INTEGRATION

WE NOW USE RESIDUE CALCULUS TO EVALUATE SOME REAL-VALUED INTEGRALS

TYPE I (TRIGONOMETRIC INTEGRALS)

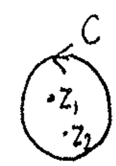
DEFINE $I = \int_0^{2\pi} F(\cos\varphi, \sin\varphi)\, d\varphi$. WE CONVERT TO A CONTOUR INTEGRAL. LET $z = e^{i\varphi}$, $\frac{dz}{iz} = d\varphi$, $\cos\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2} = \frac{(z + 1/z)}{2}$

$\sin\varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2i} = \frac{(z - 1/z)}{2i}$.

THUS $I = \int_C F\left[\frac{(z+1/z)}{2}, \frac{(z-1/z)}{2i}\right] \frac{dz}{iz}$ $\quad C: |z|=1$ counterclockwise.

DEFINE $g(z) = \frac{1}{iz} F\left(\frac{(z+1/z)}{2}, \frac{(z-1/z)}{2i}\right)$. THEN $I = \int_C g(z)\, dz$.

BY RESIDUE CALCULUS $I = 2\pi i \sum_{j=1}^{N} \operatorname{RES}[g(z); z_j]$



WHERE z_j ARE SINGULAR POINTS OF $g(z)$ INSIDE $|z| < 1$.

EXAMPLE 1 CALCULATE $I = \int_0^{2\pi} \frac{d\varphi}{2 + \cos^2\varphi}$

SOLUTION $z = e^{i\varphi}$, $\cos\varphi = \frac{1}{2}(z + 1/z)$, $\cos^2\varphi = 1/4\,(z^2 + 2 + 1/z^2)$.

NOW $2 + \cos^2\varphi = 2 + \frac{z^2}{4} + \frac{1}{2} + \frac{1}{4z^2} = \frac{5}{2} + \frac{z^2}{4} + \frac{1}{4z^2} = \frac{1}{4z^2}\left[1 + z^4 + 10z^2\right]$.

THUS $I = \int_C \frac{4z^2}{z^4 + 10z^2 + 1} \frac{dz}{iz} = -4i \int_C \frac{z}{z^4 + 10z^2 + 1}\, dz$. POLES AT

$z^4 + 10z^2 + 1 = 0$

SO $z^2 = \frac{-10 \pm \sqrt{96}}{2} = -5 \pm 2\sqrt{6}$.

NOTE: $z^2 = -5 + 2\sqrt{6}$ LIES INSIDE $|z| = 1 \longrightarrow z_{\pm} = \pm i\sqrt{5 - 2\sqrt{6}}$.

$z^2 = -5 - 2\sqrt{6}$ LIES OUTSIDE $|z| = 1 \longrightarrow$ IGNORE.

THUS
$$I = -4i\left[\operatorname{RES}\left(\frac{z}{z^4 + 10z^2 + 1}; z_+\right) + \operatorname{RES}\left(\frac{z}{z^4+10z^2+1}; z_-\right)\right] 2\pi i$$

$z_{\pm}$ ARE SIMPLE POLES. THUS,
$$I = 8\pi\left[\frac{z_+}{4z_+^3 + 20z_+} + \frac{z_-}{4z_-^3 + 20z_-}\right] = \frac{8\pi}{4}\left[\frac{1}{z_+^2 + 5} + \frac{1}{z_-^2 + 5}\right]$$

BUT $z_+^2 = z_-^2 = -5 + 2\sqrt{6}$ SO $z_+^2 + 5 = z_-^2 + 5 = 2\sqrt{6}$.

FINALLY, $I = 2\pi\left(\frac{1}{2\sqrt{6}} + \frac{1}{2\sqrt{6}}\right) = 2\pi/\sqrt{6} = \int_0^{2\pi} \frac{d\varphi}{2 + \cos^2\varphi}$.

TYPE II IMPROPER INTEGRALS OVER REAL LINE $(-\infty, \infty)$

A FUNCTION $f(x)$ IS INTEGRABLE OVER THE REAL LINE $-\infty < x < \infty$ IF THE LIMITS $\lim_{c \to +\infty} \int_0^c f(x)\, dx$ AND $\lim_{c \to -\infty} \int_c^0 f(x)\, dx$ EXIST.

IN SUCH A CASE WE HAVE $\int_{-\infty}^{\infty} f(x)\, dx = \lim_{c \to +\infty} \int_{-c}^{c} f(x)\, dx$.

(NOTE: FOR SIMPLICITY HERE WE ARE ASSUMING THAT $f(x)$ IS FINITE $\forall x$ FINITE).

HOWEVER, EVEN IF $\lim_{c \to \infty} \int_{-c}^{c} f(x)\, dx$ EXISTS, IT DOES NOT MEAN THAT THE INDIVIDUAL INTEGRALS $\lim_{c \to -\infty} \int_c^0 f(x)\, dx$ AND $\lim_{c \to +\infty} \int_0^c f(x)$ SEPARATELY EXIST.

EG LET $f(x) =$ THEN $\int_{-c}^{c} \quad dx = \Big|_{-c}^{c} = 0$

BUT $\int_0^c x\, dx = c^2/2 \to +\infty$ AS $c \to +\infty$.

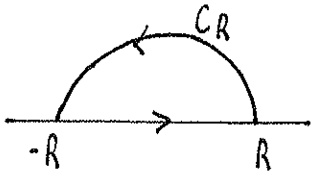
HENCE FOR ANY FUNCTION $f(x)$ WE DEFINE THE <u>CAUCHY PRINCIPAL VALUE</u> OF $f(x)$ OVER $-\infty$ TO ∞ (ABBREVIATED BY $\text{P.V.}\int_{-\infty}^{\infty} f(x)\,dx$) BY

$$\text{P.V.}\int_{-\infty}^{\infty} f(x)\,dx = \lim_{C\to+\infty}\int_{-C}^{C} f(x)\,dx.$$

THESE ARE THE INTEGRALS THAT ARISE USING RESIDUE CALCULUS.

<u>BASIC IDEA</u> LET $I = \int_{-\infty}^{\infty} f(x)\,dx$. WE WANT TO CALCULATE $\text{P.V.}\int_{-\infty}^{\infty} f(x)\,dx$.

WE CONSIDER THE CONTOUR AS SHOWN:



LET $C = C_R \cup [-R, R]$

C_R: $|z| = R$, $\operatorname{Im} z \geq 0$.

THEN $$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i \sum_{j=1}^{N} \operatorname{RES}[f; z_j]$$

where z_j is inside C.

NOW LET $R \to \infty$. THEN

$$\text{P.V.}\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{j=1}^{N} \operatorname{RES}[f; z_j] - \lim_{R\to\infty}\int_{C_R} f(z)\,dz. \quad (*)$$

where z_j IS A SINGULARITY OF $f(z)$ IN UPPER ½ plane.

Q: UNDER WHAT CONDITIONS DOES INTEGRAL OVER $C_R \to 0$ AS $R\to\infty$?

A: THE BASIC ESTIMATE IS AS FOLLOWS:

<u>LEMMA</u> LET $f(z) = P(z)/Q(z)$ P, Q polynomials in z, ($Q \neq 0$ on Real AXIS)

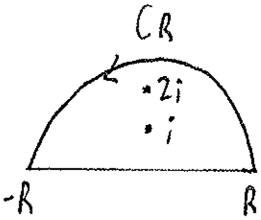
THEN IF degree $Q \geq 2 +$ degree P we have $|f(z)| \leq d/|z|^2$ FOR $|z|>>1$

AND $$\left|\int_C \frac{P(z)}{Q(z)}\,dz\right| \leq \max_{z \text{ on } C}\left|\frac{P(z)}{Q(z)}\right| \pi R \leq \frac{d}{R^2}(\pi R) \longrightarrow 0 \text{ AS } R\to\infty.$$

IN THIS CASE $(*)$ YIELDS $$\text{P.V.}\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{j=1}^{N} \operatorname{RES}[f; z_j] \quad (\operatorname{Im} z_j > 0)$$

EXAMPLE 1 EVALUATE

$$I = \int_0^\infty \frac{x^2}{(1+x^2)(4+x^2)}\,dx$$



SOLUTION NOTE $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)(4+x^2)}\,dx = 2I$. BY SYMMETRY.

CONSIDER THE CONTOUR AS SHOWN, AND DEFINE $P(z) = z^2$, $Q(z) = (1+z^2)(4+z^2)$.
SINCE $\deg Q = 2 + \deg P$, THEN $\displaystyle\left|\int_{C_R} \frac{z^2}{(1+z^2)(4+z^2)}\,dz\right| \to 0$ AS $R \to \infty$.

THE SINGULARITIES IN UPPER 1/2 PLANE ARE AT $z = i$, $z = 2i$, EACH OF WHICH IS A SIMPLE POLE.

HENCE
$$\lim_{R\to\infty}\left[\int_{-R}^{R} \frac{x^2}{(1+x^2)(4+x^2)}\,dx + \int_{C_R} \frac{z^2}{(1+z^2)(4+z^2)}\,dz\right] = 2\pi i\left[\operatorname{RES}\left[\frac{P}{Q}; i\right] + \operatorname{RES}\left[\frac{P}{Q}; 2i\right]\right].$$

THUS
$$2I = \int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)(4+x^2)}\,dx = 2\pi i\left[\frac{P(i)}{Q'(i)} + \frac{P(2i)}{Q'(2i)}\right]$$

NOW $P(z) = z^2 \longrightarrow P(i) = -1$, $P(2i) = -4$

$Q(z) = (1+z^2)(4+z^2)$, $Q'(z) = 2z(4+z^2) + 2z(1+z^2)$

$Q'(i) = 2i(3) = 6i$, $Q'(2i) = -12i$.

THUS
$$2I = \int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)(4+x^2)}\,dx = 2\pi i\left(\frac{-1}{6i} + \frac{4}{12i}\right) = 2\pi\left(\frac{-1}{6} + \frac{1}{3}\right) = \frac{\pi}{3}$$

SO $\underline{I = \pi/6}$.

EXAMPLE 2 CALCULATE $I = \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}$

SOLUTION LET $P(z) = 1$, $Q(z) = (1+z^2)^2$. $\deg Q = 4$, $\deg P = 0$

SO $\left| \int_{C_R} \frac{1}{(1+z^2)^2}\, dz \right| \to 0$ AS $R \to \infty$.

NOW $z = i$ IS A POLE OF ORDER 2 AND IS ONLY SINGULARITY IN THE UPPER ½ PLANE.

THUS $$\lim_{R\to\infty} \left(\int_{-R}^{R} \frac{dx}{(1+x^2)^2} + \int_{C_R} \frac{dz}{(1+z^2)^2} \right) = 2\pi i \ \text{RES}\left[\frac{1}{(1+z^2)^2}; i \right].$$

THIS GIVES, $$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = 2\pi i \ \text{RES}\left[\frac{1}{(z^2+1)^2}; i \right]. \quad (*)$$

TO CALCULATE THE RESIDUE WE HAVE 2 METHODS:

(i) $$\frac{1}{(z+i)^2(z-i)^2} = \frac{1}{(z-i)^2}\left[(z-i)+2i\right]^{-2} = \frac{1}{(z-i)^2}\,\frac{1}{(2i)^2\left[1+\frac{(z-i)}{2i}\right]^2}$$

SO $$\frac{1}{(z+i)^2(z-i)^2} = \frac{1}{-4(z-i)^2}\left[1 - \frac{1}{i}(z-i) + \dots\right] = \frac{-1}{4(z-i)^2} + \frac{1}{4i(z-i)} + \dots$$

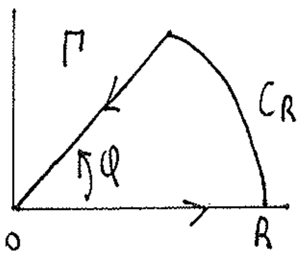
THUS $\text{RES}\left[\frac{1}{(z^2+1)^2}; i\right] = 1/4i$

(ii) $$a_{-1} = \text{RES}\left[\frac{1}{(z-i)^2(z+i)^2}; i\right] = \lim_{z\to i}\left[\frac{d}{dz}(z+i)^{-2}\right] = -2(z+i)^{-3}\Big|_i = \frac{-2}{(2i)^3} = \frac{1}{4i}$$

THIS GIVES $$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = 2\pi i\left(1/4i\right) = \pi/2 \quad \text{FROM } (*).$$

EXAMPLE 3 CALCULATE $I = \int_0^\infty \frac{dx}{x^n+1}$ $\quad n = 2, 3, 4, \ldots$

NOW CONSIDER THE CONTOUR AS SHOWN



$\varphi = 2\pi/n$. NOTE: IF $z = re^{i2\pi/n} \rightarrow z^n+1 = r^n+1$.

NOW NOTE: ON Γ: $z = re^{i\varphi}$, $0 < r <$

SO $\int_\Gamma \frac{1}{z^n+1}\, dz = \int_R^0 \frac{e^{2\pi i/n}\, dr}{r^n+1}$

THUS SINCE $z^n = -1 \longrightarrow z = e^{\pi i/n}$ INSIDE CONTOUR, WE HAVE

$$\lim_{R\to\infty} \left[\int_0^R \frac{dx}{x^n+1} + \int_R^0 \frac{e^{2\pi i/n}}{x^n+1}\, dx + \int_{C_R} \frac{1}{z^n+1}\, dz \right] = 2\pi i \,\mathrm{RES}\left[\frac{1}{z^n+1}; e^{\pi i/n}\right]$$

SINCE $\lim_{R\to\infty} \int_{C_R} \frac{1}{z^n+1}\, dz = 0$ ($n \geq 2$ ensures this),

WE HAVE
$$\int_0^\infty \frac{dx}{x^n+1} - e^{2\pi i/n} \int_0^\infty \frac{dx}{x^n+1} = 2\pi i \left. \frac{1}{n z^{n-1}} \right|_{z=e^{\pi i/n}}$$

$$\longrightarrow \quad (1 - e^{2\pi i/n})\, I = \frac{2\pi i}{n} \frac{1}{e^{\pi i/n (n-1)}} = \frac{-2\pi i}{n} e^{\pi i/n}$$

SO MULTIPLY BOTH SIDES BY $e^{-\pi i/n}$

$$\left(\frac{e^{\pi i/n} - e^{-\pi i/n}}{2i} \right) I = \frac{\pi}{n}$$

$$\longrightarrow \quad I = \frac{\pi}{n \sin(\pi/n)} = \int_0^\infty \frac{dx}{x^n+1}.$$

$n = 2, 3, \ldots$

TYPE III FOURIER-TRANSFORM TYPE INTEGRALS

NEXT WE CONSIDER INTEGRALS OF THE FORM

$$I_1 = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos(Bx)\, dx \qquad \text{OR} \qquad I_2 = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin(Bx)\, dx.$$

ASSUME THAT P, Q polynomials AND THAT Q HAS NO ZEROES ON REAL AXIS.

THE TECHNIQUE WILL BE TO WRITE

$$I_1 = \mathrm{RE}\left[\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{iBx}\, dx \right] \qquad I_2 = \mathrm{IM}\left[\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{iBx}\, dx \right]$$

AND PERFORM A CONTOUR INTEGRATION, WITH

$$J \equiv \int_C \frac{P(z)}{Q(z)} e^{iBz}\, dz.$$

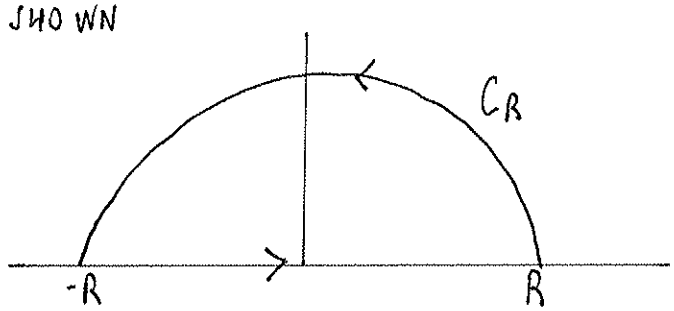
REMARK (i) IF $B > 0$ REAL THEN $|e^{iBz}| = |e^{iB(x+iy)}| = e^{-By}$.

THUS IF $\mathrm{IM}(z) = y > 0$ AND $y \to +\infty$ WE GET DECAY.

$$|e^{iBz}| \leq 1 \quad \text{IN UPPER } 1/2 \text{ PLANE.}$$

THIS SUGGESTS THAT WE TAKE A CONTOUR FOR J

AS SHOWN



THEN WE HAVE

$$(*) \quad \lim_{R\to\infty} \int_{-R}^{R} \frac{P(x)}{Q(x)} e^{iBx}\, dx + \lim_{R\to\infty} \int_{C_R} \frac{P(z)}{Q(z)} e^{iBz}\, dz = 2\pi i \sum_{j=1}^{N} \mathrm{RES}\left[\frac{P}{Q} e^{iBz}; z_j\right]$$

WHERE $\mathrm{IM}(z_j) > 0$.

HERE $z_1, \dots, z_N$ ARE ZEROES OF $Q(z) = 0$

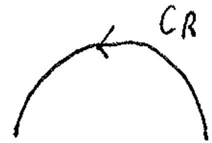
IN UPPER 1/2 PLANE. (HERE WE ASSUMED $B > 0$ REAL)

(ii) WE COULD NOT WORK WITH $\displaystyle\int_{C_R} \frac{P(z)}{Q(z)} \sin(Bz)\,dz$

SINCE $|\sin(Bz)|$ GROWS EXPONENTIALLY AS $\mathrm{Im}(z) \to +\infty$.

NOW UNDER WHAT CONDITIONS DOES $\displaystyle\left|\int_{C_R} \frac{P(z)}{Q(z)} e^{iBz}\,dz\right| \to 0$ AS $R \to \infty$?

THIS IS CONTAINED IN NEXT TWO RESULTS:



JORDON'S LEMMA LET $R > 0$ GIVEN. THEN

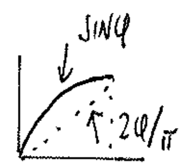
$$\left|\int_{\substack{|z|=R\\ \mathrm{Im}(z)>0}} e^{iz}\,dz\right| \le \pi(1-e^{-R}) < \pi \quad \text{FOR ALL } R > 0.$$

PROOF LET $z = Re^{i\varphi}$ $dz = iRe^{i\varphi}\,d\varphi$

SO
$$\left|\int_{C_R} e^{iz}\,dz\right| = \left|\int_0^{\pi} iRe^{i\varphi} e^{iRe^{i\varphi}}\,d\varphi\right| \le \int_0^{\pi} \left|iRe^{i\varphi} e^{i(R\cos\varphi + iR\sin\varphi)}\right| d\varphi$$

$$\le R\int_0^{\pi} e^{-R\sin\varphi}\,d\varphi = 2R\int_0^{\pi/2} e^{-R\sin\varphi}\,d\varphi.$$

NOW ON $0 \le \varphi \le \pi/2$, $\sin\varphi > 2\varphi/\pi \longrightarrow e^{-R\sin\varphi} < e^{-2\varphi/\pi\, R}$



THUS $\displaystyle\left|\int_{C_R} e^{iz}\,dz\right| \le 2R\int_0^{\pi/2} e^{-2\varphi/\pi\, R}\,d\varphi = 2R\,\frac{\pi}{2R}\left(1 - e^{-2\varphi R/\pi}\right)\Big|_0^{\pi/2}$

THUS $\displaystyle\left|\int_{C_R} e^{iz}\,dz\right| \le \pi(1-e^{-R})$ FOR ANY $R > 0$. ∎

REMARK FOR $\displaystyle\left|\int_{C_R} e^{iBz}\,dz\right|$ LET $w = Bz$ $dz = 1/B\,dw$

SO $\displaystyle\frac{1}{B}\left|\int_{C_{BR}} e^{iw}\,dw\right| \le \frac{1}{B}\pi(1-e^{-BR}) \le \pi/B.$

THUS $\displaystyle\left|\int_{C_R} e^{iBz}\,dz\right| \le \pi/B.$

KEY RESULT SUPPOSE THAT $\left|\frac{P(z)}{Q(z)}\right| \to 0$ AS $|z| \to \infty$ IN UPPER 1/2 PLANE. IN OTHER WORDS, $\deg Q \geq \deg P + 1$. THEN IN (*) WE HAVE $\lim_{R\to\infty} \int_{C_R} \frac{P(z)}{Q(z)} e^{iBz}\, dz \to 0$.

PROOF $$\left|\int_{C_R} \frac{P(z)}{Q(z)} e^{iBz}\, dz\right| \leq \max_{z \text{ on } C_R} \left|\frac{P(z)}{Q(z)}\right| \left|\int_{C_R} e^{iBz}\, dz\right|$$
$$\leq \max_{z \text{ on } C_R} \left|\frac{P}{Q}\right| \pi/B \to 0 \text{ AS } R \to \infty$$
FROM $\left|\frac{P}{Q}\right| \to 0$ AS $|z| \to \infty$.

THEREFORE, WHEN THIS HOLDS WE HAVE FROM (*) THAT

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{iBx}\, dx = 2\pi i \sum_{j=1}^{N} \text{RES}\left[\frac{P(z)}{Q(z)} e^{iBz}; z_j\right] \quad (+)$$

WHERE z_j IS A ROOT OF $Q(z) = 0$ IN UPPER 1/2 plane.

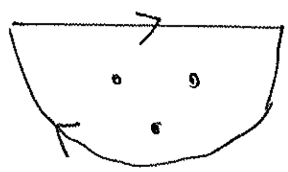
TO EVALUATE $I_1 = \int_{-\infty}^{\infty} \frac{P}{Q} \cos(Bx)\, dx$ OR $I_2 = \int_{-\infty}^{\infty} \frac{P}{Q} \sin(Bx)\, dx$

SIMPLY TAKE REAL OR IMAGINARY PART OF (+)

REMARK (i) IF $B < 0$ WE MUST ENCLOSE IN LOWER 1/2 PLANE TO GET

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{iBx}\, dx = -2\pi i \sum_{j=1}^{N} \text{RES}\left[\frac{P(z)}{Q(z)} e^{iBz}; z_j\right]$$

WITH $Q(z_j) = 0$, $j = 1, \ldots, N$ AND $\text{IM}(z_j) < 0$.



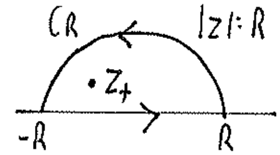
NOTICE : CLOCKWISE CONTOUR. HENCE - SIGN

EXAMPLE 1 CALCULATE

$$I = \int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^2 + 2x + 5}\, dx$$

SOLUTION WE TAKE THE CONTOUR AS SHOWN AND CONSIDER

$$J = \int_C \frac{z\, e^{i\pi z}}{z^2 + 2z + 5}\, dz \qquad I = \mathrm{IM}(J)$$



NOW $P = z,\ Q = z^2 + 2z + 5 \Rightarrow |P/Q| \to 0$ AS $|z| \to \infty$

THUS USING JORDAN'S LEMMA $\left| \int_{CR} \frac{z e^{i\pi z}}{z^2+2z+5}\, dz \right| \to 0$ AS $R \to \infty$.

POLES ARE AT $z^2 + 2z + 1 = -4 \longrightarrow z + 1 = \pm 2i \rightarrow Z_+ = -1 + 2i$ IN UPPER ½ PLANE

THUS,
$$\int_{-\infty}^{\infty} \frac{x\, e^{i\pi x}}{x^2 + 2x + 5}\, dx = 2\pi i\, \mathrm{RES}\left[\frac{z e^{i\pi z}}{z^2+2z+5} ; Z_+ \right] = 2\pi i \left[\frac{(-1+2i) e^{-i\pi - 2\pi}}{2Z_+ + 2} \right]$$

SO
$$\int_{-\infty}^{\infty} \frac{x e^{i\pi x}}{x^2+2x+5}\, dx = 2\pi i \left[\frac{(-1+2i) e^{-2\pi - i\pi}}{4i} \right] = \frac{\pi}{2}(1 - 2i) e^{-2\pi}.$$

NOW TAKE IMAGINARY PARTS OF BOTH SIDES

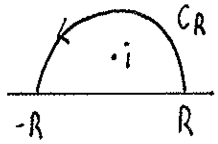
$$\int_{-\infty}^{\infty} \frac{x \sin(\pi x)}{x^2 + 2x + 5}\, dx = -\pi e^{-2\pi}.$$

BY TAKING REAL PARTS OF BOTH SIDES

$$\int_{-\infty}^{\infty} \frac{x \cos(\pi x)}{x^2 + 2x + 5}\, dx = \frac{\pi}{2} e^{-2\pi}.$$

EXAMPLE 2 EVALUATE $I = \int_{-\infty}^{\infty} \frac{e^{iBx}}{x^2+1}\, dx$ FOR $B > 0$ AND FOR $B < 0$, WITH B REAL.

SOLUTION IF $B > 0$ WE ENCLOSE IN UPPER ½ PLANE AS SHOWN:



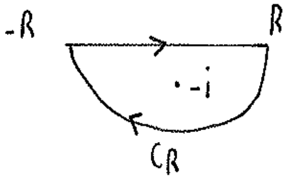
THE POLE IS AT $z = i$. IT IS A SIMPLE POLE.

$$\lim_{R \to \infty} \left[\int_{-R}^{R} \frac{e^{iBx}}{x^2+1}\, dx + \int_{CR} \frac{e^{iBz}}{z^2+1}\, dz \right] = 2\pi i\, \mathrm{RES} \left[\frac{e^{iBz}}{z^2+1} ; i \right]$$

AS BEFORE, $\left| \int_{C_R} \frac{e^{iBz}}{z^2+1}\, dz \right| \to 0$ AS $R \to \infty$, WHEN $B > 0$

HENCE, $$\int_{-\infty}^{\infty} \frac{e^{iBx}}{x^2+1}\, dx = 2\pi i \left[\frac{e^{iB(i)}}{2i} \right] = \pi e^{-B} \quad (B>0).$$

IF $B < 0$ WE MUST ENCLOSE IN LOWER 1/2 PLANE AS SHOWN.



THEN $\left| \int_{C_R} \frac{e^{iBz}}{z^2+1}\, dz \right| \to 0$ AS $R \to \infty$ WHEN $B<0$ AND C_R IS SEMI-CIRCLE IN LOWER 1/2 PLANE.

THUS $$\int_{-\infty}^{\infty} \frac{e^{iBx}}{x^2+1}\, dx = -2\pi i \,\mathrm{RES}\left[\frac{e^{iBz}}{z^2+1}; -i\right]$$
$$= -2\pi i \left(\frac{e^{B}}{-2i} \right) = \pi e^{B}. \quad (B<0).$$

WE THEN WRITE $$\int_{-\infty}^{\infty} \frac{e^{iBx}}{x^2+1}\, dx = \pi e^{-|B|} \quad \text{FOR } B>0,\ B<0.$$

<u>EXAMPLE 3</u> CALCULATE $I = \int_{-\infty}^{\infty} \frac{\sin x}{x+i}\, dx$.

<u>SOLUTION</u> NOTICE THAT WE <u>CANNOT</u> WORK WITH $\int_{-\infty}^{\infty} \frac{e^{ix}}{x+i}\, dx$ ONLY AND TAKE IMAGINARY PARTS.

WE MUST WRITE $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$ AND GET

$$I = \frac{1}{2i} \underbrace{\int_{-\infty}^{\infty} \frac{e^{ix}}{x+i}\, dx}_{\text{calculate: enclose in upper 1/2 plane}} - \frac{1}{2i} \underbrace{\int_{-\infty}^{\infty} \frac{e^{-ix}}{x+i}\, dx}_{\text{calculate: enclose in lower 1/2 plane.}} \quad (*)$$

- THUS $\int_C \frac{e^{iz}}{z+i}\, dz = 0$

$\longrightarrow \int_{-\infty}^{\infty} \frac{e^{ix}}{x+i}\, dx + \underbrace{\lim_{R\to\infty} \int_{C_R}}_{=0} = 0$

$\longrightarrow \int_{-\infty}^{\infty} \frac{e^{ix}}{x+i}\, dx = 0.$

- NEXT $\int_C \frac{e^{-iz}}{z+i}\, dz = -2\pi i \,\mathrm{RES}\left(\frac{e^{-iz}}{z+i}; -i \right) = -2\pi i\, e^{-1}. \rightarrow \int_{-\infty}^{\infty} \frac{e^{-ix}}{x+i}\, dx = -2\pi i\, e^{-1}$

FINALLY $(*)$ GIVES $$I = \frac{1}{2i}(0) - \frac{1}{2i}\left(-2\pi i\, e^{-1}\right) = \pi e^{-1}.$$

NOTICE ALSO THAT $I = \int_{-\infty}^{\infty} \frac{(x-i)\sin x}{(x+i)(x-i)}\,dx = \underbrace{\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1}\,dx}_{\text{even}} - i \underbrace{\int_{-\infty}^{\infty} \frac{\sin x}{x^2+1}\,dx}_{\text{ODD integrand.}}$

THUS $I = \int_{-\infty}^{\infty} \frac{x \sin x}{(x^2+1)}\,dx = \mathrm{IM}\left(\int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2+1}\,dx \right).$

RECALL THAT $J(B) = \int_{-\infty}^{\infty} \frac{e^{iBx}}{x^2+1}\,dx = \pi e^{-B}$ FOR $B>0$ BY EXAMPLE 2.

BY INSPECTION: $I = \mathrm{IM}\left(-i \frac{dJ}{dB}\Big|_{B=1} \right) = \mathrm{IM}\left(-i\left(-\pi e^{-B}\Big|_{B=1} \right)\right) = \pi e^{-1}.$

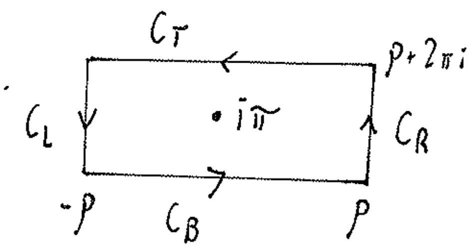
THIS GIVES A TRICKY ALTERNATIVE CALCULATION.

<u>TYPE IV</u> BOX-SHAPED CONTOURS IN THE COMPLEX PLANE ARE USEFUL WHEN $f(x)$, WHEN CONSIDERED AS $f(z)$ IS PERIODIC IN Y-DIRECTION, i.e. $f(z+ip) = f(z)$.

<u>EXAMPLE 1</u> CALCULATE $I = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx$ WITH $0<a<1$ REAL.

NOTICE THAT THIS INTEGRAL CONVERGES WHEN $0<a<1$. IN ADDITION, IF $f(z) = e^z$ THEN $f(z+2\pi i) = e^z$. NOTE: $e^z+1=0$ GIVES $z = i\pi, \ldots$

THUS, WE CONSIDER THE CONTOUR AS SHOWN



BY RESIDUE THEOREM, SINCE SIMPLE POLE IS INSIDE CONTOUR AT $z = i\pi$,

$$(*) \quad \lim_{p\to\infty} \left[\left(\int_{C_L} + \int_{C_B} + \int_{C_R} + \int_{C_T} \right) \frac{e^{az}}{1+e^z}\,dz \right] = 2\pi i \,\mathrm{RES}\left[\frac{e^{az}}{e^z+1} ; i\pi \right] = 2\pi i \left(\frac{e^{ia\pi}}{e^{i\pi}} \right)$$

NOW ESTIMATE $\left| \int_{C_R} \frac{e^{az}}{e^z+1}\,dz \right| \leq \max_{z \text{ on } C_R} \left| \frac{e^{az}}{e^z+1} \right| 2\pi \leq \frac{e^{ap}}{e^p-1} 2\pi \to 0$ AS $p \to +\infty$ SINCE $0<a<1$.

SIMILARLY $\lim\limits_{\rho \to \infty} \left| \int_{C_L} \right| = 0.$

NOW ON C_B: $z = x \implies \int_{C_B} = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\,dx$

NOW ON C_T: $z = x + 2\pi i \implies \int_{C_T} = \int_{\infty}^{-\infty} \frac{e^{a(x+2\pi i)}}{1+e^{x+2\pi i}}\,dx = -e^{2\pi i a}\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\,dx.$

THIS GIVES FROM (*) THAT

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\,dx - e^{2\pi i a}\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\,dx = 2\pi i \left(\frac{e^{ia\pi}}{e^{i\pi}}\right) = -2\pi i\, e^{ia\pi}.$$

THUS $\quad \left(e^{2\pi i a} - 1\right)\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\,dx = (2i)\,\pi\, e^{ia\pi}$

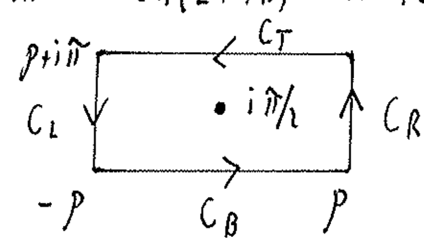
SO $\quad \left(\frac{e^{ia\pi} - e^{-ia\pi}}{2i}\right)\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\,dx = \pi \;\to\; \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\,dx = \frac{\pi}{\sin(a\pi)}$

NOTICE DIVERGENCE AS $a \to 0^+$ AND $a \to 1^-$.

EXAMPLE 2 CALCULATE $I = \int_{-\infty}^{\infty} \frac{\cos x}{e^{x}+e^{-x}}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos x}{\cosh x}\,dx = \frac{1}{2}\left(\int_{-\infty}^{\infty} \frac{e^{ix}}{\cosh x}\,dx\right)$

SOLUTION NOTICE $\cosh z = 0$ WHEN $z = i\pi/2, 3\pi i/2, \ldots$ AND $\cosh(z + i\pi) = -\cosh z$.

THIS SUGGESTS THAT WE INTEGRATE $\int_C \frac{e^{iz}}{\cosh z}\,dz$



BY RESIDUE THEOREM

$$\lim_{\rho \to \infty} \int_C \frac{e^{iz}}{\cosh z}\,dz = 2\pi i\,\mathrm{RES}\left[\frac{e^{iz}}{\cosh z}\,;\, i\pi/2\right] = 2\pi i \left(\frac{e^{-\pi/2}}{\sinh(i\pi/2)}\right) = \frac{2\pi i\, e^{-\pi/2}}{i \sin(\pi/2)}$$

THUS $\quad \lim\limits_{\rho \to \infty} \int_C \frac{e^{iz}}{\cosh z}\,dz = 2\pi\, e^{-\pi/2}.$

NOW ON C_B: $z = x$ SO $\displaystyle\lim_{\rho\to\infty}\int_{C_B} = \int_{-\infty}^{\infty}\frac{e^{ix}}{\cosh x}\,dx$

NOW ON C_T: $z = x + i\pi$ SO $\displaystyle\lim_{\rho\to\infty}\int_{C_T} = \int_{\infty}^{-\infty}\frac{e^{i(x+i\pi)}}{\cosh(x+i\pi)}\,dx = -e^{-\pi}\int_{-\infty}^{\infty}\frac{e^{ix}}{\cosh(x+i\pi)}\,dx$

$$= e^{-\pi}\int_{-\infty}^{\infty}\frac{e^{ix}}{\cosh(x)}\,dx.$$

NOW ON C_R: WE USE $|\cosh z|^2 = \sinh^2 x + \cos^2 y \geq \sinh^2 x$.

THUS $|\cosh z| \geq \sinh$ ON C_R

SO $\displaystyle\left|\int_{C_R}\right| \leq \max_{C_R}\frac{|e^{iz}|}{|\cosh z|}\,\pi \leq \frac{1}{\sinh\rho}\,\pi \to 0$ AS $\rho\to\infty$.

SIMILARLY $\displaystyle\left|\int_{C_L}\right| \to 0$ AS $\rho\to\infty$.

THUS $\displaystyle\lim_{\rho\to\infty}\left(\left(\int_{C_T}+\int_{C_R}+\int_{C_B}+\int_{C_L}\right)\left(\frac{e^{iz}}{\cosh z}\right)dz\right) = 2\pi e^{-\pi/2}$

GIVES $\displaystyle\int_{-\infty}^{\infty}\frac{e^{ix}}{\cosh x}\,dx + e^{-\pi}\int_{-\infty}^{\infty}\frac{e^{ix}}{\cosh x}\,dx = 2\pi e^{-\pi/2}$.

$$\frac{(e^{\pi/2}+e^{-\pi/2})}{2}\int_{-\infty}^{\infty}\frac{e^{ix}}{\cosh x}\,dx = \pi$$

$$\longrightarrow \int_{-\infty}^{\infty}\frac{e^{ix}}{\cosh x}\,dx = \frac{\pi}{\cosh(\pi/2)}$$

FINALLY, $\displaystyle I = \int_{-\infty}^{\infty}\frac{\cos x}{e^{x}+e^{-x}}\,dx = \frac{1}{2}\,\mathrm{RE}\left(\int_{-\infty}^{\infty}\frac{e^{ix}}{\cosh x}\,dx\right) =$

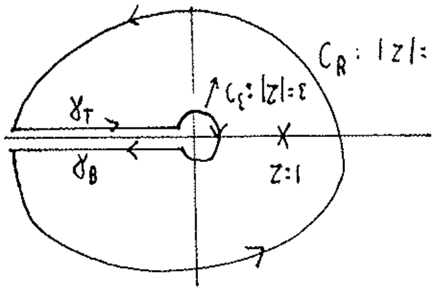
$$= \frac{\pi}{2\cosh(\pi/2)}$$

$\square$

INTEGRALS OF MULTI-VALUED FUNCTIONS

PROBLEM 1 $\quad I = \int_0^\infty \frac{x^{\alpha-1}}{1+x}\,dx \qquad 0<\alpha<1.$

TAKE THE INTEGRAL $\int_C \frac{z^{\alpha-1}}{1-z}\,dz$ WHERE C IS THE CONTOUR SHOWN BELOW. TAKE PRINCIPAL BRANCH OF $z^{\alpha-1}$. THEN,



$$\left|\int_{C_R} \frac{z^{\alpha-1}}{1-z}\,dz\right| \le a_0 R\,\frac{R^{\alpha-1}}{R} = a_0 R^{\alpha-1} \to 0 \text{ as } R\to\infty$$

SINCE $\alpha<1$

$$\left|\int_{C_\varepsilon} \frac{z^{\alpha-1}}{1-z}\,dz\right| \le a_0\,\varepsilon\,\frac{\varepsilon^{\alpha-1}}{1} = a_0\,\varepsilon^{\alpha} \to 0 \text{ as } \varepsilon\to 0$$

SINCE $\alpha>0$.

THUS
$$\lim_{R\to\infty}\int_{C_R}\frac{z^{\alpha-1}}{1-z}\,dz + \lim_{\varepsilon\to 0}\int_{C_\varepsilon}\frac{z^{\alpha-1}}{1-z}\,dz + \int_{\gamma_T}\frac{z^{\alpha-1}}{1-z}\,dz + \int_{\gamma_B}\frac{z^{\alpha-1}}{1-z}\,dz = 2\pi i\,\mathrm{RES}\left[\frac{z^{\alpha-1}}{1-z};1\right]$$

SO (X) $\displaystyle\int_{\gamma_T}\frac{z^{\alpha-1}}{1-z}\,dz + \int_{\gamma_B}\frac{z^{\alpha-1}}{1-z}\,dz = -2\pi i\,(1)^{\alpha-1} = -2\pi i.$

ON γ_T: $\quad z = \rho e^{i\pi} \qquad 0<\rho<\infty.$

SO $1-z = 1+\rho$ (this is why we took $1-z$ rather than $1+z$)

$dz = e^{i\pi}\,d\rho.$

SO
$$\int_{\gamma_T}\frac{z^{\alpha-1}}{1-z}\,dz = \int_\infty^0 \frac{e^{i\pi}\rho^{\alpha-1}}{1+\rho}\,e^{i\pi(\alpha-1)}\,d\rho = e^{i\pi(\alpha-1)}\int_0^\infty \frac{\rho^{\alpha-1}}{1+\rho}\,d\rho$$

ON γ_B: $\quad z = \rho e^{-i\pi} \qquad 0<\rho<\infty \qquad 1-z = 1+\rho \qquad dz = e^{-i\pi}\,d\rho$

$$\int_{\gamma_B}\frac{z^{\alpha-1}}{1-z}\,dz = \int_0^\infty \frac{e^{-i\pi}\rho^{\alpha-1}e^{-i\pi(\alpha-1)}}{1+\rho}\,d\rho = -e^{-i\pi(\alpha-1)}\int_0^\infty\frac{\rho^{\alpha-1}}{1+\rho}\,d\rho$$

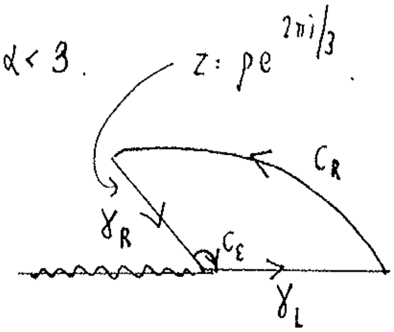
substitute these integrals into (X). Then,

$$\left[e^{i\pi(\alpha-1)} - e^{-i\pi(\alpha-1)}\right]\int_0^\infty \frac{\rho^{\alpha-1}}{1+\rho}\,d\rho = -2\pi i. \qquad \int_0^\infty\frac{\rho^{\alpha-1}}{1+\rho}\,d\rho = \frac{-2\pi i}{\left(e^{i\pi(\alpha-1)} - e^{-i\pi(\alpha-1)}\right)}$$

THUS
$$\int_0^\infty \frac{\rho^{\alpha-1}}{1+\rho}\,d\rho = \frac{-2\pi i}{2i\sin[\pi(\alpha-1)]} = \frac{\pi}{\sin[\pi(1-\alpha)]}$$

PROBLEM 2 $\quad I = \int_0^\infty \frac{x^{\alpha-1}}{1+x^3}\,dx \qquad 0<\alpha<3.$

TAKE THE CONTOUR $\displaystyle\int_C \frac{z^{\alpha-1}}{1+z^3}\,dz$



TAKE BRANCH CUT ON negative Real axis.

THEN $\displaystyle\left|\int_{C_\varepsilon} \frac{z^{\alpha-1}}{1+z^3}\,dz\right| \le \underbrace{a_0\,\varepsilon}_{\text{length of curve}}\ \varepsilon^{\alpha-1} = a_0\,\varepsilon^{\alpha} \to 0$ as $\varepsilon\to 0$ since $\alpha>0$.

$$\left|\int_{C_R} \frac{z^{\alpha-1}}{1+z^3}\,dz\right| \le a_0 R\,\frac{R^{\alpha-1}}{R^3} = a_0 R^{\alpha-3} \to 0 \quad \text{as } R\to\infty \text{ since } \alpha<3.$$

NOW inside C: $1+z^3=0$ WHEN $z=e^{i\pi/3}$.

THUS $\displaystyle \text{RES}\left[\frac{z^{\alpha-1}}{1+z^3};\ e^{i\pi/3}\right] = \frac{e^{i\pi/3(\alpha-1)}}{3e^{2\pi i/3}} = \frac{1}{3}e^{\alpha\pi i/3}e^{-i\pi} = -\frac{1}{3}e^{\alpha\pi i/3}.$

NEXT $\displaystyle \lim_{R\to\infty}\int_{C_R} + \lim_{\varepsilon\to 0}\int_{C_\varepsilon} + \int_{\gamma_R} + \int_{\gamma_L} = 2\pi i\ \text{RES}\left(\frac{z^{\alpha-1}}{1+z^3};\ e^{\pi i/3}\right) = \frac{-2\pi i}{3}e^{\alpha\pi i/3}.$

letting $R\to\infty$, $\varepsilon\to 0$ $\quad (*) \quad \displaystyle\int_{\gamma_R} + \int_{\gamma_L} = -\frac{2\pi i}{3}e^{\alpha\pi i/3}.$

NOW ON γ_R: $\quad z=\rho e^{2\pi i/3} \qquad dz = e^{2\pi i/3}\,d\rho \qquad 1+z^3 = 1+\rho^3.$

$$\text{so}\quad \int_{\gamma_R} = \int_\infty^0 \frac{\rho^{\alpha-1}}{1+\rho^3}e^{2\pi i(\alpha-1)/3}e^{2\pi i/3}\,d\rho = -e^{2\pi i\alpha/3}\int_0^\infty \frac{\rho^{\alpha-1}}{1+\rho^3}\,d\rho.$$

NOW ON γ_L $\quad z=\rho \qquad dz = d\rho \qquad \displaystyle\int_{\gamma_L}\frac{z^{\alpha-1}}{1+z^3}\,dz = \int_0^\infty \frac{\rho^{\alpha-1}}{1+\rho^3}\,d\rho.$

substituting these integrals INTO $(*)$ we get

$$\left(1-e^{+2\pi i\alpha/3}\right)\int_0^\infty \frac{\rho^{\alpha-1}}{1+\rho^2}\,d\rho = -\frac{2\pi i}{3}e^{\alpha\pi i/3}.$$

$$\text{so}\quad \left(e^{-\alpha\pi i/3}-e^{\alpha\pi i/3}\right) I = -\frac{2\pi i}{3} \longrightarrow +2i\sin\left(-\alpha\pi/3\right) I = -2\pi i/3$$

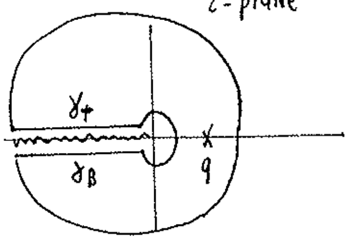
$$\text{OR}\quad \int_0^\infty \frac{\rho^{\alpha-1}}{1+\rho^3}\,d\rho = \frac{\pi}{3\sin(\pi\alpha/3)}$$

PROBLEM 3 WHAT CONTOURS SHOULD WE TAKE FOR CALCULATING

$$I_1 = \int_0^\infty \frac{x^\alpha}{(x+q)^2}\,dx \qquad I_2 = \int_0^\infty \frac{x^\alpha}{(x^2+1)^2}\,dx \qquad I_3 = \text{P.V.}\int_0^\infty \frac{x^\alpha}{x^2-1}\,dx$$

$$-1 < \alpha < 1 \qquad\qquad -1 < \alpha < 3 \qquad\qquad -1 < \alpha < 1$$

INTEGRAL I_1



$\int_C \frac{z^\alpha}{(q-z)^2}\,dz$ TAKE z^α with branch cut shown.

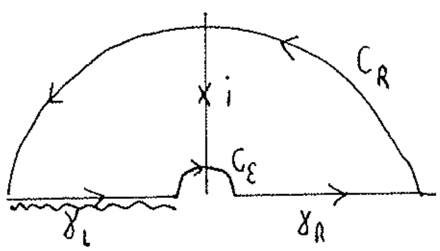
ON γ_T: $z = \rho e^{i\pi}$ so $(q-z)^2 = (q+\rho)^2$

ON γ_B: $z = \rho e^{-i\pi}$ so $(q-z)^2 = (q+\rho)^2$

this contour will work.

INTEGRAL I_2 WE could take precisely the same contour as for I_1 above. This would give poles at $z = \pm i$ when integrating $\int_C \frac{z^\alpha}{(z^2+1)^2}\,dz$.

HOWEVER, PERHAPS IT IS EASIER TO TAKE CONTOUR BELOW:



$\int_C \frac{z^\alpha}{(z^2+1)^2}\,dz$ with taking principal branch FOR z^α.

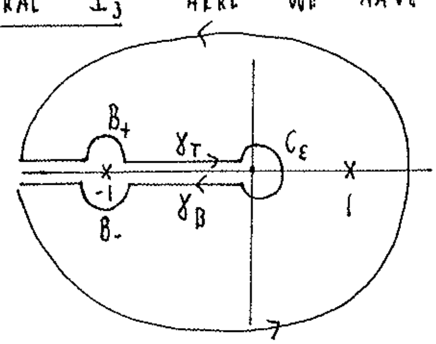
ON γ_L: $z = \rho e^{i\pi}$ so $1+z^2 = 1+\rho^2$.

ON γ_R: $z = \rho$ so $1+z^2 = \rho^2$. $\qquad i^\alpha = e^{\frac{\pi i}{2}\alpha}$

NOW $\text{RES}\left[\frac{z^\alpha}{(z^2+1)^2}; i\right] = \lim_{z\to i} \frac{d}{dz}\left[\frac{(z+i)^2 z^\alpha}{(z+i)^2(z-i)^2}\right]$

$$= \lim_{z\to i}\frac{d}{dz}\left[(z-i)^{-2} z^\alpha\right] = \alpha z^{\alpha-1}(z-i)^{-2}\Big|_i - 2(z-i)^{-3}z^\alpha\Big|_i$$

INTEGRAL I_3 HERE WE HAVE NO CHOICE BUT TO TAKE CONTOUR



$\int_C \frac{z^\alpha}{z^2-1}\,dz$ pole at ± 1 along Real axis

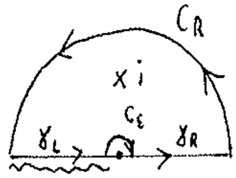
NOTE $\int_{B_\pm} \not\to 0$ AS $\varepsilon \to 0$.

ON B_+: $z = -1 + \varepsilon e^{i\varphi}$ $\quad \pi < \varphi < 0$

$$\lim_{\varepsilon\to 0}\int_{B_+}\frac{z^\alpha}{z^2-1}\,dz \to \int_\pi^0 \frac{e^{i\pi\alpha}}{\varepsilon e^{i\varphi}(-2)}\,\varepsilon i e^{i\varphi}\,d\varphi \to \frac{i\pi}{2}e^{i\pi\alpha}.$$

PROBLEM 4 $\quad I = \int_0^\infty \frac{\ln x}{x^2+1}\,dx.$

CONSIDER THE CONTOUR



$$\int_C \frac{\operatorname{LOG} z}{z^2+1}\,dz.$$

HERE $\operatorname{LOG}(z)$ is principal branch.

$$\left|\int_{C_R} \frac{\operatorname{LOG} z}{z^2+1}\,dz\right| \leq a_0 R \,\frac{\ln R}{R^2} \to 0 \text{ as } R\to\infty$$

$$\left|\int_{C_\varepsilon} \frac{\operatorname{LOG} z}{z^2+1}\,dz\right| \leq a_0\, \varepsilon\, \frac{\ln \varepsilon}{1} \to 0 \text{ AS } \varepsilon\to 0.$$

$$\text{NOW} \quad \lim_{R\to\infty}\int_{C_R} + \lim_{\varepsilon\to 0}\int_{C_\varepsilon} + \int_{\gamma_L} + \int_{\gamma_R} = 2\pi i \operatorname{RES}\left[\frac{\operatorname{LOG} z}{z^2+1};\ i\right]$$

LET $R\to\infty$, $\varepsilon\to 0$ TO GET (*) $\int_{\gamma_L} + \int_{\gamma_R} = 2\pi i \operatorname{RES}\left[\frac{\operatorname{LOG} z}{z^2+1};\ i\right] = 2\pi i\,\frac{\operatorname{LOG} i}{2i} = \pi\left(\frac{i\pi}{2}\right)$

$$= i\pi^2/2.$$

NOW ON γ_L: $\quad z = p e^{i\pi}. \quad dz = e^{i\pi}\,dp. \quad z^2+1 = p^2+1.$

$$\operatorname{LOG} z = \ln p + i\pi$$

$$\text{SO} \quad \int_{\gamma_L} = \int_\infty^0 \left(\frac{\ln p + i\pi}{p^2+1}\right) e^{i\pi}\,dp = \int_0^\infty \frac{\ln p + i\pi}{p^2+1}\,dp.$$

NOW ON γ_R: $\quad z = p$ SO $dz = dp \quad z^2+1 = p^2+1$

$$\operatorname{LOG} z = \ln p$$

$$\text{SO} \quad \int_{\gamma_R} = \int_0^\infty \frac{\ln p}{p^2+1}\,dp.$$

substitute $\int_{\gamma_L}$ AND $\int_{\gamma_R}$ INTO (*) TO get $\int_0^\infty \frac{\ln p}{p^2+1}\,dp + \int_0^\infty \frac{\ln p}{p^2+1}\,dp + i\pi\int_0^\infty \frac{1}{1+p^2}\,dp$

$$= i\pi^2/2.$$

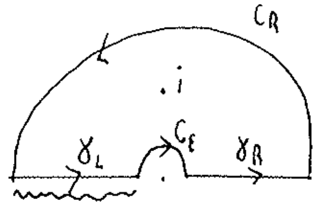
EQUATING REAL AND IMAGINARY PARTS:

$$\int_0^\infty \frac{\ln p}{p^2+1}\,dp = 0 \quad \text{AND} \quad \int_0^\infty \frac{1}{1+p^2}\,dp = \pi/2.$$

PROBLEM 5 $\quad I = \displaystyle\int_0^\infty \frac{(\ln r)^2}{1+r^2}\,dr.$

LOG Z is principal value.

TAKE $\displaystyle\int_C \frac{(\mathrm{LOG}\,z)^2}{1+z^2}\,dz$ ON



NOW $\quad \mathrm{RES}\left[\dfrac{(\mathrm{LOG}\,z)^2}{1+z^2};\ i\right] = \dfrac{(\mathrm{LOG}\,i)^2}{2i} = \dfrac{(i\pi/2)^2}{2i} = -\dfrac{\pi^2}{8i} = \dfrac{i\pi^2}{8}$

THUS AGAIN $\displaystyle\int_{C_R} \to 0$ AND $\displaystyle\int_{C_\varepsilon} \to 0$. SO

$$(*) \quad \int_{\gamma_L} + \int_{\gamma_R} = 2\pi i\ \mathrm{RES}(\ ;\ i) = 2\pi i\left(i\pi^2/8\right) = -\pi^3/4.$$

NOW ON γ_L: $\quad z = pe^{i\pi} \quad 0<p<\infty \quad 1+z^2 = 1+p^2 \quad dz = e^{i\pi}dp$

$$(\log z)^2 = (\ln p + i\pi)^2 = (\ln p)^2 + 2i\pi\ln p - \pi^2.$$

THUS

$$\int_{\gamma_L} = \int_\infty^0 \left(\frac{(\ln p)^2 + 2i\pi\ln p - \pi^2}{1+p^2}\right)e^{i\pi}\,dp = \int_0^\infty \frac{(\ln p)^2 + 2i\pi\ln p - \pi^2}{1+p^2}\,dp$$

NOW ON γ_R: $\quad z = p$

$$\int_{\gamma_R} = \int_0^\infty \frac{(\ln p)^2}{1+p^2}.$$

substitute $\displaystyle\int_{\gamma_L}, \int_{\gamma_R}$ INTO $(*)$ to get:

$$2\int_0^\infty \frac{(\ln p)^2}{1+p^2}\,dp + 2i\pi\int_0^\infty \frac{\ln p}{1+p^2}\,dp - \pi^2\int_0^\infty \frac{1}{1+p^2}\,dp = -\pi^3/4.$$

EQUATING REAL AND IMAGINARY PARTS:

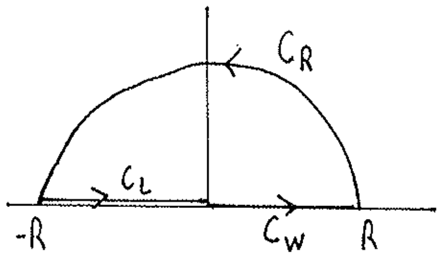
$$\int_0^\infty \frac{\ln p}{1+p^2}\,dp = 0 \qquad \text{AND} \qquad \int_0^\infty \frac{(\ln p)^2}{1+p^2}\,dp = \frac{\pi^2}{2}\int_0^\infty \frac{1}{1+p^2}\,dp - \frac{\pi^3}{8}$$

$$= \frac{\pi^2}{2}\,\mathrm{TAN}^{-1}p\,\Big|_0^\infty - \frac{\pi^3}{8} = \frac{\pi^2}{2}\left(\frac{\pi}{2}\right) - \frac{\pi^3}{8}$$

$$\text{so} \quad \int_0^\infty \frac{(\ln p)^2}{1+p^2}\,dp = \pi^3/8.$$

EXAMPLE 6 CALCULATE $I_1 = \int_0^\infty \frac{x^{1/4}}{x^2+x+1}\,dx.$

SOLUTION DEFINE $\sqrt{z}$ AS PRINCIPAL BRANCH AND CONSIDER CONTOUR SHOWN WHERE C_L IS ON TOP OF BRANCH CUT.



THEN LET $C: C_L \cup C_W \cup C_R$

WITH $C_R: |z| = R,\ \mathrm{Im}\, z \geq 0.$

NOW POLES ARE AT $z^2+z+1=0$

SO $(z+1/2)^2 = -3/4 \rightarrow z_\pm = -1/2 \pm i\sqrt{3}/2$

NOTICE ONLY z_+ IN UPPER 1/2 PLANE. $z_+ = e^{2\pi i/3}$.

THUS $\lim_{R\to\infty} \left(\int_{C_L} + \int_{C_W} + \int_{C_R} \right) \frac{z^{1/4}}{z^2+z+1}\,dz = 2\pi i\, \mathrm{RES}\left(\frac{z^{1/4}}{z^2+z+1} ; e^{2\pi i/3} \right)$ (*)

NOW $\left| \int_{C_R} \frac{\sqrt[4]{z}}{z^2+z+1}\,dz \right| \leq \frac{R^{1/4}}{|R^2-(R+1)|}\,\pi R = O\left(\frac{R^{5/4}}{R^2}\right) \rightarrow 0$ AS $R \rightarrow \infty$.

NOW ON C_L: $z = xe^{i\pi} \rightarrow dz = e^{i\pi}\,dx \qquad \frac{z^{1/4}}{z^2+z+1} = \frac{e^{i\pi/4}x^{1/4}}{x^2-x+1}$.

THUS $\int_{C_L} = \int_\infty^0 \frac{e^{i\pi}e^{i\pi/4}x^{1/4}\,dx}{x^2-x+1} = \int_0^\infty \frac{e^{i\pi/4}x^{1/4}}{x^2-x+1}\,dx.$

NOW ON C_W: $z = x \quad dz = dx \longrightarrow \int_{C_W} = \int_0^\infty \frac{x^{1/4}}{x^2+x+1}\,dx.$

THUS (*) GIVES $\int_0^\infty \frac{x^{1/4}}{x^2+x+1}\,dx + e^{i\pi/4}\int_0^\infty \frac{x^{1/4}}{x^2-x+1}\,dx = 2\pi i\left(\frac{e^{\pi i/6}}{2e^{2\pi i/3}+1}\right) = \frac{2\pi i\, e^{\pi i/6}}{2i\sqrt{3}/2}.$

DEFINE $I_1 = \int_0^\infty \frac{x^{1/4}}{x^2+x+1}\,dx,\ I_2 = \int_0^\infty \frac{x^{1/4}}{x^2-x+1}\,dx$. WE GET 1-COMPLEX EQUATION FOR 2 UNKNOWNS I_1 AND $I_2 \longrightarrow I_1 + e^{i\pi/4} I_2 = \frac{2\pi}{\sqrt{3}} e^{\pi i/6}$.

WE WANT I_1: MULTIPLY BOTH SIDES BY $e^{-i\pi/4} \rightarrow e^{-i\pi/4} I_1 + I_2 = \frac{2\pi}{\sqrt{3}} e^{-i\pi/12}$

NOW TAKE IMAGINARY PARTS: $\mathrm{IM}\left(e^{-i\pi/4} I_1\right) = \frac{2\pi}{\sqrt{3}}\, \mathrm{IM}\left(e^{-i\pi/12}\right)$

THUS $I_1 = (2\pi/\sqrt{3})\left[\sin(\pi/12)/\sin(\pi/4)\right]$

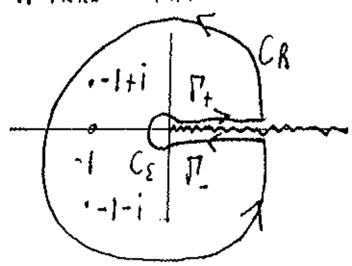
EXAMPLE 7 CALCULATE $I = \int_0^\infty \frac{1}{(x+1)(x^2+2x+2)}\,dx$.

SOLUTION WE CANNOT USE SYMMETRY HERE TO GET $\int_{-\infty}^{\infty}$. THUS, TRY

TO INTEGRATE

$$J = \int_C \frac{\log z}{(z+1)(z^2+2z+2)}$$

WHERE THE CONTOUR IS AS SHOWN



WE CHOOSE THE BRANCH OF $\log z$ WITH BRANCH CUT ON POSITIVE REAL AXIS

$$\log z = \ln|z| + i\varphi \qquad 0 \le \varphi < 2$$

POLES ARE AT $z=-1$ AND $z^2+2z+1=-1 \rightarrow (z+1)^2=-1 \rightarrow z_\pm = -1 \pm i$

CLEARLY $\left|\int_{C_R}\right| \to 0$ AS $R \to \infty$ AND $\left|\int_{C_\varepsilon}\right| \to 0$ AS $\varepsilon \to 0$.

THUS

$$\lim_{R\to\infty}\left(\int_{\Gamma_+} + \int_{\Gamma_-}\right) = 2\pi i\left[\text{RES}\left(\frac{\log z}{(z+1)(z^2+2z+2)}; -1\right) + \text{RES}\left(\frac{\log z}{(z+1)(z^2+2z+2)}; z_+\right) + \text{RES}\left(\frac{\log z}{(z+1)(z^2+2z+2)}; z_-\right)\right]$$

NOW ON Γ_+: $z = x \longrightarrow \int_{\Gamma_+} = \int_0^\infty \frac{\ln x}{(x+1)(x^2+2x+2)}\,dx$

NOW ON Γ_-: $z = xe^{2\pi i} \longrightarrow \int_{\Gamma_-} = \int_\infty^0 \frac{(\ln x + i2\pi)}{(x+1)(x^2+2x+2)}\,dx = -\int_0^\infty \frac{(\ln x + 2\pi i)}{(x+1)(x^2+2x+2)}\,dx.$

THUS ADDING $\int_{\Gamma_+} + \int_{\Gamma_-} \longrightarrow -2\pi i \int_0^\infty \frac{1}{(x+1)(x^2+2x+2)}\,dx = 2\pi i\left[\text{RES}(\ ; -1) + \text{RES}(\ ; z_+) + \text{RES}(\ ; z_-)\right].$ (*)

$\text{RES}(\ ; -1) = \frac{\log(-1)}{1} = i\pi.$ $\quad \text{RES}(\ ; z_+) = \frac{\log(z_+)}{(z_++1)(2z_++2)} = \frac{\log(\sqrt{2}e^{3\pi i/4})}{i(2i)}$

THUS $\text{RES}(\ ; z_+) = -\frac{1}{2}\left(\ln(\sqrt{2}) + 3\pi i/4\right)$. SIMILARLY $\text{RES}(\ ; z_-) = \frac{\log(\sqrt{2}e^{5\pi i/4})}{(z_-+1)(2z_-+1)}$

SO $\text{RES}(\ ; z_-) = -\frac{1}{2}\left(\ln\sqrt{2} + 5\pi i/4\right)$

THUS (*) BECOMES $-2\pi i\, I = 2\pi i\left(i\pi - \frac{1}{2}\ln\sqrt{2} - \frac{3\pi i}{8} - \frac{1}{2}\ln\sqrt{2} - \frac{5\pi i}{8}\right) \rightarrow \underline{\underline{I = -\ln(\sqrt{2})}}$

EXAMPLE 8 (INDENTED CONTOUR)

CALCULATE $\quad I = \displaystyle\int_0^\infty \frac{\sin x}{x}\,dx.$

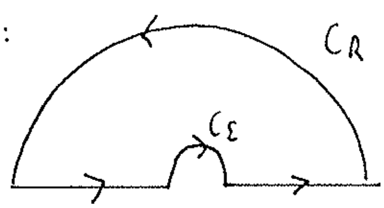
NOTICE $x=0$ IS REMOVABLE SINGULARITY AND $\sin x/x$ IS EVEN FUNCTION.

THUS $\quad I = \dfrac{1}{2}\,\mathrm{Im}\left(\text{P.V.}\displaystyle\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx\right). \quad (*)$

WE NOW CALCULATE $\quad \text{P.V.}\displaystyle\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx.$

WE ENCLOSE IN UPPER 1/2 PLANE AND MAKE INDENTATION NEAR $z=0$

AS SHOWN:



$\displaystyle\int_C \frac{e^{iz}}{z}\,dz = 0$ SINCE NO SINGULARITIES INSIDE CONTOUR.

$C_\varepsilon: \ |z| = \varepsilon \qquad C_R: \ |z| = R.$

EASILY SHOWN $\left|\displaystyle\int_{C_R} e^{iz}/z\,dz\right| \to 0$ AS $R \to \infty$.

THUS AS $\varepsilon \to 0$: $\quad \text{P.V.}\displaystyle\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx + \lim_{\varepsilon\to 0}\int_{C_\varepsilon} \frac{e^{iz}}{z}\,dz = 0.$

HENCE LET $z = \varepsilon e^{i\varphi}$: $\quad \displaystyle\int_{C_\varepsilon} \frac{e^{iz}}{z}\,dz = i\int_{\pi}^{0} e^{i\varepsilon[\cos\varphi + i\sin\varphi]}\,d\varphi$

$dz/z = i \qquad\qquad \longrightarrow -i\pi$ AS $\varepsilon \to 0$.

THUS $\quad \text{P.V.}\displaystyle\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx = -\lim_{\varepsilon\to 0}\int_{C_\varepsilon} \frac{e^{iz}}{z}\,dz = -(-i\pi) = i\pi.$

FROM $(*)$ WE OBTAIN,

$$I = \frac{1}{2}\,\mathrm{Im}\,(i\pi) = \pi/2 = \int_0^\infty \frac{\sin x}{x}\,dx.$$

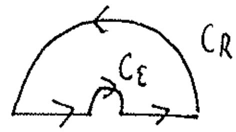
EXAMPLE 8 EXPLAIN HOW TO CALCULATE

(i) $I_1 = \int_0^{\infty} \frac{\sin^2 x}{x^2}\, dx$ (ii) $\int_0^{\infty} \frac{1-\cos x}{x^2}\, dx$. (iii) $\int_{-\infty}^{\infty} \frac{x}{\sinh x}\, dx$

SOLUTION (i) WE WRITE $\sin^2 x = \frac{1}{2}(1-\cos 2x) = \frac{1}{2} \mathrm{Re}\left(1 - e^{2ix}\right)$.

THUS DEFINE $J = \mathrm{Re}\left(\int_{-\infty}^{\infty} \frac{1-e^{2ix}}{x^2}\, dx\right) = 2\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\, dx = 4I.$ $(*)$

WE THEN CALCULATE $\int_C \left(\frac{1-e^{2iz}}{z^2}\right) dz$ OVER



THIS GIVES $\text{P.V.} \int_{-\infty}^{\infty} \left(\frac{1-e^{2ix}}{x^2}\right) dx + \lim_{\varepsilon \to 0} \int_{C_\varepsilon} \frac{1-e^{2iz}}{z^2}\, dz = 0.$

show this

$$\text{P.V. } J + (-2i)(-i\pi) = 0$$

SO $\text{P.V. } J = +2\pi$.

HENCE $(*)$ GIVES $I = \int_0^{\infty} \frac{\sin^2 x}{x^2}\, dx = \pi/2$.

(ii) FOR $\int_0^{\infty} \frac{1-\cos x}{x^2}\, dx = I$

WE WRITE $I = \frac{1}{2}\int_{-\infty}^{\infty} \left(\frac{1-\cos x}{x^2}\right) dx = \frac{1}{2}\mathrm{Re}\left(\int_{-\infty}^{\infty} \frac{1-e^{ix}}{x^2}\, dx\right)$

NOW INTEGRATE $\int_C \left(\frac{1-e^{iz}}{z^2}\right) dz$

(COMPLETED IN THE H.W.)

(iii) NOTE $\sinh(z + \pi i) = -\sinh(z)$ AND $\sinh z = 0 \to z = i\pi, 0 \ldots$

$z = 0$ IS REMOVABLE SINGULARITY FOR $\frac{z}{\sinh z}$.

WE NEED INDENTED BOX SHAPED CONTOUR: