Chapter 3

1. $r^2+2r+2=0 \Rightarrow r=-1\pm i$

$y = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t$

$y(0)=2 \Rightarrow c_1 = 2$

$y'(0)=1 \Rightarrow -c_1 + c_2 = 1 \Rightarrow c_1 = -1$

2. $y_p = t^s(A\cos t + B\sin t)$

$s=0$

$y_p' = -A\sin t + B\cos t$

$y_p'' = -A\cos t - B\sin t$

$y_p'' + 2y_p' + 2y_p = (-A+2B+2A)\cos t + (-B-2A+2B)\sin t$

$= \cos t$

$\left.\begin{array}{l} 2B + A = 1 \\ B - 2A = 0 \end{array}\right\} \Rightarrow A = \frac{1}{3} \quad B = \frac{2}{3}$

$y = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t + \frac{1}{3}\cos t + \frac{2}{3}\sin t$

$y(0)=0 \Rightarrow c_1 + \frac{1}{3} = 0 \Rightarrow c_1 = -\frac{1}{3}$

$y'(0)=0 \Rightarrow -c_1 + c_2 + \frac{2}{3} = 0 \Rightarrow c_2 = -1$

3. $y_p = t^s(A\cos t + B\sin t)e^{-t} \Rightarrow s=1$

$y_p = t(A\cos t + B\sin t)e^{-t}$

$y_p' = (A\cos t + B\sin t)e^{-t} + t(-A\sin t + B\cos t - A\cos t - B\sin t)e^{-t}$

$$y_p'' = \left(2\,(A+B)\sin t + (B-A)\cos t \right) e^{-t}$$

$$+ \left(-(A+B)\sin t + (B-A)\cos t \right) e^{-t}$$

$$+ \; t \cdots$$

$$y_p'' + 2y_p' + 2y_p = \left(2\,(B-A) + 2A \right) \cos t \, e^{-t}$$

$$+ \left(-2\,(A+B) + 2B \right) \sin t \, e^{-t}$$

$$2(B-A) + 2A = 0 \quad \Rightarrow \quad B = 0$$

$$-2(A+B) + 2B = 1 \quad \Rightarrow \quad A = -\tfrac{1}{2}$$

$$y_p = \tfrac{1}{2} e^{-t} t \cos t$$

$$y = c_1 e^{-t} \cos t + c_2 e^{-t} \sin t + \tfrac{1}{2} t e^{-t} \cos t$$

$$y(0) = 0 \Rightarrow c_1 = 0$$

$$y'(0) = 0 \Rightarrow c_2 = 0$$

$$y = \tfrac{1}{2} t e^{-t} \cos t$$

4. $r^2 + 2r + 1 = 0 \Rightarrow y_1 = e^{-t}, \; y_2 = t e^{-t}$

$$y_p = t^s e^{-t} \Rightarrow s = 2, \quad y_p = A t^2 e^{-t}$$

$$y_p'' + 2y_p' + y_p = 2Ae^{-t} = e^{-t} \Rightarrow A = \tfrac{1}{2}$$

$$y = c_1 e^{-t} + c_2 t e^{-t} + \tfrac{1}{2} t^2 e^{-t}, \quad y(0)=0,\, y'(0)=0 \Rightarrow$$

$$y = \tfrac{1}{2} t^2 e^{-t}$$

5. $r^2+3r+2=0 \Rightarrow r_1=-1,\ r_2=-2$

$y_p = A\cos t + B\sin t + Cte^{-t}$

$\Rightarrow \quad C=1,\ A=\frac{1}{6},\ B=\frac{1}{2}$

$y_p = \frac{1}{6}\cos t + \frac{1}{2}\sin t + te^{-t}$

$y = c_1 e^{-t} + c_2 e^{-2t} + \frac{1}{6}\cos t + \frac{1}{2}\sin t + te^{-t}$

$y(0)=0 \Rightarrow c_1 + c_2 + \frac{1}{6} = 0$
$y'(0)=0 \Rightarrow -c_1 - 2c_2 + \frac{1}{2} + 1 = 0$
$\Big\}\ c_2 = \frac{4}{9},\ c_1 = -\frac{11}{18}$

6. $p = \frac{t\ln t}{2t^2}$

So $W' = -pW \Rightarrow w = e^{-\int p} = e^{-\int \frac{\ln t}{2t}} = ce^{-\frac{1}{2}(\ln t)^2}$

$\int \frac{\ln t}{t} dt = \int (\ln t)\, d(\ln t) = \frac{1}{2}\ln^2 t$

7. $p = \frac{t}{1-t} \Rightarrow W = e^{-\int p\,dt} = e^{-\int \frac{t}{1-t}dt} = e^{+\int \frac{t}{t-1}dt} = e^{t+\ln(t-1)}$
$= (t-1)e^t$

$y_2 = y_1 v \Rightarrow v' = \frac{W}{y_1^2} \Rightarrow v' = \frac{(t-1)e^t}{e^{2t}} = (t-1)e^{-t}$

$v = \int (t-1)e^{-t} = -te^{-t}$

so $y_2 = e^t(-te^{-t}) = -t$

8. $y_1 = \cos 3t$, $y_2 = \sin 3t$

Use method of pariation of parameters

$y = y_1 u_1 + y_2 u_2$

$y_1 u_1' + y_2 u_2' = 0$

$y_1' u_1' + y_2' u_2' = g(t) = 9 \sec^2(3t)$

$\cos 3t \ u_1' + \sin 3t \ u_2' = 0$

$-\sin 3t \ u_1' + \cos 3t \ u_2' = 3 \sec^2(3t)$

$u_2' = 3 \sec 3t$ $\quad u_2 = \int \frac{3}{\cos 3t} dt$

$= \frac{1}{2} \ln \frac{1+\cos 3t}{1-\cos 3t}$

$u_1' = -3 \sin 3t \sec^2 3t$

$u_1 = -3 \int \frac{\sin 3t}{\cos^2 3t} dt = -\frac{1}{\cos 3t}$

9. Eu'er. $y = t^\alpha \Rightarrow$

$\alpha(\alpha-1) - 2\alpha + 2 = 0 \Rightarrow \alpha^2 - 3\alpha + 2 = 0$ $\quad \alpha = 1$ or $\alpha = 2$

So $y_1 = t$, $y_2 = t^2$

10. 1) $W = e^{-\int p}$, $p = -\frac{1+t}{t}$

$W = e^{\int \frac{1+t}{t}} = te^t$

$y_2 = y_1 v$, $v' = \frac{W}{y_1^2} = \frac{te^t}{(1+t)^2}$

So $v = \int \frac{te^t}{(1+t)^2} = -\int (te^t)\, d\left(\frac{1}{1+t}\right)$

$$= -\frac{t}{1+t}e^t + \int \frac{1}{1+t}\, d(te^t)$$

$$= -\frac{t}{1+t}e^t + e^t$$

$$= \frac{1}{1+t}e^t$$

So $y_2 = (1+t)\frac{1}{1+t}e^t = e^t$

$y_1 = 1+t,\ y_2 = e^t$

So, $y_p = (1+t)u_1 + e^t u_2$

$(1+t)u_1' + e^t u_2' = 0$

$u_1' + e^t u_2' = \frac{t^2 e^{2t}}{t} \Rightarrow te^{2t}$

$\Rightarrow tu_1' = -te^{2t} \Rightarrow u_1' = -e^{2t} \quad u_1 = -\frac{1}{2}e^{2t}$

$u_2' = -e^{-t}(1+t)(-e^{2t}) = (1+t)e^t$

$u_2 = te^t$

$y_p = (1+t)(-\frac{1}{2}e^{2t}) + e^t(te^t)$

$y = C_1(1+t) + C_2 e^t + y_p$