

# chapter 10

$$1 \text{ (a). } f(x) \sim \frac{a_0}{2} + \sum (a_n \cos \frac{n\pi}{L}x + b_n \sin \frac{n\pi}{L}x)$$

$$\text{Here } 2L = 2\pi \Rightarrow L = \pi.$$

$$\text{So } a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi}{L}x dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$= \frac{1}{\pi} \int_{-\pi}^0 x \cos nx dx.$$

$$= \frac{1}{\pi} \left( \frac{1}{n} x \sin nx + \frac{1}{n^2} \cos nx \Big|_{-\pi}^0 \right)$$

$$= \frac{1}{n^2\pi} (1 - \cos n\pi) = \frac{1 - (-1)^n}{n^2\pi}$$

$$\begin{aligned} \int x \sin \beta x dx &= -\frac{1}{\beta} x \cos \beta x + \frac{1}{\beta^2} \sin \beta x \\ \int x \cos \beta x dx &= \frac{1}{\beta} x \sin \beta x + \frac{1}{\beta^2} \cos \beta x \end{aligned}$$

~~$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi}{L}x dx$$~~

$$a_0 = \frac{1}{L} \int_{-L}^L f(x) dx = \frac{1}{\pi} \int_{-\pi}^0 x dx = \frac{\pi}{2}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^0 x \sin nx dx = \frac{1}{\pi} \left( -\frac{1}{n} x \cos \beta x + \frac{1}{n^2} \sin nx \Big|_{-\pi}^0 \right)$$

$$= -\frac{1}{n\pi} (0 - (-\pi) \cos n\pi) = -\frac{(-1)^n}{n}$$

$$\text{So } f(x) \sim \frac{\pi}{4} + \sum_{n=1}^{+\infty} \left( \frac{1 - (-1)^n}{n^2\pi} \cos nx - \frac{(-1)^n}{n} \sin nx \right)$$

(b)  $f$  is odd,  $2L=4 \Rightarrow L=2$

$$\text{So } f(x) \sim \sum_{n=1}^{+\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

$$\sim \sum_{n=1}^{+\infty} b_n \sin\left(\frac{n\pi}{2}x\right)$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{2}x\right) dx$$

$$= \int_0^2 f(x) \sin\left(\frac{n\pi}{2}x\right) dx$$

$$= \int_0^1 x \sin\left(\frac{n\pi}{2}x\right) dx$$

$$= -\frac{2}{n\pi} x \cos\frac{n\pi}{2}x + \frac{4}{n^2\pi^2} \sin\left(\frac{n\pi}{2}x\right) \Big|_0^1$$

$$= -\frac{2}{n\pi} \cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)$$

$$f(x) \sim \sum_{n=1}^{+\infty} \left( -\frac{2}{n\pi} \cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \right) \sin\left(\frac{n\pi}{2}x\right)$$

(c)  $2L=4 \Rightarrow L=2$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{+\infty} \left( a_n \cos\frac{n\pi}{2}x + b_n \sin\left(\frac{n\pi}{2}x\right) \right)$$

$$a_0 = \frac{1}{2} \int_{-2}^2 f(x) dx = \frac{1}{2} \left( \int_{-2}^0 (x+2) + \int_0^2 (2-2x) \right)$$

$$= 1$$

$$a_n = \frac{1}{2} \int_{-2}^2 f(x) \cos \frac{n\pi}{2} x dx$$

$$= \frac{1}{2} \left[ \int_{-2}^0 (x+2) \cos \frac{n\pi}{2} x dx + \int_0^2 (2-2x) \cos \frac{n\pi}{2} x dx \right]$$

$$= \frac{1}{2} \left[ \int_{-2}^0 x \cos \frac{n\pi}{2} x dx + \int_0^2 (-2x) \cos \frac{n\pi}{2} x dx \right]$$

$$= -\frac{3}{2} \int_0^2 x \cos \frac{n\pi}{2} x dx$$

$$= -\frac{3}{2} \left[ \frac{x}{n\pi} \sin \frac{n\pi}{2} x + \frac{4}{n^2\pi^2} \cos \frac{n\pi}{2} x \right] \Big|_0^2$$

$$= -\frac{3}{2} \times \frac{4}{n^2\pi^2} [\cos n\pi - 1]$$

$$= \frac{6}{n^2\pi^2} (1 - (-1)^n)$$

$$b_n = \frac{1}{2} \int_{-2}^2 f(x) \sin \left( \frac{n\pi}{2} x \right) dx$$

$$= \frac{1}{2} \left[ \int_{-2}^0 x \sin \frac{n\pi}{2} x dx + \int_0^2 (-2x) \sin \left( \frac{n\pi}{2} x \right) dx \right]$$

$$= -\frac{1}{2} \int_0^2 x \sin \left( \frac{n\pi}{2} x \right) dx$$

$$= -\frac{1}{2} \left[ -\frac{2}{n\pi} x \cos \frac{n\pi}{2} x + \frac{4}{n^2\pi^2} \sin \frac{n\pi}{2} x \right] \Big|_0^2$$

$$= \frac{1}{n\pi} (2 \cos n\pi) = \frac{(-1)^n}{n\pi}$$

2. Recall:

$$\frac{a_0}{2} + \sum_{n=1}^{+\infty} (a_n \cos(\frac{n\pi}{L}x) + b_n \sin(\frac{n\pi}{L}x)) = \frac{1}{2}(f(x+) + f(x-))$$

$$x = -\frac{\pi}{2}, \quad f(-\frac{\pi}{2}+) = -\frac{\pi}{2}, \quad f(-\frac{\pi}{2}-) = -\frac{\pi}{2}$$

$$\text{So } \frac{1}{2}(f(-\frac{\pi}{2}+) + f(-\frac{\pi}{2}-)) = -\frac{\pi}{2}$$

$$x = 0, \quad f(0+) = 0, \quad f(0-) = 0.$$

$$\text{So } \frac{1}{2}(f(0+) + f(0-)) = 0.$$

$$x = \frac{\pi}{2}, \quad f(\frac{\pi}{2}+) = 0, \quad f(\frac{\pi}{2}-) = 0$$

$$\frac{1}{2}(f(\frac{\pi}{2}+) + f(\frac{\pi}{2}-)) = 0$$

3. We have Dirichlet BC,  $L = 2\pi$ .

$$\text{So } u(x,t) \approx \sum_{n=1}^{+\infty} b_n e^{-\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right) = \sum_{n=1}^{+\infty} b_n \sin\left(\frac{n}{2}x\right) e^{-\left(\frac{n}{2}\right)^2 t}$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx = \frac{2}{2\pi} \int_0^{2\pi} f(x) \sin\left(\frac{n}{2}x\right) dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} (\sin x - \sin 4x) \sin\left(\frac{n}{2}x\right) dx$$

$$= \begin{cases} 1 & , & n=2 \\ -1 & , & n=8 \\ 0 & , & n=2,8 \end{cases}$$

$$\text{So } u(x,t) = e^{-t} \sin x - e^{-16t} \sin 4x$$

$$4. \begin{cases} u_{tt} = u_{xx}, & 0 < x < \pi \\ u(x, 0) = \cos 2x, & u_t(x, 0) = \cos^2 x \\ u_x(0, t) = 0, & u_x(\pi, t) = 0 \end{cases}$$

BC is Neumann BC

$$\text{EVP: } X'' + \lambda X = 0, \quad X'(0) = 0, \quad X'(\pi) = 0$$

$$\lambda = \lambda_n = \left(\frac{n\pi}{L}\right)^2 = \left(\frac{n\pi}{\pi}\right)^2 = n^2, \quad n = 0, 1, 2, \dots$$

$$X_n = \cos\left(\frac{n\pi}{L}x\right) = \cos nx$$

$$\begin{aligned} \cancel{u(x, t)} \\ T'' + \lambda_n T = 0 &\Rightarrow T'' + n^2 T = 0 \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \begin{array}{l} n=0 \\ T = a + b t \end{array} \\ \Rightarrow T_n = a \cos nt + b \sin nt \end{aligned}$$

$$u(x, t) = \frac{a_0 + b_0 t}{2} + \sum_{n=1}^{+\infty} (a_n \cos nt + b_n \sin nt) \cos(nx)$$

$$\text{IC: } u(x, 0) = \cos 2x \Rightarrow$$

$$\frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n \cos nx = \cos 2x$$

$$a_n = \frac{2}{L} \int_0^L (\cos 2x) \cos nx \, dx = \frac{2}{\pi} \int_0^{\pi} (\cos 2x) \cos nx \, dx$$

$$= \begin{cases} 0, & \text{if } n \neq 2 \\ 1, & \text{if } n = 2 \end{cases}$$

$$u_t(x, 0) = \cos^2 x$$

$$= \frac{1 + \cos 2x}{2} = \frac{1}{2} - \frac{1}{2} \cos 2x$$

$$u_t(x, 0) = \textcircled{2} \frac{b_0}{2} + \sum_{n=1}^{+\infty} (n b_n) \cos nx$$

$$b_0 = \frac{2}{L} \int_0^L f(x) dx = \frac{2}{\pi} \int_0^{\pi} \left( \frac{1}{2} - \frac{1}{2} \cos 2x \right) dx$$

$$= 1$$

$$n b_n = \frac{2}{\pi} \int_0^{\pi} g(x) \cos nx dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \left( \frac{1}{2} - \frac{1}{2} \cos 2x \right) \cos nx dx$$

$$= -\frac{2}{\pi} \quad \text{if } n=2.$$

$$0 \quad \text{if } n \neq 2$$

$$\Rightarrow 2b_2 = -\frac{2}{\pi} \Rightarrow b_2 = -\frac{1}{\pi}$$

$$b_n = 0 \quad \text{if } n \neq 2$$

$$u(x, t) = a_2 \cos 2t \cos 2x + \frac{t}{2} + b_2 \sin 2t \sin 2x$$

$$= \cos 2t \cos 2x + \frac{t}{2} - \frac{1}{\pi} \sin 2t \sin 2x$$

$$5. \text{ (EVP) } \begin{cases} X'' + \lambda X = 0 \\ X'(0) = 0, X(\pi) = 0 \end{cases}$$

$$\lambda_n = \left( \frac{2n-1}{2\pi} \pi \right)^2 = \left( n - \frac{1}{2} \right)^2, n=1, 2, \dots$$

$$X_n = \cos \left( n - \frac{1}{2} \right) x$$

$$T' + \left( n - \frac{1}{2} \right)^2 T = 0 \quad T_n = a e^{-\left( n - \frac{1}{2} \right)^2 t}$$

$$u(x, t) = \sum_{n=1}^{+\infty} a_n e^{-\left( n - \frac{1}{2} \right)^2 t} \cos \left( n - \frac{1}{2} \right) x$$

$$u(x, 0) = \sin^2 x = \frac{1 - \cos 2x}{2} = \sum_{n=1}^{+\infty} a_n \cos \left( n - \frac{1}{2} \right) x$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} \left( \frac{1 - \cos 2x}{2} \right) \cos \left( n - \frac{1}{2} \right) x dx$$

$$= \frac{2}{\pi} \left[ \frac{\sin \left( n - \frac{1}{2} \right) \pi}{2 \left( n - \frac{1}{2} \right)} - \frac{1}{4} \int_0^{\pi} \left( \cos \left( n + \frac{3}{2} \right) x + \cos \left( n - \frac{5}{2} \right) x \right) dx \right]$$

$$= \frac{2}{\pi} \left[ \frac{(-1)^{n+1}}{2 \left( n - \frac{1}{2} \right)} - \frac{\sin \left( n + \frac{3}{2} \right) \pi}{4 \left( n + \frac{3}{2} \right)} - \frac{\sin \left( n - \frac{5}{2} \right) \pi}{4 \left( n - \frac{5}{2} \right)} \right]$$

$$= \frac{2}{\pi} \left[ \frac{(-1)^{n+1}}{2 \left( n - \frac{1}{2} \right)} - \frac{(-1)^n}{4 \left( n + \frac{3}{2} \right)} - \frac{(-1)^{n-1}}{4 \left( n - \frac{5}{2} \right)} \right]$$