

Chapter 7

$$A = \begin{pmatrix} 1 & 2 \\ 5 & 4 \end{pmatrix}, \quad g = \begin{pmatrix} 3t-2 \\ e^{-t} \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}t - \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{-t}$$

eigenvalues $\det(A - \lambda I) = \lambda^2 - 5\lambda - 6 = 0$

$$(\lambda - 6)(\lambda + 1) = 0 \quad \lambda_1 = 6, \quad \lambda_2 = -1$$

$$\lambda_1 = 6 \Rightarrow \begin{pmatrix} -5 & 2 \\ 5 & -2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{matrix} \vdots \\ \vdots \end{matrix}^{(1)} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$\lambda_2 = -1 \quad \begin{pmatrix} 2 & 2 \\ 5 & 5 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{matrix} \vdots \\ \vdots \end{matrix}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

1 $T = \begin{pmatrix} 2 & 1 \\ 5 & -1 \end{pmatrix}$

$$x = Ty$$

$$y' = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} y + T^{-1} \begin{pmatrix} 3t-2 \\ e^{-t} \end{pmatrix}$$

$$\det T = -7 \quad T^{-1} = \frac{1}{-7} \begin{pmatrix} -1 & -1 \\ -5 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{5}{7} & -\frac{2}{7} \end{pmatrix}$$

$$T^{-1} \begin{pmatrix} 3t-2 \\ e^{-t} \end{pmatrix} = \begin{pmatrix} \frac{1}{7}(3t-2) + \frac{1}{7}e^{-t} \\ \frac{5}{7}(3t-2) - \frac{2}{7}e^{-t} \end{pmatrix}$$

$$y_1' = 6y_1 + \frac{1}{7}(3t-2) + \frac{1}{7}e^{-t}$$

$$y_1 = e^{6t} \int e^{-6t} \left(\frac{1}{7}(3t-2) + \frac{1}{7}e^{-t} \right)$$

$$= -\frac{t}{14} + \frac{1}{84} + \frac{1}{21} - \frac{1}{49}e^{-t}$$

$$= -\frac{t}{14} + \frac{1}{84} - \frac{1}{49}e^{-t}$$

$$\boxed{\begin{matrix} y' + ay = g \\ y = e^{-at} \left(ct + \int e^{at} g \right) \end{matrix}}$$

$$y_2' = -y_2 + \frac{5}{7}(3t-2) - \frac{2}{7}e^{-t}$$

$$\begin{aligned} y_2 &= e^{-t} \int e^t \left(\frac{5}{7}(3t-2) - \frac{2}{7}e^{-t} \right) \\ &= e^{-t} \left[\frac{15}{7}(t-1)e^t - \frac{10}{7}e^t - \frac{2}{7}t \right] \\ &= \frac{15}{7}(t-1) - \frac{10}{7} - \frac{2}{7}te^{-t} \end{aligned}$$

$$x = Ty = \begin{pmatrix} 2 & 1 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= \begin{pmatrix} 2y_1 + y_2 \\ 5y_1 - y_2 \end{pmatrix} = \begin{pmatrix} -\frac{t}{7} + \frac{1}{42} - \frac{2}{49}e^{-t} + \frac{15}{7}(t-1) - \frac{10}{7} - \frac{10}{7}te^{-t} \\ -\frac{5}{14}t + \frac{25}{84} - \frac{5}{49}e^{-t} - \frac{15}{7}(t-1) + \frac{10}{7} + \frac{2}{7}te^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} 2t + \frac{1}{42} - \frac{25}{7} - \frac{2}{49}e^{-t} - \frac{10}{7}te^{-t} \\ -\frac{5}{14}t + \frac{5}{84} + \frac{25}{7} - \frac{5}{49}e^{-t} + \frac{2}{7}te^{-t} \end{pmatrix}$$

2. For the first two, we use

$$x_{p1} = \vec{a}t + \vec{b}, \quad x_{p1}' = \vec{a}$$

$$\vec{a} = A\vec{a}t + A\vec{b} + \begin{pmatrix} 3 \\ 0 \end{pmatrix}t - \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$A\vec{a} + \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \vec{0}$$

$$A\vec{b} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} = -\vec{a}$$

$$\text{so } \vec{a} = \begin{pmatrix} 2 \\ -\frac{5}{2} \end{pmatrix} \quad b = \begin{pmatrix} -\frac{5}{6} \\ \frac{5}{12} \end{pmatrix}$$

For the last one

$$x_{p_2} = t\vec{c}e^{-t} + \vec{d}e^{-t}$$

$$x'_{p_2} = -\vec{c}te^{-t} + (\vec{c} - \vec{d})e^{-t}$$

$$Ax_{p_2} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{-t} = x_{p_2}$$

$$A\vec{c} = -\vec{c} \Rightarrow \vec{c} = k \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$A\vec{d} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \vec{c} - \vec{d}$$

$$(A+I)\vec{d} = \vec{c} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 \\ 5 & 5 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = k \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} k \\ -k+1 \end{pmatrix}$$

$$2d_1 + 2d_2 = k$$

$$5d_1 + 5d_2 = -k+1$$

$$5k = 2(-k+1) \quad k = \frac{2}{7}$$

$$d_1 = \frac{1}{7}, \quad d_2 = 0$$

$$x_{p_2} = \frac{2}{7} \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{-t} + \begin{pmatrix} \frac{1}{7} \\ 0 \end{pmatrix} e^{-t}$$

$$\text{So } x_p = x_{p1} + x_{p2}$$

$$= \begin{pmatrix} 2 \\ -\frac{5}{2} \end{pmatrix} t + \begin{pmatrix} -\frac{5}{6} \\ \frac{5}{12} \end{pmatrix} + \begin{pmatrix} \frac{2}{7} \\ -\frac{2}{7} \end{pmatrix} t e^{-t} + \begin{pmatrix} \frac{1}{7} \\ 0 \end{pmatrix} e^{-t}$$

$$x = c_1 \begin{pmatrix} 2 \\ 5 \end{pmatrix} e^{6t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + x_p$$

$$3. \quad x^{(1)} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} e^{6t} \quad x^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$$

$$\Psi(t) = \begin{pmatrix} 2e^{6t} & e^{-t} \\ 5e^{6t} & -e^{-t} \end{pmatrix}$$

$$x_p = \Psi(t) u(t)$$

$$\Psi(t) u'(t) = g(t)$$

$$\begin{pmatrix} 2e^{6t} & e^{-t} \\ 5e^{6t} & -e^{-t} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 3t-2 \\ e^{-t} \end{pmatrix}$$

$$2e^{6t} u_1' + e^{-t} u_2' = 3t-2$$

$$5e^{6t} u_1' - e^{-t} u_2' = e^{-t}$$

$$7e^{6t} u_1' = 3t-2 + e^{-t}$$

$$u_1' = \frac{1}{7}(3t-2) e^{-6t} + \frac{1}{7} e^{-7t}$$

$$u_1 = \frac{3}{7} \int t e^{-6t} - \frac{2}{7} \int e^{-6t} + \frac{1}{7} \int e^{-7t}$$

$$= \frac{3}{7} \left(-\frac{1}{6} t e^{-6t} + \frac{1}{36} e^{-6t} \right) - \frac{2}{7} \left(-\frac{1}{6} \right) e^{-6t} - \frac{1}{49} e^{-7t}$$

$$e^{6t} u_1 = -\frac{1}{14} t - \frac{1}{84} + \frac{1}{21} - \frac{1}{49} e^{-t}$$

Similarly

$$7e^{-t} u_2' = 5(3t-2) - 2e^{-t}$$

$$u_2' = \frac{15}{7} t e^t - \frac{10}{7} e^t - \frac{2}{7}$$

$$u_2 = \frac{15}{7} (t-1) e^t - \frac{10}{7} e^t - \frac{2}{7} t$$

$$e^{-t} u_2 = \frac{15}{7} (t-1) - \frac{10}{7} - \frac{2}{7} t e^{-t}$$

$$x_p = \begin{pmatrix} 2 \left(-\frac{1}{14} t + \frac{1}{28} - \frac{1}{49} e^{-t} \right) + \frac{15}{7} (t-1) - \frac{10}{7} - \frac{2}{7} t e^{-t} \\ 5 \left(-\frac{1}{14} t + \frac{1}{28} - \frac{1}{49} e^{-t} \right) + \frac{15}{7} (t-1) + \frac{10}{7} + \frac{2}{7} t e^{-t} \end{pmatrix}$$

4. Now $\Psi(t) = \begin{pmatrix} 2e^{6t} & e^{-t} \\ 5e^{6t} & -e^{-t} \end{pmatrix}$, $\Psi(0) = \begin{pmatrix} 2 & 1 \\ 5 & -1 \end{pmatrix}$

$$(\Psi(0))^{-1} = \begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{5}{7} & -\frac{2}{7} \end{pmatrix}$$

$$\begin{aligned} \Phi(t) &= \Psi(t) (\Psi(0))^{-1} = \begin{pmatrix} 2e^{6t} & e^{-t} \\ 5e^{6t} & -e^{-t} \end{pmatrix} \begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{5}{7} & -\frac{2}{7} \end{pmatrix} \\ &= \begin{pmatrix} \frac{2e^{6t}}{7} + \frac{5e^{-t}}{7} & \frac{2e^{6t}}{7} - \frac{2e^{-t}}{7} \\ \frac{5e^{6t}}{7} - \frac{5e^{-t}}{7} & \frac{5e^{6t}}{7} + \frac{2e^{-t}}{7} \end{pmatrix} \end{aligned}$$

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$$A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$$

$$\det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{pmatrix} = \lambda^2 - 4 + 5 = \lambda^2 + 1 = 0$$

$$\lambda = \pm i$$

$$\lambda_1 = i \Rightarrow \begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\xi_1 - (2+i)\xi_2 = 0 \quad \xi_2 = 1, \quad \xi_1 = (2+i)$$

$$\xi^{(1)} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$$

$$\text{so } \xi^{(1)} e^{\lambda_1 t} = \begin{pmatrix} (2+i)(\cos t + i \sin t) \\ \cos t + i \sin t \end{pmatrix}$$

$$x^{(1)} = \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix}, \quad x^{(2)} = \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}$$

Use variation of parameter:

$$\Psi(t) = \begin{pmatrix} 2\cos t - \sin t & \cos t + 2\sin t \\ \cos t & \sin t \end{pmatrix}$$

$$x_p = \Psi(t) u(t)$$

$$\begin{pmatrix} 2\cos t - \sin t & \cos t + 2\sin t \\ \cos t & \sin t \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

$$\begin{cases} (2\cos t - \sin t)u_1' + (\cos t + 2\sin t)u_2' = \cos t & \textcircled{1} \\ \cos t u_1' + \sin t u_2' = \sin t & \textcircled{2} \end{cases}$$

$$\textcircled{1} \times \cos t - \textcircled{2} \times (2\cos t - \sin t)$$

$$\left(\cos^2 t + 2\cos t \sin t - \sin t (2\cos t - \sin t) \right) u_2' = \cos^2 t - \sin t (2\cos t - \sin t)$$

$$u_2' = 1 - \sin 2t$$

$$u_2 = t + \frac{1}{2} \cos 2t$$

$$\textcircled{1} \times \sin t - \textcircled{2} \times (\cos t + 2\sin t)$$

$$\left((2\cos t - \sin t) \sin t - \cos t (\cos t + 2\sin t) \right) u_1' = \cos t \sin t - \sin t (\cos t + 2\sin t)$$

$$-u_1' = -2\sin^2 t$$

$$u_1' = 2\sin^2 t = 1 - \cos 2t$$

$$u_1 = t + \frac{1}{2} \sin 2t$$

~~x_p~~

$$x_p = \begin{pmatrix} 2\cos t - \sin t & \cos t + 2\sin t \\ \cos t & \sin t \end{pmatrix} \begin{pmatrix} t + \frac{1}{2} \cos 2t \\ t - \frac{1}{2} \sin 2t \end{pmatrix}$$

$$6. \quad A = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}$$

$$\det(A - \lambda I) = \lambda^2 - 16 + 16 = \lambda^2 = 0.$$

$$\lambda_1 = \lambda_2 = 0. \rightarrow \text{repeated}$$

$$\begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \zeta = 0 \Rightarrow \zeta^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\dot{x}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad x^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \eta$$

$$A\eta = \zeta \Rightarrow \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow \eta = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}$$

$$\Psi(t) = \begin{pmatrix} x^{(1)} & x^{(2)} \end{pmatrix} \\ = \begin{pmatrix} 1 & t \\ 2 & 2t - \frac{1}{2} \end{pmatrix}$$

$$x_p = \Psi(t) u(t)$$

$$\Psi(t) u'(t) = \begin{pmatrix} t^{-3} \\ -t^{-2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & t \\ 2 & 2t - \frac{1}{2} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} t^{-3} \\ -t^{-2} \end{pmatrix}$$

$$u_1' + t u_2' = t^{-3}$$

$$2u_1' + 2t u_2' - \frac{1}{2} u_2' = -t^{-2}$$

$$-\frac{1}{2} u_2' = -t^{-2} - 2t^{-3}$$

$$u_2' = 2t^{-2} + 4t^{-3}$$

$$u_2 = -2t^{-1} - 2t^{-2}$$

$$u_1' = -t u_2' + t^{-3}$$

$$= -t (2t^{-2} + 4t^{-3}) + t^{-3}$$

$$= -2t^{-1} - 4t^{-2} + t^{-3}$$

$$u_1 = -2 \ln t + 4t^{-1} - \frac{1}{2} t^{-2}$$

$$X_p = \begin{pmatrix} 1 & t \\ 2 & 2t - \frac{1}{2} \end{pmatrix} \begin{pmatrix} -2t^{-1} - 2t^{-2} \\ -2 \ln t + 4t^{-1} - \frac{1}{2} t^{-2} \end{pmatrix}$$

$$X = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \left(t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} \right) + X_p$$

7. Following 5

$$-Y(t) = \begin{pmatrix} 2 \cos t & -\sin t & \cos t + 2 \sin t \\ \cos t & & \sin t \end{pmatrix}$$

$$Y(t) u' = \begin{pmatrix} \csc t \\ \sec t \end{pmatrix}$$

$$(2 \cos t - \sin t) u_1' + (\cos t + 2 \sin t) u_2' = \csc t \quad (1)$$

$$(\cos t) u_1' + (\sin t) u_2' = \sec t \quad (2)$$

$$(1) \times \cos t - (2) \times (2 \cos t - \sin t)$$

$$u_2' = \csc t \cos t - \sec t (2 \cos t - \sin t)$$

$$= -\frac{\cos t}{\sin t} - 2 + \frac{\sin t}{\cos t}$$

$$u_2 = -\ln \sin t - 2t - \ln \cos t$$

$$\textcircled{1} \times \sin t - \textcircled{2} \times (\cos t + 2 \sin t) \Rightarrow$$

$$-u_1' = 1 - \sec t (\cos t + 2 \sin t)$$

$$= -\frac{2 \sin t}{\cos t}$$

$$u_1' = -\ln \cos t$$

$$\text{So } X_p = \begin{pmatrix} 2 \cos t - \sin t & \cos t + 2 \sin t \\ \cos t & \sin t \end{pmatrix} \begin{pmatrix} -\ln \cos t \\ -\ln \sin t - \ln \cos t - 2t \end{pmatrix}$$

$$X = C_1 \begin{pmatrix} 2 \cos t - \sin t \\ \cos t \end{pmatrix} + C_2 \begin{pmatrix} \cos t + 2 \sin t \\ \sin t \end{pmatrix} + X_p$$

$$8. \quad W' = \text{trace}(P(t))W = (t + \ln t)W$$

$$W = e^{\int (t + \ln t) dt} = C e^{\frac{t^2}{2} + t \ln t - t}$$