Chapter 2

1. $p = \frac{2}{t}$, $g = \frac{\cos t}{t^2}$

$$\mu = e^{\int p} = e^{\int \frac{2}{t} dt} = t^2$$

$$\int \mu g = \int t^2 \cdot \frac{\cos t}{t^2} dt = \sin t$$

So the general solution is

$$y = \frac{1}{\mu}\left(C + \int \mu g\right) = \frac{1}{t^2}(C + \sin t)$$

$$y(\pi) = 0 \Rightarrow 0 = \frac{1}{\pi^2}(C + 0) \Rightarrow C = 0$$

$$y = \frac{1}{t^2}\sin t$$

Interval of Existence: $0 < t < +\infty$

2. This is homogeneous, $y = xv$

$$xv' + v = \frac{x^2 + x(vx) + v^2x^2}{x^2} = 1 + v + v^2$$

$$xv' = 1 + v^2 \qquad \frac{dv}{1+v^2} = \frac{1}{x}dx$$

$$\arctan v = \ln x + C$$

$$v = \tan(\ln x + C) \qquad y(1) = 1 \Rightarrow \tan C = 1 \Rightarrow C = \frac{\pi}{4}$$

$y = \tan(\ln x + \frac{\pi}{4})$. Interval of Existence:

$$e^{-\frac{3\pi}{4}} < x < e^{\frac{\pi}{4}}$$

3. This is homogeneous.

$y = xv$

$$xv' + v = \frac{4xv - 3x}{2x - xv} = \frac{4v-3}{2-v}$$

$$xv' = \frac{4v-3}{2-v} - v = \frac{4v-3-v(2-v)}{2-v}$$

$$= \frac{v^2+2v-3}{2-v}$$

$$\frac{(2-v)\,dv}{v^2+2v-3} = \frac{1}{x}\,dx$$

$$\frac{2-v}{v^2+2v-3} = \frac{2-v}{(v+3)(v-1)} = \frac{A}{v+3} + \frac{B}{v-1}$$

$$2 - v = A(v-1) + B(v+3)$$

$$A = -\frac{5}{4}, \quad B = \frac{1}{4}$$

$$\int \frac{2-v}{v^2+2v-3}\,dv = -\frac{5}{4}\ln(v+3) + \frac{1}{4}\ln(v-1) = \int \frac{1}{x}\,dx = \ln x + C$$

$$\frac{(v-1)^{\frac{1}{4}}}{(v+3)^{\frac{5}{4}}} = cx \quad \Rightarrow \quad \frac{v-1}{(v+3)^5} = cx^4$$

Now $y(1) = 1 \Rightarrow v(1) = 1 \Rightarrow \quad c = 0$

So the only solution is $\quad v = 1 \Rightarrow y = x.$

4. This is Bernoulli

$$y' + \frac{2}{t} y = \frac{1}{t^2} y^3 \qquad p = \frac{2}{t}, \; g = \frac{1}{t^2}, \; n = 3$$

$$v = y^{1-n} = y^{-2} \Rightarrow v' + (1-n) p v = (1-n) g$$

$$v' - \frac{4}{t} v = -\frac{2}{t^2}$$

$$\mu = e^{-\int \frac{4}{t} dt} = \frac{1}{t^4}$$

$$\int \mu g = \int \frac{1}{t^4} \left(-\frac{2}{t^2}\right) = -\int \frac{2}{t^6} = \frac{2}{5} t^{-5}$$

$$v = \frac{1}{\mu} \left(c + \int \mu g\right) = t^4 \left(c + \frac{2}{5} t^{-5}\right)$$

$$y = \frac{1}{\sqrt{v}} = \frac{1}{\sqrt{t^4 \left(c + \frac{2}{5} t^{-5}\right)}} = \frac{1}{t^2} \cdot \frac{1}{\sqrt{c + \frac{2}{5} t^{-5}}}$$

5. (a) $f(y) = y(y-1)(y-3)$, $f(y) = 0 \Rightarrow$

$y = 0, 1, 3,$

At $y=0$, $f'(0) = 3 > 0 \Rightarrow 0$ is unstable

At $y=1$, $f'(1) = -2 < 0 \Rightarrow 1$ is stable

At $y=3$, $f'(3) = 6 > 0 \Rightarrow 3$ is unstable

(b) critical points: $y=0$, $\cos y = 0 \Rightarrow y = \frac{2n-1}{2}\pi$, $n = 1, 2 \ldots$

At $y=0$, $f'(0) = 1 > 0 \Rightarrow 0$ unstable

At $y = \frac{2n-1}{2}\pi$, $f'(y) = -y \sin y = -\frac{(2n-1)}{2}\pi \sin\left(\frac{2n-1}{2}\pi\right) \begin{cases} > 0, & n \text{ even} \Rightarrow \text{unstable} \\ < 0, & n \text{ odd} \Rightarrow \text{stable} \end{cases}$

(c) $f(y) = (e^y - 1)(y-2)$

$f(y) = 0 \Rightarrow e^y - 1 = 0, \quad y = 2$

$\Rightarrow y = 0 \quad \text{or } y = 2$

At $y=0$, $f'(0) = -2 < 0 \Rightarrow$ 0 is stable

At $y=2$, $f'(2) = e^2 - 1 > 0 \Rightarrow$ 2 is unstable

(d) $\cancel{f(y)} = (2-y)\ln(y+1) = 0 \Rightarrow 2-y = 0$ or $\ln(y+1) = 0$

$\Rightarrow y = 2$ or $y = 0$

At $y=2$, $f'(y) = -\ln 3 < 0 \Rightarrow$ 2 is stable

At $y=0$ $f'(0) = \frac{2}{1} > 0 \Rightarrow$ 0 is unstable

6. separable.

$(2y-5)\,dy = (3x^2 - e^x)\,dx$

$y^2 - 5y = x^3 - e^x + C$

$1 - 5 = 0 - 1 + C \Rightarrow C = -3$

$y^2 - 5y = x^3 - e^x - 3$

$y = \frac{5}{2} \pm \sqrt{x^3 - e^x - 3 + \frac{25}{4}}$

$y = \frac{5}{2} - \sqrt{x^3 - e^x \cancel{+} \frac{13}{4}}$

Interval of existence $\quad x^3 - e^x + \frac{13}{4} > 0$