0.$$

METHOD 2 WE CAN PROCEED BY PROB. 18 PAGE 203 OF THE BOOK SINCE THERE ARE NO SINGULARITIES OUTSIDE C AND $f(z) = O(|z|^{-2})$ AS $|z| \to \infty$. WE AGAIN CONCLUDE $I = 0$.

EXAMPLES OF THE USE OF RESIDUE THEOREM.

EXAMPLE 1 CALCULATE $I = \int_C \frac{dz}{z^2 \sinh z}$ $C: |z| = 1$ counterclockwise

NOW THE SINGULARITIES OF $f(z) = 1/z^2 \sinh z$ ARE A POLE OF ORDER 3 AT $z = 0$ AND SIMPLE POLES WHERE $\sinh z = 0$ FOR $z \neq 0$. THESE ARE AT $z = n\pi i$, $n = \pm 1, \pm 2, \ldots$, BUT ARE OUTSIDE C.

SO $I = 2\pi i \operatorname{Res}[f; 0]$ (*)

NOW $\sinh z = \frac{e^z - e^{-z}}{2} = \frac{(1 + z + z^2/2 + z^3/3!) - (1 - z + z^2/2 - z^3/3!) + \cdots}{2} = z + z^3/3! + \cdots$

THUS NEAR $z = 0$ $\frac{1}{z^2 \sinh z} \hat{=} \frac{1}{z^2[z + z^3/6]} \approx \frac{1}{z^3}(1 - z^2/6 + \cdots) = \frac{1}{z^3} - \frac{1}{6z} + \cdots$

THUS $\operatorname{Res}[f; 0] = -1/6$ SO $I = 2\pi i(-1/6) = -\pi i/3$.

EXAMPLE 2 CALCULATE $I = \int_C z^3 e^{1/z} dz$ WITH $C: |z| = 1$ counterclockwise.

NOTE: $z = 0$ IS AN ESSENTIAL SINGULARITY. THUS

$$I = 2\pi i \operatorname{Res}[f; 0].$$

WE CALCULATE LAURENT SERIES,

$$z^3 e^{1/z} = z^3\left(1 + 1/z + 1/2!z^2 + 1/3!z^3 + \cdots\right) = z^3 + z^2 + \frac{1}{2!}z + \frac{1}{3!} + \frac{1}{4!z} + \cdots$$

THUS $a_{-1} = 1/4!$. $\longrightarrow$ $I = 2\pi i/4! = \pi i/12$.

EXAMPLE 3 CALCULATE $I = \int_C \frac{1}{z^2}\left(\frac{1+2z}{1+z}\right) dz$ $C: |z| = 1/2$, COUNTERCLOCKWISE.

NOW THE ONLY SINGULARITY INSIDE C IS AT $z = 0$, WHICH IS A POLE OF ORDER 2. WE CALCULATE THE RESIDUE

$$\frac{1}{z^2}\frac{(1+2z)}{(1+z)} = \frac{1}{z^2}(1+2z)(1 - z + z^2 + \cdots) = \frac{1}{z^2}(1 - z + z^2 + 2z - 2z^2 + \cdots)$$

$$= \frac{1}{z^2} + \frac{1}{z} + 1 \cdots \quad \text{so} \quad a_{-1} = 1 \longrightarrow I = 2\pi i.$$

EXAMPLE 4 CALCULATE $I = \int_C \frac{z+1}{z^2 - 2z} dz$ WHERE $C: |z| = 4$ COUNTERCLOCKWISE.

BOTH $z = 0$ AND $z = 2$ ARE SIMPLE POLES. IF WE WRITE $(z+1)/z^2 - 2z = P(z)/Q(z)$

THEN $\int_C \frac{z+1}{z^2 - 2z} dz = 2\pi i\left[\frac{P(0)}{Q'(0)} + \frac{P(2)}{Q'(2)}\right] = 2\pi i\left(\frac{1}{-2} + \frac{3}{2}\right) = 2\pi i.$

WE NOW GIVE TWO EXAMPLES SHOWING HOW AN INTEGRAL OF A PERIODIC FUNCTION CAN BE CONVERTED TO A CONTOUR INTEGRAL AND THEN EVALUATED BY RESIDUES:

RECALL
$$\cos\varphi = \frac{e^{i\varphi}+e^{-i\varphi}}{2} \longrightarrow \cos\varphi = \frac{z+1/z}{2} \quad \text{WHEN } z=e^{i\varphi} \rightarrow \frac{dz}{iz} = d\varphi$$
$$\sin\varphi = \frac{e^{i\varphi}-e^{-i\varphi}}{2i} \longrightarrow \sin\varphi = \frac{z-1/z}{2i} \quad \text{WHEN } z=e^{i\varphi} \rightarrow \frac{dz}{iz} = d\varphi.$$

EXAMPLE 1 CALCULATE $\int_0^{2\pi} \frac{d\varphi}{2+\sin\varphi} = I$.

NOW LET $z=e^{i\varphi} \rightarrow 2+\sin\varphi = 2+(z-1/z)/2i = \frac{1}{2}\left[4-iz+i/z\right]$.

THEN $I = \int_C \frac{2}{(4-iz+i/z)}\frac{dz}{iz} = 2\int_C \frac{dz}{z^2+4iz-1}$ $\quad C: |z|=1$

THE POLES (SIMPLE) ARE AT $z_\pm = -2i \pm \frac{\sqrt{-16+4}}{2} = -2i \pm \sqrt{3}\,i$.

THUS $z_+ = (-2+\sqrt{3})i$ IS INSIDE $|z_+|\leq 1$, WHILE z_- IS OUTSIDE $|z_-|\geq 1$

THUS $I = 2\int_C \frac{dz}{z^2+4iz-1} = 4\pi i \,\mathrm{RES}\left(\frac{1}{z^2+4iz-1}; z_+\right) = 4\pi i \frac{1}{2z_++4i} = \frac{4\pi i}{2\sqrt{3}\,i}$

THUS $I = 2\pi/\sqrt{3}$.

EXAMPLE 2 CALCULATE $I = \int_0^{2\pi}\frac{d\varphi}{1-2a\cos\varphi+a^2}$ FOR $0<a<1$.

WE WRITE $z=e^{i\varphi}$, $\cos\varphi = (z+1/z)/2$, $dz/iz = d\varphi$.

SO $I = \int_C \frac{dz}{iz\left[1+a^2-\frac{2a}{2}(z+1/z)\right]} = \int_C \frac{dz}{iz\left[1+a^2-az-a/z\right]} = \frac{1}{i}\int_C \frac{dz}{\left[-az^2+(1+a^2)z-a\right]}$

THE ZEROES OF THE DENOMINATOR ARE AT $z=a$ AND $z=1/a$. FOR $0<a<1$, THE ONLY POLE INSIDE $|z|=1$ IS AT $z=a$.

THUS $I = \frac{1}{i}\int_C \frac{dz}{\left[-az^2+(1+a^2)z-a^2\right]} = \frac{1}{i}(2\pi i)\,\mathrm{RES}\left[\frac{1}{-az^2+(1+a^2)z-a^2}, a\right]$

THUS $I = 2\pi\left(\frac{1}{-2az+(1+a^2)}\right)\Big|_{z=a} = \frac{2\pi}{1-a^2}$, $0<a<1$

REMARK FOR AN INTEGRAL OF THE FORM
$I = \int_0^{2\pi} F(\cos\varphi,\sin\varphi)\,d\varphi$ LET $\cos\varphi = \frac{z+1/z}{2}$, $\sin\varphi = \frac{z-1/z}{2i}$, $d\varphi = dz/iz$

TO GET $I = \int_C F\left((z+1/z)/2, (z-1/z)/2i\right)\frac{dz}{iz}$. THEN CALCULATE BY RESIDUES.